CSCI 2824: Discrete Structures

Lecture 12: Set Theory and Set Operations

Reminders

Submissions:

- Homework 4: Fri 9/27 at noon – Gradescope
- Homework 5: **Mon 10/7 at noon** – 1 try on Moodle

Readings:

- Starting Ch. 2 – SETS
- Today (Wednesday): 2.1-2.2

Midterm – Tue October 1st at 6pm

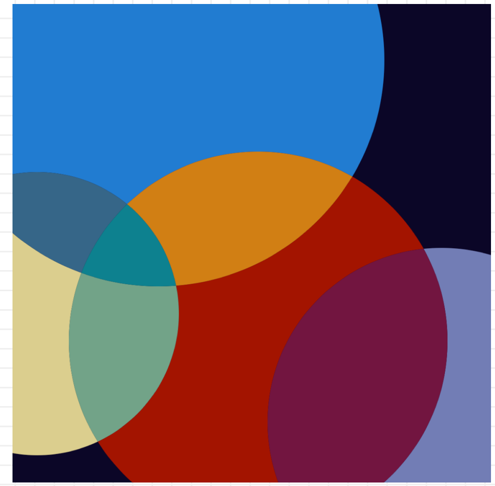
<u>Any conflicts?</u> – email csci2824@colorado.edu

Midterm I

- **Midterm 1: 6:30-8 PM, Tuesday 1 October**
  - BESC 185 – Last Names Aanvi - Chucky
  - EDUC 220 – Last Names Clay - Jordan
  - HALE 270 – Last Names Jose - Ziqi
- **Conflict?** If you emailed, **Wed 2 October from 6:00-7:30PM**, ECOT 831
- **Review:**
  - Concept guide
  - Written homework
  - “All Moodle problems” set
  - Workgroup worksheets
  - Lecture slides and examples

What did we do last time?

- Sets: definition, subsets, equality, power set
- Some special sets (empty set, singleton set)
- Cardinality



Today:

- how to cook up larger sets (unions, power sets) from smaller ones,
- and how to cook up smaller sets (intersections, subsets) from larger ones
- doing stuff with sets (proofs and manipulations)

Sets

Cartesian Product

Sometimes we want to talk about things that come from sets, but **have a defined order.**

- **Example:** The point in the *xy*-plane given by (3,5) is distinctly different from (5,3).

Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all **ordered** pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of **ordered** n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_1 \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

Sets and set operations

Example: Consider the sets A = {Jake, David} and B = {1, 2, 3}. What is A x B?

Sets and set operations

Example: Consider the sets $A = \{\text{Jake, David}\}$ and $B = \{1, 2, 3\}$. What is $A \times B$?

Answer:

$A \times B = \{ (\text{Jake}, 1), (\text{Jake},2), (\text{Jake}, 3), (\text{David},1), (\text{David}, 2), (\text{David},3)\}$

(a,b) is an ordered pair.

Not the same as {a,b}

Sets and set operations

Example: Consider the sets A = {Jake, David} and B = {} (B = Ø). What is A x B?

Example: Consider the sets A = {Jake, David} and B = {{}}. What is A x B?

Sets and set operations

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Answer:

A X Ø = Ø

Ø X A = Ø

Example: Consider the sets A = {Jake, David} and B = {{}}. What is A x B?

Answer:

A X B = { (Jake, {}), (David, {}) }

Sets and set operations

Example: What is the Cartesian product $A \times B \times C$,

where $A = \{a, b\}$, $B = \{x, y\}$ and $C = \{m, n\}$?

Answer:

A x B x C = { (a, x, m), (a, x, n), (a, y, m), (a, y, n), (b, x, m), (b, x, n), (b, y, m), (b, y, n) }

Cartesian Product

Example: Suppose that A={1, 2}. Find $A \times A$ (aka A^2) and find $A \times A \times A$ (aka A^3)

Sets and set operations

Two or more sets can be combined in different ways

Definition: Let A and B both be sets. The **union** of the sets A and B, denoted $A \cup B$, is the set that contains all elements that are either in A, or in B, or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Example: Consider the set C of all computer science majors at CU and the set A of all applied math majors at CU.

- Then the set $C \cup A$ is …

Sets and set operations

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Example: Consider the set C of all computer science majors at CU and the set A of all applied math majors at CU.

- Then the set $C \cup A$ is the set of all CU students who are majoring in either computer science, or applied math, or *double-majoring*.

Sets and set operations

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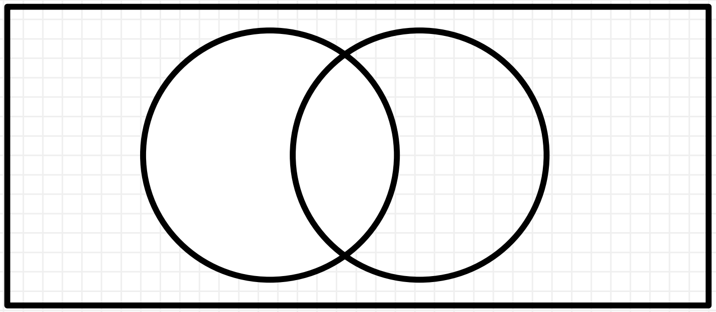
- Then the set $C \cap A$ is the set of all CU students who are *double-majoring* in computer science **and** applied math.

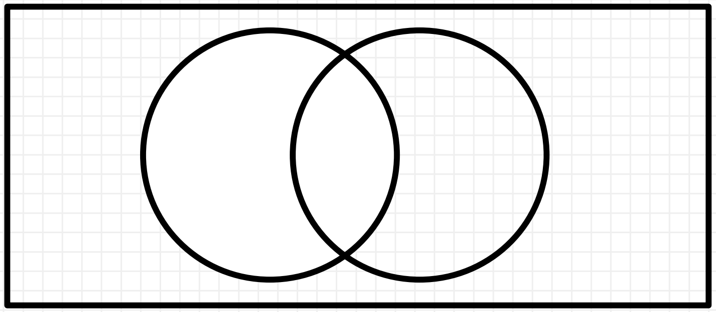
Set Operations

Venn diagrams can also be useful when trying to understand and represent set operations.

Let A and B be sets. The **union** of the sets A and B, denoted $\boldsymbol{A \cup B}$, is the set that contains those elements that are either in A or in B, or in both.



Let A and B be sets. The **intersection** of the sets A and B, denoted $\mathbf{A} \cap \boldsymbol{B}$, is the set that contains those elements that are in both A and B.



Set Operations

Example: Consider the sets: A={1, 3, 5} and B={1, 2, 3}. Find the union and the intersection of these two sets.

<u>Example</u>: Consider the sets: $\mathrm{A} = \{1, 3, 5\}$ and $\mathrm{B} = \{2, 4, 6\}$. Find the union and the intersection of these two sets.

Two sets are called **disjoint** if their intersection is the empty set.

Set Operations

Let A and B be sets. The difference of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted $\boldsymbol{A} - \boldsymbol{B}$, (or $A \backslash$B), is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

Example: What is the difference between A (the set of positive integers less than 10) and B (the set of prime numbers)?

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<u>Example</u>: What is the difference between A (the set of positive integers less than 10) and B (the set of prime numbers)?

Answer:

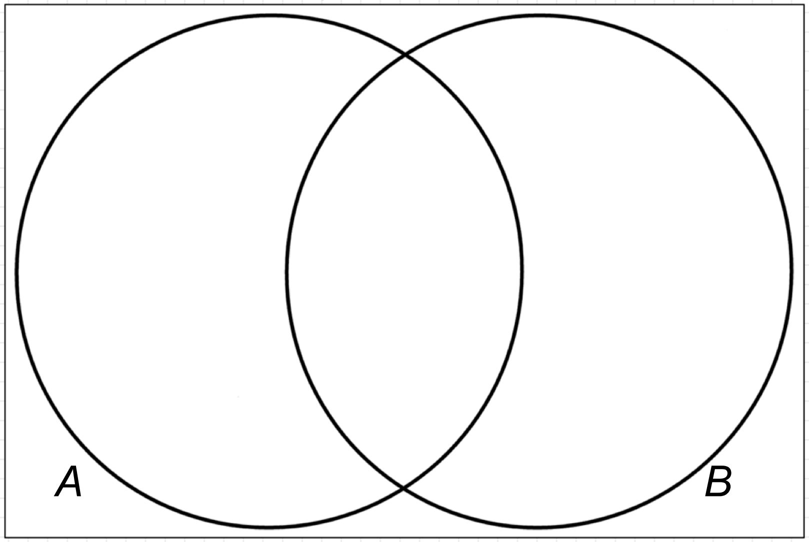
$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $\Rightarrow \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$P = \{2, 3, 5, 7, 11, \ldots\}$

$\Rightarrow A\text{-}P = \{1, 4, 6, 8, 9\}$

Sets and set operations

Question: How many elements are in the set $A \cup B$?



Sets and set operations

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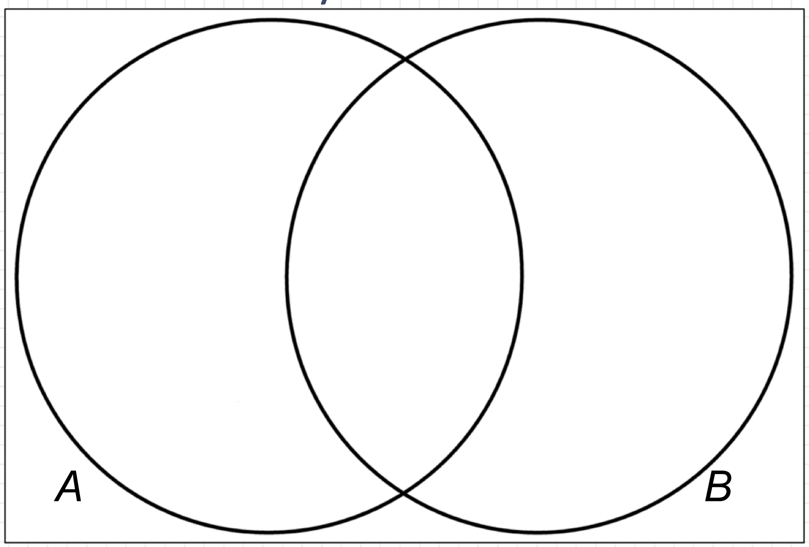

Answer: $|A \cup B| = |A| + |B| - |A \cap B|$

Things that are in both A and B (i.e., in $A \cap B$) are counted twice, by both $|A|$ and $|B|$.

So we need to remove them

Sets and set operations

Python has some nice functionality that can help you convert lists of elements into sets, and perform some operations on them.

```
In [8]: mylist = [1,2,3,1,4]

In [9]: myset = set(mylist)

In [10]: print(myset)
{1, 2, 3, 4}
```

If/when the time comes, you should feel free to explore these functions for manipulating sets...

... just saying.

```
In [15]: A = set([1,2,3,4])

In [16]: B = set([3,4,5,6])

In [17]: print(set.intersection(A,B))
{3, 4}

In [18]: print(set.union(A,B))
{1, 2, 3, 4, 5, 6}

In [19]: print(set.difference(A,B))
{1, 2}

In [20]: print(set.difference(B,A))
{5, 6}
```

Sets and set operations

Definition: The **universal set**, denoted typically by *U*, is the set containing all elements within the domain of discourse.

- You can think about the universal set as the set containing all elements under consideration.

Example: if the domain of discourse is all CU students, then *U* is the set of all CU students.

Sets and set operations

Definition: The **<u>universal set</u>**, denoted typically by U, is the set containing all elements within the domain of discourse.

- You can think about the universal set as the set containing all elements under consideration.

Example: if the domain of discourse is all CU students, then U is the set of all CU students.

Definition: Let U be the universal set. The **<u>complement</u>** of the set A, denoted $\bar{A}$, is the set $U - A$.

An element belongs to $\bar{A}$, if and only if $x \notin A$, so $\bar{A} = \{x \in U \mid x \notin A\}$, or just $\{x \mid x \notin A\}$

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

Strategy: To prove that two sets *C* and *D* are equal, we must

1. ($\Rightarrow$) Prove that $C \subseteq D$
2. ($\Leftarrow$) Prove that $D \subseteq C$

Proof: ($\Rightarrow$)

Sets and set operations

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Strategy: To prove that two sets *C* and *D* are equal, we must

1. (⇒) Prove that $C \subseteq D$
2. (⇐) Prove that $D \subseteq C$

Proof: (⇒)

1. S'pose *x* is an arbitrary element in *A* - *B*
2. $\Rightarrow x \in A \wedge x \notin B$
3. $x \notin B \Rightarrow x \in \bar{B}$
4. $\Rightarrow (x \in A) \wedge (x \in \bar{B})$
5. $\Rightarrow x \in A \cap \bar{B}$
6. Since *x* was any arbitrary element in *A* - *B*, we have shown $A - B \subseteq A \cap \bar{B}$

Sets and set operations

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Proof: ($\Leftarrow$)

1. S'pose *x* is an arbitrary element in $A \cap \bar{B}$
2. $\Rightarrow (x \in A) \wedge (x \in \bar{B})$
3. $x \in \bar{B} \Rightarrow x \notin B$
4. $(x \in A) \wedge (x \notin B) \Rightarrow (x \in A - B)$
5. Since *x* was any arbitrary element in $A \cap \bar{B}$, we have shown $A \cap \bar{B} \subseteq A - B$

Since $(A \cap \bar{B} \subseteq A - B)$ and $(A - B \subseteq A \cap \bar{B})$ it must be the case that

$$A - B = A \cap \bar{B} \quad \square$$

Sets and set operations

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Sets and set operations

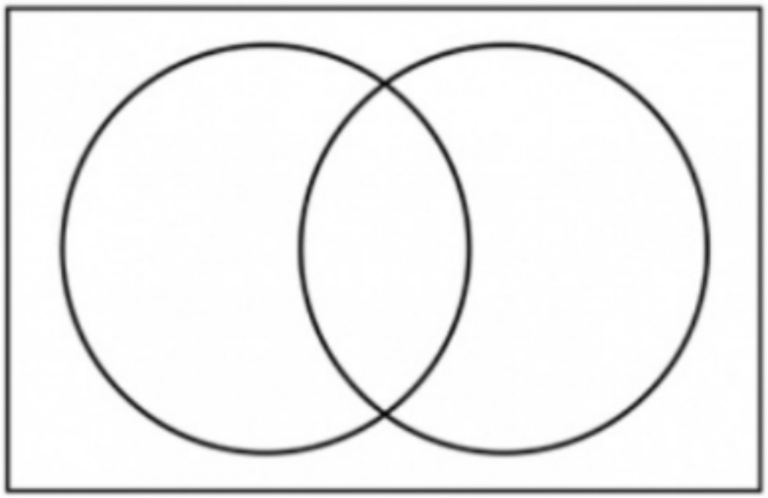
When sets are combined using only ∪, ∩, and complements, there is a set of **Set Identities** that completely mirrors the logical equivalences from last chapter.

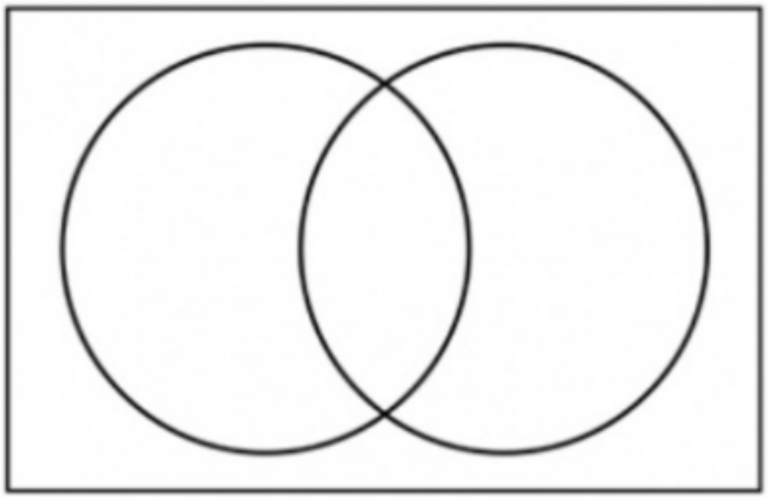
TABLE 1 **Set Identities.**

Identity	***Name***
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Sets and set operations

Example: Prove DeMorgan's Law for Sets: $\overline{A \cap B} = \overline{A} \cup \overline{B}$





Sets and set operations

<u>Example</u>: Prove (one of) De Morgan's laws for sets, using **builder notation** $\overline{A \cap B} = \overline{A} \cup \overline{B}$

(remember, builder notation looked a lot like quantifiers, so this will look at lot like our old proofs from Ch 1)

Sets and set operations

Example: Prove (one of) De Morgan's laws for sets, using **builder notation** $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof: $\overline{A \cap B}$ $= \{ x \mid x \notin A \cap B \}$ Definition of complement

$= \{ x \mid \neg(x \in A \cap B) \}$ Definition of not-in

$= \{ x \mid \neg(x \in A \wedge x \in B) \}$ Definition of intersection

$= \{ x \mid \neg(x \in A) \vee \neg(x \in B) \}$ De Morgan's

$= \{ x \mid (x \notin A) \vee (x \notin B) \}$ Definition of not-in

$= \{ x \mid (x \in \overline{A}) \vee (x \in \overline{B}) \}$ Definition of complement

$= \{ x \mid x \in (\overline{A} \cup \overline{B}) \}$ Definition of union

$= \square$

Sets and set operations

Example: Use set identities to prove

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$

Proof:

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Sets and set operations

Example: Use set identities to prove

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$

Proof:

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} && \text{De Morgan's} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{De Morgan's} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{commutative} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{commutative}
\end{aligned}$$

FYOG: Use set identities to prove

$$(A \cup \overline{B}) \cap (\overline{B \cap A}) = \overline{B}$$

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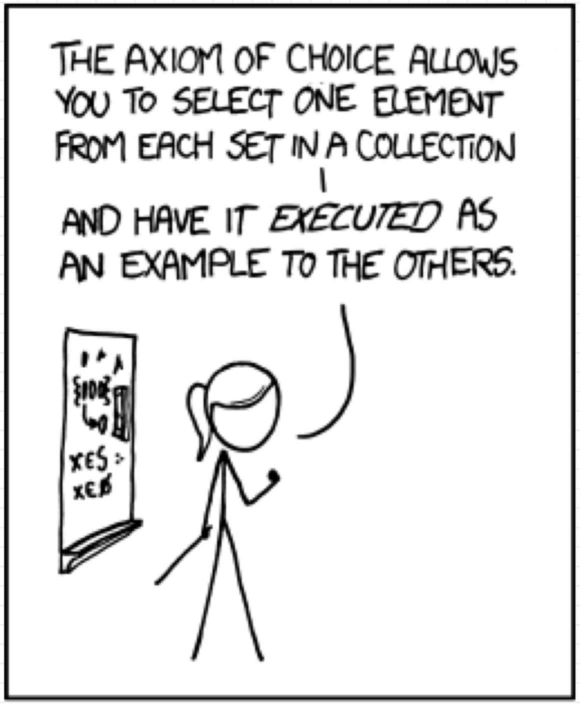
Sets and set operations - Recap

- We learned what **sets** are,
- empty set, singleton set, power set,
- how to cook up larger sets (unions, power sets) from smaller ones,
- and how to cook up smaller sets (intersections, subsets) from larger ones

Next time:

- More on ***infinite*** sets and …
- doing stuff with sets of things
  to get other sets of things
  (we call the stuff we do ***functions!***)



MY MATH TEACHER WAS A BIG BELIEVER IN PROOF BY INTIMIDATION.

Extra Practice

Example 1: Show that if *a, b* and *c* are real numbers with $a \neq 0$, then there **exists** a **unique** solution x to the equation $ax + b = c$.

Note: This is the statement that non-horizontal lines pass through each y coordinate exactly once.

Example 2: Prove DeMorgan's Law for Sets: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Set Operations

Example: Use set identities to prove $\overline{A \cup (B \cap C)} = \left(\overline{C} \cup \overline{B}\right) \cap \overline{A}$

Set Operations

Example: If P is the set of prime numbers, then what is $\overline{P}$?

Set Operations

Example: Suppose $A = \{b, c, d\}$ and $B = \{a, b\}$. Find:

(a) $(A \times B) \cap (B \times B)$

(b) $(A \times B) \cup (B \times B)$

(c) $(A \times B) - (B \times B)$

(d) $(A \cap B) \times A$

(e) $(A \times B) \cap B$

(f) $\mathscr{P}(A) \cap \mathscr{P}(B)$

(g) $\mathscr{P}(A) - \mathscr{P}(B)$

(h) $\mathscr{P}(A \cap B)$

(i) $\mathscr{P}(A) \times \mathscr{P}(B)$

Solutions

Example 1: Show that if a, b, and c are real numbers with $a \neq 0$ then there **exists** a **unique** solution x to the equation $ax + b = c$

Existence: Solve for x

$$ax + b = c \Rightarrow x = \frac{b - c}{a}$$ (where here we know that $a \neq 0$)

Uniqueness: Assume x and y are both solutions to the system, then

$$ax + b = c = ay + b \Rightarrow ax + b = ay + b \Rightarrow ax = ay \Rightarrow x = y$$

Since x and y are necessarily the same number, it follows that our solution x is unique

Example 2: Prove this identity using set builder notation

$$
\begin{aligned}
\overline{A \cup B} &= \{x \mid x \notin A \cup B\} && \text{(def. complement)} \\
&= \{x \mid \neg(x \in A \cup B)\} && \text{(def. not in)} \\
&= \{x \mid \neg(x \in A \vee x \in B)\} && \text{(def. intersection)} \\
&= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\} && \text{(DeMorgan's)} \\
&= \{x \mid x \notin A \wedge x \notin B\} && \text{(def. not in)} \\
&= \{x \mid x \in \overline{A} \wedge x \in \overline{B}\} && \text{(def. complement)} \\
&= \{x \mid x \in \overline{A} \cap \overline{B}\} && \text{(def. union)} \\
&= \overline{A} \cap \overline{B}
\end{aligned}
$$