CSCI 2824: Discrete Structures

Lecture 5: Logical Equivalences

Reminders

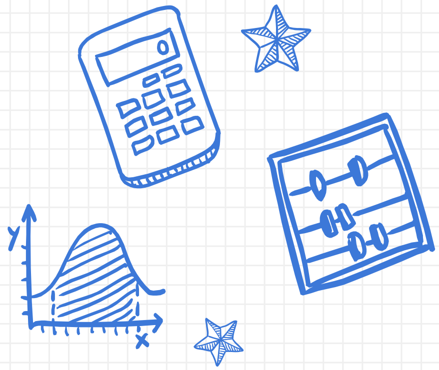
Submissions:

- Homework 1: Fri 9/6 at noon – 1 try
- Homework 2: Fri 9/13 at noon – same

Disabilities forms – turn them in by the end of 2nd week

Readings:

- 1.1-1.3 this week
- 1.4-1.6 through next week

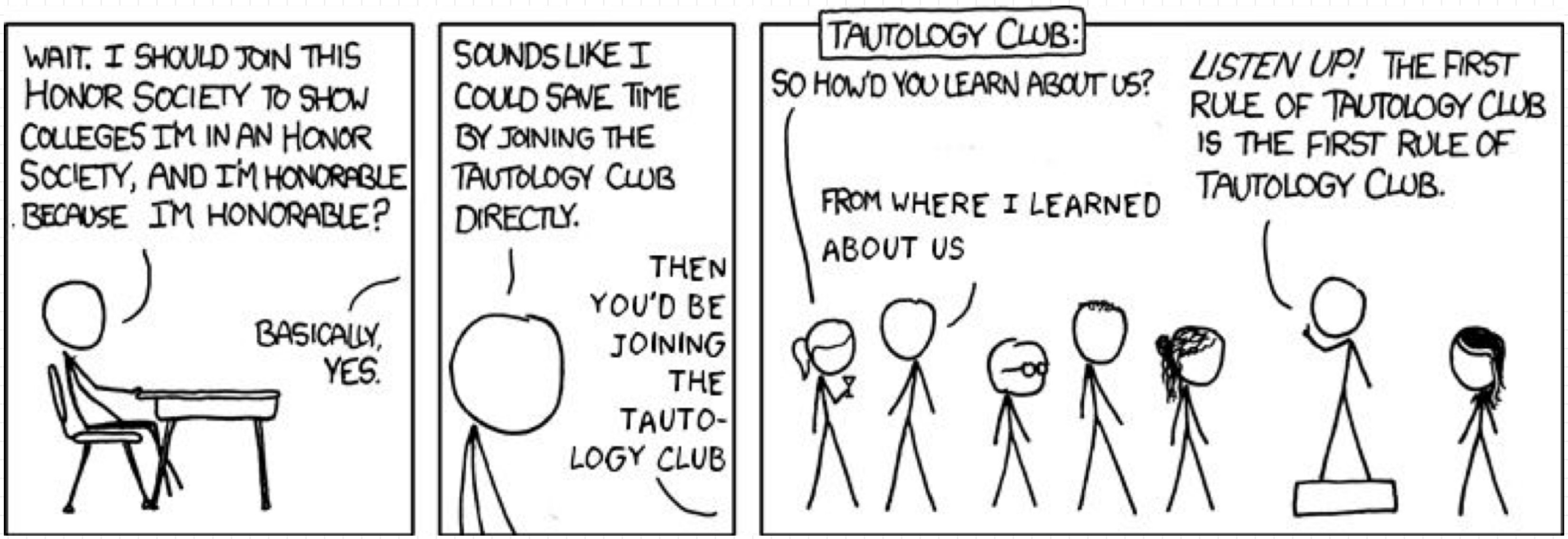


Propositional Logic

Logically Equivalent

Definition: The compound propositions p and q are called **logically equivalent** if $p \iff q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- A **tautology** is a statement that is **always true**.



Logical Equivalencies

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**. The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are **logically equivalent**.

A compound proposition that is always false is called a **contradiction**.

A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Logical Equivalencies

Example of a Tautology:

p	$\neg p$	$p \lor \neg p$

Example of a Contradiction:

p	$\neg p$	$p \land \neg p$

Logical Equivalencies

Different - but the same - ways of writing compound propositions

Example: *It is not the case that Gary is boring and rides a motorcycle.*

Can we express this statement in a different way?

Logical Equivalencies

Different - but the same - ways of writing compound propositions

Example: *It is not the case that Gary is boring and rides a motorcycle.*

Another way to express this:

Either Gary is not boring or Gary does not ride a motorcycle.

Logical Equivalencies

Different - but the same - ways of writing compound propositions

Example: *It is not the case that Gary is boring and rides a motorcycle.*

- Another way to express this:

  Either Gary is not boring or Gary does not ride a motorcycle.

- Let's break it down. Define:

  p = Gary is boring and *q = Gary rides a motorcycle*

Logical Equivalencies

Different - but the same - ways of writing compound propositions

Example: *It is not the case that Gary is boring and rides a motorcycle.*

- Another way to express this:

  Either Gary is not boring or Gary does not ride a motorcycle.

- Let's break it down. Define:

  p = *Gary is boring* and q = *Gary rides a motorcycle*

- Then the original compound proposition is: $\neg(p \land q)$
- And the revised (more normal-sounding) version is: $\neg p \lor \neg q$

  → Are they logically equivalent?

Logical Equivalencies

Example: (1) *It is not the case that Gary is boring and Gary rides a motorcycle.*
(2) *Either Gary is not boring or Gary does not ride a motorcycle.*

Are (1) and (2) logically equivalent? That is, $\neg(p \wedge q) \equiv \neg p \vee \neg q$?

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T					
T	F					
F	T					
F	F					

Logical Equivalencies

Example: (1) *It is not the case that Gary is boring and Gary rides a motorcycle.*
(2) *Either Gary is not boring or Gary does not ride a motorcycle.*

Are (1) and (2) logically equivalent? That is, $\neg(p \wedge q) \equiv \neg p \vee \neg q$?

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Example: Show that $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$

Example: Show that $\neg(p \vee q) \equiv \neg p \wedge \neg q$ - Solution

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Logical Equivalencies

We just Proved:

De Morgan's Laws

$$\neg\,(p \land q) \equiv \neg p \lor \neg q$$

$$\neg\,(p \lor q) \equiv \neg p \land \neg q$$

Logical Equivalencies

Other Equivalences (from our book)

To show that two compound propositions are logically equivalent:

- Prove it with a Truth Table
- Use Equivalence Rules to go from one to the other.

TABLE 6 **Logical Equivalences.**

Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalencies **Other Equivalences (from our book)**

TABLE 7 Logical Equivalences Involving Conditional Statements.

Equivalence	
$p \rightarrow q \equiv \neg p \vee q$	Relation by Implication (RBI)
$p \rightarrow q \equiv \neg q \rightarrow \neg p$	Contraposition
$p \vee q \equiv \neg p \rightarrow q$	
$p \wedge q \equiv \neg(p \rightarrow \neg q)$	
$\neg(p \rightarrow q) \equiv p \wedge \neg q$	
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$	
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$	
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$	
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$	

TABLE 8 Logical Equivalences Involving Biconditional Statements.

Equivalence	
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Biconditional
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$	
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$	
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$	

$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$ Alternate definition of xor

Logical Equivalencies

<u>Example</u>: Show that $p \rightarrow q \equiv \neg q \rightarrow \neg p$ (without a truth table)

Logical Equivalencies

<u>Example</u>: Show that $p \to q \equiv \neg\, q \to \neg\, p$ (without a truth table)

$p \to q \equiv \neg\neg(p \to q)$	double negation
$\equiv \neg(p \land \neg q)$	Relation by Implication (RBI)
$\equiv \neg p \lor q$	De Morgan
$\equiv q \lor \neg p$	Commutative
$\equiv \neg\neg q \lor \neg p$	double negation for q
$\equiv \neg q \to \neg p$	RBI

Logical Equivalencies

Example: Show that $(\mathrm{p} \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$ (without a truth table)

Logical Equivalencies

<u>Example</u>: Show that $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$ (without a truth table)

Satisfiability

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. If there is no such case then we say it is **unsatisfiable** (i.e. a contradiction)

Example: Show that $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$ is satisfiable.

Satisfiability

<u>Example</u>: Show that $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$ is satisfiable.

Satisfiability

Example: Show that $(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q)$ is not satisfiable.

Example: Sudoku puzzles can be written (and solved) as a satisfiability problems

Solving this:

1) First chain together the propositions with provided values: $p(1,1,5) \wedge p(1,2,3) \wedge p(1,5,7) \wedge \ldots$
2) Assert that every row contains every number: $\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i,j,n)$
3) Assert that every column contains every number: $\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$
4) Assert that every 3x3 block contains every number: $\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Let $p(i,j,n)$ denote the proposition that a number n is in the cell in row i and column j

9 rows, 9 columns, 9 numbers $= 9 \times 9 \times 9 =$ 729 propositions

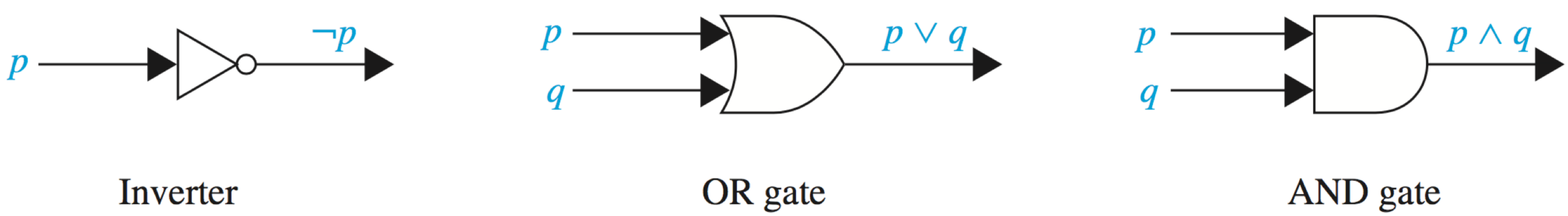
5) Assert that no cell contains more than one number: $\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigwedge_{m=1, m\neq n}^{9} (p(i,j,n) \rightarrow \neg p(i,j,m))$

6) String 1-5 together with conjunctions

Good Read on this problem!

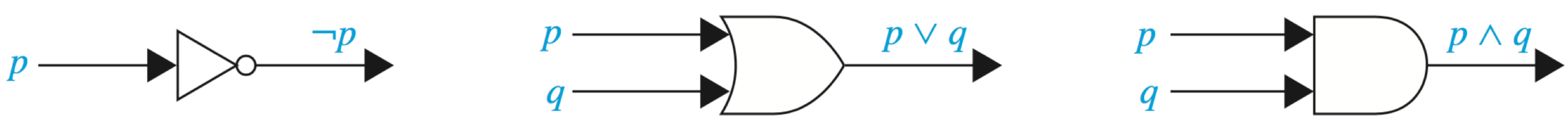
Application: logic circuits ← not required material, but interesting

- Using propositional logic to design computer hardware.
- A logic circuit is an idealized circuit where either voltage (T) or no voltage (F) travel along inputs ("wires") into logical gates with varying outputs.
- There are three main types of logical gates:



Application: logic circuits

- Using propositional logic to design computer hardware.
- A logic circuit is an idealized circuit where either voltage (T) or no voltage (F) travel along inputs ("wires") into logical gates with varying outputs.
- There are three main types of logical gates:



Inverter OR gate AND gate

Example: If the input to an *Inverter* has no voltage, that the output will have voltage.

Example: If either of the inputs to an *AND* gate has no voltage, that the output will have no voltage.

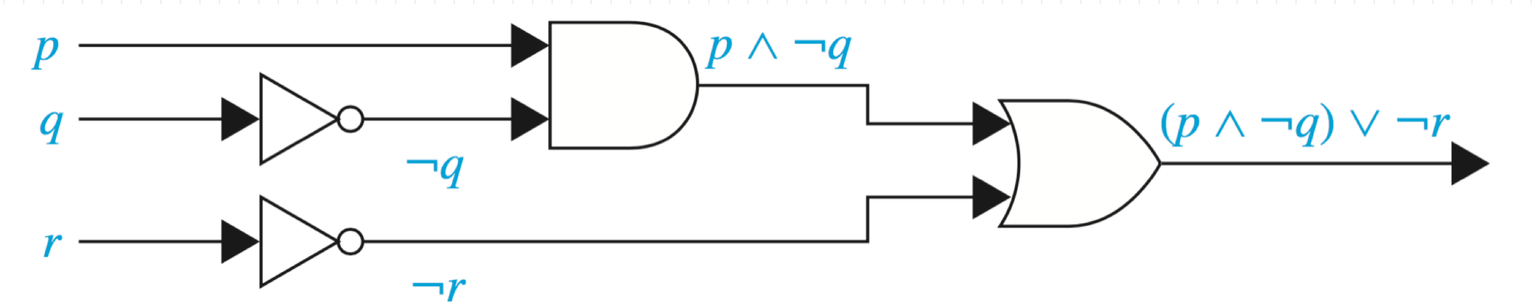
Application: logic circuits

- You can cook up more complex circuits with varying outputs, based on the flow of inputs.
- For example, *p*, *q* and *r* are each binary (voltage or no voltage).

  Say: *p* = user is pressing the "Ctrl" key

  q = user is pressing the "Alt" key

  r = user is pressing the "Delete" key



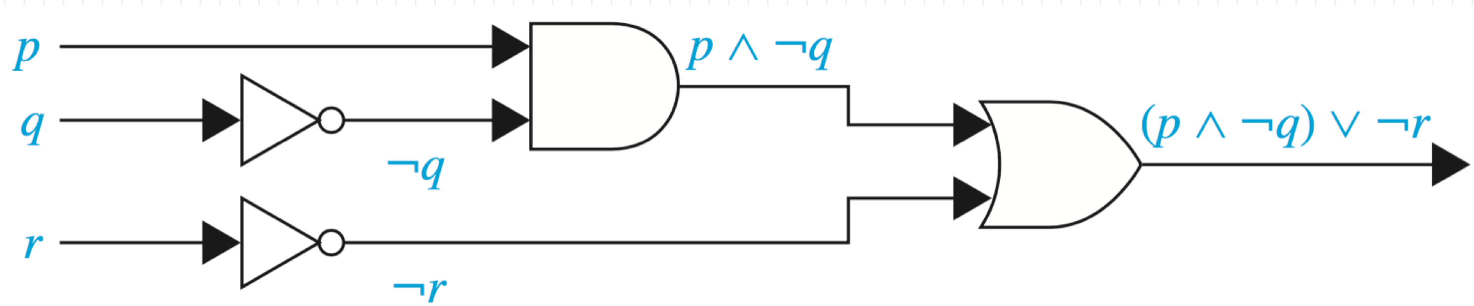
Example: What is a set of inputs for this circuit that leads to the last logic gate having an output with a voltage?

Application: logic circuits

Example: What is a set of inputs for this circuit that leads to the last logic gate having an output with a voltage?

Easy answer: Last connective has … ∨ ¬*r* on the end.
⇒ Anything where *r* has no voltage works

because r is negated at its first gate, so if it is OFF to begin, then this line has voltage after the gate, and flows into the OR gate, which is ON if either of its inputs has voltage



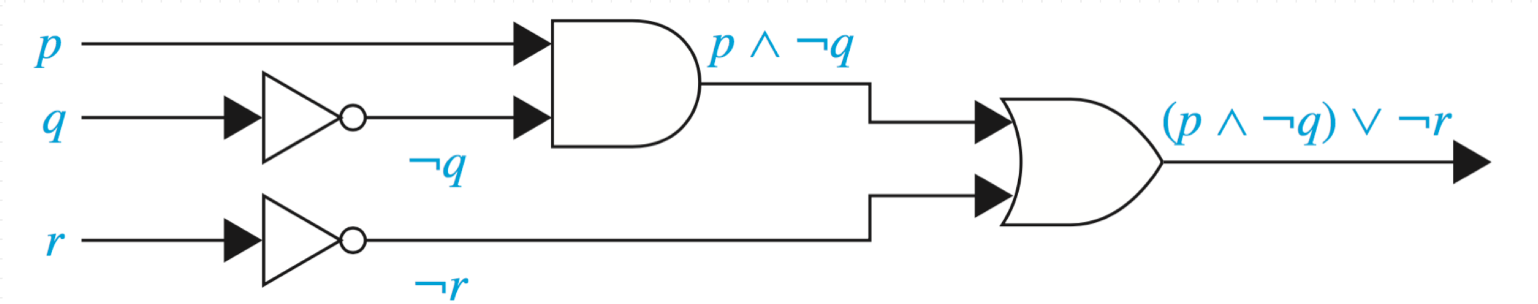
More challenging answer: Use a truth table!

Application: logic circuits

Example: What is a set of inputs for this circuit that leads to the last logic gate having an output with a voltage?

Easy answer: Last connective has … $\vee \neg r$ on the end.
$\Rightarrow$ Anything where r has no voltage works

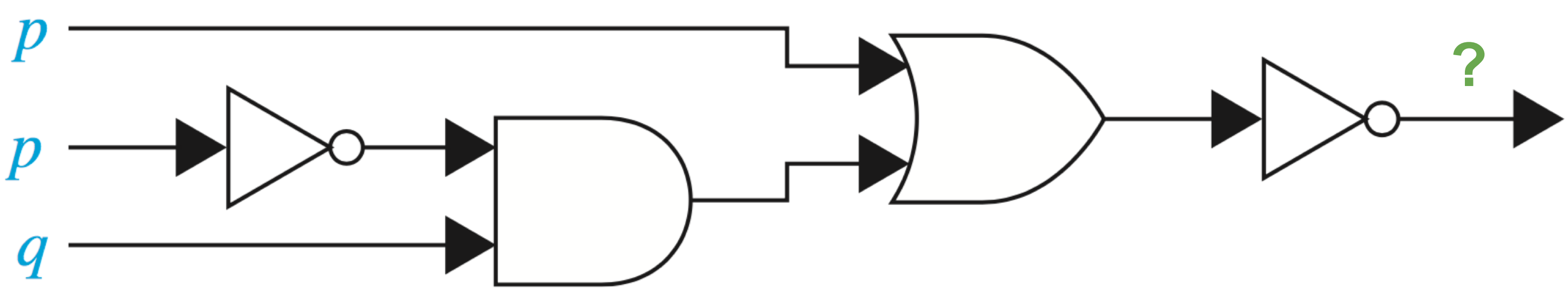
because r is negated at its first gate, so if it is OFF to begin, then this line has voltage after the gate, and flows into the OR gate, which is ON if either of its inputs has voltage



More challenging answer: Use a truth table! ***r* ON, *p* ON, *q* OFF.**

Application: logic circuits

FYOG: What is the symbolic output for the following logic circuit?



Extra Practice

Example 1: Work out the truth table to show $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Example 2: Work out the truth table to show $p \rightarrow q \equiv \neg p \vee q$

Example 3: Show that $(p \to q) \land (p \to r) \equiv p \to (q \land r)$

Example 4: Show that the following proposition is **not** satisfiable

$$(p \leftrightarrow q) \land (\neg p \leftrightarrow q)$$

Solutions

Example 1: Work out the truth table to show $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Solution:

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Example 2: Work out the truth table to show $p \to q \equiv \neg p \vee q$

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Example 3: Show that $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

Solution: This one is actually easier if we start from the second proposition

$$
\begin{aligned}
p \to (q \wedge r) &\equiv \neg p \vee (q \wedge r) && \text{(RBI)} \\
&\equiv (\neg p \vee q) \wedge (\neg p \vee r) && \text{(distribution)} \\
&\equiv (p \to q) \wedge (p \to r) && \text{(that one rule in reverse)}
\end{aligned}
$$

Example 4: Show that the following proposition is **not** satisfiable

$$(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$$

Solution: OK, this ones pretty easy

1. From the first conjunct we know that p and q must have the same truth values
2. From the second conjunct we know that p and q must have different truth values
3. This is a contradiction, thus the proposition is **not** satisfiable