Nash Equilibrium :

In a strategic game, one must form a belief about the other player's actions.

This belief can be formed from past experiences playing this game.

His experience leads him to beliefs about the actions of "typical" oponents, not any specific set of opponents.
E.g: interaction b/w buyers and sellers.

Solution theory has 2 components :

- Each player chooses his action acc. to the model of rational choice, given his belief about the other players' actions.
- ~~[illegible]~~ Every player's beliefs about the other players' action is correct.

A Nash equilibrium is an action profile a^* with the property that no player i can do better by choosing an action different from a_i^*, given that every other player j adheres to a_j^*.

A Nash equilibrium embodies a stable "social norm": if everyone else adheres to it, no individual wishes to deviate from it.

A poor fit with the idealized setting [players do not have much experience of the game] may be mitigated by other considerations. E.g: inexperienced player may be able to draw conclusions about their opponents' likely actions from their experience in other situations.

More precise definition:

Let a be an action profile, in which the action of each player i is a_i. Let a_i' be any action of player i (either equal to a_i, or diff. from it). Then (a_i', a_{-i}) denotes the action profile in which every player j except i chooses his action a_j as specified by a, whereas player i chooses a_i'. (The $-i$ subscript on a stands for "except i"). That is, (a_i', a_{-i}) is the action profile in which all players except i adhere to a while i "deviates" to a_i'. (If $a_i' = a_i$, then ~~[illegible]~~ $(a_i', a_{-i}) = (a_i, a_{-i}) = a$)

E.g.: If there are 3 players, then (a_2', a_{-2}) is the action profile in which players 1 and 3 adhere to a (player 1 chooses a_1 and 3, a_3) and player 2 chooses a_2'.

Nash equilibrium of strategic game with ordinal preferences:

The action profile a^* in a strategic game with ordinal preferences is a Nash equilibrium if, for every player i and every action a_i of player i, a^* is at least as good acc. to player i's preferences as the action profile (a_i, a_{-i}^*) in which player i chooses a_i while every other player j chooses a_j^*.
Equivalently, for every player i,

$$u_i(a^*) \geq u_i(a_i, a_{-i}^*) \text{ for every action } a_i \text{ of player } i$$

where u_i is a payoff f'n that represents player i's preferences.

No player i has any action a_i for which he prefers (a_i, a_{-i}^*) to a^*.

Examples of Nash Equilibrium :

(1) Prisoner's Dilemma: (Ex.2)

(fink, fink) is the unique Nash equilibrium.

- (Quiet, Quiet) does not satisfy Nash equi. as when P2 chooses Quiet, p1's payoff to fink > Quiet so it deviates to fink.
- (fink, Quiet) ⇒ when P1 chooses fink, p2's payoff to fink > Quiet, so P2 tends to deviate.
- Similarly with (Q, F).

(2) Both people goof off when working on a joint project. (Ex. 3.)

(3) Both duopolists charge a low price. (Ex 4.)

(4) Both countries build bombs or (Ex. 5) don't build bombs.

(5) Both farmers graze their sheep a lot. (Ex. 6)

(The overgrazing of a common thus predicted is sometimes called the "tragedy of ~~com~~ commons".)

Example 12: Altruistic Players in Prisoner's Dilemma

Each action pair results in the player's receiving amounts of money equal to the nos. corresponding to that action pair.

P1 \ P2	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

E.g. for (Quiet, Fink), P1 gets 0 and P2 gets \$3.

The players are not "selfish"; rather, the preferences of each player i are represented by the payoff f^n $m_i(a) + \alpha m_j(a)$, where $m_i(a)$ is the amount of money received by player i when action profile is a, α is non-negative.

Eg: P1's payoff to (Quiet, Quiet) $= 2 + 2\alpha$.

(a) If $\alpha = 1$, formulate a strategic game.

P1 \ P2	Quiet	Fink
Quiet	4,4	3,3
Fink	3,3	2,2

This game is not Prisoner's dilemma as if P2 chooses Quiet, then in PD, Fink is preferable to Quiet by P1, but here Quiet is preferable to fink.

(b) Range of values of α for which resulting game is Prisoner's Dilemma. Find Nash equilibria.

P1 \ P2	Quiet	Fink
Quiet	$2(1+\alpha), 2(1+\alpha)$	$3\alpha, 3$
Fink	$3, 3\alpha$	$1+\alpha, 1+\alpha$

For game to the Prisoner's Dilemma:

$\Rightarrow$ $3 > 2(1+\alpha)$ [for P2 choosing Quiet, P1 chooses Fink to Quiet]

$\Rightarrow$ $1+\alpha > 3\alpha$ [for P2 choosing Fink, P1 chooses Fink to Quiet]

$\Rightarrow$ $2(1+\alpha) > 1+\alpha$ [(Quiet, Quiet) > (Fink, Fink)]

Now, $\boxed{0 < \alpha < \frac{1}{2}}$ $\Rightarrow$ for it be to Prisoner's Dilemma

At $\alpha = \frac{1}{2}$ $\Rightarrow$ All 4 outcomes [(Q,Q), (Q,F), (F,Q), (F,F)]

are Nash equilibria.

for $\alpha < \frac{1}{2}$ $\Rightarrow$ (F, F) is the Nash equilibrium

for $\alpha > \frac{1}{2}$ $\Rightarrow$ (Q, Q) is the Nash equilibrium.

Example 13: Selfish and altruistic social behaviour

Two people enter a bus. Two adjacent cramped seats are free. Each person must decide whether to sit or stand. Sitting alone is more comfortable than sitting next to other person which is more comfortable than standing.

(a) Suppose that each person cares only about his own comfort. (Selfish)

P1 \ P2	Sit	Stand
Sit	1,1	2,0
Stand	0,2	0,0

The game is <u>not</u> Prisoner's Dilemma.

(Sit ≡ Quiet, Stand ≡ Fink

⇒ (Sit, stand) ≡ (Quiet, Fink)

$\Downarrow$ (2, 0) $\qquad$ $\Downarrow$ (0, 3)

∴ it is not Prisoner's Dil.)

<u>(Sit, Sit) ⇒ Nash equilibrium</u>

(b) Suppose that each person is altruistic, ranking the outcomes acc. to the other person's comfort, and, out of politeness, prefers to stand than to sit if the other person stands.

P1 \ P2	Sit	Stand
Sit	1,1	0,2
Stand	2,0	α,α

<u>α is positive</u>

$\boxed{0<\alpha<1}$ ⇒ Prisoner's Dilemma

<u>Nash equilibrium ⇒ (Stand, Stand)</u> [regardless of value of α]

(c) Compare the people's comfort in the equilibria of 2 games

Both people are more comfortable in the equilibrium that results when they act acc. to their selfish preferences.

(6) BoS (Ex. 8.)

- (Bach, Bach) ⇒ If P1 deviates to Stravinsky, his payoff 2 → 0
  If P2 deviates to Stravinsky, his payoff 1 → 0
  ∴ (Bach, Bach) is a Nash equilibrium.
- Similarly (Stravinsky, Stravinsky) is a Nash equilibrium.
- But (B, S) or (S, B) is not.

∴ 2 Nash equilibria ⇒ (B, B), (S, S)

(7) Matching Pennies (Ex. 9)

- for (H, H) and (T, T), P2 is better off deviating.
- for (H, T) and (T, H), P1 is better off deviating.

Thus, for this game, there is no Nash equilibrium.

(8) Stag Hunt (Ex. 11)

2 Nash equilibria: (Stag, stag), (Hare, Hare)
Many-player Stag Hunt also has the same equilibria:
(Stag, Stag ——, Stag), (Hare, Hare, —— Hare)

Example 14! Variants of the stag Hunt

Consider two variants of the n-hunter stag Hunt in which only m hunters, with $2 \le m < n$, need to pursue the stag in order to catch it. Assume that a captured stag is shared only by the hunters that catch it.

(a) Assume, that each hunter prefers the fraction $1/n$ of the stag to a hare.

Nash equilibria $\Rightarrow$ (stag, ___, stag) and (Hare, —, Hare).

Any player that deviates from the first profile obtains a hare rather than the fraction $1/n$ of the stag. Any player that deviates from the second profile obtains nothing.

An action profile in which at least 1 and at most $m-1$ hunters pursue the stag is not a Nash equi, since any one of them is better off catching a hare.

An action profile in which at least m and at most $n-1$ hunters pursue the stag is not a Nash equi, since any one of the remaining hunters is better off joining pursuit of the stag.

(b) Assume that each hunter prefers the fraction $1/k$ of the stag to a hare, but prefers the hare to any smaller fraction of stag, $m \le k \le n$.

- If hunters less than m, pursue the stag, then each obtains nothing, and is better off catching hare.
- If at least m and fewer than k hunters pursue

the stag than each one that pursues a hare is better off switching to the pursuit of the stag.

- If more than k hunters pursue the stag then the fraction of the stag that each obtains is less than $1/k$, so each of them is better off catching a hare.

Nash equilibria: (Hare, —, Hare) and

$$(\underbrace{\text{Stag}, \text{—}, \text{Stag}}_{k}, \underbrace{\text{Hare}, \text{—}, \text{Hare}}_{n-k})$$

Example 15: Extension of the Stag Hunt

Extend the n-hunter Stag Hunt by giving each hunter K (a +ve integer) units of effort, which he can allocate between pursuing the stag and catching hares. Denote the effort hunter i devotes to pursuing the stag by e_i (non-neg integer) equal to at most K. The chance that the stag is caught depends on the smallest of all the hunters' efforts, denoted $\min_j e_j$. ("A chain is as strong as its weakest link.") Hunter i's payoff to the action profile $(e_1, \text{—}, e_n)$ is $2\min_j e_j - e_i$. (He is better off the more likely the stag is caught, and worse off the more effort he devotes to pursuing the stag, which means he catches fewer hares)

Every profile $(e, \text{—}, e)$ where e is an integer from 0 to K, is a Nash equilibrium. In the equi. $(e, \text{—}, e)$, each player's payoff is e. The profile $(e, \text{—}, e)$ is a Nash equi, since if player i chooses $e_i < e$, then his payoff is $2e_i - e_i = e_i < e$, and if he chooses $e_i > e$ then his payoff $2e - e_i < e$

Consider an action profile $(e_1, \_\_\_, e_n)$ in which not all effort levels are the same. Suppose that e_i is the minimum. Consider some player j whose effort level exceeds e_i. His payoff is $2e_i - e_j < e_i$, while if he deviates to the effort level e_i his payoff is $2e_i - e_i = e_i$. Thus, he can increase the payoff by deviating, so $(e_1, \_\_\_, e_n)$ is not a Nash equilibrium.

Example 16: Hawk — Dove.

Two animals are fighting over some prey. Each can be passive or aggressive. Each prefers to be aggressive if its opponent is passive, and passive if its opponent is aggressive; given its own stance, it prefers the outcome when its opponent is passive to that in which its opponent is aggressive.

H \ D	Aggressive	Passive
Aggressive	0,0	3,1
Passive	1,3	2,2

2 Nash equilibria: (Aggressive, Passive) and (Passive, Aggressive)