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CSCI 2824: Discrete Structures

Lecture 34: Relations

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Relations

Warmup Example: Let $A = \{\alpha, \beta, \gamma\}$ and let $B = \{\pi, \tau\}$.

What is $A \times B$?

$$\{(\alpha,\pi), (\alpha,\tau), (\beta,\pi), (\beta,\tau), (\gamma,\pi), (\gamma,\tau)\}$$

Give an example of a subset of $A \times B$.

$$\{(\alpha,\pi), (\beta,\tau)\}$$

How many possible subsets of $A \times B$ are there?

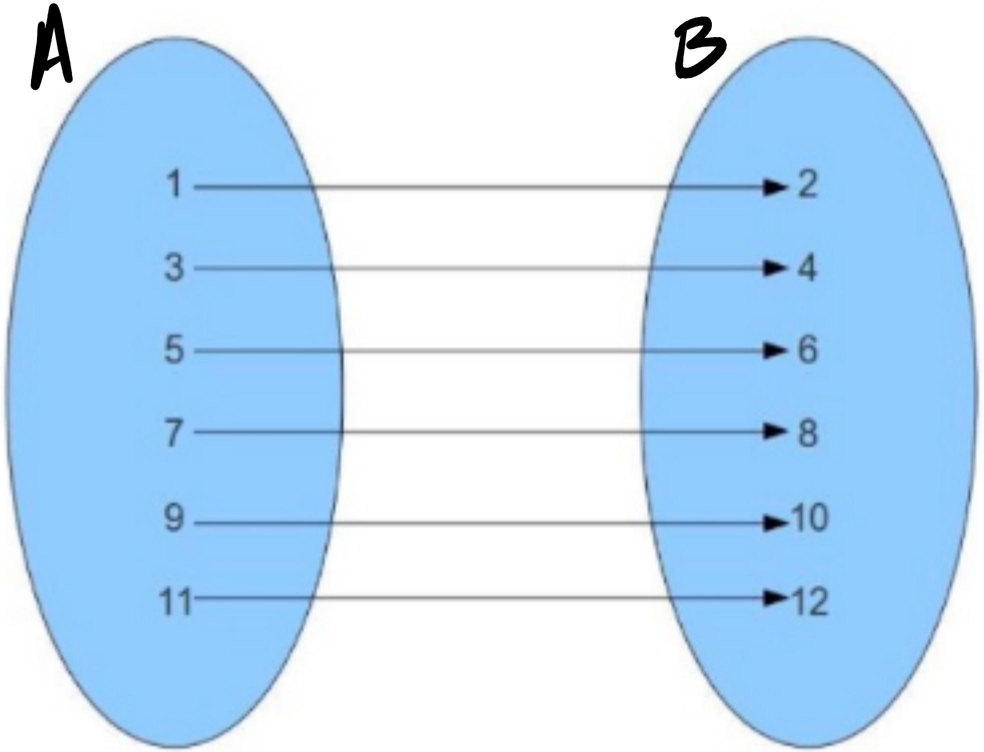
$$2^6$$

Relations

A **function** f from a set A to a set B maps every element of A to some element of B.
We write $f: A \to B$

domain codomain

Example: Let $A = \{1, 3, 5, 7, 11\}$ and $B = \{2, 4, 6, 8, 12\}$



$(1,2), (3,4), (5,6),$
$(7,8), (9,10), (11,12)$

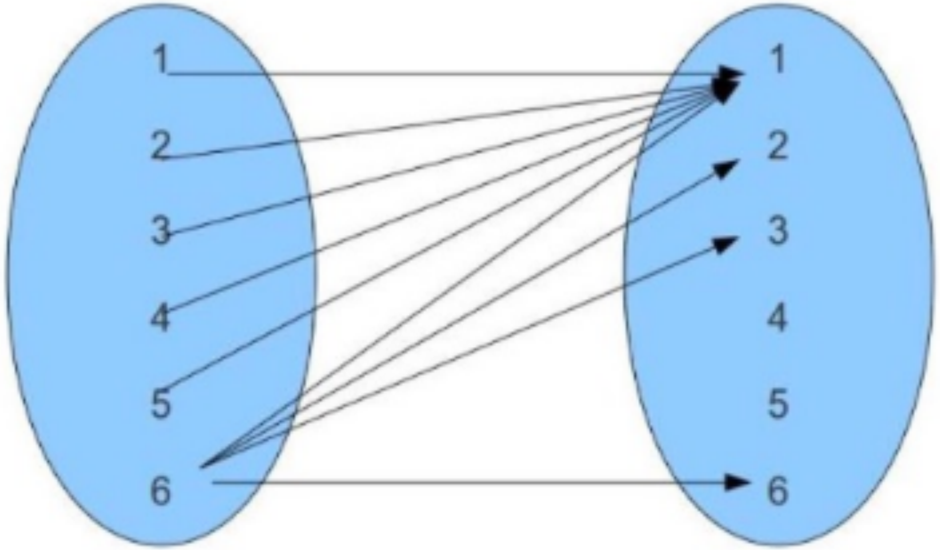
Relations

$\boxed{\text{A \textbf{relation} } R \text{ is a subset of } A \times B\text{, i.e. } R \subseteq A \times B}$

Example: Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 3, 4, 5, 6\}$

Consider the relation R defined by

$$R = \{(1,1), (2,1), (3,1), (4,1), (5,1), (6,1), (6,2), (6,3), (6,6)\}$$



All functions are relations, but not all relations are functions.

Unlike a function, in a relation it is possible that:

- $x \in A$ is not related to any element in B
- $x \in A$ could be related to multiple elements in B

Relations

$\mathbb{N}$ – natural numbers
$\{0, 1, 2, 3, \ldots\}$

Example: Let $R \subseteq \mathbb{N} \times \mathbb{N}$ be $R = \{(m, n) | \ m \text{ divides } n\}$

e.g. elements $\mathbb{N} \times \mathbb{N}$
$(0,0)$, $(1,7)$, $(7,2)$ ...

- $(2, 4) \in R$ because 2 divides 4
- $(2, 6) \in R$ because 2 divides 6
- $(6, 2) \notin R$ because 6 does not divide 2

$(4, 64)$, $(3, 3)$, $(3, 9)$ ...

$(21, 84)$ ...

Relations

Many useful relations are defined from a set to itself.

Examples:

Relations of the form $R \subseteq \text{People} \times \text{People}$

- Facebook friends
- Twitter follower and followee

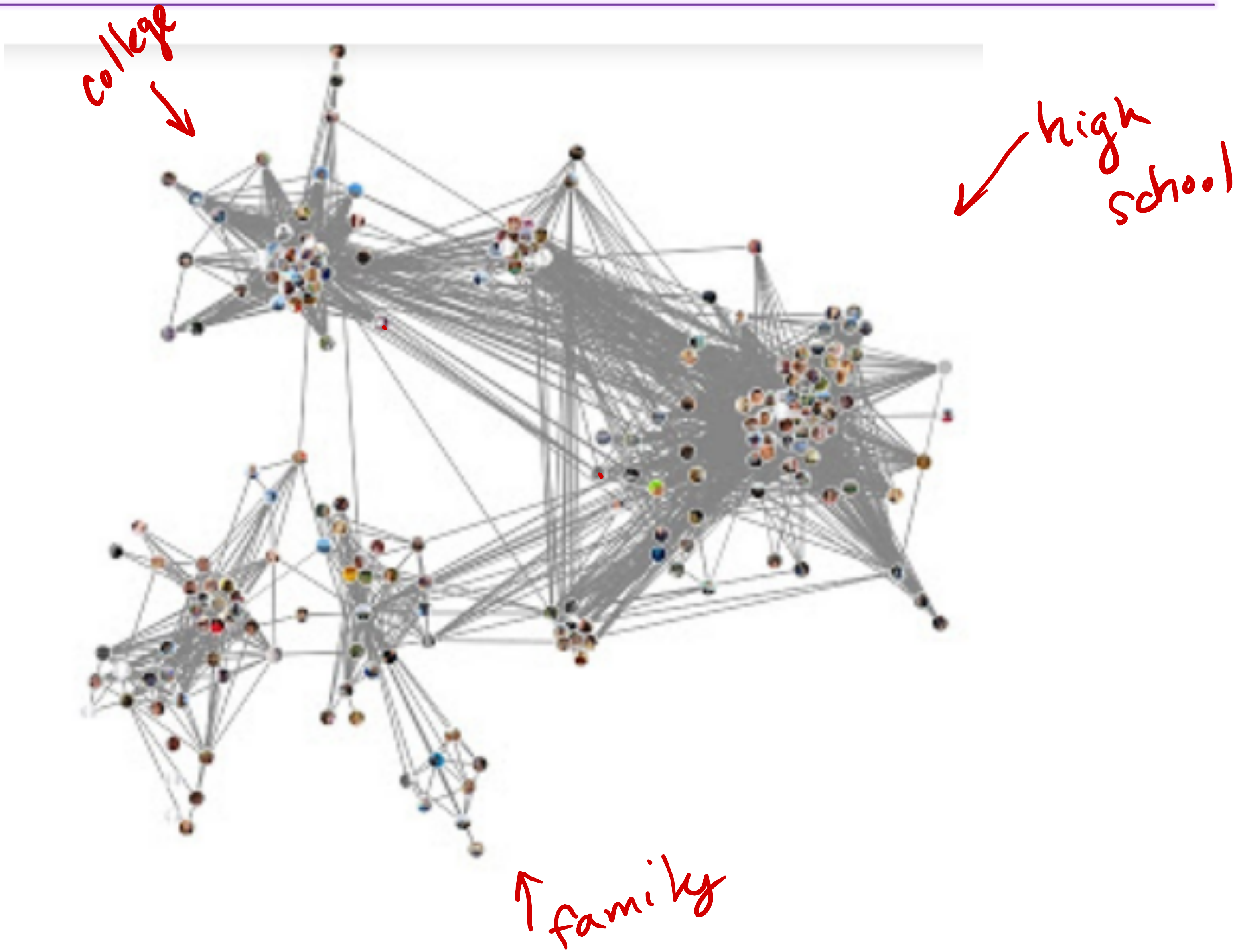
Other types of relations:

- Determining which cities are linked by airline flights
- Partitioning sets into groups of equivalent members
- Building and interacting with databases
- Analyzing graphs

Relations

Facebook friends visualization

"Lost Circles"



Relations

Example: Let A be the set of all students at CU and let B be the set of all Computer Science courses. Let R be the relation defined by

$$R = \{\, (person, class) | person \text{ is enrolled in } class \,\}$$

What are some things that are in R ?

$A = \{$ Tim, Tim 2, Sue, James, ... $\}$

$B = \{$ CSCI 1300, 2400, Discrete, Data Science.... $\}$

(Tim 2, 2400), (James, CSCI 1300) $\in R$

Relations

We now restrict ourselves to relations of the form $R \subseteq A \times A$

A ***graph*** G consists of a set A of vertices and a relation $R \subseteq A \times A$ of edges. Each edge in a graph of the form $a \to b$ corresponds to a pair $(a, b) \in R$.

A = people on facebook

people on facebook × people on facebook

(Rachel, Jacob)

(Jacob, Rachel)

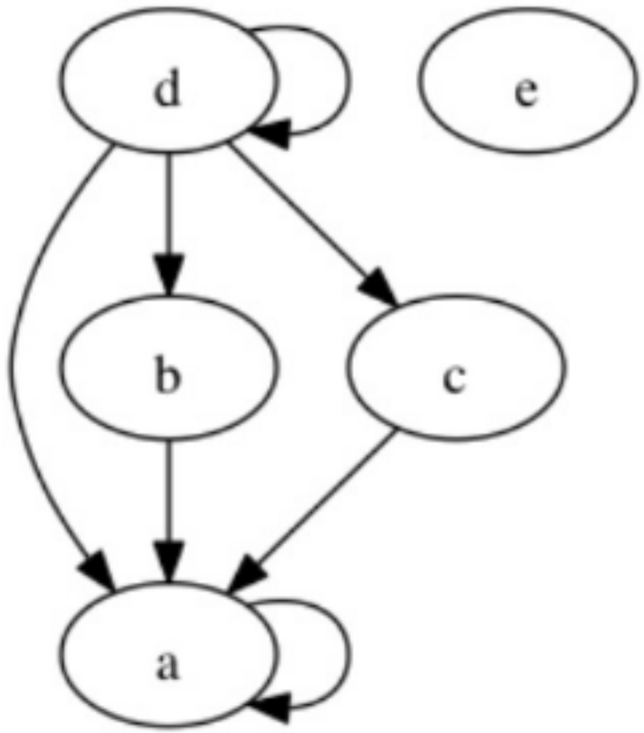
In graph theory:

- the node set is called V (for vertices)
- the relations E (for edges)

Relations

Example: Let $A = \{a, b, c, d, e\}$ be the set of vertices and R_1 be

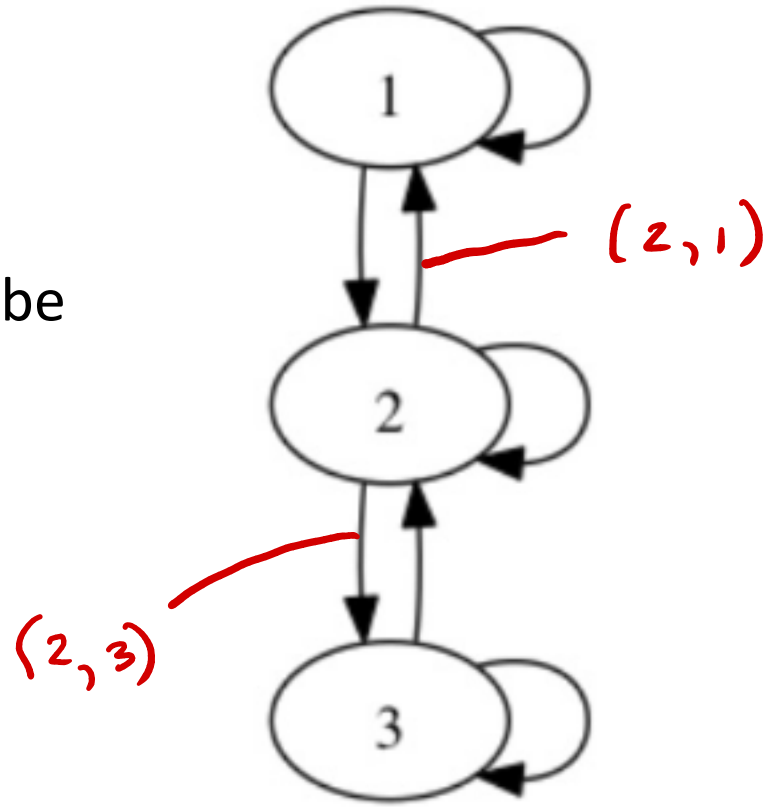
$$R_1 = \{(a,a), (b,a), (c,a), (d,a), (d,b), (d,c), (d,d)\}$$



Relations

Example: Let $A = \{1, 2, 3\}$ be the set of vertices and R_2 be

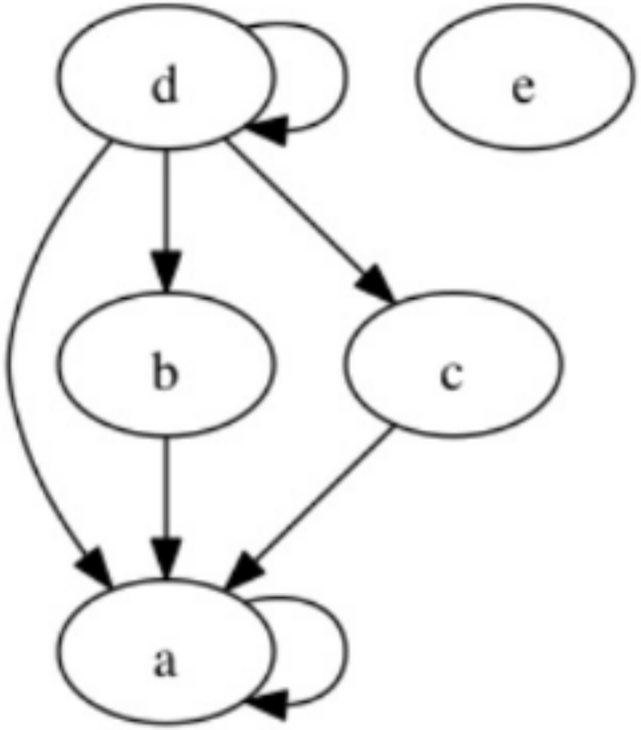
$$R_2 = \{(1,1), (1,2), (2,1), (2,3), (3,2), (2,2), (3,3)\}$$



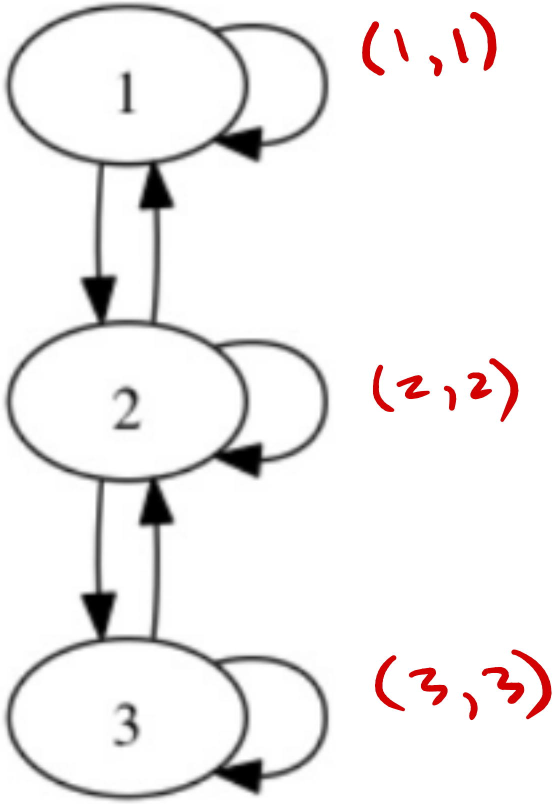
Relations

A relation is ***reflexive*** if and only if $(a, a) \in R$ for all $a \in A$

Not reflexive



Reflexive

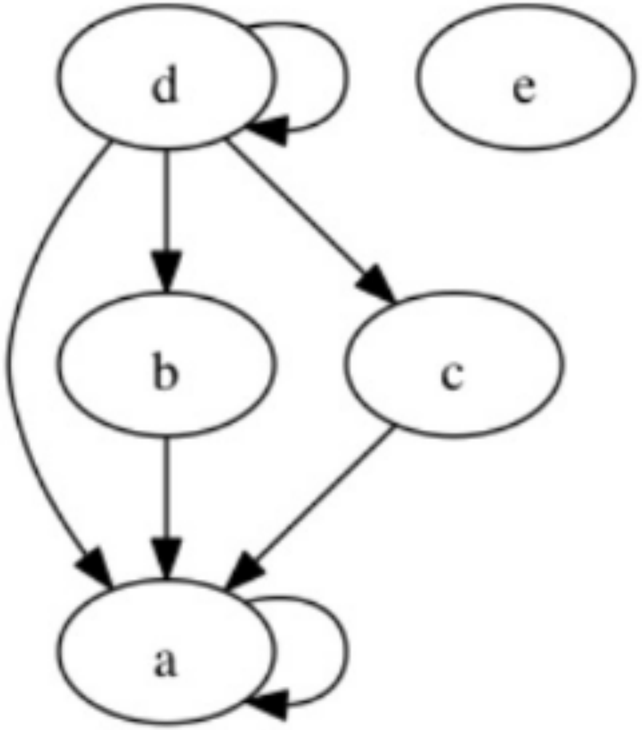


❖ A relation is reflexive if all nodes in the graph have self-loops.

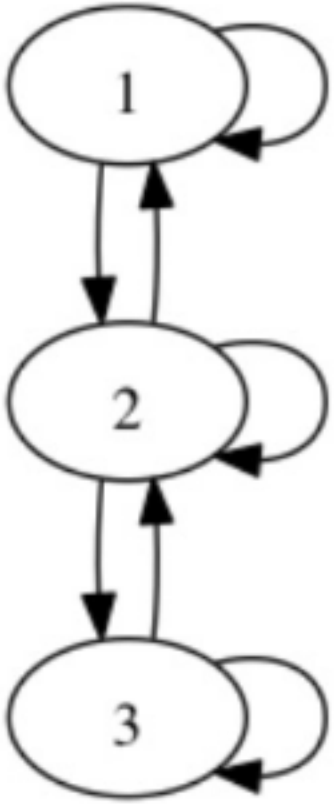
Relations

A relation is ***symmetric*** if and only if for all $(a, b) \in R, (b, a) \in R$

Not symmetric



Symmetric



❖ If there is an edge from a to b then there is also an edge from b to a

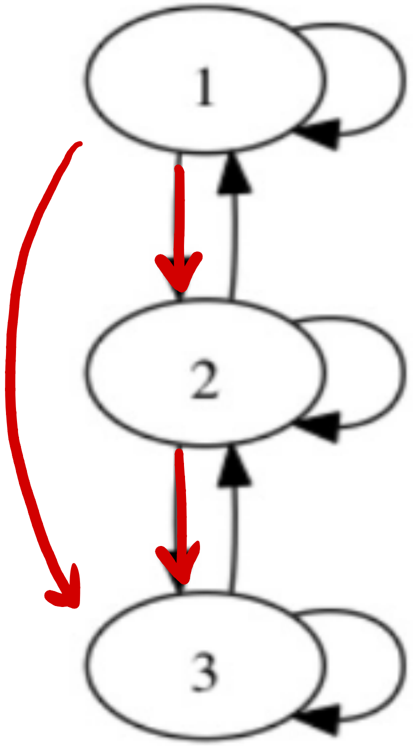
Relations

A relation is ***transitive*** if and only if for all a, b, c
IF $(a, b) \in R$ **AND** $(b, c) \in R$ **THEN** $(a, c) \in R$

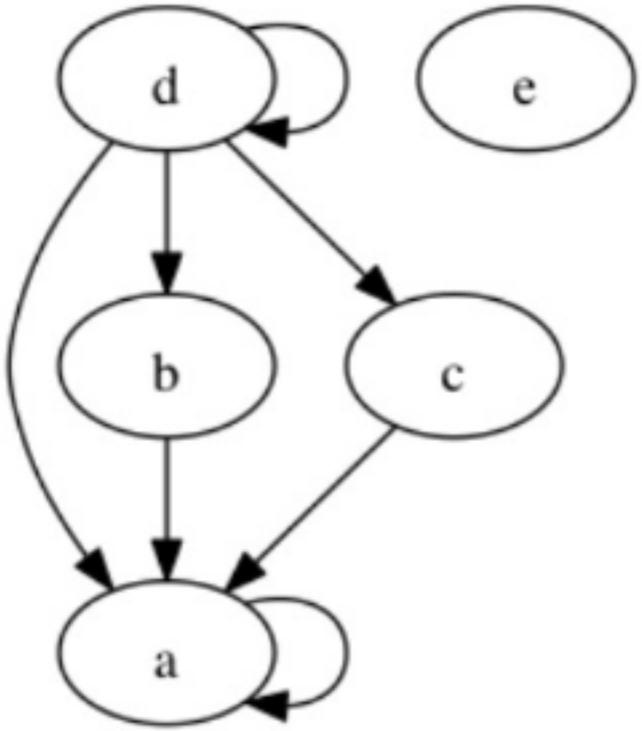
doesn't exist!

(1,3)

Not Transitive



Transitive



- If there is an edge from a to b and an edge from b to c, then there is also an edge from a to c.

Relations

A relation that is reflexive, symmetric, and transitive is called an ***equivalence relation***.

Example: Which of the following relations are equivalence relations on the set $A = \{1, 2, 3, 4\}$?

$$R_1 = \{(1,2), (2,3), (1,3), (2,2)\}$$
$$R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$$

R_1 : Reflexive? no
symmetric? no
transitive? yes!

Not an equivalence relation

R_2 : Reflexive - yes
symmetric - yes
transitive - yes

IS an equivalence relation

A way to think about these new definitions:

Reflexive = self-loops

Symmetric = two-way roads

Transitive = short-cuts

Relations

Example: Which of the following relations on the set $\{0, 1, 2, 3\}$ are/are not equivalence relations?

a) $\{(0,0),(1,1),(2,2),(3,3)\}$

Reflexive ✓ symmetric ✓ transitive ✓

$\boxed{\text{is an equivalence relation}}$

b) $\{(0,0),(0,2),(2,0),(2,2),(2,3),(3,3)\}$

not reflexive

$\boxed{\text{not an equivalence relation}}$

c) $\{(0,0),\underline{(0,1)},(0,2),(1,0),(1,1),(1,2),\underline{(2,0)},(2,2),(3,3)\}$

not symmetric (missing (2,1))

not transitive

$\boxed{\text{not an equivalence relation}}$

- Relations are a more general form of a function.
  - Elements in the domain do not need to be related to anything.
  - Elements in the domain can be related to multiple elements in the codomain

- Special kinds of relations:
  - Reflexive = self-loops
  - Symmetric = two-way roads
  - Transitive = shortcuts / bipass

Extra Practice

Ex. 1 Decide whether each of the following relations is reflexive, symmetric, and/or transitive. Is it an equivalence relation?

1. $R = \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$, where $A = \{1,2,3,4\}$

2. Facebook friends relation, where $A = set\ of\ all\ Facebook\ users$
(e.g. $(Rachel, Tony) \in R \Longrightarrow Rachel\ is\ friends\ with\ Tony\ on\ Facebook$)

3. Twitter's is-followed-by relation, where $A = set\ of\ all\ Twitter\ users$
(e.g. $(Rachel, \mathrm{Tony}) \in R \Longrightarrow Rachel\ is\ followed\ by\ Tony\ on\ Twitter$)

4. The taller-than relation
(e.g. $(Tony, Rachel) \in R \Longrightarrow Tony\ is\ taller\ than\ Rachel$)

5. $R = \{(m,n) |\ m - n\ is\ even\}$, where $A = \mathbb{N}$

6. $R = \{(m,n) | m \leq n\}$, where $A = \mathbb{N}$

Ex. 2 : Which of the following relations on the set of all people are/are not equivalence relations?

a) $\{(a, b) \mid a \text{ and } b \text{ have met}\}$

b) $\{(a, b) \mid a \text{ and } b \text{ share a common parent}\}$

c) $\{(a, b) \mid a \text{ and } b \text{ speak a common language}\}$

Solutions

Ex. 1 Decide whether each of the following relations is reflexive, symmetric, and/or transitive. Is it an equivalence relation?

1. $R = \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$, where $A = \{1,2,3,4\}$

    Not reflexive because $(1,1)$ and $(4,4)$ aren't in R

    Not symmetric because $(2,4) \in R$ but $(4,2) \notin R$

    Is transitive

2. Facebook friends relation, where $A = set\ of\ all\ Facebook\ users$

(e.g. $(Rachel, Tony) \in R \Longrightarrow Rachel\ is\ friends\ with\ Tony\ on\ Facebook$)

Not reflexive because you can't be Facebook friends with yourself

Is symmetric because being FB friends is a mutual thing

Is not transitive because you (probably) aren't friends with everyone that every one of your friends is friends with

3. Twitter's is-followed-by relation, where $A = set\ of\ all\ Twitter\ users$

(e.g. $(Rachel, \text{Tony}) \in R \Longrightarrow Rachel\ is\ followed\ by\ Tony\ on\ Twitter$)

Not reflexive because you can't follow yourself (As far as I know…)

Not symmetric because you don't necessarily follow everyone back

Not transitive because you (probably) don't follow everyone that the people you're following follow

EX. 1 (continued)

4. The taller-than relation

(e.g. $(Tony, Rachel) \in R \implies Tony\ is\ taller\ than\ Rachel$)

Not reflexive because you can't be taller than yourself

Not symmetric because if you're taller than your friend, they can't be taller than you

Is transitive because if Tony is taller than Rachel and Rachel is taller than Parker, then Tony is necessarily taller than Parker

5. $R = \{(m, n) |\ m - n\ is\ even\}$, where $A = \mathbb{N}$

Is reflexive because $\mathbf{m - n = 0}$ is even

Is symmetric

Is transitive

6. $R = \{(m, n) | m \leq n\ \}$, where $A = \mathbb{N}$

Is reflexive because $\boldsymbol{m \leq m}$

Not symmetric (e.g. $\mathbf{1 \leq 2}$ but it is not the case that $\mathbf{2 \leq 1}$)

Is transitive because if $\boldsymbol{m \leq n}$ and $\boldsymbol{n \leq p}$, then it must be the case that $\boldsymbol{m \leq p}$

Ex. 2 : Which of the following relations on the set of all people are/are not equivalence relations?

a) $\{(a, b) \mid a$ and b have met$\}$

- **Reflexive** because everyone has certainly met themselves
- **Symmetric** because if I've met you, then you've met me
- **NOT transitive** because I have not met everyone that you've met

⇒ so not an equivalence relation

b) $\{(a, b) \mid a$ and b share a common parent$\}$

- **Reflexive** because everyone shares parents with themselves
- **Symmetric** because if *a* shares a parent with *b*, then *b* must share that parent with *a*
- **NOT transitive** (*a* and *b* are half-siblings, and *b* and *c* could also be half-siblings)

⇒ so not an equivalence relation

c) $\{(a, b) \mid a$ and b are the same age$\}$

- **Reflexive** because everyone is their own age
- **Symmetric** because if *a* is same age as *b*, then *b* must be the same age as *a*
- **Transitive** because if *a* and *b* are the same age, and *b* and *c* are too, then *a* and *c* must be same age too ⇒ so it ***is*** an equivalence relation!