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CSCI 2824: Discrete Structures

Lecture 23: Basics of Counting, Permutations & Combinations, the Pigeon Hole Principle

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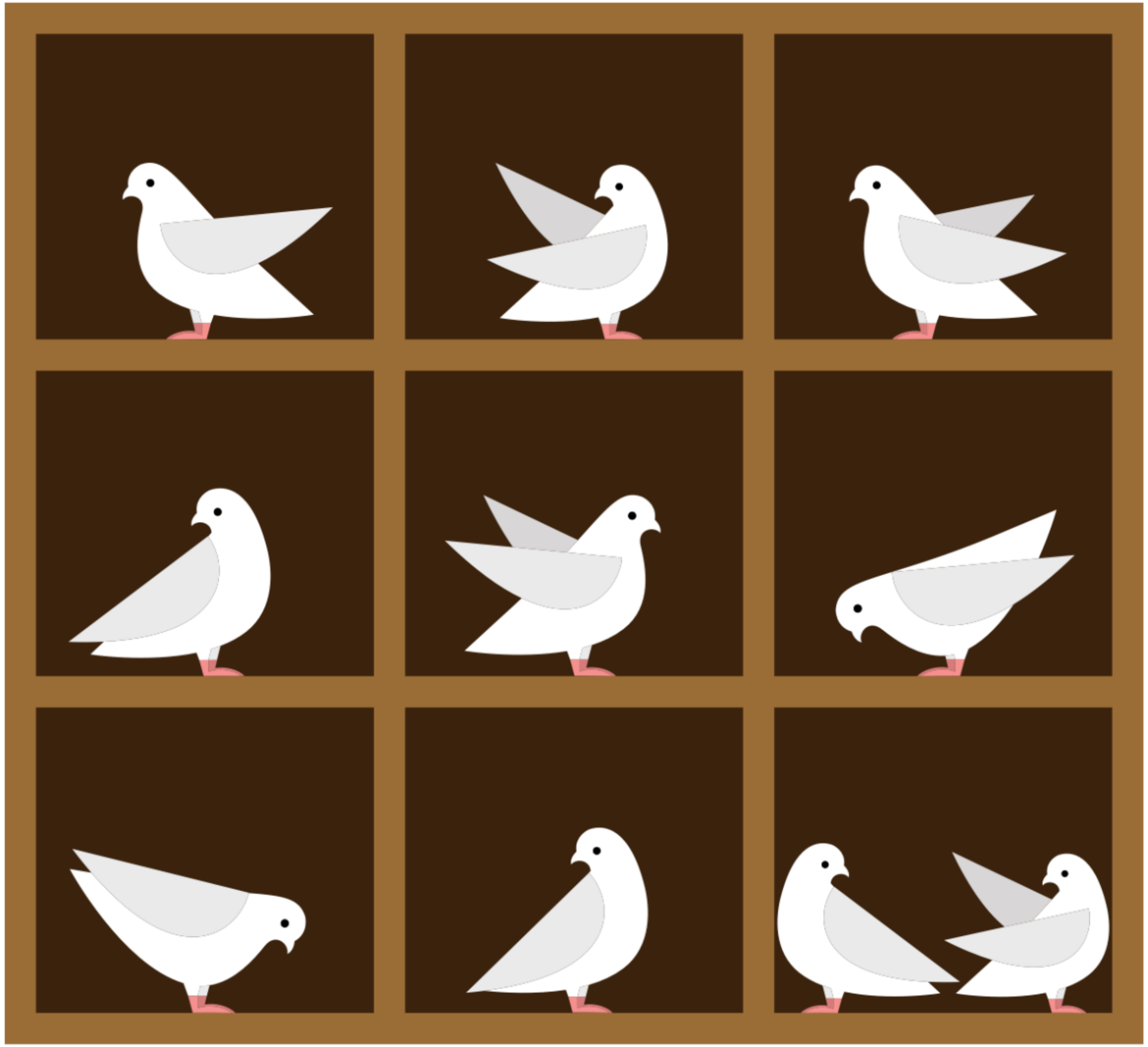

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Announcements

Written HW8 - Due at Noon

Quizlet 08 - opening today

- Due Monday at 8PM

Counting

Combinatorics is the art of counting things.

- Can a password of a particular form resist attack?
- Are there enough things to meet demand?
  - license plates
  - phone numbers
  - IP addresses
- What is the probability of a discrete event occurring?

Random phone numbers:

1 (446) 726-6436	2. (906) 346-8573	3. (621) 522-6278
4. (349) 614-2826	5. (296) 130-8079	6. (253) 586-4631
7. (615) 148-3189	8. (826) 384-5698	9. (397) 614-3092
10. (639) 120-5939	11. (284) 685-4564	12. (757) 948-4913
13. (562) 740-8815	14. (364) 750-8377	15. (292) 483-1100
16. (344) 314-0568	17. (467) 689-7248	18. (887) 812-5803
19. (690) 807-0018	20. (721) 201-1809	21. (950) 221-1061
22. (153) 646-7082	23. (820) 980-2415	24. (731) 119-6048
25. (993) 454-7168	26. (151) 209-0554	27. (974) 691-1462
28. (421) 227-7418	29. (970) 413-0808	30. (393) 668-6978
31. (641) 796-3429	32. (697) 180-8362	33. (946) 223-3319
34. (927) 243-6667	35. (352) 805-6917	36. (328) 862-1688
37. (737) 887-7300	38. (296) 752-7880	39. (543) 830-8291

Counting

The Product Rule: Suppose that a procedure can be broken down into two tasks. If there are n_1 ways to do the first task, and for each of these ways to do the first task there are n_2 ways to do thee second task, then there are $\boldsymbol{n_1}$ **x** $\boldsymbol{n_2}$ **ways** to do the procedure.

Number of ways to do Task 1 AND Task 2

The Sum Rule: If a task can be done in one of n_1 ways or in n_2 ways (where none of the n_1 or n_2 ways are the same) then there are $\boldsymbol{n_1 + n_2}$ **ways** to do the task.

Number of ways to do Task 1 OR Task 2

Counting

Example: In an auditorium-style classroom, each seat is labeled by an uppercase letter (corresponding to the row) and a single digit (corresponding to the seat in the row). How many different possible seat labels are there?

$\underline{26} \cdot \underline{10}$ $\boxed{= 260 \text{ ways to label}}$

Counting

We can generalize the product rule beyond the choice of two tasks.

If a procedure requires K tasks and the k^{th} task can be done in n_k ways, then there are n_1 x n_2 x … x n_k ways to do the procedure.

<u>Example</u>: Standard Colorado license plates consist of 3 numbers followed by 3 uppercase letters. How many distinct license plates can be made in this configuration?

$\underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26}$

$\boxed{= 17{,}576{,}000}$

1 1 1 A A A

Counting

e.g. 101 1001

1001001

Example: How many *n*-length palindromic bit-strings are there?

even # bits

$2 \cdot 2 \cdot 2 \cdot 2$

n even

$\boxed{2^{n/2} \text{ possible bit strings}}$

odd # bits

2 2 2 2 2

n odd

$\boxed{2^{\frac{n+1}{2}} \text{ possible bit strings}}$

Counting

An Product Rule Analogy Using Loops

What is the value of $\texttt{ctr}$ where $n_1, \ldots, n_m$ are positive integers?

$\texttt{ctr} = 0$
for $i_1 = 1$ **to** n_1
  for $i_2 = 1$ **to** n_2
    $\vdots$
   for $i_m = 1$ **to** n_m
    $\text{ctr} += 1$

$ctr = n_1 \cdot n_2 \cdot n_3 \cdots \cdots \cdot n_m$

Counting

Example: Suppose a student can pick a programming project from three subject areas. The three lists contain 15, 13, and 17 projects, respectively. None of the projects appear on multiple lists. How many possible projects are there to choose from?

Sum Rule

$$\#\text{possible projects} = 15 + 13 + 17 \boxed{= 45}$$

Counting

An Sum Rule Analogy Using Loops

What is the value of $\texttt{ctr}$ where $n_1, \ldots, n_m$ are positive integers?

$\texttt{ctr} = 0$
for $i_1 = 1$ **to** n_1
    $\text{ctr} += 1$
for $i_2 = 1$ **to** n_2
    $\text{ctr} += 1$
    $\vdots$
for $i_m = 1$ **to** n_m
    $\text{ctr} += 1$

$ctr = n_1 + n_2 + \cdots + n_m$

Counting

Example: Suppose your system requires you to choose a password of between 6 and 8 characters. How many passwords are there if they can be made up of digits and uppercase letters and each password must contain at least one digit?

6 character passwords $\underline{\overset{26}{\overset{+}{10}}}\cdot\underline{36}\cdot\underline{36}\cdot\underline{36}\cdot\underline{36}\cdot\underline{36} - \underline{26}\cdot\underline{26}\cdot\underline{26}\cdot\underline{26}\cdot\underline{26}\cdot\underline{26}$

$36^6 - 26^6$

7 char. pw: $36^7 - 26^7$

8 char. pw: $36^8 - 26^8$

total $= 36^6 - 26^6 + 36^7 - 26^7 + 36^8 - 26^8$

$\boxed{= 2{,}684{,}483{,}063{,}360}$

Counting

For the Sum Rule to work, the possible sets of events must be distinct. This is not always the case.

<u>The Subtraction Rule (Inclusion-Exclusion)</u>: If a task can be done in n_1 or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common between the two different ways.

project 1
grass
butterflies

project 2
butterflies
flowers
lakes

Or another way to think about this: think of *ways* as being sets of events.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting

Example: How many different bitstrings of length 10 contain either exactly five consecutive 0's or exactly five consecutive 1's?

Let's consider the five consecutive 0's case:

- 0 0 0 0 0 1 $\underset{0,1}{\text{2 ways}}$ · 2 ways · 2 ways · 2 ways $\quad 2^4$

1 0 0 0 0 0 1 2 · 2 · 2 $\quad 2^3$

_ 1 0 0 0 0 0 1 _ _ $\quad 2^3$

_ _ 1 0 0 0 0 0 1 _ $\quad 2^3$

_ _ _ 1 0 0 0 0 0 1 $\quad 2^3$

- _ _ _ _ 1 0 0 0 0 0 $\quad 2^4$

num ways to have 5 consecutive 0's $= 2 \cdot 2^4 + 4 \cdot 2^3 = 64$

Counting

Example (continued): How many different bitstrings of length 10 contain either five consecutive 0's or five consecutive 1's?

Now consider 5 consecutive 1's

1 1 1 1 1 0 _ _ _ _ 2^4

0 1 1 1 1 1 0 _ _ _ 2^3

_ 0 1 1 1 1 1 0 _ _ 2^3

_ _ 0 1 1 1 1 1 0 _ 2^3

_ _ _ 0 1 1 1 1 1 0 2^3

_ _ _ _ 0 1 1 1 1 1 2^4

double-counted

1111100000

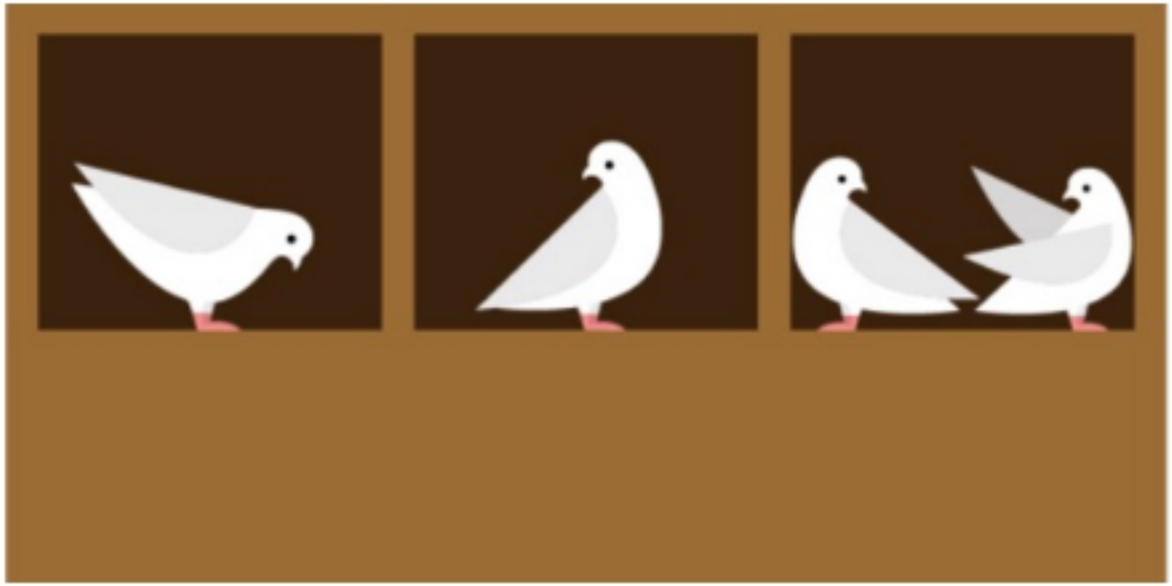
0000011111

num ways to have 5 consecutive 1's = 64

total number = 64 + 64 − 2 $\boxed{= 126}$

Pigeonhole Principle

Suppose that a flock of 4 pigeons flies into a set of 3 pigeonholes to roost. Because there are 4 pigeons and only 3 pigeonholes for them to go into, at least one of the pigeonholes must have at least two pigeons in it.



The Pigeonhole Principle: If k is a positive integer and k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Pigeonhole Principle

Example: Show that if there are 30 students in a class, then at least two of them have last names that start with the same letter.

There are 26 letters in the alphabet. Since $30 > 26$, by the pigeonhole principle, at least two people have to share a common last name letter

Pigeonhole Principle

Example: A drawer contains a dozen brown socks and a dozen black socks, all unmatched. If you take out socks in the dark, how many must you grab to ensure that you have two socks that match?

$\boxed{3}$

First two that you draw, there's a chance that 1 is black and 1 is brown.
The 3rd one you draw has to be either black or brown

thus guaranteed match

How many socks must we remove to guarantee that we get 2 black socks?

$\boxed{14}$

we could draw all 12 brown socks.

then we'd need to pick 2 more to guarantee 2 black socks

Pigeonhole Principle

Example: How many cards must be drawn from a standard deck of 52 cards to ensure that at least 3 of the cards are of the same suit?

4 suits in a deck of cards.

draw 9 cards

Pigeonhole Principle

<u>The Generalized Pigeonhole Principle</u>: If N objects are places into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

<u>Example:</u> (re-do) How many cards must be drawn from a standard deck of 52 cards to ensure that at least 3 of the cards are of the same suit?

If we have $N = 9$ cards, then there is at least one suit with at least

$$\left\lceil \frac{9}{4} \right\rceil = \lceil 2.25 \rceil = 3 \text{ cards of the same suit}$$

Pigeonhole Principle

Example: Show that there are at least seven people in California (pop. 38.8 million) with the same three initials that were born on the same day of the year. (Assume that everyone has three initials.)

Permutations and Combinations

Example: How many 4-digit PIN combinations are there?

Permutations and Combinations

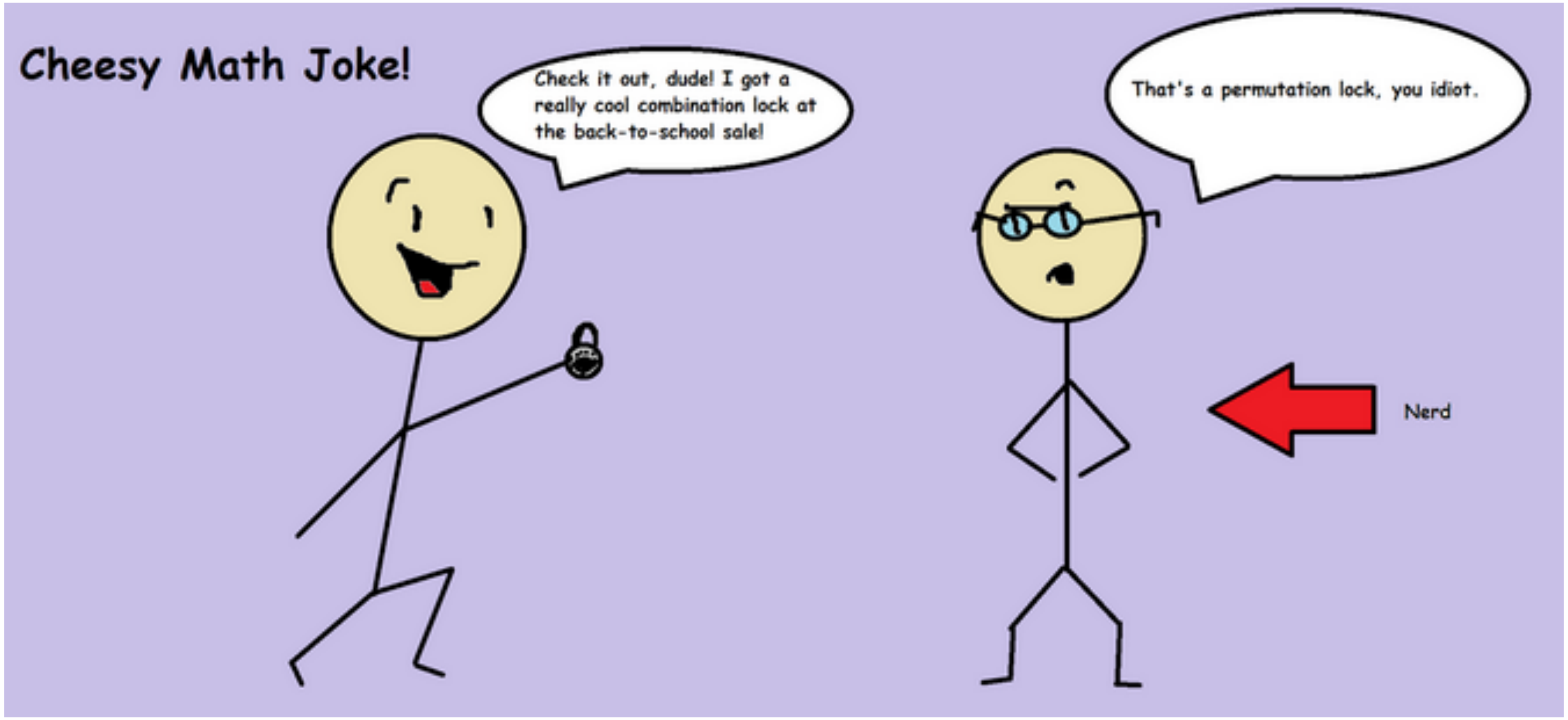
❖Think of the digits as distinct items in a bag. We reach into the bag 4 times and select an item. After each item is selected we put it back in the bag and select again.

- What IF we select items from the bag without replacing them and take note of the order they come out?
- What IF we select items from the bag without replacing them and don’t care about the order they come out?

Permutations and Combinations

The first case, where order matters, is called a **permutation**.

The second case, where order doesn't matter, is called a **combination**.



Permutations and Combinations

Example: How many three-character strings can we make if each character is a distinct letter?

Permutations and Combinations

When selecting r distinct items without replacing them, we call the possible selection an **r-permutation**.

Example: Find all 2-permutations of the set $S = \{a, b, c\}$.

Permutations and Combinations

Theorem: If n is a positive integer and r is an integer such that $1 \leq r \leq n$ then the number r-permutations from a set of size n is $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$

- Note that we define $P(n, 0) = 1$ because there is exactly one way you can select NO items from a set of size n.

Corollary: If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$

- Special Case: $P(n, n) = n!$

Extra Practice

Ex. 1: I own 9 shirts, 3 ties, and 2 pairs of shoes. Assuming that I don't care about matching, how many different outfits can I make?

Ex. 2: How many one-to-one functions are there from a set with m elements to a set with n elements if $n \geq m$?

Ex. 3: How many bitstrings of length 10 either begin with three 0's or end with two 0's?

Ex. 4: How many strings of three decimal digits

- do not contain the same digit three times
- begin with an odd digit
- have exactly two digits that are 4s

Solutions

Ex. 1: I own 9 shirts, 3 ties, and 2 pairs of shoes. Assuming that I don't care about matching, how many different outfits can I make?

Solution: $9 \times 3 \times 2 = 54$

Ex. 2: How many one-to-one functions are there from a set with m elements to a set with n elements if $n \geq m$?

Solution: Remember that a one-to-one function has to map each element of it's domain to exactly one element of the co-domain.

The first of m domain elements can be mapped to any one of n codomain elements.

The second of m domain elements can be mapped to any of the $n-1$ remaining codomain elements not already mapped to.

The third of m domain elements can be mapped to any of the $n-2$ remaining codomain elements not already mapped to.

Ex. 2: How many one-to-one functions are there from a set with m elements to a set with n elements if $n \geq m$?

Solution: We continue in this manner until we get to the last of m domain elements.

Having already mapped to $m - 1$ of the codomain elements, there are $n - m + 1$ remaining codomain elements to choose from.

Putting this all together, we have

$$n \cdot (n-1) \cdot (n-2) \cdots (n-m+1) \text{ possible 1-to-1 functions}$$

Ex. 3: How many bitstrings of length 10 either begin with three 0's or end with two 0's?

Begin with 000: $000XXXXXXX \to 2^7 = 128$

End with 00: $XXXXXXXX00 \to 2^8 = 256$

Need to subtract off strings that satisfy both

Begin with 009 and end with 00: $000XXXXX00 \to 2^5 = 32$

$$\text{Total} = 128 + 256 - 32 = 352$$

Ex. 4: How many strings of three decimal digits

Sol: First, there are $10^3 = 1000$ three digit numbers

- do not contain the same digit three times

Sol: There are 10 numbers with the same digit three times, so

$$1000 - 10 = 990$$

- begin with an odd digit

Sol: Odd$XX \rightarrow 5 \cdot 10 \cdot 10 = 500$

- have exactly two digits that are 4s

Ex. 4: How many strings of three decimal digits

- have exactly two digits that are 4s

Sol:

$44X \to 10$

$4X4 \to 10$

$X44 \to 10$

But we've counted 444 three times in this, so

$$10 + 10 + 10 - 3 = 27$$