CSCI 2824: Discrete Structures

Lecture 5: Logical Equivalences

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Logical Equivalences

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**. The compound propositions p and q are called **logically equivalent** if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are **logically equivalent**.

A compound proposition that is always false is called a **contradiction**.

A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Logical Equivalences

Example of a Tautology:

p	$\neg p$	$p \vee \neg p$

Example of a Contradiction:

p	$\neg p$	$p \wedge \neg p$

Example: Show that $\neg\,(p \wedge q) \equiv \neg\, p \vee \neg\, q$

p	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$	$\boldsymbol{\neg(p \wedge q)}$	$\boldsymbol{\neg p}$	$\boldsymbol{\neg q}$	$\boldsymbol{\neg p \vee \neg q}$	$\boldsymbol{\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q}$

Example: Show that $\neg\,(p \vee q) \equiv \neg\, p \wedge \neg\, q$

p	$\boldsymbol{q}$	$\boldsymbol{p \vee q}$	$\boldsymbol{\neg(p \vee q)}$	$\boldsymbol{\neg p}$	$\boldsymbol{\neg q}$	$\boldsymbol{\neg p \wedge \neg q}$	$\boldsymbol{\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q}$

Logical Equivalences

We just Proved:

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Other Equivalences (from our book)

TABLE 7 Logical Equivalences Involving Conditional Statements.

- $p \rightarrow q \equiv \neg p \vee q$ Relation by Implication (RBI)
- $p \rightarrow q \equiv \neg q \rightarrow \neg p$ Contraposition
- $p \vee q \equiv \neg p \rightarrow q$
- $p \wedge q \equiv \neg(p \rightarrow \neg q)$
- $\neg(p \rightarrow q) \equiv p \wedge \neg q$
- $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
- $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
- $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
- $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ Biconditional
- $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
- $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
- $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

$p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$ Alternate definition of xor

Logical Equivalences

Other Equivalences (from our book)

To show that two compound propositions are logically equivalent:

- Prove it with a Truth Table
- Use Equivalence Rules to go from one to the other.

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalences

Example: Show that $p \to q \equiv \neg\, q \to \neg\, p$ (without a truth table)

Logical Equivalences

Example: Show that $(\mathrm{p} \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$ (without a truth table)

Satisfiability

A compound proposition is **<u>satisfiable</u>** if there is an assignment of truth values to its variables that makes it true. If there is no such case then we say it is **<u>unsatisfiable</u>** (i.e. a contradiction)

<u>Example</u>: Show that $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$ is satisfiable.

Satisfiability

Example: Show that $(p \to q) \land (p \to \neg q) \land (\neg p \to q) \land (\neg p \to \neg q)$ is not satisfiable.

Example: Sudoku puzzles can be written (and solved) as a satisfiability problems

Solving this:

1) First chain together the propositions with provided values: $p(1,1,5) \wedge p(1,2,3) \wedge p(1,5,7) \wedge \ldots$
2) Assert that every row contains every number: $\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i,j,n)$

3) Assert that every column contains every number:

$$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$$

4) Assert that every 3x3 block contains every number:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$$

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Let $p(i, j, n)$ denote the proposition that a number n is in the cell in row i and column j

9 rows, 9 columns, 9 numbers $= 9 \times 9 \times 9 =$
729 propositions

5) Assert that no cell contains more than one number:

$$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigwedge_{m=1, m \neq n}^{9} (p(i,j,n) \rightarrow \neg p(i,j,m))$$

6) String 1-5 together with conjunctions

Good Read on this problem!

Extra Practice

Example 1: Work out the truth table to show $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Example 2: Work out the truth table to show $p \to q \equiv \neg p \lor q$

Example 3: Show that $(p \to q) \land (p \to r) \equiv p \to (q \land r)$

Example 4: Show that the following proposition is **not** satisfiable

$$(p \leftrightarrow q) \land (\neg p \leftrightarrow q)$$

Solutions

Example 1: Work out the truth table to show $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Solution:

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Example 2: Work out the truth table to show $p \to q \equiv \neg p \vee q$

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Example 1: Show that $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$

Solution: This one is actually easier if we start from the second proposition

$$
\begin{aligned}
p \to (q \wedge r) &\equiv \neg p \vee (q \wedge r) && \text{(RBI)} \\
&\equiv (\neg p \vee q) \wedge (\neg p \vee r) && \text{(distribution)} \\
&\equiv (p \to q) \wedge (p \to r) && \text{(that one rule in reverse)}
\end{aligned}
$$

Example 1: Show that the following proposition is **not** satisfiable

$$(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$$

Solution: OK, this ones pretty easy

1. From the first conjunct we know that p and q must have the same truth values
2. From the second conjunct we know that p and q must have different truth values
3. This is a contradiction, thus the proposition is **not** satisfiable