Preliminary Mathematics

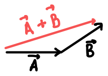
Vector Algebra

$\vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$ $\quad \vec{A} = (A_x, A_y, A_z)$ in rec. Cartesian

$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ $\quad k \; j \; i$

Magnitude $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Vector summation:

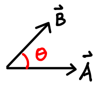


- Commutative Law of Vector Addition

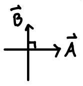
$\vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$

Dot Product

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = \vec{B}\cdot\vec{A}$

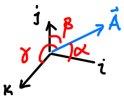


$\vec{A}\,\vec{B} = |A||B|\cos\theta$



$\vec{A}\cdot\vec{B} = 0$

$\vec{B}$ = unit Vector $|\vec{B}| = 1$



$$\begin{cases} A_x = \vec{A}\cdot i = |A|\cos\alpha \\ A_y = \vec{A}\cdot j = |A|\cos\beta \\ A_z = \vec{A}\cdot k = |A|\cos\gamma \end{cases}$$

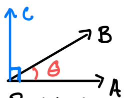
Direction cosines

$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

$$\cos\alpha = \frac{\vec{A}\cdot\hat{\imath}}{|A|}$$

Vector Multiplication (HW1 P1)

A vector cross product results in a vector



$\vec{C}$ is $\perp$ to the plane formed by $\vec{A}$ & $\vec{B}$

Right-hand System

$\boxed{|C| = |A||B|\sin\theta}$ $\quad \vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

Determinant Notation $\quad \vec{C} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

$$\vec{D}\times(\vec{A}\times\vec{B}) = (\vec{D}\times\vec{A}) + (\vec{D}\times\vec{B}) = -\left[(\vec{A}\times\vec{B})\times\vec{D}\right]$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Summation Convention

$\sum_{i=1}^{3} a_i x_i = P \longrightarrow a_i x_i = P, \; i = 1, 2, 3 \ldots$

$\Rightarrow a_1x_1 + a_2x_2 + a_3x_3 + \ldots = P$

$\left.\begin{array}{l} \vec{u} = u_i\hat{e}_i \\ \vec{v} = v_j\hat{e}_j \end{array}\right\}$ $\quad \vec{u}\cdot\vec{v} = (u_i\hat{e}_i)(v_j\hat{e}_j) = (u_iv_j)\hat{e}_i e_j$

$$\hat{e}_i\cdot\hat{e}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$\updownarrow$

δ_{ij} (knocker Delta)

$\updownarrow$

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\vec{u}\cdot\vec{v} = u_iv_j\delta_{ij} = u_iv_i$$

Permutation/Alternating Symbol (ε_{ijk})

$$\varepsilon_{ijk} = \begin{cases} +1: \text{ if } ijk \text{ is even perm. of } 1,2,3 \\ -1: \text{ if } ijk \text{ is odd perm. of } 1,2,3 \\ 0, \text{ otherwise} \end{cases}$$

$3\,\overset{2}{\circlearrowright}\,2 \;\; +1$

$3\,\overset{2}{\circlearrowleft}\,2 \;\; -1$

K: dummy index

$\vec{u}\times\vec{v} = (u_iv_j)\hat{e}_i\times\hat{e}_j = (u_iv_j)(\varepsilon_{ijk})\hat{e}_k$

$\vec{u}\times\vec{v} = \hat{e}_1(u_2v_3 - u_3v_2) + \hat{e}_2(u_3v_1 - u_1v_3) + \hat{e}_3(u_1v_2 - u_2v_1)$

$\varepsilon_{ij1} \quad \varepsilon_{ij2} \quad \varepsilon_{ij3}$

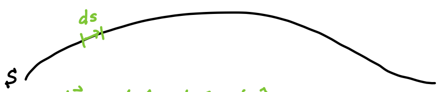
$$\det(A) = |A_{ij}| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$$

$\varepsilon_{ijk} = \varepsilon_{ijk} = 6 \quad , \quad \varepsilon_{ijk}\varepsilon_{ijL} = 2\delta_{KL}$

Scalar Point Functions

continuous functions $f(x,y,z)$ at Point $P(x,y,z)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$



$d\vec{s} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$

$|ds| = \sqrt{dx^2 + dy^2 + dz^2}$

$$\underbrace{\frac{df}{ds}}_{\text{Directional derivatives of } f} = \frac{\partial f}{\partial x}\underbrace{\frac{dx}{ds}}_{\cos\alpha} + \frac{\partial f}{\partial y}\underbrace{\frac{dy}{ds}}_{\cos\beta} + \frac{\partial f}{\partial z}\underbrace{\frac{dz}{ds}}_{\cos\gamma}$$

X-direction: $\frac{df}{ds} = \frac{\partial f}{\partial \alpha} = \frac{df}{dx}$

Gradient, Max value of directional derivative

A vector point fcn. (x,y,z) from the scalar f

$\text{grad}.f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial x}\hat{\jmath} + \frac{\partial f}{\partial x}\hat{k} \Rightarrow \boxed{\vec{n}\cdot\text{grad}f = \frac{df}{ds}}$

Unit Vector $\vec{n} = \cos\alpha\,\hat{\imath} + \cos\beta\,\hat{\jmath} + \cos\gamma\,\hat{k}$

Del Operator (nabla) $\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \Rightarrow \boxed{\text{grad } f = \nabla f}$ $\quad \boxed{\vec{n}\cdot\nabla}$

Index Notations $\quad \vec{\nabla} \equiv \hat{e}_i\frac{\partial}{\partial x_i} \rightarrow \vec{\nabla}f = \hat{e}_i\frac{\partial f}{\partial x_i} = \hat{e}_i f_{,i}$

$i = 1,2,3$

$\frac{\partial f}{\partial x_i} \equiv f_{,i}$

Laplacian of a Scalar

$\vec{\nabla}^2 f = \vec{\nabla}\cdot\vec{\nabla}f = f_{,ii}$

$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$

Laplacian Equation. $\vec{\nabla}^2 f = 0$ or $\Delta f = 0$

Laplace Operator: $\nabla^2 \equiv \Delta$

Vector Fields

$\vec{V}(x,y,t) = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$

$\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial V_j}{\partial x_i}\right)\hat{e}_i\cdot\hat{e}_j = (V_{j,i})\,\delta_{ij} = V_{i,i}$

$= V_{1,1} + V_{2,2} + V_{3,3} = \text{div } \vec{V}$ (divergence of a vector function)

Laplacian of vector function:

$\vec{V} = V_j\hat{e}_j, \; \nabla^2 = \frac{\partial^2}{\delta x_i} \longrightarrow \nabla^2 V = V_{j,ii}\,\hat{e}_j$

Concepts of Stress

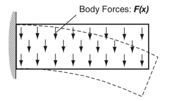
Acting forces on a continuum

$\vec{F}(\vec{x})$: body force density at x

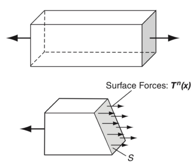
$F_R = \iiint_V F(x)\,dV \rightarrow$ over entire body

$\vec{T}^n(\vec{x})$: surface body density function at x

$F_S = \iint_S T^n(x)\,dS$



(a) Cantilever Beam Under Self-Weight Loading



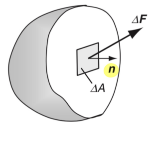
(b) Sectioned Axially Loaded Beam

Traction / Stress Vector

$\vec{T}(\vec{x},\vec{n}) = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$

$\vec{n}$ → plane

$\vec{x}$ → point



(Sectioned Body)

3 Traction Vectors or planes to define stress at a point

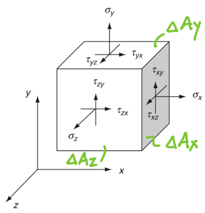
$\vec{T}(\vec{x},\vec{n}=\hat{e}_1) = \sigma_x\hat{e}_1 + \tau_{xy}\hat{e}_2 + \tau_{xz}\hat{e}_3 \rightarrow \Delta A_1 + \Delta A_x$

$\vec{T}(\vec{x},\vec{n}=\hat{e}_2) = \tau_{yx}\hat{e}_1 + \sigma_y\hat{e}_2 + \tau_{yz}\hat{e}_3 \rightarrow \Delta A_2 + \Delta A_y$

$\vec{T}(\vec{x},\vec{n}=\hat{e}_3) = \tau_{zx}\hat{e}_1 + \tau_{zy}\hat{e}_2 + \sigma_z\hat{e}_3 \rightarrow \Delta A_3 + \Delta A_z$

Stress Tensor: 9 components (6 independent)

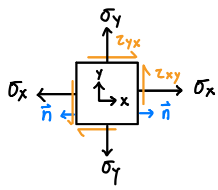
$$[\sigma] = \sigma_{ij} = \tau_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



Principal Stresses & Orientations (HW2 P1)

$$\boxed{\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} \qquad \tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}}$$

(+) Sign Convection



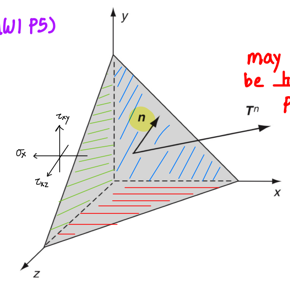
Find Traction Vector @ any Plane Orientation (HW1 P5)

unit normal vector = $\frac{\vec{n}}{|\vec{n}|}$

$\vec{T}^n_{3\times1} = [\sigma_{ij}]_{3\times3} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}_{3\times1}$ or $\vec{T}^n = [\sigma]\vec{n}$

$[N/m^2]$ $\quad \vec{T}_i = \sigma_{ji} n_i$

may not be ⊥ to plane

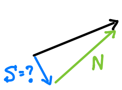


Traction Vector Decomposition

$[\sigma] = T^n$ (surface traction)

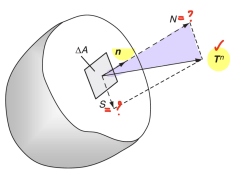
$\sigma_n = \vec{N} = \vec{T}\cdot\vec{n}$ (normal stress on plane $\vec{n}$)

$\sigma_n = \vec{N} = \vec{T}_i n_i \implies \sigma_n = N = \sigma_{ji} n_j n_i \overset{\text{sym.}}{=} \sigma_{ij} n_i n_j$



$|T|^2 = |N|^2 + |S|^2 \rightarrow \sigma_S^{\ 2} = S^2 = |T|^2 - |N|^2$

$|T^n|^2 = \vec{T}\cdot\vec{T} = T_i^n T_i^n = \sigma_{ji} n_j \cdot \sigma_{ki} n_k$



S = shear component of traction vector

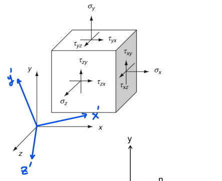
Stress Transformations

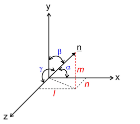
	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

$n_1 = \cos(x', z)$

OR

	x	y	z
x'	a_{11}	a_{12}	a_{13}
y'	a_{21}	a_{22}	a_{23}
z'	a_{31}	a_{32}	a_{33}





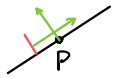
Transformation Matrix = $[A] = A_{ij}$ or $[Q] = Q_{ij}$

(Stress tensor in x', y', z') $\leftrightarrow$ $\boxed{\sigma' = [A]\,\sigma\,[A]^T}$ $\longrightarrow$ $\begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{x'z'} \\ & \sigma_{y'} & \tau_{y'z'} \\ \text{sym.} & & \sigma_{z'} \end{bmatrix}$

Expanded Equations

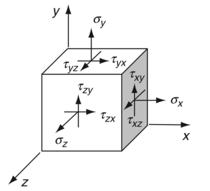
$$\sigma'_x = \sigma_x l_1^2 + \sigma_y m_1^2 + \sigma_z n_1^2 + 2(\tau_{xy} l_1 m_1 + \tau_{yz} m_1 n_1 + \tau_{zx} n_1 l_1)$$
$$\sigma'_y = \sigma_x l_2^2 + \sigma_y m_2^2 + \sigma_z n_2^2 + 2(\tau_{xy} l_2 m_2 + \tau_{yz} m_2 n_2 + \tau_{zx} n_2 l_2)$$
$$\sigma'_z = \sigma_x l_3^2 + \sigma_y m_3^2 + \sigma_z n_3^2 + 2(\tau_{xy} l_3 m_3 + \tau_{yz} m_3 n_3 + \tau_{zx} n_3 l_3)$$
$$\tau'_{xy} = \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \sigma_z n_1 n_2 + \tau_{xy}(l_1 m_2 + m_1 l_2) + \tau_{yz}(m_1 n_2 + n_1 m_2) + \tau_{zx}(n_1 l_2 + l_1 n_2)$$
$$\tau'_{yz} = \sigma_x l_2 l_3 + \sigma_y m_2 m_3 + \sigma_z n_2 n_3 + \tau_{xy}(l_2 m_3 + m_2 l_3) + \tau_{yz}(m_2 n_3 + n_2 m_3) + \tau_{zx}(n_2 l_3 + l_2 n_3)$$
$$\tau'_{zx} = \sigma_x l_3 l_1 + \sigma_y m_3 m_1 + \sigma_z n_3 n_1 + \tau_{xy}(l_3 m_1 + m_3 l_1) + \tau_{yz}(m_3 n_1 + n_3 m_1) + \tau_{zx}(n_3 l_1 + l_3 n_1)$$

Principal Stresses (no shear stress)



$\vec{T} \parallel \vec{n} \longrightarrow \sigma_{ij} = \begin{cases} \text{constant} & i=j \\ 0 & i \neq j \end{cases}$

$$\sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$



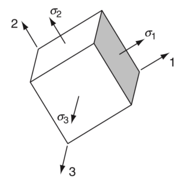
(General Coordinate System)

Method 1

Solve $\begin{vmatrix} \sigma_x - \sigma_p & \tau_{xy} & \tau_{xz} \\ & \sigma_y - \sigma_p & \tau_{yz} \\ & & \sigma_z - \sigma_p \end{vmatrix} = 0$

Method 2

$\det(\sigma - \sigma_p I) = 0$



(Principal Coordinate System)

Stress cubic Equation: $\boxed{\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0}$ (HW1 P6)

→ Solve σ_p $\underbrace{(\sigma_1 > \sigma_2 > \sigma_3)}_{\text{3 roots}}$ → principal stresses (Eigenvalues) → principal directions (Eigenvectors) → Check (HW1 P6)

Stress Invariants (HW1 P6)

$I_1 = \sigma_x + \sigma_y + \sigma_z$

$I_2 = \sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2$

$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix} = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$

Stress Invariant (principal coordinates)

$I_1 = \sigma_1 + \sigma_2 + \sigma_3$

$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3$

$I_3 = \sigma_1\sigma_2\sigma_3$

Stress Invariants & $\sigma_1, \sigma_2, \sigma_3$ are independent from choice of axes

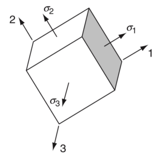
Traction Components in Principal Coordinates (No Shear)

① $\sigma_n = N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$

② $T_i T_i = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2$

③ $n_1^2 + n_2^2 + n_3^2 = 1$

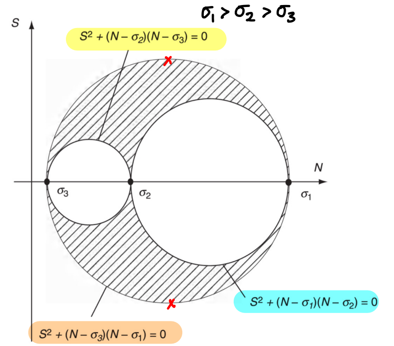
$$n_1^2 = \frac{S^2 + (N-\sigma_2)(N-\sigma_3)}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)} \geq 0$$
$$n_2^2 = \frac{S^2 + (N-\sigma_3)(N-\sigma_1)}{(\sigma_2-\sigma_3)(\sigma_2-\sigma_1)} \geq 0$$
$$n_3^2 = \frac{S^2 + (N-\sigma_1)(N-\sigma_2)}{(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)} \geq 0$$



(Principal Coordinate System)

$$\boxed{\begin{aligned} S^2 + (N-\sigma_2)(N-\sigma_3) &\geq 0 \\ S^2 + (N-\sigma_3)(N-\sigma_1) &\leq 0 \\ S^2 + (N-\sigma_1)(N-\sigma_2) &\geq 0 \end{aligned}}$$

$\sigma_1 > \sigma_2 > \sigma_3$

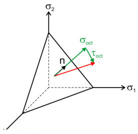


Max Shear

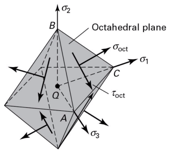
$$\boxed{\tau_{max} = S_{max} = \frac{|\sigma_1 - \sigma_3|}{2}}$$

Octahedral plane

The plane when it is equally inclined to all 3 principal directions $(\alpha=\beta=\gamma)$



$\alpha=\beta=\gamma$ & $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1 \rightarrow \hat{n}_1=\hat{n}_2=\hat{n}_3=\pm\frac{1}{\sqrt{3}}$



$N=\sigma_{oct}=\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)=\frac{I_1}{3}$

$S=\tau_{oct}=\frac{1}{3}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2}$

$\tau_{oct}=\frac{1}{3}\left(2I_1^2-6I_2\right)^{1/2}$

$\hookrightarrow$ directly related to distortion strain energy $\rightarrow$ failure in ductile materials

$$\boxed{\sigma_{oct}=\frac{\sigma_x+\sigma_y+\sigma_z}{3}\ ,\quad \tau_{oct}=\frac{1}{3}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)}}$$

Effective (von Mises) stress

$$\sigma_e=\sigma_{\substack{von\\Mises}}=\frac{3}{\sqrt{2}}\tau_{oct}=\boxed{\frac{1}{\sqrt{2}}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}}$$

Importance: $\sigma_{yield}=\sigma_{\substack{von\\Mises}} \longrightarrow$ Failure Condition

$\rightarrow \sigma_y=\frac{3}{\sqrt{2}}\tau_{oct}$

$\hookrightarrow \boxed{\tau_{oct}=\frac{\sqrt{2}}{3}\sigma_y=0.471\,\sigma_y} \rightarrow$ Yielding/Inelastic response begins

Mean & Deviatoric Stresses

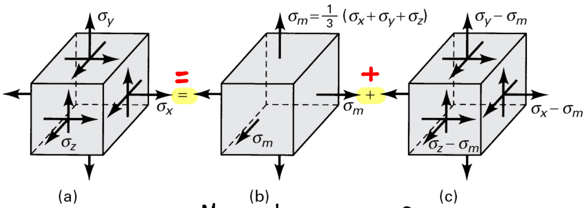
Based on experimental studies, yielding and plastic deformations of many ductile materials are independent of the applied mean stress

Mean Stress Tensor $\boxed{\sigma_m=\frac{\sigma_x+\sigma_y+\sigma_z}{3}=\frac{\sigma_1+\sigma_2+\sigma_3}{3}=\frac{I_1}{3}}$

$\hookrightarrow \sigma_m=\begin{bmatrix} I_1/3 & 0 & 0\\ 0 & I_1/3 & 0\\ 0 & 0 & I_1/3\end{bmatrix}$

$[\sigma_D]=[\sigma]-[\sigma_M]$

$[\sigma_D]=\sigma_{ij}-\frac{1}{3}\sigma_{KK}\delta_{ij}=\sigma_{ij}-\sigma_m\delta_{ij}$



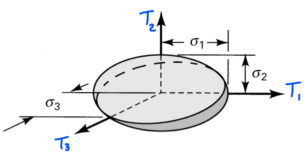
Mean Stress
Hydrostatic Stress
Dilation Stress

Deviatoric Stress
Distortion Stress

Same principal directions as stress tensor. Principal axes coincide

$\sigma_{Dp}^3-I_2'\sigma_{Dp}-I_3'=0 \rightarrow \left[\begin{array}{l} I_2'=3\sigma_m^2-I_2\\ I_3'=I_3-\sigma_m I_2+2\sigma_m^3=I_3+\sigma_m I_2'-\sigma_m^3\end{array}\right.$

Lame's Stress Ellipsoid



$$\boxed{\left(\frac{T_1}{\sigma_1}\right)^2+\left(\frac{T_2}{\sigma_2}\right)^2+\left(\frac{T_3}{\sigma_3}\right)^2=1}$$

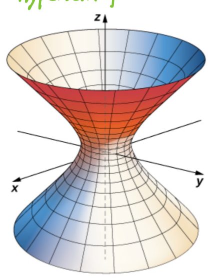
Geometric Meaning of Stress Invariants

$I_1 \propto \Sigma$ of 3 principal radii

$I_2 \propto \Sigma$ of 3 principal areas

$I_3 \propto$ Volume of ellipsoid

hyperboloid of one sheet



Cauchy's Stress Quadric

$\sigma_1(\hat{e}_1)^2+\sigma_2(\hat{e}_2)^2+\sigma_3(\hat{e}_3)^2=\pm k_o^2$

Equations of Equilibrium

$$\begin{aligned}
\Sigma F_x=0 \quad & \frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+F_x=0\\
\Sigma F_y=0 \quad & \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+F_y=0\\
\Sigma F_z=0 \quad & \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+F_z=0
\end{aligned}$$

$$\begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}_{1\times3}\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z\end{bmatrix}_{3\times3}+\begin{bmatrix}F_x\\F_y\\F_z\end{bmatrix}=0$$

OR $\vec{\nabla}\cdot[\sigma]+\vec{F}=0$

Stress Transformation in 2D (HW1P4)

$$\sigma_x'=\sigma_x\cos^2\theta+\sigma_y\sin^2\theta+2\tau_{xy}\sin\theta\cos\theta$$
$$\sigma_y'=\sigma_x\sin^2\theta+\sigma_y\cos^2\theta-2\tau_{xy}\sin\theta\cos\theta$$
$$\tau_{xy}'=-\sigma_x\sin\theta\cos\theta+\sigma_y\sin\theta\cos\theta+\tau_{xy}(\cos^2\theta-\sin^2\theta)$$

$\downarrow$ Trig. Relation

$$\sigma_x'=\frac{\sigma_x+\sigma_y}{2}+\frac{\sigma_x-\sigma_y}{2}\cos2\theta+\tau_{xy}\sin2\theta$$
$$\sigma_y'=\frac{\sigma_x+\sigma_y}{2}-\frac{\sigma_x-\sigma_y}{2}\cos2\theta-\tau_{xy}\sin2\theta$$
$$\tau_{xy}'=\frac{\sigma_y-\sigma_x}{2}\sin2\theta+\tau_{xy}\cos2\theta$$

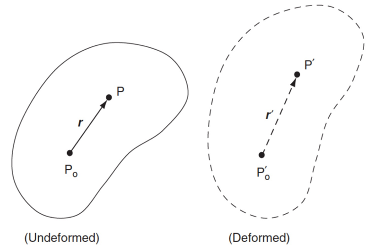
Concepts of Strain

Deformation Introduction

length (normal strain ε)

angle (shear strain γ)

1D normal strain: $\frac{\Delta L}{L_0} = \frac{\delta}{L_0}$



Strain at a Point

Displacement vector: $\vec{u} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$ or $x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$

P_0 to P: $\vec{u}$: disp. vector @ P

$\vec{u}_0$: disp. vector @ P_0

$\Delta\vec{r} = \vec{r}' - \vec{r} = \vec{u} - \vec{u}_0$

$$u = u^o + \frac{\partial u}{\partial x}r_x + \frac{\partial u}{\partial y}r_y + \frac{\partial u}{\partial z}r_z$$
$$v = v^o + \frac{\partial v}{\partial x}r_x + \frac{\partial v}{\partial y}r_y + \frac{\partial v}{\partial z}r_z$$
$$w = w^o + \frac{\partial w}{\partial x}r_x + \frac{\partial w}{\partial y}r_y + \frac{\partial w}{\partial z}r_z$$

$$\Delta r_x = \frac{\partial u}{\partial x}r_x + \frac{\partial u}{\partial y}r_y + \frac{\partial u}{\partial z}r_z$$
$$\Delta r_y = \frac{\partial v}{\partial x}r_x + \frac{\partial v}{\partial y}r_y + \frac{\partial v}{\partial z}r_z$$
$$\Delta r_z = \frac{\partial w}{\partial x}r_x + \frac{\partial w}{\partial y}r_y + \frac{\partial w}{\partial z}r_z$$

Displacement Gradient Tensor (HW2P4)

$$u_{i,j} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

$$A = \underbrace{\tfrac{1}{2}(A + A^T)}_{\text{sym.}} + \underbrace{\tfrac{1}{2}(A - A^T)}_{\text{anti-sym.}}$$

$$u_{ij} = e_{ij} + \omega_{ij} \quad \begin{cases} e_{ij} = \underbrace{\tfrac{1}{2}(u_{i,j} + u_{j,i})}_{\text{sym.}} & \text{strain tensor} \\ \omega_{ij} = \underbrace{\tfrac{1}{2}(u_{i,j} - u_{j,i})}_{\text{anti-sym.}} & \text{Rotation tensor} \end{cases}$$

Strain tensor components (HW2P4)

$$e_{i,j} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$$

$\varepsilon_{xz} = \frac{\gamma_{xz}}{2}$

— Normal & shear strains are: ε and γ

— Tensor form of normal & shear strains are: e

Normal Strain ← (HW2P2) → Shear Strain / tensor shear strain (HW2P4)

$$e_{xx} = e_x = \varepsilon_x = \frac{\partial u}{\partial x}$$
$$e_{yy} = e_y = \varepsilon_y = \frac{\partial v}{\partial y}$$
$$e_{zz} = e_z = \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\gamma_{xz} = 2\varepsilon_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$
$$\gamma_{yz} = 2\varepsilon_{yz} = \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

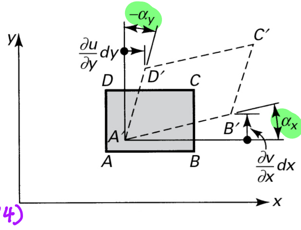
ε_{xy} is the tensor shear strain component, a measure of the half-angle change.

γ_{xy} is the engineering shear strain, which is twice that (the total change in the right angle).

Change in angle between two originally orthogonal lines

$$\gamma_{xy} = \frac{\pi}{2} - \phi = |\alpha_x| + |\alpha_y|$$

small deformation: $\alpha_x = \tan\alpha_x$, $\alpha_y = \tan\alpha_y$



Rotation tensor components (HW2P4)

$$w_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) = \begin{bmatrix} 0 & -w_{21} & w_{13} \\ w_{21} & 0 & -w_{32} \\ -w_{13} & w_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

$$w_3 = w_{21} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$
$$w_2 = w_{13} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$
$$w_1 = w_{32} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

We know $\omega_{ij} = -\omega_{ji}$

→ if $i = j$ then $\omega_{ij} = 0$

∴ 3 independent components $(\omega_1, \omega_2, \omega_3)$

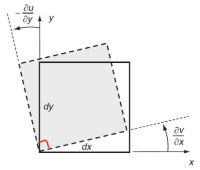
Dual Rotation vector: $\boxed{\vec{w} = w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3}$ (HW2P4)

Vector format

$$\vec{w} = \frac{1}{2}\left(\vec{\nabla} \times \vec{u}\right) = \frac{1}{2}\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

Rigid Body Motion

Rigid Body Motion can exist w/o affecting the strain field



Displacement Field (2D Rigid Body Motion)

$$\boxed{u_i = u_{i0} + e_{ij}dx_j + w_{ij}dx_j}$$ no strain

$$\begin{cases} u_R = u_0 + 0 + \omega_{12}dx_2 = u_0 - w_z y \\ V_R = V_0 + 0 + w_{21}dx_1 = V_0 + w_z x \end{cases}$$

Displacement Field (3D Rigid Body Motion)

$$\begin{cases} u_R = u_0 - w_z y + w_y z \\ V_R = V_0 + w_z x - w_x z \\ W_R = W_0 - w_y x + w_x y \end{cases}$$

w_1, w_2, w_3

$w_x \quad w_y \quad w_z$

$\hat{\imath}, \hat{\jmath}, \hat{k}$ OR $\hat{e}_1, \hat{e}_2, \hat{e}_3$

It is important to recognize the terms associated with rigid body displacements because we normally want to drop them from the analysis since they do not contribute to the strain or stress fields.

Strain Transformations in 3D (HW2P2)

$\varepsilon' = A\varepsilon A^T$ ↳ strain components in x, y, z

$$\varepsilon = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ \text{sym.} & & \varepsilon_z \end{bmatrix}$$

$\varepsilon_{xz} = \frac{\gamma_{xz}}{2}$

$$A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

↳ direction cosines

$e = \varepsilon$

$$e'_x = e_x l_1^2 + e_y m_1^2 + e_z n_1^2 + 2(e_{xy}l_1m_1 + e_{yz}m_1n_1 + e_{zx}n_1l_1)$$
$$e'_y = e_x l_2^2 + e_y m_2^2 + e_z n_2^2 + 2(e_{xy}l_2m_2 + e_{yz}m_2n_2 + e_{zx}n_2l_2)$$
$$e'_z = e_x l_3^2 + e_y m_3^2 + e_z n_3^2 + 2(e_{xy}l_3m_3 + e_{yz}m_3n_3 + e_{zx}n_3l_3)$$
$$e'_{xy} = e_x l_1l_2 + e_y m_1m_2 + e_z n_1n_2 + e_{xy}(l_1m_2 + m_1l_2) + e_{yz}(m_1n_2 + n_1m_2) + e_{zx}(n_1l_2 + l_1n_2)$$
$$e'_{yz} = e_x l_2l_3 + e_y m_2m_3 + e_z n_2n_3 + e_{xy}(l_2m_3 + m_2l_3) + e_{yz}(m_2n_3 + n_2m_3) + e_{zx}(n_2l_3 + l_2n_3)$$
$$e'_{zx} = e_x l_3l_1 + e_y m_3m_1 + e_z n_3n_1 + e_{xy}(l_3m_1 + m_3l_1) + e_{yz}(m_3n_1 + n_3m_1) + e_{zx}(n_3l_1 + l_3n_1)$$

Strain Transformations in 2D (HW2P3)

$\varepsilon' = A\varepsilon A^T$ ↳ strain components in x, y, z

$$e'_x = e_x\cos^2\theta + e_y\sin^2\theta + 2e_{xy}\sin\theta\cos\theta$$
$$e'_y = e_x\sin^2\theta + e_y\cos^2\theta - 2e_{xy}\sin\theta\cos\theta$$
$$e'_{xy} = -e_x\sin\theta\cos\theta + e_y\sin\theta\cos\theta + e_{xy}(\cos^2\theta - \sin^2\theta)$$

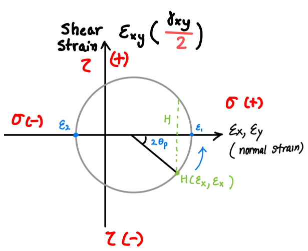
Trig. Relation

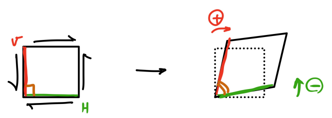
$$e'_x = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos 2\theta + e_{xy}\sin 2\theta$$
$$e'_y = \frac{e_x + e_y}{2} - \frac{e_x - e_y}{2}\cos 2\theta - e_{xy}\sin 2\theta$$
$$e'_{xy} = \frac{e_y - e_x}{2}\sin 2\theta + e_{xy}\cos 2\theta$$

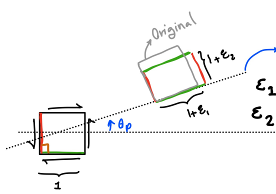
Mohr Circle for Strain

Normal strain > 0 → Elongation

Shear strain > 0 → Reduced angle







principal direction (no shear strain)

$\varepsilon_1 > 0$

$\varepsilon_2 > 0$

Principal Strains

In principal direction

$$\varepsilon = \begin{vmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{vmatrix}$$

$\Updownarrow$

$$\boxed{\begin{aligned} \delta_1 &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ \delta_2 &= \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 \\ \delta_3 &= \varepsilon_1\varepsilon_2\varepsilon_3 \end{aligned}} \leftrightarrow \boxed{\begin{aligned} \delta_1 &= \varepsilon_x + \varepsilon_y + \varepsilon_z \\ \delta_2 &= \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \varepsilon_{xy}^2 - \varepsilon_{yz}^2 - \varepsilon_{xz}^2 \\ \delta_3 &= \begin{vmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ \text{sym.} & & \varepsilon_z \end{vmatrix} \end{aligned}}$$

Find Principal Directions

$$\begin{bmatrix} \varepsilon_x - \varepsilon_i & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y - \varepsilon_i & \varepsilon_{yz} \\ \text{sym.} & & \varepsilon_z - \varepsilon_i \end{bmatrix} \begin{Bmatrix} l_i \\ m_i \\ n_i \end{Bmatrix} = 0, \quad i = 1, 2, 3$$

i.e. $i=1 \longrightarrow 3$ equations

$l_i^2 + m_i^2 + n_i^2 = 1$

$\Downarrow$

3 independent equations

2D Strain

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\gamma_{max} = \pm 2\sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = \pm(\varepsilon_1 - \varepsilon_2)$$

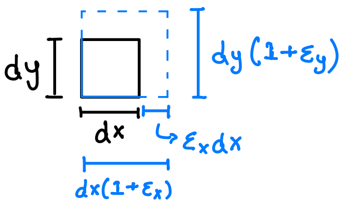
$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$

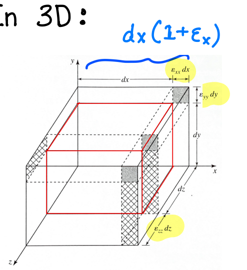
Change in Volume of a Continuum

Volume of deformed body

In 2D:



In 3D:



Unit volume change

$$\boxed{e = \frac{\Delta V}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \delta_1}$$

$$\boxed{\Delta V = V_{def} - V_0 = \left[(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) - 1\right] dx\,dy\,dz}$$

for small deformation $(\varepsilon_x)(\varepsilon_y) \approx 0$

$$\Delta V = \varepsilon_x \varepsilon_y \varepsilon_z \underbrace{dx\,dy\,dz}_{V_0}$$

Strain Compatibility (HW2P5)

Compatibility Equations for 3D

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \quad \text{for 2D}$$

$$\frac{\partial^2 e_y}{\partial z^2} + \frac{\partial^2 e_z}{\partial y^2} = 2\frac{\partial^2 e_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_x}{\partial z^2} = 2\frac{\partial^2 e_{zx}}{\partial z \partial x}$$

$$\frac{\partial^2 e_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 e_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x}\right)$$

$$\frac{\partial^2 e_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y}\right)$$

for 2D:

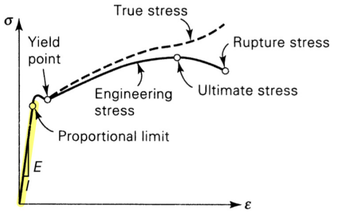
$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}$$

or

$$\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}\right)$$

Stress-Strain Relationship

Linear Elasticity



Hooke Law: $\boxed{\sigma = E \times \varepsilon}$

Initial stress-strain response is linear and "small deformation" assumption

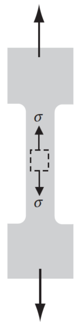
Uniaxial Tension

$\nu = -\dfrac{\text{lateral strain}}{\text{longitudinal strain}}$ (Poisson's Ratio)

1D stress

$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{strain Tensor}} \varepsilon_{ij} = \begin{bmatrix} \frac{\sigma}{E} & 0 & 0 \\ 0 & \frac{-\nu\sigma}{E} & 0 \\ 0 & 0 & \frac{-\nu\sigma}{E} \end{bmatrix}$$

3D Strain $\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix}$

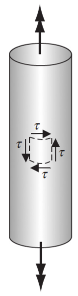


Pure Shear

$$\boxed{\tau_{xy} = \tau} \longrightarrow \sigma_{ij} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{strain} = \begin{bmatrix} 0 & \frac{\tau}{2G} & 0 \\ \frac{\tau}{2G} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\tau = G\gamma \longrightarrow \tau_{xy} = 2G\varepsilon_{xy}}$$

G or μ = Shear mod.
mod. of rigidity

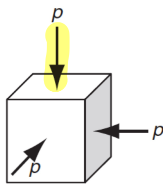


Hydrostatic Compression/Tension (HW3P1)

$\sigma_1 = \sigma_2 = \sigma_3 = -P$ (negative b/c in compression)

$$\Rightarrow \sigma_{ij} = -P\delta_{ij} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} = \boxed{-\left(\frac{1-2\nu}{E}\right)P = \sigma_{ij}}$$

Strain $\boxed{\varepsilon_x = \varepsilon_y = \varepsilon_z = -\left(\frac{1-2\nu}{E}\right)P}$ $\quad \sigma_m = -P$



Unit volume Change $\quad e = \dfrac{\Delta V}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \boxed{-3\left(\frac{1-2\nu}{E}\right)P = \frac{\Delta V}{V_0}}$

$P = -Ke$

$\hookrightarrow$ Bulk Modulus $= \dfrac{\text{Hydrostatic Pressure}}{\text{Dilation}} = \dfrac{-P}{e} = \boxed{\dfrac{E}{3(1-2\nu)}} = K$

Bulk Modulus: Measure of volumetric change when subjected to hydrostatic state of stress

Hooke's Law for Linear Elastic Materials

C_{ij} = material parameters

$$\sigma_x = C_{11}e_x + C_{12}e_y + C_{13}e_z + 2C_{14}e_{xy} + 2C_{15}e_{yz} + 2C_{16}e_{zx}$$
$$\sigma_y = C_{21}e_x + C_{22}e_y + C_{23}e_z + 2C_{24}e_{xy} + 2C_{25}e_{yz} + 2C_{26}e_{zx}$$
$$\sigma_z = C_{31}e_x + C_{32}e_y + C_{33}e_z + 2C_{34}e_{xy} + 2C_{35}e_{yz} + 2C_{36}e_{zx}$$
$$\tau_{xy} = C_{41}e_x + C_{42}e_y + C_{43}e_z + 2C_{44}e_{xy} + 2C_{45}e_{yz} + 2C_{46}e_{zx}$$
$$\tau_{yz} = C_{51}e_x + C_{52}e_y + C_{53}e_z + 2C_{54}e_{xy} + 2C_{55}e_{yz} + 2C_{56}e_{zx}$$
$$\tau_{zx} = C_{61}e_x + C_{62}e_y + C_{63}e_z + 2C_{64}e_{xy} + 2C_{65}e_{yz} + 2C_{66}e_{zx}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdot & \cdot & \cdot & C_{16} \\ C_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_{61} & \cdot & \cdot & \cdot & \cdot & C_{66} \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{bmatrix}$$

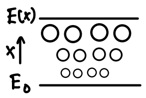
$$\underbrace{\{\sigma\}_{6\times1}}_{\text{Stress}} = \underbrace{[C]_{6\times6}}_{\text{Stiffness matrix}} \underbrace{\{\varepsilon\}_{6\times1}}_{\text{strain}} \quad \text{or} \quad \{\varepsilon\}_{6\times1} = \underbrace{[S]_{6\times6}}_{\text{Compliance matrix}} \{\sigma\}_{6\times1}$$

Homogeneous/Inhomogeneous materials

Homogeneous : C_{ij} is full of constants, measured by experiments

Non-homogeneous : Can be a function of location $C_{ij}(x) = C^{\circ}_{ij}(1+ax)$

Functionally Graded Materials (FGM)

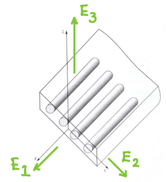


linear/exponential

Symmetry in Materials

Fully anisotropic : no plane of symmetry in material $\quad C_{ij} = C_{ji}$ 21 independent coeff.

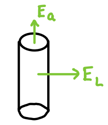
General Orthotropic : 3 planes of symmetry (2 ⊥ planes of sym.)



$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ & C_{22} & C_{23} & & 0 & \\ \text{sym.} & & C_{33} & & & \\ & & & C_{44} & 0 & 0 \\ & 0 & & 0 & C_{55} & 0 \\ & & & 0 & 0 & C_{66} \end{bmatrix}$$

9 independent coeff.

Transversely Isotropic : axial & lateral elastic modulus



$C_{11} = C_{22}$

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ & C_{22} & C_{23} & & 0 & \\ \text{sym.} & & C_{33} & & & \\ & & & C_{44} & 0 & 0 \\ & 0 & & 0 & C_{44} & 0 \\ & & & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}$$

5 independent coeff.

Isotropic Materials (HW3P2)

$$[C] = \begin{bmatrix} C_{11} & \overset{\lambda}{C_{12}} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \qquad [S] = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{44} \end{bmatrix}$$

$\mu = \dfrac{(C_{11}-C_{12})}{2} = G$

Stress comp. $\quad$ Strain Comp

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \frac{E}{(1+\upsilon)(1-2\upsilon)} \begin{bmatrix} 1-\upsilon & \upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & 1-\upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & \upsilon & 1-\upsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\upsilon}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\upsilon}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\upsilon}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{Bmatrix}$$

Stiffness Lamé's Constants (HW3P2)

1st, 2nd
λ, μ

$$C_{ij} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

$$\boxed{\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = C_{12}}$$

$$\boxed{\mu = G = \frac{E}{2(1+\nu)} = C_{44}}$$

Expanded Form (HW3P2)

$$\sigma_x = \lambda(e_x + e_y + e_z) + 2\mu e_x$$
$$\sigma_y = \lambda(e_x + e_y + e_z) + 2\mu e_y$$
$$\sigma_z = \lambda(e_x + e_y + e_z) + 2\mu e_z$$
$$\tau_{xy} = 2\mu e_{xy}$$
$$\tau_{yz} = 2\mu e_{yz}$$
$$\tau_{zx} = 2\mu e_{zx}$$

Strain terms of stress (Isotropic Materials)

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ 2\varepsilon_{xy}\\ 2\varepsilon_{xz}\\ 2\varepsilon_{yz}\end{Bmatrix}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\upsilon & -\upsilon & 0 & 0 & 0\\ -\upsilon & 1 & -\upsilon & 0 & 0 & 0\\ -\upsilon & -\upsilon & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\upsilon) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\upsilon) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\upsilon)\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{Bmatrix}$$

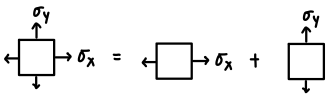
$$\boxed{\sigma_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}}$$

$$\boxed{\varepsilon_{ij}=\frac{1}{2\mu}\left(\sigma_{ij}-\frac{\lambda}{3\lambda+2\mu}\sigma_{kk}\delta_{ij}\right)=\left(\frac{1+\nu}{E}\right)\sigma_{ij}-\frac{\nu}{E}\sigma_{kk}\delta_{ij}}$$

Six Equations for stress-strain relations in isotropic materials

$$\begin{cases}\varepsilon_x=\dfrac{1}{E}\left[\sigma_x-\upsilon(\sigma_y+\sigma_z)\right]\\ \varepsilon_y=\dfrac{1}{E}\left[\sigma_y-\upsilon(\sigma_x+\sigma_z)\right]\\ \varepsilon_z=\dfrac{1}{E}\left[\sigma_z-\upsilon(\sigma_x+\sigma_y)\right]\end{cases}\qquad \text{Shear Strain}\begin{cases}\gamma_{xy}=\dfrac{\tau_{xy}}{G}\\ \gamma_{xz}=\dfrac{\tau_{xz}}{G}\\ \gamma_{yz}=\dfrac{\tau_{yz}}{G}\end{cases}$$

Strain-Stress Relations in 2D



Shear:

$$\boxed{\gamma_{xy}=\frac{\tau_{xy}}{G}=2\varepsilon_{xy}}$$

in x-direction:

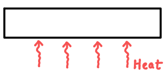
$$\boxed{\varepsilon_x=\frac{\sigma_x}{E}-\nu\varepsilon_y=\frac{\sigma_x}{E}-\frac{\nu\sigma_y}{E}}$$

in y-direction:

$$\boxed{\varepsilon_x=\frac{\sigma_y}{E}-\frac{\nu\sigma_x}{E}}$$

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ 2\varepsilon_{xy}\end{Bmatrix}=\underbrace{\begin{bmatrix}\frac{1}{E} & \frac{-\nu}{E} & 0\\ \frac{-\nu}{E} & \frac{1}{E} & 0\\ 0 & 0 & \frac{1}{G}\end{bmatrix}}_{[S]}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix};\quad \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\frac{E}{1-\nu^2}\underbrace{\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{G(1-\nu^2)}{E}\end{bmatrix}}_{[C]}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

Effect of Temperature



For isotropic materials:

$$\varepsilon_x^T=\alpha\Delta T\qquad \alpha=\text{coeff. of thermal expansion}=\frac{\Delta L/L}{\Delta T}=\frac{\varepsilon}{\Delta T}$$

$$\varepsilon_z^T=\alpha\Delta T$$

$$\varepsilon_y^T=\alpha\Delta T$$

This temperature change will cause a change in the volume of solid, but not a change of shape for isotropic materials.

$$\begin{cases}\varepsilon_x=\dfrac{1}{E}\left[\sigma_x-\upsilon(\sigma_y+\sigma_z)\right]+\alpha\Delta T\\ \varepsilon_y=\dfrac{1}{E}\left[\sigma_y-\upsilon(\sigma_x+\sigma_z)\right]+\alpha\Delta T\\ \varepsilon_z=\dfrac{1}{E}\left[\sigma_z-\upsilon(\sigma_x+\sigma_y)\right]+\alpha\Delta T\end{cases}\qquad \begin{cases}\gamma_{xy}=\dfrac{\tau_{xy}}{G}\\ \gamma_{xz}=\dfrac{\tau_{xz}}{G}\\ \gamma_{yz}=\dfrac{\tau_{yz}}{G}\end{cases}$$

$$\boxed{\varepsilon_{ij}=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\sigma_{kk}\delta_{ij}+\alpha\Delta T\delta_{ij}}$$

Thermoelastic constitutive Relations

- $\varepsilon_{ij}=\varepsilon_{ij}^{(m)}+\varepsilon_{ij}^{(T)}$

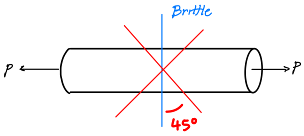
$$\varepsilon_{ij}^{(T)}=\alpha_{ij}(T-T_0)=\overbrace{\alpha_{ij}}^{\text{Coeff. of thermal expansion tensor}}\Delta T$$

for isotropics:

- $\alpha_{ij}=\alpha\Delta T\delta_{ij}$
- $\varepsilon_{ij}=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}\sigma_{kk}\delta_{ij}+\alpha\Delta T\delta_{ij}$
- $\sigma_{ij}=C_{ijkl}\varepsilon_{kl}-\beta_{ij}\Delta T$ $\quad\beta_{ij}$: Thermoelastic tensor

Failure Theories

Simple Tension

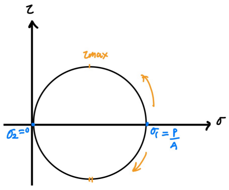


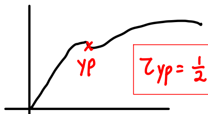
Ductile: max shear stress

$$\boxed{\tau_{max} = \frac{P}{2A}}$$

Brittle: max normal stress

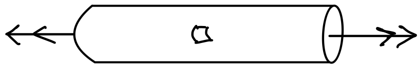
$$\boxed{\sigma_{max} = \frac{P}{A}}$$





$$\boxed{\tau_{yp} = \tfrac{1}{2}\sigma_{yp}}$$

Simple Torsion



$$\boxed{\tau = \frac{Tc}{\delta} = \tau_{yp}}$$

HW3P3 For a solid shaft

$$\tau_t = \frac{Tc}{\delta} = \frac{T\frac{d}{2}}{\frac{\pi}{32}D^4} = \frac{16T}{\pi d^3}$$

$$\sigma_b = \frac{Mc}{I} = \frac{M\frac{d}{2}}{\frac{\pi}{64}D^4} = \frac{32M}{\pi d^3}$$

Ductile

1 Max Shear stress theory (Coulomb Theory or Tresca Yield Criteria) (HW3P4)

Max shear stress in a solid = Max shear stress in yielding

$$\tau^{3D}_{Max} = \tau^{1D}_{yp}$$

$$\frac{|\sigma_1 - \sigma_3|}{2} = \tau_{yp} = \frac{\sigma_{yp}}{2} \longrightarrow \boxed{|\sigma_1 - \sigma_3| = \sigma_{yp}} \text{ FAIL}$$

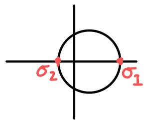
-OR-

$$\sigma_1 - \sigma_3 = \pm\sigma_{yp} \xrightarrow{\div \sigma_{yp}} \boxed{\frac{\sigma_1}{\sigma_{yp}} - \frac{\sigma_3}{\sigma_{yp}} = \pm 1}$$

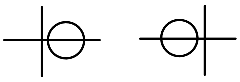
For plane stress ($\sigma_3 = 0$), when σ_1 & σ_2 have opposite sign

$$\boxed{\tau_{Max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_{yp}}{2}} \text{ FAIL}$$

$$\sigma_1 - \sigma_2 = \sigma_{yp} \longrightarrow \boxed{\frac{\sigma_1}{\sigma_{yp}} - \frac{\sigma_2}{\sigma_{yp}} = \pm 1} \ ①$$



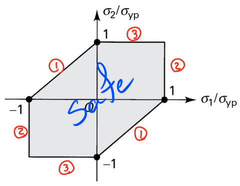
When σ_1 & σ_2 have the same sign



$$\tau_{Max} = \frac{\sigma_1 - \cancel{\sigma_3}^{\,0}}{2} = \frac{\sigma_1}{2} = \frac{\sigma_{yp}}{2}$$

for $|\sigma_1| > |\sigma_2| \longrightarrow |\sigma_1| = \sigma_{yp} \longrightarrow \frac{\sigma_1}{\sigma_{yp}} = \pm 1$ ②

$|\sigma_2| > |\sigma_1| \longrightarrow |\sigma_2| = \sigma_{yp} \longrightarrow \frac{\sigma_2}{\sigma_{yp}} = \pm 1$ ③



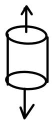
2 Maximum Distortion Energy (Von Mises) Theory (HW3P4)

Distortion energy per volume = Distortion energy with yielding in simple tension

$$U^{3D}_{od} = \frac{3}{4G}\tau^2_{oct} = \frac{3}{2}\left(\frac{1+\nu}{E}\right)\tau^2_{oct}$$

$$\tau^{3D}_{oct} = \frac{1}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau^2_{xy} + \tau^2_{yz} + \tau^2_{zx})\right]^{1/2}$$

$$\tau^{3D}_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$



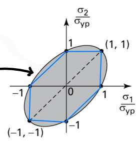
$$\boxed{\sigma_x = \sigma_{yp} \longrightarrow \tau_{oct} = \frac{\sqrt{2}}{3}\sigma_{yp}}$$

$$U^{1D}_{od} = \frac{3}{2}\frac{(1+\nu)}{E}\tau^2_{oct} = \frac{3}{2}\frac{(1+\nu)}{E}\frac{(2\sigma^2_{yp})}{9} \quad ; \quad U^{3D}_{od} = U^{1D}_{od}$$

$$\boxed{2\sigma^2_{yp} = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \qquad \boxed{\sigma_{yp} = \sigma_e = \frac{3}{\sqrt{2}}\tau_{oct}}$$

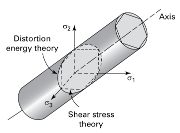
For plane stress: $\boxed{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma^2_{yp}}$

$$\boxed{\left(\frac{\sigma_1}{\sigma_{yp}}\right)^2 - \left(\frac{\sigma_1}{\sigma_{yp}}\right)\left(\frac{\sigma_2}{\sigma_{yp}}\right) + \left(\frac{\sigma_2}{\sigma_{yp}}\right)^2 = 1}$$



Yield Surface in 3D: $\boxed{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$

only difference is hydrostatic stress has no effect



Ductile

3 Octahedral Shear Stress Theory (Mises-Hencky Von-Mises Criteria)

$$U_{od} = \frac{3}{2}\left(\frac{1+\nu}{E}\right)\tau^2_{oct} \quad , \quad \tau_{oct} = \frac{1}{3}\left(2\sigma^2_{yp}\right)^{1/2}$$

Work with τ_{oct} (instead of energy)

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_e = \frac{\sqrt{2}}{3}\sigma_{yp} \longrightarrow \boxed{\tau_{oct} = 0.47\sigma_{yp}}$$

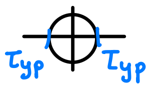
$$\boxed{\tau_{oct} = \frac{1}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau^2_{xy} + \tau^2_{yz} + \tau^2_{zx})\right]^{1/2}}$$

Comparison of yielding theories

① Max shear stress (Tresca) $\boxed{\tau_{yp} = \frac{1}{2}\sigma_{yp}}$

Simple tension (σ_{yp}): $\sigma_1 = \sigma_{yp}, \sigma_2 = 0 \longrightarrow \boxed{\tau_{yp} = \frac{\sigma_{yp} - 0}{2}}$

Simple torsion (τ_{yp}): $\sigma_1 = \tau_{yp}, \sigma_2 = -\tau_{yp} \longrightarrow \boxed{\sigma_{yp} = \sigma_1 - \sigma_2 = 2\tau_{yp}}$



$\boxed{\tau_{yp} = 0.5\sigma_{yp}}$ (Simple & useful Safer)

② Maximum Distortion Energy / ③ Octahedral Shear Stress Theory

Simple tension (σ_{yp}): $\sigma_1 = \sigma_{yp}, \sigma_2 = 0 \implies$

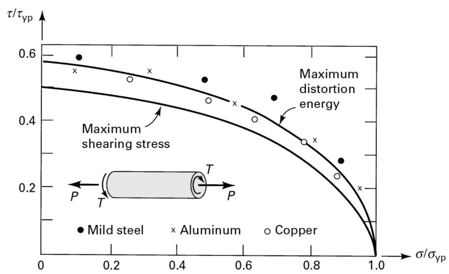
$$\sigma_1^2 - 0 + 0 = \sigma^2_{yp} \quad \& \quad \boxed{\tau_{oct} = \frac{1}{3}\left[\sigma^2_{yp} + \sigma^2_{yp}\right]^{1/2} = \frac{\sqrt{2}}{3}\sigma_{yp}}$$

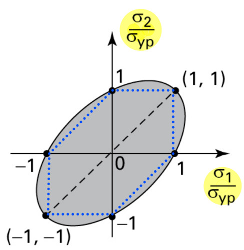
Simple torsion (τ_{yp}): $\sigma_1 = \tau_{yp}, \sigma_2 = -\tau_{yp} \implies$

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_{yp} \longrightarrow 3\tau^2_{yp} = \sigma_{yp} \longrightarrow \boxed{\tau_{yp} = \frac{1}{\sqrt{3}}\sigma_{yp}}$$

②/③ : Max distortion Energy (Von-Mises)

$\boxed{\tau_{yp} = 0.577\sigma_{yp}}$ (a better match)



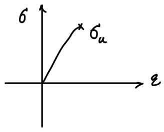


Brittle

① Max principal stress theory

largest principal stress = ultimate strength → fracture

$\sigma_{ij} \rightarrow \sigma_1 > \sigma_2 > \sigma_3$



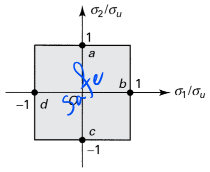
For plane stress ($\sigma_3 = 0$)

$|\sigma_1| = \sigma_u \longrightarrow \dfrac{\sigma_1}{\sigma_u} = \pm 1$

or

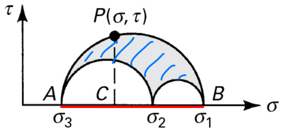
$|\sigma_2| = \sigma_u \longrightarrow \dfrac{\sigma_2}{\sigma_u} = \pm 1$

4 line equations



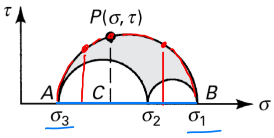
② Mohr's Theory

- Mat. behavior tension ≠ compression



- @ any point normal & shear components

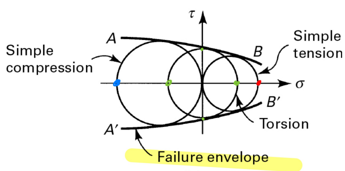
$|\tau| = f(\sigma)$



- Points on outer circle corresponds to Weakest planes → only need to know A & B (σ_1 & σ_3)

1) tension 2) torsion 3) compression

- Much more resistance in compression



Brittle

(HW3P5)

③ Coulomb - Mohr Theory (internal friction theory)

Critical shear stress is related to internal friction

If the frictional force is regarded as a function of the normal stress acting on a shear plane, the critical shearing stress and normal stress can be connected by an equation of the following form:

$\boxed{\tau = a\sigma + b}$

Shear stress is a linear fcn of σ

It depends on material prop. (a & b)

Plane stress ($\sigma_3 = 0$)

$\sigma_1 > 0$ (tensile) & $\sigma_2 < 0$ (compressive)

$\rightarrow \tau = \dfrac{\sigma_1 - \sigma_2}{2}$ & $\sigma = \dfrac{\sigma_1 + \sigma_2}{2}$ (on max shear plane)

$\tau = a\sigma + b \longrightarrow \dfrac{\sigma_1 - \sigma_2}{2} = a\left(\dfrac{\sigma_1 + \sigma_2}{2}\right) + b$

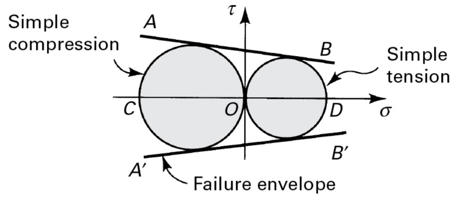
$\boxed{\therefore\ \sigma_1(1-a) - \sigma_2(1+a) = 2b}$ ①

Tension : $\sigma_1 = \sigma_{uT}$, $\sigma_2 = 0$

Compression : $\sigma_2 = -\sigma_{uc}$, $\sigma_1 = 0$

$\xrightarrow{\text{sub ①}}$ $\left[\begin{array}{l} \sigma_{uT}(1-a) = 2b \\ \sigma_{uc}(1+a) = 2b \end{array}\right.$

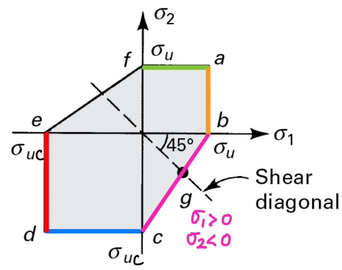
Solve for a & b → $\left[\begin{array}{l} a = \dfrac{\sigma_{uT} - \sigma_{uc}}{\sigma_{uT} + \sigma_{uc}} \\[2ex] b = \dfrac{\sigma_{uT}\,\sigma_{uc}}{\sigma_{uT} + \sigma_{uc}} \end{array}\right.$ $\xrightarrow{\text{sub ①}}$ $\boxed{\dfrac{\sigma_1}{\sigma_{uT}} - \dfrac{\sigma_2}{\sigma_{uc}} = 1}$ ($\sigma_1 > 0, \sigma_2 < 0$)



If $\sigma_1 > 0, \sigma_2 > 0 \rightarrow \underbrace{\sigma_1 = \sigma_{uT}}_{ab}$, $\underbrace{\sigma_2 = \sigma_{uT}}_{af} \rightarrow$ diameter of tensile circle

If $\sigma_1 < 0, \sigma_2 < 0 \rightarrow \underbrace{\sigma_1 = -\sigma_{uc}}_{ed}$, $\underbrace{\sigma_2 = -\sigma_{uc}}_{cd} \rightarrow$ diameter of comp. circle

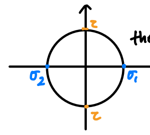
Line ef : $\dfrac{\sigma_2}{\sigma_{uT}} - \dfrac{\sigma_1}{\sigma_{uc}} = 1$



For Pure Shear (HW3P5)

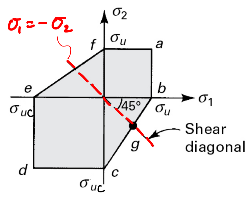
$\boxed{\sigma_1 = -\sigma_2}$

$\dfrac{\sigma_1}{\sigma_{uT}} - \dfrac{\sigma_2}{\sigma_{uc}} = 1$



the limiting shear

$\boxed{\tau_u = \sigma_1}$



$\rightarrow \dfrac{\sigma_1}{\sigma_{uT}} + \dfrac{\sigma_1}{\sigma_{uc}} = 1 \longrightarrow \boxed{\sigma_1 = \tau_u = \dfrac{\sigma_{uT}\,\sigma_{uc}}{\sigma_{uT} + \sigma_{uc}}}$