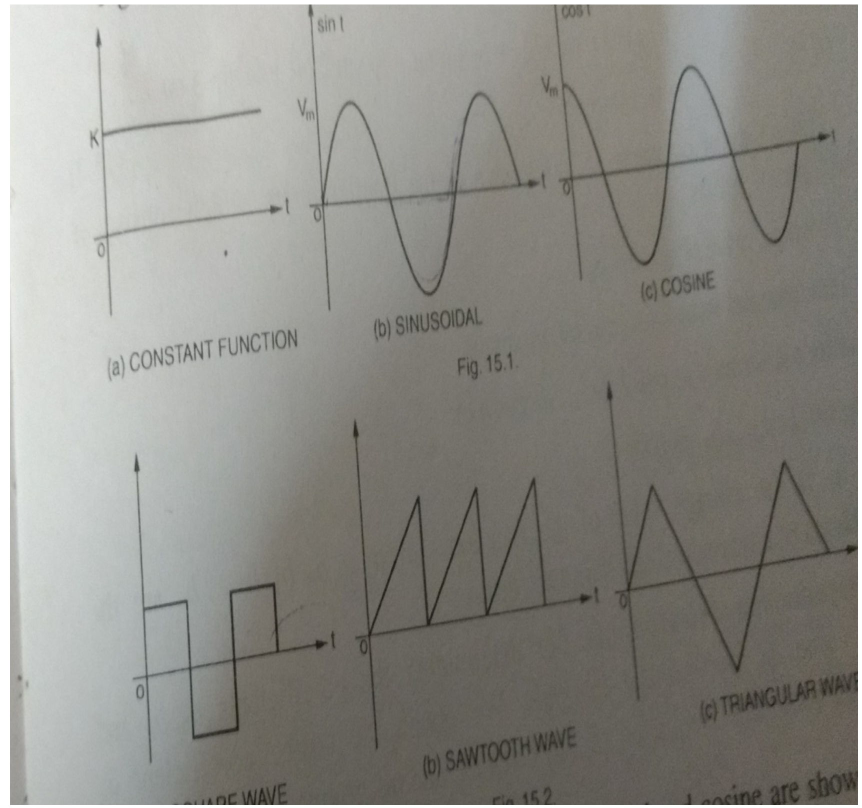
Fourier series analysis---

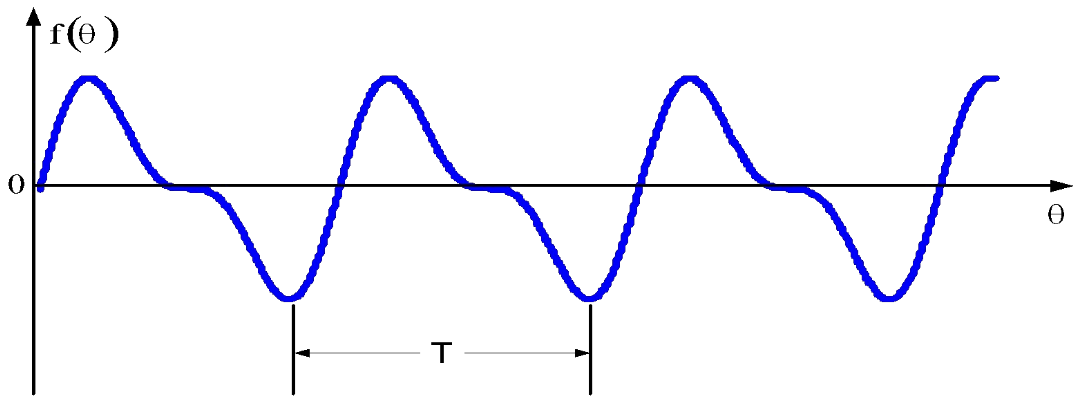
French Mathematician

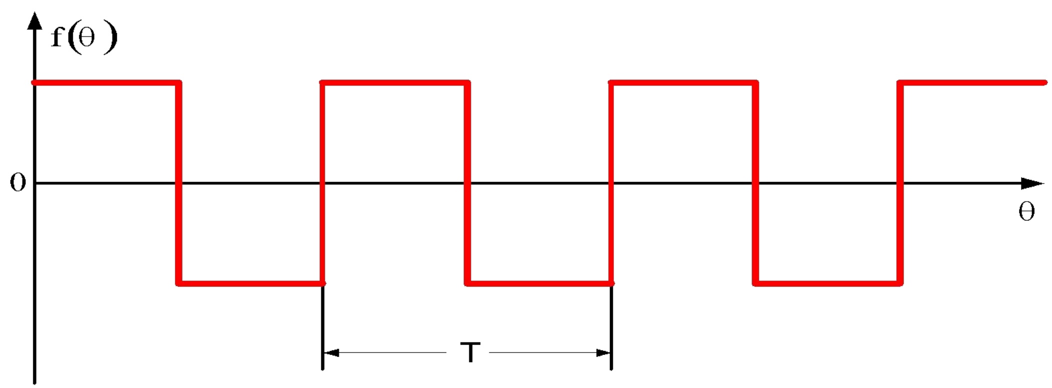
J B J Fourier introduced the concept of Fourier series for the analysis of arbitrary periodic functions:



Fourier series analysis---

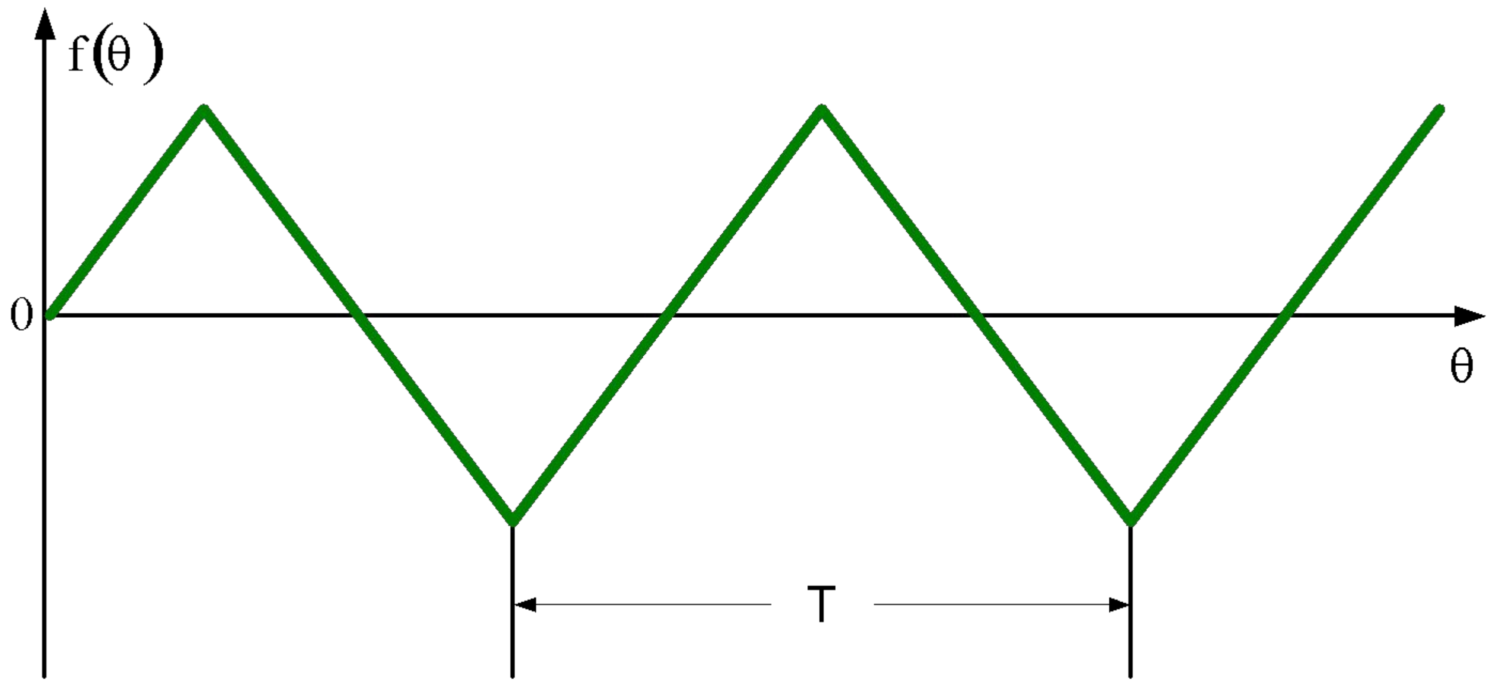
Some Periodic Functions





Fourier series analysis---

Some Periodic Functions



Fourier series analysis---

Fourier Series: A Fourier series is a representation of a function using a series of sinusoidal functions of different "*frequencies*".

They are extremely useful to be used to represent functions of phenomena that are periodic in nature.

Fourier series analysis---

- **Fourier Integral & Transform**: Similar to Fourier series but extend its application to both periodic and non-periodic functions and phenomena.

- First developed by the French mathematician Joseph Fourier (1768-1830)

Fourier series analysis---

Why using Fourier analysis? Analyze the phenomena from time-basis to a frequency-basis analysis. It is simpler to describe a periodic function using Fourier description. Ex: a periodic time series can always be described by using its frequency and amplitude.

A periodic function $f(t)$ has a period of T will satisfy $f(t)=f(t+T)$, i.e. the function repeats itself after each interval of time period T.

Fourier series analysis---

- Trigonometric series, i.e. sine or cosine series with fundamental frequency $f = \omega/2\pi$ and its harmonics (combined), will have the time period of $2\pi/\omega$.

Any periodic function can be expressed by a Fourier series, if that function satisfy following conditions which are also known as Dirichlet conditions:

Fourier series analysis---Dirichlet conditions

1. If periodic function is discontinuous, then there should be finite number of discontinuities in one period.
2. The average value of the periodic function over one period must be finite.
3. Function should have finite no. of maxima and minima over one period.

Fourier series analysis---

- Fourier Series: Trigonometric form;

Any periodic function, which satisfy the Dirichlet conditions can be expressed in the form of series of infinite sines and cosine terms as given below;

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t) \text{ ---- (1)}$$

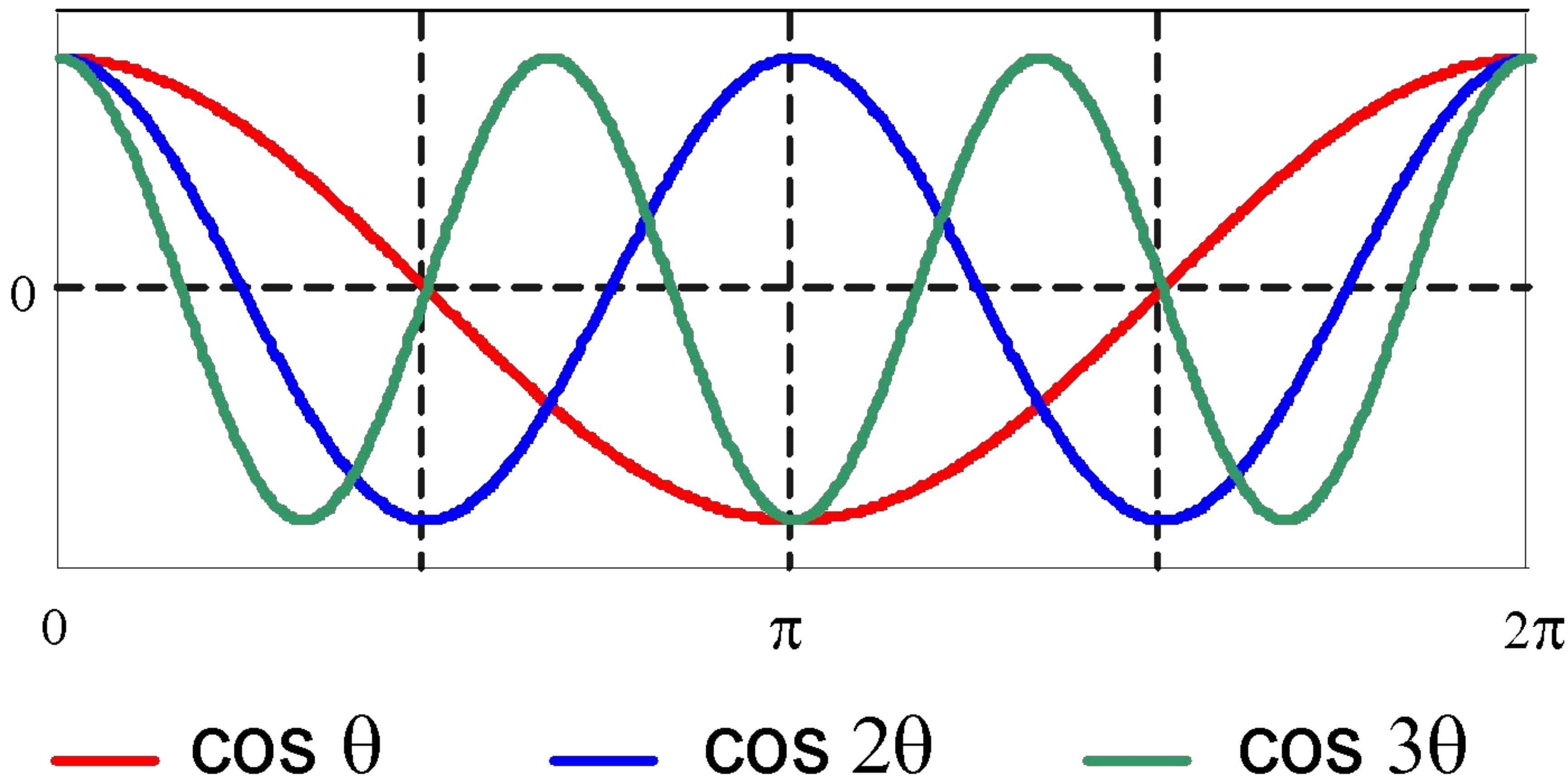
Above equation is known as the Fourier series, where

$a_0 = Average\ value\ of\ periodic\ function\ over\ one\ period$

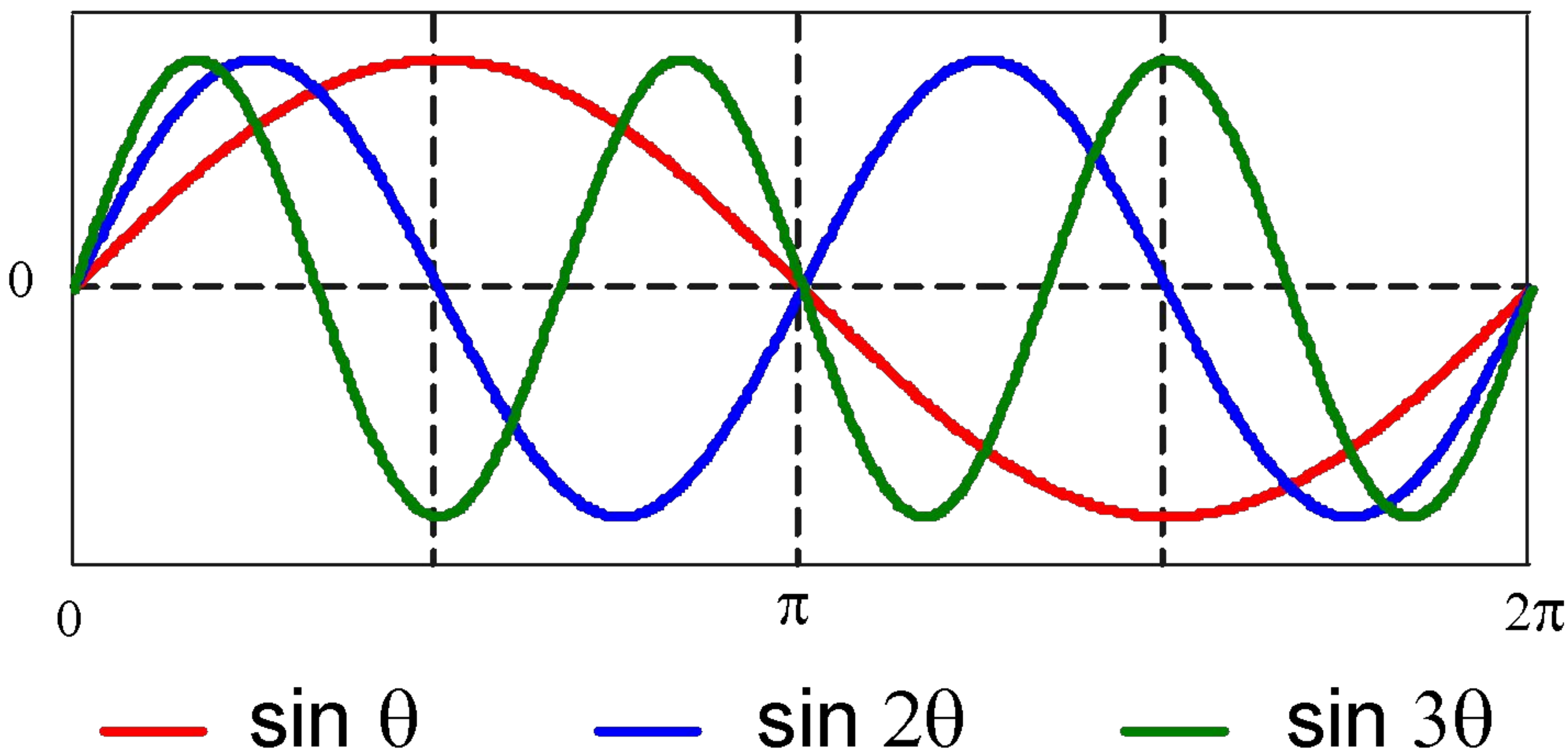
$a_1 \cos \omega t$ and $b_1\ sin\ \omega t$ = fundamental frequency terms

All other terms are harmonic terms.

Fourier series analysis---



Fourier series analysis---



Fourier series analysis---

- Determination of coefficients of Fourier series;

To determine coefficients $a_0, a_{1\ldots.}$ and $b_{1\ldots.}$ we have to first refer following Trigonometric relations;

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)\theta \, d\theta + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)\theta \, d\theta$$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta d\theta = 0 \qquad n \neq m$$

Fourier series analysis---

$$\int_{-\pi}^{\pi} \cos \boldsymbol{n}\Theta \cos \boldsymbol{m}\Theta \boldsymbol{d}\Theta = \pi \qquad \boldsymbol{n = m}$$

$$\int_{-\pi}^{\pi} \sin n\theta \cos m\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)\theta \, d\theta + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)\theta \, d\theta$$

$$\int_{-\pi}^{\pi} \sin \boldsymbol{n}\Theta \cos \boldsymbol{m}\Theta \boldsymbol{d}\Theta = \mathbf{0}$$

for all values of *m*.

Fourier series analysis---

$$\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)\theta \, d\theta - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)\theta \, d\theta$$

$$\int_{-\pi}^{\pi} \sin n\Theta \sin m\Theta d\Theta = 0 \quad n \neq m$$

$$\int_{-\pi}^{\pi} \sin n\Theta \sin m\Theta d\Theta = \pi \quad n = m$$

Fourier series analysis---

- Determination of a_0:

By Integrating eq. (1) both side with respect to t for one period we get;

$$\int_0^T f(t)dt = \int_0^T \{a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)\}dt$$

All the terms of sines and cosines will be zero except for one constant term a_0 results in;

$\int_0^T f(t)dt = a_0 T$ or

$a_0 = \frac{1}{T}\int_0^T f(t)dt$ -------------(2)

Fourier series analysis---

- Determination of a_n and b_n :

Multiplying both side of equation (1) by $\cos n\omega t$ and integrating with respect to t over one period we get;

$$\int_0^T f(t)\cos n\omega t\,dt = \int_0^T [\{a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t\,\omega t)\}\cos n\omega t]\,dt$$

All the terms will vanish except for one term in the RHS i.e.,

$$\int_0^T f(t)\cos n\omega t\,dt = \int_0^T a_n cos^2\, n\omega t\,dt = a_n\frac{T}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt \text{----------(3)}$$

Similarly we can write;

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt \text{----------(4)}$$

n is varying 1 to ∞.

Fourier series analysis---

- Another Trigonometric Form:

From equation (1) the like frequency terms of sine and cosine can be combined as;

$$a_n \cos n\omega t + b_n \sin n\omega t$$

Divide and multiply by $\sqrt{({a_n}^2 + {b_n}^2)} = c_n$ we have;

$$c_n[\frac{a_n}{c_n} \cos \mathrm{n}\omega t + \frac{b_n}{c_n} \sin n\, \omega \mathrm{t}]$$ can be written as

$$c_n[\cos n\omega t\ cos\theta_n + \sin n\omega t \sin \theta_n] = c_n \cos(n\omega t - \theta_n)$$

Here $c_n = \sqrt{({a_n}^2 + {b_n}^2)}$ and $\theta_n = tan^{-1}\ \frac{b_n}{a_n}$

Fourier series analysis---

- Equations (2), (3) and (4) are also written in following form;

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)d(\omega t) \text{ ------ (5)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos n\omega t\, d(\omega t) \text{-------(6)}$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin n\omega t\, d(\omega t) \text{------(7)}$$

Fourier series analysis---

- Hence Fourier series in cosine form can be written as;

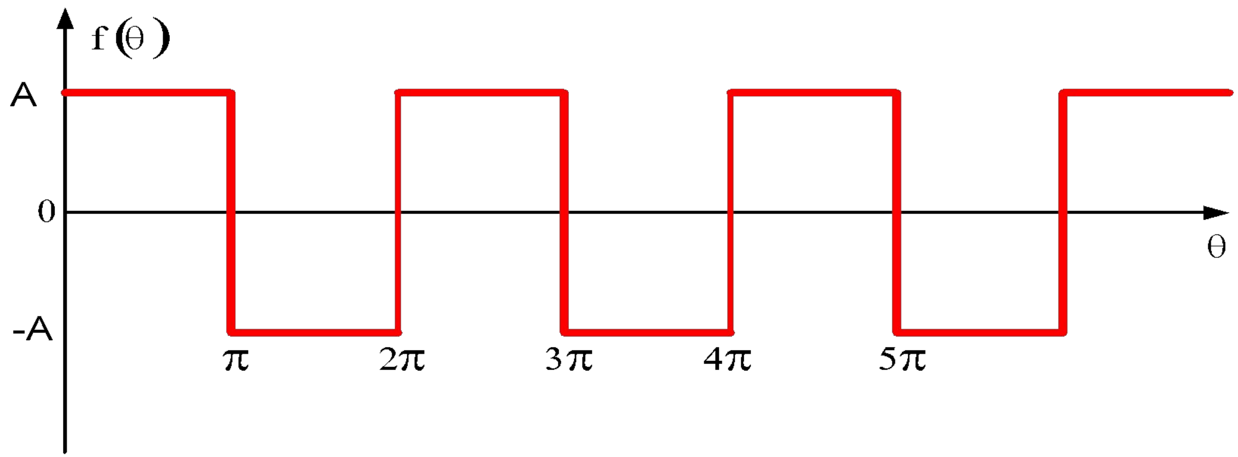
$$f(t) = a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega t - \theta_n) \text{ ----(8)}$$

Doing similar manipulation we can also write Fourier series in sine form as;

$$f(t) = a_0 + \sum_{n=1}^{\infty} c_n \sin(n\omega t + \emptyset_n) \text{ ------(9)}$$

Here $c_n = \sqrt{(a_n{}^2 + b_n{}^2)}$ and $\emptyset_n = tan^{-1} \frac{a_n}{b_n}$

Example 1. Find the. Fourier series of the following periodic function



$$f(\theta) = A \quad \textbf{\textit{when}} \quad 0 < \theta < \pi$$

$$= -A \quad \textbf{\textit{when}} \quad \pi < \theta < 2\pi$$

$$f(\theta + 2\pi) = f(\theta)$$

Example 1.----

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} f(\theta)d\theta + \int_{\pi}^{2\pi} f(\theta)d\theta\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A\,d\theta + \int_{\pi}^{2\pi} -A\,d\theta\right]$$

$$= 0$$

Example 1.----

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\cos n\theta \, d\theta + \int_{\pi}^{2\pi} (-A)\cos n\theta \, d\theta\right]$$

$$= \frac{1}{\pi}\left[A\frac{\sin n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[-A\frac{\sin n\theta}{n}\right]_{\pi}^{2\pi} = 0$$

Example 1.----

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A \sin n\theta \, d\theta + \int_{\pi}^{2\pi} (-A) \sin n\theta \, d\theta\right]$$

$$= \frac{1}{\pi}\left[-A\frac{\cos n\theta}{n}\right]_0^{\pi} + \frac{1}{\pi}\left[A\frac{\cos n\theta}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

Example 1.----

$$b_n = \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$= \frac{A}{n\pi}\left[1 + 1 + 1 + 1\right]$$

$$= \frac{4A}{n\pi} \quad \text{when n is odd}$$

$$b_n = \frac{A}{n\pi}\left[-\cos n\pi + \cos 0 + \cos 2n\pi - \cos n\pi\right]$$

$$= \frac{A}{n\pi}\left[-1 + 1 + 1 - 1\right]$$

$$= 0 \quad \text{when n is even}$$

Example 1.----

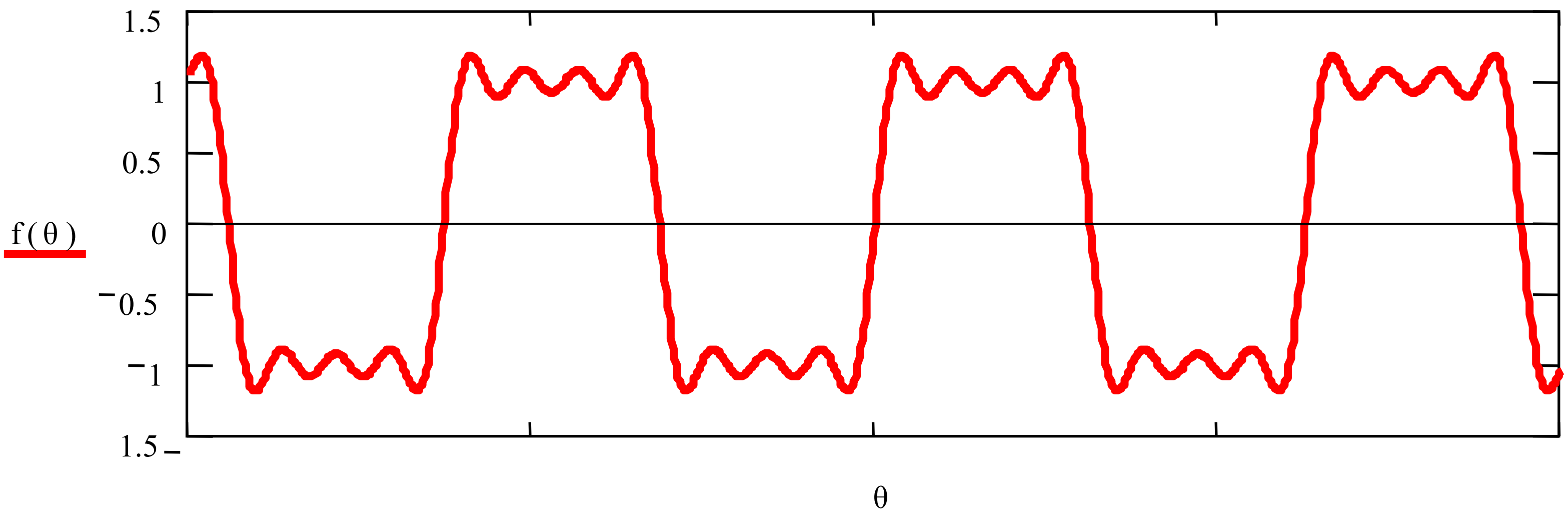
Therefore, the corresponding Fourier series is

$$\frac{4A}{\pi}\left(\sin\theta+\frac{1}{3}\sin 3\theta+\frac{1}{5}\sin 5\theta+\frac{1}{7}\sin 7\theta+\cdots\right)$$

In writing the Fourier series we may not be able to consider infinite number of terms for practical reasons. The question therefore, is – how many terms to consider?

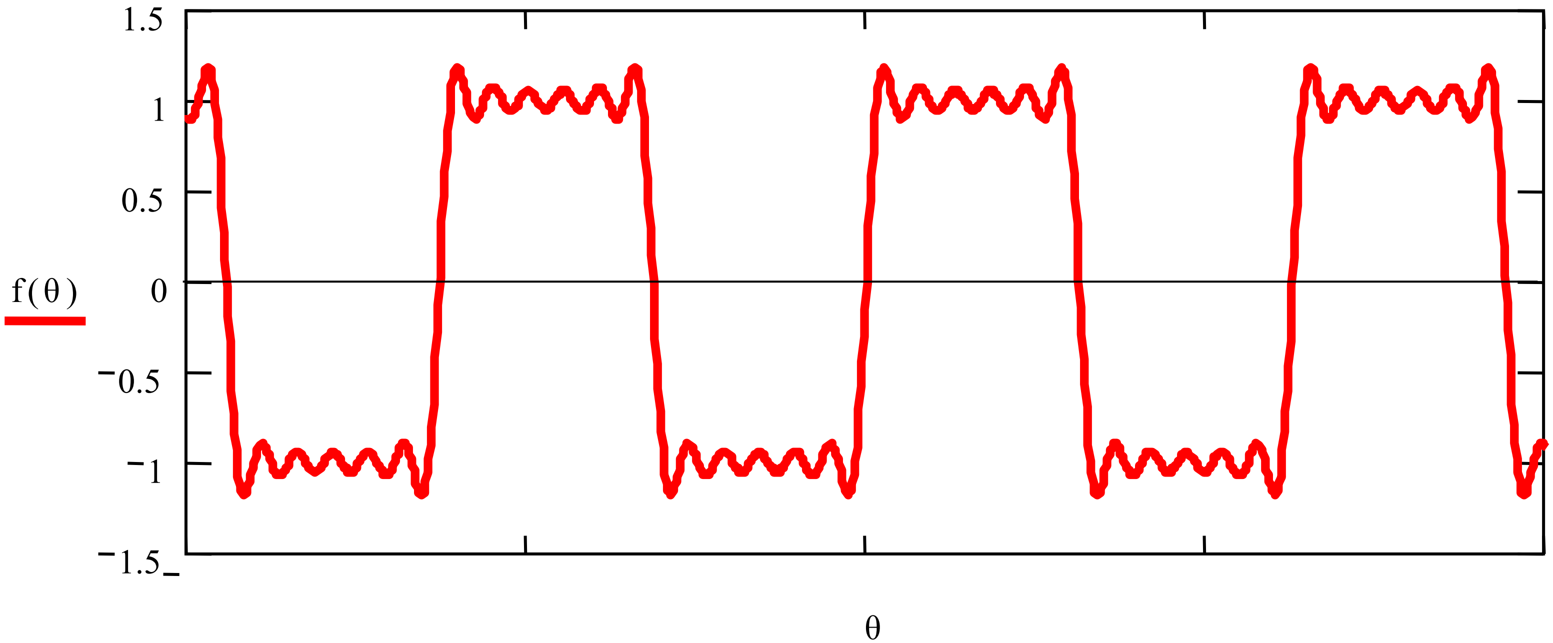
Example 1.----

When we consider 4 terms as shown in the previous slide, the function looks like the following.



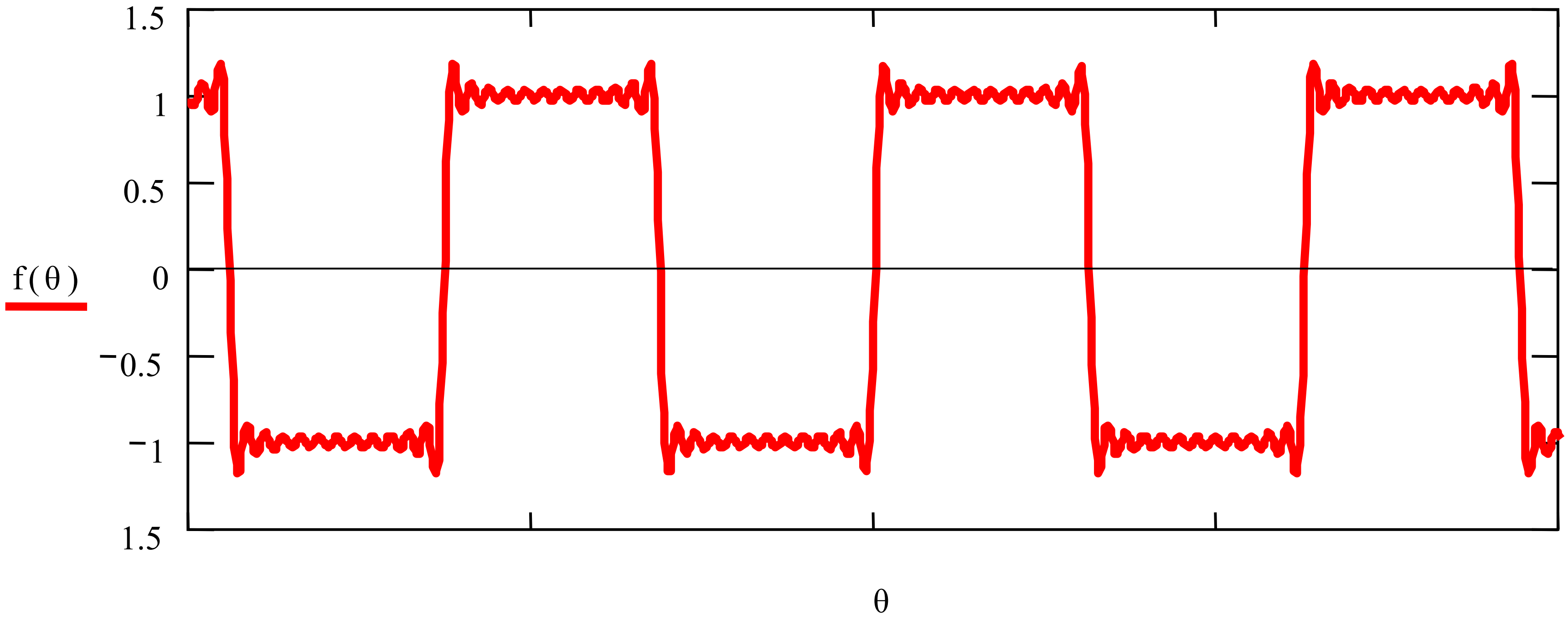
Example 1.----

When we consider 6 terms, the function looks like the following.



Example 1.----

When we consider 12 terms, the function looks like the following.



Example 1.----

The red curve was drawn with 20 terms and the blue curve was drawn with 4 terms.