CSCI 2824: Discrete Structures

Lecture 26: Intro to Discrete Probability

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Written HW 10 – posted

– Due Friday Nov. 15th at noon

Quizlet 09 – Due Monday at 8pm

Discrete Probability

Discrete Probability: concerned with the probability of a finite event occurring

Probability of an Event: $\dfrac{\text{number of ways the event can occur}}{\text{number of possible outcomes}}$

Example: Suppose the experiment is to roll a die and the event is that an even number is rolled. Find the probability of this event.

$$P(\text{even roll on a fair die}) = \frac{|E|}{|S|} = \frac{3}{6} \boxed{= \frac{1}{2}}$$

possible outcomes = aka the "Sample Space"

$= \{1, 2, 3, 4, 5, 6\} = S$

event $= \{2, 4, 6\} = E$

Discrete Probability

Experiment: A procedure that yields one of a given set of possible outcomes.

Sample Space: The set of possible outcomes of an experiment. S

Event: A subset of the sample space. E

Example: Suppose the experiment is flipping 3 coins. We want to know the probability that all three coins show the same side of the coin. $P = \frac{|E|}{|S|} = \frac{2}{8} = \frac{1}{4}$

Sample Space: The set of all possible combos of the three flips:

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Event: The set of all combos where all three coins are the same: $E = \{HHH, TTT\}$

Discrete Probability

If S is a finite nonempty sample space of *equally likely* outcomes, and E is an event, that is, a subset of S, then the **probability of E**, written $p(E)$, is

$$p(E) = \frac{|E|}{|S|}$$

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Note that: $0 \leq p(E) \leq 1$ ← very important!

- So from the coin flip example on the previous slide: $p(E) = \frac{|E|}{|S|} = \frac{2}{8} = \frac{1}{4}$

Discrete Probability

Example: What is the probability that when two dice are rolled, the numbers on the two dice sum to 7?

$$P(\text{two dice sum to } 7) = \frac{6}{36} \boxed{= \frac{1}{6}}$$

Alternate way to visualize

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5			
3						
4						
5						
6						

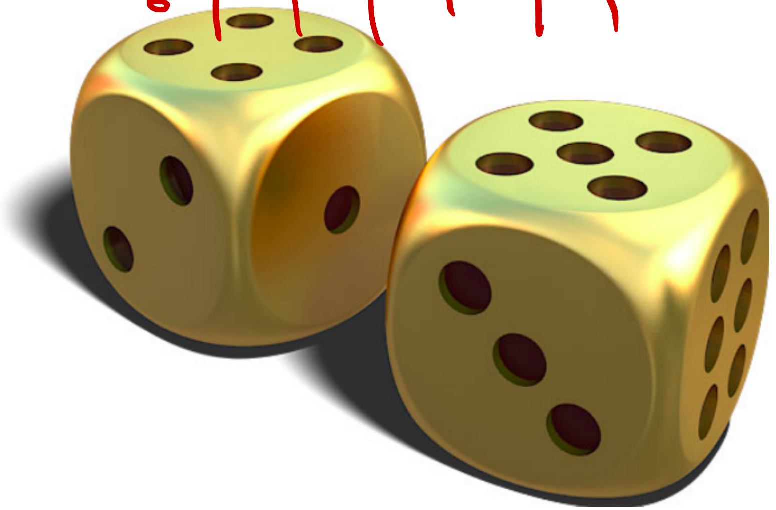
$E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$S = \{(1,1), (1,2), (1,3) \ldots (1,6)$

$(2,1) (2,2), (2,3) \ldots (2,6)$

$\vdots$

$\}$



Discrete Probability

2 3 1 20 21 17

Example: Consider a lottery that awards a large prize to anyone that correctly chooses a set of 6 distinct numbers out of a possible n numbers. What is the probability of winning a lottery that selects 6 numbers out of 40?

$$P(\text{winning the lottery}) = \frac{\text{total number of ways to win}}{\text{total number of ways to choose 6 numbers from 40}}$$

$$= \frac{1}{\binom{40}{6}}$$

$$= \frac{1}{3{,}838{,}380} \quad \boxed{\approx .00000026}$$

Discrete Probability

Example: In Powerball, contestants select 5 white balls and 1 Powerball. The white balls are picked at random from a drum that holds 69 balls numbered from 1 to 69. The Powerball number is a single ball that is picked from a second drum that has 26 numbers ranging from 1 to 26. What is the probability of winning the Jackpot?

$$P(\text{Jackpot winner}) = \frac{1}{C(69,5)\cdot C(26,1)}$$

$$= \frac{1}{292{,}201{,}338}$$

$$\boxed{\approx .000000003422}$$

Discrete Probability

4 suits : ♦ ♣ ♥ ♠ ; 13 values: 2, 3, 4 ... 10
J, Q, K, A

Example: What is the probability that a person is dealt a flush in a 5-card poker game? A flush is 5 cards of the same suit, but not in sequential order.

ways to have the suit   # ways to choose 5 values

$$C(4,1) \cdot C(13,5)$$

How many ways to get cards in sequential order?

A2345, 23456, 34567, 45678, 56789, 678910, 78910J, 8910JQ, 910JQK, 10JQKA

$* 4$ suits $= 4 * 10$ ways $=$ 40 ways to have the cards in sequential order.

ways to get a flush $= C(4,1) \cdot C(13,5) - 40 = 5108$

$$\boxed{P(\text{flush}) = \frac{5108}{C(52,5)}}$$

$$\boxed{\approx .001965}$$

Discrete Probability

Example: What is the probability of drawing a Full House in a 5-card poker hand? A Full House is 3 cards of one value and 2 cards of another value.

e.g. 2♦, 2♥, 5♠, 5♦, 5♣

2♦, 2♥, 2♣, 5♦, 5♣

- calculate the number of ways to distribute the two values.
- calculate the number of ways to be dealt 3 cards of same value
- calculate the number of ways to be dealt 2 cards of same value.

ways to be dealt a full house $= P(13,2) \cdot C(4,3) \cdot C(4,2) = 3744$

$$\boxed{P(\text{full house}) = \frac{3744}{C(52,5)} \boxed{\approx .001441}}$$

Discrete Probability

The probability of an event E is the sum of the probabilities of the outcomes in E.

$$p(E) = \sum_{s \in E} p(s)$$

The probability of an event occurring is inversely proportional to its desirability.

Discrete Probability

What if we want to **model events that aren't equally likely**? e.g. a loaded die or an unfair coin

Let S be a sample space of an experiment with a finite number of outcomes. We assign a probability $p(s)$ to each outcome s. We require that two conditions be met:

1. $0 \leq p(s) \leq 1$ for any $s \in S$
2. $\sum_{s \in S} p(s) = 1$

The function p from the set of all outcomes of the sample space S is called a **probability distribution**.

Extra Practice

EX. 1 What is the probability that when a coin is flipped 6 times it comes up heads all 6 times?

EX. 2 What is the probability that a five-card poker hand contains at least one ace?

EX. 3 What is the probability that a five-card poker hand contains a straight, that is, five cards that have consecutive values?

EX. 4 What is the probability that a five-card poker hand contains cards of five different values and does not contain a flush or a straight?

EX. 5 What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

Solutions

EX. 1 What is the probability that when a coin is flipped 6 times it comes up heads all 6 times?

There are $2^6 = 64$ ways to flip 6 coins

In only 1 of those ways do the coins come up all heads

So $p(E) = \dfrac{1}{64} \approx 0.016$

EX. 2 What is the probability that a five-card poker hand contains at least one ace?

Easier to compute the probability that no aces appear and then use the complement rule. There are 48 cards in the deck that *aren't* aces, so the number of ways to choose 5 non-aces is $C(48, 5)$

Then the probability of picking no aces is

$$p(E) = \frac{C(48, 5)}{C(52, 5)}$$

The probability of picking at least one ace is

$$p(\bar{E}) = 1 - p(E) = 1 - \frac{C(48, 5)}{C(52, 5)} \approx 0.34$$

EX. 3 What is the probability that a five-card poker hand contains a straight, that is, five cards that have consecutive values?

We can specify the hand by first choosing the lowest card in the straight from the set $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (Note, there is a straight that starts with 10 because 10-J-K-Q-A is a straight)

There are $C(10, 1)$ ways to do this

Next we need to choose the suits of each card. Since there are 5 cards and 4 suits there are $C(4, 1)^5 = 4^5$ ways to do this

So by the product rule this gives $10 \cdot 4^5 = 10, 240$ hands

The probability is then $p(\mathrm{E}) = \frac{10{,}240}{2{,}598{,}960} \approx 0.00394$

Note: Excluding straight flushes yields a slightly smaller probability

EX. 4 What is the probability that a five-card poker hand contains cards of five different values and does not contain a flush or a straight?

First we need to compute the number of hands that contain cards of 5 different values, then we'll subtract off the flushes and straights

There are $C(13, 5)$ ways to choose 5 values, and for each card there are $C(4, 1)$ ways to choose the suit. Thus there are

$$C(13, 5) \cdot C(4, 1)^5 = 1317888 \text{ hands with } 5 \text{ different values}$$

Of those, 10240 of them are straights and 5148 of those are flushes. But of those, 40 of them are straight flushes. So the number of hands of the desired type is

$$1317888 - (10240 + 5148 - 40) = 1302540$$

EX. 4 What is the probability that a five-card poker hand contains cards of five different values and does not contain a flush or a straight?

The probability of drawing a 5-card hand with 5 different values and does not contain a flush or a straight is then

$$p(\mathrm{E}) = \frac{1,302,540}{2,598,960} \approx 0.501$$

EX. 5 What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

There are 20 numbers between 1 and 100 that are divisible by 5

There are 14 numbers between 1 and 100 that are divisible by 7

But 2 of those numbers (35 and 70) are divisible by both 5 and 7

Thus the probability is

$$p(E) = \frac{20 + 14 - 2}{100} = \frac{32}{100} = 0.32$$