CSCI 2824: Discrete Structures

Lecture 11: Proof Methods and Strategies

Reminders

Submissions:

- Homework 3: Fri 9/20 at noon – 1 try
- Homework 4: Fri 9/27 at noon – Gradescope

Readings:

- 1.6 – 1.8 this week
- Starting Ch. 2 – SETS - next week

Midterm – Tue October 1st at 6pm

Any conflicts? – email csci2824@colorado.edu

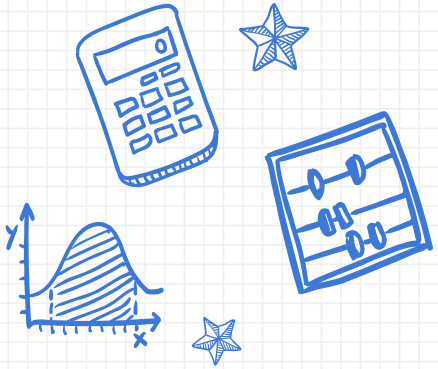
What did we do last time?

We learned about and saw some examples using:

- Direct proof
- Contrapositive proof
- Proof by contradiction

Today:

- How do we prove that something exists? (or does not exist?)
- How do we prove that something that does exist is **unique**?
- How can we **exhaustively** prove something?
- What are common mistakes/missteps/blunders in proving stuff?



Proof Methods and Strategies

Proof methods and strategies

Proof by Cases (Exhaustive Proof): benefit is that we may have more information about each specific case than we would have about just some general n.

Proof by Construction (Existence Proof): prove the existence of a solution by explicitly constructing it.

Existence and Uniqueness Proofs:

1) Show existence by construction
2) Show uniqueness by supposing there are two such objects that exist but then show they must be equal to each other.

Proof methods and strategies

Proof by Cases

Proof methods and strategies

Example: Suppose you want to prove $p \rightarrow q$ if p is some statement that is true for all CU undergraduates.

"If a *student studies*, then *they are cool*."

Proof methods and strategies

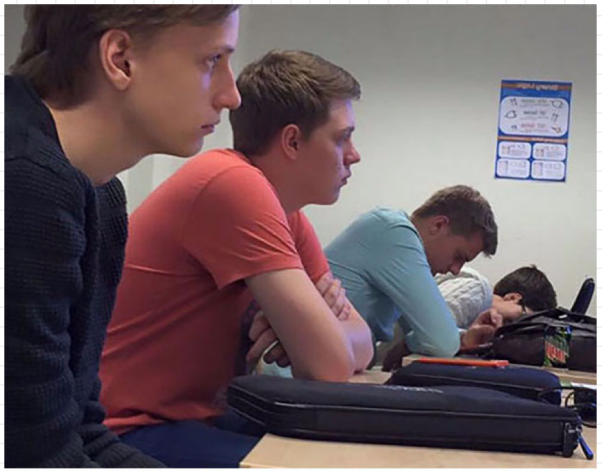
Example: Suppose you want to prove $p \rightarrow q$ if p is some statement that is true for all CU undergraduates.

"If a *student studies*, then *they are cool*."

Break up into smaller cases:

"If a ***Fr**eshman studies* **or** *a **So**phomore studies* **or** *a **Ju**nior studies* **or** *a **Se**nior studies*, then *they are cool*."

Which is: $(p(Fr) \vee p(So) \vee p(Ju) \vee p(Se)) \rightarrow q$



Proof methods and strategies

Example: Suppose you want to prove $p \rightarrow q$ if p is some statement that is true for all CU undergraduates.

"If a *student studies*, then *they are cool*."

Break up into smaller cases:

"If a ***Fr**eshman studies* **or** *a **So**phomore studies* **or** *a **Ju**nior studies* **or** *a **Se**nior studies*, then *they are cool.*"

Which is: $(p(Fr) \vee p(So) \vee p(Ju) \vee p(Se)) \rightarrow q$

Can we break this up into smaller statements?

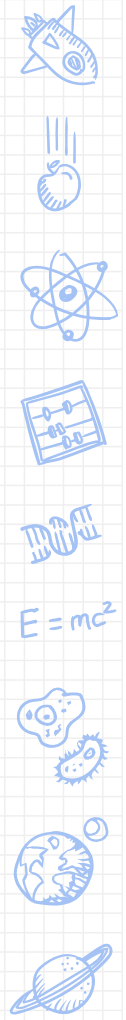
$(p(Fr) \rightarrow q) \wedge (p(So) \rightarrow q) \wedge (p(Ju) \rightarrow q) \wedge (p(Se) \rightarrow q)$

This is **proof by cases**

Proof methods and strategies

Example: Prove that if n is any integer not divisible by 5, then n^2 leaves a remainder of 1 or 4 when divided by 5.

How can we represent an integer that is **not** divisible by 5?



Proof methods and strategies

Example: Prove that if n is any integer not divisible by 5, then n^2 leaves a remainder of 1 or 4 when divided by 5.

How can we represent an integer that is **not** divisible by 5?

- n leaves a remainder of 1: $n = 5a + 1$ (for some integer a)
- n leaves a remainder of 1: $n = 5a + 2$
- n leaves a remainder of 1: $n = 5a + 3$
- n leaves a remainder of 1: $n = 5a + 4$

Show for each case above that n^2 divided by 5 leaves a remainder of 1 or 4.

The benefit of using **proof by cases** here is that we have **more information** about each specific case (the four bullet points above) than we would have about just some general n.

Proof methods and strategies

Example: Prove that if n is any integer not divisible by 5, then n^2 leaves a remainder of 1 or 4 when divided by 5.

Proof (by cases):

Case 1: S'pose $n/5$ leaves a remainder of 1: $n = 5a + 1$ (for some integer a)

Proof methods and strategies

Example: Prove that if n is any integer not divisible by 5, then n^2 leaves a remainder of 1 or 4 when divided by 5.

Proof (by cases):

Case 1: S'pose $n/5$ leaves a remainder of 1: $n = 5a + 1$ (for some integer a)

$\Rightarrow n^2 = (5a + 1)^2 = 25a^2 + 10a + 1 = 5(5a^2 + 2a) + 1$

$\Rightarrow n^2/5$ leaves a remainder of 1

Proof methods and strategies

Example: Prove that if n is any integer not divisible by 5, then n^2 leaves a remainder of 1 or 4 when divided by 5.

Proof (by cases):

Case 2: S'pose $n/5$ leaves a remainder of 2: $n = 5a + 2$ (for some integer a)

Proof methods and strategies

Example: Prove that if n is any integer not divisible by 5, then n^2 leaves a remainder of 1 or 4 when divided by 5.

Proof (by cases):

Case 2: S'pose $n/5$ leaves a remainder of 2: $n = 5a + 2$ (for some integer a)

$\Rightarrow n^2 = (5a + 2)^2 = 25a^2 + 20a + 4 = 5(5a^2 + 4a) + 4$

$\Rightarrow n^2/5$ leaves a remainder of 4

Proof methods and strategies

Example: Prove that if n is any integer not divisible by 5, then n^2 leaves a remainder of 1 or 4 when divided by 5.

Proof (by cases):

Case 3: $n = 5a + 3$

Case 4: $n = 5a + 4$

Proof methods and strategies

Example: Let's open the hood on the logic here.

Proof by cases logic: We're using the fact (which we still need to show) that

$$(p_1 \vee p_2 \vee p_3 \vee p_4) \rightarrow q \equiv (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge (p_3 \rightarrow q) \wedge (p_4 \rightarrow q)$$

cover all p

So let's prove this logical equivalence.

- By truth table ….nah, too long!
- Using equivalence relations

Proof methods and strategies

Show that:

$$(p_1 \vee p_2 \vee p_3 \vee p_4) \rightarrow q \;\equiv\; (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge (p_3 \rightarrow q) \wedge (p_4 \rightarrow q)$$

Proof methods and strategies

Show that:

$$(p_1 \vee p_2 \vee p_3 \vee p_4) \rightarrow q \;\equiv\; (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge (p_3 \rightarrow q) \wedge (p_4 \rightarrow q)$$

$(p_1 \vee p_2 \vee p_3 \vee p_4) \rightarrow q$

$\equiv \neg(p_1 \vee p_2 \vee p_3 \vee p_4) \vee q$ (RBI)

$\equiv (\neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge \neg p_4) \vee q$ (De Morgan)

$\equiv (\neg p_1 \vee q) \wedge (\neg p_2 \vee q) \wedge (\neg p_3 \vee q) \wedge (\neg p_4 \vee q)$ (distribution)

$\equiv (p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge (\neg p_3 \rightarrow q) \wedge (\neg p_4 \rightarrow q)$ (RBI)

□

Proof methods and strategies

FYOG: Use a proof by cases to show that for real numbers x and y, $\max(x, y) + \min(x, y) = x + y$.

Hint: You could use the cases: (1) $x \geq y$ and (2) $x < y$.

Note that you need one to be "or equal to" and the other to be strict inequality, otherwise there might be overlap between the two cases!

Proof methods and strategies

Example: Suppose you have two water jugs: one holds 5 gallons and the other holds 3 gallons. Assume you have an endless supply of water. Prove that an algorithm exists that allows you to measure out exactly 4 gallons of water just by transferring water between the two jugs (or pouring it down the drain, if that helps).

Think about it for a while!

Try to come up with an algorithm that will work.

Proof methods and strategies

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Proof:

1.
2.
3.
4.
5.
6.

Proof methods and strategies

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Proof:

1. Pour water in the 3G jug. Then add it to the 5G jug.
2. Fill the 3G jug again. The add it to the 5G jug (full). 1G remains in the 3G jug
3. Empty the 3G jug. Pour 3G from the 5G jug. 2G remain in the 5G jug.
4. Empty the 3G jug.
5. Pour the 2G from the 5G jug to the 3G jug. 5G jug is now empty
6. Fill the 5G jug. Pour 1G in the 3G jug. Now we have 4G in the 5G jug. QED

Proof methods and strategies

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Proof:

1. Pour 5G into the 5G jug.
2. Pour 3G. from the 5G jug into the 3G jug (leaving 2G in the 5G jug).
3. Pour the 3G in the 3G jug down the drain.
4. Pour the 2G from the 5G jug into the 3G jug.
5. Pour 5G into the 5G jug.
6. Pour 1G from the 5G jug into the 3G jug.

At this point, the 3G jug is full and **5G jug has 4G in it.** □

Proof methods and strategies

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5. Pour 5G into the 5G jug.
6. Pour 1G from the 5G jug into the 3G jug. At this point, the 3G jug is full and **5G jug has 4G in it.** □

Proved the **existence** of a solution to the problem by explicitly **constructing** it. Called a **proof by construction**.

Existence and uniqueness

Example: Show that if n is an odd integer, then there **exists** a **unique** integer k such that n is the sum of $k - 2$ and $k + 3$.

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This one is asking for two things:

1. Show that such an integer k **exists**.
2. Show that **there is only one** such k that does this.

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1. Show that such an integer k **exists**.
2. Show that **there is only one** such k that does this.

We typically tackle these **existence and uniqueness** proofs in two steps:

1) show existence by construction (i.e., actually find it).
2) show uniqueness by supposing that there are two such k, but then we do math and find out that they must be equal to each other

Existence and uniqueness

Example: Show that if n is an odd integer, then there **exists** a **unique** integer k such that n is the sum of k - 2 and k + 3.

1. Show that such an integer k **exists**.

Proof of existence:

- *Show* that such a k exists directly using our old friend, Algebra:

Existence and uniqueness

Example: Show that if n is an odd integer, then there **exists** a **unique** integer k such that n is the sum of $k - 2$ and $k + 3$.

1. Show that such an integer k **exists**.

Proof of existence: *Show* that such a k exists directly using our old friend, Algebra:

S'pose n is an odd integer $\Rightarrow n = 2a + 1$, for some integer a

$\Rightarrow n = 2a + 1 = (k - 2) + (k + 3) = 2k + 1$

$\Rightarrow k = a$

$\Rightarrow$ so to find our k for any given odd $n = 2a + 1$, **take $k = (n-1)/2$**

Existence and uniqueness

Example: Show that if n is an odd integer, then there **exists** a **unique** integer k such that n is the sum of k - 2 and k + 3.

1. Show that such an integer k **exists**.
2. Show that **there is only one** such k that does this.

Proof of uniqueness:

Existence and uniqueness

Example: Show that if n is an odd integer, then there **exists** a **unique** integer k such that n is the sum of $k - 2$ and $k + 3$.

1. Show that such an integer k **exists**.
2. Show that **there is only one** such k that does this.

Proof of uniqueness: S'pose two such numbers exist, k and m. That is:

$$n = (k - 2) + (k + 3) \quad \text{and} \quad n = (m - 2) + (m + 3)$$

$\Rightarrow n = (k - 2) + (k + 3) = (m - 2) + (m + 3)$

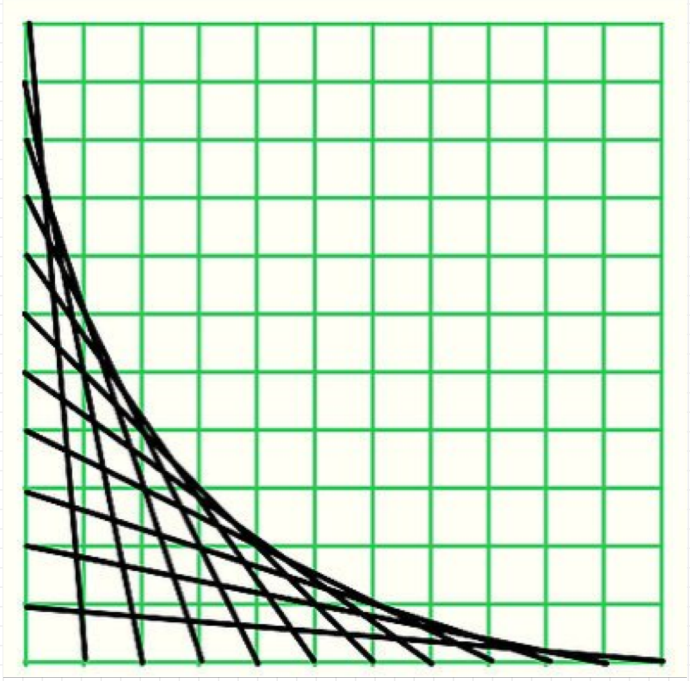
$\Rightarrow 2k + 1 = 2m + 1$

$\Rightarrow k = m$ $\Rightarrow$ Since any numbers that satisfy this problem are necessarily the same, the solution is **unique**. □

Existence and uniqueness

FYOG: Show that if *a, b* and *c* are real numbers with $a \neq 0$, then there **exists** a **unique** solution *x* to the equation $ax + b = c$.

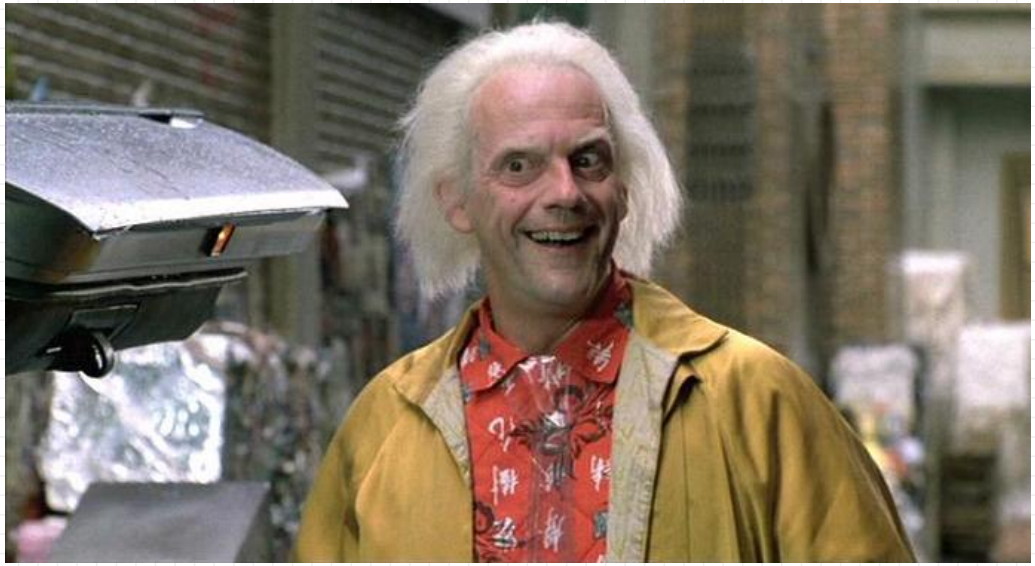
Note: This is the statement that non-horizontal lines pass through each y coordinate exactly once.



Conditional proof (specific kind of direct proof)

Example: S'pose that anyone who is mad is evil. Let the domain be all people. Prove that all mad scientists are evil.

What propositions do we need?



Conditional proof (specific kind of direct proof)

Example: S'pose that anyone who is mad is evil. Let the domain be all people. Prove that all mad scientists are evil.

Proof:

Let *S(x)* denote "*x* is a scientist"

Let *M(x)* denote "*x* is mad"

Let *E(x)* denote "*x* is evil"

How do we write the premise and the conclusion?

Conditional proof (specific kind of direct proof)

Example: S'pose that anyone who is mad is evil. Let the domain be all people. Prove that all mad scientists are evil.

	Step	Justification
1.	$\forall x\,(M(x) \rightarrow E(x))$	premise
	$\therefore \forall x\,[\,(M(x) \wedge S(x)) \rightarrow E(x)\,]$	

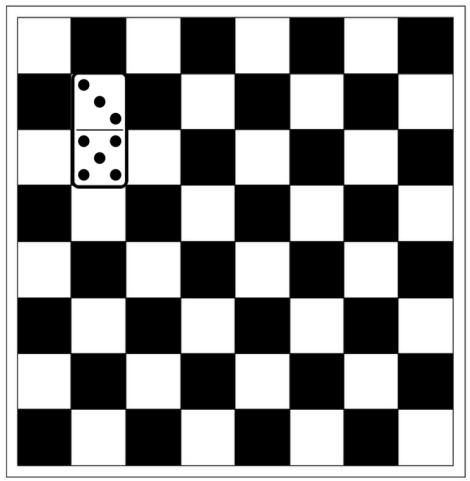
Conditional proof (specific kind of direct proof)

Example: S'pose that anyone who is mad is evil. Let the domain be all people. Prove that all mad scientists are evil.

	Step	Justification
1.	$\forall x\,(M(x) \rightarrow E(x))$	premise
2.	$M(a) \rightarrow E(a)$	universal instantiation (1) (arb. *a*)
3.	$M(a) \wedge S(a)$	**assumption for conditional proof**
4.	$M(a)$	simplification (3)
5.	$E(a)$	modus ponens (2), (4)
6.	$[\,(M(a) \wedge S(a)) \rightarrow E(a)\,]$	**by conditional proof (2-5)**
7.	$\therefore \forall x\,[\,(M(x) \wedge S(x)) \rightarrow E(x)\,]$	universal generalization (6)

Disproving all things (or: looking for counterexamples)

Example: Consider a standard 8x8 chessboard.



Can you completely cover the board in dominos that are the size of two squares?

Disproving all things (or: looking for counterexamples)

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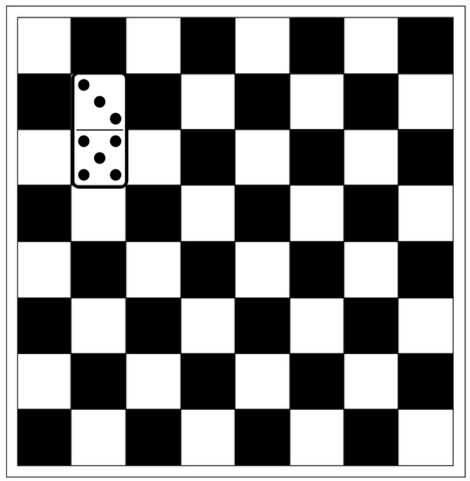
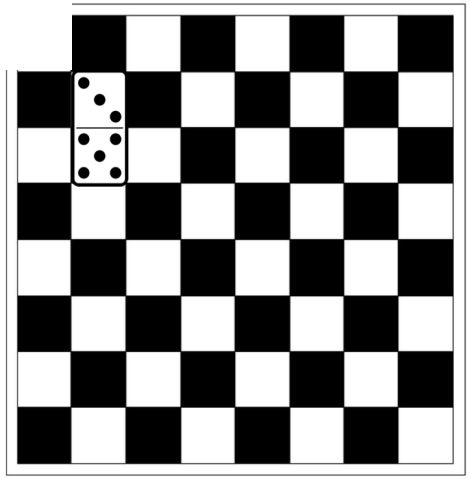


Can you completely cover the board in dominos that are the size of two squares?

⇒ **Yes.** There are **many** ways!

Disproving all things (or: looking for counterexamples)

Example: What about if we removed one of the corners?



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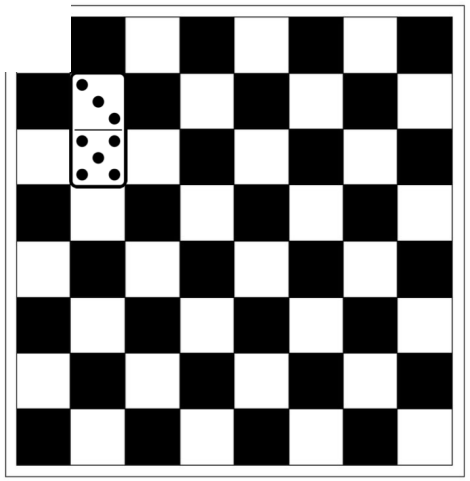
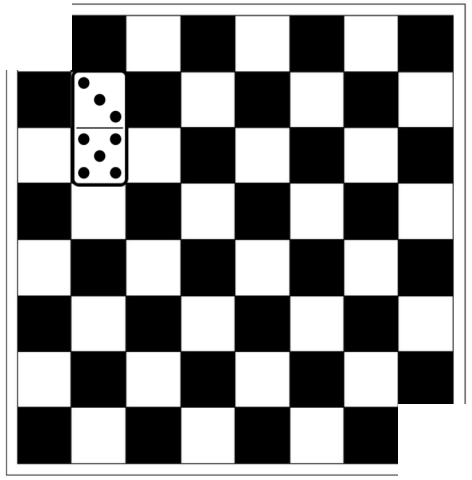


It might take a second ……. but of course we can't!

- The domino tiles have 2 squares each, so we can only **tile** an even number of squares
- But with only 1 square removed, the chess board now has an *odd* number of squares

Disproving all things (or: looking for counterexamples)

Example: What about if we removed the opposite corner as well?



Disproving all things (or: looking for counterexamples)

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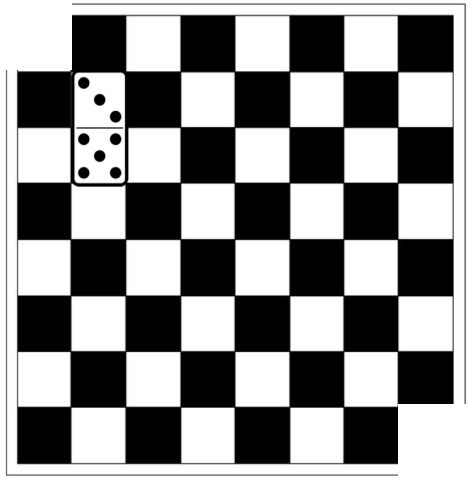

This one is also tricky. Note that each domino must cover both a white and a black square.

Disproving all things (or: looking for counterexamples)

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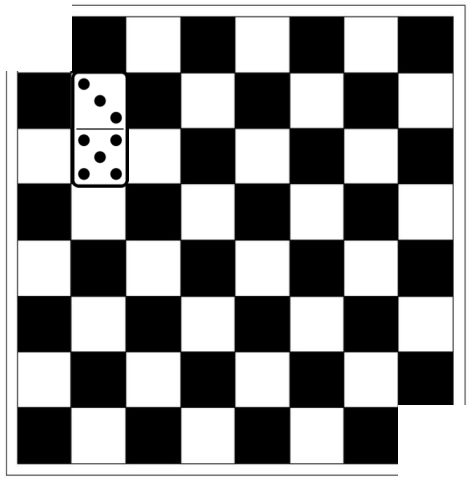

This one is also tricky. Note that each domino must cover both a white and a black square.

Nope! Because we have fewer white squares than black ones now, and each domino must cover both a white and a black square.

Proofs Methods and strategies - Recap

Recap: We've seen now:

- **Proof by cases** (exhaustive)
- **Proof by construction** (existence) and the de facto method for proving **uniqueness** (s'pose two of them exist and show they must be the same thing)

  - We're going to keep coming back to proofs, so don't purge it from your memory yet!

Next time: Sets

- We have talked about "the set of all integers" for example… but what does that actually mean?
- Could we make sets of arbitrary things? The set of all gray pants?

Extra Practice

Example 1: Show that if a, b and c are real numbers with $a \neq 0$, then there **exists** a **unique** solution x to the equation $ax + b = c$.

Note: This is the statement that non-horizontal lines pass through each y coordinate exactly once.

Example 2: Use a proof by cases to show that for real numbers x and y,

$$\max(x, y) + \min(x, y) = x + y.$$

Hint: You could use the cases: (1) $x \geq y$ and (2) $x < y$.

Note that you need one to be "or equal to" and the other to be strict inequality, otherwise there might be overlap between the two cases!

Solutions

Example 1: Show that if a, b, and c are real numbers with $a \neq 0$ then there **exists** a **unique** solution x to the equation $ax + b = c$

Existence: Solve for x

$$ax + b = c \Rightarrow x = \frac{b - c}{a}$$ (where here we know that $a \neq 0$)

Uniqueness: Assume x and y are both solutions to the system, then

$$ax + b = c = ay + b \Rightarrow ax + b = ay + b \Rightarrow ax = ay \Rightarrow x = y$$

Since x and y are necessarily the same number, it follows that our solution x is unique

Example 2: Prove that for real numbers x and y,

$$\max(x, y) + \min(x, y) = x + y$$

Case 1: Assume $x \geq y$. Then $\max(x, y) = x$ and $\min(x, y) = y$
(Here we realize that if $x = y$ then we can decide to choose either)

Thus $\max(x, y) + \min(x, y) = x + y$

Case 2: Assume $x < y$. Then $\max(x, y) = y$ and $\min(x, y) = x$

Thus $\max(x, y) + \min(x, y) = y + x = x + y$

Since the cases cover all possible combinations of x and y and both yield the conclusion, we are done.