CSCI 2824: Discrete Structures

Lecture 17: Sequences

Reminders

Submissions:

- Homework 5: **Mon 10/7 at noon** – 1 try on Moodle
- Homework 6: **Fri 10/11 at noon** – Gradescope
- **Quizlet 6: due Wednesday 10/9 at 8pm**

Readings:

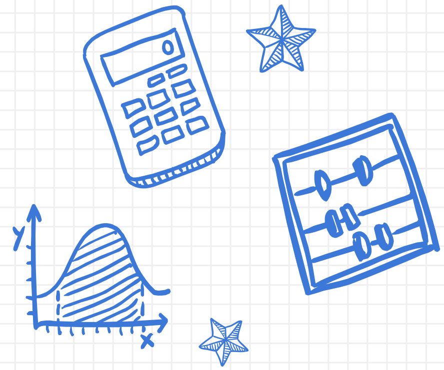
- This week Ch. 2 – SETS (2.4 – Sequences and Summations)

What did we do last time?

- Functions
  - What are they?
  - Special kinds of functions (floor, ceiling, inverse, composition…)
  - Special properties functions can have (onto, 1-1…)
  - That we shouldn't fear them!
- Cardinality of sets (countable, uncountable)

Today:

- Sequences
  - What are they?
  - Special kinds of sequences…
  - Different ways to define them…



Sequences

Sequences

A **sequence** is a discrete structure used to store an *ordered* list of elements.
It is one of the most fundamental structures in computer science.

Examples:

- 0, 1, 2, 3, 4, 5, …
- 2, 4, 8, 16, 32, …
- 1, 4, 78, 109, 4, 25, …

Definition: A **sequence** is a function that maps from a subset of integers (usually {0, 1, 2, 3, … } or {1, 2, 3, … }) to a set S.
We use the notation $\{a_n\}$ to represent a sequence where a_n represents the n^{th} individual term in the sequence.

Sequences

Fun fact: This very definition of a sequence implies that if we can define a ***set*** in terms of a sequence, then that set must be countable.

Example, rebooted: Even integers

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Sequences

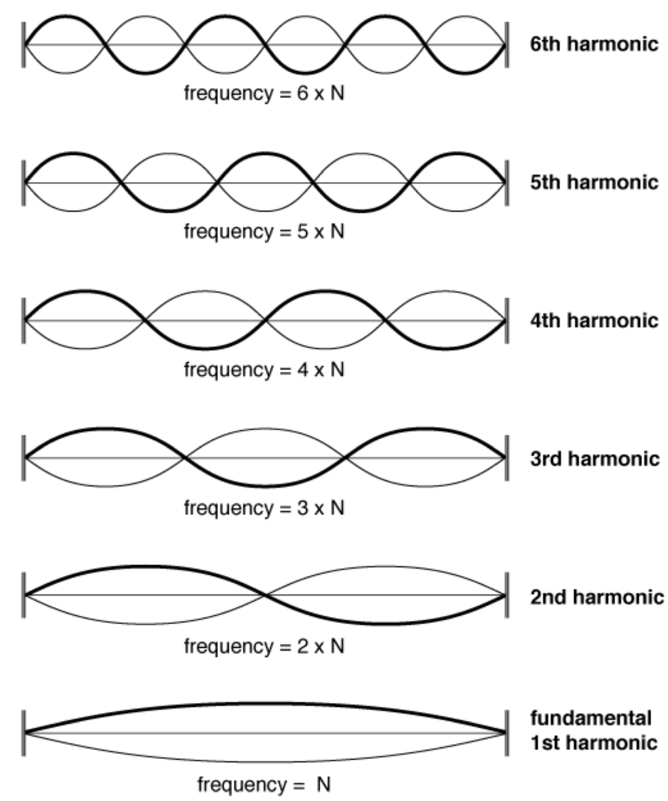
Example: Consider the sequence $\{a_n\}$,
where $a_n = 1/n$

- In this case, the sequence starts with $n = 1$

  (do you see why?)
- The first three terms are:

  $a_1 = 1$ $a_2 = ½$ $a_3 = ⅓$

- This is one of several sequence patterns that is so common, we have a special name for it: ***the harmonic sequence***



wave image of the harmonic series

Sequences

Definition: A **geometric progression** is a sequence of the form

$$a, ar, ar^2, ar^3, ar^4, ar^5, \ldots$$

where the **initial term** a and the **common ratio** r are real numbers.

Note that this can be rewritten to show the pattern explicitly as:

$$ar^0, ar^1, ar^2, ar^3, ar^4, ar^5, \ldots$$

Sequences

Example: What are a and r in the following geometric sequence? What is the next term?

3, 6, 12, 24, …

Sequences

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3, 6, 12, 24, …

Solution: We know a geometric sequence follows the pattern: $a, ar, ar^2, ar^3, ar^4, \ldots$, so we can use the a_0 and a_1 terms to pick out a and r.

- $a_0 = a = 3$
- $a_1 = ar = 3r = 6 \quad \Rightarrow r = 2$

- The next term would be $a_4 = ar^4 = 3 \times 2^4 = 3 \times 16 = 48$

Sequences

Definition: Another common sequence is an **arithmetic progression**, which has the form

$$a, a + d, a + 2d, a + 3d, a + 4d, a + 5d, \ldots$$

where the **initial term** a and the **common difference** d are real numbers.

Note that this can be rewritten to show the pattern explicitly as:

$$a + 0d, a + 1d, a + 2d, a + 3d, a + 4d, a + 5d, \ldots$$

Example: What are a and d in the following arithmetic sequence? What is the next term?

$$5, 9, 13, 17, \ldots$$

Sequences

Example: What are *a* and *d* in the following arithmetic sequence? What is the next term?

2, 9, 16, 23, …

Sequences

Example: What are *a* and *d* in the following arithmetic sequence? What is the next term?

2, 9, 16, 23, …

Solution: We know an arithmetic sequence follows the pattern $a, a + d, a + 2d, a + 3d, a + 4d, a + 5d, \dots$ so we can use the a_0 and a_1 terms to pick out *a* and *d*.

- $a_0 = a = 2$
- $a_1 = a + d = 2 + d = 9 \quad \Rightarrow d = 7$

- The next term would be $a_4 = a + 4d = 2 + 4 * 7 = 30$

Fun fact: This is the discrete generalization of a linear function $f(x) = a + dx$

Sequences

Sometimes we don't give an explicit formula for the terms a_n

Instead, we might define the sequence in terms of a **recurrence relation**, where the later terms are a function of the previous ones, and specifying the first few terms.

Example: Let $a_0 = 1$, $a_1 = 3$, and $a_n = 2a_{n-1} - a_{n-2}$. *Let's derive the next few terms*

Sequences

Sometimes we don't give an explicit formula for the terms a_n

Instead, we might define the sequence in terms of a **recurrence relation**, where the later terms are a function of the previous ones, and specifying the first few terms.

Example: Let $a_0 = 1$, $a_1 = 3$, and $a_n = 2a_{n-1} - a_{n-2}$

Then $a_2 = 2a_1 - a_0 = 2(3) - 1 = 5$

$a_3 = 2a_2 - a_1 = 2(5) - 3 = 7$

$a_4 = 2a_3 - a_2 = 2(7) - 5 = 9$ … and so on…

So the sequence is: 1, 3, 5, 7, 9, …

If we want a ***closed form*** version of a_n, we can figure it out:

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So the sequence is: 1, 3, 5, 7, 9, …

If we want a ***closed form*** version of a_n, we can figure it out:

$$a_n = 2n + 1$$

$a_n = 2n + 1$ is called a ***solution*** of the recurrence relation defined above.

Sequences

Example: Show that $a_n = 4^n$ is a solution to the recurrence $a_n = 8a_{n-1} - 16a_{n-2}$

Strategy: Plug a_n into the recurrence and show both sides are equal

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Strategy: Plug a_n into the recurrence and show both sides are equal

$a_n = 4^n$ $\qquad a_{n-1} = 4^{n-1}$ $\qquad a_{n-2} = 4^{n-2}$

We have: $\quad 8a_{n-1} - 16a_{n-2} = 8(4^{n-1}) - 16(4^{n-2})$

$$= 8(4^n)(4^{-1}) - 16(4^n)(4^{-2})$$
$$= 8(4^n)(1/4) - 16(4^n)(1/16)$$
$$= 2(4^n) - 4^n$$
$$= 4^n$$
$$= a_n$$

Sequences

FYOG: Show that $a_n = n\,4^n$ is also a solution to the recurrence $a_n = 8a_{n-1} - 16a_{n-2}$

FYOG: Determine a recurrence relation with solution $a_n = n + (-1)^n$

FYOG: Determine a recurrence relation with solution $a_n = n^2 + n$

FYOG: Determine a recurrence relation with solution $a_n = n!$

Sequences

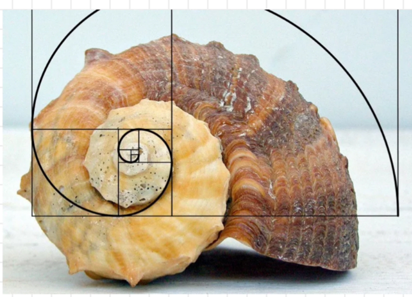
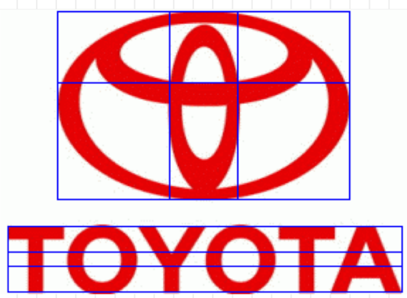
Probably the most famous sequence is the ***Fibonacci sequence***:

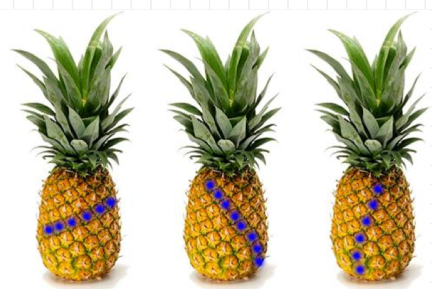
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

It has the recurrence relation: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$

Lots of cool stuff hidden in the Fibonacci sequence

- Ratio of the terms approaches the **golden ratio:** $\frac{1+\sqrt{5}}{2}$
- Pattern found all over the place in nature




Sequences

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Notice any interesting patterns?

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Notice any interesting patterns?

➢ **Parity pattern:** even, odd, odd, even, odd, odd, even, …

$\Rightarrow$ Could we find the sequence of just the *even* Fibonacci numbers?

Sequences

Probably the most famous sequence is the ***Fibonacci sequence***:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Notice any interesting patterns?

- **Parity pattern:** even, odd, odd, even, odd, odd, even, …

  ⇒ Could we find the sequence of just the *even* Fibonacci numbers?

  ⇒ Seems like the evens are the Fibonacci numbers with indices that are multiples of 3.

$$F_0 = 0,\ F_3 = 2,\ F_6 = 8,\ \ldots$$

Sequences

Tougher Example: Determine a recurrence relation for the even Fibonacci numbers, $\{E_n\}$

$$E_0 = F_0 = 0, \quad E_1 = F_3 = 2, \quad E_2 = F_6 = 8, \quad E_3 = F_9 = 34, \quad \ldots$$

Sequences

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Strategy: 1. How should our recurrence relation look like? What previous terms are involved?

Sequences

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Strategy: We want a recurrence of the form $F_n = a\,F_{n-3} + b\,F_{n-6}$

- As long as we start it off with the first two evens, this recurrence will stick to the even Fibonaccis

Tougher Example: Determine a recurrence relation for the even Fibonacci numbers, $\{E_n\}$

$$E_0 = F_0 = 0, \quad E_1 = F_3 = 2, \quad E_2 = F_6 = 8, \quad E_3 = F_9 = 34, \quad \ldots$$

Strategy: We want a recurrence of the form $F_n = a\,F_{n-3} + b\,F_{n-6}$

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$$\begin{aligned}
F_n &= F_{n-1} + F_{n-2} \\
&= (F_{n-2} + F_{n-3}) + (F_{n-3} + F_{n-4}) \\
&= ((F_{n-3} + F_{n-4}) + F_{n-3}) + (F_{n-3} + (F_{n-5} + F_{n-6})) \\
&= 3F_{n-3} + F_{n-6} + F_{n-4} + F_{n-5} \\
&= 3F_{n-3} + F_{n-6} + F_{n-3} \\
&= 4F_{n-3} + F_{n-6}
\end{aligned}$$

Sequences

Tougher Example: Determine a recurrence relation for the even Fibonacci numbers, $\{E_n\}$

$$E_0 = F_0 = 0, \quad E_1 = F_3 = 2, \quad E_2 = F_6 = 8, \quad E_3 = F_9 = 34, \quad \ldots$$

Strategy: We want a recurrence of the form $F_n = a\,F_{n-3} + b\,F_{n-6}$

- As long as we start it off with the first two evens, this recurrence will stick to the even Fibonaccis

$F_n \quad = 4F_{n-3} + F_{n-6}$

Solution:

$F_0 = 0$, $F_3 = 2$, and $F_n = 4\,F_{n-3} + F_{n-6}$

We can express the recurrence relation using only terms of the "evens" sequence $E_0 = 0$, $E_1 = 2$, and $E_n = 4\,E_{n-1} + E_{n-2}$

Sequences

Solving recurrence relations using **iteration**:

Example: Let $a_n = a_{n-1} + 3$, with $a_1 = 2$. Find a closed form solution to this recurrence.

Solution: (using iteration) Any ideas?

Sequences

Solving recurrence relations using **<u>iteration</u>**:

Example: Let $a_n = a_{n-1} + 3$, with $a_1 = 2$. Find a closed form solution to this recurrence.

Solution: (using iteration)

We write out the first few terms by plugging successively back into the recurrence:

Sequences

Solving recurrence relations using **iteration**:

Example: Let $a_n = a_{n\text{-}1} + 3$, with $a_1 = 2$. Find a closed form solution to this recurrence.

Solution: (using iteration) We write out the first few terms by plugging successively back into the recurrence:

$a_1 = 2$

$a_2 = a_1 + 3 = 2 + 3$

$a_3 = a_2 + 3 = (2 + 3) + 3 = 2 + 3{\cdot}2$

$a_4 = a_3 + 3 = (2 + 3{\cdot}2) + 3 = 2 + 3{\cdot}3$

…

$a_n = 2 + 3{\cdot}(n\text{-}1)$

Countable and uncountable sets

Fun fact: This very definition of a sequence implies that if we can define a ***set*** in terms of a sequence, then that set must be countable.

Example, rebooted: Even integers

Reminder:
THIS happened

Definition: A **sequence** is a function that maps from a subset of integers (usually {0, 1, 2, 3, … } or {1, 2, 3, … }) to a set S.

We use the notation $\{a_n\}$ to represent a sequence where a_n represents the n^{th} individual term in the sequence.

Countable and uncountable sets

Theorem: If A and B are countable sets, then $A \cup B$ is countable as well.

Proof: (by cases)

3 cases: (i) A and B are both finite

(ii) A is infinite and B is finite (without loss of generality, covers the case of B being infinite and A finite)

(iii) A and B both infinite

Countable and uncountable sets

Theorem: If A and B are countable sets, then $A \cup B$ is countable as well.

Proof: (by cases)

Case (i): A and B are both finite

⇒ We can write the elements of A as $A = \{a_1, a_2, \ldots , a_n\}$ and the elements of B as $B = \{b_1, b_2, \ldots , b_m\}$ for some positive integers n and m (equal to $|A|$ and $|B|$, resp.)

⇒ We can write the elements of $A \cup B$ as $\{a_1, a_2, \ldots , a_n, b_1, b_2, \ldots , b_m\}$, which has $n+m$ elements (at most)

Countable and uncountable sets

Theorem: If A and B are countable sets, then $A \cup B$ is countable as well.

Proof: (by cases)

<u>Case (ii): *A* infinite and *B* finite</u>

$\Rightarrow$ Because A is countably infinite, we can write its elements as an infinite sequence: $a_1, a_2, \ldots, a_n, a_{n+1}, \ldots$

$\Rightarrow$ Because B is finite we can write its elements as $B = \{b_1, b_2, \ldots, b_m\}$ for some positive integer m (equal to |B|)

$\Rightarrow$ We can write the elements of $A \cup B$ as an infinite sequence as:

$b_1, b_2, \ldots, b_m, a_1, a_2, \ldots, a_n, \ldots$

Countable and uncountable sets

Theorem: If A and B are countable sets, then $A \cup B$ is countable as well.

Proof: (by cases)

<u>Case (iii): *A* and *B* both infinite</u>

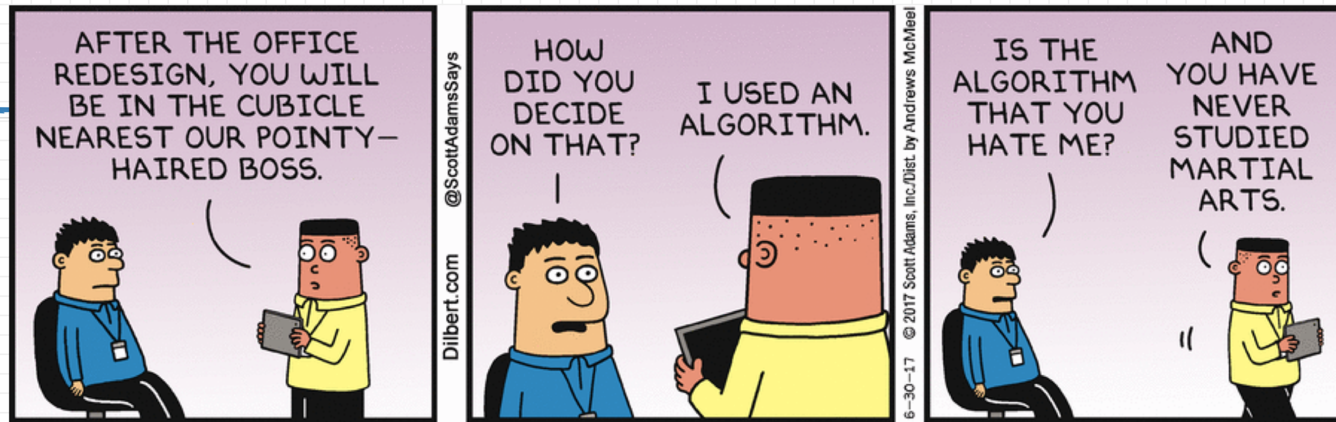
$\Rightarrow$ Because A and B are countably infinite, we can write their elements as infinite sequences:

$$a_1, a_2, \ldots, a_n, a_{n+1}, \ldots \quad \text{and} \quad b_1, b_2, \ldots, b_n, b_{n+1}, \ldots$$

$\Rightarrow$ We can write the elements of $A \cup B$ as an infinite sequence by alternating between the sequences $\{a_n\}$ and $\{b_n\}$:

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots$$

Recap



Sequences…

- What are they?
- Special kinds
- Nifty ways to define sequences

Next time:

- we learned about *functions*
- we learned about *sequences*…
- ***algorithms*** are basically applying a function to all the elements along a sequence
- … and pretty much the backbone of computational science.