CSCI 2824: Discrete Structures

Lecture 12: Set Theory and Set Operations

Reminders

Submissions:

- Homework 4: Fri 9/27 at noon – Gradescope

Readings:

- Starting Ch. 2 – SETS
- Today (Monday): 2.1-2.2

Midterm – Tue October 1st at 6pm

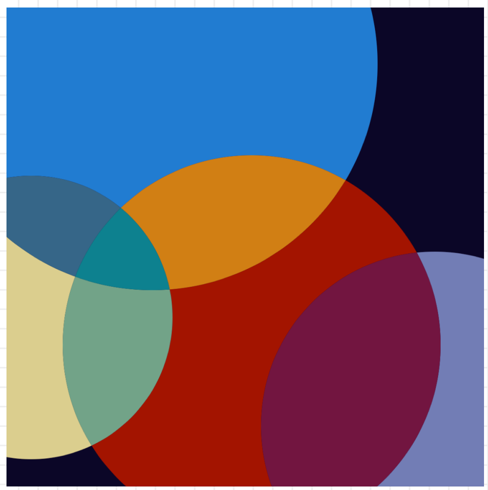
<u>Any conflicts?</u> – email csci2824@colorado.edu

What did we do last time?

- Proofs, proofs, proofs
- Used logical equivalences,
  rules of inference and propositional logic
  (you know, everything we've done so far)
  to prove things

Today:

- Introducing **Sets**



Sets

Sets and Set Operations

Definition: A **<u>set</u>** is a collection of objects, usually called **elements** or **members** of the set. A set is said to **contain** its elements.

- We write $a \in A$ to denote that a is an element of set A.
- We write $a \notin A$ to denote that a is not an element of set A.

Notation: For sets with a small number of elements, that we can actually list, we write the set with its members inside curly braces: $A = \{a, b, c, d\}$

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Example: The set of all vowels is $V =$

Example: The set of all prime numbers less than 10 is $P =$

Example: The set of all positive integers less than 100: $A =$

Sets and Set Operations

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Notation: For sets with a small number of elements, that we can actually list, we write the set with its members inside curly braces: $A = \{a, b, c, d\}$

Example: The set of all vowels is $V = \{a, e, i, o, u\}$

Example: The set of all prime numbers less than 10 is $P = \{2, 3, 5, 7\}$

Example: The set of all positive integers less than 100: $A = \{1, 2, \ldots, 98, 99\}$

Sets and Set Operations

Example: The set of all positive integers less than 100: $A = \{1, 2, \ldots, 98, 99\}$

Ø aka { }

- This is an example of using the **roster method** for conveying what elements are in the set *A*.
- Equally popular and sometime more compact is the **set builder method**:

$$A = \{x \in \mathbf{Z}^+ \mid x < 100\}$$

Sets and Set Operations

Example: The set of all positive integers less than 100: $A = \{1, 2, \ldots, 98, 99\}$

- This is an example of using the **roster method** for conveying what elements are in the set A.
- Equally popular and sometime more compact is the **set builder method**:

$$A = \{x \in \mathbf{Z}^{+} \mid x < 100\}$$

Note: this looks a lot like a quantifier statement!

But it's inside curly braces because it's a set.

Sets and Set Operations

Example: The **set builder method**:

$A = \{x \in \mathbf{Z}^+ \mid x < 100\}$

Popular sets:

$\mathbf{N} = \{0, 1, 2, \ldots\}$	natural numbers
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	integers
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$	positive integers
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$	rational numbers

R, the set of **real** numbers

R+, the set of positive real numbers

C, the set of complex numbers.

Sets and Set Operations

Important: Recall the notation for **intervals** of real numbers.

When a and b are real numbers with $a < b$, we write

- $[a, b] = \{x \mid a \leq x \leq b\}$
- $[a, b) = \{x \mid a \leq x < b\}$
- $(a, b] = \{x \mid a < x \leq b\}$
- $(a, b) = \{x \mid a < x < b\}$

$[a, b]$ is called the **closed interval** from a to b

(a, b) is called the **open interval** from a to b

Sets and Set Operations

Sets can have pretty much anything in them. Even other sets

Example: $A = \{\mathbf{N}, \mathbf{Q}, \mathbf{Z}^+\}$

Fun fact: Sets have no ordering.

- So $\{1, 2, x, y\} = \{x, 1, y, 2\}$ (or however you want to rearrange it)

Fun fact: It doesn't matter if an element of a set is repeated

- So $\{1, 2, x, y\} = \{1, 1, 1, 2, x, y\}$

Definition: Two sets are **equal** if and only if they contain the same elements. So if A and B are sets, we say A and B are equal if and only if

$$\forall x\, (x \in A \Leftrightarrow x \in B)$$

Sets and Set Operations

Definition: The set A is a **<u>subset</u>** of another set B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of B.

Question: How can we use quantifiers to denote $A \subseteq B$?

Example: Finish the sentences:

a. The set of all integers **Z** is a subset of …
b. The set of all rational numbers **Q** is a subset of …
c. The set of all CSCI 2824 students is a subset of …

Sets and Set Operations

Definition: The set A is a **subset** of another set B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of B.

Question: How can we use quantifiers to denote $A \subseteq B$?

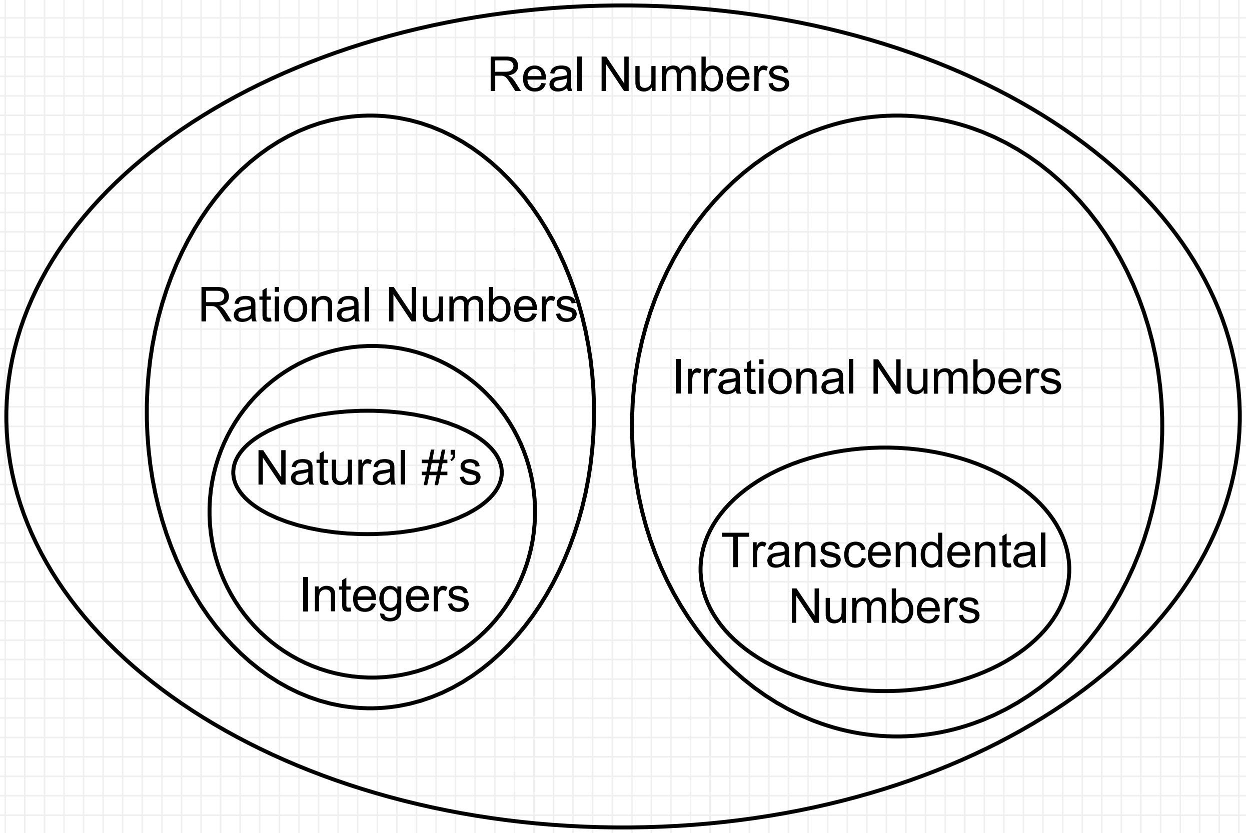
$$\forall x \; ((x \in A) \to (x \in B))$$

Example: Finish the sentences:

a. The set of all integers **Z** is a subset of … **Q, R**
b. The set of all rational numbers **Q** is a subset of … **R**
c. The set of all CSCI 2824 students is a subset of … {all CU students}

{all people}

Sets and Set Operations



Definition: The set A is a **subset** of another set B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of B.

Sets and Set Operations

Strategy: To show $A \subseteq B$, we must show that every element of A is also an element of B

To show $A \nsubseteq B$, we must find at least one element of A that is not an element of B

Question: How do you feel about the following?

$$\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$$

Sets and Set Operations

Strategy: To show $A \subseteq B$, we must show that every element of A is also an element of B

To show $A \nsubseteq B$, we must find at least one element of A that is not an element of B

Question: How do you feel about the following?

$$\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$$

Definition: If we want to emphasize the fact that $A \subseteq B$ but $A \neq B$, we say that A is a **<u>proper subset</u>** of B and we write $A \subset B$.

Example: The set of all even integers is a **proper subset** of the integers.

$$\{x \in \mathbf{Z} \mid x \text{ is even}\} \subset \mathbf{Z}$$

Sets and Set Operations

To show $A \subseteq B$, we must show that every element of A is also an element of B

To show $A \nsubseteq B$, we must find at least one element of A that is not an element of B

Definition: If we want to emphasize the fact that $A \subseteq B$ but $A \neq B$, we say that A is a **<u>proper subset</u>** of B and we write $A \subset B$.

- denoted: $A \subset B$
- means: $\forall x(x \in A \rightarrow x \in B) \land \exists x(x \in B \land x \notin A)$

Sets and Set Operations

Venn Diagrams: drawing a picture can help when thinking about sets and subsets

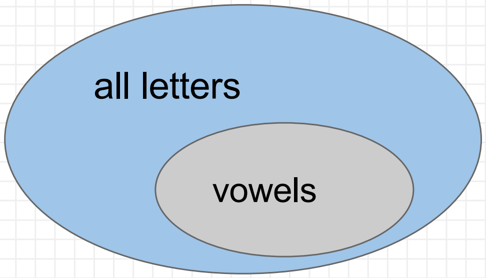
Example: Draw a Venn Diagram relating the set of all vowels to the set of all letters in the English alphabet.

Example: Draw a Venn Diagram relating the set of all prime numbers and the set of odd numbers. *(prime numbers are numbers that are only divisible by 1 and itself)*

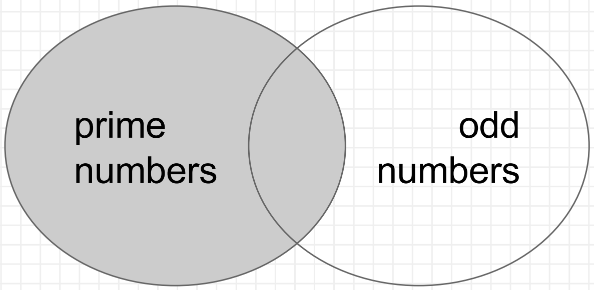
Sets and Set Operations

Venn Diagrams: drawing a picture can help when thinking about sets and subsets

Example: Draw a Venn Diagram relating the set of all vowels to the set of all letters in the English alphabet.



Example: Draw a Venn Diagram relating the set of all prime numbers and the set of odd numbers. *(prime numbers are numbers that are only divisible by 1 and itself)*



Three Special Sets

The empty set: the set with no elements, written as either Ø or { }

The singleton set: a set with only one element, for example:

{42} or {Tony} or {Ø}

The power set: ***P***(*S*) is the set of all subsets of set *S*

- Confused? That's normal.

Three Special Sets

The empty set: the set with no elements, written as either Ø or { }

The singleton set: a set with only one element, for example:

{42} or {Tony} or {Ø}

The power set: ***P***(S) is the set of all subsets of set S

Example: ***P***({0, 1, 2}) = {Ø, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2}}

Every nonempty set has at least two subsets:

Theorem: For every set S, $\emptyset \subseteq S$ and $S \subseteq S$.

Power Sets

Question: How many elements does ***P***(*S*) have?

P({0, 1, 2}) = {∅, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2}}

⇒ Each of *n* elements has 2 possibilities: in or out

Power Sets … and Cardinality

Question: How many elements does $\boldsymbol{P}(S)$ have?

$\boldsymbol{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

$\Rightarrow$ Each of n elements has 2 possibilities: in or out

Answer: If S has n elements, then $\boldsymbol{P}(S)$ has 2^n elements. (2 x 2 x 2 x … x 2 (n times))

Definition: The number of elements in a set is called its **cardinality**. If a set's cardinality is a finite number, then we say the set is **finite**. We denote the cardinality of a set A by $|A|$.

Power Sets … and Cardinality

Question: How many elements does ***P***(*S*) have?

Answer: If *S* has *n* elements, then ***P***(*S*) has 2^n elements. (2 x 2 x 2 x … x 2 (*n* times))

Definition: The number of elements in a set is called its **cardinality**. If a set's cardinality is a finite number, then we say the set is **finite**. We denote the cardinality of a set *A* by $|A|$.

Example: What is the cardinality of the English alphabet?

Example: What is the cardinality of the English vowels?

Example: What is the cardinality of the natural numbers? → **We'll come back to this!**

Cartesian Product

Let A and B be sets. The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all **ordered** pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of **ordered** n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_1 \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

Example: Think about ordered pairs of integers representing points in the xy-plane. $(3, 5)$ is very different from $(5, 3)$

Sets and set operations

Example: Consider the sets A = {Jake, David} and $B = \{1, 2, 3\}$. What is $A \times B$?

Note: $A \times B \neq B \times A$. Why?

Example: What is the Cartesian product $A \times B \times C$,
where $A = \{a, b\}$, $B = \{x, y\}$ and $C = \{m, n\}$?

Sets and set operations

Example: Consider the sets A = {Jake, David} and B = {1, 2, 3}. What is A x B?

Note: $A \times B \neq B \times A$. Why?

- unless A = ∅ or B = ∅ (so that A × B = ∅)
- or A = B

Sets and set operations

Example: Consider the sets A = {Jake, David} and B = {1, 2, 3}. What is A x B?

Note: $A \times B \neq B \times A$. Why?

Example: What is the Cartesian product $A \times B \times C$,
where $A = \{a, b\}$, $B = \{x, y\}$ and $C = \{m, n\}$?

Answer:

A x B x C = { (a, x, m), (a, x, n), (a, y, m), (a, y, n), (b, x, m), (b, x, n), (b, y, m), (b, y, n) }

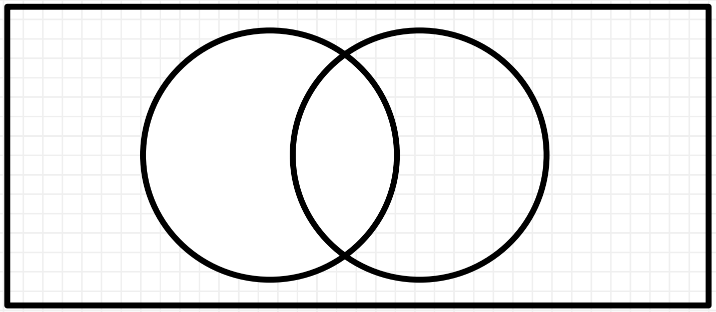
Cartesian Product

Example: Suppose that A={1, 2}. Find $A \times A$ (aka A^2) and find $A \times A \times A$ (aka A^3)

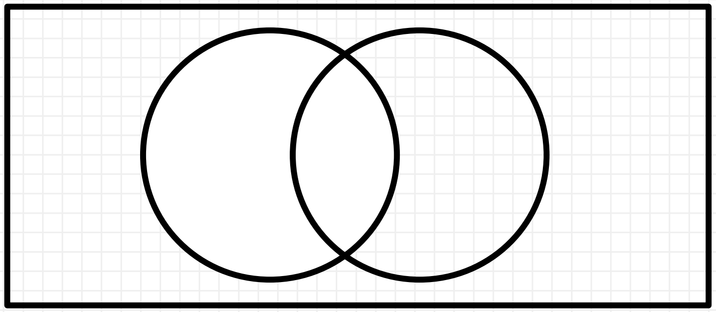
Set Operations

Venn diagrams can also be useful when trying to understand and represent set operations.

Let A and B be sets. The **union** of the sets A and B, denoted $\boldsymbol{A \cup B}$, is the set that contains those elements that are either in A or in B, or in both.



Let A and B be sets. The **intersection** of the sets A and B, denoted $\mathbf{A} \cap \boldsymbol{B}$, is the set that contains those elements that are in both A and B.



Set Operations

Example: Consider the sets: A={1, 3, 5} and B={1, 2, 3}. Find the union and the intersection of these two sets.

Example: Consider the sets: $\mathrm{A} = \{1, 3, 5\}$ and $\mathrm{B} = \{2, 4, 6\}$. Find the union and the intersection of these two sets.

Two sets are called **disjoint** if their intersection is the empty set.

Set Operations

Let A and B be sets. The difference of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted $\boldsymbol{A}-\boldsymbol{B}$, (or $A\backslash$B), is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

Example: What is the difference between A (the set of positive integers less than 10) and B (the set of prime numbers)?

Set Operations

Let A and B be sets. The difference of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted $\boldsymbol{A}-\boldsymbol{B}$, (or A\B), is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

Example: What is the difference between A (the set of positive integers less than 10) and B (the set of prime numbers)?

Sets and set operations

Definition: The **universal set**, denoted typically by U, is the set containing all elements within the domain of discourse.

- You can think about the universal set as the set containing all elements under consideration.

Example: if the domain of discourse is all CU students, then U is the set of all CU students.

Definition: Let U be the universal set. The **complement** of the set A, denoted $\bar{A}$, is the set $U - A$.

An element belongs to $\bar{A}$, if and only if $x \notin A$, so $\bar{A} = \{x \in U \mid x \notin A\}$, or just $\{x \mid x \notin A\}$

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

Strategy: To prove that two sets C and D are equal, we must

1. ($\Rightarrow$) Prove that $C \subseteq D$
2. ($\Leftarrow$) Prove that $D \subseteq C$

Proof: ($\Rightarrow$)

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

Strategy: To prove that two sets *C* and *D* are equal, we must

1. (⇒) Prove that $C \subseteq D$
2. (⇐) Prove that $D \subseteq C$

Proof: (⇒)

1. S'pose *x* is an arbitrary element in *A - B*
2. $\Rightarrow x \in A \ \wedge \ x \notin B$
3. $x \notin B \Rightarrow x \in \bar{B}$
4. $\Rightarrow (x \in A) \wedge (x \in \bar{B})$
5. $\Rightarrow x \in A \cap \bar{B}$
6. Since *x* was any arbitrary element in *A - B*, we have shown $A - B \subseteq A \cap \bar{B}$

Sets and set operations

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Proof: ($\Leftarrow$)

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1. (⇒) Prove that $C \subseteq D$
2. (⇐) Prove that $D \subseteq C$

Proof: (⇐)

1. S'pose *x* is an arbitrary element in $A \cap \bar{B}$
2. $\Rightarrow\ (x \in A) \wedge (x \in \bar{B})$
3. $x \in \bar{B}\ \Rightarrow\ x \notin B$
4. $(x \in A) \wedge (x \notin B) \Rightarrow (x \in A - B)$
5. Since *x* was any arbitrary element in $A \cap \bar{B}$, we have shown $A \cap \bar{B} \subseteq A - B$

Since $(A \cap \bar{B} \subseteq A - B)$ and $(A - B \subseteq A \cap \bar{B})$ it must be the case that

$$A - B = A \cap \bar{B} \quad \square$$

Sets and set operations

Example: Prove that $A - B = A \cap \bar{B}$

Strategy: To prove that two sets *C* and *D* are equal, we must

1. (⇒) Prove that $C \subseteq D$
2. (⇐) Prove that $D \subseteq C$

Since *x* was any arbitrary element in $A \cap \bar{B}$, we have shown $A \cap \bar{B} \subseteq A - B$

Since $(A \cap \bar{B} \subseteq A - B)$ and $(A - B \subseteq A \cap \bar{B})$ it must be the case that

$$A - B = A \cap \bar{B} \square$$

Sets and set operations

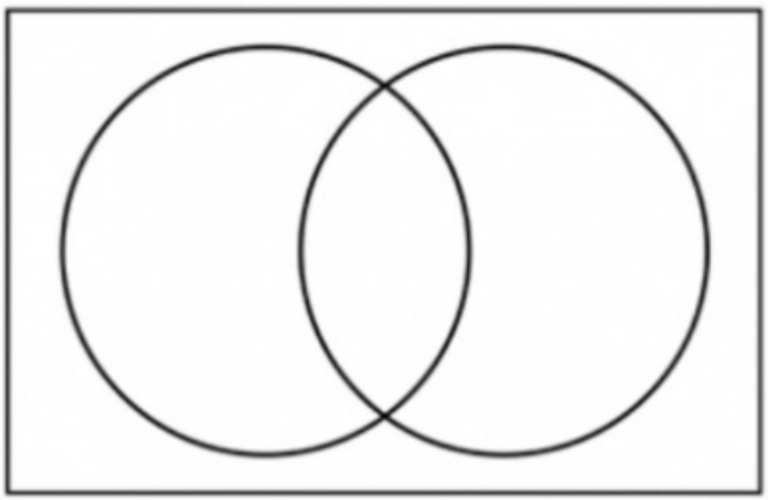
When sets are combined using only ∪, ∩, and complements, there is a set of **Set Identities** that completely mirrors the logical equivalences from last chapter.

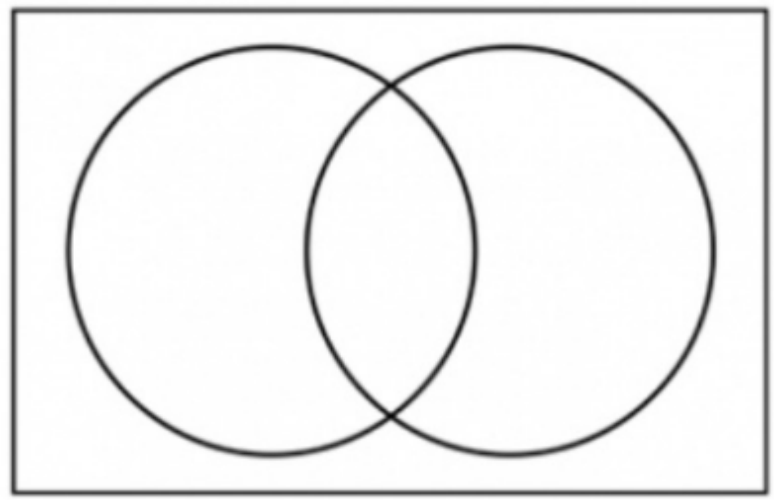
TABLE 1 **Set Identities.**

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Sets and set operations

Example: Prove DeMorgan's Law for Sets: $\overline{A \cap B} = \overline{A} \cup \overline{B}$





Sets and set operations

Example: Prove (one of) De Morgan's laws for sets, using **builder notation** $\overline{A \cap B} = \overline{A} \cup \overline{B}$
(remember, builder notation looked a lot like quantifiers, so this will look at lot like our old proofs from Ch 1)

Sets and set operations

<u>Example</u>: Prove (one of) De Morgan's laws for sets, using **builder notation** $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof: $\overline{A \cap B}$	$= \{ x \mid x \notin A \cap B \}$	Definition of complement
	$= \{ x \mid \neg(x \in A \cap B) \}$	Definition of not-in
	$= \{ x \mid \neg(x \in A \land x \in B) \}$	Definition of intersection
	$= \{ x \mid \neg(x \in A) \lor \neg(x \in B) \}$	De Morgan's
	$= \{ x \mid (x \notin A) \lor (x \notin B) \}$	Definition of not-in
	$= \{ x \mid (x \in \overline{A}) \lor (x \in \overline{B}) \}$	Definition of complement
	$= \{ x \mid x \in (\overline{A} \cup \overline{B}) \}$	Definition of union
	$= \square$	

Sets and set operations

Example: Use set identities to prove

$$\overline{A\cup(B\cap C)} = (\overline{C}\cup\overline{B})\cap\overline{A}$$

Proof:

$\overline{A\cup(B\cap C)} = \overline{A}\cap\overline{(B\cap C)}$	De Morgan's
$= \overline{A}\cap(\overline{B}\cup\overline{C})$	De Morgan's
$= (\overline{B}\cup\overline{C})\cap\overline{A}$	commutative
$= (\overline{C}\cup\overline{B})\cap\overline{A}$	commutative

FYOG: Use set identities to prove

$$(A\cup\overline{B})\cap(\overline{B\cap A}) = \overline{B}$$

TABLE 1 Set Identities.

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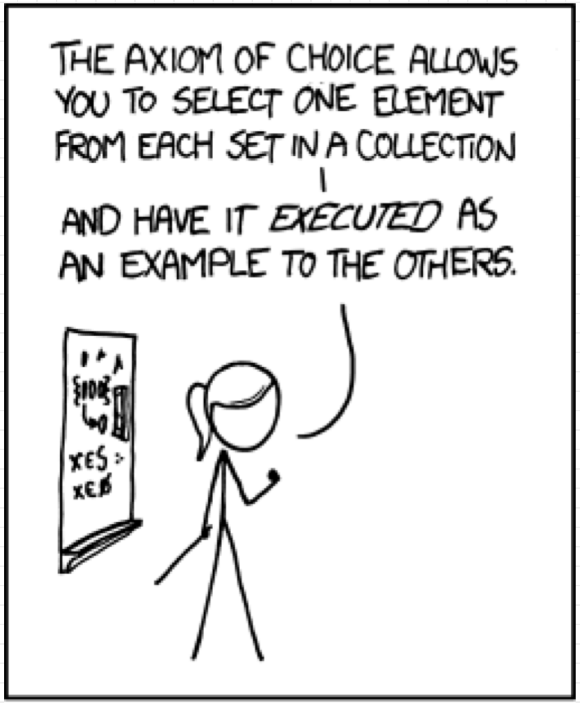
$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$ $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$	Distributive laws
$\overline{A\cap B} = \overline{A}\cup\overline{B}$ $\overline{A\cup B} = \overline{A}\cap\overline{B}$	De Morgan's laws
$A\cup(A\cap B) = A$ $A\cap(A\cup B) = A$	Absorption laws
$A\cup\overline{A} = U$ $A\cap\overline{A} = \emptyset$	Complement laws

Sets and set operations - Recap

- We learned what **sets** are,
- empty set, singleton set, power set,
- how to cook up larger sets (unions, power sets) from smaller ones,
- and how to cook up smaller sets (intersections, subsets) from larger ones

Next time:

- More on ***infinite*** sets and …
- doing stuff with sets of things
  to get other sets of things
  (we call the stuff we do ***functions!***)



MY MATH TEACHER WAS A BIG BELIEVER IN PROOF BY INTIMIDATION.

Extra Practice

Example 1: Show that if *a, b* and *c* are real numbers with $a \neq 0$, then there **exists** a **unique** solution x to the equation $ax + b = c$.

Note: This is the statement that non-horizontal lines pass through each y coordinate exactly once.

Example 2: Use a proof by cases to show that for real numbers x and y,

$\max(x, y) + \min(x, y) = x + y$.

Hint: You could use the cases: (1) $x \geq y$ and (2) $x < y$.

Note that you need one to be "or equal to" and the other to be strict inequality, otherwise there might be overlap between the two cases!

Solutions

Example 1: Show that if a, b, and c are real numbers with $a \neq 0$ then there **exists** a **unique** solution x to the equation $ax + b = c$

Existence: Solve for x

$$ax + b = c \Rightarrow x = \frac{b - c}{a} \text{ (where here we know that } a \neq 0\text{)}$$

Uniqueness: Assume x and y are both solutions to the system, then

$$ax + b = c = ay + b \Rightarrow ax + b = ay + b \Rightarrow ax = ay \Rightarrow x = y$$

Since x and y are necessarily the same number, it follows that our solution x is unique

Example 2: Prove that for real numbers x and y,
$\max(x, y) + \min(x, y) = x + y$

Case 1: Assume $x \geq y$. Then $\max(x, y) = x$ and $\min(x, y) = y$
(Here we realize that if $x = y$ then we can decide to choose either)

Thus $\max(x, y) + \min(x, y) = x + y$

Case 2: Assume $x < y$. Then $\max(x, y) = y$ and $\min(x, y) = x$

Thus $\max(x, y) + \min(x, y) = y + x = x + y$

Since the cases cover all possible combinations of x and y and both yield the conclusion, we are done.