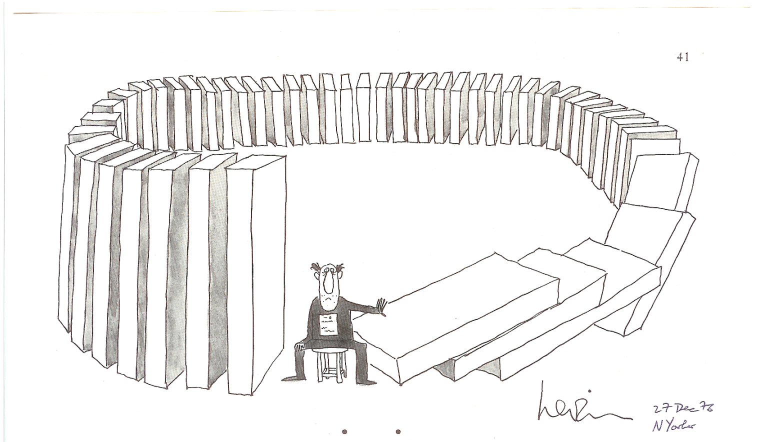

CSCI 2824: Discrete Structures

Lecture 23: Weak Induction. Strong Induction

Reminders

Submissions:

- Homework 8: **Fri 10/25 at noon –** Gradescope

Readings:

- Ch. 5 – Induction and Recursion
  - 5.1 Mathematical Induction

Last time

Induction:

- Definition
- Weak Induction
- Examples

Today:

- Strong Induction

Recursive Relations - recap

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Example: $a_0 = 4$, $a_n = a_{n-1} + 5$

By forward substitution

Recursive Relations - recap

Example: Suppose that a person deposits $10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

$P_n = P_{n-1} + 0.11 * P_{n-1}$

$P_0 = 10\ 000$

By backward substitution:

$P_1 = (1.11)P_0$
$P_2 = (1.11)P_1 = (1.11)^2P_0$
$P_3 = (1.11)P_2 = (1.11)^3P_0$
…
$P_n = (1.11)P_{n-1} = (1.11)^nP_0$

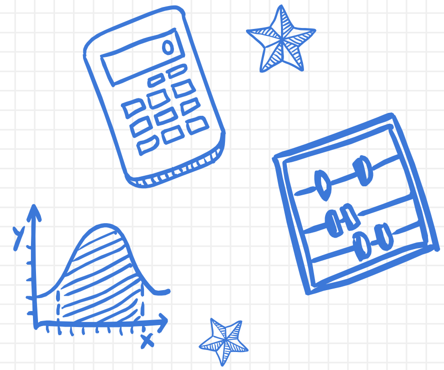
Recursive Relations - recap

Example: Suppose that a person deposits $10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. **He/she deposits $1000 at the end of every month.** How much will be in the account after 30 years?

$P_n = P_{n-1} + 0.11 * P_{n-1} + 1000.$

$P_0 = 10\ 000$

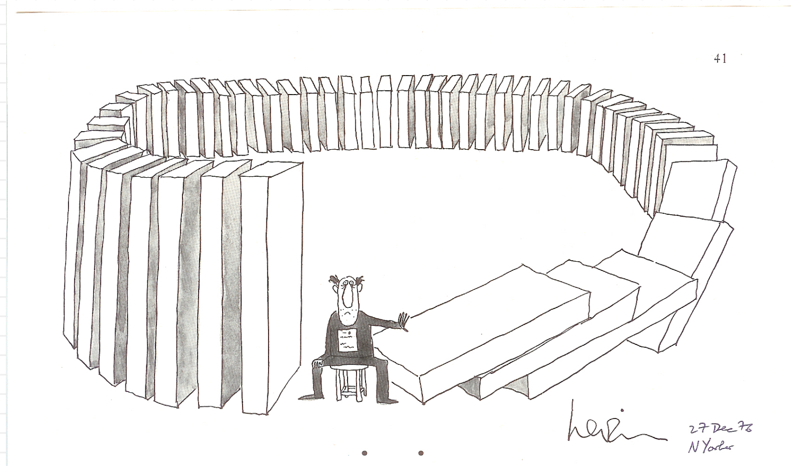
By backward substitution:



Induction

Mathematical induction

Your argument could go something like this:



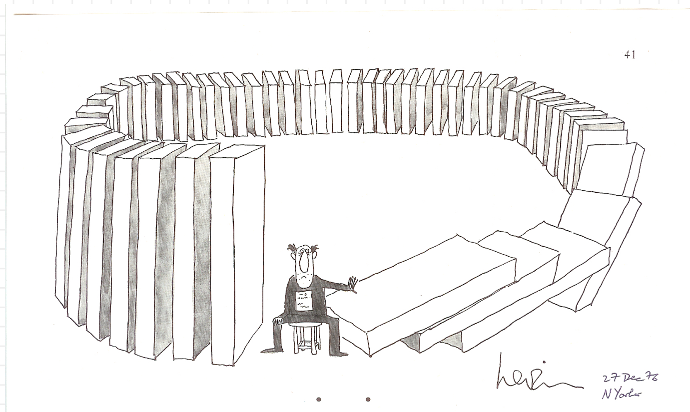
Base case: The first domino falls (because you knocked it over)
Inductive Step: Whenever domino number *k* falls, the one after it numbered *k*+1 also will fall (because domino *k* knocks it over).
Therefore: We conclude that all of the dominoes will fall over.

Mathematical induction

That kind of argument is the crux of **induction**. To prove a property holds for all natural numbers *k*, we argue as follows:

1. The property is true for $k = 0$ (or $k = 1$, or some other **base case**)
2. If the property is true for some natural number *k*, then it is true for natural number $k + 1$

(note that this is asking us to prove a conditional: [property true for *k*] → [true for *k*+1])



Mathematical induction

So if we let *P(n)* be the property that we're trying to prove, where *n* is some natural number, then an inductive argument goes like this:

1. Base case: Verify that *P(0)* holds (or *P(1)*, or *P(whatever)*)

2. Induction step: $(\forall k \geq 0)$ **if** $P(k)$ **then** $P(k+1)$

3. Conclusion: $(\forall n \geq 0)\ P(n)$

There are two slightly different kinds of inductive arguments...

Mathematical induction

There are two slightly different kinds of inductive arguments...

If the argument is of the form

- Verify that $P(1)$ is true
- Assume $P(k)$ is true, and show that $P(k+1)$ must be true (the proof part)

then we call it **weak induction**, or **ordinary induction**.

If the argument is of the form

- Verify that $P(1)$ is true
- Assume $P(k)$ is true for all $k = 1, 2, \ldots, n$, and show $P(n+1)$

then we call it **strong induction**.

Mathematical induction

Example: Prove that if n is an integer and $n \geq 4$, then $2^n < n!$

Base case:

Inductive step:

Conclusion:

Mathematical induction

Example: Prove that if n is an integer and $n \geq 4$, then $2^n < n!$

Base case: $n=4$, we have $2^{(4)} = 16$ and $4! = 24$, so it is true that $2^4 < 4!$

Inductive step: S'pose the formula holds for $n=k$ $(2^k < k!)$

$$\Rightarrow \quad 2^k < k!$$

$$\Rightarrow \quad 2^k \cdot 2 < k! \cdot 2$$

$$\Rightarrow \quad 2^{k+1} < k! \cdot 2 < k! \cdot (k+1) \qquad (k+1 \text{ definitely} > 2\text{, since } k \geq n \geq 4)$$

$$\Rightarrow \quad 2^{k+1} < (k+1)!$$

Conclusion: We've shown that the inequality holds for $n=k+1$, thus, by induction, we've proved the inequality. ⧅

Mathematical induction

Example: (geometric progressions) Prove that when $r \neq 1$,

$$a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

Base case:

Inductive step:

Mathematical induction

Example: (geometric progressions) Prove that when $r \neq 1$,

$$a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

Base case: n=0, $a = \dfrac{ar^1 - a}{r - 1} = \dfrac{a(r-1)}{r-1} = a$

Inductive step: S'pose the formula holds for n=k. Then…

$$\Rightarrow \quad a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a}{r - 1} + ar^{k+1}$$

$$= \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+1}(r-1)}{r - 1}$$

$$= \frac{ar^{k+1} - a + ar^{k+2} - ar^{k+1}}{r - 1} = \frac{ar^{k+2} - a}{r - 1}$$

Conclusion:
Thus, by induction,
the formula is true ⧅

Mathematical induction

Example: Prove that if $n \geq 1$ is an integer, then $n^3 - n$ is divisible by 3

Mathematical induction

Example: Prove that if $n \geq 1$ is an integer, then $n^3 - n$ is divisible by 3

Base case: $n=1 \Rightarrow n^3-n = 1^3-1 = 0$, and $0=3(0)$ so 1^3-1 is divisible by 3.

Inductive step: S'pose k^3-k is divisible by 3, for some integer $k > 1$ (**inductive hypothesis**)
To show: $(k+1)^3 - (k+1)$ must also be divisible by 3.

$$\begin{aligned}\Rightarrow \quad (k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - k - 1 \\ &= k^3 - k + 3k^2 + 3k \\ &= 3(\text{some integer}) + 3k^2 + 3k \qquad \leftarrow \text{by } \textbf{inductive hypothesis} \\ &= 3(\text{some integer} + k^2 + k)\end{aligned}$$

Therefore, $(k+1)^3 - (k+1)$ is divisible by 3, and the hypothesis is true by **weak induction** ?

Mathematical induction

Example: Let F_n by the n^{th} Fibonacci number. Prove that $$\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}$$

Mathematical induction

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Base case: $n=0 \Rightarrow \sum_{k=0}^{0} F_k^2 = F_0^2 = 1^1 = 1$, and $F_0 \cdot F_1 = 1 \cdot 1$

Inductive step: S'pose the formula is true for some $m > 0$: $\sum_{k=0}^{m} F_k^2 = F_m F_{m+1}$
(ind. hypothesis)

To show: formula holds for $m+1$: $\sum_{k=0}^{m+1} F_k^2 = F_{m+1} F_{m+2}$

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$$\sum_{k=0}^{m+1} F_k^2 = F_{m+1}^2 + \sum_{k=0}^{m} F_k^2 = F_{m+1}^2 + F_m F_{m+1} \qquad \leftarrow \text{by induction hypothesis}$$

$$= F_{m+1}(F_{m+1} + F_m)$$

$$= F_{m+1} F_{m+2} \qquad \leftarrow \text{by definition of Fibonacci sequence}$$

Therefore, the formula holds in general, by **weak induction QED**

Mathematical induction

S'pose you have an infinite line of dominos…

Now, we learn **strong induction:**

- If *all of the previous* dominoes fall, then *this* domino will fall
- **Base case:** Prove $P(1)$
- **Inductive step:** Prove: [if $P(l)$ for $1 \leq l \leq k$, then $P(k+1)$]

Strong induction

Example: Prove the Fundamental Theorem of Arithmetic. That is, prove that any integer $n \geq 2$ is either prime or can be written as the product of prime numbers.

Strong induction

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Proof: (by strong induction)

- **Base case:** Let $n=2$. Then n is indeed prime
- **Inductive step**: Assume that for each l s.t. $2 \leq l \leq k$, l is either prime or can be written as the product of primes. (this is the **inductive hypothesis**)
- **To show:** That k+1 is either prime or can be written as a product of primes.

  Two cases to consider:

  - Case 1: k+1 itself is prime
  - Case 2: k+1 is composite

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  $\Rightarrow$ we're done!

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Strong induction

- **To show:** That k+1 is either prime or can be written as a product of primes.
- **Case 1:** k+1 itself is prime

  ⇒ we're done!
- **Case 2:** k+1 is composite

If k+1 is not prime then it is composite and (from the definition of composite) can be written as a product of two integers a and b, with $a, b < k$

But by the **induction hypothesis**, since a and $b < k$, they can be written as the product of primes.

Multiply the prime factorization of a by the prime factorization of b, and we have the prime factorization of k+1

Conclusion: We've shown that k+1 is either prime or can be written as the product of primes. Thus, we've proved the theorem by **strong induction**. **QED**

Strong induction

Tougher Example: Prove: If n is a positive integer, then n can be written as the sum of distinct Fibonacci numbers.

Strong induction

Tougher Example: Prove: If *n* is a positive integer, then *n* can be written as the sum of distinct Fibonacci numbers.

Proof: (by strong induction)

- **Base case:** Let $n=1$. Then $n = F_0 + F_1 = 0 + 1 = 1$
- **Inductive step**: Assume that for each l s.t. $1 \leq l \leq k$, l can be written as the sum of distinct Fibonacci numbers. (inductive hypothesis)
- **To show:** That $k+1$ can be written as the sum of distinct Fibonacci numbers.

  Two cases to consider:

  - Case 1: $k+1$ itself is a Fibonacci number
  - Case 2: $k+1$ is not a Fibonacci number

Strong induction

To show: That k+1 can be written as the sum of distinct Fibonacci numbers.

- Case 1 If k+1 itself is a Fibonacci number, then for some $m \in \mathbf{N}$, we have

  $k+1 = F_m = F_{m-1} + F_{m-2}$

- Case 2: If k+1 is not a Fibonacci number, then

Strong induction

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- Case 1 If k+1 itself is a Fibonacci number, then for some $m \in \mathbf{N}$, we have

$$k+1 = F_m = F_{m-1} + F_{m-2}$$

- Case 2: If k+1 is not a Fibonacci number, then we can let F_j be the largest Fibonacci number that is less than k+1.

⇒ So we have: $F_j < k+1 < F_{j+1}$

⇒ Subtract F_j everywhere to find: $0 < (k+1) - F_j < F_{j+1} - F_j = F_{j-1} < k+1$

But now $(k+1) - F_j \leq k$, so by the **inductive hypothesis**, $(k+1) - F_j$ can be written as the sum of distinct Fibonacci numbers. Furthermore, since $(k+1) - F_j < F_{j-1}$, we know that F_j is not part of that sum.

Thus, adding F_j to that sum expresses k+1 as the sum of distinct Fibonacci numbers. Thus, we've proved the claim by **strong induction**. QED

Strong induction

Example: Consider a game in which two players take turns removing any positive number of matches they want from one of two piles. The player who removes the last match wins the game. Prove that if the two piles contain the same number of matches initially, then the second player can always guarantee a win.

Strong induction

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Solution:

Let $P(n)$ = proposition that Player 2 can always win if each pile starts out with n matches

Base case: If n=1, then Player 1 is forced to take 1 match from 1 of the piles. Then Player 2 can take the last match from the other pile

Inductive step: Suppose that Player 2 can always win if each pile starts out with l matches, where $1 \leq l \leq k$ (inductive hypothesis)

To show: Player 2 can always win if the piles start out with k+1 matches

Strong induction

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To show: Player 2 can always win if the piles start out with k+1 matches

There are, again, two cases, depending on how many matches Player 1 takes (call it r):

Case 1: Player 1 takes $r = k+1$ matches (all) from one pile

Case 2: Player 1 takes $1 \leq r \leq k$ matches from one pile

Strong induction

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There are, again, two cases, depending on how many matches Player 1 takes (call it r):

Case 1: Player 1 takes $r = k+1$ matches (all) from one pile

⇒ Player 2 can take all of the matches from the other pile and win!

Strong induction

To show: Player 2 can always win if the piles start out with k+1 matches

Case 2: Player 1 takes $1 \leq r \leq k$ matches from one pile

⇒ There are $k - r + 1$ matches left in this pile, and k+1 matches in the other pile

⇒ Player 2 should take r matches out of the other pile, leaving $k - r + 1$ matches in each pile

⇒ Now there are $k - r + 1 < k+1$ matches in each pile, and it's Player 1's turn

⇒ By the **inductive hypothesis**, there must be a strategy for Player 2 to guarantee a win

⇒ Thus, we've proved (by **strong induction**) that Player 2 can always win the game if the piles start out with equal integer numbers of matches. ◇

Induction: *bad* examples

(Bad) Example: "Prove" that $6n = 0$ for all $n \geq 0$

Base case: Let $n = 0$. Then $6n = 6(0) = 0$

Induction step: Assume that $6l = 0$ for all l s.t. $0 \leq l \leq k$

To show: $6(k+1) = 0$

$\Rightarrow$ Write $k+1 = a + b$, where a and b are integers s.t. $0 \leq a, b \leq k$

$\Rightarrow$ By **induction hypothesis**, $6a = 6b = 0$

$\Rightarrow$ $6(k+1) = 6a + 6b = 0$ " "

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Mistake: We relied on our ability to write n as the sum of two non-negative integers. What happens when $k=0$?

$\Rightarrow$ $0 \leq a, b \leq k = 0$, so $a = b = 0$… so it turns out we can't write $k+1 = a+b$

Induction: *bad* examples

(Bad) Example: “Prove” that all Fibonacci numbers are even

Base case: Let $n = 0$. $F_0 = 0$, which is even

Induction step: Assume that F_l is even for all l s.t. $0 \leq l \leq k$

To show: F_{k+1} is even

$\Rightarrow F_{k+1} = F_k + F_{k-1}$

$\Rightarrow$ By **induction hypothesis**, F_k and F_{k-1} are both even

$\Rightarrow F_{k+1}$ is the sum of two evens, and therefore even “ ”

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$\Rightarrow$ $F_{k+1} = F_k + F_{k-1}$

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Mistake: We only showed F_0 is even, but then we used $F_{k+1} = F_k + F_{k-1}$ in our proof

Induction: *bad* examples

(Bad) Example: "Prove" that all Fibonacci numbers are even

Mistake: We only showed F_0 is even, but then we used $F_{k+1} = F_k + F_{k-1}$ in our proof

⇒ If you think of our "proof" as a line of dominoes, each domino falling in this case requires **two** previous ones to fall

⇒ Here, we only did one base case, so we only knocked over **one** domino

Rule of thumb: If your proof requires going back *s* steps, then you need *s* base cases.

Strong induction

Example (a real one this time): Prove that for any integer $n \geq 1$, there exist numbers $a, b \geq 1$ such that $5^n = a^2 + b^2$

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Important and sorta philosophical note: Proofs almost never "just plain work out".

- Usually: flail around, figure out what you need, …
- Then go back and put it all together in the right order.

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→ Tackle this one *without any base cases* to begin,

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Probable inductive step: Suppose that for each l s.t. $1 \leq l \leq k$, there exist integers $a, b \geq 1$ s.t. $5^l = a^2 + b^2$ (← inductive hypothesis)

To show: There exist integers $a, b \geq 1$ s.t. $5^{k+1} = a^2 + b^2$

Mathematical induction

Recap:

- ***Strong* induction** -- the proof technique that's a proof by cases on **for real** steroids
  - **Base case:** show that the hypothesis is true for the first case
  - **Inductive step:** name your inductive hypothesis
    $\Rightarrow$ *S'pose* true at all stages $j \leq k$, **To Show** true at k+1

Next time:

- We talk **re-re-re-re-recursively**