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CSCI 2824: Discrete Structures

Lecture 37: Graph Coloring

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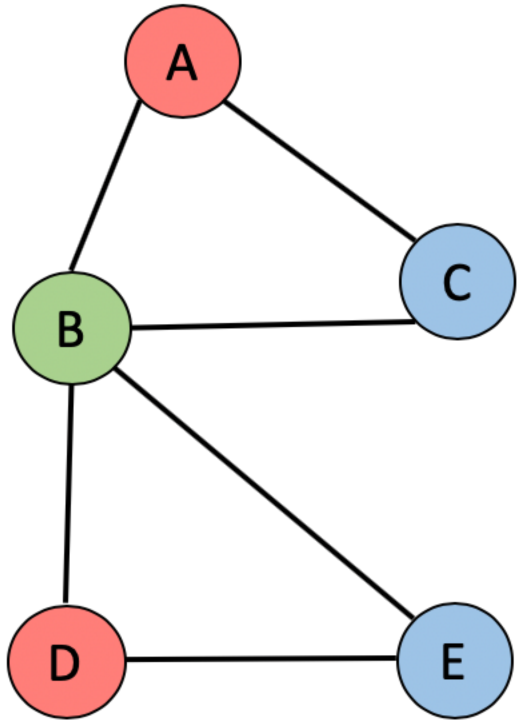

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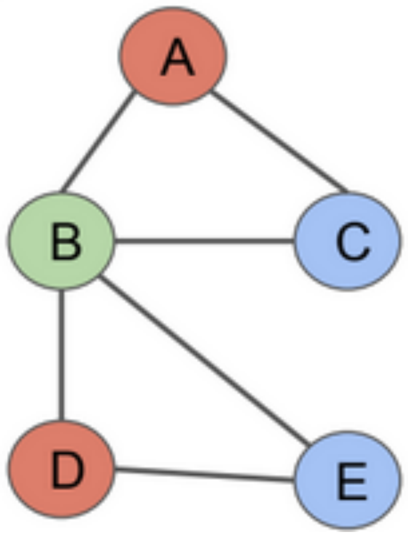
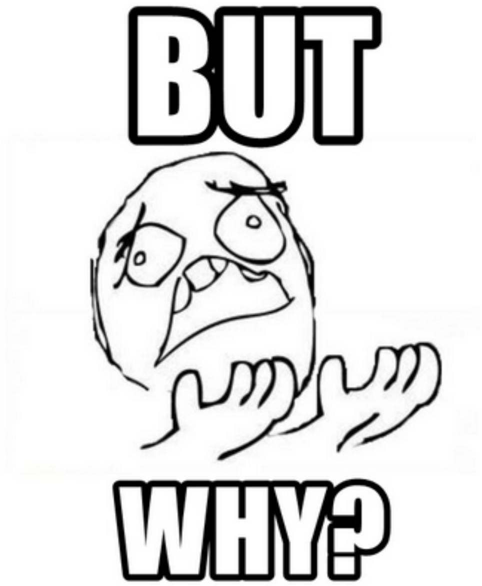
Graph Coloring

A ***coloring*** of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

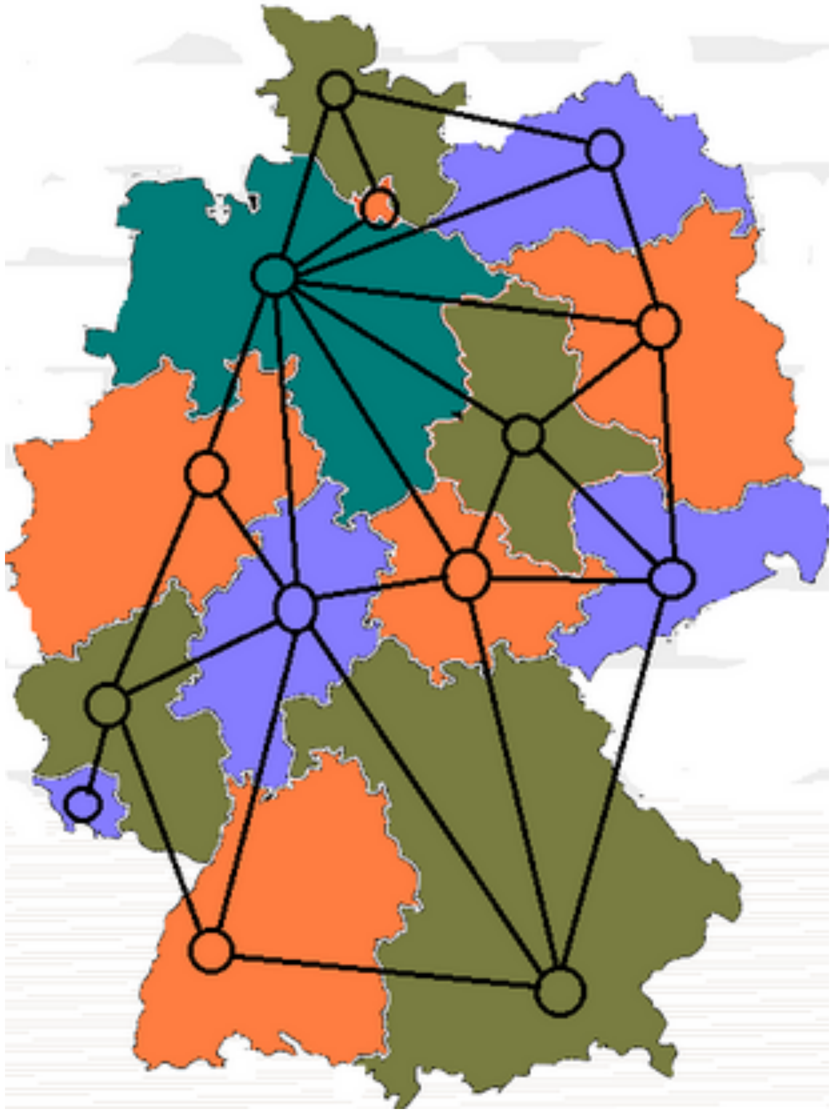
in other words: goal of graph coloring is to assign labels (colors) to vertices in the graph so that no vertices of the same color share an edge.



Graph Coloring



We can do neat things like schedule final exams and color maps so that no two adjacent countries have the same color.



Graph Coloring

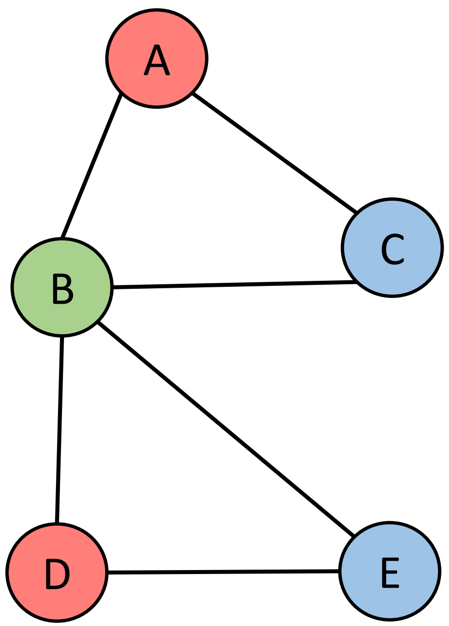
Let $G = (V, E)$ be a graph (directed or undirected). A ***cycle*** of a graph is a sequence of alternating vertices and edges that starts and ends on the same vertex but doesn't repeat any other vertices. The number of edges traversed is called the ***length*** of the cycle.

Example: Find two cycles and specify their length.

$A \to C \to B \to A$

$B \to E \to D \to B$

length 3 cycles

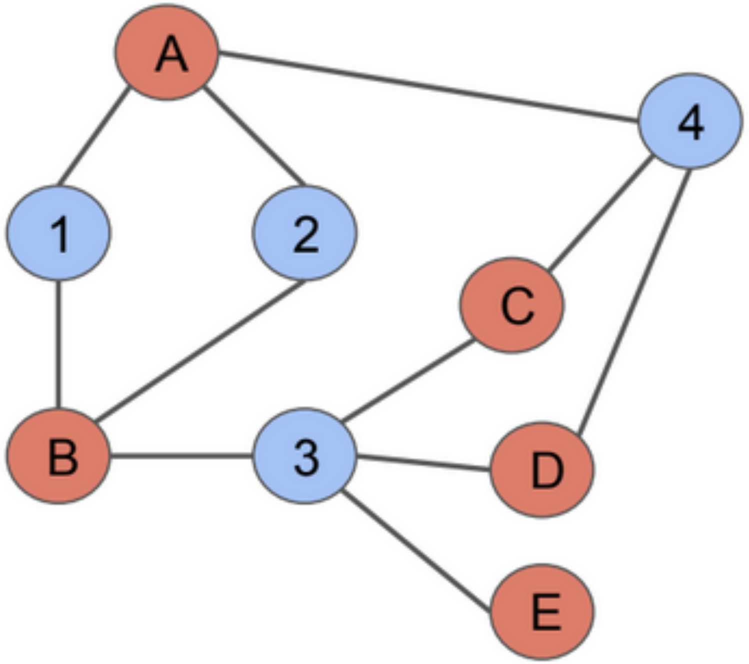


Graph Coloring

A graph $G = (V, E)$ is called ***bipartite*** if and only if the vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that

- $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$
- Any edge in E goes from a vertex in V_1 to one in V_2, and vice versa.

This graph is bipartite.



Graph Coloring

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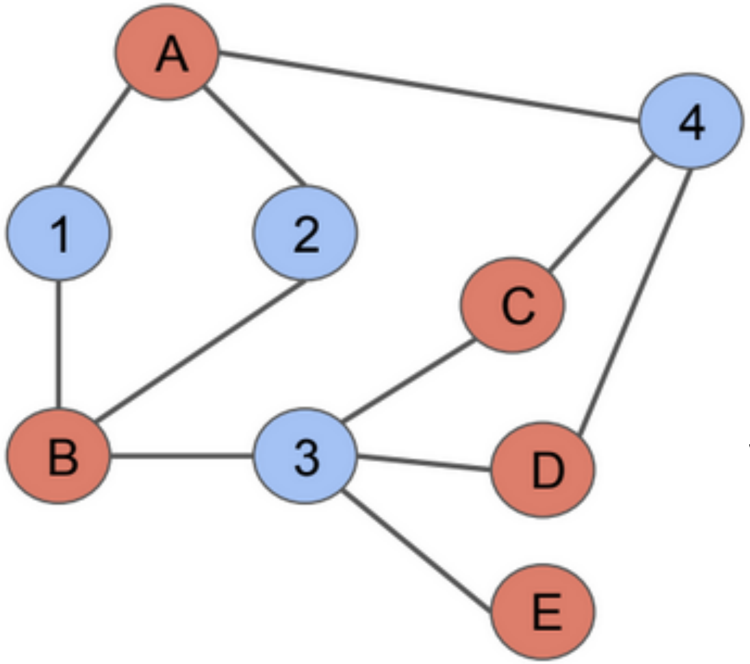
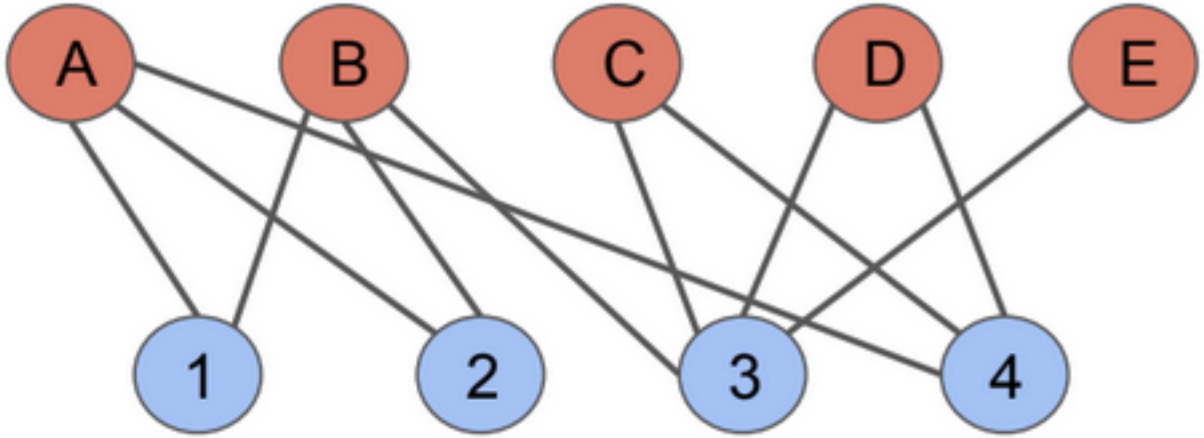


Easier to see the "bipartite-ness" (not a real word) if we rearrange it.



Graph Coloring

- It's difficult to decide whether a graph is bipartite or not simply based on a picture (which may be quite complicated.

Theorem: A graph G is bipartite if and only if it has no odd length cycles.

Proof: (One direction, by contradiction) $p \wedge \neg q$

Suppose G is bipartite and has an odd length cycle, $\underbrace{\boldsymbol{v_1 \to v_2 \to v_3 \to \cdots \to v_{2n} \to v_{2n+1} \to v_1}}$

Since G is bipartite, we must be able to color the vertices along this cycle alternating blue/red.

Without loss of generality, we can start with blue:

$$\boldsymbol{v_1 \to v_2 \to v_3 \to \cdots \to v_{2n} \to \underbrace{v_{2n+1} \to v_1}}$$

But the cycle must connect back to $\boldsymbol{v_1}$, which was blue, from $\boldsymbol{v_{2n+1}}$, which is also blue.

→← contradiction! Thus, if G is bipartite, then it must not have any odd length cycles.

Graph Coloring

Example: Suppose you are responsible for scheduling final exams at a university. You want to make sure that any two courses with a common student are scheduled at different times to avoid a conflict.

Class	Astro.	CS	Greek	Hist.	Ital.	Latin	Math	Phys.	Span.
Astronomy		X	X	X			X		
Comp Sci	X								X
Greek	X			X		X	X	X	
History	X		X			X			X
Italian						X	X		X
Latin			X	X	X		X	X	X
Math	X		X		X	X			
Physics			X			X			
Spanish		X		X	X	X			

This means there is a student taking a math class + an italian class

Graph Coloring

Example: Suppose you are responsible for scheduling final exams at a university. You want to make sure that any two courses with a common student are scheduled at different times to avoid a conflict.

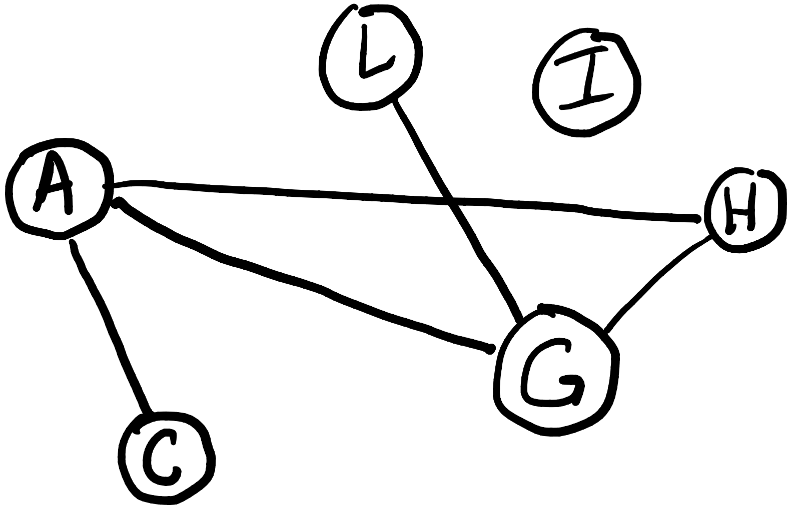
We can encode the information in a graph by assigning each course a vertex, and connecting two vertices by an edge if they have a common student.

Class	Astro.	CS	Greek	Hist.	Ital.	Latin	Math	Phys.	Span.
Astronomy		X	X	X			X		
Comp Sci	X								X
Greek	X			X		X	X	X	
History	X		X			X			X
Italian						X	X		X
Latin			X	X	X		X	X	X
Math	X		X		X	X			
Physics			X			X			
Spanish		X		X	X	X			

Graph Coloring

Class	Astro.	CS	Greek
Astronomy		X	X
Comp Sci	X		
Greek	X		
History	X		X
Italian			
Latin			X

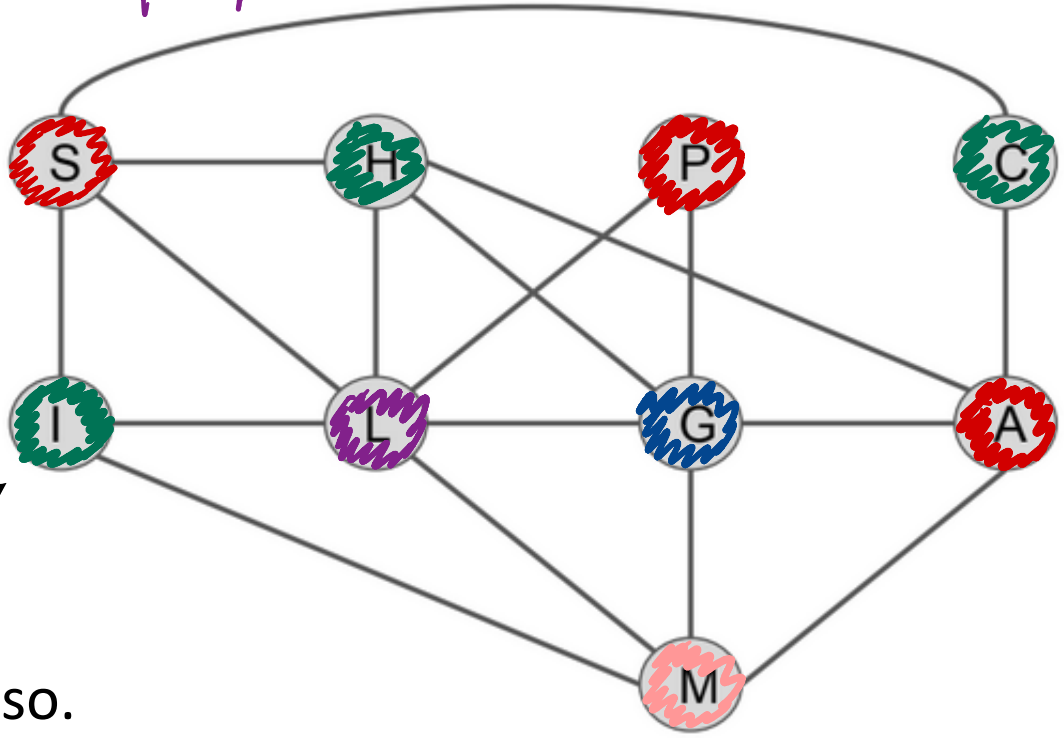
e.g. Just taking a piece of the table, we can graph in the following way.



Graph Coloring

Class	Astro.	CS	Greek	Hist.	Ital.	Latin	Math	Phys.	Span.
Astronomy		X	X	X			X		
Comp Sci	X								X
Greek	X			X		X	X	X	
History	X		X			X			X
Italian						X	X		X
Latin			X	X	X		X	X	X
Math	X		X		X	X			
Physics			X			X			
Spanish		X		X	X	X			

Red: 8-9 am
green: 9-10am
purple: 10-11am
blue: 11-12pm
pink: 12-1pm



Looking at the whole data set, we can graph it as so.

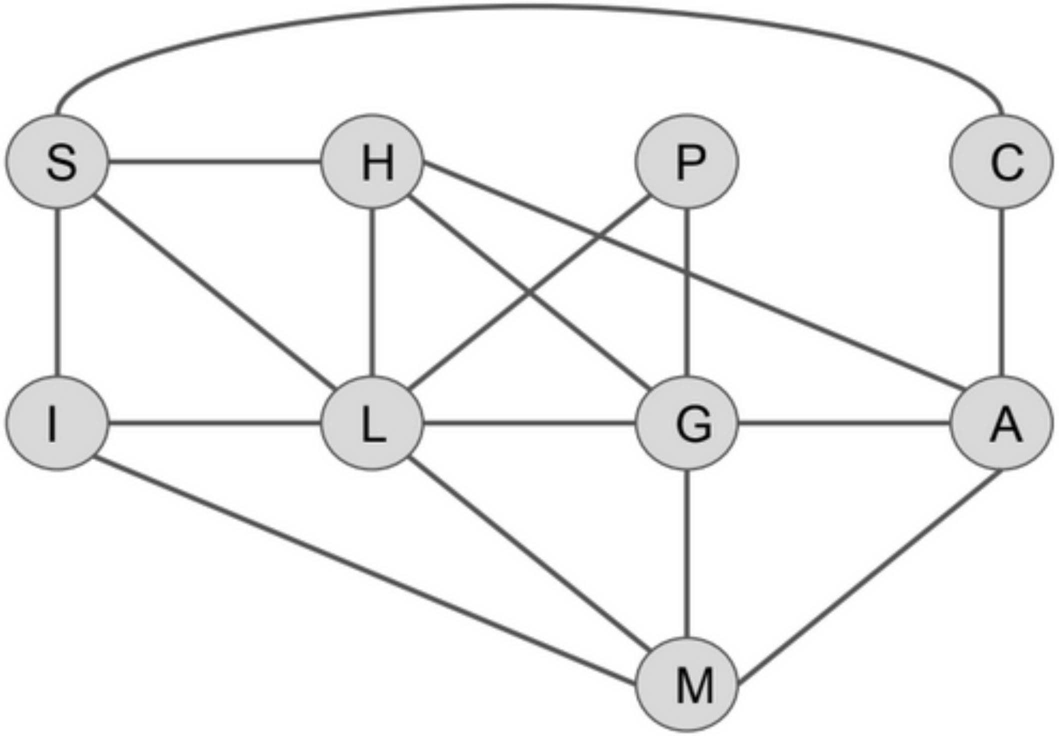
Graph Coloring

practice!

Example: Suppose you are responsible for scheduling final exams at a university. You want to make sure that any two courses with a common student are scheduled at different times to avoid a conflict.

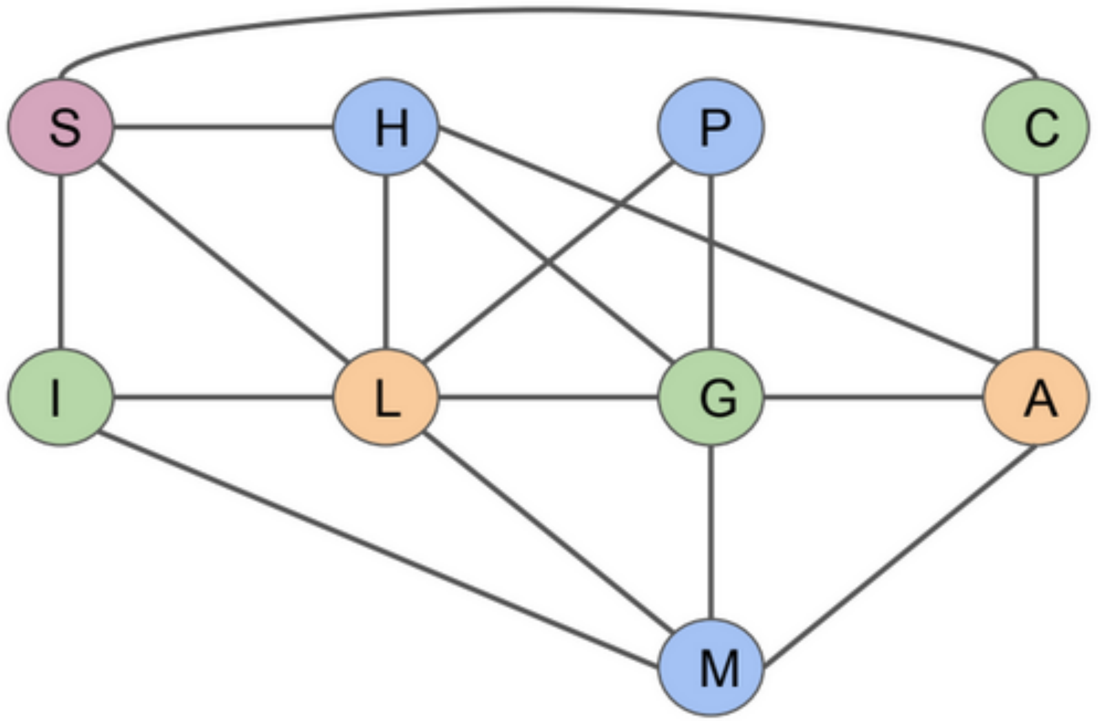
Now, think about finding a coloring of the graph, such that no adjacent vertices have the same color.

Then associating a final exam time with each color solves the problem!



Graph Coloring

Example: Suppose you are responsible for scheduling final exams at a university. You want to make sure that any two courses with a common student are scheduled at different times to avoid a conflict.



Here is one coloring of the graph.
It requires 4 colors.

Some questions:

- Can I always find a coloring of a graph?
- How many colors do I need?
- Are there certain colorings that are "better" than others?

Graph Coloring

The ***chromatic number*** of a graph, $\chi(G)$, is the minimum number of colors required to color a graph.

- A graph with n vertices can always be colored using n colors.
- The scheduling problem motivates us to use as few colors as possible.

Graph Coloring – Greedy Algorithm

How can we find a graph coloring?

Here is a greedy algorithm that is ***not guaranteed to find the minimum coloring*** of a graph, but it is guaranteed to find a coloring that works.

GREEN = color 1
RED = color 2
BLUE = color 3

The procedure requires us to number the colors consecutively.

1. Order the vertices in some (arbitrary) way.
2. Color the first vertex using Color 1
3. Pick the next uncolored vertex v.
   a. Color it using the lowest-numbered color that has not been used on any of the vertices adjacent to v.
   b. If you run out of colors, then add a new color.
4. Repeat Step 3 until all vertices are colored.

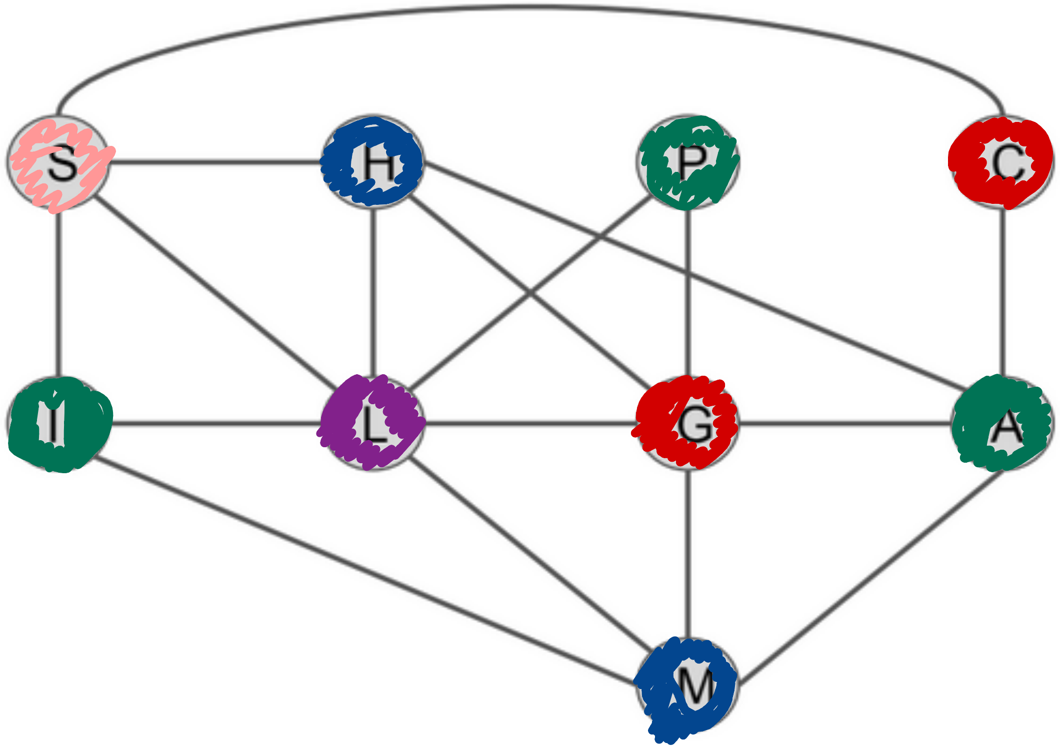
Graph Coloring

Example: Let's test it out on the Exam Scheduling Problem.

Order the vertices as $A, C, G, H, I, L, M, P, S$ and start with colors GREEN, RED, BLUE (in that order).

1 2 3 4 PURPLE 5 PINK

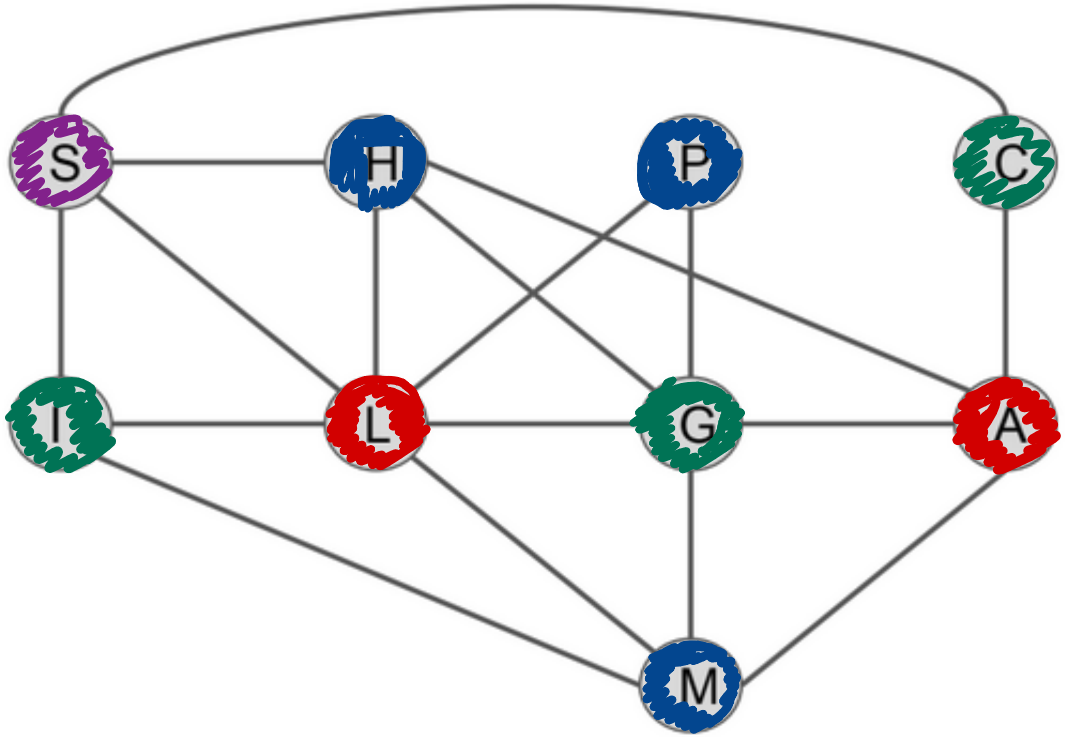
Need 5 colors



Graph Coloring

Example: Let's test it out on the Exam Scheduling Problem.

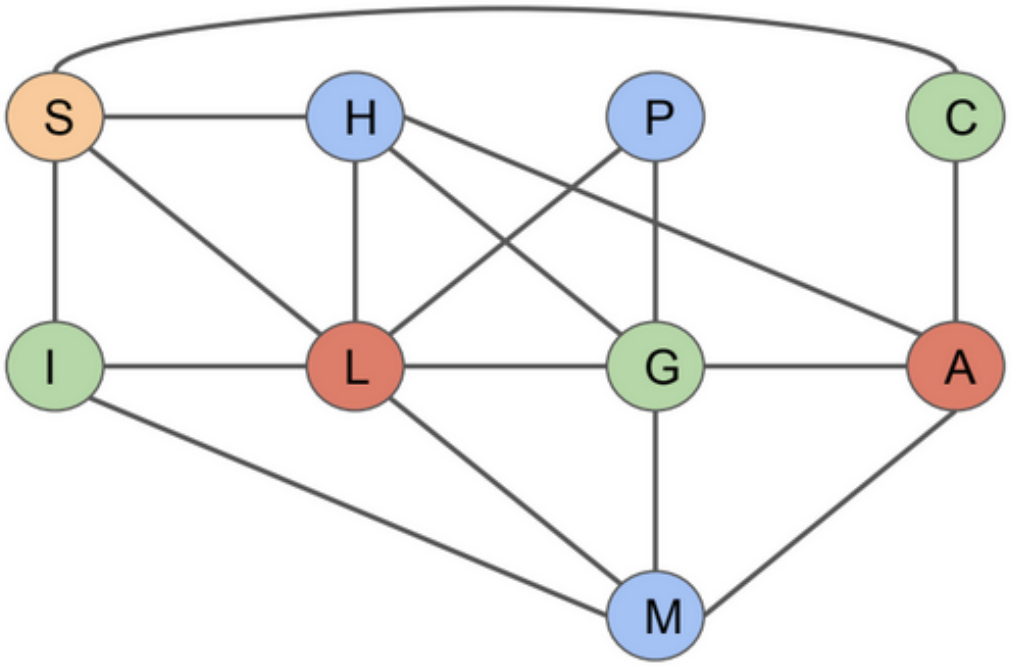
This time, let's try the order $G, L, H, P, M, A, I, S, C$ and again start with colors GREEN, RED, BLUE (in that order). PURPLE GRAY



Graph Coloring

Summary:

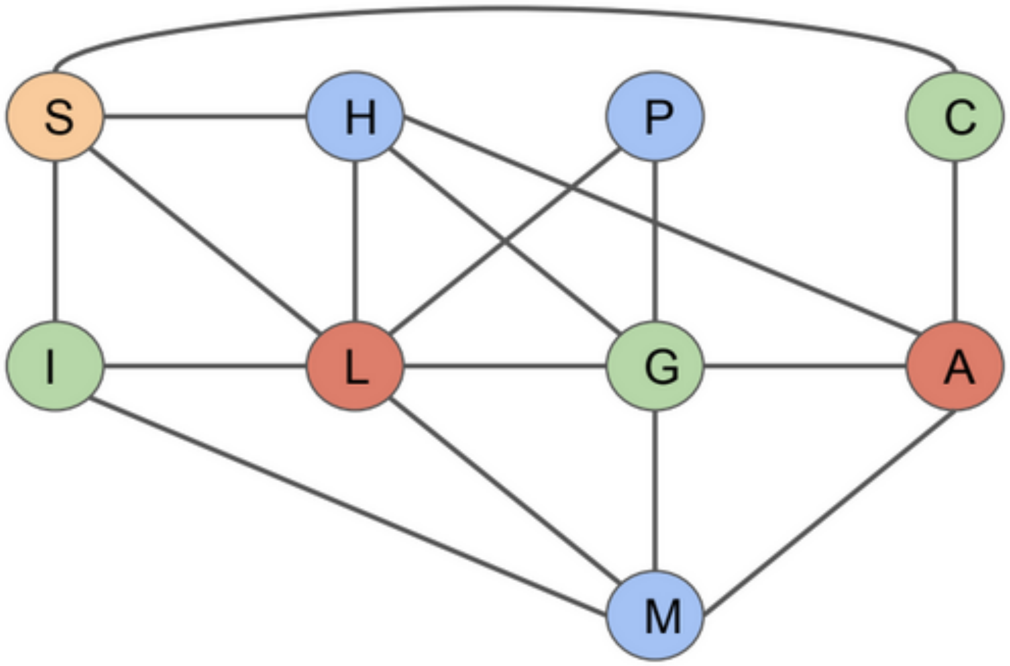
- We can get different colorings based on the initial ordering of the vertices.
- This is common with greedy algorithms.
- In practice: run the algorithm a few times with different vertex orderings and choose the one that provides the minimal coloring.



Graph Coloring

<u>Greedy Coloring Theorem</u>: If d is the largest degree of any vertex in graph G, then G has a coloring with $d + 1$ or fewer colors. In other words, the chromatic number of G is at most $d + 1$.

In the Exam Scheduling Problem, the largest degree is 6, so we need no more than 7 colors.



Graph Coloring

Greedy Coloring Theorem: If d is the largest degree of any vertex in graph G, then G has a coloring with $d+1$ or fewer colors. In other words, the chromatic number of G is at most $d+1$.

Proof sketch: Suppose that d is the largest degree in the graph.

- All vertices have d or fewer connections.
- When we color a new vertex, it is connected to at most d other vertices, which may already be colored.
- There are at most d colors we must avoid using.
- In the worst case, we need $d+1$ colors.

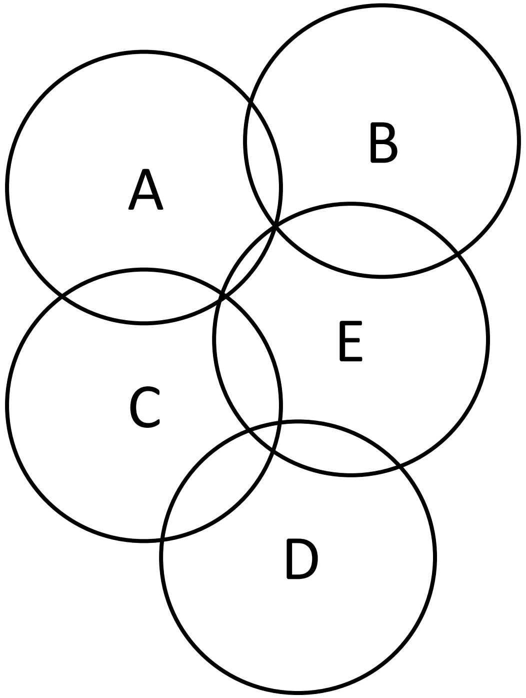
Graph Coloring

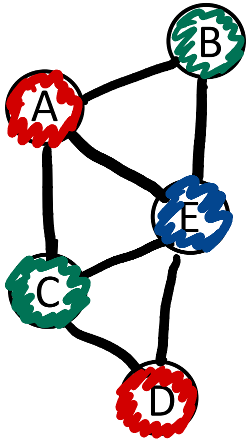
Example: Radio towers that have overlapping coverage must be assigned different frequencies to avoid interference with each other.

But companies have to pay the government to use a particular frequency. The question is: given a coverage map for several radio towers, how many frequencies do you need to lease from the government?

A B C D E

Red, green, blue



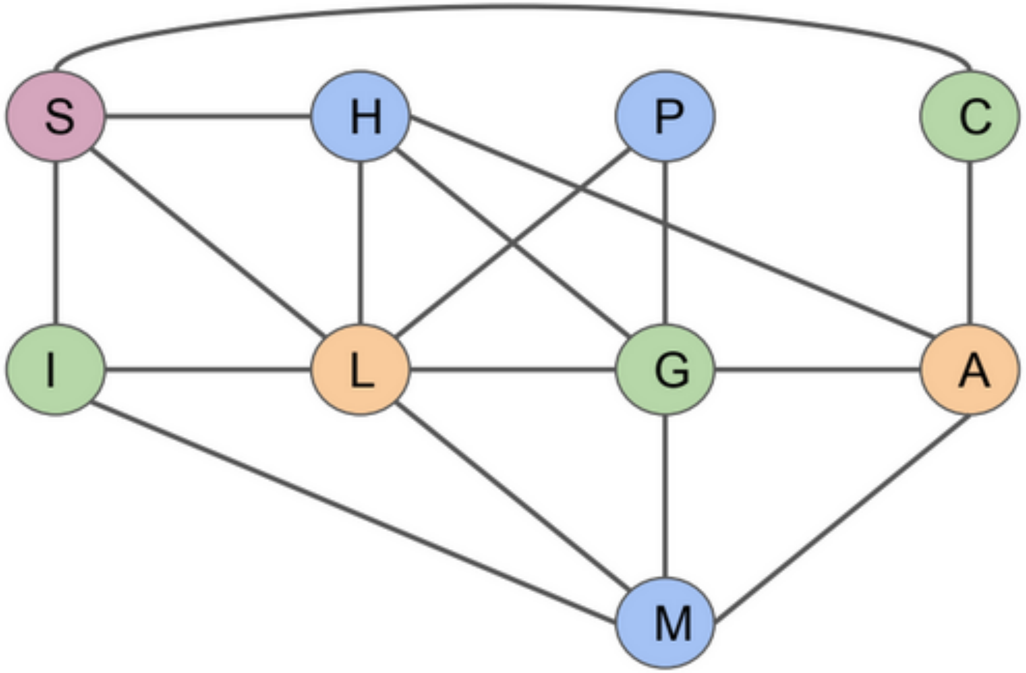


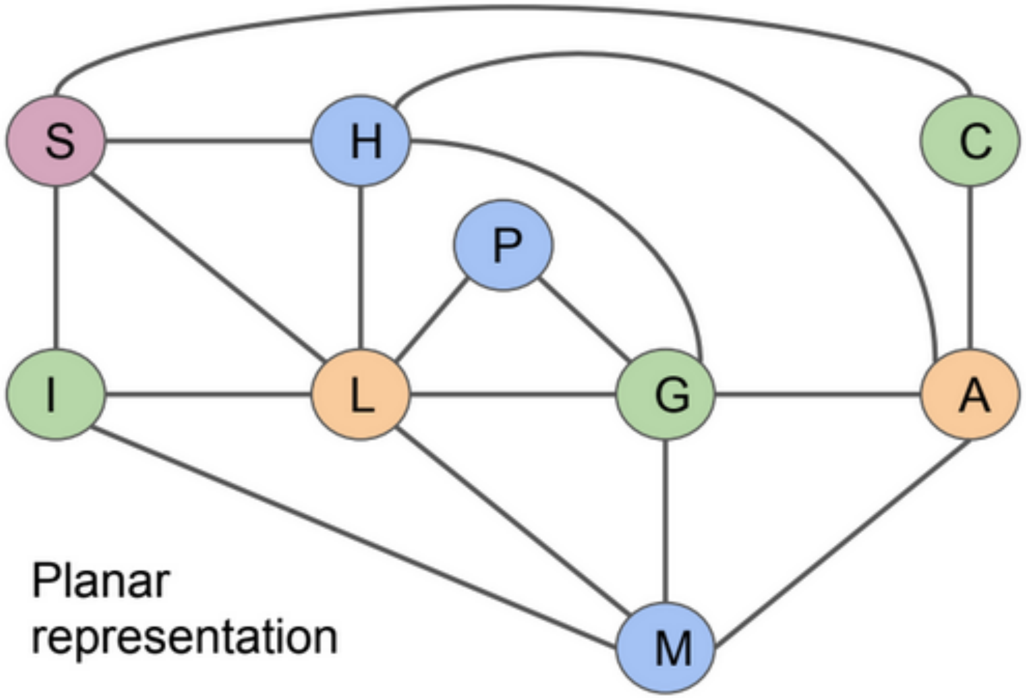
$\boxed{\text{3 colors}}$

Graph Coloring

A graph is a ***planar graph*** if it can be drawn in the plane without any edges crossing.

The 4-color Theorem: Any planar graph can be colored using at most 4 colors. [$\chi(G) \leq 4$]





Next time: Trees!