Partial Differential Equation Modeling: Temperature Modeling of Rectangular Hot Plate

Exercise 3–2: Solve for the equilibrium temperature distribution using the 2D Laplace equation on an $L \times H$ sized rectangular domain with the following boundary conditions:

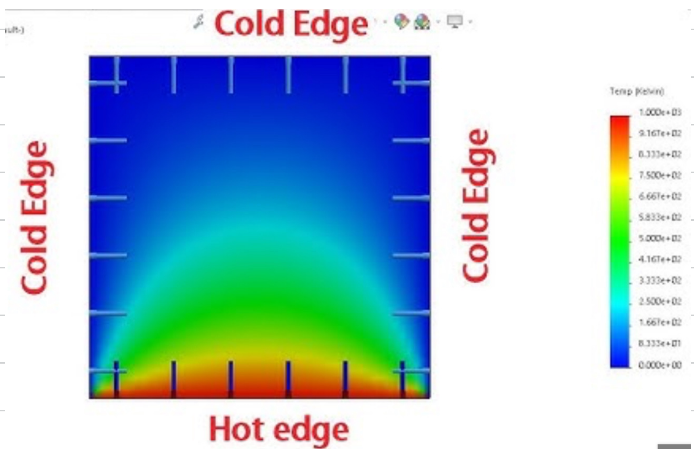
1. Left: $u_x(0, y) = 0$ (insulated)
2. Right: $u_x(L, y) = 0$ (insulated)
3. Bottom: $u(x, 0) = 0$ (fixed temperature)
4. Top: $u(x, H) = f(x)$ (prescribed temperature profile)

(a) Solve for a general boundary temperature $f(x)$. Hint: use an inner product to determine the most general form of your undetermined coefficient. At this point, the expression for your coefficient may still include an integral.

(b) Solve for the particular temperature distribution $f(x) = \cos(4\pi x/L)$.

(c) Sketch the solution you found in part (b). You can do this using a surface or waterfall plot on a computer, or by carefully hand-drawing a series of curves from top to bottom.

(d) Without too much extra work, tell me how this solution would change if we made the left and right boundary conditions fixed at a temperature of zero?



a) $u(x,y) = F(x)G(y)$

$F_{xx}(x)\,G(y) + F(x)\,G_{yy}(y) = 0 = \nabla^2 u$

$F_{xx}(x)\,G(y) = -F(x)\,G_{yy}(y)$

$$\frac{F_{xx}}{F} = \frac{-G_{yy}}{G} = -\lambda_n$$

step 1) ODE for x

$F_{xx} + \lambda F = 0$

$F(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$ $\qquad F_x(0) = F_x(L) = 0$

$F'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\,x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$

For $\lambda > 0$ $\qquad F'(0) = B\sqrt{\lambda} = 0 \qquad \sqrt{\lambda} \neq 0$ $\qquad$ Apply BC #1 : $u_x(0,y) = 0$

$B = 0$

$F'(L) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\,L) = 0$ $\qquad$ Apply BC #2 : $u_x(L,y) = 0$

$\sqrt{\lambda}\,L = n\pi$ to satisfy BC

$\lambda = \left(\frac{n\pi}{L}\right)^2$ for $n = 0, 1, 2, 3, \ldots$

only special λ satisfy the BC

$F(x) = A\cos\left(\frac{n\pi}{L}x\right)$ for an integer value of n

If $\lambda < 0$, we will not be able to satisfy both BC $(F_x(0) = F_x(L) = 0)$

Step 2) ODE for y

$G_{yy} = -\lambda G$

$G_{yy} = \left(\frac{n\pi}{L}\right)^2 G$

$G(y) = Ce^{\frac{n\pi}{L}y} + De^{\frac{n\pi}{L}y}$

$G(0) = C_n + D_n = 0 \rightarrow C_n = -D_n$ $\qquad$ Apply BC #3 : $u(x,0) = 0 \rightarrow G(0) = 0$

$G(y) = \underbrace{Ce^{\frac{n\pi}{L}y} - Ce^{\frac{n\pi}{L}y}}_{\text{Hyperbolic sine function}}$

$G(y) = 2C_n \sinh\left(\frac{n\pi}{L}y\right)$

Global sol'n : $u(x,y) = F(x)G(y) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi}{L}y\right)$

Apply BC #4 : $u(x,H) = f(x) \rightarrow G(H) = f(x)$

$$u(x,H) = f(x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi}{L}H\right)$$ Breaking up f(x) into Fourier Coefficients

Solve coefficient by computing inner product of f(x)

$$\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L \sum \underbrace{A_n \underbrace{\sinh\left(\frac{n\pi}{L}H\right)}_{\text{sinh (numbers)}} \cos\left(\frac{n\pi}{L}x\right)}_{f(x)} \cos\left(\frac{m\pi}{L}x\right)dx$$

Only non-zero when $n = m \longrightarrow$ drop summation term

$$\therefore \int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx = \int_0^L \sum A_m \sinh\left(\frac{m\pi}{L}H\right)\underbrace{\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx}_{\text{Integral when } n=m \text{ is } L/2}$$

$$= \frac{A_m L}{2}\sinh\left(\frac{m\pi H}{L}\right)$$ Brea

$$A_m = \underbrace{\frac{2}{L\sinh(m\pi H/L)}}_{\text{constants}} \underbrace{\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx}_{\text{FT of data from BC}}$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty}\left[\frac{2}{L\sinh(n\pi H/L)}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx\right]\cos\left(\frac{n\pi}{L}x\right)\sinh\left(\frac{n\pi}{L}y\right)}$$

b) Solve particular temperature distribution $f(x) = \cos\left(\frac{4\pi x}{L}\right)$

$$f(x) = \cos\left(\frac{4\pi x}{L}\right) = \sum_{n=0}^{\infty} A\cos\left(\frac{4\pi x}{L}\right)$$

$F(x) = A\cos\left(\frac{n\pi}{L}x\right)$ for an integer value of n

$$A = \frac{2}{L\sinh(4\pi H/L)}\int_0^L \cos\left(\frac{4\pi x}{L}\right)\cos\left(\frac{4\pi}{L}x\right)dx$$

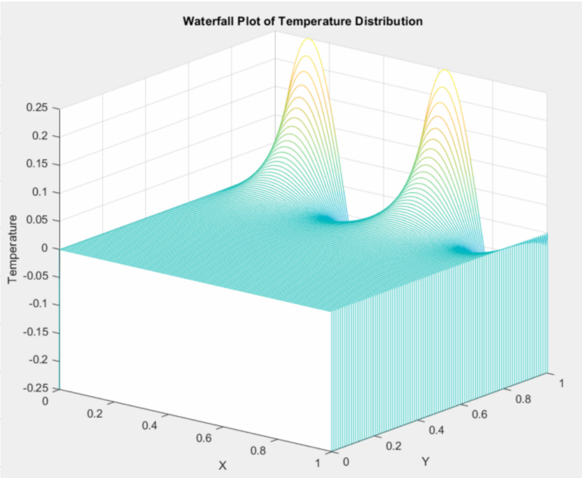
$$A = \frac{2}{L\sinh(4\pi H/L)}\int_0^L \cos^2\left(\frac{4\pi x}{L}\right)dx$$

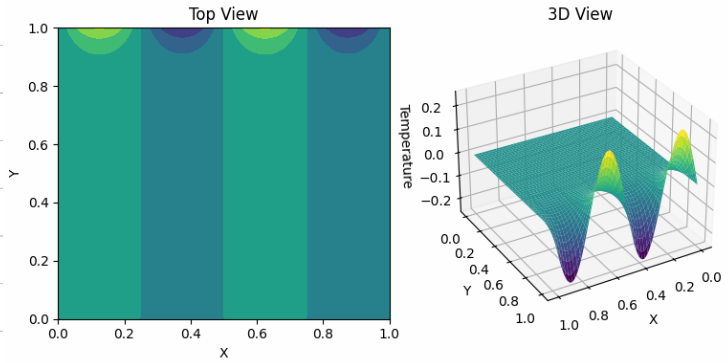
$$A = \frac{2}{L\sinh(4\pi H/L)} \times \frac{L}{2} = \frac{1}{\sinh(4\pi H/L)}$$

$\int_0^L \cos^2\left(\frac{n\pi x}{L}\right)dx = L/2$ for all integers n

$$\boxed{u(x,y) = \frac{1}{\sinh(4\pi H/L)}\cos\left(\frac{4\pi}{L}x\right)\sinh\left(\frac{4\pi}{L}y\right)}$$

c)





d) If boundary conditions on left and right side are : $u(0,y) = u(L,y) = 0$

$$F(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$
$$F(0) = A = 0$$
$$F(L) = B\sin(\underbrace{\sqrt{\lambda}\,L}_{0,\pi,2\pi,3\pi}) = 0 \qquad \lambda = \left(\frac{2\pi n}{L}\right)^2$$

The solution would involve a Fourier sine series instead of a cosine series

Code

```python
import numpy as np
import matplotlib.pyplot as plt

L = 1.0
H = 1.0

x = np.linspace(0, L, 100)
y = np.linspace(0, H, 100)
X, Y = np.meshgrid(x, y)

u = (1 / (4 * np.sinh(4 * np.pi * H/L))) * np.sin(4 * np.pi * X/L) * np.sinh(4 * np.pi * Y/L)

fig = plt.figure(figsize=(8, 4))

# Top View
ax1 = fig.add_subplot(121)
ax1.contourf(X, Y, u, cmap='viridis')
ax1.set_title('Top View')
ax1.set_xlabel('X')
ax1.set_ylabel('Y')

# 3D View (elev=30, azim=60)
ax2 = fig.add_subplot(122, projection='3d')
ax2.plot_surface(X, Y, u, cmap='viridis', edgecolor='none')
ax2.set_title('3D View')
ax2.set_xlabel('X')
ax2.set_ylabel('Y')
ax2.set_zlabel('Temperature')
ax2.view_init(elev=30, azim=60)

plt.tight_layout()
plt.show()
```

Partial Differential Equation Modeling of Vibrating String System

Exercise 3–1: Consider the following PDE for a vibrating string of finite length L:

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x \le L$$

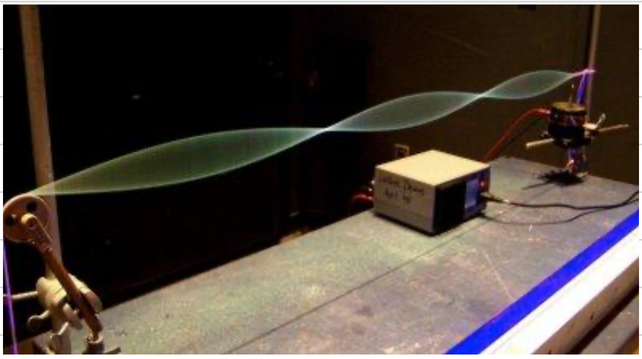
with the following initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0;$$

and boundary conditions

$$u(0,t) = 0, \qquad u_x(L,t) = f(t).$$

Show that you cannot solve this PDE using separation of variables. In other words, begin with the separation assumption, and work through the problem until there is a contradiction with the ICs and BCs.



Initial Conditions:

- $u(x,0) = 0$ No displacement at $t=0$
- $u_t(x,0) = 0$ No initial velocity at $t=0$

Boundary Conditions:

- $u(0,t) = 0$ Position of string is fixed at $x=0$
- $u_x(L,t) = f(t)$ Slope of string is determined by $f(t)$ at $x=L$

0) Seperation Assumption:

$$u(x,t) = F(x)G(t)$$

1) $u_{tt} = F(x)G''(t)$

$u_{xx} = F''(x)G(t)$

Apply $u_{tt} = c^2 u_{xx}$: $F(x)G''(t) = c^2 F''(x)G(t)$

$$\frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{G''(t)}{G(t)} = -\lambda^2$$

2a) $\frac{F''(x)}{F(x)} = -\lambda^2 \longrightarrow F'' + \lambda^2 F = 0$

Characteristic eqn: $r^2 + \lambda^2 = 0 \rightarrow r = \pm i\sqrt{\lambda}$

$F(x) = \alpha\cos(\lambda x) + \beta\sin(\lambda x)$

$F'(x) = -\lambda\alpha\sin(\lambda x) + \lambda\beta\cos(\lambda x)$

$F(0) = \alpha\cos(0) + \beta\sin(0) = 0$ Apply first BC: $u(0,t) = 0$

$\alpha = 0$

$f(t) = -\lambda\alpha\sin(\lambda L) + \lambda\beta\cos(\lambda L)$ Apply second BC: $u_x(L,t) = f(t)$

$$\underbrace{f(t)}_{\text{function of time}} = \underbrace{\lambda\beta\cos(\lambda L)}_{\text{constant term}}$$

$$F(x) = \frac{f(t)}{\cos(\lambda L)}\sin(\lambda x)$$

2b) $G''(t) = -c^2\lambda^2 G \longrightarrow G'' + c^2\lambda^2 G = 0$

$G(t) = K\cos(c\lambda t) + b\sin(c\lambda t)$

$G'(t) = -Kc\lambda\sin(c\lambda t) + bc\lambda b\cos(c\lambda t)$

$G(0) = K\cos(0) + b\sin(0) = 0$ Apply first IC $u(x,0) = 0$

$K = 0$

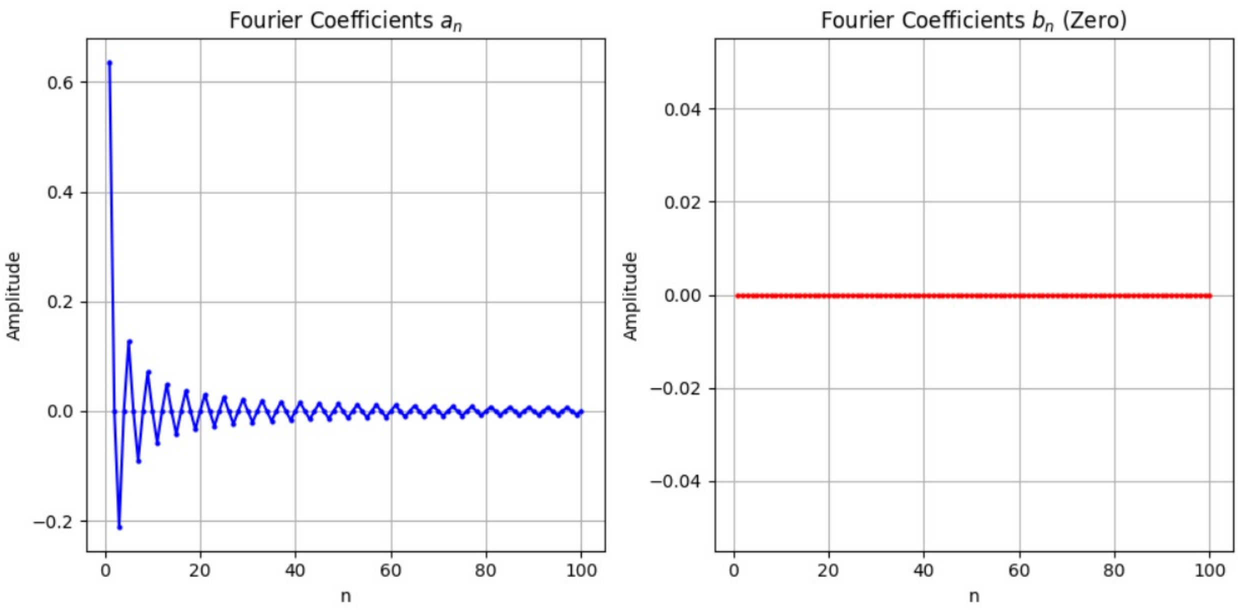
$G'(0) = -Kc\lambda\sin(0) + bc\lambda\cos(0) = 0$ Apply second IC $u_t(L,t) = f(t)$

$b = 0$

$G(t) = 0$ Violates assumption that F and G are seperable and non-zero.

The initial and boundary conditions are contradicting each other. In part 0, we assume f and g are separable and **non-zero functions**. In part 2a, f(t) is equal to a constant by applying the Boundary Conditions. In Part 2b, G(t) is equal to zero by applying the Initial Conditions.

b)



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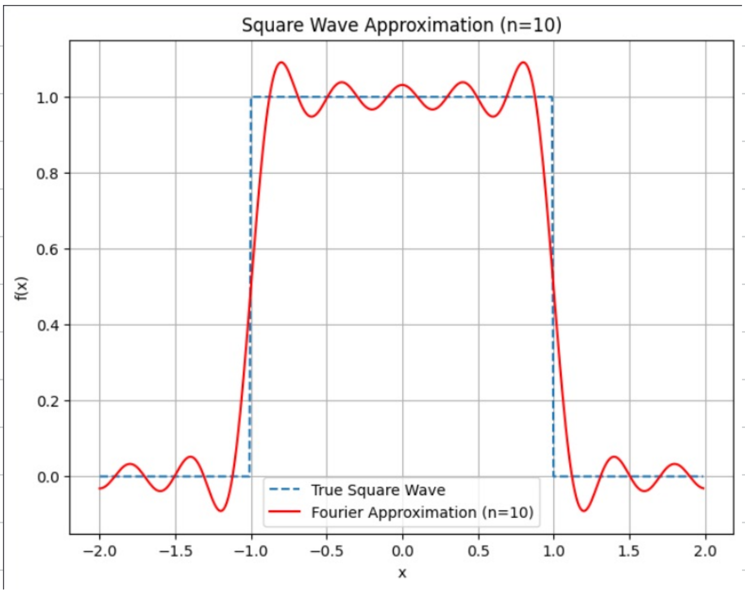

d) $$\hat{f}(\omega) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x} dx = \int_{-1}^{1} 1\, e^{-i\omega x} dx$$

$$= \frac{-e^{-i\omega x}}{i\omega}\Bigg|_{-1}^{1} = \frac{2\sin\omega}{\omega}$$

$$\boxed{\hat{f}(\omega) = \frac{2\sin\omega}{\omega}}$$

e) Fourier transform can be thought of as a Fourier series where you take the periodic domain from minus L to L and you take the limit as L goes to infinity

The difference between the Fourier transform and the Fourier series is that the Fourier transform is applicable for non-periodic signals, while the Fourier series is applicable to periodic signals