CSCI 2824: Discrete Structures

Lecture 4: Propositional Logic

Reminders

Submissions:

- Homework 1: Fri 9/6 at noon – 1 try
- Quizlet 1: Wed 9/4 at 8 AM

Disabilities forms – turn them in by the end of 2nd week

Readings:

- 1.1-1.3 through next week

Propositional Logic

Propositional Logic – Connectives, Logical Operations

Definition: Let *p* and *q* be two propositions. The **biconditional** "*p* if and only if *q*", denoted by $p \Leftrightarrow q$, or *p* **iff** *q*, is true when *p* and *q* have the same truth value, and false otherwise.

- The conditional describes an *if-and-only-if* relationship between the two propositions.

Example: Let p = "A polygon has 3 sides." and q = "It is a triangle."

A polygon has 3 sides **if and only if** it is a triangle

Truth Table for a biconditional: $p \Leftrightarrow q$

p	q	$p \Leftrightarrow q$

Truth Table for a biconditional: $p \iff q$

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Propositional Logic – Compound Propositions

Compound propositions are constructed by linking together multiple simple propositions using connectives.

Example: Determine the truth table for the compound proposition

$$(p \rightarrow q) \land (q \rightarrow p)$$

For example: “If I think I see a ghost, then I am superstitious.”

and “If I am superstitious, then I think I see ghosts.”

Propositional Logic – Compound Propositions

Compound propositions are constructed by linking together multiple simple propositions using connectives.

Order of Operations/Precedence of Logical Operators

1. Negation
2. Conjunction over disjunction $p \land q \lor r$ means $(p \land q) \lor r$
3. Conditionals, biconditionals $p \lor q \longrightarrow r$ means $(p \lor q) \longrightarrow r$

Propositional Logic – Compound Propositions

<u>Example:</u> Construct a truth table for $(p \longrightarrow q) \land (\neg p \longrightarrow r)$

p	q	r	$\neg p$	$p \longrightarrow q$	$\neg p \longrightarrow r$	$(p \longrightarrow q) \land (\neg p \longrightarrow r)$

Propositional Logic – Compound Propositions

Compound propositions are constructed by linking together multiple simple propositions using connectives.

Example: Determine the truth table for the compound proposition $(p \to q) \land (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \land (q \to p)$
T	T			
T	F			
F	T			
F	F			

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Example: Determine the truth table for the compound proposition $(p \to q) \land (q \to p)$

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T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Propositional Logic – Compound Propositions

Does the truth table for $(p \rightarrow q) \wedge (q \rightarrow p)$ look familiar?

p	q	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T
T	F	F
F	T	F
F	F	T

Propositional Logic – Compound Propositions

Does the truth table for $(p \to q) \wedge (q \to p)$ look familiar?

… maybe to the biconditional, $p \Leftrightarrow q$?

p	q	$(p \to q) \wedge (q \to p)$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Because the two propositions have the same truth values for all possible values of p and q, they are **logically equivalent**. (more on that later)

Definition: Let *p* and *q* be two propositions. The **conditional** "if *p* then *q*", denoted by $p \to q$, is false when *p* is true but *q* is false, and true otherwise.

- The conditional describes an *if-then* relationship between the two propositions.
- Think of the conditional $p \to q$ as defining a rule. What are the cases where the rule holds or where the rule is broken.

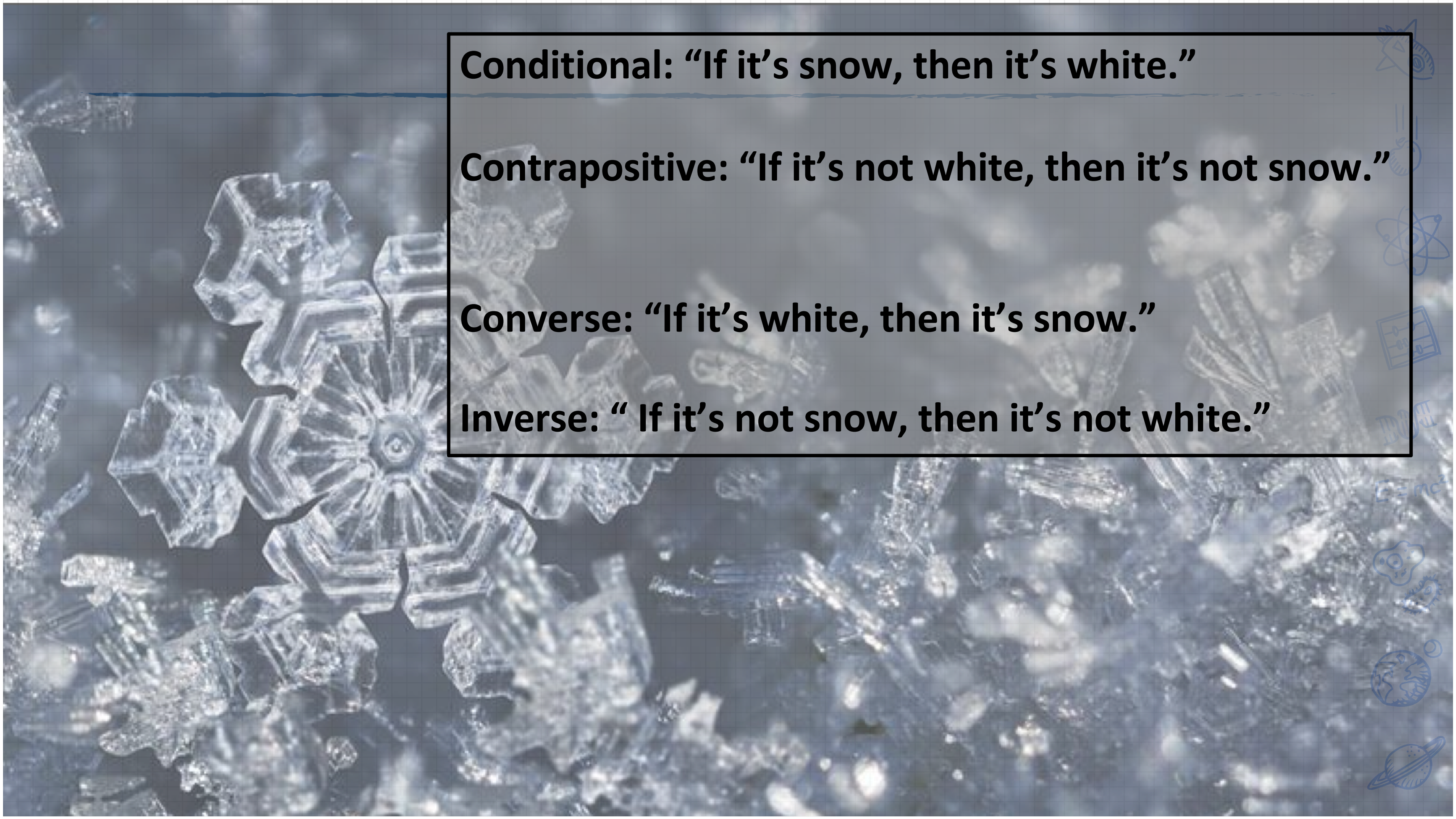
$$p \longrightarrow q$$

$q \longrightarrow p$ **converse**

$\neg p \longrightarrow \neg q$ **inverse**

$\neg q \longrightarrow \neg p$ **contrapositive**

p	q	$p \longrightarrow q$


Conditional: “If it’s snow, then it’s white.”
Contrapositive: “If it’s not white, then it’s not snow.”
Converse: “If it’s white, then it’s snow.”
Inverse: “ If it’s not snow, then it’s not white.”

Logically Equivalent

				Conditional	Converse	Inverse	Contrapositive
p	**q**	$\neg p$	$\neg q$	$p \longrightarrow q$	$q \longrightarrow p$	$\neg p \longrightarrow \neg q$	$\neg q \longrightarrow \neg p$

Logically Equivalent

Definition: The compound propositions p and q are called **logically equivalent** if $p \Leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- A **tautology** is a statement that is **always true**.

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Logically Equivalent

Definition: The compound propositions p and q are called **logically equivalent** if $p \iff q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

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p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
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$p \longrightarrow q$	$q \longrightarrow p$	$\neg p \longrightarrow \neg q$	$\neg q \longrightarrow \neg p$	$(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$	$(q \rightarrow p) \Leftrightarrow (\neg p \rightarrow \neg q)$
T	**T**	**T**	**T**		
F	**T**	**T**	**F**		
T	**F**	**F**	**T**		
T	**T**	**T**	**T**		

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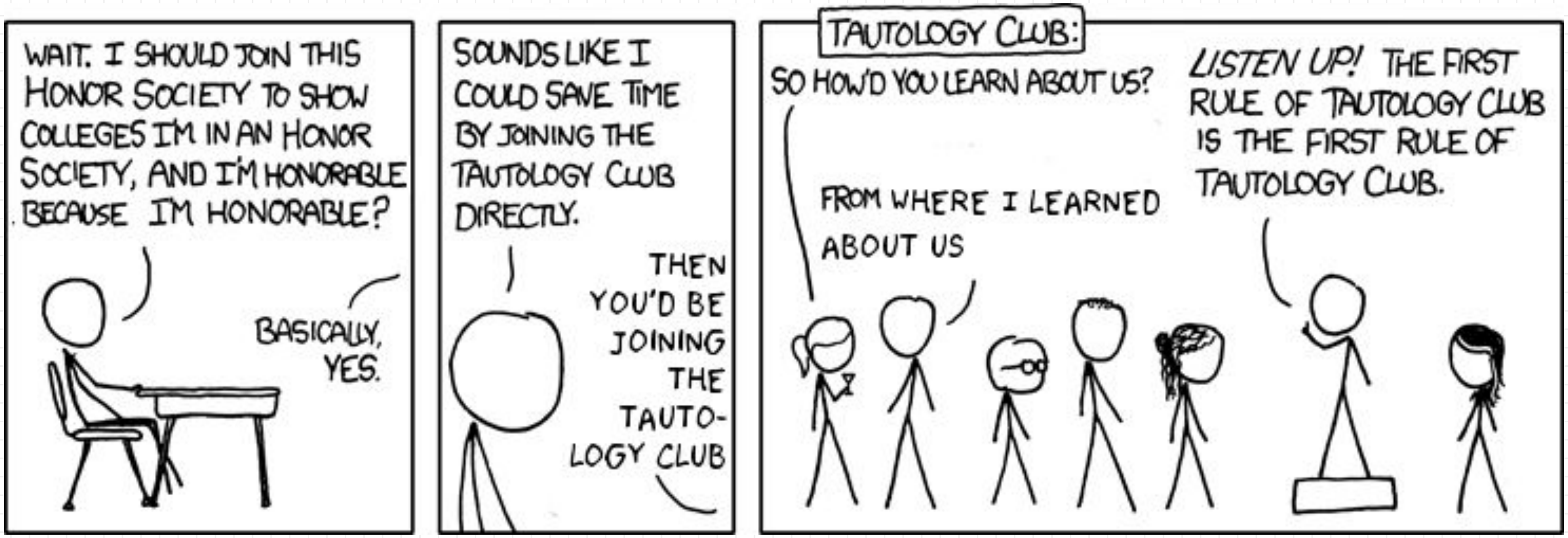
$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$	$(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$	$(q \rightarrow p) \Leftrightarrow (\neg p \rightarrow \neg q)$
T	**T**	**T**	**T**	**T**	
F	**T**	**T**	**F**	**T**	
T	**F**	**F**	**T**	**T**	
T	**T**	**T**	**T**	**T**	

Logically Equivalent - Tautologies

Thomas is either wearing shoes or he is not wearing shoes



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- A **tautology** is a statement that is **always true**.

$p \longrightarrow q$	$q \longrightarrow p$	$\neg p \longrightarrow \neg q$	$\neg q \longrightarrow \neg p$	$(p \to q) \Leftrightarrow (\neg q \to \neg p)$	$(q \to p) \Leftrightarrow (\neg p \to \neg q)$
T	T	T	T	T	T
F	T	T	F	T	T
T	F	F	T	T	T
T	T	T	T	T	T

Logically Equivalent – Conditional and Contrapositive

The conditional and the contrapositive are logically equivalent.

The converse and inverse are logically equivalent.

$(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$	$(q \rightarrow p) \Leftrightarrow (\neg p \rightarrow \neg q)$

Propositional Logic – Knights and Knaves

<u>Example:</u> The island of Knights and Knaves. Suppose you are on an island where there are two types of people: Knights always tell the truth, and Knaves always lie.

Suppose on this island, you encounter two people, Alfred and Batman. Let's call them A and B for short. Suppose A tells you "I am a Knave or B is a Knight." Use a truth table to determine what kind of people A and B are.

p: Alfred is a Knight.

q: Batman is a Knight.

A's statement: ?

Propositional Logic – Knights and Knaves

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p: Alfred is a Knight.

q: Batman is a Knight.

A's statement: $\neg p \vee q$

New Strategy: We'd like to know the combinations of truth values of p and q that ensure that statements made by A and B are consistent with their nature as Knights or Knaves. (i.e. we don't want A to be a Knight but utter a False statement.)

Here: test that p (the statement that "A is a Knight") is **equivalent** in truth value to the statement that he uttered (i.e. $\neg p \vee q$)

Propositional Logic – Knights and Knaves

$\boldsymbol{p} \Leftrightarrow (\neg \boldsymbol{p} \vee \boldsymbol{q})$

It must be True on this island that if A is a Knight, then his statement is true and if A's statement is true, then he must be a knight.

p	q	$\neg p$	$\neg p \vee q$	$p \Leftrightarrow (\neg p \vee q)$

So any truth table row corresponding to ($p \Leftrightarrow \neg p \vee q$) = T is a possibility

Propositional Logic – Knights and Knaves

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p	q	$\neg \boldsymbol{p}$	$\neg \boldsymbol{p} \vee \boldsymbol{q}$	$\boldsymbol{p} \Leftrightarrow (\neg \boldsymbol{p} \vee \boldsymbol{q})$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	F
F	F	T	T	F

So any truth table row corresponding to ($p \Leftrightarrow \neg p \vee q$) = T is a possibility

Necessary and sufficient

Example: Let n be a natural number. It is **sufficient** that n be divisible by 12 for n to be divisible by 6

Let $r = n$ *is divisible by 12* and $s = n$ *is divisible by 6*

How could we represent this claim using a conditional?

Necessary and sufficient

Example: Let n be a natural number. It is **necessary** that n^2 be divisible by 9 for n to be divisible by 6

Let $q = n^2$ *is divisible by 9* and $s = n$ *is divisible by 6*

How could we represent this claim using a conditional?

Wason Selection Class

Consider the following four cards. They have letters on one side and numbers on the other. Suppose I tell you the following rule:

If a card has an odd number, then its letter is a vowel.

A	B	1	2

Question: What card(s) do you need to turn over in order to verify that the given rule is true?

Wason Selection Class

Consider the following four cards. They have letters on one side and numbers on the other. Suppose I tell you the following rule:

If a card has an odd number, then its letter is a vowel.

A	B	1	2

Question: What card(s) do you need to turn over in order to verify that the given rule is true?

odd -> vowel
not vowel -> not odd
consonant -> even

Wason Selection Class

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If a card has an odd number, then its letter is a vowel.

A	B	1	2

Question: What card(s) do you need to turn over in order to verify that the given rule is true?

Answer: Say p = card has an odd number and q = card has a vowel letter.

- Then... the rule is the conditional p → q.
- This only has a truth value of F if p is T but q is F
- So we definitely need to check the card where p is T. **So need to flip over the 1.**

Wason Selection Class

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If a card has an odd number, then its letter is a vowel.

A	B	1	2

Question: What card(s) do you need to turn over in order to verify that the given rule is true?

Answer: Say p = card has an odd number and q = card has a vowel letter.

- But the rule p → q is logically equivalent to another rule: ¬q → ¬p
- So we need to check the card where ¬q is T to verify that ¬p is also true.
- **So we need to flip over B as well as the 1.**

Wason Selection Class

- Experiment designed by Peter Wason in the 1960s.
- Demonstrates that if-then relationships may be intuitive to us, but contrapositives are not.
- If you did not immediately get it correct, you are in good company - **less than 10%** of Wason's original sample got it correct.

- Interestingly, later work found it might be context-specific.
    - For example: If you are drinking a beer, you must be over 21.
    - Most participants successfully answer these "social rules" versions.

Logical Equivalences – Proofs!

- We have found that $p \rightarrow q \equiv \neg q \rightarrow \neg p$. So. Who cares?
- Turns out, this can be **very** useful in proving things.
- Mathematical arguments/proofs:
  - progressing from a set of assumptions to useful/interesting conclusions
  - **logical equivalences** link the steps together
- To prove $p \rightarrow q$, you might suppose p is true, then work your way forward to show that it must be the case that q is true.
- **But** it might be easier to suppose that q is *false*, then work your way toward showing that it must be the case that p is also false.
  - And because $p \rightarrow q \equiv \neg q \rightarrow \neg p$, either way is valid.

Logical Equivalences – Proofs!

<u>Example</u>: Suppose n is an integer. Prove that if n^2 is even, then n must be even.

Logical Equivalences – Proofs!

Example: Suppose *n* is an integer. Prove that if n^2 is even, then *n* must be even.

Direct approach first:

- n^2 is even
- this means n^2 = 2k
- n = sqrt(2k) …. STUCK!

Logical Equivalences – Proofs!

Example: Suppose *n* is an integer. Prove that if n^2 is even, then *n* must be even.

Contrapositive approach:

- Suppose n is odd
- this means $n = 2k + 1$
- $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$
- $n^2 =$ even + even + odd
- $n^2 =$ odd

We have proven that if n is odd then n^2 is also odd

$$\neg \boldsymbol{q} \longrightarrow \neg \boldsymbol{p}$$

Extra Practice

Propositional Logic – Knights and Knaves

Example: The island of Knights and Knaves. Suppose you are on an island where there are two types of people: Knights always tell the truth, and Knaves always lie.

Suppose on this island, you encounter two people, Alfred and Batman. Let's call them A and B for short. Suppose instead that A tells you "B is a Knight" and B tells you "The two of us are of different types". Use a truth table to determine the sorts of people that A and B are. (aside from confusing)

(Extra tricky because you need to incorporate both person's statements into a test proposition, which needs to ensure that each statement is consistent with that person's type.

Propositional logic

FYOG: Suppose instead that *A* tells you "*B* is a Knight" and *B* tells you "The two of us are of different types". Use a truth table to determine the sorts of people that *A* and *B* are. (aside from confusing)

Let p = "A is a knight" and let q = "B is a knight"

Then A's statement is q, which holds if and only if A is telling the truth: $p \Leftrightarrow q$

and B's statement can be represented in a few different ways.

- The simplest way is probably: $(p \wedge \neg q) \vee (\neg p \wedge q)$
- The most compact way is with the exclusive-or: $p \oplus q$

B's statement is true **if and only if** B is a knight (i.e., telling the truth), we have: $q \Leftrightarrow p \oplus q$

Now, *both* A and B have made statements, and we want to include *both* of them in our truth table test. Since we want to test both, we link them together with a conjunction:

FYOG (continued): Suppose instead that *A* tells you "*B* is a Knight" and *B* tells you "The two of us are of different types". Use a truth table to determine the sorts of people that *A* and *B* are.

→ Test when this compound proposition is true: $(p \Leftrightarrow q) \wedge (q \Leftrightarrow p \oplus q)$

p	q	$p \Leftrightarrow q$	$p \oplus q$	$q \Leftrightarrow p \oplus q$	$(p \Leftrightarrow q) \wedge (q \Leftrightarrow p \oplus q)$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	T	T	F
F	F	T	F	T	T

The only row where our compound proposition is T is the last one, so both A and B must be knaves.

Propositional Logic – Knights and Knaves

Example: The island of Knights and Knaves. Suppose you are on an island where there are two types of people: Knights always tell the truth, and Knaves always lie.

Suppose on this island, you encounter two people, Alfred and Batman. Let's call them A and B for short. Person A tells you that "B and I are of opposite types." Person B tells you that "A is a knave and I am a knight." Use a truth table to determine what type of people A and B are.