Fourier Series analysis---

Fourier Series, Exponential Form:

Starting with Trigonometric form of Fourier series;

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

We can write;

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n (\frac{e^{jn\omega t} + e^{-jn\omega t}}{2}) + b_n (\frac{e^{jn\omega t} - e^{-jn\omega t}}{2j})]$$

Or $f(t) = a_0 + \sum_{n=1}^{\infty} [a_n (\frac{e^{jn\omega t} + e^{-jn\omega t}}{2}) - jb_n (\frac{e^{jn\omega t} - e^{-jn\omega t}}{2})]$

Or $f(t) = a_0 + \sum_{n=1}^{\infty} [(\frac{a_n + jb_n}{2}) e^{-jn\omega t} + (\frac{a_n - jb_n}{2}) e^{jn\omega t}]$

Fourier Series analysis---

- Or $f(t) = A_0 + \sum_{n=1}^{\infty}[A_{-n}\, e^{-jn\omega t} + A_n e^{jn\omega t}] - -(1)$

Here: $A_0 = a_0,\ A_{-n} = \frac{(a_n + jb_n)}{2},\ A_n = \frac{(a_n - jb_n)}{2}$

Equation (1) can be written in compact form as;

$$f(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega t} - - -(2)$$

Here $A_n = \frac{(a_n - jb_n)}{2} = \frac{1}{2}\left[\frac{2}{T}\int_0^T f(t)\cos n\omega t\, dt - j\frac{2}{T}\int_0^T f(t)\sin n\omega t\, dt\right]$

Or $\quad A_n = \frac{1}{T}\int_0^T f(t)\,(\cos n\omega t - j\sin n\omega t)\, dt$

Or $\quad A_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega t} dt - - - -(3)$

Equation (2) is exponential form of Fourier series.

From above we can also write;

$$a_n = A_n + A_{-n}\ and\ b_n = j(A_n - A_{-n}) \text{-----}(4)$$

Fourier Series analysis---

- Waveform Symmetry:

(i) Odd Symmetry: A function f(t) is said to be having odd symmetry if

$$f(-t) = -f(t)$$

where t is independent variable.

Some example of odd functions are;

$f(t) = \sin t, f(t) = t^3 + t$ etc.

(ii) Even Symmetry: A function f(t) is said to be having even
` symmetry if

$$f(-t) = f(t)$$

Some example of even functions are;

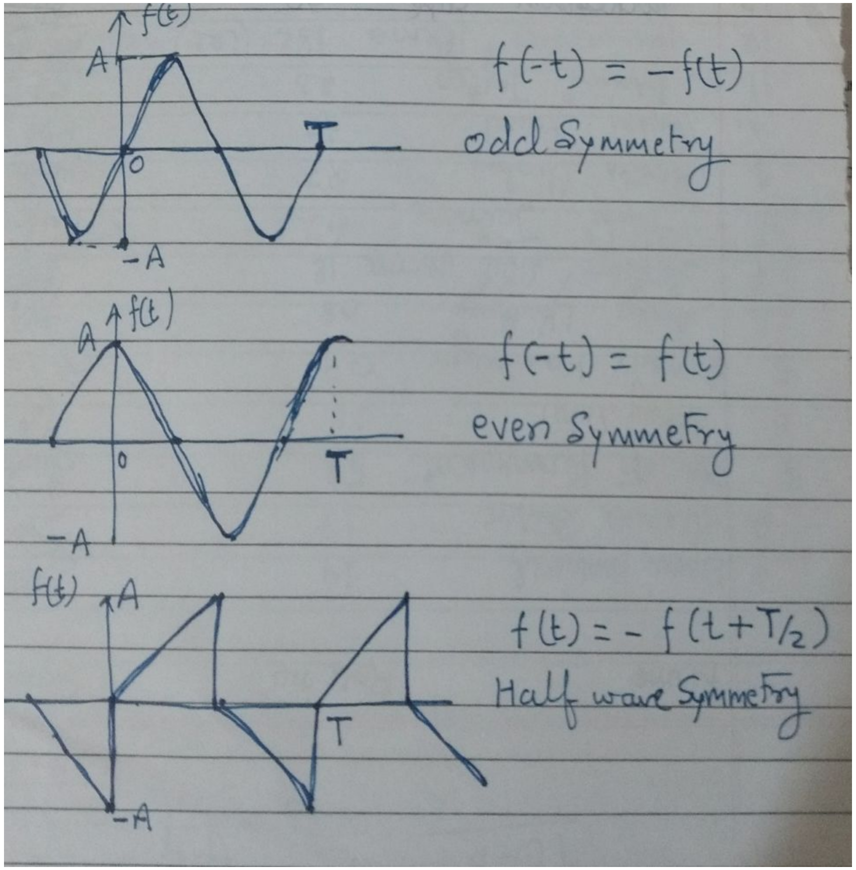
$f(t) = \cos t, f(t) = t^4 + t^2$ etc.

Waveform Symmetry

(iii) Half wave symmetry:
A function f(t) is said to be having half wave symmetry if

$$f(t) = -f(t + \frac{T}{2})$$

where t is independent variable.



Fourier Series analysis---

- Determination of Fourier series coefficients for Even, Odd and Half wave symmetry functions:

(i) Even Symmetry Function:

Where $f(-t) = f(t)$, we have for period of time $-\frac{T}{2}$ $to \frac{T}{2}$ we can write;

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos n\omega t \, dt$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{0} f(t) \cos n\omega t \, dt + \frac{2}{T} \int_{0}^{\frac{T}{2}} f(t) \cos n\omega t \, dt$$

Now in first part of integral in above equation by putting $t = -t$

And thus $dt = -dt$ we have

$$a_n = \frac{2}{T} \int_{\frac{T}{2}}^{0} f(-t) \cos(-n\omega t)(-\,dt) + \frac{2}{T} \int_{0}^{\frac{T}{2}} f(t) \cos n\omega t \, dt$$

Fourier Series analysis---

- $$a_n = -\frac{2}{T}\int_{\frac{T}{2}}^{0} f(t)\cos(n\omega t)(\,dt) + \frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\cos n\omega t\,dt$$

$$a_n = \frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\cos n\omega t\,dt + \frac{2}{T}\int_{0}^{\frac{T}{2}} f(t)\cos n\omega t\,dt$$

$$a_n = \frac{4}{T}\int_{0}^{\frac{T}{2}} f(t)\cos n\omega t\,dt \;---(5)$$

With similar manipulation we can show that all the values of b_n with even symmetry function will be equal to 0.

Value of a_0 will depend upon symmetry with respect to horizontal axis.

Fourier Series analysis---

(ii) Odd Symmetry Function: With similar mathematical manipulations as applied to even symmetry, here we can show that;

All a_n will be equal to zero, and value of b_n will be;

$$b_n = \frac{4}{T} \int_0^{\frac{T}{2}} f(t) \sin n\omega t \, dt \text{ ---(6)}$$

Value of a_0 will depend upon symmetry with respect to horizontal axis.

(iii) Half wave symmetry: In similar way we can write in this case;

$a_0 = 0$,

and for odd n;

$$a_n = \frac{4}{T} \int_0^{\frac{T}{2}} f(t) \cos n\omega t \, dt \text{ and } \quad b_n = \frac{4}{T} \int_0^{\frac{T}{2}} f(t) \sin n\omega t \, dt$$

and for even n

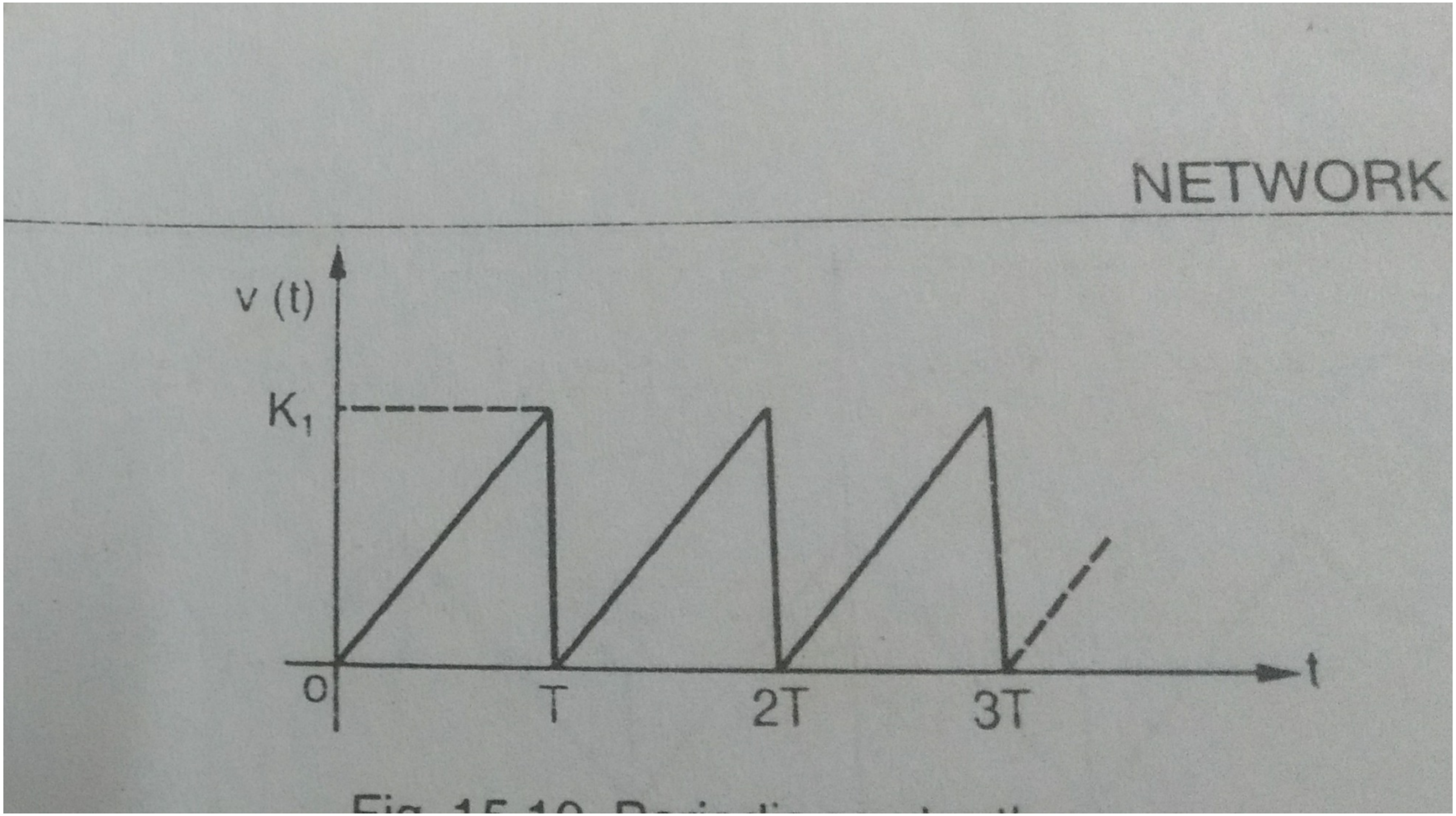
$$a_n = b_n = 0$$

Fourier Series analysis---

The above results are summarized in Table 15.1.

Type of function	Property	Co-efficients of Fourier series		
		a_0	a_n	b_n
Odd	$f(-t) = -f(t)$	*	0	$\frac{4}{T}\int_0^{T/2} f(t)\ \sin n\,\omega t\, dt$
Even	$f(-t) = f(t)$	*	$\frac{4}{T}\int_0^{T/2} f(t) \cos n\,\omega t\, dt$	0
Half wave symmetry	$f(t) = -f(t \pm T/2)$	0	$\frac{4}{T}\int_0^{T/2} f(t) \cos n\,\omega t\, dt$ for $n =$ odd	$\frac{4}{T}\int_0^{T/2} f(t)\ \sin n\,\omega t\, dt$ for $n =$ odd

Example 2. Find the. Fourier series of the following periodic function



Example 2.....

Maximum value $= k_1$

The expression for function for $0 \le t \le T$

$$v(t) = \frac{k_1}{T} t$$

So the average value :

$$a_0 = \frac{1}{T}\int_0^T v(t)\, dt = \frac{1}{T}\int_0^T \frac{k_1}{T} t\, dt = \frac{k_1}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{k_1}{2}.$$

Since the wave has the odd symmetry so, the coefficients of cosine terms will be zero.

$$a_n = \frac{2}{T}\int_0^T v(t) \cos n\,\omega t\, dt = 0$$

The coefficient b_n is given by

$$b_n = \frac{2}{T}\int_0^T v(t) \sin n\,\omega t\, dt = \frac{2}{T}\int_0^T \frac{K_1}{T} t \sin n\,\omega t\, dt$$

$$= \frac{2\,k_1}{T^2}\left[-t\,\frac{\cos n\,\omega t}{n\,\omega} + \frac{\sin n\,\omega t}{n^2\,\omega^2}\right]_0^T$$

$$= \frac{2\,k_1}{T^2}\left[-T\,\frac{\cos n\,\omega T}{n\,\omega} + \frac{\sin n\,\omega T}{n^2\,\omega^2}\right]$$

put $\quad \omega = \dfrac{2\pi}{T}$

$$b_n = \frac{2\,k_1}{T^2}\left[-T^2\,\frac{\cos n\,2\pi}{n\,2\pi} + T^2\,\frac{\sin n\,2\pi}{(n\,2\pi)^2}\right] = -\frac{k_1}{n\,\pi}$$

So the Fourier series of given function $v(t)$ is

$$v(t) = \frac{k_1}{2} - \frac{k_1}{\pi}\sin \omega t - \frac{k_1}{2\pi}\sin 2\,\omega t - \frac{k_1}{3\pi}\sin 3\,\omega t \ldots$$

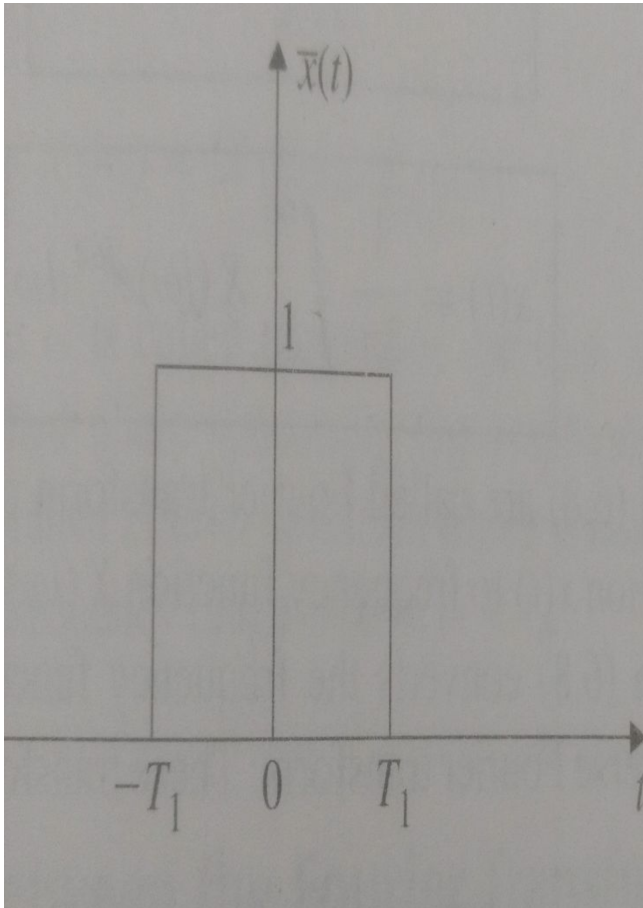
Fourier Transform analysis------

- Fourier transform defines a relationship between a signal in the time domain and its representation in the frequency domain. It is mainly used for aperiodic signals:

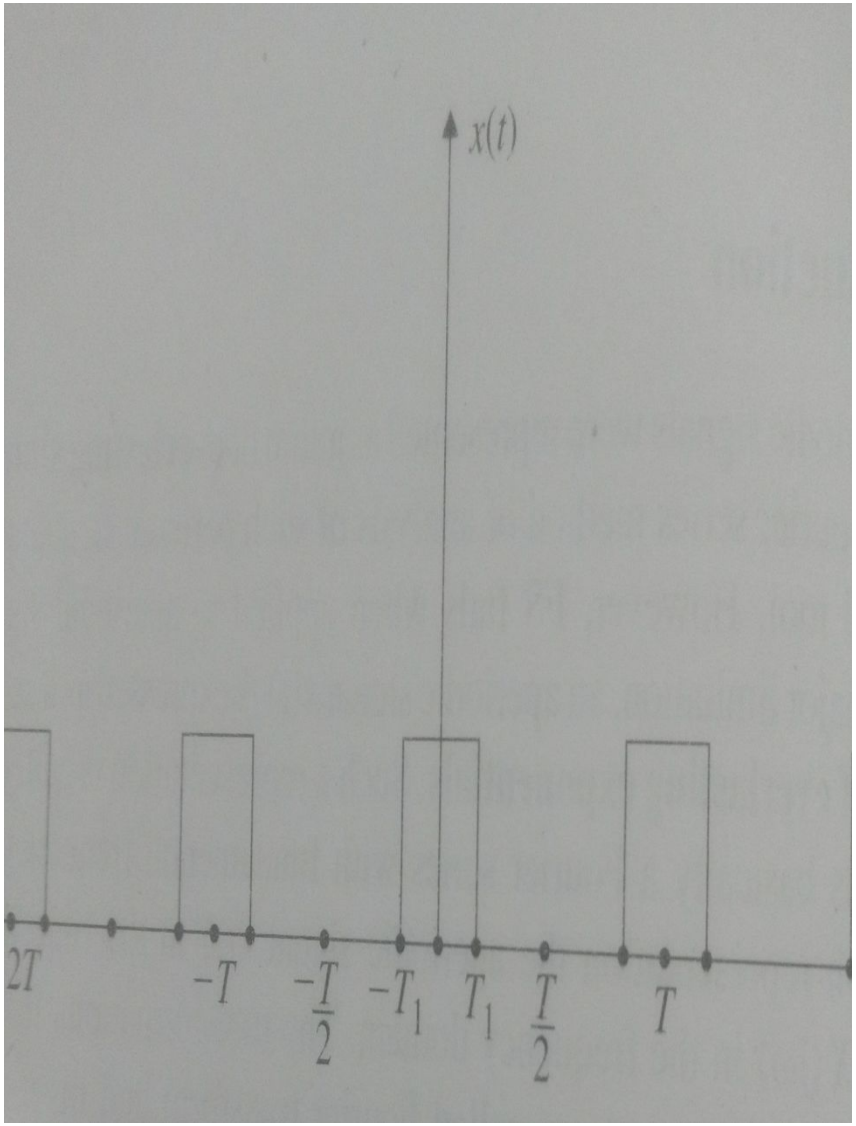
Now start with a non periodic square wave signal x(t) as shown in next slide and also the equivalent periodic signal $\bar{x}(t)$ with fundamental period T is also shown in the next slide. Now in this if $T \to \infty$ $i.e.,$

$$\lim_{T\to\infty} \bar{x}(t) = x(t)$$

Fourier Transform analysis------



A continuous time aperiodic square wave.



Fourier Transform analysis------

- Starting with exponential form of Fourier series i.e., from eq.(2) and eq.(3):

$$x(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega_0 t} \quad and \quad A_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\omega_0 t}dt$$

From above we can write:

$$TA_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\omega_0 t}dt$$

and also $Tx(t) = \sum_{n=-\infty}^{\infty} TA_n e^{jn\omega_0 t}$

Let $X(n\omega_0) = TA_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\omega_0 t}dt$

And $x(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} TA_n e^{jn\omega_0 t} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(n\omega_0)e^{jn\omega_0 t}$--(7)

Fourier Transform analysis------

- As $T \to \infty$ $\omega_0 = \frac{2\pi}{T} \to 0$ negligebly small and hence $n\omega_0 = \omega$ will become continuous frequency. Thus in equation (7), Summation can be replaced by Integration with respect to ω .

Hence we can write

$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ for all ω and

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ for all t. -----(8)

above two equations (8) are known as Fourier Transform pair.

Difference between Fourier series and Fourier Transform.

- **Fourier Series:**
- Fourier series is an expansion of a periodic function using infinite sum of sines and cosines terms.
- Fourier series can be used to solve a large set of mathematical problems specially the problems that involve linear differential equations with constant coefficients.
- Fourier series has applications in large number of fields including electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics and econometrics.

Difference between Fourier series and Fourier Transform.

- Fourier series uses the orthogonality relationships of sine and cosine functions. The calculation and the study of Fourier series is known as the harmonic analysis.
- It is very useful when working with arbitrary periodic functions, since it allows to break the function in to simple terms that can be used to obtain a solution to the original problem.

Difference between Fourier series and Fourier Transform.

- <u>Fourier Transform:</u>
- Fourier transform defines a relationship between a signal in the time domain and its representation in the frequency domain.
- The Fourier transform decomposes a function into oscillatory functions. Since this is a transformation, the original signal can be obtained from knowing the transformation, thus no information is created or lost in the process.
- Fourier transform has some basic properties such as linearity, translation, modulation, scaling, conjugation, duality and convolution. Fourier transform is applied in solving differential equations

Difference between Fourier series and Fourier Transform.

- Fourier transform is also used in nuclear magnetic resonance (NMR) and in other kinds of spectroscopy.

  Based on above points we can summaries the differences as;

Fourier series is an expansion of periodic signal as a linear combination of sines and cosines while Fourier transform is the process or function used to convert signals from time domain in to frequency domain. Fourier series is defined for periodic signals and the Fourier transform can be applied to both periodic and aperiodic signals.