CSCI 2824: Discrete Structures

Lecture 24: Permutations & Combinations

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Exam 2 Tuesday 6:30 – 8 pm

Homework 9 – Due Saturday at 6pm

Definition	Formula	Example
A *combination* is a grouping of outcomes in which the order does not matter.	The number of *combinations* of n things chosen r at a time is found using ${}_nC_r = \frac{n!}{r!(n-r)!}$. **Combinations**	How many pairs can be made from a group of 6 people? (Jin and Tom are the same pair as Tom and Jin) ${}_6C_2 = \frac{6!}{2!(6-2)!} = \frac{720}{2(24)} = 15$
Definition A *permutation* is an arrangement of outcomes in which the order does matter.	**Permutations** **Formula** The number of *permutations* of n things chosen r at a time is found using ${}_nP_r = \frac{n!}{(n-r)!}$.	**Example** 6 people are in a contest. How many ways can 1st and 2nd place be awarded? (Jin first and Tom second is different than Tom first and Jin second) ${}_6P_2 = \frac{6!}{(6-2)!} = \frac{720}{24} = 30$

Permutations and Combinations

Theorem: If n is a positive integer and r is an integer such that $1 \leq r \leq n$ then the number r-permutations from a set of size n is $P(n, r) = n(n-1)(n-2) \cdots (n-r+1)$

- Note that we define $P(n, 0) = 1$ because there is exactly one way you can select NO items from a set of size n.

$$P(n,0) = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$$

Corollary: If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$

- Special Case: $P(n, n) = n!$ $= \frac{n!}{(n-n)!} = \frac{n!}{0!}$ $= n!$

We define $0! = 1$

Permutations and Combinations

Example: How many different 1st-2nd-3rd place permutations can occur in a race with 5 runners?

$$5 \cdot 4 \cdot 3 = \boxed{60 \text{ ways}}$$

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}}$$

$$= 5 \cdot 4 \cdot 3 \quad \checkmark$$

Permutations and Combinations

Example: How many permutations of the letters ABCDEFGH contain the string ABC?

order matters.

8 characters in the string

$\underline{ABC}\ \underline{5}\cdot\underline{4}\cdot\underline{3}\cdot\underline{2}\cdot\underline{1}$ $5!$

$\underline{5}\ \underline{ABC}\cdot\underline{4}\cdot\underline{3}\cdot\underline{2}\cdot\underline{1}$ $5!$

$\underline{5}\cdot\underline{4}\ \underline{ABC}\ \underline{3}\cdot\underline{2}\ \underline{1}$ $5!$

$\underline{5}\cdot\underline{4}\cdot\underline{3}\ \underline{ABC}\cdot\underline{2}\cdot\underline{1}$ $5!$

$\underline{5}\cdot\underline{4}\cdot\underline{3}\cdot\underline{2}\ \underline{ABC}\cdot\underline{1}$ $5!$

$\underline{5}\cdot\underline{4}\cdot\underline{3}\cdot\underline{2}\cdot\underline{1}\ \underline{ABC}$ $5!$

$\#$ permutations $= 6 \cdot 5!$

$= 6!$

$\boxed{= 720 \text{ ways}}$

Permutations and Combinations

Example: How many ways are there to arrange *n* dogs and *n* cats in a row if the dogs and the cats alternate?

e.g. Suppose we have 4 dogs + 4 cats.

$\underline{D_1}\ \underline{C_1}\ \underline{D_2}\ \underline{C_2}\ \underline{D_3}\ \underline{C_3}\ \underline{D_4}\ \underline{C_4}$ ← one permutation starting with a dog

$\underline{C_1}\ \underline{D_1}\ \underline{C_2}\ \underline{D_2}\ \underline{C_3}\ \underline{D_3}\ \underline{C_4}\ \underline{D_4}$ ← one permutation starting with a cat.

$\left.\begin{matrix}\text{either start with} \\ \text{dog OR cat}\end{matrix}\right\}$ 2 ways

Final answer

$$\boxed{2 \cdot n! \, n!}$$

_ _ _ _ _ _ _ _ . . . . _ _ _ _ _ _

number of ways to place dogs: $P(n,n)$

number of ways to place cats: $P(n,n)$

n blue spaces – dogs

n purple spaces – cats

Permutations and Combinations

When we make an ***unordered*** selection of distinct items it's called a combination.

<u>**Example**</u>: Find all 2-letter combinations from the set $S = \{a, b, c\}$

In the previous example, we listed all of the 2-permutations of S.

$$ab, ac, ba, bc, ca, cb$$

$P(3,2) = 6$

But now we want to throw out items that are unordered repeats:

ab, ac, bc repeats: ba, ca, cb

$$C(3,2) = 3 = \frac{P(3,2)}{2!}$$

Permutations and Combinations

Let's derive a formula for the number of r-combinations by starting with the number of r-permutations and then throwing out duplicates.

$$P(n,r) = \frac{n!}{(n-r)!}$$

The number of duplicates depends on the number of selected items.

E.g. Suppose we selected three letters from the alphabet. The permutations involving abc are:

$$abc, bca, cba, acb, cab, bca$$

Number of r −combinations $= \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$

Note that we are overcounting the combinations of abc by a factor of 6 which exactly equals the number of permutations of 3 items.

Permutations and Combinations

Thereom: If n is a nonnegative integer and r is an integer with $0 \leq r \leq n$ then the number of r −combinations is

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Alternate notation: $C(n,r) = \underbrace{\binom{n}{r}}$

n "choose" r

also ${}_nC_r = C(n,r)$

${}_nP_r = P(n,r)$

Permutations and Combinations

Example: How many poker hands of five cards can be drawn from a 52-card deck?

Note: We don't care about order here, so are interested in combinations instead of permutations.

$$C(52,5) = \binom{52}{5} = \frac{52!}{(52-5)!\cdot 5!}$$

$$= \frac{52\cdot 51\cdot 50\cdot 49\cdot 48\cdot 47!}{47!\cdot 5!}$$

$$= \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5!}\cdot \cancel{\frac{47!}{47!}}^{\,1}$$

$$\boxed{= 2{,}598{,}960 \text{ hands}}$$

Permutations and Combinations

Example: A club has 25 members. How many ways are there to choose 4 members to serve on the club's executive committee?

order does not matter.

→ we use combinations.

$$C(25,4) = \binom{25}{4} = \frac{25!}{21!\,4!} = \boxed{12{,}650 \text{ ways}}$$

Permutations and Combinations

Example: How many bit strings of length 10:

a) Contain exactly four 1's

b) Contain at most four 1's

101

101

1111000000

1010101000

0010001110

a) _1_ _1_ __ __ __ __ __ __ _1_ _1_

$$C(10,4) = C(10,6)$$

$$\frac{10!}{6!\,4!} = \frac{10!}{4!\,6!} = 210 \text{ bitstrings}$$

b) $C(10,4) + C(10,3) + C(10,2) + C(10,1) + C(10,0)$

$= 386$ bitstrings

Permutations and Combinations

The hardest part in working with permutations and combinations is knowing if you should use permutations, combinations, or both.

Example: The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain exactly 2 vowels?

Permutations and Combinations

How many 6 character passwords are there that have at least 1 digit and all of the rest of the spots are capital letters?

$36^6 - 26^6$

Extra Practice

EX. 1 A club has 25 members. How many ways are there to choose a president, vice president, secretary, and treasurer?

EX. 2 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain exactly 2 vowels.

EX. 3 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain *at least* one vowel?

EX. 4 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain *at least* two vowels?

EX. 5 Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men as women?

EX. 1 A club has 25 members. How many ways are there to choose a president, vice president, secretary, and treasurer?

This is similar to the example in the lecture, but this time we actually care about order since we're now differentiating between positions on the executive committee.

Instead of computing the combinations we now compute the permutations. So we have

$$P(25,4) = \frac{25!}{21!} = 25 \times 24 \times 23 \times 22 = 303,600$$

EX. 2 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain exactly 2 vowels.

First pick the position of the vowels, $C(6, 2) = 15$

For each vowel-positions there are 5 possible vowels, so for both vowel positions there are 5^2 possible combinations

There are 21^4 ways the consonants can be arranged

By the product rule we have

$$15 \times 5^2 \times 21^4 = 72,930,375 \text{ strings}$$

EX. 3 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain *at least* one vowel?

The easiest way to do this one is to compute the total possible number of strings and then subtract off the ones that don't have any vowels.

$$26^6 - 21^6 = 223,149,655 \text{ strings}$$

EX. 4 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain *at least* two vowels?

Similar to the previous problem, we could compute the total number of strings and then subtract off the number of strings with no vowels and the number of strings with exactly 1 vowel (which we've already computed)

$$26^6 - 21^6 - 6 \times 5 \times 21^5 = 100,626,625$$

EX. 5 Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men as women?

Note that order does not matter here, so we'll be using combinations.

We need to choose the 3 male members of the committee and the 3 female members of the committee separately.

There are $C(10, 3) = 120$ ways to choose the men

There are $C(15, 3) = 455$ ways to choose the women

Then by the product rule, there are $C(10, 3) \cdot C(15, 3) = 54,600$ ways to choose the committee