CSCI 2824: Discrete Structures

Lecture 20: Weak Induction

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HW7 Due Friday at Noon

Quizlet 07 Due Friday at 8 PM

Weak Induction

Suppose we have an infinite line of dominoes.

Prove that if we tip over the first domino, the rest of them will fall.

Argument could go like this:
Base Case: The first domino falls (because we knock it over)
Induction Step: Whenever the kth domino falls, then its successor k+1 also falls.
Therefore, we conclude that all the dominoes will fall.

Weak Induction

This argument is an example of **induction**!

To prove a property over all natural numbers k, we may argue as follows:

Base case

The property is true for *k=0* (or *k=1*, etc).

If the property is true for some natural number *k*, then it is true for natural number *k+1*

induction step: $S(k) \to S(k+1)$

Weak Induction

Example: Prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \geq 1$.

Base Case: $n = 1$ Does this formula hold for the base case?

$$1 = \frac{1(1+1)}{2}$$

$$1 = \frac{1 \cdot 2}{2}$$

$$1 = 1 \quad \checkmark$$

Induction Step: For some $K \geq 1$, we assume that

$$1 + 2 + 3 + \cdots + K = \frac{K(K+1)}{2}.$$

This is the induction hypothesis.

The next step is to show that the formula is true for $K+1$.

Weak Induction

$k+1$

$$\frac{(k+1)(k+1+1)}{2}$$

Example(continued): Prove that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

By IH

$$\underbrace{1+2+3+4+\cdots+(k-1)+k}+(k+1) = \underbrace{\frac{k(k+1)}{2}} + k+1$$

$$= \frac{k(k+1)}{2} + \frac{(k+1)\cdot 2}{2}$$

$$= \frac{k(k+1)+2k+2}{2}$$

$$= \frac{k^2+k+2k+2}{2}$$

$$= \frac{k^2+3k+2}{2}$$

$$= (k+1)(k+2)/2$$

Thus, by weak induction the sum of the first n integers is $n(n+1)/2$ for all $n \geq 1$.

Mathematical Induction

Let $P(n)$ be the property that we're trying to prove.

An inductive argument goes as follows:

Base Case: Verify that $P(0)$ holds

Induction Step: $(\forall k \geq 0)$ If $P(k)$, then $\mathrm{P}(\mathrm{k}+1)$

$P(k) \rightarrow P(k+1)$

Conclusion: $(\forall n \geq 0)$ $P(n)$

Direct proof!

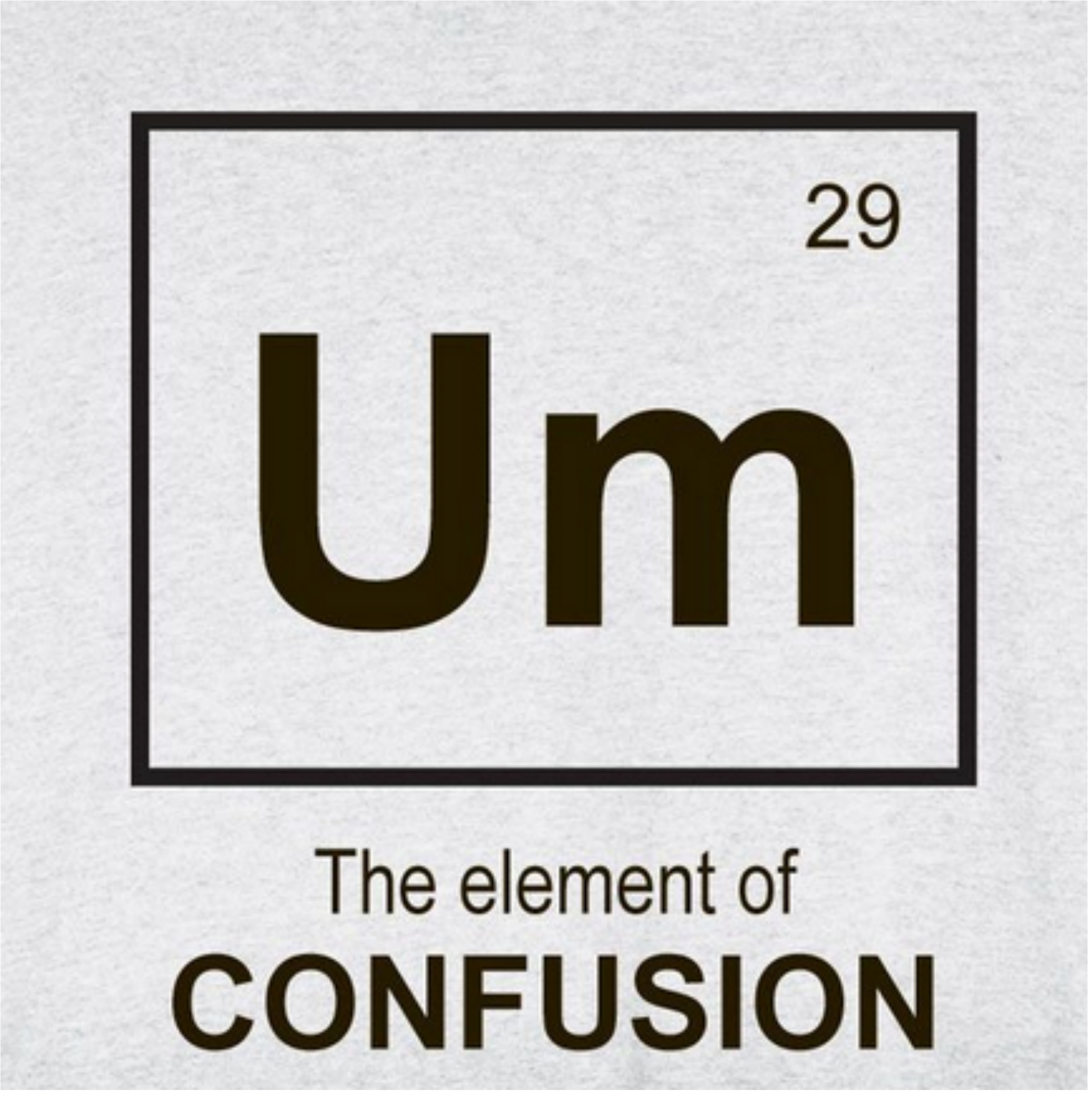
Mathematical Induction

Weak Induction:

- Verify that $P(1)$ is true.
- Assume $P(k)$ is true and show that $P(k+1)$ is true.

Strong Induction:

- Verify that $P(1)$ is true.
- Assume $P(k)$ for all $k = 1, 2, \dots, n$ and show $P(n+1)$



Mathematical Induction

In both strong and weak induction, you must prove that the first domino in the line falls. i.e. the first logical proposition is true – base case.

Analogy 1: To prove that ALL the other dominoes fall, you either show why (1) each falling domino by itself causes the next one to fall, or (2) all the dominoes that have fallen up to a point will cause the next one to fall. Tactic 1 is called weak induction, tactic 2 is called strong induction.

Analogy 2: Weak induction only cares about the ladder rung you are currently standing on. As long as that one exists, you know that you can step up to the next rung. For strong induction, you need all the previous rungs to still exist before you can safely move up to the next rung.

Weak Induction

Example: Propose a formula for the sum of the first n odd positive integers. Then prove it with induction.

$1 = 1 = 1^2$

$1 + 3 = 4 = 2^2$

$1 + 3 + 5 = 9 = 3^2$

$1 + 3 + 5 + 7 = 16 = 4^2$

$1 + 3 + 5 + 7 + 9 = 25 = 5^2$

$1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$

$\vdots$

Conjecture: The sum of the first n odd positive integers is n^2.

$$1 + 3 + 5 + \cdots + 2n - 1 = n^2$$

Weak Induction

Example (continued): Propose a formula for the sum of the first n odd positive integers. Then prove it with induction.

$1+3+5+\cdots+2n-1 = n^2.$

Base Case: $n=1$ $\quad 1 = 1^2$ ✓

Induction Step: For some $k \geq 1$, $\underbrace{1+3+5+\cdots+2k-1 = k^2}_{\text{I.H.}}$.

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1) = k^2 + (2(k+1)-1) \quad \text{By I.H}$$

$$= k^2+2k+2-1$$

$$= k^2+2k+1$$

$$= (k+1)^2$$

Thus by weak induction, we've proven our conjecture.

Weak Induction

$$e.g.\ 5! = 5\cdot4\cdot3\cdot2\cdot1 = 5\cdot4! = 5\cdot4\cdot3!$$

exponent property

$$a^m a^n = a^{m+n}$$

Example: Prove that if n is an integer and $n \geq 4$ then $2^n < n!$.

Base Case: $n = 4$ $\quad 2^4 = 16 \quad 4! = 4\cdot3\cdot2\cdot1 = 24$

$$16 < 24 \Rightarrow 2^4 < 4!$$

Induction Step: Assume for some $k \geq 4$ that $2^k < k!$ } Inductive Hypothesis

We want to show that $2^{k+1} < (k+1)!$

$$(k+1)! = (k+1)k! \qquad k! > 2^k$$

$$> (k+1)2^k \quad \text{by the Inductive Hypothesis.}$$

Notice, since $k \geq 4$, then $k+1 \geq 5 > 2$

$$\Rightarrow (k+1)! \geq 5\cdot2^k > 2^1\cdot2^k = 2^{k+1}.$$

Weak Induction

Example (continued): Prove that if n is an integer and $n \geq 4$ then $2^n < n!$.

Since we've shown that $2^k < k!$ implies that $2^{k+1} < (k+1)!$, by weak induction $2^n < n!$ for all $n \geq 4$.

Weak Induction

e.g: $\frac{-3}{3} = -1$, $\frac{0}{3} = 0$

$\frac{27}{3} = 9$

Example: Prove that if $n \geq 1$ is an integer then $n^3 - n$ is divisible by 3.

Base Case: $n = 1$ $\quad n^3 - n = 1^3 - 1 = 0$

0 is divisible by 3 ✓

Inductive Step: Assume for some $k \geq 1$ that $k^3 - k$ is divisible by 3. I.H.

We want to show that $(k+1)^3 - (k+1)$ is also divisible by 3.

$\longrightarrow$

Weak Induction

Example (continued): Prove that if $n \geq 1$ is an integer then $n^3 - n$ is divisible by 3.

$$(K+1)^3 - (K+1) = (K+1)(K+1)^2 - K - 1$$
$$= (K+1)(K^2+2K+1) - K - 1$$
$$= K^3 + 2K^2 + K + K^2 + 2K + 1 - K - 1$$
$$= K^3 + 3K^2 + 3K + 1 - K - 1$$
$$= K^3 + 3K^2 + 3K - K$$
$$= K^3 - K + 3K^2 + 3K$$
$$= K^3 - K + 3(K^2 + K)$$

By weak induction we have proven that $n^3 - n$ is divisible by 3 for $n \geq 1$

By the inductive hypothesis, $K^3 - K$ is divisible by 3.
$3(K^2+K)$ is also divisible by 3. $\Rightarrow$ $(K+1)^3 - (K+1)$ is divisible by 3.

What we’ve done:

- ❖ Weak Induction

Next:

Strong Induction!

Extra Practice

Example 1: Geometric Progressions. Prove that when $r \neq 1$

$$a + ar + ar^2 + \ldots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

Example 2: Prove that if n is odd, then $n^2 - 1$ is divisible by 8.

Solutions

Example 1: Geometric Progressions. Prove that when $r \neq 1$

$$a + ar + ar^2 + \ldots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

Pf: Base Case n=0: $a = \frac{ar^1 - a}{r-1} = \frac{a(r-1)}{(r-1)} = a$

Induction Step: Assume that the claim is true for n=k

$$a + ar + ar^2 + \ldots + ar^k = \frac{ar^{k+1} - a}{r - 1}$$

Now add ar^{k+1} to both sides of the induction hypothesis.

$$\begin{aligned} a + ar + ar^2 + \ldots + ar^k + ar^{k+1} &= \frac{ar^{k+1} - a}{r - 1} + ar^{k+1} \\ &= \frac{ar^{k+1} - a}{r-1} + \frac{ar^{k+1}(r-1)}{r-1} \\ &= \frac{ar^{k+1} - a + ar^{k+2} - ar^{k+1}}{r-1} \\ &= \frac{ar^{k+2} - a}{r-1} \end{aligned}$$

Thus, by induction, the formula must be true.

$\square$

Example 2: Prove that if n is odd, then $n^2 - 1$ is divisible by 8.

Since n is odd, let's write the claim in a better way. Let $n = 2k - 1$,

Claim: $(2k-1)^2 - 1$ is divisible by 8.

Pf: Base Case: k=1. $(2-1)^2 - 1 = 0$ and 0 is divisible by 8.

Induction Step: Assume the claim is true for $k = m$, $i.e. (2m-1)^2 - 1$ is divisible by 8.

Now we must show this is true for $k = m + 1$.

$$
\begin{aligned}
(2(m+1)-1)^2 - 1 &= (2m+1)^2 - 1 \\
&= 4m^2 + 4m + 1 - 1 \\
&= 4m^2 + 4m + 1 - 1 - 4m + 4m \\
&= 4m^2 - 4m + 1 - 1 + 8m \\
&= [(2m-1)^2 - 1] + 8m
\end{aligned}
$$

This term is divisible by 8 by the induction hypothesis.

This term is divisible by 8.

Thus, the claim holds for m+1 and is therefore true by weak induction. $\square$