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# CSCI 2824: Discrete Structures

## Lecture 36: Graph Theory and Eulerian Circuits

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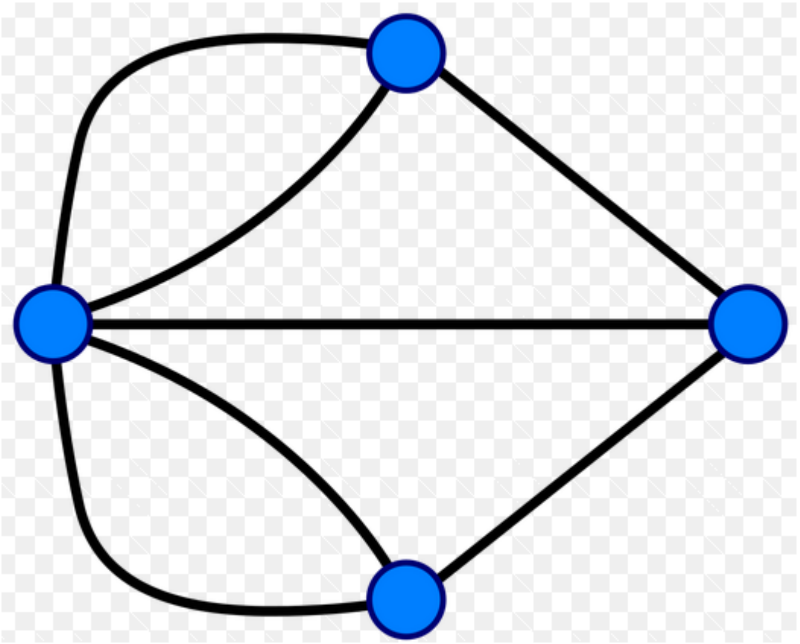

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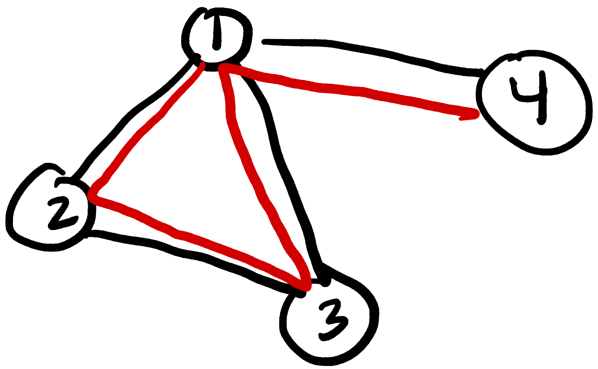
# Graph Theory

Let $G = (V, E)$ be a graph (directed or undirected). A walk of graph $G$ is a sequence of alternating vertices (in $V$) and edges (in $E$) such that

- we start on any vertex and end on any vertex .
- a single step in the walk proceeds upon an outgoing edge from the current vertex .

❖ A walk has to respect edge direction in a directed graph. In an undirected graph, it doesn't matter.



# Graph Theory

**Example**: Consider the directed graph below:

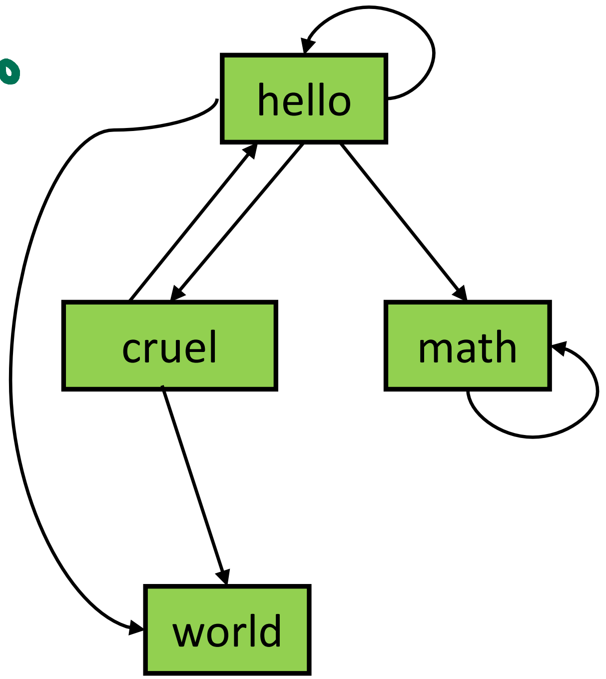
Valid walk: hello → hello → cruel → hello

Valid walk: hello → math

Valid walk: hello → cruel → world

Invalid walk: world → world

Invalid walk: world → cruel

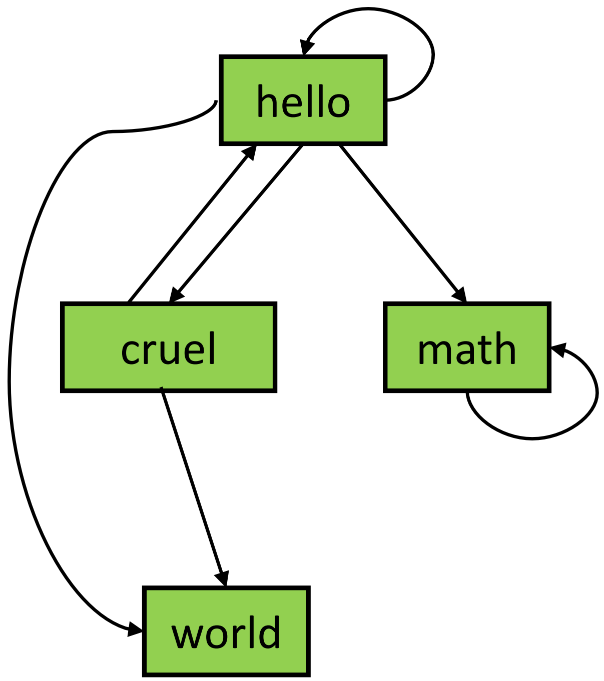


## Graph Theory

Let $G = (V, E)$ be a graph (directed or undirected). A ***path*** of a graph is a walk in which no vertex is repeated. The ***length*** of a path is the number of edges that it traverses.

☆ Valid path: hello → cruel → world (path of length 2)

invalid path: hello → cruel → hello



# Graph Theory

named after the mathematician Euler

Let $G = (V, E)$ be a graph (directed or undirected). An ***Eulerian Circuit*** of a graph is a special kind of walk which starts and ends at the same vertex and traverses each edge exactly once.

- A **walk** is pretty much anything, going from vertex to vertex.
- A **path** is a walk that does not repeat vertices.
- An **Eulerian Circuit** is a walk that does not repeat edges and starts and ends at the same vertex.

and traverses each edge

# Walks, paths, and Eulerian circuits

**Example**: Can there possibly be an Eulerian circuit on the graph below?

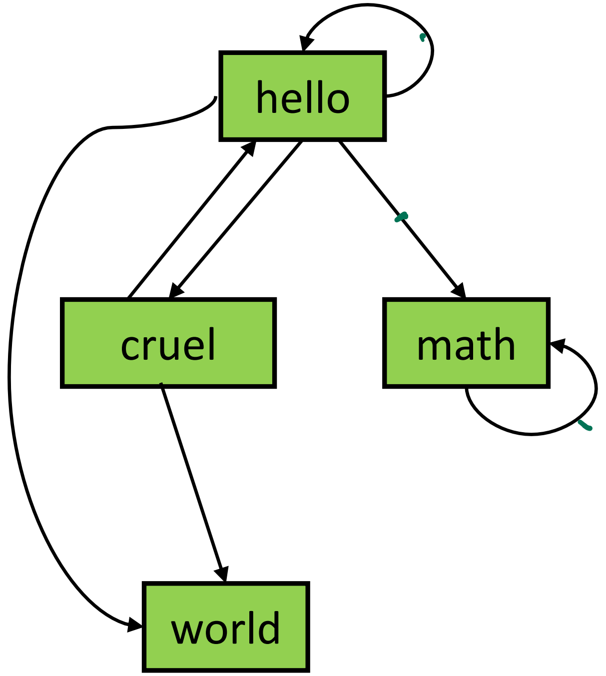
No!

Prove this by cases

Case 1: Start at hello

Case 2: Start at cruel

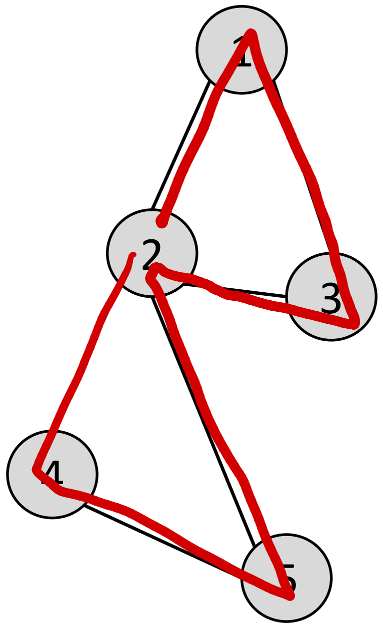
Case 3: Start at world




Case: Start at math

# Walks, paths, and Eulerian circuits

**Example**: Can there possibly be an Eulerian circuit on the graph below?

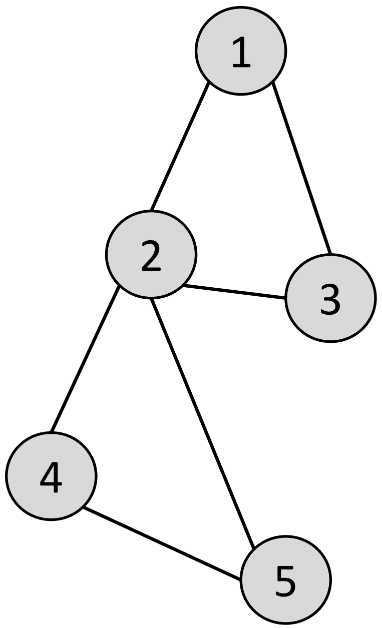



$2 \to 4 \to 5 \to 2 \to 3 \to 1 \to 2$

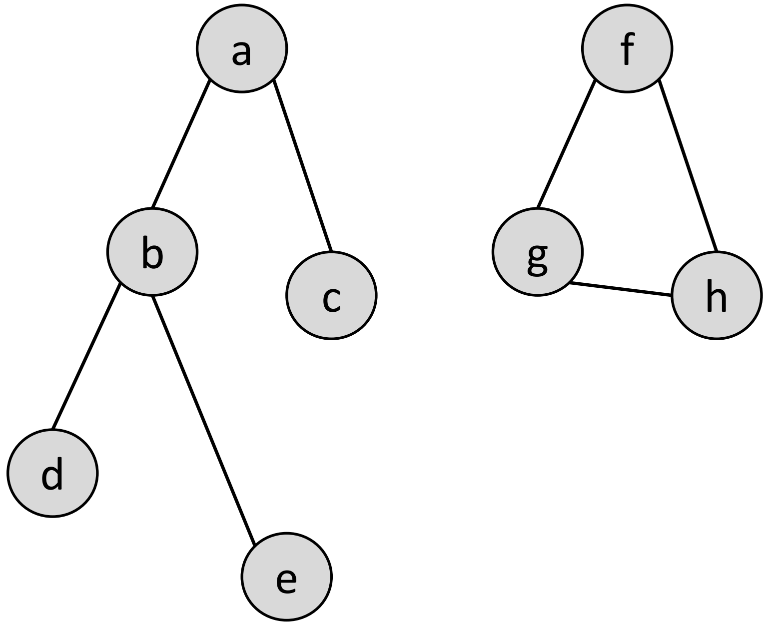
# Walks, paths, and Eulerian circuits

An undirected graph $G$ is ***connected*** if there is a path between every pair of distinct vertices in the graph. An undirected graph is called disconnected if it is not connected.

This graph is connected.



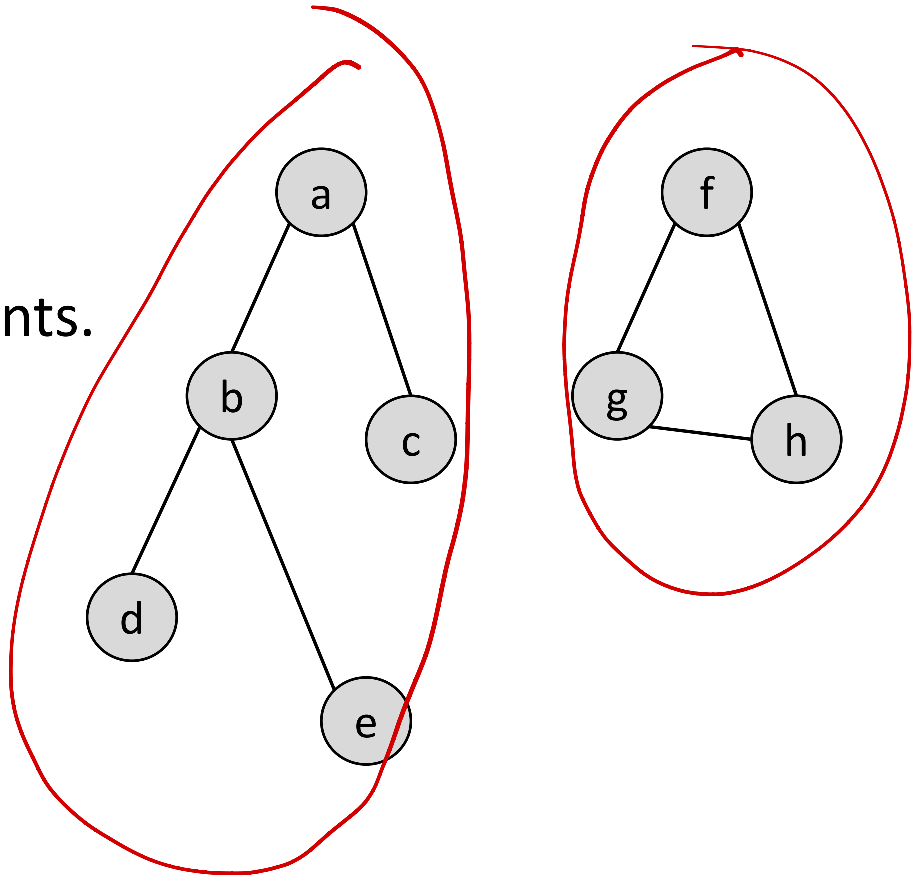

This graph is disconnected.



# Walks, paths, and Eulerian circuits

A subgraph of a disconnected graph $G$ is called a ***connected component*** of $G$ if it is a maximally connected subgraph of $G$.

$\{a, b, c, d, e\}$ and $\{f, g, h\}$ are connected components.




$\{a, b, c, d\}$ are connected

# Walks, paths, and Eulerian circuits

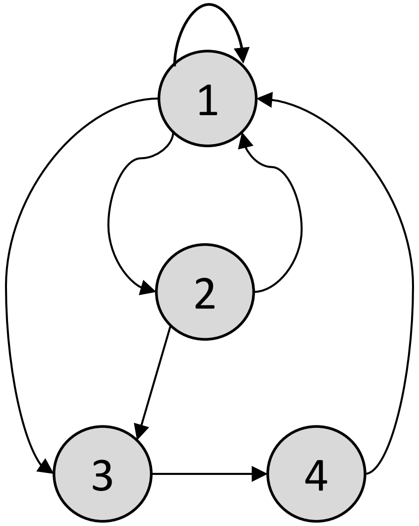
A directed graph (digraph) $G$ is ***strongly connected*** if for each pair of distinct vertices $a$ and $b$ there is a path from $a$ to $b$ AND a path from $b$ to $a$.

For example:

1 4

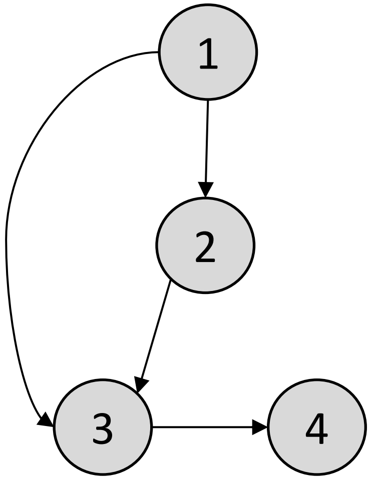
$1 \to 2 \to 3 \to 4$

$4 \to 1$ ✓



# Walks, paths, and Eulerian circuits

A directed graph (digraph) $G$ is ***weakly connected*** if for each pair of distinct vertices $a$ and $b$ there is a path from $a$ to $b$ OR a path from $b$ to $a$.

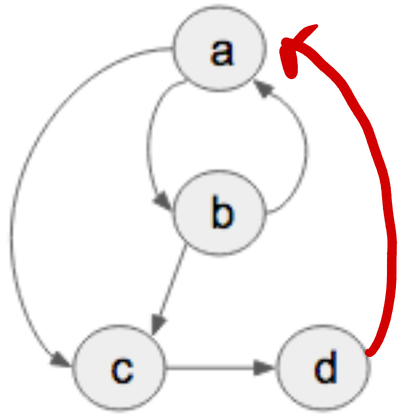


# Walks, paths, and Eulerian circuits

A subgraph of a digraph $G$ is a strongly connected component if it is a maximal strongly connected subgraph of $G$.

What is the strongly connected component of the graph $G$ to the right?

$\{a, b\}$




What is the smallest number of directed edges you could add to $G$ to make it a strongly connected graph?

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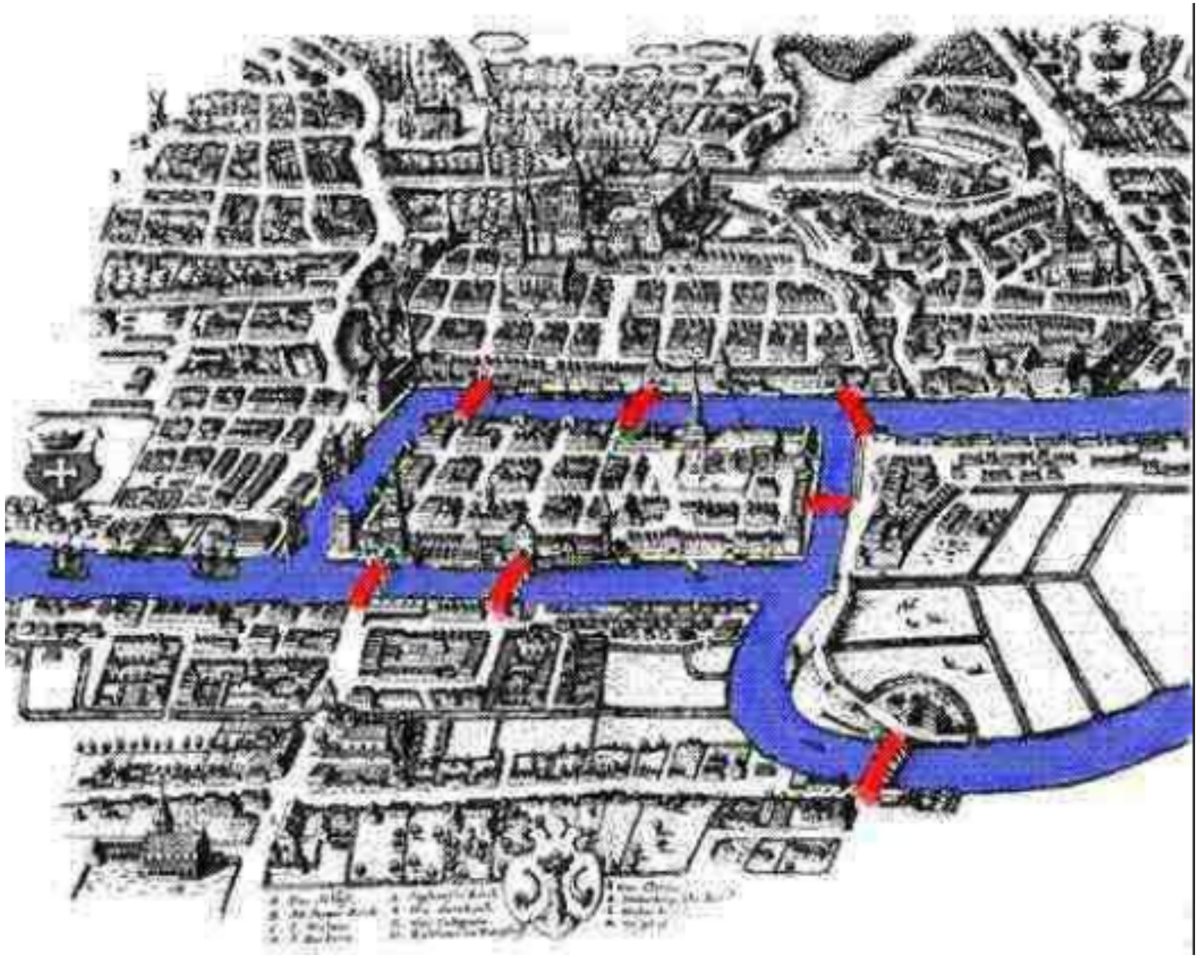
# Walks, paths, and Eulerian circuits

**Puzzle**: The city of Konigsberg has two islands formed by a river with seven bridges connecting the islands and the mainland.

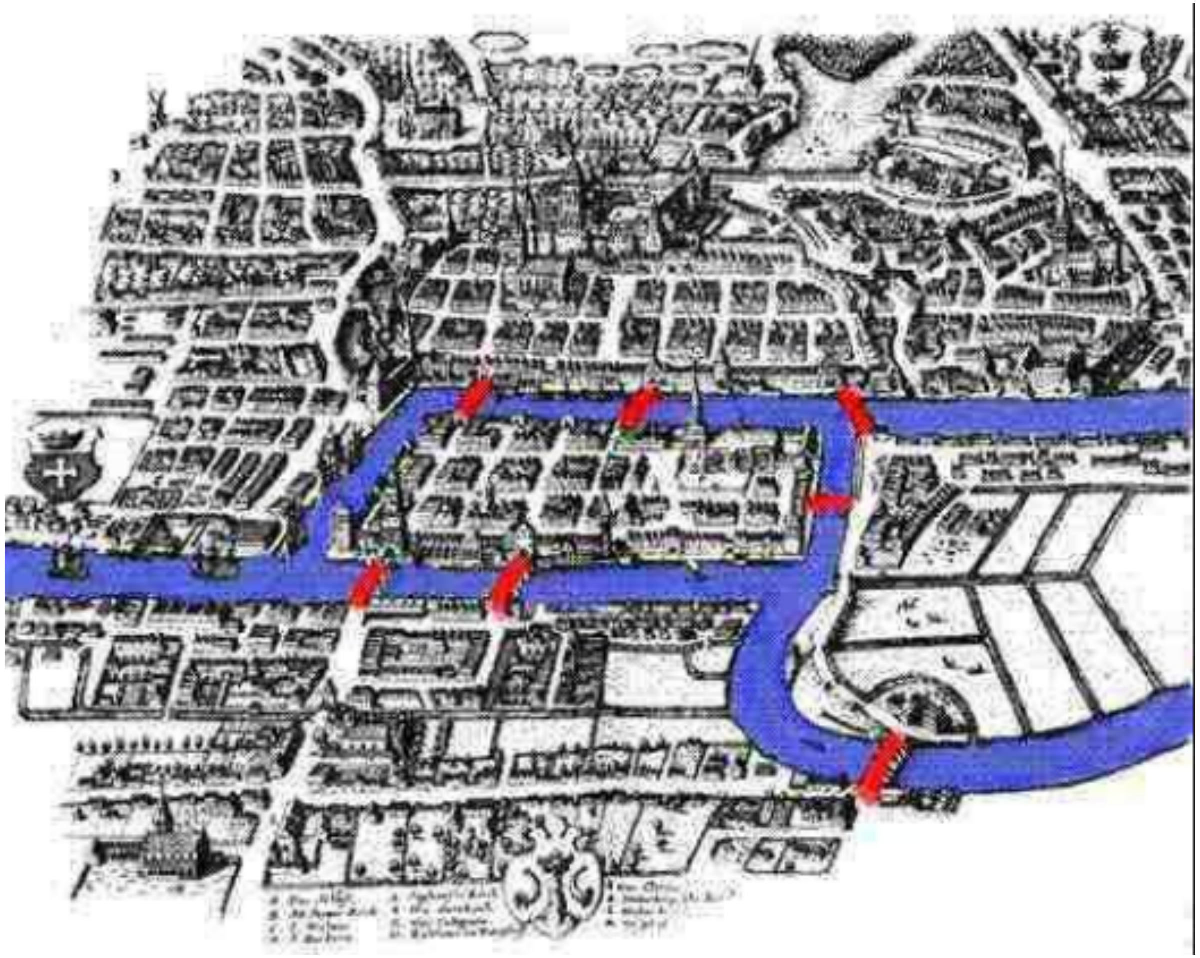
Is there a circuit that traverses each bridge exactly once?



# Walks, paths, and Eulerian circuits

Is there a circuit that traverses each bridge exactly once?

We can represent this as a graph, and solve the riddle by checking if it has an Eulerian circuit!

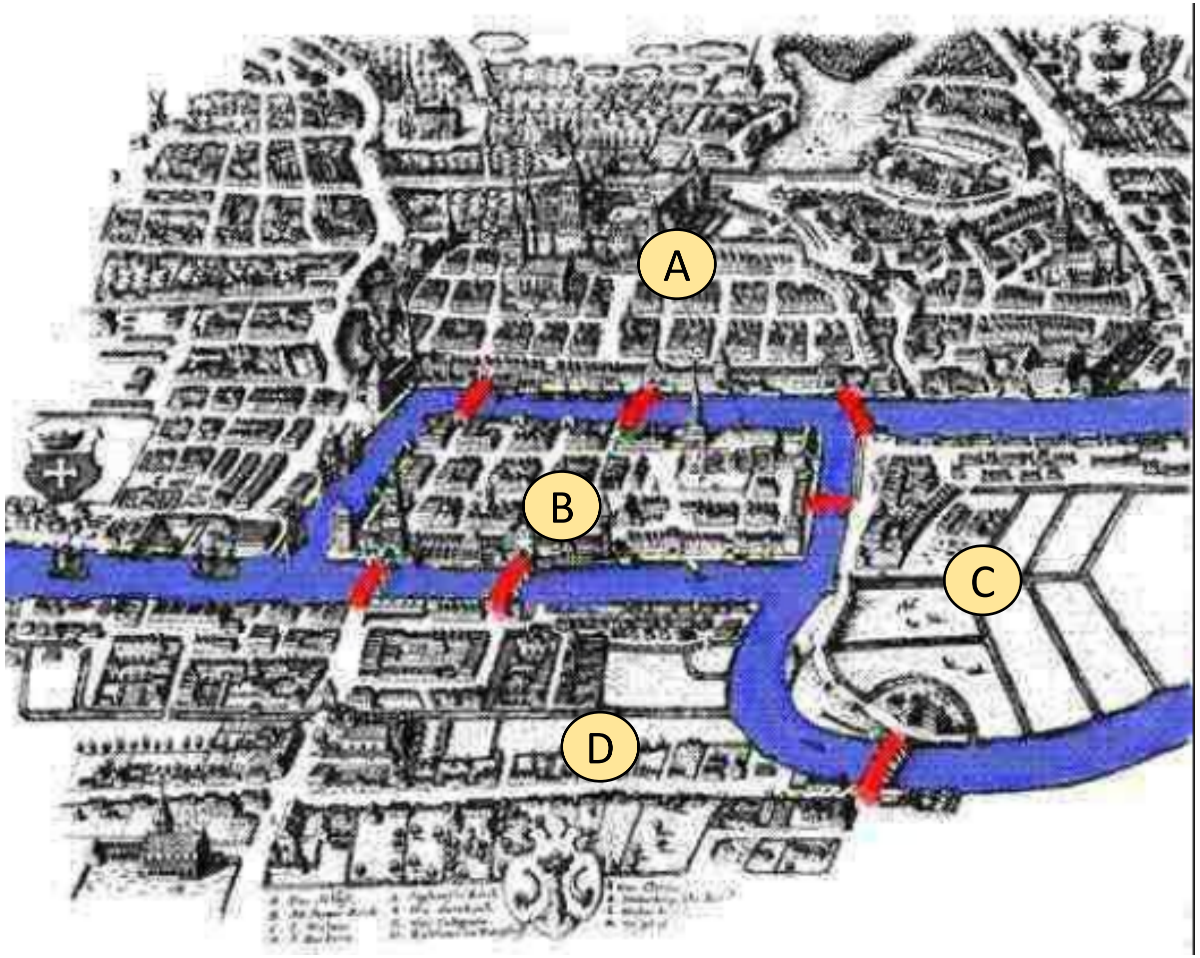


# Walks, paths, and Eulerian circuits

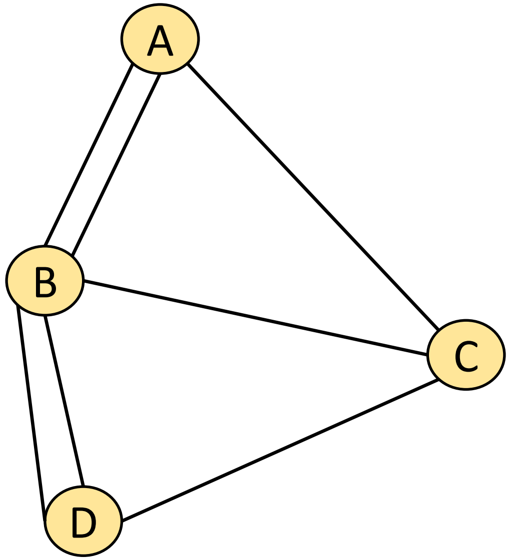
Let’s formalize this as a graph.

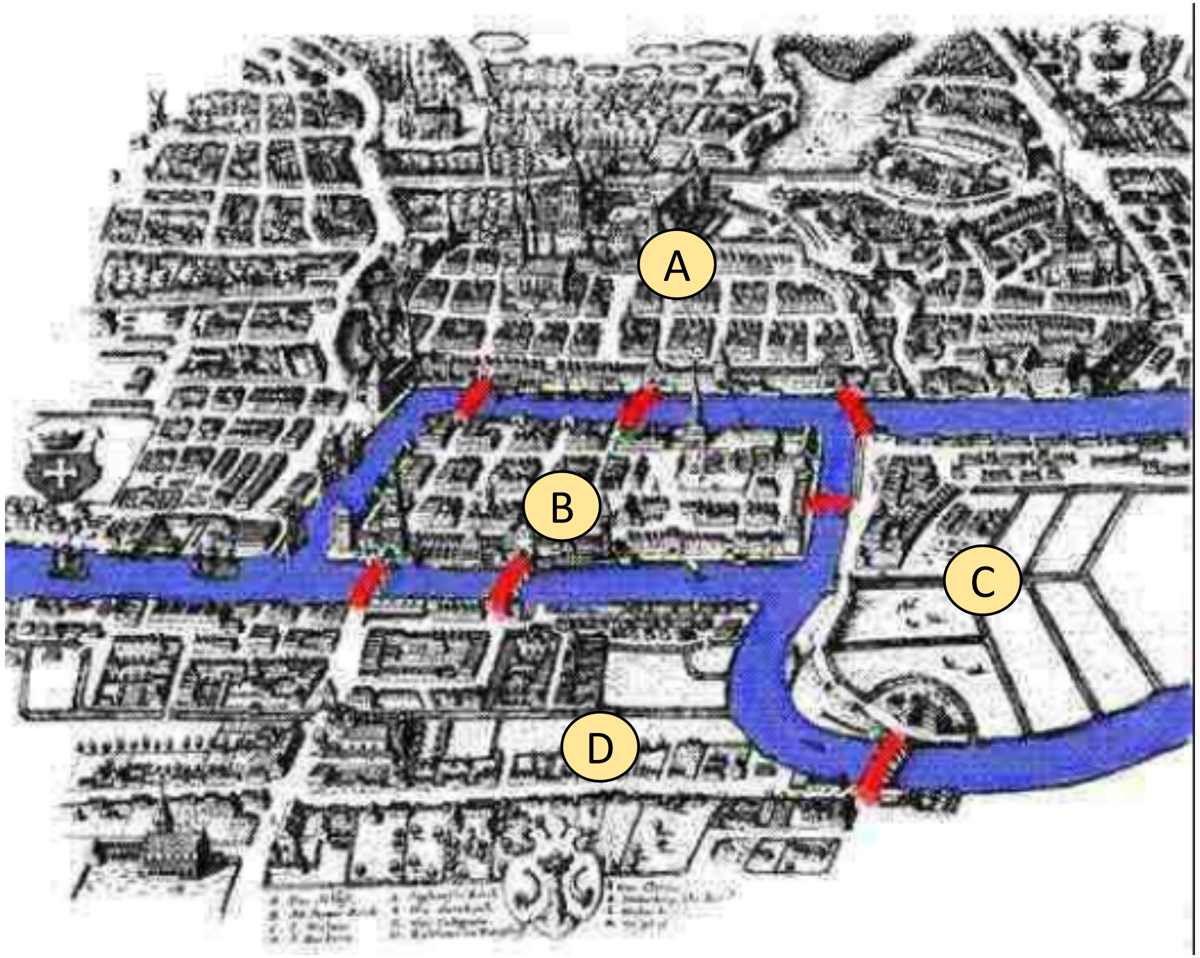
We will make the nodes (aka vertices) the separate landmasses.

The edges will be the bridges.



# Walks, paths, and Eulerian circuits

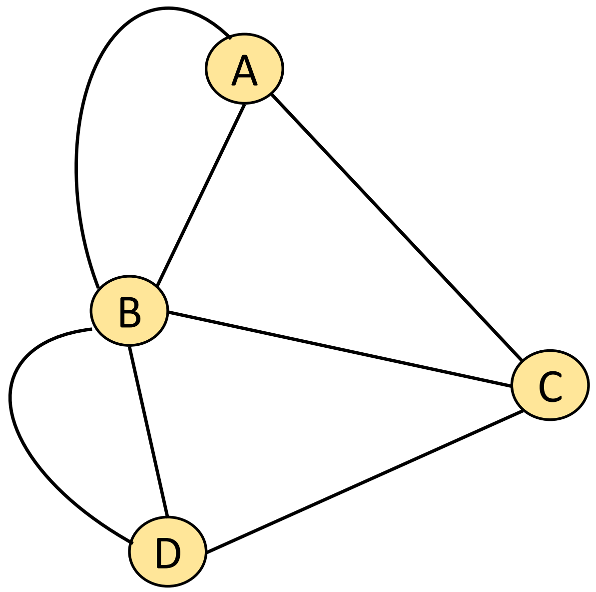



This is called a *multigraph*, because pairs of vertices have multiple edges connecting them.

# Walks, paths, and Eulerian circuits

Does this have an Eulerian circuit?



# Walks, paths, and Eulerian circuits

**Theorem**: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Seven Bridges of Konigsberg: Since the graph has at least one vertex with odd degree, there cannot be an Eulerian circuit.