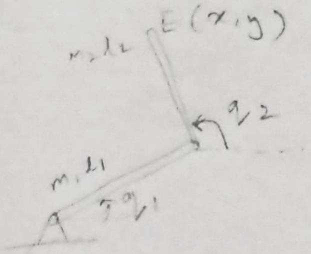
2R Manipulator



T_1 : Trajectory following

T_2 : Apply a force on wall.

T_3 : Act like a spring.

now,

$$x = l_1 \cos q_1 + l_2 \cos q_2$$
$$y = l_1 \sin q_1 + l_2 \sin q_2$$

[Forward kinematics]

$$x = l_1 cq_1 + l_2 cq_2$$
$$y = l_1 sq_1 + l_2 sq_2$$

Differentiating (1), we get

$$\dot{x} = -l_1 sq_1 \dot{q}_1 - l_2 sq_2 \dot{q}_2$$
$$\dot{y} = l_1 cq_1 \dot{q}_1 + l_2 cq_2 \dot{q}_2$$

End effector velocity,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1 sq_1 & -l_2 sq_2 \\ l_1 cq_1 & l_2 cq_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \rightarrow (2)$$

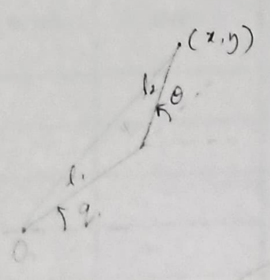
transformation from joints space to end effector space.

for T_1 we need inverse relationships
Given x, y as fns of time, we need to find q_1 & q_2

Option 1. Solve numerically (simple

Option 2. Derive a closed form, expression (hard to derive in general)
(substitute all unknowns you will get answer directly, no extra steps)
→ multiple solutions

[Inverse kinematics]



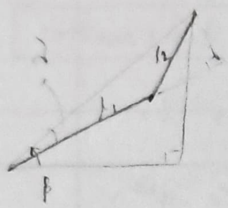
(cosine rule)

$$x^2+y^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos\phi$$

$$x^2+y^2 = l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta$$

$$\boxed{\theta = \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2l_1 l_2}\right)}$$

$q_2 = q_1 + \theta$



$$q_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta}{l_1 + l_2\cos\theta}\right)$$

$$q_2 = q_1 + \theta$$

$\rightarrow$ (3)

$\beta - \gamma = q_1$

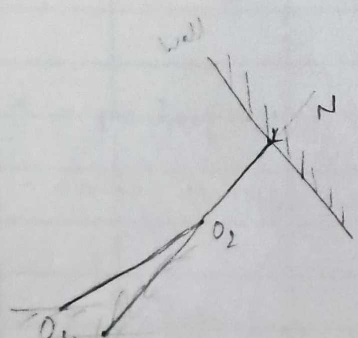
First level ans. to T1

We will later start using the notation x_d and y_d (and q_{1d} & q_{2d}) here for desired values.

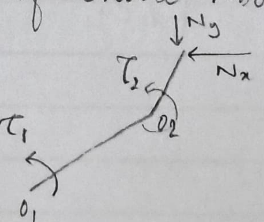
[x, y, q_1, q_2 → actual values the robot can take practically

Task - (2) :

Reach the wall → T_1



FBD of entire robot



(Neglect gravity)

Forces applied by manipulator

$F_x = -N_x$

$F_y = -N_y$

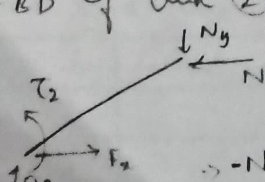
Given N_x N_y,

τ_1, τ_2 unknown

(and reaction forces at the links)

Static equilibrium,

FBD of link (2).

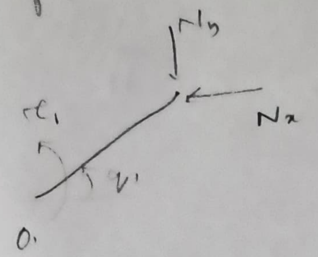


$\Sigma M_{O_2} = 0$ ccw +ve

$$\Rightarrow -N_y \cdot l_2 \cos q_2 + N_x l_2 s q_2 + \tau_2 = 0$$

$$\tau_2 = -N_x l_2 s q_2 + N_y l_2 c q_2$$

FBD of link (1)



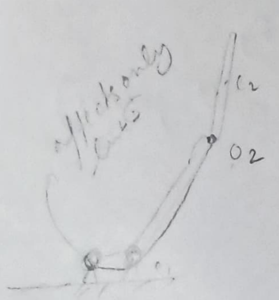
$\Sigma M_{O_1} = 0$

$$\Rightarrow \quad N_y l_1 c q_1 - N_x l_1 s q_1 = \tau_1$$
$$N_y l_2 c q_2 - N_x l_2 s q_2 = \tau_2 \quad \rightarrow (4)$$

here we have assumed τ_2 can be applied wrt inertial frame w/o affecting the link 1.

If motor was fixed onto O_2, there will be an equal & opp torque on link (1)

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 s q_1 & l_1 c q_1 \\ -l_2 s q_2 & l_2 c q_2 \end{bmatrix} \begin{bmatrix} N_x \\ N_y \end{bmatrix}$$



(3) along w/ (4) solves τ_2.

Task 3:

For T3 and next-level answer to τ_1, need to understand dynamics

Langrange's Equations

Langragian: $\boxed{L = K - V}$

K - kinetic energy
V - potential energy

$$(5) \rightarrow \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i'}$$

Q_i are generalised forces derived using principle of virtual work. (applied torque)

i - relating to the number of links. (here, i = 1, 2)

V_{c_2} - velocity of CoM of link 2

$$K = \frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right)\dot{q}_2^2 + \frac{1}{2} m_2 v_{c_2}^2$$

pure rotation of l_1 — for l_2

$$v_{c_2}^2 = (l_1 \dot{q}_1)^2 + \left(\frac{l_2}{2}\dot{q}_2\right)^2 + 2 l_1 \dot{q}_1 \cdot \frac{l_2}{2}\dot{q}_2 \cos(q_2 - q_1)$$

$$V = m_1 g \frac{l_1}{2} s q_1 + m_2 g \left(l_1 s q_1 + \frac{l_2}{2} s q_2\right)$$

IMP

$$\frac{1}{3} m_1 l_1^2 \ddot{q}_1 + m_2 l_1^2 \ddot{q}_1 + m_2 l_1 \frac{l_2}{2} \ddot{q}_2 \cos(q_2 - q_1) - m_2 l_1 \frac{l_2}{2} \dot{q}_2 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1)$$
$$+ m_1 g \frac{l_1}{2} c q_1 + m_2 g l_1 c q_1 = \tau_1$$

$\rightarrow$ (6)

$$\frac{1}{3} m_2 l_2^2 \ddot{q}_2 + m_2 \frac{l_2^2}{4} \ddot{q}_2 + m_2 \frac{l_1 l_2}{2} \ddot{q}_1 \cos(q_2 - q_1) - m_2 \frac{l_1 l_2}{2} \dot{q}_1 (\dot{q}_2 - \dot{q}_1) \sin(q_2 - q_1)$$
$$+ m_2 g \frac{l_2}{2} c q_2 = \tau_2$$

Dynamic eqns of motion of robot. ↑

We note that (4) is valid for any forces F_x, F_y.
(not just wall forces)

We want,

Generally
$$F_x = k_x (x - x_0)$$
$$F_y = k_y (y - y_0)$$

Assume
$$F_x = K x$$
$$F_y = K y$$

From (1),
$$F_x = K (l_1 c q_1 + l_2 c q_2)$$
$$F_y = K (l_1 s q_1 + l_2 s q_2)$$

From (4),

$$\boxed{\begin{aligned} K(l_1 s q_1 + l_2 s q_2) l_2 c q_2 - K(l_1 c q_1 + l_2 c q_2) l_2 s q_2 &= \tau_{2s} \\ K(l_1 s q_1 + l_2 s q_2) l_1 c q_1 - K(l_1 c q_1 + l_2 c q_2) l_1 s q_1 &= \tau_{1s} \end{aligned}} \rightarrow (7)$$

Set motor torques to be $\tau_1 + \tau_{1s}$ and $\tau_2 + \tau_{2s}$ respectively

Answer to T3

$\tau_1, \tau_2 \rightarrow$ to compensate for inertia and motion of the robot (basic motion)

τ_{1s}, τ_{2s} not enough (for robot functions)

account for spring behaviour

Task - 1 (alternate, better way)

Solve for q_{1d} & q_{2d} from (3).

↳ Find $\dot{q}_{1d}$, $\ddot{q}_{1d}$, $\dot{q}_{2d}$, $\ddot{q}_{2d}$

↳ Solve for τ_1 and τ_2 from (6).

Works better when dynamic effects are significant.

Still needs feedback control.