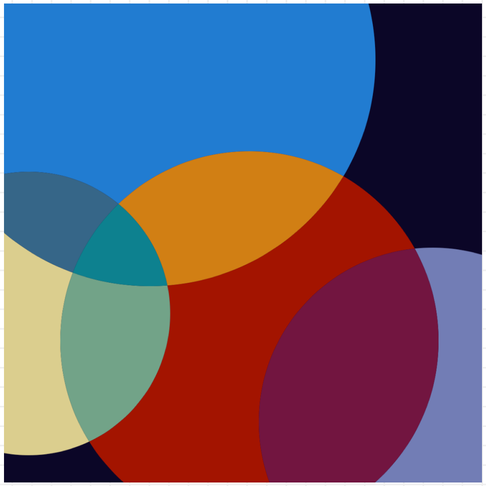


CSCI 2824: Discrete Structures

Lecture 14: Set Operations

Reminders

Submissions:

- Homework 4: Fri 9/27 at noon – Gradescope
- Homework 5: **Mon 10/7 at noon** – 1 try on Moodle
- **Quizlet 4: Due Mon 9/30 at 8pm**

Readings:

- This week Ch. 2 – SETS (2.1-2.2)
- Next week: cont. 2.3 - …

Midterm – Tue October 1st at 6:30pm

Midterm I

- **Midterm 1: 6:30-8 PM, Tuesday 1 October**
  - BESC 185 – Last Names   Abbasi - Finley
  - EDUC 220 – Last Names Fitze - Mackillip
  - HALE 270 – Last Names   Mahre - Zuyus

Conflict? If you emailed, **Wed 2 October from 6:00-7:30PM**, ECOT 831

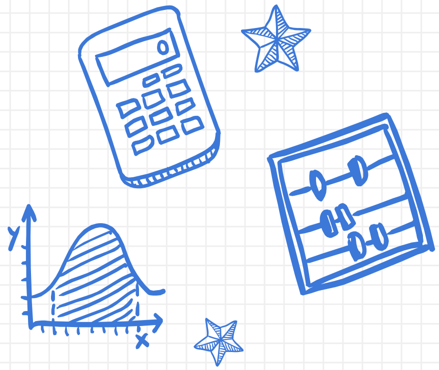
- **Review: TAs: Monday 9/30 @ 6:30pm – room TBD**
  - Concept guide
  - Written homework
  - “All Moodle problems” set
  - Workgroup worksheets
  - Lecture slides and examples

What did we do last time?

- how to cook up larger sets (unions, power sets) from smaller ones,
- and how to cook up smaller sets (intersections, subsets) from larger ones

Today:

- doing stuff with sets (proofs and manipulations)



Sets

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

Strategy: To prove that two sets *C* and *D* are equal, we must

1. ($\Rightarrow$) Prove that $C \subseteq D$
2. ($\Leftarrow$) Prove that $D \subseteq C$

Proof: ($\Rightarrow$)

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

Strategy: To prove that two sets *C* and *D* are equal, we must

1. (⇒) Prove that $C \subseteq D$
2. (⇐) Prove that $D \subseteq C$

Proof: (⇒)

1. S'pose *x* is an arbitrary element in *A* - *B*
2. $\Rightarrow x \in A \ \wedge \ x \notin B$
3. $x \notin B \ \Rightarrow \ x \in \bar{B}$
4. $\Rightarrow \ (x \in A) \wedge (x \in \bar{B})$
5. $\Rightarrow \ x \in A \cap \bar{B}$
6. Since *x* was any arbitrary element in *A* - *B*, we have shown $A - B \subseteq A \cap \bar{B}$

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

Strategy: To prove that two sets *C* and *D* are equal, we must

1. ($\Rightarrow$) Prove that $C \subseteq D$
2. ($\Leftarrow$) Prove that $D \subseteq C$

Proof: ($\Leftarrow$)

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

Strategy: To prove that two sets *C* and *D* are equal, we must

1. ($\Rightarrow$) Prove that $C \subseteq D$
2. ($\Leftarrow$) Prove that $D \subseteq C$

Proof: ($\Leftarrow$)

1. S'pose *x* is an arbitrary element in $A \cap \bar{B}$
2. $\Rightarrow$ $(x \in A) \wedge (x \in \bar{B})$
3. $x \in \bar{B}$ $\Rightarrow$ $x \notin B$
4. $(x \in A) \wedge (x \notin B) \Rightarrow (x \in A - B)$
5. Since *x* was any arbitrary element in $A \cap \bar{B}$, we have shown $A \cap \bar{B} \subseteq A - B$

Since $(A \cap \bar{B} \subseteq A - B)$ and $(A - B \subseteq A \cap \bar{B})$ it must be the case that

$$A - B = A \cap \bar{B} \quad \square$$

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

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1. ($\Rightarrow$) Prove that $C \subseteq D$
2. ($\Leftarrow$) Prove that $D \subseteq C$

Since *x* was any arbitrary element in $A \cap \bar{B}$, we have shown $A \cap \bar{B} \subseteq A - B$

Since $(A \cap \bar{B} \subseteq A - B)$ and $(A - B \subseteq A \cap \bar{B})$ it must be the case that

$$A - B = A \cap \bar{B} \square$$

Sets and set operations

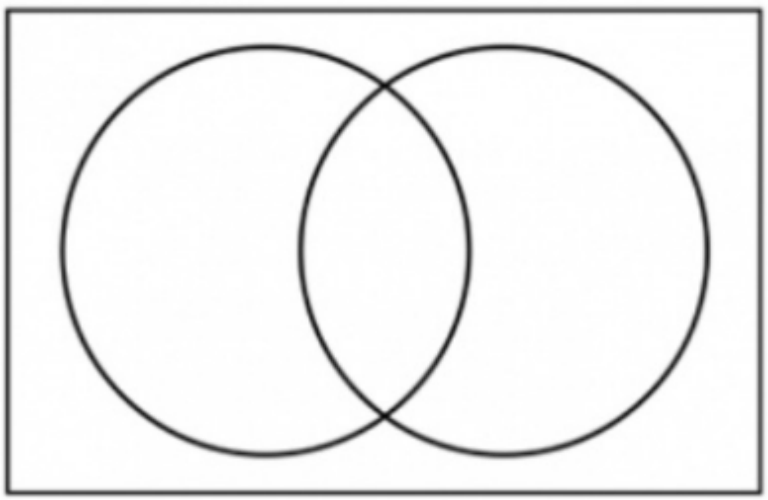
When sets are combined using only ∪,∩ , and complements, there is a set of **Set Identities** that completely mirrors the logical equivalences from last chapter.

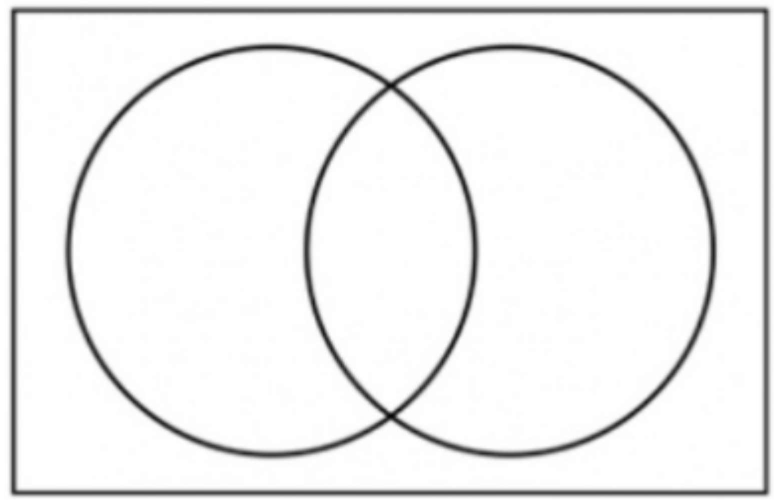
TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Sets and set operations

Example: Prove DeMorgan's Law for Sets: $\overline{A \cap B} = \overline{A} \cup \overline{B}$





Sets and set operations

<u>Example</u>: Prove (one of) De Morgan's laws for sets, using **builder notation** $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (remember, builder notation looked a lot like quantifiers, so this will look at lot like our old proofs from Ch 1)

Sets and set operations

<u>Example</u>: Prove (one of) De Morgan’s laws for sets, using **builder notation** $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof: $\overline{A \cap B}$ $= \{ x \mid x \notin A \cap B \}$ Definition of complement

$= \{ x \mid \neg(x \in A \cap B) \}$ Definition of not-in

$= \{ x \mid \neg(x \in A \wedge x \in B) \}$ Definition of intersection

$= \{ x \mid \neg(x \in A) \vee \neg(x \in B) \}$ De Morgan’s

$= \{ x \mid (x \notin A) \vee (x \notin B) \}$ Definition of not-in

$= \{ x \mid (x \in \overline{A}) \vee (x \in \overline{B}) \}$ Definition of complement

$= \{ x \mid x \in (\overline{A} \cup \overline{B}) \}$ Definition of union

$= \square$

Sets and set operations

Example: Use set identities to prove

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$

Proof:

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$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
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$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
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Sets and set operations

Example: Use set identities to prove

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$$

Proof:

$$\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{(B \cap C)}$$ De Morgan's

$$= \overline{A} \cap (\overline{B} \cup \overline{C})$$ De Morgan's

$$= (\overline{B} \cup \overline{C}) \cap \overline{A}$$ commutative

$$= (\overline{C} \cup \overline{B}) \cap \overline{A}$$ commutative

FYOG: Use set identities to prove

$$(A \cup \overline{B}) \cap (\overline{B \cap A}) = \overline{B}$$

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Sets and set operations

Example: Use set identities to prove

$A \cap (B \cup C) = (A \cap B) \cup C$

Proof:

TABLE 1 Set Identities.

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Sets and set operations

Example: Use set identities to prove

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: 1. ($\Rightarrow$) Prove that Lhs$\subseteq$ *Rhs*

Suppose that $x \in A \cap (B \cup C)$.

Then $x \in A$ and $x \in B \cup C$

$\Rightarrow (x \in A) \land ((x \in B) \lor (x \in C))$

$\Rightarrow ((x \in A) \land (x \in B)) \lor ((x \in A) \land (x \in C))$

$\Rightarrow x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

TABLE 1 Set Identities.

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Sets and set operations

Example: Use set identities to prove

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: 2. (⇒) Prove that Rhs⊆ *Lhs*

Suppose that $x \in (A \cap B) \cup (A \cap C)$

Then $x \in A \cap B$ or $x \in A \cap C$

$\Rightarrow ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$

$\Rightarrow (x \in A) \wedge ((x \in B) \vee (x \in C))$

$\Rightarrow x \in A$ and $x \in B \cup C$

$\Rightarrow x \in A \cap (B \cup C)$.

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Sets and set operations

Example: Use set identities to prove

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof: using **Membership Tables** (1 – element is in, 0– element is not in)

TABLE 2 **A Membership Table for the Distributive Property.**

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Because the columns for A∩(B ∪C) and (A∩B)∪(A∩C) are the same, the identity is valid.

Sets and set operations

FYOG: Use set identities to prove

$$(A \cup \overline{B}) \cap (\overline{B \cap A}) = \overline{B}$$

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Sets and set operations - Recap

- We learned what <u>**sets**</u> are,
- empty set, singleton set, power set,
- how to cook up larger sets (unions, power sets) from smaller ones,
- and how to cook up smaller sets (intersections, subsets) from larger ones

Next time:

- More on ***infinite*** sets and …
- doing stuff with sets of things
  to get other sets of things
  (we call the stuff we do ***functions****!*)

Extra Practice

Example 2: Prove DeMorgan's Law for Sets: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Set Operations

Example: Use set identities to prove $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

Set Operations

Example: If P is the set of prime numbers, then what is $\overline{P}$?

Set Operations

Example: Suppose $A = \{b, c, d\}$ and $B = \{a, b\}$. Find:

(a) $(A \times B) \cap (B \times B)$

(b) $(A \times B) \cup (B \times B)$

(c) $(A \times B) - (B \times B)$

(d) $(A \cap B) \times A$

(e) $(A \times B) \cap B$

(f) $\mathscr{P}(A) \cap \mathscr{P}(B)$

(g) $\mathscr{P}(A) - \mathscr{P}(B)$

(h) $\mathscr{P}(A \cap B)$

(i) $\mathscr{P}(A) \times \mathscr{P}(B)$

Solutions

Example 2: Prove this identity using set builder notation

$$
\begin{aligned}
\overline{A \cup B} &= \{x \mid x \notin A \cup B\} && \text{(def. complement)} \\
&= \{x \mid \neg(x \in A \cup B)\} && \text{(def. not in)} \\
&= \{x \mid \neg(x \in A \vee x \in B)\} && \text{(def. intersection)} \\
&= \{x \mid \neg(x \in A) \wedge \neg(x \in B)\} && \text{(DeMorgan's)} \\
&= \{x \mid x \notin A \wedge x \notin B\} && \text{(def. not in)} \\
&= \{x \mid x \in \overline{A} \wedge x \in \overline{B}\} && \text{(def. complement)} \\
&= \{x \mid x \in \overline{A} \cap \overline{B}\} && \text{(def. union)} \\
&= \overline{A} \cap \overline{B}
\end{aligned}
$$