CSCI 2824: Discrete Structures

Lecture 28: Count-ing. Permutations. Combinations

Reminders

Submissions:

- Homework 9: **Fri 11/01 at noon -** Moodle

Readings:

- Ch. 6 – Counting

Midterm II

- Tuesday November 5th

Last time

Recap:

- **Generalized Pigeonhole Principle** (PHP) -- If you want to sort n objects into k bins, at least one of the bins will contain $\lceil n/k \rceil$ objects.

Today:

- Permutations and Combinations.

Permutations and combinations

Example: How many 4-digit PIN combinations are there

(10 choices for first digit) $\times$ (10 choices for second digit) $\times$ (10 …) $\times$ (10 …) = **10,000**

Permutations and combinations

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- Then put digit back into bag. Do this 4 times (digits)

Permutations and combinations

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What if we picked from the bag **without replacement** and **don't care about order**?

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⇒ ***permutations***

What if we picked from the bag **without replacement** and **don't care about order**?
⇒ ***combinations***

Permutations

When selecting r distinct items without replacing them, we call the possible selection an ***r*-permutation**

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ab, ac, ba, bc, ca, cb

Note that there are $3 \times 2 = 6$ such 2-permutations, and the list contains both *ab* and *ba* (for example) because **order matters**

Permutations

Example: How many three-character strings can we make if each character is a distinct lowercase letter?

Permutations

Example: How many three-character strings can we make if each character is a distinct lowercase letter?

Solution: 26 choices for the first letter, 25 choices for second, 24 choices for third

$\Rightarrow$ 26 $\times$ 25 $\times$ 24 = **15,600 strings**

Theorem: If n is a positive integer and r is an integer such that $0 \leq r \leq n$, then the number of r-permutations from a set of size n is

$$P(n, r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-r+1)$$

Special cases: $P(n, 0) = 1$ and $P(n, n) = n!$

Permutations

Example: How many different 1st-2nd-3rd place permutations can occur in a race with 5 runners?

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC*?

Permutations

Example: How many different 1st-2nd-3rd place permutations can occur in a race with 5 runners?

Solution: Let's use the factorial formula for $P(5,3)$...

$$P(5, 3) = 5! / (5-3)! = 5! / 2! = 120 / 2 = \mathbf{60}$$

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC*?

Think of *ABC* as a block, a single character. In addition to *ABC*, there are 5 other characters, for a total of 6 discrete items.

$\Rightarrow$ Total number of permutations is $P(6, 6) = 6! = \mathbf{720}$

Combinations

When we make an *unordered* selection of distinct items, it's called a **combination**

Example: Find all 2-letter combinations of the set $S = \{a, b, c\}$

The 2-permutations were: *ab, ac, ba, bc, ca, cb*

But now we throw out all the repeats, now that we don't care about order;

The 2-combinations are: *ab, ac, bc*

Repeats: *ba, ca, cb*

⇒ there are **3** 2-combos of the letters in S

Combinations

We can come up with a formula for the number of r-combinations ($C(n,r)$) by starting with the formula for the number of r-permutations ($P(n,r)$), and the **product rule**:

In English:

[# r-permutations of n elements] = [# pick r elements from n] $\times$ [# ways to arrange them]

Combinations

We can come up with a formula for the number of *r*-combinations ($C(n,r)$) by starting with the formula for the number of *r*-permutations ($P(n,r)$), and the **product rule**:

In English:

[# *r*-permutations of *n* elements] = [# pick *r* elements from *n*] × [# ways to arrange them]

$$P(n, r) \quad = \quad C(n, r) \quad \times \quad P(r, r)$$

Which means:

$$C(n, r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!\, r!} = \binom{n}{r}$$

Alternative notation: Since we are *choosing* *r* elements from *n* elements, sometimes we say "*n* choose *r*" and denote it using the **binomial coefficient** (on the right)

Combinations

Example: How many 5-card poker hands can be drawn from a 52-card deck?

Example: There are about [650 computer science] majors at CU. How many ways are there to choose 6 of them to serve as on an undergraduate CS committee?

Combinations

Example: How many 5-card poker hands can be drawn from a 52-card deck?

We don't care about the order in which the cards are received so we want **5-combinations**

$$C(52,5) = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1}$$

= … = **2,598,960 distinct 5-card poker hands**

Example: There are about 650 computer science majors at CU. How many ways are there to choose 6 of them to serve as on an undergraduate CS committee?

Solution: We want $C(650, 6)$ = … = (a very large number)

Combinations

Example: How many bit-strings of length 10:

a) Contain exactly four 1's?

b) Contain at most four 1's?

Combinations

Example: How many bit-strings of length 10:

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b) Contain at most four 1's?

Solution:

a) Choose where the four 1s go:

$$C(10,4) = \frac{10!}{6!\ 4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

b) Count up the # ways to have zero 1s, one 1, two 1s, three 1s, or four 1s:

$C(10, 0) + C(10, 1) + C(10, 2) + C(10, 3) + C(10, 4) = \ldots = \mathbf{386}$

Permutations and combinations

It can be tricky to know when to use permutations, or combinations, or both.

Example: The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain exactly 2 vowels?

Permutations and combinations

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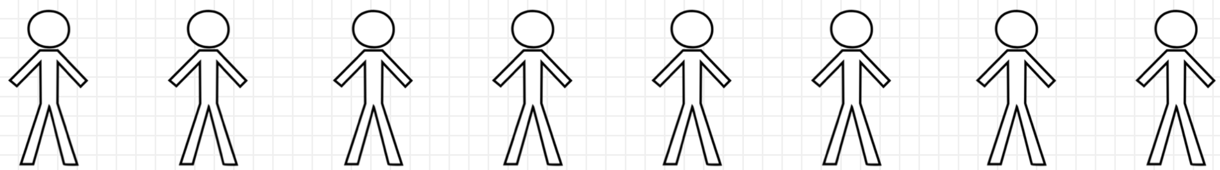
Solution: by **product rule**, we have 3 tasks:

1. Pick where the vowels go $\Rightarrow$ $C(6, 2)$
2. Pick what the vowels are $\Rightarrow$ 5×5
3. Pick what the consonants are $\Rightarrow$ $21 \times 21 \times 21 \times 21$

$\Rightarrow$ There are $C(6, 2) \times 5^2 \times 21^4 = 15 \times 25 \times 21^4 = \ldots =$ **72,930,375 such strings**

Permutations and combinations

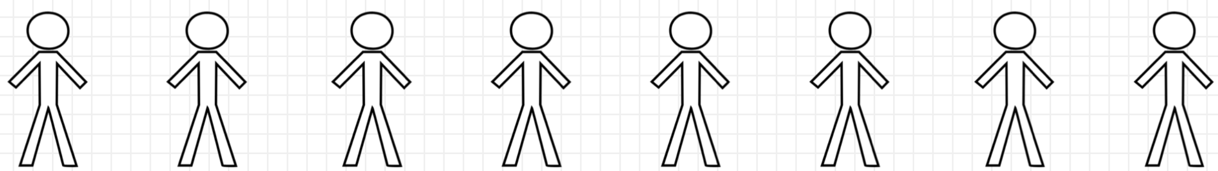
Example: How many ways are there for eight regular people and the five dysfunctional members of the Scooby Doo gang to stand in a line so that no two members of the Scooby Doo gang stand next to each other?



Permutations and combinations

Example: How many ways are there for eight regular people and the five dysfunctional members of the Scooby Doo gang to stand in a line so that no two members of the Scooby Doo gang stand next to each other?

Solution:



Product rule, 2 tasks:

- 9 places SDG can stand, need to pick 5, and order matters $\Rightarrow$ $P(9, 5)$
- Arrange the 8 regular people $\Rightarrow$ $P(8, 8)$

$\Rightarrow$ Have $P(8, 8) \times P(9, 5) = (8!) \times (9! / 4!) =$ **609,638,400 possible arrangements**

Permutations and combinations

FYOG: A club has 25 members. How many ways are there to choose a president, a vice president, a secretary and a treasurer? (Assume each member can hold at most one position)

FYOG: How many strings of six letters contain *at least* one vowel?

FYOG: How many strings of six letters contain *at least* two vowels?

FYOG: S'pose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men as women?

Permutations and combinations

Previously: calculate # of **permutations** and **combinations** of *r* objects from a set of *n* total objects **without repetition**

→ Care about order? **permutations:**

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

→ Order doesn't matter? **combinations:**

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Now: we want to count permutations and/or combinations **with repetition**

Permutations with repetition

… **return of the product rule!**

Example: How many strings of length 4 can be formed using the uppercase letters of the English alphabet?

Permutations with repetition

… **return of the product rule!**

Example: How many strings of length 4 can be formed using the uppercase letters of the English alphabet?

Solution: Since order matters (*AMCE* is different from *MACE*, e.g.), we want **permutations**

⇒ We can bring back the analogy of filling slots:

[26 choices for 1st char.] x [26 choices for 2nd char.] x [26 choices for 3rd char.] x [26 choices for 4th char.]

$$= 26^4$$

Theorem: The number of *r*-permutations of *n* objects with repetition is n^r

Combinations with repetition

Example: S'pose that a bakery in the Engineering Center Lobby has 4 different kinds of donuts. How many different ways can I select 6 delicious, ***delicious*** donuts?

Combinations with repetition

Example: S'pose that a bakery in the Engineering Center Lobby has 4 different kinds of donuts. How many different ways can I select 6 delicious, ***delicious*** donuts?

Solution: Since order does *not* matter (I am for sure going to eat them all anyway), we want **combinations**

Theorem: There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements where repetition of elements is allowed.

So we have: $C(n+r-1, r) = C(9,6) = \dfrac{9!}{(9-6)!\,6!} = \dfrac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1} = 84$

Okay… but why does this work?

Combinations with repetition

Theorem: There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements where repetition of elements is allowed.

This problem is asking us to distribute our 6 donuts choices among the 4 donut types. We could represent each of our donut choices with a ★, like so:

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★★

★

★★★

|

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And we only really need to represent where the *divisions* between donut categories are. Let's use a bar (|) for this:

★★ | ★ | ★★★ |

This gives 2 glazed, 1 chocolate-frosted, 3 pink sprinkles, and 0 jelly-filled

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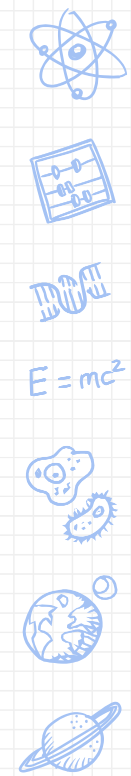
Upshot: the problem is reduced to asking how many different ways can we arrange r stars and n-1 bars.

Combinations with repetition

Upshot: the problem is reduced to asking how many different ways can we arrange *r* stars and *n*-1 bars.



Each unique arrangement of *r* stars (the things we're selecting) and *n*-1 bars (the divisions between the bins) gives a different distribution of our selections.

We have *n*+*r*-1 total objects (stars and bars) and need to pick *r* of them to be stars:

$C(n+r-1, r)$

Similarly, we could also pick *n*-1 of our objects to be bars:

$C(n+r-1, n-1)$

| ★ |★★★★★ |

Combinations with repetition

And these two quantities are equal:

$$C(n+r-1,r) = \frac{(n+r-1)!}{(n+r-1-(r))!\ r!} = \frac{(n+r-1)!}{(n-1)!\ r!}$$

... and...

$$C(n+r-1,n-1) = \frac{(n+r-1)!}{(n+r-1-(n-1))!\ (n-1)!} = \frac{(n+r-1)!}{r!\ (n-1)!}$$

Combinations with repetition

Example: How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2 and x_3 are all nonnegative integers?

Combinations with repetition

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Solution: Think of each of x_1, x_2 and x_3 as the bins into which we must sort 11 items. Then this is stars 'n' bars with 11 stars and 2 bars. For example, one possibility is

★★★ | ★★ | ★★★★★★

which represents $x_1 = 3$, $x_2 = 2$, $x_3 = 6$.

So we have $C(n+r-1,r) = C(13,11) = \frac{13!}{(13-11)!\,11!} = \frac{13 \cdot 12}{2 \cdot 1} = 78$

Combinations with repetition

Example: How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, if x_1, x_2 and x_3 are all nonnegative integers that satisfy $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$?

Combinations with repetition

Example: How many solutions does the equation

$x_1 + x_2 + x_3 = 11$

have, if x_1, x_2 and x_3 are all nonnegative integers that satisfy $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$?

Solution: This is the same as the previous problem, except we first need to allocate 1 star to the left-most bin, 2 stars to the center bin, and 3 stars to the right-most bin.

★ | ★★ | ★★★

So we're left with 5 stars to distribute among the 3 bins (separated by 2 bars) as we wish.

So we have $$C(n+r-1, r) = C(7,5) = \frac{7!}{(7-5)!\,5!} = \frac{7 \cdot 6}{2 \cdot 1} = 21$$

Permutations/combinations with repetition

Useful Recap:
(Rosen, p. 427)

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Next time:

- More permutations and combinations, and the Binomial Theorem
- More ***counting***, and moving toward ***probability!***

Pigeonhole principle

Example: Show that for any number *n* there is a multiple of *n* that has only 0s and 1s in its decimal representation.

Pigeonhole principle

Example: Show that for any number n there is a multiple of n that has only 0s and 1s in its decimal representation.

Solution: Consider the n+1 numbers 1, 11, 111, …, $111^{\cdots}11$, where the last number has n+1 ones in it.

Note that if we divide a number by n, there are only n possible remainders:

$$0, 1, 2, \ldots, n-1$$

Since there are n+1 numbers above but only n possible remainders, **by PHP** at least 2 of them must have the same remainder when divided by n.

⇒ Let those numbers be a and b, where $b > a$

⇒ $n \mid (b-a)$, which means that $b-a$ is a multiple of n

⇒ $b-a$ is a number made up of only 0s and 1s (e.g., b=111 and a=11 give $b-a$ = 100)

▨

Pigeonhole principle

Example: Show that for any number *n* there is a multiple of *n* that has only 0s and 1s in its decimal representation.

An example of the example: S'pose we let *n* = 7

We check: 1 11 111 1,111 11,111 111,111 1,111,111 11,111,111

Then (it turns out) 1,111,111 ≡ 1 **mod** 7 so 1 and 1,111,111 have the same remainder when divided by 7

⇒ 1,111,111 - 1 = 1,111,110 is divisible by 7

(1,111,110 / 7 = **158,730**)

```
In [5]: n=7
        numbers = [1]
        for i in range(1,n):
            numbers.append(10**i + numbers[i-1])

        remainders = []
        for i in range(0,n):
            remainders.append(numbers[i]%n)

In [6]: print(remainders)

        [1, 4, 6, 5, 2, 0, 1]
```