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Euler's equations (rigid body dynamics)

In classical mechanics, **Euler's rotation equations** are a vectorial quasilinear first-order ordinary differential equation describing the rotation of a rigid body, using a rotating reference frame with its axes fixed to the body and parallel to the body's principal axes of inertia. Their general form is:

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) = \mathbf{M}.$$

where M is the applied torques, I is the inertia matrix, and ω is the angular velocity about the principal axes.

In three-dimensional principal orthogonal coordinates, they become:

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= M_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= M_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= M_3 \end{aligned}$$

where M_k are the components of the applied torques, I_k are the principal moments of inertia and ω_k are the components of the angular velocity about the principal axes.

Contents

Motivation and derivation

Starting from Newton's second law, in an inertial frame of reference (subscripted "in"), the time derivative of the angular momentum **L** equals the applied torque

$$\frac{d\mathbf{L}_{\text{in}}}{dt} \overset{\text{def}}{=} \frac{d}{dt}\left(\mathbf{I}_{\text{in}}\boldsymbol{\omega}\right) = \mathbf{M}_{\text{in}}$$

where $\mathbf{I}_{\text{in}}$ is the moment of inertia tensor calculated in the inertial frame. Although this law is universally true, it is not always helpful in solving for the motion of a general rotating rigid body, since both $\mathbf{I}_{\text{in}}$ and **ω** can change during the motion.

Therefore, we change to a coordinate frame fixed in the rotating body, and chosen so that its axes are aligned with the principal axes of the moment of inertia tensor. In this frame, at least the moment of inertia tensor is constant (and diagonal), which simplifies calculations. As described in the moment of inertia, the angular momentum **L** can be written

$$\mathbf{L} \stackrel{\text{def}}{=} L_1\mathbf{e}_1 + L_2\mathbf{e}_2 + L_3\mathbf{e}_3 = I_1\omega_1\mathbf{e}_1 + I_2\omega_2\mathbf{e}_2 + I_3\omega_3\mathbf{e}_3$$

where M_k, I_k and ω_k are as above.

In a *rotating* reference frame, the time derivative must be replaced with (see time derivative in rotating reference frame)

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\mathrm{rot}} + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{M}$$

where the subscript "rot" indicates that it is taken in the rotating reference frame. The expressions for the torque in the rotating and inertial frames are related by

$$\mathbf{M}_{\mathrm{in}} = \mathbf{Q}\mathbf{M},$$

where **Q** is the rotation tensor (not rotation matrix), an orthogonal tensor related to the angular velocity vector by

$$\boldsymbol{\omega} \times \boldsymbol{v} = \dot{\mathbf{Q}}\mathbf{Q}^{-1}\boldsymbol{v}$$

for any vector **v**.

In general, **L** = **Iω** is substituted and the time derivatives are taken realizing that the inertia tensor, and so also the principal moments, do not depend on time. This leads to the general vector form of Euler's equations

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) = \mathbf{M}.$$

If principal axis rotation

$$L_k \stackrel{\text{def}}{=} I_k\omega_k$$

is substituted, and then taking the cross product and using the fact that the principal moments do not change with time, we arrive at the Euler equations in components at the beginning of the article.

Torque-free solutions

For the RHSs equal to zero there are non-trivial solutions: torque-free precession. Notice that since **I** is constant (because the inertia tensor is a 3×3 diagonal matrix (see the previous section), because we work in the intrinsic frame, or because the torque is driving the rotation around the same axis $\hat{\mathbf{n}}$ so that **I** is not changing) then we may write

$$\mathbf{M} \stackrel{\text{def}}{=} I\frac{d\omega}{dt}\hat{\mathbf{n}} = I\alpha\,\hat{\mathbf{n}}$$

where

> α is called the angular acceleration (or rotational acceleration) about the rotation axis $\hat{\mathbf{n}}$.

However, if $\mathbf{I}$ is not constant in the external reference frame (i.e. the body is moving and its inertia tensor is not constantly diagonal) then we cannot take the $\mathbf{I}$ outside the derivative. In this case we will have torque-free precession, in such a way that $\mathbf{I}(t)$ and $\boldsymbol{\omega}(t)$ change together so that their derivative is zero. This motion can be visualized by Poinsot's construction.

Generalizations

It is also possible to use these equations if the axes in which

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\mathrm{relative}}$$

is described are not connected to the body. Then $\boldsymbol{\omega}$ should be replaced with the rotation of the axes instead of the rotation of the body. It is, however, still required that the chosen axes are still principal axes of inertia. This form of the Euler equations is useful for rotation-symmetric objects that allow some of the principal axes of rotation to be chosen freely.

See also

- Euler angles
- Dzhanibekov effect
- Moment of inertia
- Poinsot's construction
- Rigid rotor

References

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