CSCI 2824: Discrete Structures

Lecture 21: Strong Induction

Rachel Cox

Department of Computer Science

HW 8 is posted

→ written, turn into Gradescope

→ Due Friday at noon

Mathematical Induction

Weak Induction:

- Verify that $P(1)$ is true.
- Assume $P(k)$ is true and show that $P(k+1)$ is true.

Strong Induction:

- Verify that $P(1)$ is true.
- Assume $P(k)$ for all $k = 1, 2, \ldots, n$ and show $P(n+1)$

Argument: $P(1) \wedge P(2) \wedge \ \ldots \wedge P(k) \rightarrow P(k+1)$

Strong Induction

Example: For any integer $n \geq 1$ there exist numbers $a, b \geq 1$ such that $5^n = a^2 + b^2$

Examining the claim:

$n=1 \quad 5^1 = 5 = 1+4 = 1^2+2^2$

$n=2 \quad 5^2 = 25 = 9+16 = 3^2+4^2$

$n=3 \quad 5^3 = 125 = 100+25 = 10^2+5^2$

Note $4a^2+4b^2 = (2a)^2+(2b)^2$

$25a^2+25b^2 = (5a)^2+(5b)^2$

Base Case: $n=1 \quad 5^1 = 5 = 1+4 = 1^2+2^2$ ✓

Induction Step: (try weak ~~induction~~) Assume for some $k \geq 1$ that $5^k = a^2+b^2$ ✓

$5^{k+1} = 5 \cdot 5^k = \cancel{5(a^2+b^2)} = 5a^2+5b^2$

won't work! stuck!

Strong Induction

Example (continued1): For any integer $n \geq 1$ there exist ~~numbers~~ integers $a, b \geq 1$ such that $5^n = a^2 + b^2$

Inductive Step: [Strong Induction]

Assume for $1 \leq m \leq k$ that $5^m = a^2 + b^2$

Now consider 5^{k+1}

$5^{k+1} = 5 \cdot 5^k = 5^2 \cdot 5^{k-1} = 5^3 \cdot 5^{k-2} = 5^4 \cdot 5^{k-3} \cdots$] Note to self.

$5^{k+1} = 5^2 \cdot 5^{k-1}$

$= 5^2 (a_1^2 + b_1^2)$ by the inductive hypothesis because $k-1 < k$

$= 5^2 a_1^2 + 5^2 b_1^2$

$= (5a_1)^2 + (5b_1)^2$ let $c = 5a_1$ and $d = 5b_1$

Strong Induction

Example (continued2): For any integer $n \geq 1$ there exist numbers $a, b \geq 1$ such that $5^n = a^2 + b^2$

Then $5^{k+1} = c^2 + d^2$

By strong induction, $5^n = a^2 + b^2$ for any integer $n \geq 1$.

Strong Induction

Example: It used to be that throughout the world McDonald's sold chicken nuggets in 4, 6, 9, and 20 piece boxes. Prove that back in the good old days, for any $n \geq 12$ you could buy exactly n nuggets.

Base Case: $n = 12 = 6 + 6$
$n = 13 = 4 + 9$
$n = 14 = 4 + 4 + 6$
$n = 15 = 9 + 6$

Induction Hypothesis: For $12 \underbrace{\leq m \leq k}_{\text{Strong.}}$ we can make exactly m nuggets.

Strong Induction

Example (continued): It used to be that throughout the world McDonald's sold chicken nuggets in 4, 6, 9, and 20 piece boxes. Prove that back in the good old days, for any $n \geq 12$ you could buy exactly n nuggets.

$$k + 1 = (k - 3) + 4$$

$k+1 = 16$

$16 = 12 + 4$

$k - 3 \leq k$ By the induction hypothesis we can make exactly $k - 3$ nuggets.

Then, if we buy a 4-piece…we are good to go!

Strong Induction

Example: If n is a positive integer then it can be written as the sum of distinct Fibonacci numbers.

Base Case: Let $n = 1$.

$$n = 1 = F_1$$

Induction Step: Assume that for each m such that $1 \leq m \leq k, m$ can be written as the sum of distinct Fibonacci numbers.

We want to show that $k + 1$ can be written as the sum of distinct Fibonacci numbers.

Strong Induction

$k+1$ is either a Fibonacci number, or it's not.

Case 1: If $k+1$ is a Fibonacci number, then we're done.

Case 2: If $k+1$ isn't a Fibonacci number, then let F_j be the largest Fibonacci number less than $k+1$

$$F_j < k+1 < F_{j+1}$$

Now, subtract F_j from each piece to derive the following:

$$0 < (k+1) - F_j < F_{j+1} - F_j = F_{j-1} < k+1$$

Strong Induction

and $F_j \geq 1$

Since $(k+1) - F_j < k+1$, then $(k+1) - F_j \leq k$ because k and F_j are both integers.

The induction hypothesis tells us that $(k+1) - F_j$ can be written as a sum of distinct Fibonacci numbers.

Furthermore, since $(k+1) - F_j < F_{j-1}$ we know that F_j is not part of that sum.

Therefore, adding F_j to $(k+1) - F_j$ expresses $k+1$ as the sum of distinct Fibonacci's.

Thus we've proved the claim by induction.

Strong Induction

Example: Prove the Fundamental Theorem of Arithmetic, that any number $n \geq 2$ is either prime or can be written as the product of prime numbers.

Induction: *bad* examples

(Bad) Example: "Prove" that all Fibonacci numbers are even

Base case: Let $n = 0$. $F_0 = 0$, which is even ✔

Induction step: Assume that F_ℓ is even for all ℓ s.t. $0 \leq \ell \leq k$

To show: F_{k+1} is even

$\Rightarrow F_{k+1} = F_k + F_{k-1}$

$\Rightarrow$ By **induction hypothesis**, F_k and F_{k-1} are both even

$\Rightarrow F_{k+1}$ is the sum of two evens, and therefore even "✔"

Induction: *bad* examples

(Bad) Example: "Prove" that all Fibonacci numbers are even

Mistake: We only showed F_0 is even, but then we used $F_{k+1} = F_k + F_{k-1}$ in our proof

⇒ If you think of our "proof" as a line of dominoes, each domino falling in this case requires **two** previous ones to fall

⇒ Here, we only did one base case, so we only knocked over **one** domino

Rule of thumb: If your proof requires going back *s* steps, then you need *s* base cases.

Next up: **Recursion**!

Extra Practice

Example 1: Use Strong Induction to prove that if you have $n > 0$ squares of chocolate combined in a rectangular fashion, the minimum number of breaks necessary to separate the n squares is $n - 1$.

Solution

Example 1: Use Strong Induction to prove that if you have $n > 0$ squares of chocolate combined in a rectangular fashion, the minimum number of breaks necessary to separate the n squares is $n - 1$.

Base Case: Let $n = 1$. Then it takes 0 breaks to separate the pieces ✓

Induction Step: Assume it takes $\ell - 1$ breaks to separate ℓ pieces of chocolate where $1 \leq \ell \leq k$.

We want to show it takes k breaks to separate $k + 1$ pieces of chocolate.

Break the size $k+1$ piece into two pieces of size k_1 and k_2 so that

$$k+1 = k_1 + k_2$$

Since necessarily $k_1 \leq k$ and $k_2 \leq k$, the Induction Hypothesis tells us that it takes $k_1 - 1$ breaks to separate the first piece, and $k_2 - 1$ breaks to separate the second piece

Remembering that we already made 1 break, we have

$$\begin{aligned}\textbf{Total Breaks} &= 1 + (k_1 - 1) + (k_2 - 1) \\ &= (k_1 + k_2) - 1 \\ &= (k+1) - 1 = k \ \checkmark\end{aligned}$$

Thus we've proved the claim via Strong Induction