ENPM 808M: Homework #2

Due on Wednesday, October 7, 2015

Dr. William Levine 4:00 PM

Kanishka Ganguly

Contents

Problem 1

Given A is a rotation matrix. So, $A \in SO(2)$, or the **special orthogonal group** of dimension 2 whose determinant is 1 or -1. Here the 2 in $SO(2)$ denotes that A belongs to the $2-D$ rotation group.
Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We know (from identity) that since $A \in SO(2)$, $A \in SO(3)$ also. Using this identity and Cramer's Rule, we have

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

This implies that $a = d$ and $b = -c$. Thus we have

$$A = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$$

Since it is given that $det(A) = 1$, we have $a^2 + c^2 = 1$.

Let us define $\theta = tan^{-1}(\frac{c}{a})$. This gives us

$$cos(\theta) = a \tag{1}$$
$$sin(\theta) = c \tag{2}$$

Thus, there exists a unique θ such that A is of the form

$$A = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

Problem 2

Let us consider the following sequence of rotations:

1. Rotate by ϕ about world x-axis. Let this be represented by $R_{x,\phi}$.
2. Rotate by θ about current z-axis. Let this be represented by $R_{z,\theta}$.
3. Rotate by ψ about current x-axis. Let this be represented by $R_{x,\psi}$.
4. Rotate by α about world z-axis. Let this be represented by $R_{z,\alpha}$.

By convention, we apply the rotations in the following order:

1. Apply all the rotations in the world coordinate axis in the **last in, first out** order.
2. Apply all the rotations in the current coordinate axis in the **first in, first out** order.

So, we have

$$R_{final} = R_{z,\alpha} \times R_{x,\phi} \times R_{z,\theta} \times R_{x,\psi} \tag{3}$$

Problem 3

The Euler angles are given as $(\phi, \theta, \psi) \equiv (\frac{\pi}{2}, 0, \frac{\pi}{4})$ We have the general rotation matrix A given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Also, we have the following component matrices:

$$D \equiv \left.\begin{bmatrix} cos(\phi) & sin(\phi) & 0 \\ -sin(\phi) & cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} \phi \text{ about z-axis} \tag{4}$$

$$C \equiv \left.\begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & sin(\theta) \\ 0 & -sin(\theta) & cos(\theta) \end{bmatrix}\right\} \theta \text{ about former x-axis} \tag{5}$$

$$B \equiv \left.\begin{bmatrix} cos(\psi) & sin(\psi) & 0 \\ -sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} \psi \text{ about former z-axis} \tag{6}$$

This gives us the following:

$$a_{11} = cos(\psi)cos(\phi) - cos(\theta)sin(\phi)sin(\psi) = -\frac{1}{\sqrt{2}} \tag{7}$$

$$a_{12} = cos(\psi)sin(\phi) + cos(\theta)cos(\phi)sin(\psi) = -\frac{1}{\sqrt{2}} \tag{8}$$

$$a_{13} = sin(\psi)sin(\theta) = 0 \tag{9}$$

$$a_{21} = -sin(\psi)cos(\phi) - cos(\theta)sin(\phi)cos(\psi) = \frac{1}{\sqrt{2}} \tag{10}$$

$$a_{22} = -sin(\psi)sin(\phi) + cos(\theta)cos(\phi)cos(\psi) = -\frac{1}{\sqrt{2}} \tag{11}$$

$$a_{23} = cos(\psi)sin(\theta) = 0 \tag{12}$$

$$a_{31} = sin(\theta)sin(\phi) = 0 \tag{13}$$

$$a_{32} = -sin(\theta)cos(\phi) = 0 \tag{14}$$

$$a_{33} = cos(\theta) = 1 \tag{15}$$

Thus we get

$$R_1^0 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The direction of the x_1 axis relative to the base frame is given by the first column of R_1^0 as $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

Problem 4

It is given that

$$H = Rot_{x,\alpha} Trans_{x,b} Trans_{z,d} Rot_{z,\theta} \tag{16}$$

I. Consider the first pair of transformation $Rot_{x,\alpha} Trans_{x,b}$.
Here, by rotating by α around the x-axis, i.e. $Rot_{x,\alpha}$, we have a change in the y and z axes only, while the x-axis is preserved. Or, $(x, y, z) \to (x, y', z')$.
Now, with the translation $Trans_{x,b}$, which is the translation along the x-axis by a distance b, we have $(x, y, z) \to (x + b, y, z)$. Thus, all axes are preserved in translation.

II. So, by similar logic, we have $Trans_{z,d} Rot_{z,\theta}$ giving the same result, i.e. translation and rotation about the same axis is commutative, since the orientation of the axis is preserved.

III. Moving on to $Rot_{x,\alpha} Rot_{z,\theta}$, since the rotations are first α around x-axis and then θ around z-axis, we have y, z axes being affected in the former and x, y axes being affected in the latter. In such a case, the order of transformation matters. Thus, $Rot_{x,\alpha}$ and $Rot_{z,\theta}$ are not commutative since they are transformations on two different axes.

IV. Taking $Rot_{x,\alpha}$ and $Trans_{z,d}$, we have the case of rotation and translation on different axes, where $Rot_{x,\alpha}$ is along the x-axis which affects the y, z orientation while $Trans_{z,d}$ is along the z-axis which affects the z orientation. Thus, in this case also, the order in which the transformations are applied matter since they are applied on different axes. So, rotation and translation along different axes are not commutative, since orientation of axes are not preserved.

V. The same logic applies to the case of $Trans_{x,b}$ and $Rot_{z,\theta}$, where the x-axis is affected by the former transformation and the y, z axes are affected by the latter.

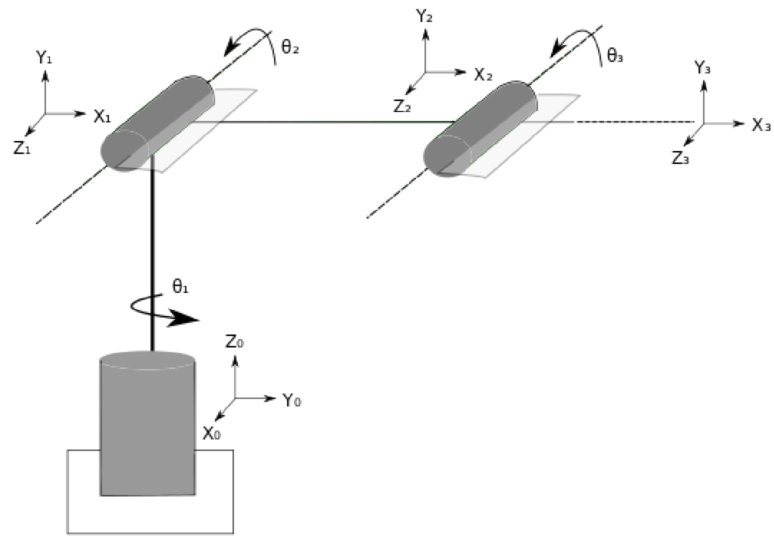
Thus, we have the following conclusions:

$$Rot_{x,\alpha} Trans_{x,b} = Trans_{x,b} Rot_{x,\alpha} \tag{17}$$
$$Trans_{z,d} Rot_{z,\theta} = Rot_{z,\theta} Trans_{z,d} \tag{18}$$
$$Trans_{x,b} Trans_{z,d} = Trans_{z,d} Trans_{x,b} \tag{19}$$
$$Rot_{x,\alpha} Rot_{z,\theta} \neq Rot_{z,\theta} Rot_{x,\alpha} \tag{20}$$
$$Rot_{x,\alpha} Trans_{z,d} \neq Trans_{z,d} Rot_{x,\alpha} \tag{21}$$
$$Trans_{x,b} Rot_{z,\theta} \neq Rot_{z,\theta} Trans_{x,b} \tag{22}$$

From this, we have the following possible permutations of $H = Rot_{x,\alpha} Trans_{x,b} Trans_{z,d} Rot_{z,\theta}$:

$$H = Rot_{x,\alpha} Trans_{z,d} Trans_{x,b} Rot_{z,\theta} \tag{23}$$
$$H = Trans_{x,b} Rot_{x,\alpha} Trans_{z,d} Rot_{z,\theta} \tag{24}$$
$$H = Trans_{x,b} Rot_{x,\alpha} Rot_{z,\theta} Trans_{z,d} \tag{25}$$
$$H = Rot_{x,\alpha} Trans_{x,b} Rot_{z,\theta} Trans_{z,d} \tag{26}$$

Problem 5



Three-link Articulated Robot

We have, for each link i in the robot, the following homogenous transformation

$$A_i = R_{z,\theta_i} Trans_{z,d_i} Trans_{x,a_i} R_{x,\alpha_i} \tag{27}$$

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{28}$$

Thus, we have the following table for each of the links:

Link (i)	a_i	α_i	d_i	θ_i
1	0	90	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

So, we have three homogenous matrices for each of the three links as follows:

$$A_1 = \begin{bmatrix} c_{\theta_1} & 0 & s_{\theta_1} & 0 \\ s_{\theta_1} & 0 & -c_{\theta_1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

$$A_2 = \begin{bmatrix} c_{\theta_2} & -s_{\theta_2} & 0 & a_2 c_{\theta_2} \\ s_{\theta_2} & c_{\theta_2} & 0 & a_2 s_{\theta_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{30}$$

$$A_3 = \begin{bmatrix} c_{\theta_3} & -s_{\theta_3} & 0 & a_3 c_{\theta_3} \\ s_{\theta_3} & c_{\theta_3} & 0 & a_3 s_{\theta_3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{31}$$

Combining the three, we get:

$$T_3^0 = A_1 \times A_2 \times A_3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{32}$$

where

$$r_{11} = c_1 c_2 c_3 - c_1 s_2 s_3 = c_1 c_{23} \tag{33}$$

$$r_{12} = -c_1 c_2 c_3 - c_1 s_3 c_2 = -c_1 s_{23} \tag{34}$$

$$r_{13} = s_1 \tag{35}$$

$$d_x = a_2 c_1 c_2 + a_3 c_1 c_2 c_3 - a_3 c_1 s_2 s_3 = a_2 c_1 c_2 + a_3 c_1 c_{23} \tag{36}$$

$$r_{21} = c_2 c_3 s_1 - s_1 s_2 s_3 = s_2 c_{23} \tag{37}$$

$$r_{22} = -c_2 s_1 s_3 - c_3 s_1 s_2 = -s_1 s_2 3 \tag{38}$$

$$r_{23} = -c_1 \tag{39}$$

$$d_y = a_2 c_2 s_1 + a_3 c_2 c_3 s_1 - a_3 s_1 s_2 s_3 = a_2 c_2 s_1 + a_3 s_1 c_{23} \tag{40}$$

$$r_{31} = c_2 s_3 + c_3 s_2 = s_2 3 \tag{41}$$

$$r_{32} = c_2 c_3 - s_2 s_3 = c_{23} \tag{42}$$

$$r_{33} = 0 \tag{43}$$

$$d_z = a_2 s_2 + a_3 c_2 s_3 + a_3 c_3 s_2 = a_2 s_2 + a_3 s_{23} \tag{44}$$

Problem 6

The six-link chain can be broken up into two types of chains, namely an elbow manipulator and a spherical wrist.
Thus, we have

Link (i)	a_i	α_i	d_i	θ_i
1	0	90	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3
4	0	-90	0	θ_4
5	0	90	0	θ_5
6	0	0	d_6	θ_6

The 6 homogenous matrices for each of the links are as follows:

$$A_1 = \begin{bmatrix} c_{\theta_1} & 0 & s_{\theta_1} & 0 \\ s_{\theta_1} & 0 & -c_{\theta_1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{45}$$

$$A_2 = \begin{bmatrix} c_{\theta_2} & -s_{\theta_2} & 0 & a_2 c_{\theta_2} \\ s_{\theta_2} & c_{\theta_2} & 0 & a_2 s_{\theta_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{46}$$

$$A_3 = \begin{bmatrix} c_{\theta_3} & -s_{\theta_3} & 0 & a_3 c_{\theta_3} \\ s_{\theta_3} & c_{\theta_3} & 0 & a_3 s_{\theta_3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{47}$$

$$A_4 = \begin{bmatrix} c_{\theta_4} & 0 & -s_{\theta_4} & 0 \\ s_{\theta_4} & 0 & c_{\theta_4} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{48}$$

$$A_5 = \begin{bmatrix} c_{\theta_5} & 0 & s_{\theta_5} & 0 \\ s_{\theta_5} & 0 & -c_{\theta_5} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{49}$$

$$A_6 = \begin{bmatrix} c_{\theta_6} & -s_{\theta_6} & 0 & 0 \\ s_{\theta_6} & c_{\theta_6} & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{50}$$

$$\tag{51}$$

Upon multiplication we have:

I For the elbow manipulator portion

$$T_3^0 = A_1 \times A_2 \times A_3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{52}$$

where

$$r_{11} = c_1c_2c_3 - c_1s_2s_3 = c_1c_{23} \tag{53}$$
$$r_{12} = -c_1c_2c_3 - c_1s_3c_2 = -c_1s_{23} \tag{54}$$
$$r_{13} = s_1 \tag{55}$$
$$d_x = a_2c_1c_2 + a_3c_1c_2c_3 - a_3c_1s_2s_3 = a_2c_1c_2 + a_3c_1c_{23} \tag{56}$$
$$r_{21} = c_2c_3s_1 - s_1s_2s_3 = s_2c_{23} \tag{57}$$
$$r_{22} = -c_2s_1s_3 - c_3s_1s_2 = -s_1s_23 \tag{58}$$
$$r_{23} = -c_1 \tag{59}$$
$$d_y = a_2c_2s_1 + a_3c_2c_3s_1 - a_3s_1s_2s_3 = a_2c_2s_1 + a_3s_1c_{23} \tag{60}$$
$$r_{31} = c_2s_3 + c_3s_2 = s_23 \tag{61}$$
$$r_{32} = c_2c_3 - s_2s_3 = c_{23} \tag{62}$$
$$r_{33} = 0 \tag{63}$$
$$d_z = a_2s_2 + a_3c_2s_3 + a_3c_3s_2 = a_2s_2 + a_3s_{23} \tag{64}$$

II For the spherical wrist portion

$$T_6^3 = A_4 \times A_5 \times A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{65}$$

where

$$r_{11} = c_4c_5c_6 - s_4s_6 \tag{66}$$
$$r_{12} = -c_4c_5s_6 - s_4c_6 \tag{67}$$
$$r_{13} = c_4s_5 \tag{68}$$
$$d_x = c_4s_5d_6 \tag{69}$$
$$r_{21} = s_4c_5c_6 + c_4s_6 \tag{70}$$
$$r_{22} = -s_4c_5s_6 + c_4c_6 \tag{71}$$
$$r_{23} = s_4s_5 \tag{72}$$
$$d_y = s_4s_5d_6 \tag{73}$$
$$r_{31} = -s_5c_6 \tag{74}$$
$$r_{32} = s_5s_6 \tag{75}$$
$$r_{33} = c_5 \tag{76}$$
$$d_z = c_5d_6 \tag{77}$$

Finally, combining the two portions together, we have

$$T_6^0 = T_3^0 \times T_6^3 \tag{78}$$