CSCI 2824: Discrete Structures

Lecture 6: Predicates and Quantifiers

Reminders

Submissions:

- Homework 2: Fri 9/13 at noon – 1 try
- Quizlet 2: due Wednesday 9/11 at **8pm**

Disabilities forms – please bring them by Friday

Readings:

- 1.4-1.6 this week
- 1.6-1.8 through next week

Propositional Logic

Satisfiability – last time

Example: Show that $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$ is satisfiable.

Satisfiability – last time

<u>Example</u>: Show that $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$ is satisfiable.

$(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$ -> All three components must be True

Option 1: $(p \vee \neg q)$ True if **p is True**, but then $(\neg p \vee q) \wedge (\neg p \vee \neg q)$ can never be True

Option 2: $(p \vee \neg q)$ True if **p is False** and **q is False.** Let's check the other two propositions:

$(\neg p \vee q)$ will be True

$(\neg p \vee \neg q)$ will be True

$(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$ True if **p is False** and **q is False**

Satisfiability

<u>Example</u>: Show that $(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q)$ is not satisfiable.

Satisfiability

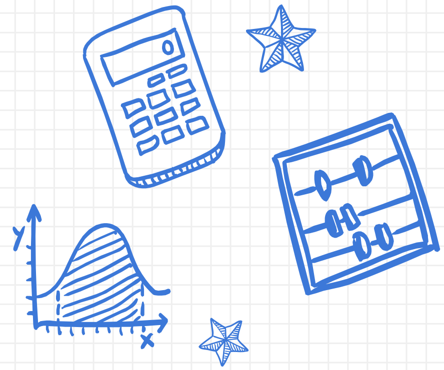
Example: Show that $(p \to q) \wedge (p \to \neg q) \wedge (\neg p \to q) \wedge (\neg p \to \neg q)$ is not satisfiable.

Similar strategy-> All four components must be True

Option 1: $(p \to q)$ True if **p and q are True**, but then $(p \to \neg q)$ will be False

Any combination of truth values for p and q makes one proposition True and another False.

Conclusion: Not Satisfiable



Predicate Logic

Predicate Logic

There are deficiencies in propositional logic. Today - a more flexible framework for constructing arguments and proofs! We will talk about two new constructs: **Predicates & Quantifiers**

From Merriam-Webster:

Definition of PREDICATE

1 **a** : something that is affirmed or denied of the subject in a proposition in logic
b : a term designating a property or relation

2 : the part of a sentence or clause that expresses what is said of the subject and that usually consists of a verb with or without objects, complements, or adverbial modifiers

Predicate Logic

Consider the following statements:

- All of my dogs love ham.
- Corky is one of my dogs.

You can probably figure out that Corky loves ham…

… but propositional logic lacks the flexibility to show this easily. You would need to do something like

$$\textit{All of my dogs love ham} = \textit{Corky loves ham} \wedge \textit{Citra loves ham} \wedge \textit{Archer loves ham}$$

This gets ridiculous as the number of dogs you are talking about grows.
… And lots of dogs love ham!

Predicate Logic

Two new constructs in **predicate logic** help us achieve this flexibility:
predicates and **quantifiers**

Consider the statement $x > 3$.

- x is a variable or a placeholder
- > 3 is a **predicate**

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Predicate Logic

Two new constructs in **predicate logic** help us achieve this flexibility:
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Consider the statement $x > 3$.

- x is a variable or a placeholder
- > 3 is a **predicate**

Let $P(x)$ represent $x > 3$

We call $P(x)$ a **propositional function**. When we assign a value to x, then $P(x)$ becomes a proposition and has a truth value.

Example: $P(4)$ is T, and $P(1)$ is F.

Predicate Logic

<u>Example:</u> If we let F be the name of the predicate, then we can think of $F(x)$ as a sentence that asserts an object (or subject) **is fast.**

$F(x)$ is read as "x is fast." where x represents the **domain** or "**domain of discourse**"

Rockets are fast. Usain Bolt is fast. Kikkan Randall is fast.

The **predicate** can be thought of as "is fast".

Predicate Logic

Example: Rachel makes lunch for Murray.
Anna makes necklaces for Naomi.

Predicate "template" is: ____ makes ____ for ____

The predicate describes a relationship between three variables or objects.

$Makes(x, y, z)$ or $M(x, y, z)$

x: who makes something
y: things being made
z: people that something is being made for

Predicate Logic

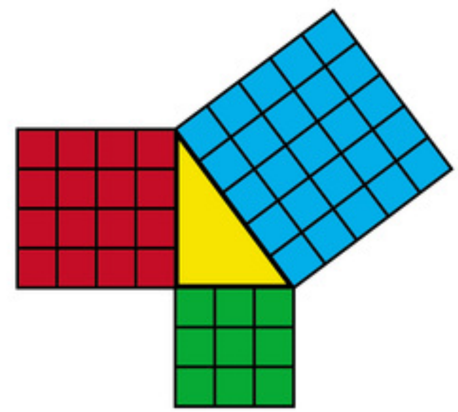
Propositional functions can have multiple variables.

<u>Example:</u> Let $Q(x, y)$ represent $x + 1 = y$

- What is the truth value of $Q(1,2)$?
- What is the truth value of $Q(3,2)$?

<u>Example:</u> Let $\mathrm{R}(x, y, z)$ represent $x^2 + y^2 = z^2$

- What is the truth value of $\mathrm{R}(1,1,1)$?
- What is the truth value of $\mathrm{R}(3,4,5)$?



Predicate Logic

We see propositional functions in computer science all the time.

- **If / Then statements**

```
if ( x > 10 ) then BREAK
```

- **While loops**

```
while ( i ≤ 10 ) do [stuff]
```

- **Error checking**

```
assert ( len(variable) ≥ 7 )
```

Predicate Logic

When using predicates we have to think about what values we input.

The set of values we intend to plug in is called the **domain of discourse** or commonly just the **domain.**

Example: All babies love to sleep. Parker is a baby.

Let $S(x)$ denote: x likes to sleep.

$S(Parker)$ is a true statement.

What is $S(8)$?

Predicate Logic

Suppose we fix a domain for *S(x)*. Let the domain of babies be

$$\{Parker, Madeline, Nocona, Tatum\}$$

Then *S(Parker)*, *S(Madeline)*, *S(Nocona)*, and *S(Tatum)* have truth values and make sense. S(8), S(★) don't have truth values and can't really be defined.

Predicate Logic

<u>Example:</u> [with Propositional Logic] "All babies love to sleep." might be represented as

Parker loves to sleep $\wedge$ **Madeline** loves to sleep $\wedge$
Nocona loves to sleep $\wedge$ **Tatum** loves to sleep …

But what if we re-define our domain to be All Babies in the USA…. using only propositional logic would be pretty inefficient.

Predicate Logic

Example: [with Propositional Logic] "All babies love to sleep." might be represented as

Parker loves to sleep $\wedge$ **Madeline** loves to sleep $\wedge$ **Nocona** loves to sleep $\wedge$ **Tatum** loves to sleep …

Instead, we introduce the universal quantifier: $\forall$
$\forall x\, S(x)$ means "for all x in my domain, P(x)"
So for our example, $\forall x\, S(x)$ means "for all babies in the set {Parker, Madeline, Nocona, Tatum} (i.e., all babies), these babies love to sleep" $\Rightarrow$ So, "all babies love to sleep"

Note: the quantifier $\forall$ turns $\forall x\, S(x)$ into a proposition

Predicate Logic

Universal Quantifier: $\forall$

$\forall x\ P(x)$ reads "For all x, P(x)." or "For every x, P(x)." for some general predicate *P(x)*

The domain must always be specified when a universal quantifier is used; without it, the universal quantification of a statement is not defined.

Example: [with Quantifiers] "All babies love to sleep." might be represented as

$$\forall x\ S(x)$$

S(x) is true for all x in the domain

Note that with the quantifier $\forall \boldsymbol{x}\ \boldsymbol{S}(\boldsymbol{x})$ becomes a proposition

Predicate Logic

Question: When is the statement $\forall x\ P(x)$ true?

Question: When is the statement $\forall x\ P(x)$ false?

Predicate Logic

Question: When is the statement $\forall x\ P(x)$ true?

- **Answer**: It is true when P(x) is true for all x in the domain.
- **Example**: Let the domain be all integers. $\forall$x ($x^2 \geq 0$) is true.
  (Any integer squared is non-negative.)

Question: When is the statement $\forall x\ P(x)$ false?

Predicate Logic

Question: When is the statement $\forall x\, P(x)$ true?

- **Answer**: It is true when P(x) is true for all x in the domain.
- **Example**: Let the domain be all integers. $\forall x\ (x^2 \geq 0)$ is true.
  (Any integer squared is non-negative.)

Question: When is the statement $\forall x\, P(x)$ false?

- **Answer**: It is false if there is any x in the domain such that P(x) is false.
- **Example**: Let the domain be all integers. $\forall x\ (x^2 > x)$ is false.

The case where $x = 0$ "breaks" the universal statement that for all integers x, $x^2 > x$

A case that demonstrates the breaking of a universal statement is called a **counterexample.**

Predicate Logic

Important take-aways:

To **disprove** a universal proposition, all you need is one specific **counterexample** that makes the statement not work.

- "All computer science instructors have gray hair."

To **prove** a universal proposition, one **specific** example **does nothing.** Usually, you need to work much, much harder.

- Even if you find a hundred computer science instructors with gray hairs, you have not proved that all of us have gray hair.
- **Never** try to prove a universal statement by demonstrating its truth for a set of specific examples and then asserting P(x) to be true in general.
- (that's why we have gray hairs)

Predicate Logic

So that's how you talk about **all things**

What if we just want to talk about **something**?

- Often, we just want to show that *something* is possible. In mathematical terms, that means showing that *there exists* an element in the domain that has some certain property in which we are interested.
  - "Are there any flights that will get me home on time?"
  - "Does there exist a green men's size 11 Croc in this store?"

For this, we introduce the **existential quantifier**: $\exists$

- $\exists x\ P(x)$ means "there exists an x in the domain, such that $P(x)$"
- **Example**: $\exists x\ P(x)$ could mean "there exists a dog [in my domain] such that that dog loves ham"

Predicate Logic

Question: When is the statement $\exists x\ P(x)$ true?

Question: When is the statement $\exists x\ P(x)$ false?

Predicate Logic

Question: When is the statement $\exists x\ P(x)$ true?

- **Answer:** It is true if you can find at least one x in the domain such that P(x) is true.
- **Example:** Let the domain be the integers. ∃x (x2 > x) is true.

  x = 2 works, because 22 = 4 > 2

Question: When is the statement $\exists x\ P(x)$ false?

Predicate Logic

Question: When is the statement $\exists x\, P(x)$ true?

- **Answer:** It is true if you can find at least one x in the domain such that P(x) is true.
- **Example:** Let the domain be the integers. $\exists x\ (x2 > x)$ is true.

  $x = 2$ works, because $22 = 4 > 2$

Question: When is the statement $\exists x\, P(x)$ false?

- **Answer:** It is false if there is no x in the domain that makes P(x) is true.
- **Example:** Let the domain be the integers. $\exists x\ (x2 < 0)$ is false.

Predicate Logic

TABLE 1 Quantifiers.

Statement	***When True?***	***When False?***
$\forall x\, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Predicate Logic

Special case to consider. **Empty Domain**

- $\forall x\ P(x)$ is true
- $\exists x\ P(x)$ is false

Example 1: All of the Olympic skiing medals I have won are gold.

Example 2: One of the Olympic skiing medals I have won is gold.

Predicate Logic

Scope of Quantifiers

Quantifiers have the narrowest scope of all logical operands.

Example: $\forall x\ \big(P(x) \wedge Q(x)\big)$ is not the same as $\forall x\ P(x) \wedge Q(x)$

Operator Precedence

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.

Example: $\forall x\ P(x) \vee Q(x)$ means $\big(\forall x P\ (x)\big) \vee\ Q(x)$,

rather than $\forall x(P\ (x)\ \vee\ Q(x))$.

Predicate Logic

Logical Equivalence involving Quantifiers

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates you use and which domain they're defined over.

Example: Are these equivalent?

$$\forall x\,\big(P(x) \land Q(x)\big) \equiv \forall x\, P(x) \land \forall x\, Q(x)$$

Predicate Logic

Example: Let our domain be $\{a, b, c\}$. Prove that the following is true:

$$\forall x \left(P(x) \wedge Q(x)\right) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$$

Predicate Logic

<u>Example:</u> Let our domain be $\{a, b, c\}$. Prove that the following is true:

$$\forall x\,\big(P(x) \land Q(x)\big) \equiv \forall x\, P(x) \land \forall x\, Q(x)$$

Intuition: let …

domain = all of my dogs
P(x) = "loves ham"
Q(x) = "has a short tail"

What do you think?

Predicate Logic

Example: Let our domain be $\{a, b, c\}$. Prove that the following is true:

$$\forall x \left(P(x) \wedge Q(x)\right) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$$

Predicate Logic

Example: Are these equivalent? $\forall x\, (P(x) \land Q(x)) \equiv \forall x\, P(x) \land \forall x\, Q(x)$

They are! Let's prove it.

- Let domain = "all of my dogs"
- Let *i* index the set of all of my dogs. Like, $x_1 = Citra$, $x_2 = Corky$, $x_3 = Archer$, …

Then …

$$\forall x\, P(x) \equiv P(x_1) \land P(x_2) \land P(x_3) \land P(x_4) \land \ldots$$

and

$$\forall x\, P(x) \land \forall x\, Q(x) \equiv (P(x_1) \land P(x_2) \land P(x_3) \land \ldots) \land (Q(x_1) \land Q(x_2) \land Q(x_3) \land \ldots)$$

Rearrange the order (associativity):

$$\forall x\, P(x) \land \forall x\, Q(x) \equiv (P(x_1) \land Q(x_1)) \land (P(x_2) \land Q(x_2)) \land (P(x_3) \land Q(x_3)) \land \ldots$$

Predicate Logic

Example: Are the statements below equivalent?

$$\forall x \left(P(x) \vee Q(x)\right) \equiv \forall x\, P(x) \vee \forall x\, Q(x)$$

Cool! So you can distribute ∀ across conjunctions (∧)
Can you do the same thing with disjunctions? (∨)

Predicate Logic

Example: Are the statements below equivalent?

$$\forall x \left(P(x) \lor Q(x)\right) \equiv \forall x\, P(x) \lor \forall x\, Q(x)$$

Cool! So you can distribute ∀ across conjunctions (∧)
Can you do the same thing with disjunctions? (∨)

Answer: No!

Counterexample:

Predicate Logic

Example: Are the statements below equivalent?

$$\exists x \left(P(x) \lor Q(x)\right) \equiv \exists x\, P(x) \lor \exists x\, Q(x)$$

Predicate Logic

Example: Are the statements below equivalent?

$$\exists x \left(P(x) \vee Q(x) \right) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$$

Answer: They are! *Demonstration* for the domain {*a, b, c*} (**not** a proof!)

Recall the original idea behind the existential quantifier:

$\exists x\, P(x) \equiv P(a) \vee P(b) \vee P(c)$

Predicate Logic

Example: Are the statements below equivalent?

$$\exists x\left(P(x) \vee Q(x)\right) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$$

LHS: $\exists x\,(P(x) \vee Q(x)) \equiv (P(a) \vee Q(a)) \vee (P(b) \vee Q(b)) \vee (P(c) \vee Q(c))$

$\equiv (P(a) \vee P(b) \vee P(c)) \vee (Q(a) \vee Q(b) \vee Q(c))$

(associativity)

RHS: $\exists x\, P(x) \vee \exists x\, Q(x) \equiv (P(a) \vee P(b) \vee P(c)) \vee (Q(a) \vee Q(b) \vee Q(c))$

Predicate Logic

Example: Are the statements below equivalent?

$$\exists x\,\big(P(x) \land Q(x)\big) \equiv \exists x\,P(x) \land \exists x\,Q(x)$$

Predicate Logic

Example: Are the statements below equivalent?

$$\exists x\,\big(P(x) \wedge Q(x)\big) \equiv \exists x\, P(x) \wedge \exists x\, Q(x)$$

Answer: No - (**FYOG**, think about it!)

Predicate Logic

Question: What is the negation of $\forall x\ P(x)$?

$$\neg \forall x\ P(x)$$

Example: What is the negation of the statement: All babies like to sleep?

Predicate Logic

Question: What is the negation of $\exists x\ P(x)$?

$$\neg\exists x\ P(x)$$

Example: What is the negation of the statement: There exists a baby that loves sleep?

Predicate Logic

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

And then we had the distribution laws

- $\forall x\,(P(x) \land Q(x)) \equiv \forall x\,P(x) \land \forall x\,Q(x)$
- $\exists x\,(P(x) \lor Q(x)) \equiv \exists x\,P(x) \lor \exists x\,Q(x)$

With these rules, and the logical equivalences we found for regular propositions, we can prove all kinds of equivalences of quantifier propositions

Predicate Logic

<u>Example:</u> Prove that $\neg \forall x\,(P(x) \to Q(x)) \equiv \exists x(P(x) \land \neg Q(x))$

Predicate Logic

Translating from English into Logical Expressions

Example: Translate the following into symbols: “Every student in CSCI 2824 has passed Calculus 1.” Let the domain be all the students in CSCI 2824.

Predicate Logic

Translating from English into Logical Expressions

Example: Translate the following into symbols: “Every student in CSCI 2824 has passed Calculus 1.” Let the domain be all the students in CSCI 2824.

Rewrite: For every student in CSCI 2824, that student has passed Calc1.

We introduce **C(x),** which is the statement “**x has studied calculus.**”

If the domain for x consists of the students in the class, we can translate our statement as **$\forall x\, C(x)$**

Predicate Logic

Translating from English into Logical Expressions

<u>Example:</u> Translate the following into symbols: “Every student in CSCI 2824 has passed Calculus 1.” but use the domain, all students at CU.

Predicate Logic

Translating from English into Logical Expressions

Example: Translate the following into symbols: "Every student in CSCI 2824 has passed Calculus 1." but use the domain, all students at CU.

Rewrite: "For every CU student x, if x is a student in CSCI 2824, then x has passed Calc 1."

We introduce **S(x),** which is the statement "**x is in CSCI 2824.**"

If the domain for x consists of the CU students, we can translate our statement as $\forall x\, (S(x) \rightarrow C(x))$

Predicate Logic

Translating from English into Logical Expressions

<u>Example:</u> Let the domain be the set of all CU students, and translate: "Every student in CSCI 2824 is either taking Data Structures, or has already passed it."

Predicate Logic

Translating from English into Logical Expressions

<u>Example:</u> Let the domain be the set of all CU students, and translate: “Every student in CSCI 2824 is either taking Data Structures, or has already passed it.”

“For every CU student x, if x in this class, x has the property that x has taken Data Structures or x has passed Data Structures.”

Extra Practice

Example 1: What is the truth value of $\forall x\,(x^2 \geq x)$ when the domain is all real numbers?

Example 2: Think of a domain, and specific propositional functions $P(x)$ and $Q(x)$ to illustrate this equivalence

$$\neg\forall x\,(P(x) \to Q(x)) \equiv \exists x\,(P(x) \land \neg Q(x))$$

Example 3: Translate the following and then find the negation.

Let the domain be of all CSCI 2824 students, and translate:

- "There exists a student in CSCI 2824 that has taken Calculus 3 but not Differential Equations"

Example 4: Translate the following and then find the negation.

Let the domain be the set of all CU students, and translate:

- "There exists a student in CSCI 2824 that has taken Calculus 3 but not Differential Equations"

Solutions

Example 1: What is the truth value of $\forall x\,(x^2 \geq x)$ when the domain is all real numbers?

Solution: The statement is false. A counter example is $x = \frac{1}{2}$ because $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < \frac{1}{2}$

Example 2: Think of a domain, and specific propositional functions $P(x)$ and $Q(x)$ to illustrate this equivalence

$$\neg\forall x\,(P(x) \rightarrow Q(x)) \equiv \exists x\,(P(x) \wedge \neg Q(x))$$

Solution: Let x be the set of all shapes, $P(x)$ mean x is a polygon, and $Q(x)$ mean x is a rectangle.

The first proposition says it's not the case that if x is a polygon then it is necessarily a rectangle

The second proposition says that there exists a shape that is a polygon and is not a rectangle (e.g. a triangle)

Example 3: Translate the following and then find the negation.

Let the domain be of all CSCI 2824 students, and translate:

- "There exists a student in CSCI 2824 that has taken Calculus 3 but not Differential Equations"

Solution: Let $C(x)$ mean x has taken Calc 3 and $E(x)$ mean x has taken DiffEq. In symbols the statement as $\exists x(C(x) \wedge \neg E(x))$

The negation is $\neg\exists x(C(x) \wedge \neg E(x)) \equiv \forall x(\neg C(x) \vee E(x))$

In English,the negation is

- "All 2824 students have taken DiffEq or not taken Calc 3"

Example 4: Translate the following and then find the negation.

Let the domain be the set of all CU students, and translate:

- "There exists a student in CSCI 2824 that has taken Calculus 3 but not Differential Equations"

Solution Let $D(x)$ mean x is a 2824 student. In symbols, we have

$$\exists x \; [D(x) \wedge (C(x) \wedge \neg E(x)]$$

It's negation is $\forall x \; [\neg D(x) \vee (\neg C(x) \vee E(x))]$, which in English is

- "All CU students either aren't 2824 students or haven't taken Calc 3 or have taken Differential Equations"