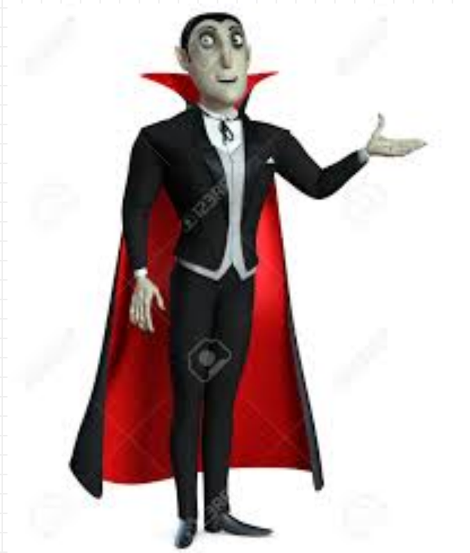
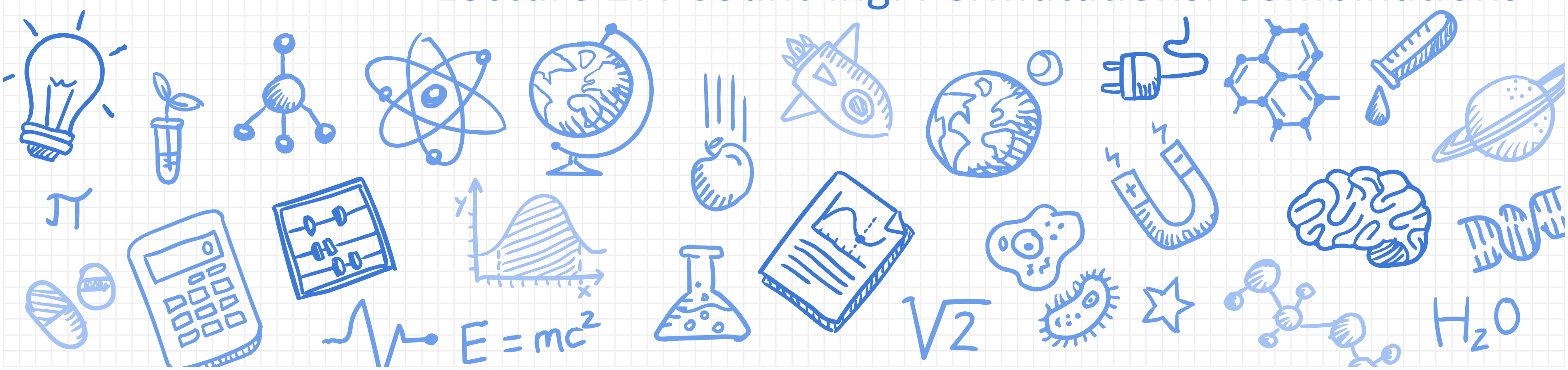


CSCI 2824: Discrete Structures

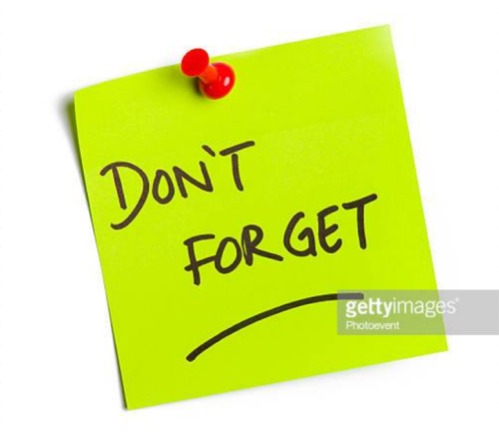
Lecture 27: Count-ing. Permutations. Combinations



Reminders

Submissions:

- Homework 9: **Fri 11/01 at noon -** Moodle



Readings:

- Ch. 6 – Counting

Midterm II

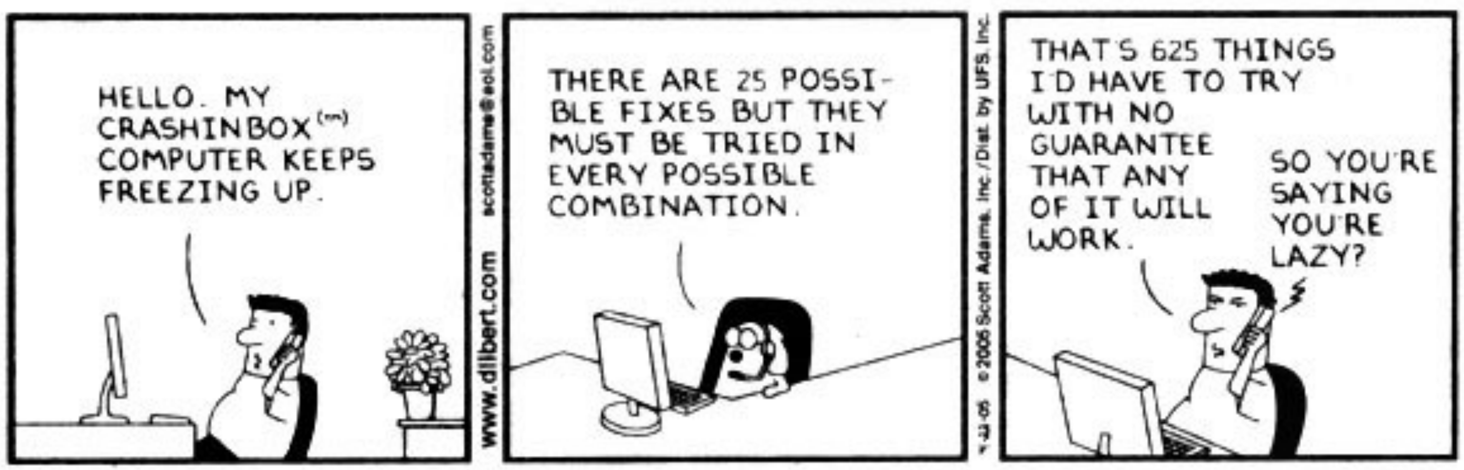
- Tuesday November 5th

Last time

- Counting
  - **Product Rule**
  - **Sum Rule**

Today:

- Permutations and Combinations. Pigeonhole Principle



Counting basics

Many of the combinatorial concepts we'll talk about depend on two simple rules:

The Product Rule. S'pose that a procedure can be broken down into two tasks. If there are n_1 ways to do the first task, and for each of these ways to do the first task, there are n_2 ways to do the second task, then there are $n_1 \times n_2$ ways to do the whole procedure.

Shorthand: Number of ways to do both *Task 1* **and** *Task 2*

The Sum Rule. If a goal can be achieved in one of n_1 ways using *Task 1* or in n_2 ways using *Task 2* (where none of the n_1 or n_2 ways are the same), then there are $n_1 + n_2$ ways total to achieve the goal.

Shorthand: Number of ways to do *Task 1* **or** *Task 2*

Counting basics

Example: How many different bit-strings of length 7 either begin with two 0s or end with three 1s?

Counting basics

Example: How many different bit-strings of length 7 either begin with two 0s or end with three 1s?

Begin with two 0s: 00xxxxx $\Rightarrow$ 5 tasks, 2 choices each $\Rightarrow$ 2^5 ways = 32

End with three 1s: xxxx111 $\Rightarrow$ 4 tasks, 2 choices each $\Rightarrow$ 2^4 ways = 16

But we need to subtract off the strings with **both** (the intersection):

Begin with two 0s **and** end with three 1s: 00xx111 $\Rightarrow$ 2^2 ways = 4

Total = 32 + 16 - 4 = **44**

FYOG: How many bitstrings of length 10 either begin with three 0s or end with two 0s?

Counting basics

Example: How many bit-strings of length 10 contain either five consecutive 0s or five consecutive 1s?

Counting basics

Example: How many bit-strings of length 10 contain either five consecutive 0s or five consecutive 1s?

6 positions that the 5 0s can start:

1. 00000xxxxx $\Rightarrow$ 2^5 choices
2. 100000xxxx $\Rightarrow$ 2^4
3. x100000xxx $\Rightarrow$ 2^4
4. xx100000xx $\Rightarrow$ 2^4
5. xxx100000x $\Rightarrow$ 2^4
6. xxxx100000 $\Rightarrow$ 2^4

Total from 0s $= 2^5 + 5 \cdot 2^4 =$ **112**

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Total from 0s $= 2^5 + 5 \cdot 2^4 =$ **112**

Same deal for the 1s: Total from 1s $=$ **112**

But: 2 bitstrings with both:

0000011111 and 1111100000

Subtract those off to get our total:

Total = 112 + 112 - 2 = **222**

Counting basics

Example: How many positive integers not exceeding 100 are divisible by both 4 and 6?

Counting basics

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There are **25** integers divisible by 4:

$$1 \times 4,\ 2 \times 4,\ 3 \times 4, \ldots, 25 \times 4 \quad \leftarrow \text{because } 25 \times 4 = 100$$

Counting basics

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… and there are **16** integers divisible by 6:

$1 \times 6, 2 \times 6, 3 \times 6, \ldots, 16 \times 6$ ← because $16 \times 6 = 96$

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… and there are **16** integers divisible by 6:

$1 \times 6, 2 \times 6, 3 \times 6, \ldots, 16 \times 6$ ← because $16 \times 6 = 96$

… and there **8** integers divisible by both 4 and 6:

$1 \times 12, 2 \times 12, \ldots, 8 \times 12$ ← because $8 \times 12 = 96$, and 12 is the LCM of 4 and 6

Total = 25 + 16 - 8 = **33**

Counting basics

FYOG: How many positive integers between 100 and 999 (inclusive):

a) are divisible by 7
b) are odd
c) have the same three decimal digits
d) are not divisible by 4
e) are divisible by 3 or 4
f) are not divisible by either 3 or 4
g) are divisible by 3 but not 4
h) are divisible by 3 and 4

FYOG: How many strings of three decimal digits:

a) do not contain the same digit three times
b) begin with an odd digit
c) have exactly two digits that are 4s

Pigeonhole principle

S'pose that a flock of 10 pigeons flies into a set of 9 pigeonholes to roost.

Because there are 10 pigeons but only 9 pigeonholes for them to go into, at least one of the pigeonholes must have at least two pigeons in it.

The Pigeonhole Principle: If k is a positive integer and k+1 or more objects are placed into k boxes, then there is at least one box containing two or mor of the objects.

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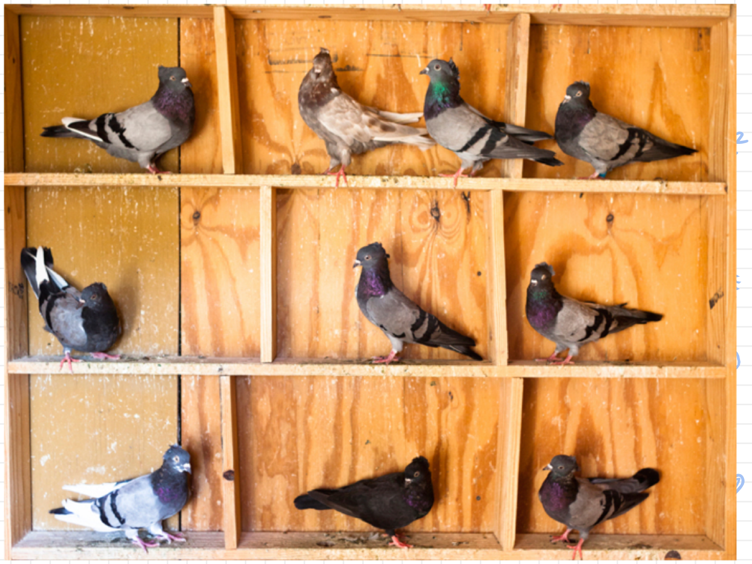
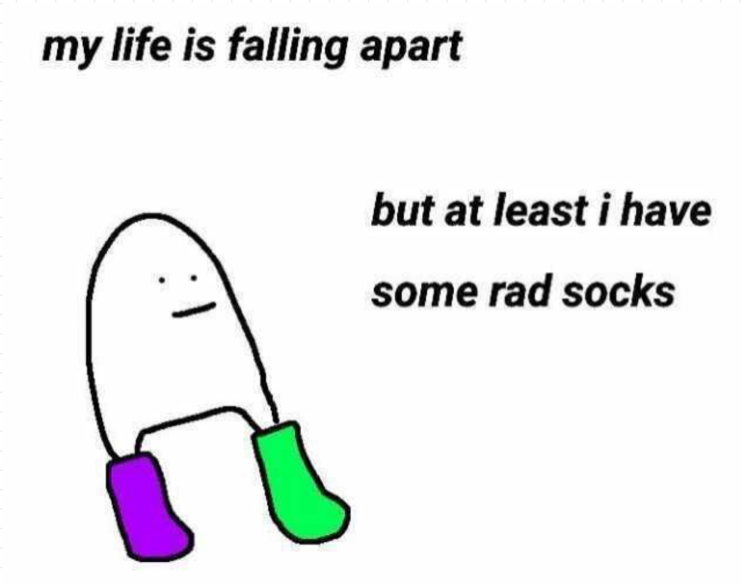
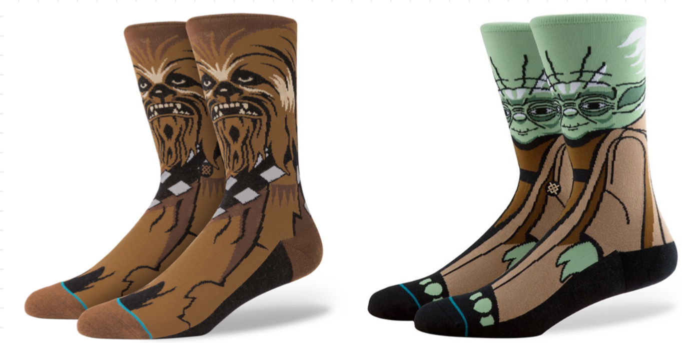


Pigeonhole principle

Example: A drawer contains a dozen brown socks and a dozen green socks, all unpaired. If you take out socks in the dark, how many must you grab to ensure that you have two socks that match?





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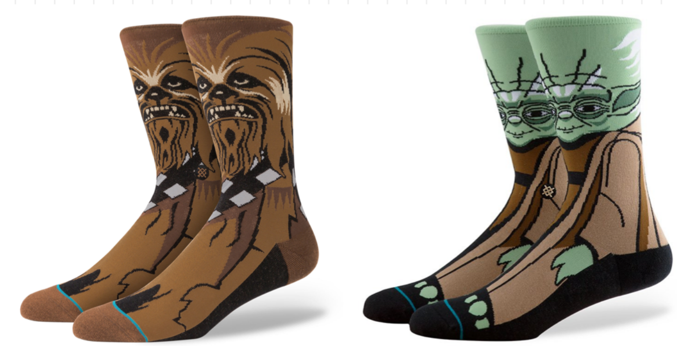
Answer: Since there are 2 bins (colors of socks), the Pigeonhole Principle tells us that we need to pull out 2+1 = **3 socks** in order to ensure that at least 2 of them are in the same bin.

Example: How many socks must we remove in order to ***guarantee*** that we get at least 2 green socks?



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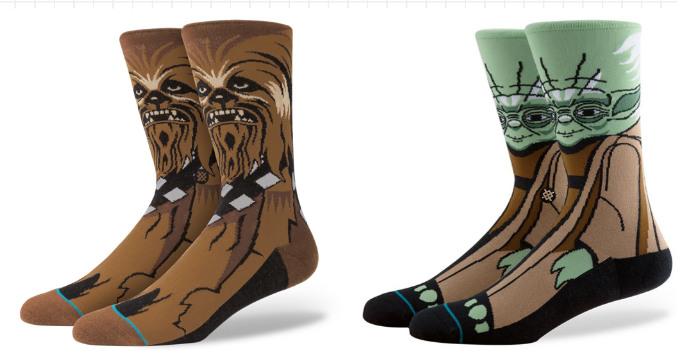
Example: How many socks must we remove in order to ***guarantee*** that we get at least 2 green socks?
Answer: It's possible that the first 12 socks we pull out of the drawer are all brown. So we need to pull out **14 socks** in order to **guarantee** 2 green ones.



Pigeonhole principle

Example: How many cards must be drawn from a standard deck of 52 cards to ensure that at least 3 of the cards are of the same suit?

Pigeonhole principle

Example: How many cards must be drawn from a standard deck of 52 cards to ensure that at least 3 of the cards are of the same suit?

Strategy: How can we distribute cards in such a way that *no 3 cards* are of the same suit.

⇒ We could have 2 clubs, 2 diamonds, 2 hearts and 2 spades

⇒ Then the next card we draw *must* fall into one of these four bins (suits)

⇒ **9 cards**

Pigeonhole principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Card example, rebooted: The 4 suits are the boxes.

⇒ If we have N=9 cards, then there is at least one box (suit) containing at least

$$\lceil 9/4 \rceil = \lceil 2.25 \rceil = 3 \text{ cards}$$

Pigeonhole principle

Example: The CU Boulder undergrad student enrollment is about 27,000. Show that there are at least 2 students with the same three initials. (Assume that everyone has three initials.)

Pigeonhole principle

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Solution:

There are $26 \times 26 \times 26 = 17{,}576$ possible combinations of initials

By the Generalized Pigeonhole Principle, there are at least

$$\lceil 27{,}000 / 17{,}576 \rceil = \lceil \text{more than 1, less than 2} \rceil = 2$$

students with the same three initials.

Pigeonhole principle

Example: Show that for any number *n* there is a multiple of *n* that has only 0s and 1s in its decimal representation.

Pigeonhole principle

Example: Show that for any number n there is a multiple of n that has only 0s and 1s in its decimal representation.

Solution: Consider the n+1 numbers 1, 11, 111, …, $111^{\cdots}11$, where the last number has n+1 ones in it.

Note that if we divide a number by n, there are only n possible remainders:

$0, 1, 2, \ldots, n-1$

Since there are n+1 numbers above but only n possible remainders, **by PHP** at least 2 of them must have the same remainder when divided by n.

⇒ Let those numbers be a and b, where $b > a$

⇒ $n \mid (b-a)$, which means that $b-a$ is a multiple of n

⇒ $b-a$ is a number made up of only 0s and 1s (e.g., b=111 and a=11 give $b-a$ = 100)

▨

Pigeonhole principle

Example: Show that for any number *n* there is a multiple of *n* that has only 0s and 1s in its decimal representation.

An example of the example: S'pose we let $n = 7$

We check: 1 11 111 1,111 11,111 111,111 1,111,111 11,111,111

Then (it turns out) 1,111,111 ≡ 1 **mod** 7 so 1 and 1,111,111 have the same remainder when divided by 7

⇒ 1,111,111 - 1 = 1,111,110 is divisible by 7

(1,111,110 / 7 = **158,730**)

```
In [5]: n=7
        numbers = [1]
        for i in range(1,n):
            numbers.append(10**i + numbers[i-1])

        remainders = []
        for i in range(0,n):
            remainders.append(numbers[i]%n)

In [6]: print(remainders)

[1, 4, 6, 5, 2, 0, 1]
```

Pigeonhole principle

FYOG: Show that if a function f maps elements from a set of k+1 or more elements to a set with k elements, then f is not one-to-one.

FYOG: Show that if a **function** f maps elements from a set of k elements to a set with k+1 or more elements, then f is not onto.

FYOG: Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

FYOG: What is the minimum number of students, each of whom comes from one of the 50 U.S. states, who must be enrolled in a university in order to guarantee that there are at least 100 who come from the same state?

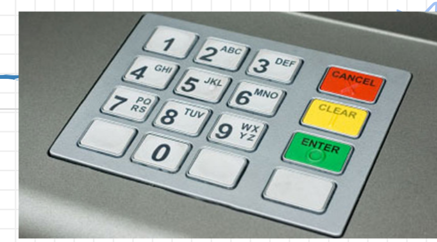
FYOG: There are 38 different time slots during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

Permutations and combinations

Example: How many 4-digit PIN combinations are there

(10 choices for first digit) $\times$ (10 choices for second digit) $\times$ (10 …) $\times$ (10 …) = **10,000**

Permutations and combinations



Example: How many 4-digit PIN combinations are there?

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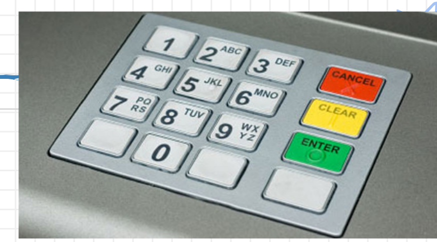
Digits 0-9 like 10 distinct items in a bag

- Pick first PIN digit: pull a number out of the bag
- Then put digit back into bag. Do this 4 times (digits)

Permutations and combinations



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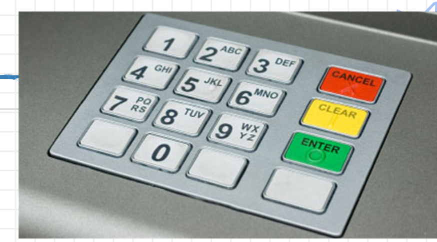
What if we picked from the bag **without replacement** and (still) **care about their order**?

What if we picked from the bag **without replacement** and **don't care about order**?

Permutations and combinations



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⇒ ***permutations***

What if we picked from the bag **without replacement** and **don't care about order**?
⇒ ***combinations***

Permutations

When selecting *r* distinct items without replacing them, we call the possible selection an ***r*-permutation**

Example: Find all 2-permutations of the set $S = \{a, b, c\}$

Permutations

When selecting *r* distinct items without replacing them, we call the possible selection an ***r*-permutation**

Example: Find all 2-permutations of the set $S = \{a, b, c\}$

ab, ac, ba, bc, ca, cb

Note that there are $3 \times 2 = 6$ such 2-permutations, and the list contains both *ab* and *ba* (for example) because **order matters**

Permutations

Example: How many three-character strings can we make if each character is a distinct lowercase letter?

Permutations

Example: How many three-character strings can we make if each character is a distinct lowercase letter?

Solution: 26 choices for the first letter, 25 choices for second, 24 choices for third

$\Rightarrow$ $26 \times 25 \times 24 =$ **15,600 strings**

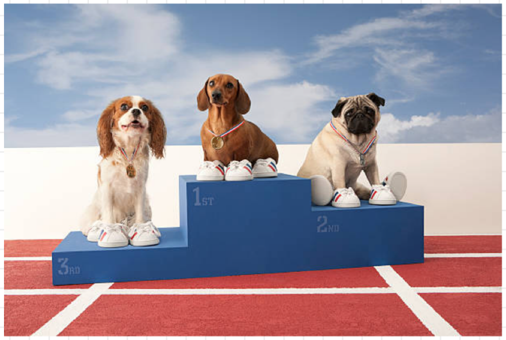
Theorem: If n is a positive integer and r is an integer such that $0 \le r \le n$, then the number of r-permutations from a set of size n is

$$P(n, r) = \frac{n!}{(n-r)!} = n\,(n-1)\,(n-2) \ldots (n-r+1)$$

Special cases: $P(n, 0) = 1$ and $P(n, n) = n!$

Permutations

Example: How many different 1st-2nd-3rd place permutations can occur in a race with 5 runners?



Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC*?

Think of *ABC* as a block, a single character. In addition to *ABC*, there are 5 other characters, for a total of 6 discrete items.

Permutations

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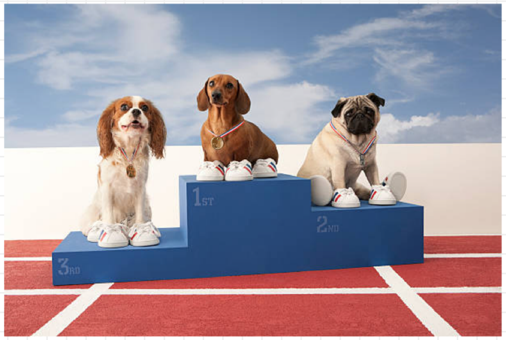

Solution: Let's use the factorial formula for $P(5,3)$...

$$P(5, 3) = 5! \,/\, (5-3)! = 5! \,/\, 2! = 120 \,/\, 2 = \mathbf{60}$$

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC*?

Think of *ABC* as a block, a single character. In addition to *ABC*, there are 5 other characters, for a total of 6 discrete items.

$\Rightarrow$ Total number of permutations is $P(6, 6) = 6! = \mathbf{720}$

Permutations

Example: How many ways are there to arrange *n* men and *n* women in a row so that the men and women alternate?

Permutations

Example: How many ways are there to arrange n men and n women in a row so that the men and women alternate?

Note that we can arrange the men and women separately:

There are $P(n, n) = n!$ permutations of the men,

and $P(n, n) = n!$ permutations of the women

Permutations

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and $P(n, n) = n!$ permutations of the women

… *and* for each ordering, we can start the row off with either a man or a woman

$\Rightarrow$ This gives $2 \times n! \times n! = \mathbf{2(n!)^2}$ **permutations**

Pigeonhole principle and permutations

Recap:

- **Generalized Pigeonhole Principle** -- If you want to sort n objects into k bins, at least one of the bins will contain $\lceil n/k \rceil$ objects.
- **Permutations** -- $P(n, k)$ = the number of ways to pick k objects from n objects, without replacement, when **order matters**
- n^k = the number of ways to pick k objects from n objects, **with replacement**, when order does not matter

Next time:

- More permutations and **combinations**, and the **Binomial Theorem!**
- More ***counting***, and moving toward ***probability!***