CSCI 2824: Discrete Structures

Lecture 31: Solving Recurrence Relations

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* HW11 Due at Noon on Friday

Solving Recurrence Relations

Recurrence relations are used often in Discrete Structures

Last time, we saw how to use recurrence relations to count things.

E.g. **The Handshake Problem/Round-Robin Tournament**:

$$a_2 = 1$$
$$a_n = a_{n-1} + (n-1) \text{ for } n > 2$$

E.g. **Number of Bit Strings with No Consecutive Zeros**:

$$a_1 = 2$$
$$a_2 = 3$$
$$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 3$$

Solving Recurrence Relations

Developing a Systematic Approach to finding Closed-Form Solutions

1) Let's look at recurrences of the form: $\boldsymbol{a_n = Ka_{n-1} + L}$, where K and L are constants.

2) Let's not forget the sum rules we've already talked about:

$$\sum_{k=0}^{n} k = 0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{n} a^k = 1 + a + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}$$

Solving Recurrence Relations

Example: Find a closed form for the Tower of Hanoi Recurrence

$$M_1 = 1$$
$$M_n = 2M_{n-1} + 1 \text{ for } n > 2$$

Strategy: Start with the n^{th} term and continuously plug in earlier terms until we get all the way down to the first. Then apply sum formulas as necessary.

$$\begin{aligned} M_n &= 2M_{n-1} + 1 \\ &= 2(2M_{n-2} + 1) + 1 &&= 4M_{n-2} + 2 + 1 \\ &= 2(2(2M_{n-3} + 1) + 1) + 1 &&= 8M_{n-3} + 4 + 2 + 1 \end{aligned}$$

Solving Recurrence Relations

If we keep going…

$$M_n = 2^{n-1}M_1 + 2^{n-2} + 2^{n-3} + \cdots + 1$$

$$M_1 = 1$$

$$\Longrightarrow M_n = 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 1 \quad = \frac{2^{n-1+1} - 1}{2 - 1}$$

geometric sequence formula

$$= \mathbf{2^n - 1}$$

Solving Recurrence Relations

Now derive the closed form solution for the general case: $\boldsymbol{a_n = Ka_{n-1} + L}$
(Assume a_0 is known.)

$$
\begin{aligned}
a_n &= Ka_{n-1} + L \\
&= K(Ka_{n-2} + L) + L \\
&= K^2 a_{n-2} + KL + L \\
&= K^2(Ka_{n-3} + L) + KL + L \\
&= K^3 a_{n-3} + K^2 L + KL + L \\
&\vdots \\
&= K^n a_{n-n} + K^{n-1} L + K^{n-2} L + \cdots + K^2 L + KL + L \\
&= K^n a_0 + L(K^{n-1} + K^{n-2} + \cdots + K^2 + K + 1)
\end{aligned}
$$

sum of a geometric seq.

Solving Recurrence Relations

$$a_n = K^n a_0 + L\left(\frac{K^{n-1+1} - 1}{K-1}\right)$$

geometric sum formula

Closed Form Solution for $a_n = Ka_{n-1} + L$

$$\boldsymbol{a_n = K^n a_0 + L\left(\frac{K^n - 1}{K - 1}\right)}$$

Solving Recurrence Relations – Linear Homogeneous

Example: Find the closed-form solution to the recurrence

$$a_0 = 1$$
$$a_n = 3a_{n-1} + 2$$

we just derived

$$a_n = K^n a_0 + L\left(\frac{K^n - 1}{K - 1}\right)$$

Here $K = 3$, $L = 2$

$a_0 = 1$

$$a_n = 3^n \cdot (1) + \cancel{2}\left(\frac{3^n - 1}{\cancel{3-1}}\right)$$

$$= 3^n + 3^n - 1$$

$$\boxed{a_n = 2 \cdot 3^n - 1}$$

Solving Recurrence Relations – Linear, Homogeneous

Let $c_1, c_2, \ldots, c_k$ be constants. A recurrence of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called a **linear homogeneous recurrence relation of degree k.**

It's called **linear** because the coefficients are constants and none of the terms have powers.

- linear: $a_n = a_{n-1} + 2a_{n-2}$
- not linear: $a_n = na_{n-1} + (a_{n-2})^2$

It's called **homogeneous** because all of the terms have an a_k.

- Homogeneous: $a_n = a_{n-1} + 5a_{n-2}$
- Not Homogeneous: $a_n = a_{n-1} + 5a_{n-2} + \underbrace{2n}_{\text{not homogeneous piece}}$

Solving Recurrence Relations – Linear Homogeneous

Let $c_1, c_2, \ldots, c_k$ be constants. A recurrence of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called a **linear homogeneous recurrence relation of degree k**.

To solve these types of recurrence relations, we use an educated guess and check method.

1. Let $a_n = r^n$, plug this guess into the recurrence.
2. Solve the resulting characteristic polynomial. •
3. Use the roots of the characteristic polynomial to create a general solution.
   e.g. $a_n = A \cdot r_1^n + B \cdot r_2^n$ where A, B are constants and r_1, r_2 are the roots of the characteristic polynomial.
4. Solve for the constants using the initial conditions.

Solving Recurrence Relations – Linear, Homogeneous

Recall $\frac{a^m}{a^n} = a^{m-n}$

Example: Find the solution to

$$a_0 = 2 \text{ and } a_1 = 7$$
$$a_n = a_{n-1} + 2a_{n-2}$$

educated guess

Step 1 Let $a_n = r^n$, $a_{n-1} = r^{n-1}$, $a_{n-2} = r^{n-2}$

Plug this into the recurrence: $r^n = r^{n-1} + 2r^{n-2}$

Next, simplify by dividing out by the smallest power of r.

$$\frac{r^n}{r^{n-2}} = \frac{r^{n-1}}{r^{n-2}} + \frac{2r^{n-2}}{r^{n-2}}$$

$$r^{n-(n-2)} = r^{(n-1)-(n-2)} + 2r^{(n-2)-(n-2)}$$

$$r^2 = r + 2r^0$$

$$r^2 = r + 2 \longrightarrow \underline{r^2 - r - 2} = 0$$

This is our characteristic polynomial!

Solving Recurrence Relations – Linear, Homogeneous

Example (continued): Find the solution to

$$a_0 = 2 \text{ and } a_1 = 7$$
$$a_n = a_{n-1} + 2a_{n-2}$$

Step 2: Find the roots of the characteristic polynomial.

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

$$\Rightarrow r = 2, -1$$

Step 3: Use the Roots to write out the general solution.

$$a_n = A \cdot 2^n + B.(-1)^n$$

general solution.

Step 4: Solve for the constants using initial conditions.

$$a_0 = A \cdot 2^0 + B \cdot (-1)^0 = 2$$

$$A + B = 2 \rightarrow A = 2 - B$$

$$a_1 = A \cdot 2^1 + B(-1)^1 = 7$$

$$2A - B = 7$$

So $2(2-B) - B = 7$

$$4 - 2B - B = 7$$

$$4 - 3B = 7 \rightarrow -3B = 3 \rightarrow B = -1$$

Solving Recurrence Relations – Linear, Homogeneous

Example (continued): Find the solution to

$a_0 = 2$ and $a_1 = 7$

$a_n = a_{n-1} + 2a_{n-2}$

$B = -1$

$A + B = 2$

$A - 1 = 2$

$A = 3$

general solution: $a_n = A \cdot 2^n + B \cdot (-1)^n$

final solution: $a_n = 3 \cdot 2^n + (-1) \cdot (-1)^n$

$$\boxed{a_n = 3 \cdot 2^n + (-1)^{n+1}}$$

Solving Recurrence Relations – Linear, Homogeneous

Example: Find the solution to

$$f_0 = 0 \text{ and } f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

Let's find the closed form of the Fibonacci Recurrence!

Let $f_n = r^n$

$$r^n = r^{n-1} + r^{n-2}$$
$$r^2 = r + 1$$
$$\underbrace{r^2 - r - 1}_{\text{characteristic polynomial}} = 0$$

Next, solve $r^2 - r - 1 = 0$

use the quadratic formula

$$ar^2 + br + c = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$
$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

Solving Recurrence Relations – Linear, Homogeneous

Example (continued): Find the solution to

$$f_0 = 0 \text{ and } f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

general solution: $f_n = A \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + B \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$

Now we solve for the constants using the initial conditions.

$$f_0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$$

$$A + B = 0 \longleftarrow \underbrace{A = -B}$$

$$f_1 = A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

$$A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Solving Recurrence Relations – Linear, Homogeneous

Example (continued): Find the solution to

$f_0 = 0$ and $f_1 = 1$

$f_n = f_{n-1} + f_{n-2}$

$$A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1 \qquad A = -B$$

$$2\cdot\left[-B\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)\right] = 1\cdot 2$$

$$-B(1+\sqrt{5}) + B(1-\sqrt{5}) = 2$$

$$-\cancel{B} - B\sqrt{5} + \cancel{B} - B\sqrt{5} = 2$$

$$-2B\sqrt{5} = 2$$

$$-B\sqrt{5} = 1$$

$$B = -\frac{1}{\sqrt{5}} \Rightarrow A = \frac{1}{\sqrt{5}}$$

$$\boxed{f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

use python to check for $n = 2 \ldots 10$

Solving Recurrence Relations – Linear, homogeneous

Example: Find a closed-form solution to the following recurrence.

$a_0 = 1$ and $a_1 = 6$

$a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$

↑

Find the characteristic eq.

$r^n = 6r^{n-1} - 9r^{n-2}$

$r^2 = 6r - 9$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$a_n = A \cdot 3^n + Bn\,3^n$

$a_0 = A \cdot 3^0 + B \cdot 0 \cdot 3^0 = 1$

$A = 1$

$a_1 = A \cdot 3^1 + B \cdot 1 \cdot 3^1 = 6$

$3A + 3B = 6$

Since $A = 1$

$3 + 3B = 6$

$3B = 3$

$B = 1$

$$\boxed{a_n = 3^n + n \cdot 3^n}$$

Solving Recurrence Relations – Linear, homogeneous

<u>**Example**</u> **(continued)**: Find a closed-form solution to the following recurrence.

$a_0 = 1$ and $a_1 = 6$

$a_n = 6a_{n-1} - 9a_{n-2}$ for $n \geq 2$

Solving Recurrence Relations – Linear, homogeneous

characteristic eq.

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $\boxed{r^2 - c_1 r - c_2 = 0}$ has *two distinct roots* r_1 *and* r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \underline{\alpha_1 r_1^n + \alpha_2 r_2^n}$ for $n = 0, 1, 2, \ldots,$ where α_1 and α_2 are constants.

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has *only one root* r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $\underline{a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n}$, for $n = 0, 1, 2, \ldots,$ where α_1 and α_2 are constants.

Solving Recurrence Relations – Linear, homogeneous

In summary:

- Find the roots of the characteristic polynomial.
- If r_1 and r_2 are different, $a_n = Ar_1^n + Br_2^n$
- If $r_1 = r_2 = r$, $a_n = Ar^n + Bnr^n$

Solving Recurrence Relations

- The solution process for most recurrence relations is an educated version of guess and check.

- We've seen that many recurrence solutions involve a number to the n^{th} power, so we guess that the solution is of that form, plug it in, and see what happens!