ODE

1 Lyapunov Stability

In this chapter, we introduce the ODE stability concepts and useful criterions to check them.

Definition 1.1. ***(Autonomous Ordinary Differential Equation and Equilibrium Point)***

$$\dot{x} = f(x) \quad x \in \mathbb{R}^n. \tag{1.2}$$

where ***equilibrium point*** $x_{\mathrm{eq}} \in \mathbb{R}^n$ *is a solution to the ODE which satisfy* $f\left(x_{\mathrm{eq}}\right) = 0$ *and such* $x(t) = x_{\mathrm{eq}}, \forall\, t \geq 0$ *.*

1.1 Stability

1.1.1 $\epsilon - \delta$ Definition

Definition 1.3. *The equilibrium point* $x_{\mathrm{eq}} \in \mathbb{R}^n$ *is* ***(Lyapunov) stable****if*

$$\forall\, \epsilon > 0, \exists\, \delta > 0 : \left\|x\left(t_0\right) - x_{\mathrm{eq}}\right\| \leq \delta \Longrightarrow \left\|x(t) - x_{\mathrm{eq}}\right\| \leq \epsilon, \forall\, t \geq t_0 \geq 0$$

Remark 1.4. ***(explanation)*** *See also Fig. 1 for the visualization.*

1. *If the solution starts close to* x_{eq} *it will remain close to it forever.*
2. ϵ *can be made arbitrarily small by choosing* δ *sufficiently small.*

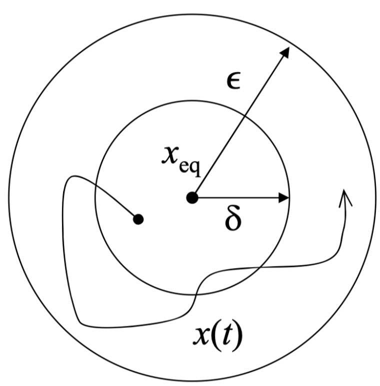


Fig. 1: $\epsilon - \delta$ definition of stability.

1.1.2 Continuity Definition

Definition 1.5. *We first list several concepts:*

- $\mathcal{X}_{sig} \equiv$ *set of all piecewise continuous signals taking values in* $\mathbb{R}^n$
- *Signal norm: Given a signal* $x \in \mathcal{X}_{\mathrm{sig}}$, $\|x\|_{\mathrm{sig}} := \sup_{t \geq 0} \|x(t)\|$

- *ODE can be seen as an operator* $\mathrm{T}: \mathbb{R}^n \to \mathcal{X}_{\mathrm{sig}}$ *that maps* $x_0 \in \mathbb{R}^n$ *into the solution that starts at* $x(0) = x_0$

The equilibrium point $x_{\mathrm{eq}} \in \mathbb{R}^n$ *is* ***(Lyapunov) stable*** *if* T *is continuous at* x_{eq} *:*

$$\forall \epsilon > 0, \exists \delta > 0 : \|x_0 - x_{\mathrm{eq}}\| \leq \delta \Longrightarrow \underbrace{\|\mathrm{T}(x_0) - \mathrm{T}(x_{\mathrm{eq}})\|_{\mathrm{sig}} \leq \epsilon}_{\sup_{t \geq 0}\|x(t) - x_{\mathrm{eq}}\| \leq \epsilon}$$

Remark 1.6. ***(stability of arbitrary solutions)*** *It can be extended to* ***nonequilibrium solutions*** *as shown below in Fig. 2 and defined below:*

A solution x^* *is* ***(Lyapunov) stable*** *if* T *is continuous at* ${x^*}_0 := x^*(0)$*, i.e.,*

$$\forall \epsilon > 0, \exists \delta > 0 : \|x_0 - {x^*}_0\| \leq \delta \Longrightarrow \underbrace{\|\mathrm{T}(x_0) - \mathrm{T}(x_0^*)\|_{\mathrm{sig}} \leq \epsilon}_{\sup_{t \geq 0}\|x(t) - x^*(t)\| \leq \epsilon}$$

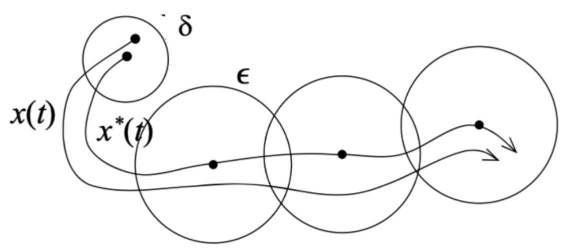


Fig. 2: $\epsilon - \delta$ definition of stability.

1.1.3 Class $\mathcal{K}$ Function Definition

We first give the definition of class $\mathcal{K}$ function.

Definition 1.7. ***(Class*** $\mathcal{K}$ ***Function)*** *Class* $\mathcal{K} \equiv$ set *of functions* $\alpha : [0, \infty) \to [0, \infty)$ *that are*

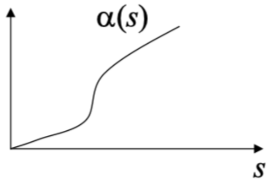
1. *continuous*
2. *strictly increasing*
3. $\alpha(0) = 0$

Definition 1.8. *The equilibrium point* $x_{\mathrm{eq}} \in \mathbb{R}^n$ *is* ***(Lyapunov) stable*** *if*

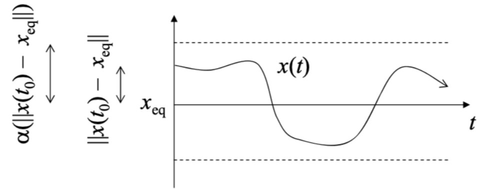
$$\exists\, \alpha \in \mathcal{K}, \|x(t) - x_{\mathrm{eq}}\| \leq \alpha\left(\|x(t_0) - x_{\mathrm{eq}}\|\right), \forall\, t \geq t_0 \geq 0 \text{ and } \|x(t_0) - x_{\mathrm{eq}}\| \leq \mathrm{c}$$

Remark 1.9. ***(some explanation)***

1. *Strictly increasing is because if* $c_1 < c_2$*, we have* $\alpha(c_1) < \alpha(c_2)$ *to show the region of* $x(t)$ *should be strictly smaller with starting points* x_0 *which has smaller distance from* x_{eq}*. Think in a subset way.*
2. *Continuous is because ode is about integration and* x *is continuous? (Not sure. Need to look back again). Why* α *cannot be piece wise continuous with some discontinuous points? It seems intuitive and reasonable but need some proof.*
3. $\alpha(0) = 0$ *is for equilibrium.*
4. *The function* α *can be constructed directly from the* $\delta(\epsilon)$ *in the* $\epsilon - \delta$ *(or continuity) definitions.*



(a) function α



1.2 Asymptotic Stability

Asymptotic stability is stronger than stability since we need it to converge to 0 when $t \to \infty$. For simplicity, here we care about the globally asymptotically stable instead of the local one.

1.2.1 Convergence Definition

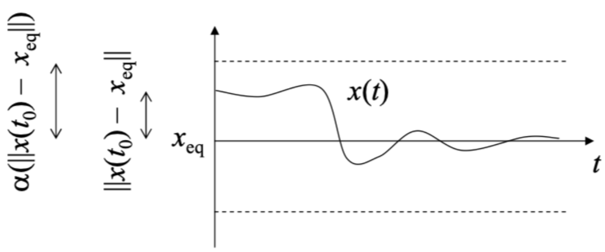


Fig. 4: asymptotically stable.

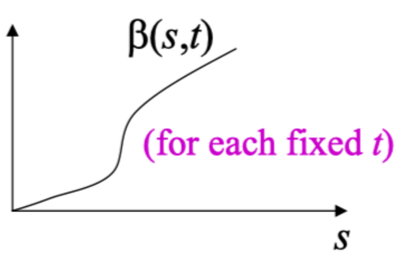
Definition 1.10. *The equilibrium point $x_{\text{eq}} \in \mathbb{R}^n$ is (globally) asymptotically stable if it is Lyapunov stable and for every initial state the solution exists on $[0, \infty)$ and*

$$x(t) \to x_e \text{ as } t \to \infty.$$

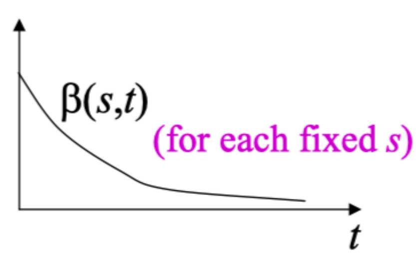
1.2.2 Class $\mathcal{KL}$ Function Definition

Definition 1.11. ***(Class $\mathcal{KL}$ Function)*** *Class $\mathcal{K}L \equiv$ set of functions $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ s.t.*

1. *For each fixed t, $\beta(\cdot, t) \in \mathcal{K}$,*
2. *For each fixed s, $\beta(s, \cdot)$ is monotone decreasing and $\beta(s, t) \to 0$ as $t \to \infty$*



(a) function β with fixed t



(b) function β with fixed s

Definition 1.12. *The equilibrium point $x_{\text{eq}} \in \mathbb{R}^n$ is **(globally) asymptotically stable** if $\exists\, \beta \in \mathcal{KL}$:*

$$\|x(t) - x\| < \beta(\|x(t) - x\|, t - t_0), \forall\, t \geq t_0 \geq 0$$

Remark 1.13. ***(exponential stability)*** *We have exponential stability when $\beta(s,t) = ce^{-\lambda t}s$ with $c, \lambda > 0$. It is linear in s and negative exponential in t.*

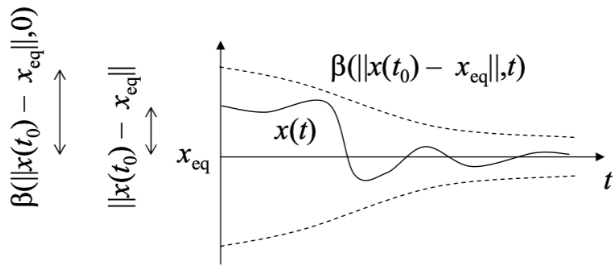


Fig. 6: asymptotically stable using $\mathcal{KL}$ function class.

1.3 Lyapunov's Stability Theorem

Suppose we could show that $\|x(t) - x_{\text{eq}}\|$ always decreases along solutions to the ODE. Then $\|x(t) - x_{\text{eq}}\| \leq \|x(t_0) - x_{\text{eq}}\| \quad \forall t \geq t_0 \geq 0$, i.e. we could pick $\alpha(s) = s \Rightarrow$ Lyapunov stability. We can draw the same conclusion by using other measures of how far the solution is from x_{eq}.

• **Lyapunov Function Is a Distance Measure**

Definition 1.14. ***(Positive Definite*** V***)*** *$V : \mathbb{R}^n \to \mathbb{R}$* ***positive definite****, i.e. $V(x) \geq 0, \forall x \in \mathbb{R}^n$ with $= 0$ only for $x = 0$:*

$$V\left(x - x_{\text{eq}}\right) \quad \begin{cases} = 0 & x = x_{\text{eq}} \\ > 0 & x \neq x_{\text{eq}} \end{cases}$$

Remark 1.15. *It provides a measure of how far x is from x_{eq} (not necessarily a metric-may not satisfy triangular inequality). Note V is not only* ***positive semi-definite*** *(avoid $V(x) = 0$ for some $x \neq 0$).*

Definition 1.16. ***(Radially Unbounded*** V***)*** *We call $V : \mathbb{R}^n \to \mathbb{R}$ is radially unbounded, if additionally $x \to \infty \Rightarrow V(x) \to \infty$:*

$$V\left(x - x_{\text{eq}}\right) \quad \begin{cases} = 0 & x = x_{\text{eq}} \\ > 0 & x \neq x_{\text{eq}} \\ \to \infty & \|x - x_{\text{eq}}\| \to \infty \end{cases}$$

Remark 1.17. ***(explanation and gradient of*** V***)***

Q: How to check if $V\left(x(t) - x_{eq}\right)$ decreases along solutions?

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V\left(x(t) - x_{\text{eq}}\right) &= \frac{\partial V}{\partial x}\left(x(t) - x_{\text{eq}}\right) \dot{x}(t) \\ &= \frac{\partial V}{\partial x}\left(x(t) - x_{\text{eq}}\right) f(x(t)) \end{aligned}$$

$A : V\left(x(t) - x_{eq}\right)$ will decrease if gradient of V

$$\frac{\partial V}{\partial x}\left(z - x_{\text{eq}}\right) f(z) \leq 0 \quad \forall z \in \mathbb{R}^n$$

Note here it can be computed without actually computing the solution $x(t)$ of the ODE.

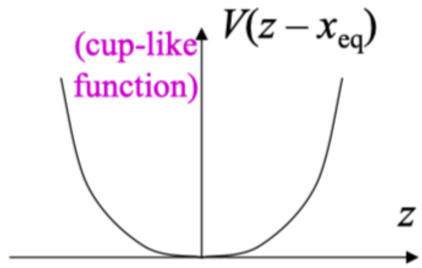


Fig. 7: cup-like function V.

Theorem 1.18. ***(Lyapunov Stable Theorem I)*** *Suppose there exists a continuously differentiable, positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\frac{\partial V}{\partial x}\left(z - x_{\text{eq}}\right) f(z) \leq 0 \quad \forall z \in \mathbb{R}^n$$

Then x_{eq} is a ***Lyapunov stable equilibrium.***

Proof. V non increasing $\Rightarrow V\left(x(t) - x_{\text{eq}}\right) \leq V\left(x\left(t_0\right) - x_{\text{eq}}\right) \forall t \geq t_0$ Thus, by making $x\left(t_0\right) - x_{\text{eq}}$ small we can make $V\left(x(t) - x_{\text{eq}}\right)$ arbitrarily small $\forall t \geq t_0$ So, by making $x\left(t_0\right) - x_{\text{eq}}$ small we can make $x(t) - x_{\text{eq}}$ arbitrarily small $\forall t \geq t_0$. We can actually compute α from V explicitly and take $c = +\infty$ using Definition 1.8. □

Theorem 1.19. ***(Lyapunov Stable Theorem II)*** *Suppose there exists a continuously differentiable, positive definite,* ***radially unbounded function*** $V : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\frac{\partial V}{\partial x}\left(z - x_{\text{eq}}\right) f(z) \leq 0 \quad \forall z \in \mathbb{R}^n$$

Then x_{eq} is a Lyapunov stable equilibrium and ***the solution always exists globally****. Moreover, if $= 0$ only for $z = x_{\text{eq}}$ then x_{eq} is a globally* ***asymptotically stable equilibrium.***

Proof. V can only stop decreasing when $x(t)$ reaches x_{eq}, but V must stop decreasing because it cannot become negative Thus, $x(t)$ must converge to x_{eq}. See "Laypunov stable_2.pdf" for details, where we also need to use compactness from radially unboundness to prove global stable. □

Theorem 1.20. ***(Lyapunov Stable Theorem: A Summary)***

1. *The equilibrium point is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite (i.e. $\dot{V}(x) \leq 0$).*
2. *It is (not necessarily globally) asymptotically stable if $\dot{V}(x)$ is negative definite (i.e. $\dot{V}(x) = 0$ only if $x = 0$).*
3. *It is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded.*

Proof. See supp Laypunov stable_2.pdf where the compact and continuous of V is used. □

1.4 LaSalle's Invariance Principle

What if

$$\frac{\partial V}{\partial x}\left(z - x_{\text{eq}}\right) f(z) = 0 \quad \forall z \in \mathbb{R}^n$$

for other z than x_{eq} ? Can we still claim some form of convergence?

Definition 1.21. ***(Invariant Set)*** *We still consider* (1.2)*: $\dot{x} = f(x) \quad x \in \mathbb{R}^n$. We call $M \in \mathbb{R}^n$ is an* ***invariant set*** $\equiv x\left(t_0\right) \in M \Rightarrow x(t) \in M, \forall t \geq t_0$

Theorem 1.22. ***(LaSalle Invariance Principle)*** *Suppose there exists a continuously differentiable, positive definite,* ***radially unbounded*** *function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f(z) \le W(z) \le 0 \quad \forall z \in \mathbb{R}^n$$

Then x_{eq} is a Lyapunov stable equilibrium and the solution always exists globally. Moreover, $x(t)$ converges to the largest invariant set M contained in

$$E := \{z \in \mathbb{R}^{\text{n}} : W(z) = 0\}$$

Remark 1.23. ***(compare to Lyapunov Theorem)*** *Note that:*

1. *When $W(z) = 0$ only for $z = x_{\text{eq}}$ then $E = \{x_{\text{eq}}\}$. Since $M \subset E, M = \{x_{\text{eq}}\}$ and therefore $x(t) \to x_{eq} \Rightarrow$ asympt. stability. That means Lyapunov Theorem is a speical case LaSalle Invariance Principle.*
2. *Even when E is larger then $\{x_{\text{eq}}\}$ we often have $M = \{x_{\text{eq}}\}$ and can conclude asymptotic stability.*

1.5 Linear systems

Consider the following linear ODE:

$$\dot{x} = Ax \quad x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$$

We have the solution:

$$x(t) = e^{A(t-t_0)} x(t_0) \quad t \ge t_0 \quad e^{A\tau} := \sum_{k=0}^{\infty} \frac{\tau^k}{k!} A^k$$

Theorem 1.24. *The origin $x_{\text{eq}} = 0$ is an equilibrium point. It is*

- ***Lyapunov stable*** *if and only if all eigenvalues of A have negative or zero real parts and for each eigenvalue with zero real part there is an independent eigenvector.*
- ***Asymptotically stable*** *if and only if all eigenvalues of A have negative real parts. In this case the origin is actually exponentially stable*

Theorem 1.25. *The origin $x_{\text{eq}} = 0$ is an equilibrium point. It is asymptotically stable if and only if for every positive symmetric definite matrix Q the* ***Lyapunov equation***

$$A'P + PA = -Q$$

has a unique solutions P that is symmetric and positive definite.

Proof. We have

1. asympt. stable $\Rightarrow$ P exists and is unique (constructive proof)

$$P := \lim_{T\to\infty} \int_0^T e^{A'\tau} Q e^{A\tau} d\tau$$
$$= \lim_{T\to\infty} \int_0^T e^{A'(T-s)} Q e^{A(T-s)} ds$$

A is asympt. stable $\Rightarrow$ e^{At} decreases to zero exponentiall fast $\Rightarrow$ *P* is well defined (limit exists and is finite)

change of integration variable $\tau = T - s$

2. *P* exists $\Rightarrow$ asymp. stable
   Consider the quadratic Lyapunov equation: $V(x) = x' P x$
   V is positive definite & radially unbounded because *P* is positive definite
   V is continuously differentiable: $\frac{\partial V}{\partial x}(x) = 2x'P$

$$\frac{\partial V}{\partial x}(x)Ax = x'(A'P + PA)x = -x'Qx < 0 \quad \forall x \ne 0$$

thus system is asymptotically stable by Lyapunov Theorem

$\square$

2 Basic Theorem

2.1 Basic Thorems from Real Analysis

2.1.1 Basic Concepts

"Function" below generally means a map from some specified set of a vector space $\mathbf{R}^e$ into a space $\mathbf{R}^d$, not always of the same dimension. $\mathbf{R}^d$ denotes a normed, real d-dimensional vector space of elements $y=\left(y^1, \ldots, y^d\right)$ with norm $|y|$. Unless otherwise specified, $|y|$ will be the norm

$$|y|=\max \left(\left|y^1\right|, \ldots,\left|y^d\right|\right)$$

and $\|y\|$ the Euclidean norm.

If y_0 is a point and E a subset of $\mathbf{R}^d$, then dist $\left(y_0, E\right)$, the distance from y_0 to E, is defined to be inf $\left|y_0-y\right|$ for $y \in E$. If E_1, E_2 are two subsets of $\mathbf{R}^d$, then dist $\left(E_1, E_2\right)$ is defined to be inf $\left|y_1-y_2\right|$ for $y_1 \in E_1, y_2 \in E_2$, and is called the distance between E_1 and E_2. If E_1 (or E_2) is compact and E_1, E_2 are closed and disjoint, then dist $\left(E_1, E_2\right)>0$.

If E is an open set or a closed parallelepiped in $\mathbf{R}^d$, $f \in C^n(E), 0 \leq n<\infty$, means that $f(y)$ is continuous on E and that the components of f have continuous partial derivatives of all orders $k \leq n$ with respect to $y^1, \ldots, y^d$. A function $f(y, z)=f\left(y^1, \ldots, y^d, z^1, \ldots, z^e\right)$ defined on a (y, z)-set E, where $y \in \mathbf{R}^d$, is said to be uniformly Lipschitz continuous on E with respect to y if there exists a constant K satisfying

$$\left|f\left(y_1, z\right)-f\left(y_2, z\right)\right| \leq K\left|y_1-y_2\right| \quad \text{for all} \quad \left(y_j, z\right) \in E$$

with $j=1,2$. Any constant K satisfying (3.5) is called a Lipschitz constant (for f on E). (The admissible values of K depend, of course, on the norms in the f - and y-spaces.)

A family F of functions $f(y)$ defined on some y-set $E \subset \mathbf{R}^d$ is said to be equicontinuous if, for every $\epsilon>0$, there exists a $\delta=\delta_\epsilon>0$ such that $\left|f\left(y_1\right)-f\left(y_2\right)\right| \leq \epsilon$ whenever $y_1, y_2 \in E,\left|y_1-y_2\right| \leq \delta$ and $f \in F$. The point of this definition is that δ_ϵ does not depend on f but is admissible for all $f \in F$. The most frequently encountered equicontinuous families F below will occur when all $f \in F$ are uniformly Lipschitz continuous on E and there exists a $K>0$ which is a Lipschitz constant for all $f \in F$; in which case, δ can be chosen to be $\delta=\epsilon / K$.

Lemma 2.1. *If a sequence of continuous functions on a compact set E is uniformly convergent on E, then it is uniformly bounded and equicontinuous.*

2.1.2 Cantor Selection

Theorem 2.2. ***(Cantor Selection)*** *Let $f_1(y), f_2(y), \ldots$ be a uniformly bounded sequence of functions on a y-set E. Then for any countable set $D \subset E$, there exists a subsequence $f_{n(1)}(y), f_{n(2)}(y), \ldots$ convergent on D.*

2.1.3 Ascoli

Theorem 2.3. ***(Ascoli)*** *On a compact y-set E, let $f_1(y), f_2(y), \ldots$ be a sequence of functions which is equicontinuous and convergent on a dense subset of E. Then $f_1(y), f_2(y), \ldots$ converges uniformly on E.*

2.1.4 Arzela

Theorem 2.4. ***(Arzela)*** *On a compact y-set $E \subset \mathbf{R}^d$, let $f_1(y), f_2(y), \ldots$ be a sequence of functions which is pointwise bounded and equicontinuous.*

1. *They are uniformly bounded.*
2. *Then there exists a subsequence $f_{n(1)}(y), f_{n(2)}(y), \ldots$ which is uniformly convergent on E.*

The follwing remarks are also import:

Remark 2.5. *If, in Theorem 2.4 $y_0 \in E$ and f_0 is a limit point of the sequence $f_1(y_0), f_2(y_0), \ldots$, then the subsequence $f_{n(1)}(y), f_{n(2)}(y), \ldots$ in the assertion can be chosen so that the limit function $f(y)$ satisfies $f(y_0) = f_0$.*

Remark 2.6. *If, in Theorem 2.4, it is known that all (uniformly) convergent subsequences of $f_1(y), f_2(y), \ldots$ have the same limit, say $f(y)$, then a selection is unnecessary and $f(y)$ is the uniform limit of $f_1(y), f_2(y), \ldots$.*

2.1.5 Convergence of ODE I

Theorem 2.7. *Let $y, f \in \mathbf{R}^d$ and $f_0(t,y), f_1(t,y), f_2(t,y), \ldots$ be a sequence of continuous functions on the parallelepiped $R : t_0 \leq t \leq t_0 + a, \mid y - y_0 \mid \leq b$ such that*

$$f_0(t,y) = \lim_{n\to\infty} f_n(t,y) \text{ uniformly on } R. \tag{2.8}$$

Let $y_n(t)$ be a solution of

$$y' = f_n(t,y), \quad y(t_n) = y_n \tag{2.9}$$

on $[t_0, t_0 + a]$, where $n = 1, 2, \ldots$, and

$$t_n \to t_0, y_n \to y_0 \quad \text{as} \quad n \to \infty \tag{2.10}$$

Then

1) *there exists a subsequence $y_{n(1)}(t), y_{n(2)}(t), \ldots$ which is* ***uniformly convergent*** *on $[t_0, t_0 + a]$.*
2) *For any such subsequence, the limit*
$$y_0(t) = \lim_{k\to\infty} y_{n(k)}(t) \tag{2.11}$$
is a solution of (2.9) with $n = 0$ on $[t_0, t_0 + a]$.
3) *In particular, if (2.9) with $n = 0$ possesses a unique solution $y = y_0(t)$ on $[t_0, t_0 + a]$, then*
$$y_0(t) = \lim_{n\to\infty} y_n(t) \quad \text{uniformly on } [t_0, t_0 + a]. \tag{2.12}$$

Proof. **proof of 1).:** Since $f_1, f_2, \ldots$ are continuous and (2.8) holds uniformly on R, there is a constant K such that $|f_n(t,y)| \leq K$ for $n = 0, 1, \ldots$ and $(t,y) \in R$. Since $|y_n'(t)| \leq K$, from [1, Theorem 5.19] it is clear that K is a Lipschitz constant for **all** $y_1, y_2, \ldots$. We therefore have the sequence $y_1, y_2, \ldots$ is equicontinuous. It is also uniformly bounded since $|y_n(t) - y_0| \leq b$. Thus the existence of uniformly convergent subsequences follows from Theorem 2.4.

proof of 2).: By (2.8), Lemma 2.1, and the uniformity of (2.11), it is easy to see that

$$f_{n(k)}\left(t, y_{n(k)}(t)\right) \to f_0(t, y_0(t))$$

uniformly on $[t_0, t_0 + a]$ as $k \to \infty$. Thus term-by-term integration is applicable to

$$y_n(t) = y_n + \int_{t_n}^{t} f_n(s, y_n(s))\, ds$$

where $n = n(k)$ and $k \to \infty$. It follows that the limit (2.11) is a solution of (2.9) with $n = 0$.

proof of 3).: Note that the assumed uniqueness of the solution $y_0(t)$ of (2.9) with $n = 0$ shows that the limit of every (uniformly) convergent subsequence of $y_1(t), y_2(t), \ldots$ is the solution $y_0(t)$. Hence a selection is unnecessary and (2.12) holds by Remark 2.6. $\square$

2.1.6 Implicit Function

Theorem 2.13. ***(Implicit Function)*** *Let x, y, f, g be d-dimensional vectors and z an e-dimensional vector. Let $f(y,z)$ be continuous for (y,z) near a point (y_0, z_0) and have continuous partial derivatives with respect to the components of y. Let the Jacobian $\det\left(\partial f^j / \partial y^k\right) \neq 0$ at $(y,z) = (y_0, z_0)$. Let $x_0 = f(y_0, z_0)$. Then there exist positive numbers, ϵ and δ, such that if x and z are fixed, $|x - x_0| < \delta$ and $|z - z_0| < \delta$, then the equation $x = f(y,z)$ has a unique solution $y = g(x,z)$ satisfying $|y - y_0| < \epsilon$. Furthermore, $g(x,z)$ is continuous. for $|x - x_0| < \delta, |z - z_0| < \delta$ and has continuous partial derivatives with respect to the components of x.*

2.1.7 Smooth Approximations

In some situations, it will be convenient to extend the definition of a function f, say, given continuous on a closed parallelepiped, or to approximate it uniformly by functions which are smooth $\left(C^1\right.$ or $\left.C^\infty\right)$ with respect to certain variables. The following approach can be used to obtain such extensions or approximations (which have the same bounds as f).

Let $f(t,y)$ be defined on $R: t_0 \le t \le t_1, |y| \le b$ and let $|f(t,y)| \le M$.

• **Extension:**

Case I: We can extend f to a continuous f^* as:

1. Let $f^*(t,y)$ be defined for $t_0 \le t \le t_1$ and all y by placing $f^*(t,y) = f(t,y)$ if $|y| \le b$.
2. Let $f^*(t,y) = f(t, by/|y|)$ if $|y| > b$.

It is clear that $f^*(t,y)$ is continuous for $t_0 \le t \le t_1, y$ arbitrary, and that $|f^*(t,y)| \le M$.

Case II: In some cases, we need a f^* by an extension of f which is 0 **for large** $|y|$:

1. Let $f^*(t,y)$ be defined as above.
2. Let $f^0(t,y) = f^*(t,y)\varphi^0(|y|)$, where $\varphi^0(s)$ is a continuous function for $t \ge 0$ satisfying $0 \le \varphi^0(s) \le 1$ for $s \ge 0$:
   (a) $\varphi^0(s) = 1$ for $0 \le s \le b$
   (b) $\varphi^0(s) = 0$ for $s \ge b+1$.

• **Uniformly C^∞ Approximate:**

Similar to [2, P139 Lemma 4.1.1]. In order to approximate $f(t,y)$ uniformly on R by functions $f^\epsilon(t,y)$ which are, say, smooth with respect to the components of y, let $\varphi(s)$ be **a function of class** C^∞ for $s \ge 0$ satisfying

1. $\varphi(s) > 0$ for $0 \le s < 1$ and
2. $\varphi(s) = 0$ for $s \ge 1$.

Then there is a constant $c > 0$ (we can also cooperate it to φ) depending only on $\varphi(s)$ and the dimension d, such that for every $\epsilon > 0$,

$$c\epsilon^{-d} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \varphi\left(\epsilon^{-2}\|y\|^2\right) dy^1 \ldots dy^d = 1 \tag{2.14}$$

where $\|y\| = \left(\Sigma\left|y^k\right|^2\right)^{1/2}$ is the Euclidean length of y. Put

$$f^\epsilon(t,y) = c\epsilon^{-d} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^0(t,\eta)\varphi\left(\epsilon^{-2}\|y-\eta\|^2\right) d\eta^1 \ldots d\eta^d \tag{2.15}$$

where $\eta = \left(\eta^1, \ldots, \eta^d\right)$, which is equivalent to

$$f^\epsilon(t,y) = c\epsilon^{-d} \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} f^0(t,y-\eta)\varphi\left(\epsilon^{-2}\|\eta\|^2\right) d\eta^1 \ldots d\eta^d. \tag{2.16}$$

Since $f^\epsilon(t,y)$ is an "average" of the values of f^0 in a sphere $\|\eta - y\| \le \epsilon$ for a fixed t, it is intuitively true that $f^t \to f^0$ as $\epsilon \to 0$ uniformly on $t_0 \le t \le t_1, y$ arbitrary. **Lebesgue Dominated Convergent Theorem** can be used to prove $f^\epsilon \in C^\infty$ w.r.t. y and uniformly convergence follows from the continuous of f^0 and the compact support of ϕ. We have $|f^\epsilon| \le M$ for all $\epsilon > 0$ and that $f^*(t,y) = 0$ for $|y| \ge b+1+\epsilon$.

2.2 The Picard-Lindelöf Theorem

Various types of existence proofs will be given. One of the most simple and useful is the following.

Theorem 2.17. *Let $y, f \in \mathbf{R}^d$; $f(t,y)$ continuous on a parallelepiped $R : t_0 \le t \le t_0+a, |y-y_0| \le b$ and **uniformly Lipschitz continuous** with respect to y. Let M be a bound for $|f(t,y)|$ on R; $\alpha = \min(a, b/M)$. Then*

$$y' = f(t,y), \quad y(t_0) = y_0 \tag{2.18}$$

*has a **unique** solution $y = y(t)$ on $[t_0, t_0+\alpha]$.*

2.3 Peano's Existence Theorem

The next theorem to be proved drops the assumption of Lipschitz continuity and the assertion of uniqueness.

Theorem 2.19. *We have the following conditions:*

1. *Let $y, f \in \mathbb{R}^d$; $f(t,y)$ continuous on set*

$$R = \{t_0 \le t \le t_0 + a, |y - y_0| \le b\}.$$

2. *We also have that M is a bound for $|f(t,y)|$ on R.*

Let $\alpha = \min(a, b/M)$. We then have the following conclusion:

- (2.18) *possesses **at least one solution** $y = y(t)$ on $[t_0, t_0+\alpha]$.*

Remark 2.20. *In this theorem, $|y|$ can be any convenient norm on $\mathbf{R}^d$.*

An important consequence of Peano's existence theorem will often be used:

Corollary 2.21. *Let $f(t,y)$ be continuous on an **open** (t,y)-set E and satisfy $|f(t,y)| \le M$. Let E_0 be a **compact subset** of E. Then there exists an $\alpha > 0$, depending on E, E_0 and M, with the property that if $(t_0, y_0) \in E_0$, then* (2.18) *has a solution on $|t - t_0| \le \alpha$.*

Remark 2.22. *In fact, if $a = \operatorname{dist}(E_0, \partial E) > 0$, where ∂E is the boundary of E, then $\alpha = \min(a, a/M)$. In applications, when f is not bounded on E, the set E in this corollary is replaced by an open subset E^0 having compact closure in E and containing E_0.*

2.4 Extension Theorem

Let $f(t,y)$ be continuous on a (t,y)-set E and let $y = y(t)$ be a solution of

$$y' = f(t,y) \tag{2.23}$$

on an interval J.

Remark 2.24. *When we talk about maximal or minimal interval, we restrict to the domain E, and consider intervals inside E.*

Definition 2.25. ***(maximal interval)*** *The interval J is called a right maximal interval of existence for y if there **does not exist** an extension of $y(t)$ over an interval J_1 so that*

1. *$y = y(t)$ remains a solution of* (2.23)*;*
2. *J is a **proper subset** of J_1;*
3. *J, J_1 have different right endpoints.*

A left maximal interval of existence for y is defined similarly. A maximal interval of existence is an interval which is both a left and right maximal interval.

Theorem 2.26. *Let $f(t,y)$ be continuous on an **open** (t,y)-set E and let $y(t)$ be a solution of* (2.23) *on some interval. Then*

- *$y(t)$ can be **extended** (as a solution) over a maximal interval of existence (ω_-, ω_+).*
- *Also, if $(\omega_-, \omega_+)$is a maximal interval of existence and if E is **bounded**, then $y(t)$ **tends to the boundary** ∂E of E as $t \to \omega_-$ and $t \to \omega_+$.*

Remark 2.27. *The extension of $y(t)$ **need not be unique** and, correspondingly, $\omega_\pm$ depends on the extension.*

$y(t)$ tends to ∂E as $t \to \omega_+$ is interpreted to mean that:
if E^0 is any compact subset of E, then $(t, y(t)) \notin E^0$ when t is near ω_+.

Proof. Let $E_1, E_2, \ldots$ be open subsets of E such that $E = \bigcup E_n$; the closures $E_1, E_2, \ldots$ are compact, and $E_n \subset E_{n+1}$. For example let

$$E_n = \{(t, y) : (t, y) \in E, |t| < n, |y| < n \text{ and } \operatorname{dist}((t, y), \partial E) > 1/n\}).$$

Corollary 2.21 implies that there exists an $\epsilon_n > 0$ such that if (t_0, y_0) is any point of E_n, then all solutions of (2.23) through (t_0, y_0) exist on $|t - t_0| \leq \epsilon_n$.

We next construct a sequence as follows:

Start from $(b_0, y(b_0)) \in E$. Let $n(1)$ be so large that $(b_0, y(b_0)) \in \bar{E}_{n(1)}$. Then $y(t)$ can be extended over an interval $\left[b_0, b_0 + \epsilon_{n(1)}\right]$. If $\left(b_0 + \epsilon_{n(1)}, y\left(b_0 + \epsilon_{n(1)}\right)\right) \in E_{n(1)}$, then $y(t)$ can be extended over another interval $\left[b_0 + \epsilon_{n(1)}, b_0 + 2\epsilon_{n(1)}\right]$ of length $\epsilon_{n(1)}$. Continuing this argument, it is seen that there is an integer $j(1) \geq 1$ such that $y(t)$ can be extended over $a \leq t \leq b_1$, where $b_1 = b_0 + j(1)\epsilon_{n(1)}$ and $(b_1, y(b_1)) \notin \bar{E}_{n(1)}$.

Let $n(2)$ be so large that $(b_1, y(b_1)) \in E_{n(2)}$. Then there exists an integer $j(2) \geq 1$ such that $y(t)$ can be extended over $a \leq t \leq b_2$, where $b_2 = b_1 + j(2)\epsilon_{n(2)}$ and $(b_2, y(b_2)) \notin \bar{E}_{n(2)}$.

Repetitions of this argument lead to sequences of integers $n(1) < n(2) < \ldots$ and numbers $b_0 < b_1 < \ldots$ such that $y(t)$ has an extension over $[a, \omega_+)$, where $\omega_+ = \lim b_k$ as $k \to \infty$, (here ω_+ could be ∞) and that $(b_k, y(b_k)) \notin \bar{E}_{n(k)}$.

The sequence $(b_1, y(b_1)), (b_2, y(b_2)), \ldots$ is either **unbounded** over y or has a limit point on the boundary ∂E of E:

1. E is unbounded for t, then $\omega_+ = \infty$. In this case, $y(t)$ can be bounded or unbounded, we don't case.
2. E is bounded for t, $\omega_+ < \infty$. In this case, there are two cases to consider:
   (a) If E is unbounded for y, there is still no boundary. $y(t)$ can be bounded or unbounded, we don't case.
   (b) If E is bounded for y, there is boundary, and $y(t)$ only be bounded. We next only consider this case.

To see that $y(t)$ tends to ∂E as $t \to \omega_+$on a right maximal interval $[a, \omega_+)$, it must be shown that no limit point of a sequence $(t_1, y(t_1))$, $(t_2, y(t_2)), \ldots$, where $t_n \to \omega_+$, can be an interior point of E. This is a consequence of the following: □

Lemma 2.28. *Let $f(t, y)$ be continuous on a (t, y)-set E. Let $y = y(t)$ be a solution of (2.23) on an interval $[a, \delta), \delta < \infty$, for which there exists a sequence $t_1, t_2, \ldots$ such that*

$$a \leq t_n \to \delta \text{ as } n \to \infty \text{ and } y_0 = \lim y(t_n) \text{ exists.} \tag{2.29}$$

*If $f(t, y)$ is **bounded on the intersection** of E and a neighbour of the point (δ, y_0), then*

$$y_0 = \lim y(t) \text{ as } t \to \delta. \tag{2.30}$$

*If, in addition, $f(\delta, y_0)$ is or can be defined so that $f(t, y)$ is **continuous at** (δ, y_0), then $y(t) \in C^1[a, \delta]$ and is a solution of (2.23) on $[a, \delta]$.*

Proof. Let $\epsilon > 0$ be so small and $M_\epsilon > 1$ so large that $|f(t, y)| \leq M_\epsilon$ for (t, y) on the **intersection of E and the parallelepiped** $0 \leq \delta - t \leq \epsilon$, $|y - y_0| \leq \epsilon$. If n is so large that $0 < \delta - t_n \leq \epsilon/2M_\epsilon$ and $|y(t_n) - y_0| \leq \epsilon/2$, then

$$|y(t) - y(t_n)| < M_\epsilon (\delta - t_n) \leq \frac{1}{2}\epsilon \quad \text{for all } t_n \leq t < \delta \tag{2.31}$$

Otherwise, since $y(t)$ is continuous, there is **smallest** t^1 such that $t_n < t^1 < \delta, \left|y\left(t^1\right) - y\left(t_n\right)\right| = M_\epsilon\left(\delta - t_n\right) \leq \frac{1}{2}\epsilon$. And for all $t_n < t < t^1$,

$$|y(t) - y(t_n)| \leq M_\epsilon\left(\delta - t_n\right) \leq \frac{1}{2}\epsilon$$

Hence $|y(t) - y_0| \leq \frac{1}{2}\epsilon + |y\left(t_n\right) - y_0| \leq \epsilon$ for $t_n \leq t < t^1$; thus $|y'(t)| \leq M_\epsilon$ for $t_n \leq t \leq t^1$. Consequently, $\left|y\left(t^1\right) - y\left(t_n\right)\right| \leq M_\epsilon\left(t^1 - t_n\right) < M_\epsilon\left(\delta - t_n\right)$. A contradiction.

This proves (2.31), hence (2.30). The last part of the lemma follows from $y'(t) = f(t, y(t)) \to f\left(\delta, y_0\right)$ as $t \to \delta$. $\square$

Corollary 2.32. *Let $f(t, y)$ be continuous on a strip $t_0 \leq t \leq t_0 + a\ (< \infty), y \in \mathbb{R}^d$ arbitrary. Let $y = y(t)$ be a solution of* (2.18) *on a right maximal interval J. Then either the following cases can happen:*

1. $J = [t_0, t_0 + a]$
2. $J = [t_0, \delta), \delta \leq t_0 + a$, *and* $|y(t)| \to \infty$ *as* $t \to \delta$.

More generally,

Corollary 2.33. *Let $f(t, y)$ be continuous on the* ***closure*** *E of an open (t, y)-set E and let* (2.18) *possess a solution $y = y(t)$ on a maximal right interval J. Then either the following cases can happen:*

1. $J = [t_0, \infty)$
2. $J = [t_0, \delta]$ *with* $\delta < \infty$ *and* $(\delta, y(\delta)) \in \partial E$
3. $J = [t_0, \delta)$ *with* $\delta < \infty$ *and* $|y(t)| \to \infty$ *as* $t \to \delta$.

A somewhat different, but very useful, result is given by the following theorem as an extension of Theorem 2.7.

Theorem 2.34. ***(Convergence of ODE II)*** *Let $f(t, y)$ and $f_1(t, y), f_2(t, y), \ldots$ be a sequence of continuous functions defined on an open (t, y)-set E such that*

$$f_n(t, y) \to f(t, y) \quad \textit{as} \quad n \to \infty \tag{2.35}$$

holds uniformly on every compact subset *of E. Let $y_n(t)$ be a solution of*

$$y' = f_n(t, y), \quad y\left(t_n\right) = y_{n0} \tag{2.36}$$

$(t_n, y_{n0}) \in E$, and let $(\omega_{n-}, \omega_{n+})$ be its maximal interval of existence. If

$$(t_n, y_{n0}) \to (t_0, y_0) \in E \quad \textit{as} \quad n \to \infty. \tag{2.37}$$

Then there exist

1) *a solution $y(t)$ of*

$$y' = f(t, y), \quad y\left(t_0\right) = y_0, \tag{2.38}$$

having a maximal interval of existence (ω_-, ω_+), and

2) *a sequence of positive integers $n(1) < n(2) < \ldots$* ***with the property that***
if $\omega_- < t^1 < t^2 < \omega_+$, then $\omega_{n-} < t^1 < t^2 < \omega_{n+}$ for $n = n(k)$ and k large, and

$$y_{n(k)}(t) \to y(t) \quad \textit{as} \quad k \to \infty \tag{2.39}$$

uniformly *for $t^1 \leq t \leq t^2$.*

3) *In particular,*

$$\limsup \omega_{n-} \leq \omega_- < \omega_+ \leq \liminf \omega_{n+} \quad \textit{as} \quad n = n(k) \to \infty. \tag{2.40}$$

Proof. **proof of 1).:** Let $E_1, E_2, \ldots$ be open subsets of E such that $E = \cup E_n$, the closures $E_1, E_2, \ldots$ are compact and $E_n \subset E_{n+1}$. Suppose that $(t_0, y_0) \in E_1$ and hence that $(t_n, y_{n0}) \in E_1$ for large n since E_1 is open.

In the proof, $y(t)$ will be constructed only on a **right** maximal interval of existence $[t_0, \omega_+)$. The construction for a left maximal interval is similar.

By Corollary 2.21, there exists an ϵ_1**, independent of** n for large n, such that any solution of (2.36) [or (2.38)] for any point $(t_n, y_{n0}) \in E_1$ [or $(t_0, y_0) \in E_1$] exists on an interval of length $3\epsilon_1$ centered at $t = t_n$ [or $t = t_0$]. Note the boundness condition in Corollary 2.21 is satisfied, see Theorem 2.7.

By Theorem 2.4 or Theorem 2.7, it follows that if $n(1) < n(2) < \ldots$ are suitably chosen, then the limit (2.39) exists uniformly for $t_0 \leq t \leq t_0 + \epsilon_1$ and is a solution of (2.38). This already shows 1). is correct and additionally (2.38) gives the solution over a interval. (2.37) can of course be extended to maximal interval of existence. We next show how to get the maximal interval use subsequences iteratively.

proof of 2).: If the point $(t_0 + \epsilon_1, y(t_0 + \epsilon_1)) \in E_1$, the sequence $n(1) < n(2) < \ldots$ can be replaced by a subsequence, again called $n(1) < n(2) < \ldots$, such that the limit (2.39) exists uniformly for $t_0 + \epsilon_1 \leq t \leq t_0 + 2\epsilon_1$ and is a solution of (2.23). This process can be repeated j times, where $(t_0 + m\epsilon_1,\ y(t_0 + m\epsilon_1)) \in E_1$ for $m = 0, \ldots, j-1$ but not for $m = j$. However, please note that $(t_1, y(t_1))$ is still in E according to Corollary 2.21.

In this case, let $t_1 = t_0 + j\epsilon_1$ and choose the integer $r > 1$ so that $(t_1, y(t_1)) \in E_r$. Repeat the procedure above using a suitable $\epsilon_r > 0$ (depending on r but independent of n for large n) to obtain $y(t)$ on an interval $t_1 \leq t \leq t_1 + j_1\epsilon_r$, where $(t_1 + m\epsilon_r, y(t_1 + m\epsilon_r)) \in E_r$ for $m = 0, \ldots j_1 - 1$ but not for $m = j_1$. Put $t_2 = t_1 + j_1\epsilon_r$.

Repetitions of these arguments lead to a sequence of t-values $t_0 < t_1 < \ldots$ and a sequence of successive subsequences of integers:

$$\begin{aligned} &n_1(1) < n_1(2) < \ldots \\ &n_2(1) < n_2(2) < \ldots \\ &\ldots\ldots\ldots \end{aligned}$$

such that (2.39) holds uniformly for $t_0 \leq t \leq t_m$ if $n(k) = n_m(k)$. Put $\omega_+ = \lim t_m (\leq \infty)$.

Since $(t_{m+1}, y(t_{m+1})) \notin E_m$, for $m = 1, 2, \ldots$, we have that $t_{m+1} < w_{n(k)}$ and $[t_0, \omega_+)$ is the right maximal interval of existence for $y(t)$. Finally, the usual diagonal process supplies the desired sequence $n(1) < n(2) < \ldots$ This proves the theorem.

proof of 3).: We immediately get 3). ☐

2.5 Connectness of Nonunique Solutions

The following theorem concerning the case of nonunique solutions of initial value problems will be proved in this section.

Theorem 2.41. *Let $f(t, y)$ be continuous on $R : t_0 \leq t \leq t_0 + a, |y - y_0| \leq b$. Let $|f(t, y)| \leq M; \alpha = \min(a, b/M)$ and $t_0 < c \leq t_0 + \alpha$. Finally, let S_c be the set of points y_c for which there is a solution $y = y(t)$ of*

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{2.42}$$

on $[t_0, c]$ such that $y(c) = y_c$; i.e., $y_c \in S_c$ means that y_c is a point reached at $t = c$ by some solution of (2.42)*. Then S_c is a* ***continuum, i.e., a closed connected set.***

3 Differential Inequalities and Uniqueness

In this chapter u, v, U, V are scalars; y, z, f, g are d-dimensional vectors.

3.1 Gronwall's Inequality

One of the simplest and most useful results involving an integral inequality is the following.

Theorem 3.1. *Let $u(t), v(t)$ be **non-negative, continuous** functions on $[a, b]$. Let $C \geq 0$ be a constant; and*

$$v(t) \leq C + \int_a^t v(s)u(s)ds \quad \textit{for } a \leq t \leq b \tag{3.2}$$

Then

$$v(t) \leq C \exp \int_a^t u(s)ds \quad \textit{for } a \leq t \leq b \tag{3.3}$$

in particular, if $C = 0$, then $v(t) \equiv 0$.

Proof. Case (i), $C > 0$. Let $V(t)$ denote the right side of (3.2), so that $v(t) \leq V(t), V(t) \geq C > 0$ on $[a, b]$. Also, $V'(t) = u(t)v(t) \leq u(t)V(t)$. Since $V > 0, V'/V \leq u$, and $V(a) = C$, an integration over $[a, t]$ gives $V(t) \leq C \exp \int_a^t u(s)ds$. Thus (3.3) follows from $v(t) \leq V(t)$.

Case (ii), $C = 0$. If (3.2) holds with $C = 0$, then Case (i) implies (3.3) for every $C > 0$. The desired result follows by letting C tend to 0 .

□

3.2 Maximal and Minimal Solutions

Definition 3.4. ***(Maximal and Minimal Solutions)*** *Let $U(t, u)$ be a continuous function on a plane (t, u)-set E. By a **maximal** solution $u = u^0(t)$ of*

$$u' = U(t, u), \quad u(t_0) = u_0 \tag{3.5}$$

is meant a solution of (3.5) *on a maximal interval of existence such that if $u(t)$ is any solution of* (3.5)*, then*

$$u(t) \leq u^0(t) \tag{3.6}$$

holds on the common interval of existence of u, u^0. A **minimal** solution is similarly defined.

Lemma 3.7. *We have the following conditions:*

1. *Let $U(t, u)$ be continuous on a rectangle $R : t_0 \leq t \leq t_0 + a$, $|u - u_0| \leq b$.*
2. *Let $|U(t, u)| \leq M$ and $\alpha = \min(a, b/M)$.*

We then have that (3.5) *has a solution $u = u^0(t)$ on $[t_0, t_0 + \alpha]$ with the property that every solution $u = u(t)$ of*

$$u' = U(t, u), u(t_0) \leq u_0$$

satisfies (3.6) *on $[t_0, t_0 + \alpha]$.*

This lemma implies existence theorems for maximal and minimal solutions (which will be stated only for an open set E):

Theorem 3.8. ***(Existence of Maximal and Minimal Solutions)*** *Let $U(t, u)$ be continuous on an open set E and $(t_0, u_0) \in E$. Then* (3.5) *has a maximal and a minimal solution.*

Proof. of Lemma 3.7. Let $0 < \alpha' < \alpha$. Then, by Theorem 2.19,

$$u' = U(t, u) + 1/n, \quad u(t_0) = u_0 \tag{3.9}$$

has a solution $u = u_n(t)$ on an interval $[t_0, t_0 + \alpha']$ if n is sufficiently large. By Theorem 2.7, there is a sequence $n(1) < n(2) < \cdots$ such that

$$u^0(t) = \lim_{k \to \infty} u_{n(k)}(t) \tag{3.10}$$

exists uniformly on $[t_0, t_0 + \alpha']$ and is a solution of (3.5). To verify that (3.6) holds on $[t_0, t_0 + \alpha']$, it is sufficient to verify

$$u(t) \leq u_n(t) \quad \text{on } [t_0, t_0 + \alpha'] \tag{3.11}$$

for all large fixed n. We use the similar proof technique as shown in the proof of Lemma 2.28:

If (3.11) does not hold, there is a $t = t_1, t_0 < t_1 < t_0 + \alpha'$ such that $u(t_1) > u_n(t_1)$. Hence there is a **largest** t_2 on $[t_0, t_1)$, where $u(t_2) = u_n(t_2)$, so that $u(t) > u_n(t)$ on $(t_2, t_1]$.

But (3.9) implies that $u'_n(t_2) = u'(t_2) + 1/n$, so that $u_n(t) > u(t)$ for $t\,(> t_2)$ near t_2. This contradication proves (3.11). Since $\alpha' < \alpha$ is arbitrary, the lemma follows. □

3.3 Right Derivatives

The following simple lemmas will be needed subsequently.

Lemma 3.12. *Let $u(t) \in C^1[a,b]$ be the scalar function. Then $|u(t)|$ has a* ***right derivative*** *$D_R|u(t)|$ for $a \le t < b$, where*

$$D_R|u(t)| = \lim h^{-1}(|u(t+h)| - |u(t)|) \quad \text{as } 0 < h \to 0, \tag{3.13}$$

and

1. *$D_R|u(t)| = u'(t) \operatorname{sgn} u(t)$ if $u(t) \neq 0$, and*
2. *$D_R|u(t)| = |u'(t)|$ if $u(t) = 0$.*

In particular, $\big|D_R|u(t)|\big| = |u'(t)|$.

Proof. The assertion concerning $D_R|u(t)|$ is clear if $u(t) \neq 0$. The case when $u(t) = 0$ follows from $u(t+h) = h\,(u'(t) + o(1))$ as $h \to 0$, so that $|u(t+h)| = h\,(|u'(t)| + o(1))$ as $0 < h \to 0$. □

Lemma 3.14. *Let $y = y(t) \in C^1[a,b]$ be the* ***high dimension function****. Then*

1. *$|y(t)|$ has a* ***right derivative*** *$D_R|y(t)|$, and*
2. *$\big|D_R|y(t)|\big| \le |y'(t)|$ for $a \le t < b$.*

Remark 3.15. *It is also correct if $|y|$ is replaced by the Euclidean length of y.*

Proof. Since $|y(t)| = \max\left(\left|y^1(t)\right|, \ldots, \left|y^d(t)\right|\right)$, there are indices k such that $\left|y^k(t)\right| = |y(t)|$. In the following, k denotes any such index. By the last lemma, $\left|y^k(t)\right|$ has a right derivative, so that

$$\left|y^k(t+h)\right| = |y(t)| + h\left(D_R\left|y^k(t)\right| + o(1)\right) \quad \text{as } 0 < h \to 0$$

For small $h > 0$, $|y(t+h)| = \max_k \left|y^k(t+h)\right|$, so that by taking the $\max_k$ in the last formula line,

$$|y(t+h)| = |y(t)| + h\left(\max_k D_R\left|y^k(t)\right| + o(1)\right) \quad \text{as} \quad 0 < h \to 0$$

Thus $D_R|y(t)|$ exists and is $\max_k D_R\left|y^k(t)\right|$. Since $|D_R|\,y^k(t)|| = \left|y^{k\prime}(t)\right| \le |y'(t)|$, Lemma 3.14 follows. □

3.4 Differential Inequalities

The next theorem concerns the integration of a differential inequality. It is one of the results which is used most often in the theory of differential equations.

Theorem 3.16. *Let $U(t,u)$ be continuous on an open (t,u)-set E and $u = u^0(t)$ the* ***maximal*** *solution of* (3.5)*. Let $v(t)$ be a continuous function on $[t_0, t_0 + a]$ satisfying the conditions*

1. *$v(t_0) \le u_0$,*
2. *$(t, v(t)) \in E$, and*
3. *$v(t)$ has a* ***right derivative*** *$D_R v(t)$ on $t_0 \le t < t_0 + a$ such that*

$$D_R v(t) \le U(t, v(t)) \tag{3.17}$$

Then, on a common interval of existence of $u^0(t)$ and $v(t)$,

$$v(t) \leq u^0(t). \tag{3.18}$$

Remark 3.19. *Correspondingly, we have the following*

- *If we have*

  1. $D_R v(t) \geq U(t, v(t))$*, and*
  2. $v(t_0) \geq u_0$

  we have the conclusion
  $$v(t) \geq u_0(t),$$
  *where $u = u_0(t)$ is the **minimal** solution of* (3.5)

- *If in Theorem 3.16 the function $v(t)$ is continuous on an interval $t_0 - \alpha \leq t \leq t_0$ with a **left derivative** $D_L v(t)$ on $(t_0 - \alpha, t_0]$ satisfying*

  1. $D_L v(t) \leq U(t, v(t))$*, and*
  2. $v(t_0) \geq u_0$

  then again (3.18) *must be replaced by $v(t) \geq u_0(t)$, the **minimal** solution.*

- *For **left derivative** and **maximal** solution:*

  1. $D_L v(t) \geq U(t, v(t))$*, and*
  2. $v(t_0) \leq u_0$

  then again (3.18) *must be replaced by $v(t) \leq u^0(t)$, the **maximal** solution.*

Remark 3.20. *It will be clear from the proof that Theorem 3.16 holds if the "right derivative" D_R is replaced by the "upper right derivative" where the latter is defined by replacing "$\lim$" by "$\limsup$" in* (3.13).

Proof. It is sufficient to show that there exists a $\delta > 0$ such that (3.18) holds for $[t_0, t_0 + \delta]$.

Let $n > 0$ be large and let $\delta > 0$ be chosen independent of n such that (3.9) has a solution $u = u_n(t)$ on $[t_0, t_0 + \delta]$. In view of the proof of Lemma 3.7, it is sufficient to verify that $v(t) \leq u_n(t)$ on $[t_0, t_0 + \delta]$, but the proof of this is identical to the proof of Lemma 3.7. □

Corollary 3.21. *Let $v(t)$ be continuous on $[a, b]$ and possess a right derivative $D_R v(t) \leq 0$ on $[a, b]$. Then $v(t) \leq v(a)$.*

Corollary 3.22. *Let $U(t, u), u^0(t)$ be as in Theorem 3.16. Let $V(t, u)$ be continuous on E and satisfy*

$$V(t, u) \leq U(t, u) \tag{3.23}$$

Let $v = v(t)$ be a solution of

$$v' = V(t, v), \quad v(t_0) = v_0 \,(\leq u_0) \tag{3.24}$$

on an interval $[t_0, t_0 + a]$. Then (3.18) *holds on any common interval of existence of $v(t)$ and $u^0(t)$ to the right of $t = t_0$.*

Corollary 3.25. *We have the conditions:*

1. *Let $u^0(t)$ be the maximal solution of $u' = U(t, u), u(t_0) = u^0$;*

2. *Let $u = u_0(t)$ the minimal solution of*
   $$u' = -U(t, u), \quad u(t_0) = u_0 \geq 0.$$

3. *Let $y = y(t)$ be a C^1 vector-valued function on $[t_0, t_0 + \alpha]$ such that $u_0 \leq |y(t_0)| \leq u^0$, $(t, |y(t)|) \in E$ and*
   $$|y'(t)| \leq U(t, |y(t)|) \tag{3.26}$$
   on $[t_0, t_0 + \alpha]$.

Then the first [second] of the two inequalities

$$u_0(t) \leq |y(t)| \leq u^0(t) \tag{3.27}$$

holds on any common interval of existence of $u_0(t)$ *and* y *[*$u^0(t)$ *and* y*].*

Remark 3.28. *This corollary remains valid if* $|y|$ *denotes the Euclidean norm.*

Proof. This is an immediate consequence of Theorem 3.16 and Remark 3.19, since $|y(t)|$ has a right derivative satisfying $-|y'(t)| \leq D_R|y(t)| \leq |y'(t)|$ by Lemma 3.14. □

Theorem 3.16 has an "integrated" analogue which, however, requires the **monotony** of U with respect to u. This theorem is a generalization of Theorem 3.1:

Theorem 3.29. *Let* $U(t,u)$ *be continuous and nondecreasing with respect to u for* $t_0 \leq t \leq t_0 + a, u$ *arbitrary. Let the maximal solution* $u = u^0(t)$ *of* (3.5) *exist on* $[t_0, t_0 + a]$*. On* $[t_0, t_0 + a]$*, let* $v(t)$ *be a continuous function satisfying*

$$v(t) \leq v_0 + \int_{t_0}^{t} U(s, v(s)) ds, \tag{3.30}$$

where $v_0 \leq u_0$*. Then* $v(t) \leq u^0(t)$ *holds on* $[t_0, t_0 + a]$*.*

Proof. Let $V(t)$ be the right side of (3.30), so that $v(t) \leq V(t)$, and $V'(t) = U(t, v(t))$. By the monotony of $U, V'(t) \leq U(t, V(t))$. Hence Theorem 3.16 implies that $V(t) \leq u^0(t)$ on $[t_0, t_0 + a]$; thus $v(t) \leq u^0(t)$ holds. □

3.5 A Theorem of Wintner

Theorem 3.16 and its corollaries can be used to help **find intervals of existence** of solutions of some differential equations.

Theorem 3.31. *We have conditions:*

1. *Let* $U(t,u)$ *be continuous for* $t_0 \leq t \leq t_0 + a, u \geq 0$*.*
2. *Let the maximal solution of* (3.5)*, where* $u_0 \geq 0$*,* ***exist*** *on* $[t_0, t_0 + a]$*. One possbile condition is that:*
   - For an autonomous system, let $U(t,u) = \psi(u)$, where $\psi(u)$ is a positive, continuous function on $u \geq 0$ such that

$$\int^{\infty} du/\psi(u) = \infty \tag{3.32}$$

Let $f(t,y)$ *be continuous on the strip* $t_0 \leq t \leq t_0 + a, y$ *arbitrary, and satisfy*

$$|f(t,y)| \leq U(t, |y|) \tag{3.33}$$

Then the ***maximal interval*** *of existence of solutions of*

$$y' = f(t,y), \quad y(t_0) = y_0, \tag{3.34}$$

where $|y_0| \leq u_0$*, is* $[t_0, t_0 + a]$*.*

Remark 3.35. *It is clear that* (3.33) *is only required for large* $|y|$*. Admissible choices of* $\psi(u)$ *are, for example,* $\psi(u) = Cu, Cu \log u, \ldots$ *for large u and a constant C.*

Proof. (3.33) implies the inequality (3.26) on any interval on which $y(t)$ exists. Hence, if assume the existence of maximal solution, by Corollary 3.25, the second inequality in (3.27) holds on such an interval and so the main assertion follows from Corollary 2.32.

In order to complete the proof, it has to be shown that the function $U(t,u) = \psi(u)$ satisfies the condition that the existence of maximal solution on $[t_0, t_0 + a]$ given

$$u' = \psi(u), \quad u(t_0) = u_0 (\geq 0) \tag{3.36}$$

and (3.32). Since $\psi > 0$, (3.36) implies that for any solution $u = u(t)$,

$$t - t_0 = \int_{t_0}^{t} u'(s)ds/\psi(u(s)) = \int_{u_0}^{u(t)} du/\psi(u). \tag{3.37}$$

Note that $\psi > 0$ implies that $u'(t) > 0$ and $u(t) > 0$ for $t > t_0$. By Corollary 2.32, the solution $u(t)$ can fail to exist on $[t_0, t_0 + a]$ only if it exists on some interval $[t_0, \delta)$ and satisfies $u(t) \to \infty$ as $t \to \delta(\leq a)$. If this is the case, however, $t \to \delta$ in (3.37) gives a contradiction for the left side tends to $\delta - t_0$ and the right side to ∞ by (3.32). This completes the proof. □

Remark 3.38. *The type of argument in the proof of Theorem 3.31 supplies a* ***priori estimates for solutions*** $y(t)$ *of* (3.34)*. For example, if $\psi(u)$ is the same as in the last part of Theorem 3.31, let*

$$\Psi(u) = \int_{u_0}^{u} ds/\psi(s) \quad \text{for } u \geq u_0$$

and let $u = \Phi(v)$ be the function inverse to $v = \Psi(u)$. Then $|f(t, y)| \leq \psi(|y|)$ implies that a solution $y(t)$ of (3.34) *satisfies $|y(t)| \leq \Phi(t - t_0)$ for $t_0 \leq t \leq t_0 + a$ from* (3.37)*.*

Corollary 3.39. *Let $f(t, y)$ be continuous on the strip $t_0 \leq t \leq t_0 + a$, y arbitrary. Let $|f(t, y)| \leq \varphi(t)\psi(|y|)$, where $\varphi(t) \geq 0$ is integrable on $[t_0, t_0 + a]$ and $\psi(u)$ is a positive continuous function on $u \geq 0$ satisfying* (3.32)*. Show that the assertion of Theorem 3.31 is valid.*

Proof. Similar to [1]. □

Corollary 3.40. *If $A(t)$ is a continuous $d \times d$ matrix function and $g(t)$ a continuous vector function for $t_0 \leq t \leq t_0 + a$, then the (linear) initial value problem*

$$y' = A(t)y + g(t), \quad y(t_0) = y_0 \tag{3.41}$$

has a ***unique*** *solution $y = y(t)$, and $y(t)$ exists on $t_0 \leq t \leq t_0 + a$.*

Proof. This is a consequence of Theorem 2.17 and Theorem 3.31 with the choice of $\psi(u) = C(1+u)$ for some large C. More specifically, Theorem 2.17 shows uniqueness and existence over a sub interval, and then Theorem 3.31 show that the existence is actually over the full interval $(t_0, t_0 + a)$. The uniqueness of the solution over the full interval comes from that the b/M in Theorem 2.17 can be selected to be a from the existence. □

In a scalar case, Theorem 3.31 can be "read backwards":

Corollary 3.42. *Let $U(t, u), V(t, u)$ be continuous functions satisfying* (3.23) *on $t_0 \leq t \leq t_0 + a, u$ arbitrary. Let some solution $v = v(t)$ of* (3.24) *on $[t_0, \delta), \delta \leq t_0 + a$, satisfy $v(t) \to \infty$ as $t \to \delta$. Then the maximal solution $u = u^0(t)$ of* (3.5) *has a maximal interval of existence $[a, \omega_+)$, where $\omega_+ \leq \delta$, and $u^0(t) \to \infty$ as $t \to \omega_+$.*

3.6 Uniqueness Theorems

One of the principal uses of Theorem 3.16 and its corollaries is to obtain uniqueness theorems.

Theorem 3.43. ***(Kamke's General Uniqueness Theorem)***

1. *Let $f(t, y)$ be continuous on the parallelepiped $R : t_0 \leq t \leq t_0 + a, |y - y_0| \leq b$.*

2. *Let $\omega(t, u)$ be a continuous (scalar) function on $R_0 : t_0 < t \leq t_0 + a, 0 \leq u \leq 2b$, with the properties that $\omega(t, 0) = 0$ and that the only solution $u = u(t)$ of the differential equation*

$$u' = \omega(t, u) \tag{3.44}$$

on any interval $(t_0, t_0 + \epsilon]$ satisfying

$$u(t) \to 0 \text{ and } \frac{u(t)}{t - t_0} \to 0 \quad \text{as } t \to t_0 + 0 \tag{3.45}$$

is $u(t) \equiv 0$.

For $(t, y_1), (t, y_2) \in R$ *with* $t > t_0$, *let*

$$|f(t, y_1) - f(t, y_2)| \leq \omega(t, |y_1 - y_2|). \tag{3.46}$$

Then the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{3.47}$$

has ***at most one solution*** *on* ***any*** *interval* $[t_0, t_0 + \epsilon]$.

Proof. The fact that

$$\omega(t, 0) = 0 \quad \text{for } t_0 < t \leq t_0 + a \tag{3.48}$$

implies of course that $u(t) \equiv 0$ is a solution of (3.44).

Suppose that, for some $\epsilon > 0$, (3.47) has two distinct solutions $y = y_1(t)$ and $y_2(t)$ on $t_0 \leq t \leq t_0 + \epsilon$. Let $y(t) = y_1(t) - y_2(t)$. By decreasing ϵ,if necessary, it can be supposed that $y(t_0 + \epsilon) \neq 0$ and $|y(t_0 + \epsilon)| < 2b$. Also $y(t_0) = y'(t_0) = 0$ since they are solutions of the same equation.

By (3.46), $|y'(t)| \leq \omega(t, |y(t)|)$ on $(t_0, t_0 + \epsilon]$. It follows from Theorem 3.16 (and the remark follows) and Corollary 3.25 that if $u = u_0(t)$ is the minimal solution of the initial value problem $u' = \omega(t, u)$, $u(t_0 + \epsilon) = |y(t_0 + \epsilon)|$, where $0 < |y(t_0 + \epsilon)| < 2b$, then

$$|y(t)| \geq u_0(t) \tag{3.49}$$

on **any subinterval** of $(t_0, t_0 + \epsilon]$ on which $u_0(t)$ **exists**.

By the proofs of Theorem 2.26 and Lemma 2.28, $u_0(t)$ can be extended, as the minimal solution, to the left until $(t, u_0(t))$ approaches arbitrarily close to a point of ∂R_0 for some t-values. During the extension (3.49) holds, so that $(t, u_0(t))$ comes arbitrarily close to some point $(\delta, 0) \in \partial R_0$ for certain t-values, where $\delta \geq t_0$. If $\delta > t_0$, then (3.48) shows that $u_0(t)$ has an extension over $(t_0, t_0 + \epsilon]$ with $u_0(t) = 0$ for $(t_0, \delta]$. Thus, in any case, the left maximum interval of existence of $u_0(t)$ is $(t_0, t_0 + \epsilon]$.

It follows from (3.49) that $u_0(t) \to 0$ and $u_0(t)/(t - t_0) \to 0$ as $t \to t_0 + 0$. By the assumption concerning (3.45), $u_0(t) \equiv 0$. Since this contradicts $u_0(t_0 + \epsilon) = |y(t_0 + \epsilon)| \neq 0$, the theorem follows. ☐

Remark 3.50. *Theorem 3.43 is false if* (3.45) *is replaced by* $u(t), u'(t) \to 0$ *as* $t \to t_0 + 0$ *because we does not have* $u_0(t_0) = 0$.

4 Linear Differential Equations

In this chapter, u, v, p are scalars; c, y, z, f, g are (column) d-dimensional vectors; and A, B, Y, Z are matrices. The scalars, components of the vectors, and elements of the matrices will be supposed to be complex-valued.

4.1 Linear Systems

- **homogeneous case:**

$$y' = A(t)y. \tag{4.1}$$

- **inhomogeneous case:**

$$y' = A(t)y + f(t). \tag{4.2}$$

Throughout this chapter, $A(t)$ is a continuous $d \times d$ matrix and $f(t)$ a continuous vector on a t-interval $[a, b]$. Recall the following fundamental fact stated in Corollary 3.40:

Lemma 4.3. *The initial value problem* (4.2) *and*

$$y(t_0) = y_0 \tag{4.4}$$

$a \leq t_0 \leq b$, *has a unique solution* $y = y(t)$ *and* $y(t)$ *exists on* $a \leq t \leq b$.

Remark 4.5. *The fact that the elements of $A(t)$ and components of y are complex-valued does not affect the applicability of Corollary 3.40:. For example,* (4.2) *is equivalent to a real linear system for a $2d$-dimensional vector made up of the real and imaginary parts of the components of y.*

The uniqueness of solutions implies that

Corollary 4.6. *If $y = y(t)$ is a solution of* (4.1) *and $y(t_0) = 0$ for some $t_0, a \leq t_0 \leq b$, then $y(t) \equiv 0$.*

Theorem 4.7. ***((Principles of Superposition)***

1. *Let $y = y_1(t), y_2(t)$ be solutions of* (4.1)*, then any linear combination $y = c_1 y_1(t) + c_2 y_2(t)$ with constant coefficients c_1, c_2 is a solution of* (4.1).
2. *If $y = y_1(t)$ and $y = y_0(t)$ are solutions of* (4.1) *and* (4.2)*, respectively, then $y = y_0(t) + y_1(t)$ is a solution of* (4.2)*; conversely, if $y = y_0(t), y^0(t)$ are solutions of* (4.2)*, then $y = y_0(t) - y^0(t)$ is a solution of* (4.1).

The vector equation (4.1) can be replaced by a **matrix differential equation**,

$$Y' = A(t)Y, \tag{4.8}$$

where Y is matrix with d rows and k (arbitrary) columns. It is clear that a matrix $Y = Y(t)$ is a solution of (4.8) if and only if each column of $Y(t)$, when considered as a column vector, is a solution of (4.1).

Corollary 4.9. *Corollary 4.6 and the principle of superposition imply that if $Y = Y(t)$ is a $d \times k$ matrix solution of* (4.8)*, then rank $Y(t)$ does not depend on t. That is, if $y_1(t), \ldots, y_r(t)$ are r solutions of* (4.1)*, then the constant vectors $y_1(t_0), \ldots, y_r(t_0)$ are linearly independent for some t_0 if and only if they are linearly independent for* ***every*** *t_0 on $a \leq t_0 \leq b$.*

Below, unless otherwise specified only $d \times d$ matrix solutions $Y = Y(t)$ of (4.8) will be considered. In this case, either det $Y(t) \equiv 0$ or det $Y(t) \neq 0$ for all t. This fact can be strengthened as follows:

Theorem 4.10. ***(Liouville)*** *Let $Y = Y(t)$ be a $d \times d$ matrix solution of* (4.8)*, $\Delta(t) = \det Y(t)$, and $a \leq t_0 \leq b$. Then, on $[a, b]$,*

$$\Delta(t) = \Delta(t_0) \exp \int_{t_0}^{t} \operatorname{tr} A(s) ds_0 \tag{4.11}$$

Proof. Let $A(t) = (a_{jk}(t)), j, k = 1, \ldots, d$. The usual expansion for the determinant $\Delta(t) =$ det $Y(t)$, where $Y(t) = (y_k j(t))$, and the rule for differentiating the product of d scalar functions show that

$$\Delta'(t) = \sum \det Y_j(t)$$

where $Y_j(t)$ is the matrix obtained by replacing the j-th row $\left(y_1^j(t), \ldots, y_d^j(t)\right)$ of $Y(t)$ by its derivative $\left(y_1^{j\prime}(t), \ldots, y_d^{j\prime}(t)\right)$. Since $y_k^{j\prime}(t) = \sum a_{ji} y_k^i$ by (4.1), it is seen that the j th row of $Y_j(t)$ is the sum of $a_{jj}(t)$ times the j th row of $Y(t)$ and a linear combination of the other rows of $Y_j(t)$. Hence $\det Y_j(t) = a_{jj}(t) \det Y(t)$ and so, $\Delta'(t) = (\operatorname{tr} A(t))\Delta(t)$. □

Remark 4.12. *We could use [3, eq. 0.8.10.1 and eq. 0.8.2.2] to get a more tidy statement of the proof.*

Definition 4.13. ***(Fundamental Matrix)*** *A fundamental matrix $Y(t)$ of* (4.1) *or* (4.8) *means a solution of* (4.8) *such that* $\det Y(t) \neq 0$.

Question: How to obtain a fundamental matrix $Y(t)$?

Answer: Let $Y(t)$ be a matrix with columns $y_1(t), \ldots, y_d(t)$, where $y = y_j(t)$ is a solution of (4.1) belonging to a given initial condition $y_j(t_0) = y_{j0}$, where $y_{10}, \ldots, y_{d0}$ are **linearly independent** vectors. It is clear that all fundamental matrices $Y(t)$ can be obtained in this fashion. We may denote is as $Y(t, t_0)$.

Definition 4.14. ***(State Transition Matrix)*** *Let $\Phi(t, t_0) = Y(t, t_0)$ denote the particular fundamental matrix satisfying*

$$\Phi(t_0, t_0) = I. \tag{4.15}$$

In particularly, we can get it by: $\Phi(t, t_0) = Y(t)Y^{-1}(t_0)$.

Corollary 4.16. *The state transition matrix is the* ***unique solution*** *of*

$$Y' = A(t)Y$$

with initial condition $Y(t_0) = I$.

Corollary 4.17.

$$\begin{aligned} \Phi(t,t) &= I \\ \Phi^{-1}(t, t_0) &= \left[Y(t)Y^{-1}(t_0)\right]^{-1} = Y(t_0)Y^{-1}(t) = \Phi(t_0, t) \\ \Phi(t, t_0) &= \Phi(t, t_1)\Phi(t_1, t_0) \end{aligned}$$

for every t, t_0 and t_1.

5 Dependence on Initial Conditions and Parameters

5.1 Preliminaries

Let $f(t, y)$ be defined on an open (t, y)-set E with the property that if $(t_0, y_0) \in E$, then the initial value problem

$$y' = f(t, y) \text{ and } y(t_0) = y_0 \tag{5.1}$$

has a unique solution $y(t) = \eta(t, t_0, y_0)$ which is then defined on a maximal t-interval (ω_-, ω_+), where $\omega_\pm$depends on (t_0, y_0).

In this chapter, the problem of the **smoothness** (i.e., of the continuity or differentiability properties) of $\eta(t, t_0, y_0)$ will be considered.

Often, a more general situation is encountered in which (5.1) is replaced by a family of initial value problems depending on a set of parameters $z = \left(z^1, \ldots, z^e\right)$

$$y' = f(t, y, z) \text{ and } y(t_0) = y_0 \tag{5.2}$$

where for each fixed z, (5.2) has a unique solution $y(t) = \eta(t, t_0, y_0, z)$.

• **(5.1) $\Rightarrow$ (5.2):**

In most cases, the question of the dependence of solutions of (5.1) on t and initial conditions can be reduced to the question of the dependence on t, z of solutions of a family of initial value problems (5.2) for **fixed initial conditions** $y(t_0) = y_0$.

The reduction is accomplished by the change of variables $t, y \to t + t_0, y + y_0$ which changes (5.1) to

$$y' = f(t + t_0, y + y_0) \text{ and } y(0) = 0, \tag{5.3}$$

in which $z = (t_0, y_0) = \left(t_0, {y_0}^1, \ldots, {y_0}^d\right)$ can be considered as a set of parameters (and the initial condition $y(0) = 0$ is fixed).

• **(5.2) $\Rightarrow$ (5.1):**

conversely, the question of the dependence of solutions of (5.2) on t, t_0, y_0, z can be reduced to the question of smoothness of solutions of initial value problems in which extra parameters z do not occur.

The reduction is obtained by replacing (5.2) by an initial value problem for a $(d + e)$-dimensional vector (y, z), in which no extra parameters occur:

$$y' = f(t, y, z), z' = 0 \text{ and } y(t_0) = y_0, z(t_0) = z_0 \tag{5.4}$$

where (t_0, y_0, z_0) denotes any of the possible choices for (t, y, z).

For the above reason, some of the theorems which follow will be stated for (5.2) but proved for (5.1).

Below $x, y, \eta, f \in \mathbf{R}^d$ and $z \in \mathbf{R}^e$, where $d, e \geq 1$.

5.2 Continuity

The assumption of **uniqueness** implies the **continuity** of the general solution $y = \eta(t, t_0, y_0, z)$ of $y' = f(t, y, z)$:

Theorem 5.5. *Let $f(t, y, z)$ be continuous on an open (t, y, z)-set E with the property that for* ***every*** *$(t_0, y_0, z) \in E$, the initial value problem (5.2), with z fixed, has a unique solution $y(t) \equiv \eta(t, t_0, y_0, z)$. Let $\omega_- < t < \omega_+$ be the maximal interval of existence of $y(t) = \eta(t, t_0, y_0, z)$. We then have*

1) *$\omega_+ = \omega_+(t_0, y_0, z)$ (or $\omega_- = \omega_-(t_0, y_0, z)$) is a* ***lower (or upper) semicontinuous*** *function of $(t_0, y_0, z) \in E$ and*

2) *$\eta(t, t_0, y_0, z)$ is continuous on the set $\omega_- < t < \omega_+, (t_0, y_0, z) \in E$. Furthermore, it is* ***uniform continuous*** *over a compact subinterval.*

Remark 5.6. *It is understood that $\omega_+ [\omega_-]$ can assume the value $+\infty[-\infty]$. The lower semicontinuity of ω_+ at (t_0, y_0, z_0) means that $\omega_+(t_0, y_0, z_0) \leq \liminf \omega_+(t_1, y_1, z_1)$ as $(t_1, y_1, z_1) \to (t_0, y_0, z_0)$. The upper semicontinuity of ω_- is similarly defined.*

$\omega_\pm(t_0, y_0, z)$ ***need not be continuous.*** *For suppose that $(t_1, y_1, z_1) \in E$ and $t_1 < t^0 < \omega_+(t_1, y_1, z_1)$. If E is replaced by the set obtained from E by deleting the point $(t, y, z) = \left(t^0, \eta\left(t^0, t_1, y_1, z_1\right), z_1\right)$, then $\omega_+(t_1, y_1, z_1)$ now takes the value t^0, but ω_+ is not altered for all points (t_0, y_0, z) near (t_1, y_1, z_1).*

Remark 5.7. *Let $f(t, y, z), \eta(t, t_0, y_0, z)$ be as in Theorem 5.5.*

$\diamond$ ***continuous one-to-one map*** *$y_0 \mapsto y$:*

For ***fixed*** *(t, t_0, z), the relation $y = \eta(t, t_0, y_0, z)$ can be considered as a map carrying y_0 into y. The assumption that the solution of (5.2), for $(t_0, y_0, z) \in E$, is unique implies that this map is one-to-one. A consequence of Theorem 5.5 is that* ***"the map*** *$y_0 \to y$* ***is continuous."***

$\diamond$ ***inverse map*** *$y \mapsto y_0$:*

In fact the ***inverse map*** *is given by $y_0 = \eta(t_0, t, y, z)$.*

$\diamond$ ***Some thinking:***

Consider (5.1). The solution of the initial value problem $y' = f, y(t_0) = \eta(t_0, t_0 + h, y_0)$ is the same as the solution of $y' = f, y(t_0 + h) = y_0$. In other words, $\eta(t, t_0 + h, y_0) = \eta(t, t_0, \eta(t_0, t_0 + h, y_0))$

See also Remark 5.17 for C^1 continuous.

Proof. Since (5.2) can be replaced by (5.4) and (y, z) by y, there is no loss of generality in supposing that f does not depend on z. Thus, it will be supposed that $f(t, y)$ is defined on an open (t, y)-set E and that (5.1) has a unique solution $y(t) = \eta(t, t_0, y_0)$ on a maximal interval of existence $\omega_- < t < \omega_+$, where $\omega_\pm = \omega_\pm(t_0, y_0)$. It will be shown that, in this form, Theorem 5.5 is merely a corollary of Theorem 2.34 Convergence of ODE II for the case $f_n(t, y) \equiv f(t, y), n = 1, 2, \ldots$.

proof of 1).: In order to verify that $\omega_+(t_0, y_0)$ is lower semicontinuous, for any sequence of points $(t_1, y_{10}), (t_2, y_{20}), \ldots$ in E such that $(t_n, y_{n0}) \to (t_0, y_0) \in E$ and $c = \liminf \omega_+(t_n, y_{n0})$ as $(t_n, y_{n0}) \to (t_0, y_0)$. Since the solution of (5.1) is unique, it follows from Theorem 2.34 Convergence of ODE II that $c \geq \omega_+ = \omega_+(t_0, y_0)$; i.e., $\omega_+(t_0, y_0)$ is lower semicontinuous. The proof of the upper semi-continuity of $\omega_-(t_0, y_0)$ is the same.

proof of 2).: Note that, for the case where $f_n = f, n = 1, 2, \ldots$, and the solution of (5.1) is unique, a selection of a subsequence in Theorem 2.34 Convergence of ODE II is unnecessary from Remark 2.6. Thus it follows that $\eta(t, t_0, y_0)$ is a **continuous function of** (t_0, y_0) **for every fixed** $t, \omega_-(t_0, y_0) < t < \omega_+(t_0, y_0)$.

In fact, this continuity is **uniform** for fixed $t^1 \leq t \leq t^2, \omega_-(t_0, y_0) < t^1 < t^2 < \omega_+(t_0, y_0)$. In other words, if $\epsilon > 0$, there exists a $\delta_{\epsilon_0} > 0$, depending on $\left(t_0, y_0, t^1, t^2\right)$, such that

$$|\eta(t, t_0, y_0) - \eta(t, t_1, y_1)| < \epsilon \quad \text{if} \quad |t_0 - t_1|, |y_0 - y_1| < \delta_{\epsilon 0}$$

for $t^1 \leq t \leq t^2$. But since $\eta(t, t_0, y_0)$ is a continuous function of t for fixed (t_0, y_0), there is a $\delta_{\epsilon 1} > 0$, depending on $\left(t_0, y_0, t^1, t^2\right)$, such that

$$|\eta(t, t_0, y_0) - \eta(s, t_0, y_0)| < \epsilon \quad \text{if} \quad |t - s| < \delta_{\epsilon 1}, \quad t^1 \leq s, t \leq t^2.$$

Hence, if $\delta_\epsilon = \min(\delta_{\epsilon_0}, \delta_{\epsilon 1})$ and $|t - s|, |t_0 - t_1|, |y_0 - y_1| < \delta_\epsilon$, then

$$|\eta(s, t_0, y_0) - \eta(t, t_1, y_1)| < 2\epsilon.$$

This completes the proof of Theorem 5.5. □

5.3 Differentiability

If it is assumed that $f(t, y, z)$ is of class C^1, it follows that the general solution $y = \eta(t, t_0, y_0, z)$ of (5.2) is of class C^1. In fact, even more is contained in the following theorem. Here when differential w.r.t. t we write as η', while differential w.r.t. t_0 we write as $\partial\eta/\partial t_0$.

5.3.1 Main Theorem

Theorem 5.8. ***(Peano)*** *Let $f(t, y, z)$ be in class C^1, i.e. be continuous on an open (t, y, z)-set E and possess continuous first order partials $\partial f/\partial y^k, \partial f/\partial z^j$ with respect to the components of y and z. We then have*

1) *Then the unique solution $y = \eta(t, t_0, y_0, z)$ of* (5.2) *is of class C^1 on its open domain of definition $\omega_- < t < \omega_+, (t_0, y_0, z) \in E$, where $\omega_\pm = \omega_\pm(t_0, y_0, z)$.*

2) *Furthermore, if $J(t) = J(t, t_0, y_0, z)$ is the **Jacobian matrix** $(\partial f/\partial y)$ of $f(t, y, z)$ with respect to y at $y = \eta(t, t_0, y_0, z)$,*

$$J(t) = J(t, t_0, y_0, z) = \left(\frac{\partial f}{\partial y}\right) \text{ at } y = \eta(t, t_0, y_0, z) \tag{5.9}$$

   then

   (a) *$x = \partial\eta(t, t_0, y_0, z)/\partial y_0^k$ is the solution of the initial value problem,*

   $$x' = J(t)x, \quad x(t_0) = e_k, \tag{5.10}$$

   where $e_k = \left(e_k^1, \ldots, e_k^d\right)$ with $e_k^j = 0$ if $j \neq k$ and $e_k^k = 1$;

   (b) *$x = \partial\eta(t, t_0, y_0, z)/\partial z^j$ is the solution of*

   $$x' = J(t)x + g_j(t), \quad x(t_0) = 0, \tag{5.11}$$

   where $g_j(t) = g_j(t, t_0, y_0, z)$ is the vector $\partial f(t, y, z)/\partial z^j$ at $y = \eta(t, t_0, y_0, z)$;

   (c) *$\partial\eta(t, t_0, y_0, z)/\partial t_0$ is given by*

   $$\frac{\partial\eta}{\partial t_0} = -\sum_{k=1}^{d} \frac{\partial\eta}{\partial y_0^k} f^k(t_0, y_0, z) \tag{5.12}$$

5.3.2 More Explanation

(a). (b). The uniqueness of the solution of (5.2) is assured, e.g., by Theorem 2.17. Note that

(a) the assertion concerning $\partial\eta/\partial y_0^k$ and (5.10), or

(b) the assertion concerning $\partial\eta/\partial z^j$ and (5.11)

result from "formally" differentiating both equations in (5.2), i.e., both equations in

$$\eta'(t, t_0, y_0, z) = f(t, \eta, z), \quad \eta(t_0, t_0, y_0, z) = y_0$$

with respect to y_0^k or z^j.

(note here we need to assume **symmetric property of partial derivative**. This property is implied by the formal proof in Section 5.3.3 and Corollary 5.18):

(a)

$$\frac{\partial \eta'(t,t_0,y_0,z)}{\partial y_0^k} = \frac{\partial f(t,\eta,z)}{\partial \eta}\frac{\partial \eta}{\partial y_k^0}, \quad \frac{\partial \eta(t_0,t_0,y_0,z)}{\partial y_0^k} = \frac{\partial y_0}{\partial y_0^k} = e_k$$

(b)

$$\frac{\partial \eta'(t,t_0,y_0,z)}{\partial z^j} = \frac{\partial f(t,\eta,z)}{\partial \eta}\frac{\partial \eta}{\partial z^j} + \frac{\partial f(t,\eta,z)}{\partial z^j}, \quad \frac{\partial \eta(t_0,t_0,y_0,z)}{\partial z^j} = \frac{\partial y_0}{\partial z^j} = 0$$

(c). Similarly, differentiating these equations formally with respect to t_0 shows that $x = \partial\eta/\partial t_0$ is also a solution of

$$x' = J(t)x, \quad x(t_0) = -\eta'(t_0,t_0,y_0,z) = -f(t_0,y_0,z). \tag{5.13}$$

Note the initial condition is from

$$\partial\eta(t_0,t_0,y_0,z)/\partial t_0 = \eta' + \partial\eta/\partial t_0 = \partial y_0/\partial t_0 = 0.$$

Writing $f(t_0,y_0,z) = \sum f^k(t_0,y_0,z)\,e_k$, it follows that (5.12) is a formal consequence of (5.10) and the linear property of linear system Theorem 4.7, since $e_k = x(t_0)$ in (5.12).

Remark 5.14. ***(equation of variation)*** *More generally, if $y(t,s)$ is a 1-parameter family of solutions of $y' = f(t,y,z)$ for fixed z, if $y(t_0,s_0) = y_0$, and if $y(t,s)$ is of class C^1 in (t,s), then the partial derivative $x = \partial y(t,s)/\partial s$ at $s = s_0$ is also a solution of the system $x' = J(t)x$. For this reason $x' = J(t)x$ is called the* ***equation of variation*** *of* (5.2) *along the solution $y = \eta(t,t_0,y_0,z)$.*

The assertion concerning $x = \partial\eta/\partial y_0^k$ and (5.10) shows that the Jacobian matrix $(\partial\eta/\partial y_0)$ is the **fundamental matrix** for $x' = J(t)x$ with $x(t_0) = I$. In particular, Theorem 4.10 implies

Corollary 5.15. *Under the conditions of Theorem 5.8*

$$\det\left(\frac{\partial\eta(t,t_0,y_0,z)}{\partial y_0}\right) = \exp\int_{t_0}^{t}\sum_{j=1}^{d}\frac{\partial f^j}{\partial y^j}ds \tag{5.16}$$

for $\omega_- < t < \omega_+$, where the argument of the integrand is $(s,\eta(s,t_0,y_0,z),z)$.

Remark 5.17. *By* (5.16), $\det(\partial\eta/\partial y_0) \neq 0$*. Thus, the continuous one-to-one map $y_0 \to y = \eta(t,t_0,y_0,z)$ for* ***fixed*** *(t,t_0,z), considered in Remark 5.7, is of class C^1 and has an inverse of class C^1 with respect to (t,t_0,y,z). This statement about the inverse is also clear from the explicit formula $y \to y_0 = \eta(t_0,t,y,z)$.*

Corollary 5.18. *Under the conditions of Theorem 5.8, the second mixed derivatives $\partial^2\eta/\partial y_0^k\partial t = \partial^2\eta/\partial t\partial y_0^k, \partial^2\eta/\partial z^j\partial t = \partial^2\eta/\partial t\partial z^j, \partial^2\eta/\partial t_0\partial t = \partial^2\eta/\partial t\partial t_0$ exist and are continuous.*

Proof. Note that the assertion that $x = \partial\eta/\partial y_0^k$ is a solution of (5.10) implies that the iterated derivative $\partial\left(\partial\eta/\partial y_0^k\right)/\partial t$ exists and is $J(t)\partial\eta/\partial y_0^k$. The last expression is a continuous function of t, t_0, y_0, z; hence, Schwarz's theorem implies that $\partial(\partial\eta/\partial t)/\partial y_0^k$ exists and is $\partial\left(\partial\eta/\partial y_0^k\right)/\partial t$. A similar remark applies to $\partial\eta/\partial z^j$. Also, note that on the right side of (5.12) the variable t only occurs in $\partial\eta/\partial y_0^k$.

□

5.3.3 Proof

We first state and prove a lemma which is a convenient substitute for the **mean value theorem** of differential calculus when dealing with vectors, for it avoids some awkwardness in the fact that θ_k depends on k in $y(b) - y(a) = (b-a)\left(y^{1\prime}(\theta_1),\ldots,y^{d\prime}(\theta_d)\right)$, where $a < \theta_k < b$.

Lemma 5.19. *Let $f(t,y)$ be continuous on a product set $(a,b)\times K$, where K is an* ***open convex*** *y-set, and let f have continuous partials $\partial f/\partial y^k$ with respect to the components of y. Then there exist continuous functions $f_k(t,y_1,y_2)$, $k = 1,\ldots,d$, on the product set $(a,b)\times K\times K$ such that*

$$f_k(t,y,y) = \frac{\partial f(t,y)}{\partial y^k} \tag{5.20}$$

and that if $(t, y_1, y_2) \in (a, b) \times K \times K$*, then*

$$f(t, y_2) - f(t, y_1) = \sum_{k=1}^{d} f_k(t, y_1, y_2)\left(y_2^k - y_1^k\right).$$

In fact, $f_k(t, y_1, y_2)$ *is given by*

$$f_k(t, y_1, y_2) = \int_0^1 \frac{\partial f(t, sy_2 + (1-s)y_1)}{\partial y^k} ds$$

Proof. Put $F(s) = f(t, sy_2 + (1-s)y_1)$ for $0 \le s \le 1$. The convexity of K implies that $F(s)$ is defined. Then $dF/ds = \sum\left(y_2^k - y_1^k\right)\partial f(t, sy_2 + (1-s)y_1)/\partial y^k$. Note that $F(1) = f(t, y_2)$ and $F(0) = f(t, y_1)$, the lemma follows. □

Proof. **of Theorem 5.8**. Since (5.2) can be replaced by (5.4) and (y, z) by y, there is no loss of generality in supposing that f does not depend on z when proving the existence and continuity of the partial derivatives of η. Thus we consider the initial value problem (5.1) with the solution $y = \eta(t, t_0, y_0)$ on $\omega_- < t < \omega_+$.

In order to simplify the domain on which the function η must be considered, let a, b be arbitrary numbers satisfying $\omega_- < a < b < \omega_+$, where $\omega_\pm = \omega_\pm(t_1, y_1)$. Then, by Theorem 5.5, $\eta(t, t_0, y_0)$ is defined and continuous for $a \le t \le b$ and (t_0, y_0) near (t_1, y_1). In the following only such (t, t_0, y_0) will be considered. Since the assertions of Theorem 5.8 are "local," it clearly suffices to prove the assertions on the interior of such a (t, t_0, y_0)-set. We only need to prove 2) since the conclusion C^1 of 1) is implied in 2).

proof of 2)(a):.

$\diamond$ **existence of** $\partial\eta/\partial y_0^k$:

Let h be a scalar, e_k the vector in (5.10) and, for small $|h|$, define

$$y_h(t) = \eta(t, t_0, y_0 + he_k) \tag{5.21}$$

This is defined on $a \le t \le b$ and, by Theorem 5.5,

$$y_h(t) \to y_0(t) \quad \text{as } h \to 0 \tag{5.22}$$

uniformly on $[a, b]$. By (5.1), $(y_h(t) - y_0(t))' = f(t, y_h(t)) - f(t, y_0(t))$. Applying Lemma 5.19 with $y_2 = y_h(t), y_1 = y_0(t)$, we have

$$[y_h(t) - y_0(t)]' = \sum_{k=1}^{d} f_k(t, y_0(t), y_h(t))\left[y_h^k(t) - y_0^k(t)\right] \tag{5.23}$$

Define the abbreviation x_h as

$$x_h = \frac{y_h(t) - y_0(t)}{h}, \quad h \ne 0. \tag{5.24}$$

The existence of $\partial\eta(t, t_0, y_0)/\partial y_0^k$ **is equivalent to the existence** of $\lim x_h(t)$ as $h \to 0$.

By (5.1) and (5.21), $y_h(t_0) = y_0 + he_k$, and so $x_h(t_0) = e_k$. Thus, by dividing both sides of (5.23) by h and (5.24), $x = x_h(t)$ is the solution of the initial value problem

$$x' = J(t; h)x, \quad x(t_0) = e_k, \tag{5.25}$$

where $J(t; h)$ is a $d \times d$ matrix in which the k th column is the vector $f_k(t, y_0(t), y_h(t))$.

By Lemma 5.19, the continuity of $f_k(t, y_1, y_2)$ and (5.22), it follows that $J(t; h) \to J(t; 0)$ as $h \to 0$ uniformly on $[a, b]$ (continuous function of uniformly convergence argument), where $J(t; 0) = J(t)$ is the matrix defined by (5.9).

Consider (5.25) to be a family of initial value problem depending on a parameter h, where the right side $J(t; h)x$ of the differential equation is continuous on the open-set $a < t < b, |h|$ small, x arbitrary. Since the solutions of (5.25) are unique, Theorem 5.5 implies that the general solution is a

continuous function of h [for fixed (t, t_0)]. In particular, $x(t) = \lim x_h(t), h \to 0$, exists and is the solution of (5.10) on $a < t < b$. Hence $\partial\eta(t, t_0, y_0) / \partial y_0^k$ **exists**.

$\diamond$ **continuous of** $\partial\eta/\partial y_0^k$:

In order to verify that this partial derivative is continuous with respect to all of its arguments, rewrite (5.10) as

$$x' = J(t, t_0, y_0)\, x, \quad x(t_0) = e_k, \tag{5.26}$$

a family of initial value problems depending on parameters (t_0, y_0). Since $J(t, t_0, y_0)$ is a continuous function of (t, t_0, y_0) and initial value problems associated with linear differential equations have unique solutions, Theorem 5.5 implies that the solution $x = \partial\eta(t, t_0, y_0) / \partial y_0^k$ of (5.26) is a continuous function of its arguments.

proof of 2)(b): No need to prove since we have convert to the problem (5.1).

proof of 2)(c):.

The existence and continuity of $\partial\eta(t, t_0, y_0) / \partial t_0$ will now be considered. Put

$$x_h(t) = \frac{\eta(t, t_0 + h, y_0) - \eta(t, t_0, y_0)}{h}, \quad h \neq 0$$

The solution of the initial value problem $y' = f, y(t_0) = \eta(t_0, t_0 + h, y_0)$ is the same as the solution of $y' = f, y(t_0 + h) = y_0$. In other words, $\eta(t, t_0 + h, y_0) = \eta(t, t_0, \eta(t_0, t_0 + h, y_0))$ (see also the inverse map in Remark 5.7). Thus we have

$$h x_h(t) = \eta(t, t_0, \eta(t_0, t_0 + h, y_0)) - \eta(t, t_0, y_0)$$

Since $\eta(t, t_0, y_0)$ has continuous partial derivatives with respect to the components of y_0 and $\eta(t_0, t_0 + h, y_0) \to \eta(t_0, t_0, y_0) = y_0$ as $h \to 0$, it follows that

$$h x_h(t) = \sum \left[\frac{\partial\eta(t, t_0, y_0)}{\partial y_0^k} + o(1)\right] \left[\eta^k(t_0, t_0 + h, y_0) - y_0^k\right]$$

as $h \to 0$. By the mean value theorem and $y_0 = \eta(t_0 + h, t_0 + h, y_0)$, there is a $\theta = \theta_k$ such that

$$\eta^k(t_0, t_0 + h, y_0) - y_0^k = -h\eta^{k\prime}(t_0 + \theta h, t_0 + h, y_0), 0 < \theta < 1.$$

Note that $\eta^{k\prime}(t_0 + \theta h, t_0 + h, y_0) = f^k(t_0 + \theta h, \eta(t_0 + \theta h, t_0 + h, y_0))$ is $f^k(t_0, y_0) + o(1)$ as $h \to 0$. Thus, as $h \to 0$

$$x_h(t) = -\sum \left[\frac{\partial\eta(t, t_0, y_0)}{\partial y_0^k} + o(1)\right] \left[f^k(t_0, y_0) + o(1)\right]$$

This shows that $\partial\eta/\partial t_0 = \lim x_h(t)$ exists as $h \to 0$ and satisfies the analogue of the relation (5.12) . This relation implies that $\partial\eta/\partial t_0$ is a continuous function of (t, t_0, y_0).

$\square$

The proof of Theorem 5.8 has the following consequence.

Corollary 5.27. *Let $f(t, y, z, z^*)$ be a continuous function on an open (t, y, z, z^*)-set E, where z^* is a vector of any dimension. Suppose that f has continuous first order partial derivatives with respect to the components of y and z. Then*

$$y' = f(t, y, z, z^*), \quad y(t_0) = y_0$$

has a unique solution $\eta = \eta(t, t_0, y_0, z, z^*)$ for fixed z, z^* with $(t_0, y_0, z, z^*) \in E; \eta$ has first order partials with respect to t, t_0, the components of y and of z, and the second order partials $\partial^2\eta/\partial t\partial t_0, \partial^2\eta/\partial t\partial y_0^k, \partial^2\eta/\partial t\partial z^j$; finally, these partials of η are continuous with respect to (t, t_0, y_0, z, z^*).

Remark 5.28. *Compare with Theorem 5.8, Corollary 5.27 does not require partial derivative w.r.t. z^*.*

5.4 Higher Order Difierentiability

The question of higher order differentiability of the general solution is easily settled by the use of Theorem 5.8 and Corollary 5.27.

Theorem 5.29. *Let $f(t, y, z, z^*)$ be a continuous function on an open (t, y, z, z^*)-set E such that f has continuous partial derivatives of all orders not exceeding $m, m \geq 1$, with respect to the components of y and z. Then*

$$y' = f(t, y, z, z^*), \quad y(t_0) = y_0 \tag{5.30}$$

has a unique solution $\eta = \eta(t, t_0, y_0, z, z^*)$, for fixed z, z^* with $(t_0, y_0, z, z^*) \in E$, and η has all continuous partial derivatives of the form

$$\frac{\partial^{i+i_0+\alpha_1+\cdots+\alpha_d+\beta_1+\cdots+\beta_0}\eta}{\partial t^i \partial t_0^{i_0} \partial\left(y_0^1\right)^{\alpha_1} \ldots \partial\left(y_0^d\right)^{\alpha_d} \partial\left(z^1\right)^{\beta_1} \ldots \partial\left(z^e\right)^{\beta_e}} \tag{5.31}$$

where $i \leq 1, i_0 \leq 1$ and $i_0 + \sum \beta_k + \sum \alpha_j \leq m$.

Proof. The proof will be given first with $i_0 = 0$ by induction on m. The case $m = 1$ is correct by Corollary 5.27 for z^* of any dimension. Assume the validity of the theorem if m is replaced by $m - 1 (\geq 1)$.

Consider the analogue of (5.10),

$$x' = J(t, t_0, y_0, z, z^*)\, x, \quad x(t_0) = e_k \tag{5.32}$$

where $J = (\partial f / \partial y)$ at $y = \eta(t, t_0, y_0, z, z^*)$. By the assumption on f and by the induction hypothesis, the right side, $J(t, t_0, y_0, z, z^*)\, y$, of the differential equation in (5.32) has continuous partial derivatives of order $\leq m-1$ with respect to the components of y, y_0, and z. Hence, **appling the induction hypothesis to (5.32)**, the solution $x = \partial\eta(t, t_0, y_0, z, z^*) / \partial y_0^k$ of (5.32) has continuous partial derivatives of all orders $\leq m - 1$ with respect to the components of y_0 and each of these partials has a continuous partial derivative with respect to t.

Similarly, the analogue of (5.11) shows that $\partial\eta/\partial z^j$ has continuous partial derivatives of all orders $\leq m - 1$ with respect to the components of y_0, and z, each of these partials has a continuous partial derivative with respect to t. This completes the induction and shows that $\eta(t, t_0, y_0, z, z^*)$ has continuous partial derivatives of the form (5.31) with $i_0 = 0, i \leq 1, \sum \alpha_k + \sum \beta_j \leq m$.

The existence and continuity of derivatives of the form (5.31) with $i_0 = 1, i \leq 1, \sum_{\alpha_k} + \sum \beta_j \leq m - 1$ follows from the analogue of (5.12). This completes the proof.

□

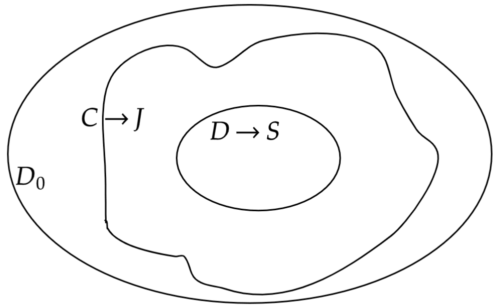
Corollary 5.33. *Let $f(t, y, z)$ be of class $C^m, m \geq 1$, on an open $(t, y, z)-$ set. Then the solution $y = \eta(t, t_0, y_0, z)$ of (5.2)i s of class C^m on its domain of existence.*

5.5 Exterior Derivatives

Several useful concepts will be introduced in this section. All concepts are of a "local" nature.

Definition 5.34. *(2-**surface**) By a (piece of) **2-dimensional surface** S of class $C^m, m \geq 1$, in a Euclidean y-space R^d is meant a set S of points y in R^d which can be put into one-to-one correspondence with an open set D of points (u, v) in a Euclidean plane by a function $y = y(u, v)$ of class C^m on D such that the two vectors $\partial y/\partial u, \partial y/\partial v$ are **linearly independent** at every point of D. The function $y = y(u, v)$ is called an **admissible parametrization** of S.*

Definition 5.35. ***(Jordan Curve)*** *Consider a piece of surface S_0 of class C^1 with an admissible parametrization $y = y(u, v)$ defined on a simply connected, bounded open set D_0 and a piecewise C^1 Jordan curve C in D_0 **bounding** an open subset D of D_0. Let S, J be the y-image of D, C, respectively. This situation will be described briefly by saying **"a piece of C^1 surface S bounded by piecewise C^1 Jordan curve J"**.*



• **differential r-form**

A differential r-form on an open set E is a formal expression

$$\omega = \sum_{i_1=1}^{d} \dots \sum_{i_r=1}^{d} p_{i_1} \dots i_r(y) dy^{i_1} \wedge \dots \wedge dy^{i_r}$$

with real-valued coefficients defined on E.

Remark 5.36. *Some notes:*

1. *$p_{i_1} \dots i_r(y) = \pm p_{j_1} \dots_r(y)$ according as $(j_1, \dots, j_r)$ is an even or odd permutation of $(i_1, \dots, i_r)$. In particular, $p_{i_1 \dots i_r}(y) = 0$ if two of the indices $i_1, \dots, i_r$ are equal.*

2. *Differential r-forms can be added in the obvious way. Differential r-and s-forms can be multiplied to give an $(r+s)$-form by the usual associative, distributive laws and anticommutative law $dy^i \wedge dy^j = -dy^j \wedge dy^i$.*

Definition 5.37. ***(Continuous or Class C^m k-form)*** *The form ω is called continuous [or of class C^m or 0] if its coefficients are continuous [or of class C^m or identically 0] on E.*

Definition 5.38. ***(Uniformly Bounded [or Uniformly Convergent] C^m k-form)*** *A sequence of differential r-forms on E is said to be* ***uniformly bounded [or uniformly convergent]*** *if the sequences of the corresponding coefficients are uniformly bounded [or uniformly convergent].*

Definition 5.39. ***(Continuous Exterior Derivative)*** *A continuous linear differential form (1-form or Pfaffian)*

$$\omega = \sum_{j=1}^{d} p_j(y) dy^j \tag{5.40}$$

on E is said to possess a ***continuous exterior derivative*** *$d\omega$ if there exists a continuous differential 2-form*

$$d\omega = \sum_{i=1}^{d} \sum_{j=1}^{d} p_{ij}(y) dy^i \wedge dy^j, \quad p_{ij} = -p_{ji} \tag{5.41}$$

on E such that Stokes' formula

$$\int_J \omega = \iint_S d\omega \tag{5.42}$$

holds for ***every piece*** *of C^1 surface S in E bounded by a C^1 piecewise Jordan curve J in E.*

Remark 5.43. ***(equivalent statement)*** *It is clear that if S is the image of D on the surface $S_0 : y = y(u, v)$ for $(u, v) \in D$, and J is the image of the Jordan curve C, then* (5.42) *means that*

$$\int_C \sum_{j=1}^{d} p_j(y(u, v)) dy^j(u, v) = \int_D \sum_{i=1}^{d} \sum_{j=1}^{d} p_{ij}(y(u, v)) \frac{\partial \left(y^i, y^j\right)}{\partial(u, v)} dudv \tag{5.44}$$

with the usual convention as to orientation of C around D.

Example 5.45. *If the coefficients $p_j(y)$ of* (5.40) *are of class C^1, then ω has a continuous exterior derivative with $d\omega = \Sigma dp_j(y) \wedge dy^j$ or*

$$d\omega = \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{1}{2}\left(\frac{\partial p_j}{\partial y^i} - \frac{\partial p_i}{\partial y^j}\right) dy^i \wedge dy^j.$$

The fundamental lemma about the existence of continuous exterior derivatives is the following:

Lemma 5.46. *Let* (5.40) *be a continuous linear differential* 1*-form on an open set E. Then* (5.40) *has a continuous exterior derivative* (5.41) *on E if and only if, on every open subset E^0 with compact closure $E^0 \subset E$, there exists a sequence of 1-forms $\omega^1, \omega^2, \dots$ of class C^1 such that*

1. *$\omega^n \to \omega$ as $n \to \infty$ uniformly on E^0, and*
2. *$d\omega^1, d\omega^2, \dots$ is uniformly convergent on E^0*

Proof. **proof of "$\Leftarrow$":** Denote the limit of $d\omega^n$ as $d\omega$. For any p_i as in (5.44), from uniform convergence and the exchange of integral and limit, we could get (5.42).

proof of "$\Rightarrow$": If (5.40) is a continuous 1-form on E with a continuous exterior derivative (5.41), approximate the coefficients of ω, $d\omega$ by the method in Section 2.1.7:

Let $\varphi(t)$ be as in Section 2.1.7 and put, for $\epsilon = 1/n$,

$$p_i^{(n)}(y) = c\epsilon^{-d}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty} p_i(y-\eta)\varphi\left(\epsilon^{-2}\|\eta\|^2\right) d\eta^1 \dots d\eta^d$$
$$p_{ij}^{(n)}(y) = c\epsilon^{-d}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} p_{ij}(y-\eta)\varphi\left(\epsilon^{-2}\|\eta\|^2\right) d\eta^1 \dots d\eta^d.$$

Since the integrals are actually integrals over spheres $\|\eta\| \le \epsilon, p_i^{(n)}$ and $p_{ij}^{(n)}$ are defined on the sets E_n consisting of points y whose (Euclidean) distance from the boundary of E not exceeds $\epsilon = 1/n$. In particular, they are defined on E^0 for large n and tend uniformly to p_i, p_{ij}, respectively, on E^0, as $n \to \infty$.

Define the C^∞ forms $\omega^n = \Sigma p_j^{(n)}(y)dy^j$ and $\alpha^n = \Sigma\Sigma p_{ij}^{(n)}(y)dy^i \wedge dy^j$ on E^0 for large n. Let S be a piece of C^1 surface in E^0 bounded by a C^1 piecewise Jordan curve J in E^0. Then if $\epsilon = 1/n$ is sufficiently small and $\|\eta\| \le \epsilon$, the translation $S(\eta)$ of S by the vector $-\eta$ is in E and (5.42) is valid if S is replaced by $S(\eta)$. This can be written in a form analogous to (5.44),

$$\int_C \sum p_j(y(u,v)-\eta)dy^j(u,v) = \iint_D \sum\sum p_{ij}(y(u,v)-\eta)\frac{\partial\left(y^i, y^j\right)}{\partial(u,v)}dudv$$

Let this relation be multiplied by $c\epsilon^{-d}\varphi\left(\epsilon^{-2}\|\eta\|^2\right)$ and integrated over $\|\eta\| \le \epsilon$ with respect to $d\eta^1 \dots d\eta^d$. An obvious change of the order of integration shows that the result can be interpreted as the Stokes' relation $\int_J \omega^n = \iint_S \alpha^n$. Thus ω^n has the continuous exterior derivative $d\omega^n = \alpha^n$ in E_0. This completes the proof. □

In deciding whether or not a continuous 1 -form (5.40) has the continuous exterior derivative (5.41), it suffices to verify Stokes' formula (5.42) for rectangles S on coordinate 2-planes $y^i =$ const. for $i \ne j, k$, where $1 \le j < k \le d$. This is a consequence of the following lemma.

Lemma 5.47. *A continuous differential* 1*-form* (5.40) *on an open set E has a continuous exterior derivative.*

1. *if and only if there exists a continuous differential* 2*-form* (5.41) *such that for every pair $j, k (1 \le j < k \le d)$ and fixed y^i, with $i \ne j, i \ne k$, the* 1*-form $p_j(y)dy^j + p_k(y)dy^k$ has the continuous exterior derivative $p_{jk}dy^j \wedge dy^k + p_{kj}dy^k \wedge dy^j$; in other words,*
2. *if and only if Stokes' formula* (5.42) *holds for all rectangle S on 2-planes $y^i =$ const. for $i \ne j, k$ with closure $\bar{S} \subset E$.*

Proof. I think the proof is based on the composition of the integration trail. □

• Notation

For the sake of brevity, "vector" and "matrix" notation will be used in connection with 1-forms and their exterior derivatives. For example, an ordered set of e 1-forms $\omega_1, \ldots, \omega_e$ will be abbreviated $\omega = (\omega_1, \ldots, \omega_e)$; analogously if these forms have continuous exterior derivatives, $d\omega$ denotes the ordered set of 2 -forms $d\omega = (d\omega_1, \ldots, d\omega_e)$. Finally, if $A = (a_{ij}(y))$ is an $e \times d$ matrix function on E, by $\omega = A(y)dy$ will be meant the ordered set of 1-forms $\omega = (\omega_1, \ldots, \omega_e)$, where $\omega_i = \sum_{j=1}^{d} a_{ij}(y)dy^j$ for $i = 1, \ldots, e$.

5.6 Another Difierentiability Theorem

The main result Theorem 5.8 on the differentiability of general solutions has the following generalization.

Theorem 5.48. *Let $f(t, y, z)$ be continuous on an open (t, y, z)-set E. The initial value problem*

$$y' = f(t, y, z), \quad y(t_0) = y_0$$

*have a unique solution $y = \eta(t, t_0, y_0, z)$ for all $(t_0, y_0, z) \in E$ which is of class C^1 with respect to (t, t_0, y_0, z) on its domain of definition **if and only if** every point of E have an open neighborhood E^0 on which there exist a **continuous nonsingular** $d\times d$ matrix $A(t, y, z)$ and a continuous $d\times e$ matrix $C(t, y, z)$ such that the d differential 1-forms*

$$\omega = A(dy - f dt) + C dz \tag{5.49}$$

in the variables $dt, dy^1, \ldots, dy^d, dz^1, \ldots, dz^e$ have continuous exterior derivatives on E^0.

Remark 5.50. ***(compare with Theorem 5.8 and more explanation)*** *In contrast to Theorem 5.8, the conditions of Theorem 5.48 are invariant under C^1 changes of the variables t, y, z.*

It is understood that if $A = (a_{ij}(t, y, z))$ and $C = (c_{ik}(t, y, z))$, then (5.49) *represents an ordered set of 1 -forms, the i-th one of which is*

$$\omega_i = \sum_{j=1}^{d} a_{ij} dy^j - \left(\sum_{k=1}^{d} a_{ij} f^k\right) dt + \sum_{k=1}^{e} c_{ik} dz^k$$

$i = 1, \ldots, d$. *If this has a continuous exterior derivative, the latter is a differential 2 -form of the type*

$$\begin{aligned} d\omega_i = & \sum_{j=1}^{d}\sum_{k=1}^{d} \alpha_{ijk} dy^j \wedge dy^k + \sum_{j=1}^{d} \beta_{ij} dt \wedge dy^j + \sum_{k=1}^{e} \gamma_{ik} dt \wedge dz^k \\ & + \sum_{j=1}^{d}\sum_{k=1}^{e} \delta_{ijk} dz^k \wedge dy^j + \sum_{j=1}^{e}\sum_{k=1}^{e} \epsilon_{ijk} dz^j \wedge dz^k \end{aligned} \tag{5.51}$$

where $\alpha_{ijk} = -\alpha_{ikj}, \beta_{ij}, \gamma_{ik}, \delta_{ijk}, \epsilon_{ijk} = -\epsilon_{ikj}$ are continuous function of (t, y, z).

Theorem 5.48 will be proved later. The proof of Theorem 5.48 will have the following consequence.

Definition 5.52. *Define a $d \times d$ matrix $F(t, y, z) = (f_{ij}(t, y, z))$ and a $d \times e$ matrix $N(t, y, z) = (n_{ij}(t, y, z))$ by*

$$f_{ij} = \beta_{ij} - 2\sum_{k=1}^{d} \alpha_{ijk} f^k$$

$$n_{ij} = \gamma_{ij} + \sum_{k=1}^{d} \delta_{ikj} f^k$$

Corollary 5.53. *Let $f(t, y, z)$ be as in Theorem 5.48, let $A(t, y, z), C(t, y, z)$ exist on E^0 as specified, and consider only $(t, y, z) \in E^0$. Then $x = \partial\eta/\partial y_0^k$ is the solution of*

$$[A(t, \eta, z)x]' = F(t, \eta, z)x, \quad x(t_0) = e_k \tag{5.54}$$

and $x = \partial\eta/\partial z^i$ *is the solution of*

$$[A(t, \eta, z)x + c_i(t, \eta, z)]' = F(t, \eta, z)x + n_i(t, \eta, z), \quad x(t_0) = 0, \tag{5.55}$$

where $c_i(t, y, z), n_i(t, y, z)$ *are the* i*-th columns of* $C(t, y, z), N(t, y, z)$*, respectively, for* $i = 1, \ldots,$ e *and* $\eta = \eta(t, t_0, y_0, z)$*.*

Remark 5.56. *Note that a solution* $x = x(t)$ *of* (5.54) *does not necessarily have a derivative, but* $A(x, \eta, z)x(t)$ *has a derivative satisfying* (5.54)*.*

Remark 5.57. ***(equivalent statement)*** *The statements concerning* (5.54) *and* (5.55) *can be written more conveniently as matrix equations*

$$\left[A(t, \eta, z)\frac{\partial\eta}{\partial y_0}\right]' = F(t, \eta, z)\frac{\partial\eta}{\partial y_0}, \quad \frac{\partial\eta}{\partial y_0} = I \text{ at } t = t_0 \tag{5.58}$$

$$\left[A(t, \eta, z)\frac{\partial\eta}{\partial z} + C(t, \eta, z)\right]' = F(t, \eta, z)\frac{\partial\eta}{\partial z} + N(t, \eta, z), \quad \frac{\partial\eta}{\partial z} = 0 \text{ at } t = t_0 \tag{5.59}$$

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