CSCI 2824: Discrete Structures

Lecture 8: Rules of Inference

Reminders

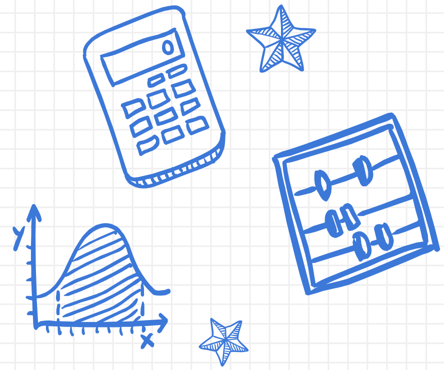
Submissions:

- Homework 2: Fri 9/13 at noon – Gradescope
- Homework 3: Fri 9/20 at noon – 1 try

Disabilities forms – please bring them by Friday

Readings:

- 1.6 today Friday
- 1.6-1.8 through next week



Rules of Inference

What did we do last time?

- We can represent standard propositions with quantifiers.
- We can translate statement from English to symbolic logical statements with quantifiers.
- We can prove and derive logical equivalences.

Today:

We will learn about:

1. the structure of arguments
2. identifying valid, sound and fallacious arguments
3. rules of inferences

Rules of inference

Next time we will discuss how to construct **mathematical proofs**.

- Mathematical proofs are **valid arguments** that establish the truth of a mathematical statement.

But first, we need to learn how to construct **valid arguments**

Think of an argument as a symbolic template that starts with some assumptions (called **premises**) and proceeds along a path of logical inferences to reach a **conclusion**.

Rules of inference

But first, we need to learn how to construct **valid arguments.**

Think of an argument as a symbolic template that starts with some assumptions (called **premises**) and proceeds along a path of logical inferences to reach a **conclusion**.

Example: If Osiris can bleed, then Osiris is a mortal.
Osiris can bleed.

Therefore, Osiris is a mortal.

This is an example of a specific valid argument.

$E = mc^2$

Rules of inference

So now we need to cast this argument into a symbolic template. We can use our previous experience abstracting propositions from English to logical symbols.

- p denote "Xerxes can bleed"
- q denote "Xerxes is mortal"

Then in symbolic logic, our argument becomes

$$p \to q$$

$$p$$

$$\therefore \quad q$$

Note: the symbol ∴ means **therefore**.

Use this to denote the conclusion of the argument.

Rules of inference

An **argument** is a set of **premises** coupled with a **conclusion**.

A **valid** argument is an argument such that **there is no circumstance in which the premises could be true and the conclusion be false**.

Rules of inference

"Valid Arguments"

valid: the conclusion must follow from the truth of the preceding statements (premises)

argument: a sequence of statements that end with a conclusion

The Distinction between truth and validity	
TRUTH	**VALIDITY**
Concerned with **what is the case**	Concerned with whether **conclusions** follows from **premises**
	The validity of an argument is independent of the truth or falsity of the premises it contains.

Rules of Inference are the basic tools for establishing the truth of statements.

Rules of inference

If Osiris can bleed, then Osiris is a mortal. Osiris can bleed. Therefore, Osiris is a mortal.

$$
\begin{array}{ll}
 & p \to q \\
 & p \\
\hline
\therefore & q
\end{array}
$$

Consider the compound proposition: $((p \to q) \land p) \to q$

Note that this is a conditional:

- The hypothesis is the conjunction of the premises of our argument, and
- the conclusion is the conclusion of our argument.

Rules of inference

If Osiris can bleed, then Osiris is a mortal. Osiris can bleed. Therefore, Osiris is a mortal.

For our argument to be valid, it must be the case that there is no situation (i.e., truth values for p and q) in which the premises of the argument are **true** but the conclusion **false**.

We joined the premises with the conclusion in a conditional (as *premises* → *conclusion*)

- So for the argument to be valid, the conditional describing it must be **always true** (i.e., it needs to be a tautology)

Check with a truth table!

Rules of inference

If Osiris can bleed, then Osiris is a mortal. Osiris can bleed. Therefore, Osiris is a mortal.

Check with a truth table!

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q$
T	T			
T	F			
F	T			
F	F			

Rules of inference

If Osiris can bleed, then Osiris is a mortal. Osiris can bleed. Therefore, Osiris is a mortal.

Check with a truth table!

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$((p \rightarrow q) \wedge p) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Rules of inference

This general form of argument is so useful and common that it has a special and fancy name, and designate it as a **rule of inference**:

Modus Ponens:

Latin for "mode that affirms"

"the way that affirms by affirming"

1. $p \rightarrow q$
2. p

$\therefore \quad q$

If a conditional statement and the hypothesis of the conditional statement are both true, then the conclusion must also be true.

Rules of inference are common, valid mini-arguments that we can link together to construct more complex valid arguments.

Rules of inference

Modus Tolens:

Latin for “mode that denies”

“the way that denies by denying”

1. $\neg q$
2. $p \to q$

$\therefore$ $\neg p$

If a conditional statement is true, then so is its contrapositive.

Example: If it rains today, then my basement will flood.

My basement did not flood.

$\therefore$ It did not rain today.

Rules of inference

Example: If it rains today, then my basement will flood.
My basement did not flood.

∴ It did not rain today.

1. $\neg q$
2. $p \rightarrow q$

∴ $\neg p$

This other way will also be useful to prove arguments that are too unwieldy for a truth table (recall that you need 2^N rows, where N is the number of propositions).

	Step	Justification
1.	$p \rightarrow q$	premise
2.	$\neg q$	premise
3.		
4.		

Rules of inference

Example: If it rains today, then my basement will flood.
My basement did not flood.

∴ It did not rain today.

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	Step	Justification
1.	$p \rightarrow q$	premise
2.	$\neg q$	premise
3.	$\neg q \rightarrow \neg p$	contraposition of (1)
4.		

Rules of inference

Example: If it rains today, then my basement will flood.
My basement did not flood.

∴ It did not rain today.

1. $\neg q$
2. $p \rightarrow q$

$\therefore \quad \neg p$

This other way will also be useful to prove arguments that are too unwieldy for a truth table (recall that you need 2^N rows, where N is the number of propositions).

	Step	Justification
1.	$p \rightarrow q$	premise
2.	$\neg q$	premise
3.	$\neg q \rightarrow \neg p$	contraposition of (1)
4.	$\therefore \neg p$	Modus Ponens of (3) and (2)

Rules of inference

Disjunctive Syllogism:

Historically: *Modus Tollendo Ponens*

1. $p \vee q$
2. $\neg p$

$\therefore$ q

Example: My foot is disfigured or there is a rock in my shoe

My foot is not disfigured

$\therefore$ I have a rock in my shoe

Other sort-of example: *"Once you eliminate the impossible, whatever remains, no matter how improbable, must be the truth."*

- Sherlock Holmes (Sir Arthur Conan Doyle, 1890: The Sign of the Four, ch. 6)

Rules of inference

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	Step	Justification
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2.	$\neg p$	premise
3.		
4.	$\therefore$	

Rules of inference

Disjunctive Syllogism:

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1. $p \vee q$
2. $\neg p$

$\therefore \quad q$

	Step	Justification
1.	$p \vee q$	premise
2.	$\neg p$	premise
3.	$\neg p \rightarrow q$	relation by implication, using (1)
4.	$\therefore$	

Rules of inference

Disjunctive Syllogism:

Historically: *Modus Tollendo Ponens*

1. $p \vee q$
2. $\neg p$

$\therefore$ q

	Step	Justification
1.	$p \vee q$	premise
2.	$\neg p$	premise
3.	$\neg p \rightarrow q$	relation by implication, using (1)
4.	$\therefore q$	modus ponens, using (2) and (3)

Rules of inference

Example: What can you conclude from the following?

If it is sunny outside, then I will go to the park.

If I go to the park, then I will get ice cream

$\therefore$?

Rules of inference

Example: What can you conclude from the following?

If it is sunny outside, then I will go to the park.

If I go to the park, then I will get ice cream

∴ If it is sunny outside, then I will get ice cream

This is called **hypothetical syllogism**:

1. $p \rightarrow q$
2. $q \rightarrow r$

∴ $p \rightarrow r$

FYOG: Show that hypothetical syllogism is a valid rule of inference by showing that $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$ is a tautology.

Rules of inference

Here are a few more simple ones…

Simplification:

$$\frac{p \wedge q}{\therefore\ p}$$

Addition:

$$\frac{p}{\therefore\ p \vee q}$$

Conjunction:

$$\begin{array}{c} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$$

Rules of inference

… and one tricky one

Resolution:

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

Rules of inference

… and one tricky one

Resolution:

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

Intuition: *p* can either be true or false

- If *p* is true, then *r* must be true.
- If *p* is false, then *q* must be true.
- So either way, at least one of *q* or *r* must be true (or both).

Rules of inference

Can we derive resolution with the rules we already know?

	Step	Justification
1.	$p \vee q$	premise
2.	$\neg p \vee r$	premise
3.		
4.		
5.		
6.		
7.	$\therefore q \vee r$	

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

Rules of inference

Can we derive **resolution** with the rules we already know?

$$\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

	Step	Justification
1.	$p \vee q$	premise
2.	$\neg p \vee r$	premise
3.	$q \vee p$	commutativity, using (1)
4.	$\neg q \rightarrow p$	relation by implication, using (3)
5.	$p \rightarrow r$	relation by implication, using (2)
6.	$\neg q \rightarrow r$	hypothetical syllogism, using (4) and (5)
7.	$\therefore q \vee r$	relation by implication, using (6)

Use the rules of inference to show that the following argument is valid

	Step	Justification
1.	$p \vee q \rightarrow r$	premise
2.	$\neg r \rightarrow s$	premise
3.	p	premise
4.		
5.		
6.		
7.		

$$
\begin{array}{ll}
 & (p \vee q) \rightarrow \neg r \\
 & \neg r \rightarrow s \\
 & p \\
\hline
\therefore & s
\end{array}
$$

Use the rules of inference to show that the following argument is valid

	Step	Justification
1.	$p \vee q \rightarrow r$	premise
2.	$\neg r \rightarrow s$	premise
3.	p	premise
4.	$p \vee q$	addition, using (3)
5.	$\neg r$	modus ponens, using (1) and (4)
6.	$\therefore\ s$	modus ponens, using (2) and (5)

$$(p \vee q) \rightarrow \neg r$$
$$\neg r \rightarrow s$$
$$p$$
$$\therefore\ s$$

Rules of inference

FYOG: Use the rules of inference that you know so far to show that the following argument is valid.

$$\begin{array}{rl} & p \wedge q \\ & p \rightarrow \neg r \\ & q \rightarrow \neg s \\ \hline \therefore & \neg r \wedge \neg s \end{array}$$

Rules of Inference

Example: Which argument form is used by the following:

Socrates is immortal or Socrates is a man.
Socrates is not immortal.

Therefore, Socrates is a man.

Rules of inference

What valid argument form is present in the following?

If n is a real number with $n > 3$, then $n^2 > 9$.

Suppose that $n^2 \leq 9$. Then $n \leq 3$.

Rules of inference

What valid argument form is present in the following?

> If n is a real number with $n > 3$, then $n^2 > 9$.
>
> Suppose that $n^2 \leq 9$. Then $n \leq 3$.

Solution:

Let p represent the statement n is a real number with $n > 3$

Let q represent the statement $n^2 > 9$

Then we have: $p \to q$

$\neg q$

$\therefore \neg p$ $\Leftarrow$ This is **Modus Tollens**

Rules of inference

What valid argument form is present in the following?

If $\sqrt{2} > 3/2$, then $(\sqrt{2})^2 > (3/2)^2$. We know that $\sqrt{2} > 3/2$. Consequently, $(\sqrt{2})^2 = 2 > (3/2)^2 = 9/4$.

Rules of inference

What valid argument form is present in the following?

> If $\sqrt{2} > 3/2$, then $(\sqrt{2})^2 > (3/2)^2$. We know that $\sqrt{2} > 3/2$. Consequently, $(\sqrt{2})^2 = 2 > (3/2)^2 = 9/4$.

Solution:

- This is demonstrating "if *p*, then *q*; *p*, therefore *q*"
  - Use p = the statement $\sqrt{2} > 3/2$
  - Use q = the statement $(\sqrt{2})^2 > (3/2)^2$
- That is **Modus Ponens** at work!

Question: Does something here not look quite right…?

Rules of inference

A **valid** argument is one where there is no way the conclusion can be false **if the premises are true**

Valid arguments are patterns of logical reasoning.

Rules of inference

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Valid arguments are patterns of logical reasoning.

But just because an argument is valid does not mean you can trust the conclusion.

In the previous example, the conclusion that $2 > 9/4$ is very, very **false**.

Rules of inference

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But just because an argument is valid does not mean you can trust the conclusion.

In the previous example, the conclusion that $2 > 9/4$ is very, very **false**.

The problem arises because the premise that $\sqrt{2} > 3/2$ is **false**.

Rules of inference

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Valid arguments are patterns of logical reasoning.

But just because an argument is valid does not mean you can trust the conclusion.

In the previous example, the conclusion that $2 > 9/4$ is very, very **false**.

The problem arises because the premise that $\sqrt{2} > 3/2$ is **false**.

So even though this argument is valid, it is not "useful" or "nice".

- We want to be able to tell which arguments are not only valid, but "nice" too.

Rules of inference

Recap:

- We have learned the rules of inference, and how to use them to construct **valid** arguments. (good arguments)
- We have learned how to identify a **sound** argument. (great arguments)
- We have learned how to recognize common **fallacious** arguments. (awful arguments)

Next time:

- Rules of inference, continued
- We bring **quantifiers** into the mix!

  (*"for all"*, *"there exist"*)

Extra Practice

Ex. 1 Show that $p \wedge q, p \rightarrow \neg r, q \rightarrow \neg s$, therefore $\neg r \wedge \neg s$ is a valid argument.

Solution

Ex. 1 Show that $p \wedge q, p \to \neg r, q \to \neg s$, therefore $\neg r \wedge \neg s$ is a valid argument.

	step	justification
1.	$p \wedge q$	premise
2.	$p \to \neg r$	premise
3.	$q \to \neg s$	premise
4.	p	simplification of (1)
5.	$\neg r$	modus ponens (2), (4)
6.	q	simplification of (1)
7.	$\neg s$	modus ponens (3), (6)
6.	$\therefore \neg r \wedge \neg s$	conjunction (5) and (7)