CSCI 2824: Discrete Structures

Lecture 16: Functions. Cardinality.

Reminders

Submissions:

- Homework 5: **Mon 10/7 at noon** – 1 try on Moodle
- Homework 6: **Fri 10/11 at noon** – Gradescope
- **Quizlet 5: due Friday 10/4 at 8pm**

Readings:

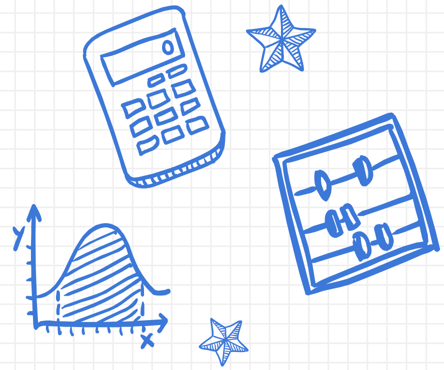
- This week Ch. 2 – SETS (2.3-2.4)

What did we do last time?

- functions with sets: definitions, domani, codomain, range
- one-to-one (injective)
- onto (surjective)
- bijective

Today:

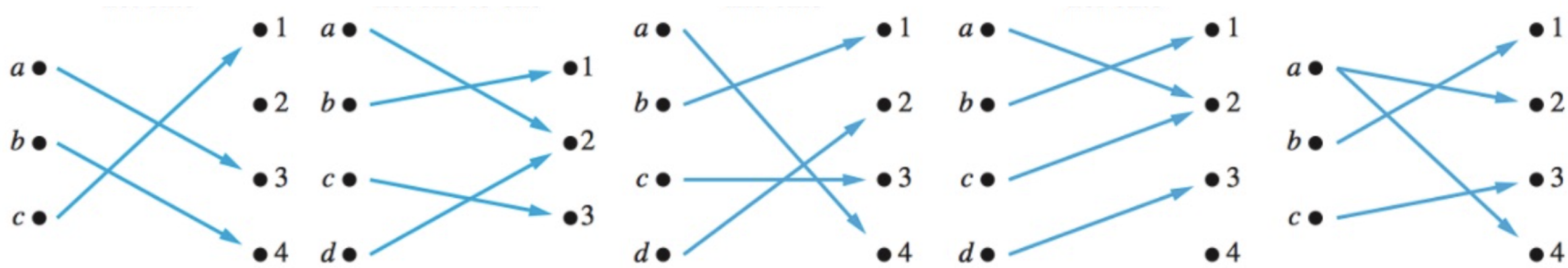
- Special properties functions can have – cont.
- Should we fear them?



Functions

One-to-one and *Onto* Functions

Example: Classify these functions as one-to-one (1-1), onto, both or neither.



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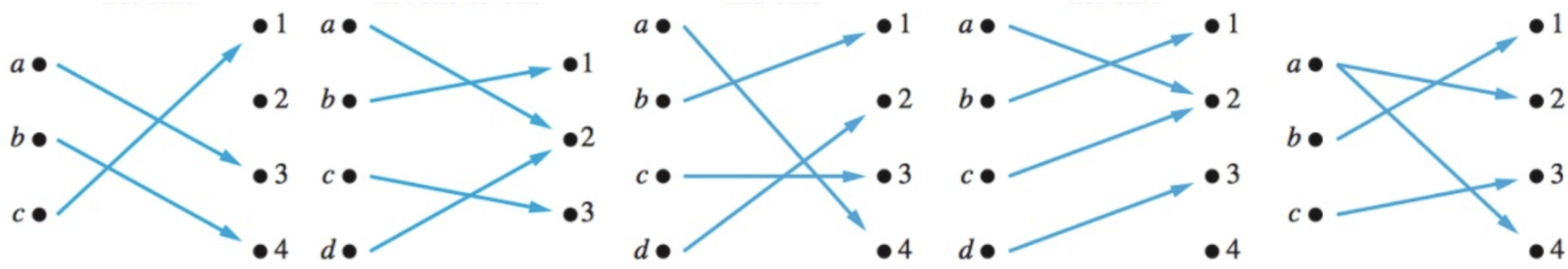

1-1	not 1-1	1-1	not 1-1	not a function
not onto	onto	onto	not onto	

Inverse Functions

Nice things happen when a function is both 1-1 and onto.

- This is called a ***bijective*** function.

  The function can be called a ***bijection*** (special kind of function that is 1-1 and onto).

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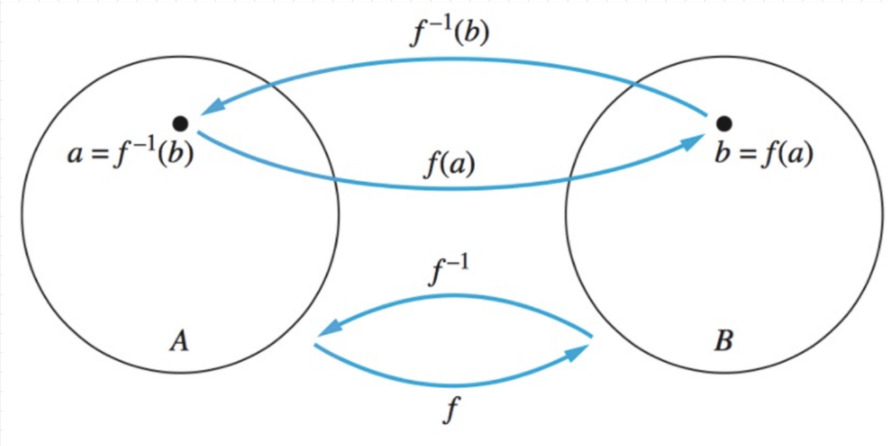
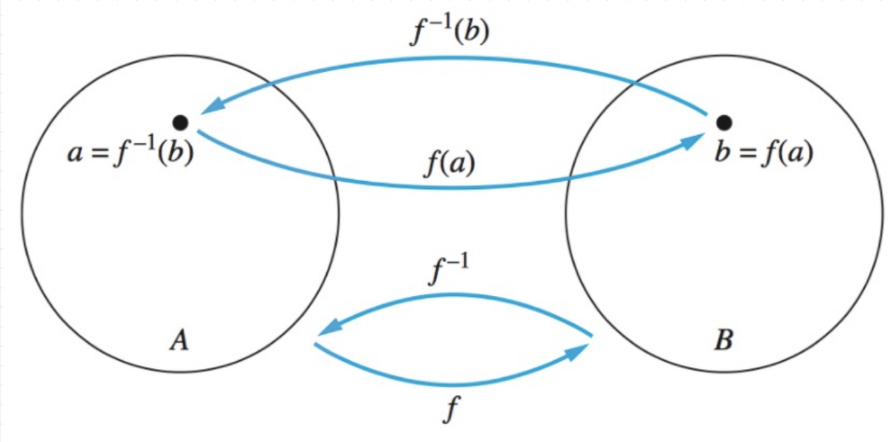


Inverse Functions

The function can be called a ***bijection*** (special kind of function that is 1-1 and onto).



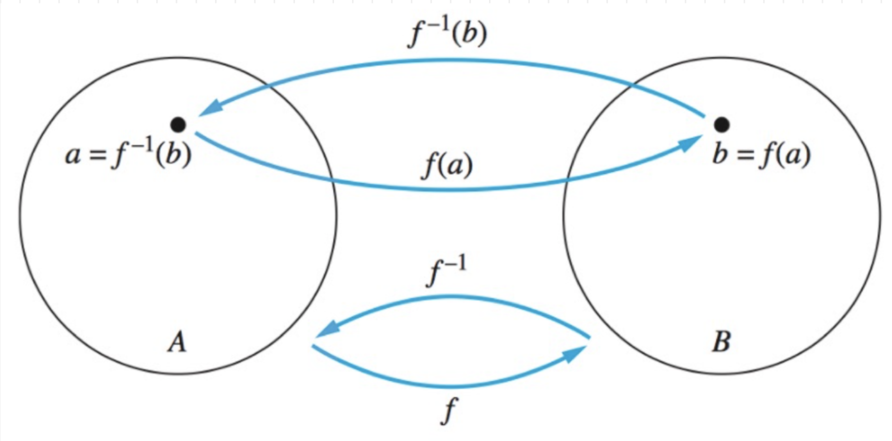
If $f : A \to B$ is 1-1 and onto, then:

- f maps to each of the elements of B (because f is onto)
- But f is 1-1 as well, so each element of B has a unique element in A that maps to it.

Inverse Functions

This implies that there is a unique **one-to-one correspondence** between elements in *A* and elements in *B*.

When this happens, we can go back and forth between *A* and *B* via *f* and an **inverse** function f^{-1}



Definition: Let *f* be a 1-1 and onto function from *A* to *B*. Then there exists an **inverse function**, f^{-1}, such that $f^{-1}(b) = a$ when $f(a) = b$.

Inverse Functions

Example: The inverse of $f(x) = x^3$, where $f : \mathbf{R} \to \mathbf{R}$ is $f^{-1}(y) = y^{1/3}$.

- f cubes stuff, and f^{-1} "un-cubes" stuff.

Example: $f(\mathrm{x}) = x^2$ where $f : \mathbf{R} \to \mathbf{R}$ does $\boldsymbol{f}$ have an inverse?

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Example: $f(\mathrm{x}) = x^2$ where $f : \mathbf{R} \to \mathbf{R}$ does ***not*** have an inverse.

- One reason: f is not 1-1. e.g., $f(-2) = f(2) = 4$
- Another reason: f is not onto. e.g., there is no x in the domain s.t. $f(\mathrm{x}) = -1$

Question: could we redefine f such that it does have an inverse?

Inverse Functions

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Question: could we redefine f such that it does have an inverse?

Answer: sure!

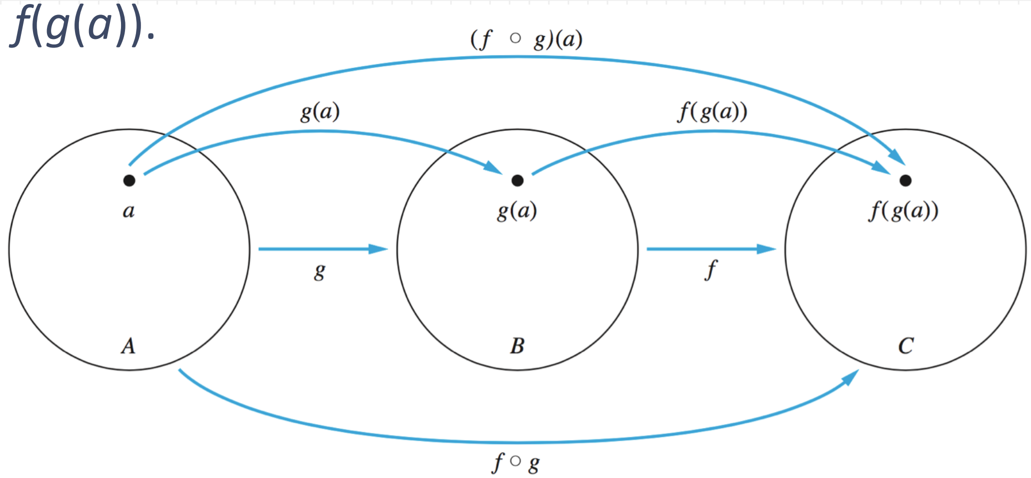
- The problem was those pesky negative numbers. So redefine

$f : (\mathbf{R} \geq 0) \rightarrow (\mathbf{R} \geq 0)$

- Then $f^{-1}(y) = y^{1/2}$

Composition of Functions

Definition: Let g be a function from set A to set B, and let f be a function from set B to set C. The **composition** of f and g, denoted $f \circ g$, is defined for $a \in A$ by $(f \circ g)(a) = f(g(a))$.



Example: What are the domain, codomain and range of $(f \circ g)(a)$?

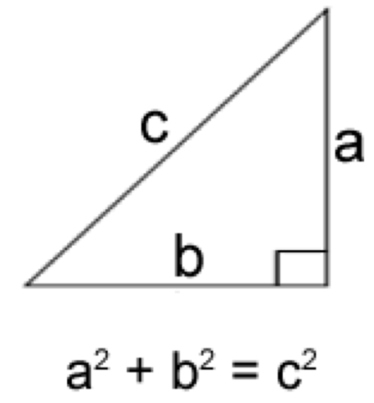
- Domain = domain of g Codomain = codomain of f
- Range = range of f if you use the range of g as the domain of f???

Composition of Functions

We use composition of functions all the time in the wild.

Example: $c = \sqrt{a^2 + b^2}$

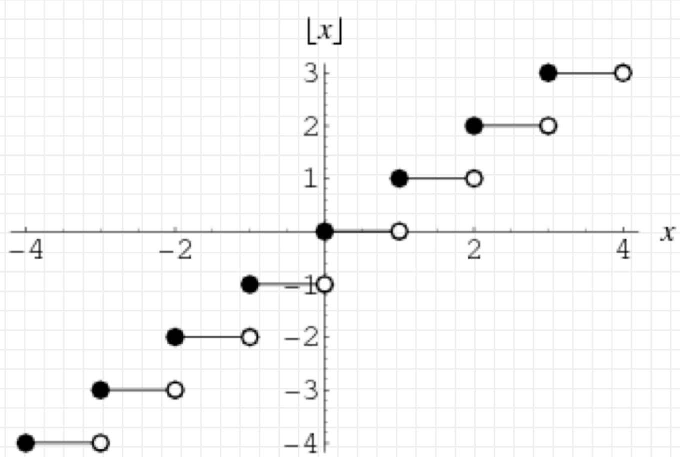
```
In [14]: # a function for adding
    ...: def Add(x_in, y_in):
    ...:
    ...:     sum_out = x_in + y_in
    ...:
    ...:     return (sum_out)
    ...:
    ...: # a function for squaring
    ...: def Square(x_in):
    ...:
    ...:     x2_out = x_in * x_in
    ...:
    ...:     return (x2_out)
    ...: c = pow( Add( Square(3), Square(4) ) , 0.5)
    ...: print(c)
    ...:
5.0
```

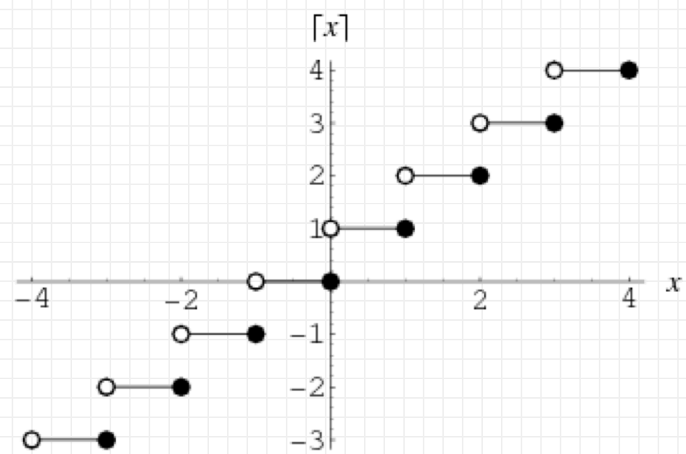


Some important functions

Definition: The **floor** function, denoted $\lfloor x \rfloor$, assigns to the real number x the largest integer that is less than or equal to x. The **ceiling** function, denoted $\lceil x \rceil$, assigns to the real number x the smallest integer that is greater than or equal to x.

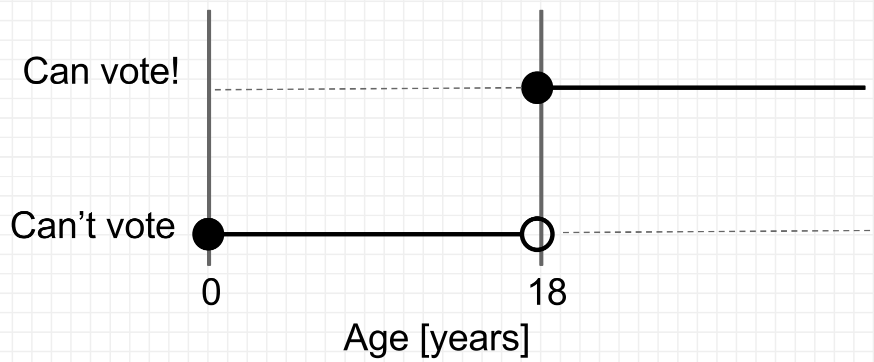
Remark: Both of these come up frequently in CS applications.





Some important functions

Example: Being able to vote follows a floor function:



More examples:

$\lfloor 3.5 \rfloor = 3$ $\lfloor 5 \rfloor = 5$ $\lfloor -3.5 \rfloor = -4$

$\lceil 3.5 \rceil = 4$ $\lceil 5 \rceil = 5$ $\lceil -3.5 \rceil = -3$

Some important functions

Example: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

Some important functions

Example: Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

Turns out, this is *not* true. We just need to find a single counterexample that breaks it.

(Dis)proof:

- Note that each of $\lceil x \rceil$ and $\lceil y \rceil$ will round up to x+1 and y+1 if x and y are just a tiny bit larger than an integer.
- But $\lceil x + y \rceil$ will only round up once, to x+y+1, if x and y are only a tiny bit larger
- So try: $x = 1.1$ and $y = 2.1$
- Then: $\lceil x + y \rceil = \lceil 1.1 + 2.1 \rceil = \lceil 3.2 \rceil = 4$

  But $\lceil x \rceil + \lceil y \rceil = \lceil 1.1 \rceil + \lceil 2.1 \rceil = 2 + 3 = 5.$ $\square$

Countable and uncountable sets

So $|\mathbf{N}| = |\mathbf{Z}| = |\mathbf{Q}| = |\mathbf{R}| = \infty$, and we're done, right?

Wrong!

- Turns out, it is useful to break these up into cases of ***how infinite*** a set is.
- These are described roughly as follows:
  - **Countably infinite** (or **countable**): We could count up each member of the set if we had infinite time.
  - **Uncountably infinite** (or **uncountable**): We could never count even list each element of the set, even if we had infinite time.

Definition: A set A is called **countable** or **countably infinite** if it is not finite and there is a one-to-one function between each element of A and the natural numbers. A is called **uncountable** if it is infinite and not countable. (Finite sets are **countable**.)

Countable and uncountable sets

Example: Show that the set of positive even integers is countably infinite.

We need to find a one-to-one map

from the positive even integers **E** = {2, 4, 6, 8, …}

to the natural numbers **N** = {0, 1, 2, 3, …}

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to the natural numbers **N** = {0, 1, 2, 3, …}

Well, if we divide all the elements in **E** by 2, we get {1, 2, 3, 4, …}

… and notice that those are each a corresponding element of **N**, but + 1

So we could establish a pattern as:

N		**E**
0	⇔	2
1	⇔	4
2	⇔	6
	…	

Or define a relationship:

$f(n) = 2(n + 1)$, where $n \in$ **N**, and f maps from **N** to **E**

$g(m) = m/2 - 1$, where $m \in$ **E**, and g maps from **E** to **N**

Countable and uncountable sets

Example: Show that the set of all integers is countable.

Need a map from the set of natural numbers {0, 1, 2, 3, …}

to the set of all integers { …, -2, -1, 0, 1, 2, … }

Countable and uncountable sets

Example: Show that the set of all integers is countable.

Need a map from the set of natural numbers {0, 1, 2, 3, …}

to the set of all integers { …, -2, -1, 0, 1, 2, … }

The natural way to line them up might be:

N		**Z**
0	⇔	0
1	⇔	1
2	⇔	-1
3	⇔	2
4	⇔	-2
5	⇔	3
	…	

Does this give rise to a relationship $f(n)$ we can define?

Yes! Easiest to break into cases:

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

Countable and uncountable sets

Example: Show that the set of positive rational numbers is countable.

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$$\begin{array}{ccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\[1em]
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\[1em]
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
 & \vdots & & &
\end{array}$$

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$p + q = 2$

$p + q = 3$

$p + q = 4$

Countable and uncountable sets

Example: Show that the set of real numbers is **un**countable.

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Note: intervals like [0, 1] or (0, 1) are **always** assumed to be subsets of $\mathbb{R}$ unless otherwise stated.

Countable and uncountable sets

Example: Show that the set of real numbers is **un**countable.
Let's just look at the interval [0, 1]

Note: intervals like [0, 1] or (0, 1) are **always** assumed to be subsets of $\mathbb{R}$ unless otherwise stated.

We will try for a **proof by contradiction**.

So s'pose we have a list of all of the real numbers between 0 and 1.

0.	4	8	7	5	4	6	9	0	1	…
0.	9	1	9	6	5	5	3	4	8	…
0.	0	9	7	4	3	3	7	9	9	…
0.	1	2	3	4	5	6	7	8	9	…
0.	2	5	3	0	0	4	2	1	7	…

(and so on…)

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0.	0	9	7	4	3	3	7	9	9	…
0.	1	2	3	4	5	6	7	8	9	…
0.	2	5	3	0	0	4	2	1	7	…

(and so on…)

Let's **construct** a number m that **can't be on the list**.
Strategy: Let the k^{th} digit of m depend on the k^{th} digit of the k^{th} element in our list, according to a rule that ensures m will be different from the k^{th} list element.

Countable and uncountable sets

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0.	1	2	3	4	5	6	7	8	9	…
0.	2	5	3	0	0	4	2	1	7	…

(and so on…)

Rule: - If the k^{th} digit of the k^{th} digit of the k^{th} number in our list is a 1, then the k^{th} digit of m is a 9.
- If the k^{th} digit of the k^{th} digit of the k^{th} number is not a 1, then the k^{th} digit of m is a 1.

Countable and uncountable sets

Example: Show that the set of real numbers is uncountable.

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0.	9	**1**	9	6	5	5	3	4	8	…
0.	0	9	**7**	4	3	3	7	9	9	…
0.	1	2	3	**4**	5	6	7	8	9	…
0.	2	5	3	0	**0**	4	2	1	7	…

(and so on…)

m = 0. 1 9 1 1 1 …

Rule: - If the k^{th} digit of the k^{th} digit of the k^{th} number in our list is a 1, then the k^{th} digit of m is a 9.

- If the k^{th} digit of the k^{th} digit of the k^{th} number is not a 1, then the k^{th} digit of m is a 1.

Countable and uncountable sets

Claim: Our constructed number *m* can't already be on the list.

Argument: 1. *m* is not the 1st number because their 1st digits are different.

2. *m* is not the 2nd number because their 2nd digits are different.

3. *m* is not the 3rd number because their 3rd digits are different.

(and so on…)

$\Rightarrow$ our number *m* can't be on the list.

$\Rightarrow$ contradicts assumption that the real numbers between 0 and 1 are countable.

$\Rightarrow$ Thus, the real numbers between 0 and 1 are uncountable.

$\Rightarrow$ ***All*** real numbers are uncountable. □

(this proof is called **Cantor's diagonal argument**)

Countable and uncountable sets

A non-exhaustive summary:

Countable	**Uncountable**
N (natural numbers)	[0, 1] (or **any** interval of **R**)
Z (integers)	**R**
$\mathbb{Q}$ (the rational numbers)	**R**-$\mathbb{Q}$ (the irrational numbers)
$\{1.234, \pi\}$ (**finite** sets of anything)	

Functions

Recap: We learned about ***functions***…

- What are they?
- Special kinds of functions (floor, ceiling, inverse, composition…)
- Special properties functions can have (onto, 1-1…)
- How we can use functions to see how large a **set** is…

Next time:

- We learn about ***sequences!***

… no, not that

Extra Practice

Some important functions

- **Example:** Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + ½ \rfloor$

- **Typical strategy:** (1) Write x as $x = n + \epsilon$, where n is an integer and $0 \le \epsilon < 1$.
  (2) Plug in stuff and "do math"
  (3) Wealth and fame!

- **Proof:** We notice that depending on what ϵ is, the $\lfloor x + ½ \rfloor$ term might do different things.
  $\Rightarrow$ so let's split this up into two cases:

  (i) $0 \le \epsilon < ½$

  (ii) $½ \le \epsilon < 1$

Some important functions

- **Example:** Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + ½ \rfloor$

Proof (cont.)

- **Case (i):** S'pose $x = n + \epsilon$, where $0 \le \epsilon < ½$
- Then $\lfloor 2x \rfloor = \lfloor 2(n + \epsilon) \rfloor = \lfloor 2n + 2\epsilon \rfloor$

  And we know that $0 \le 2\epsilon < 1$

  $\Rightarrow \lfloor 2x \rfloor = \lfloor 2n + 2\epsilon \rfloor = \lfloor 2n \rfloor = 2n$ *(on the left-hand side)*
- Similarly, $\lfloor x \rfloor + \lfloor x + ½ \rfloor = \lfloor n + \epsilon \rfloor + \lfloor n + \epsilon + ½ \rfloor = \lfloor n \rfloor + \lfloor n \rfloor = n + n = 2n$
- Since we proved both the LHS and the RHS = $2n$, we've proved they are equal

$0 \le \epsilon < ½$, so $0 \le \epsilon + ½ < 1$

Some important functions

- **Example:** Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + ½ \rfloor$

Proof (cont.)

- **Case (ii):** S'pose $x = n + \epsilon$, where $½ \leq \epsilon < 1$
  - Then $\lfloor 2x \rfloor = \lfloor 2(n + \epsilon) \rfloor = \lfloor 2n + 2\epsilon \rfloor$

    And we know that $1 \leq 2\epsilon < 2$

    $\Rightarrow \lfloor 2x \rfloor = \lfloor 2n + 2\epsilon \rfloor = \lfloor 2n + 1 \rfloor = 2n + 1$
  - Similarly, $\lfloor x \rfloor + \lfloor x + ½ \rfloor = \lfloor n + \epsilon \rfloor + \lfloor n + \epsilon + ½ \rfloor = \lfloor n \rfloor + \lfloor n+1 \rfloor = n + n + 1 = 2n + 1$
  - Since we proved both the LHS and the RHS = $2n$+1, we've proved they are equal

$½ \leq \epsilon < 1$, so $1 \leq \epsilon+½ < 3/2$

$\square$