ENPM 808M: Exam #1

Due on Sunday, November 1, 2015

Dr. William Levine 4:00 PM

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Contents

Problem 1

To prove that the following homogenous transformations are correct:

(a)

$$H = \begin{bmatrix} \cos(\theta_1) & 0 & -sin(\theta_1) & -sin(\theta_1)d_3 \\ \sin(\theta_1) & 0 & cos(\theta_1) & cos(\theta_1)d_3 \\ 0 & 1 & 0 & d_1 - d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

This matrix is a combination of a rotational matrix H_R and a translational matrix H_{R_d} and is of the form

$$\begin{bmatrix} H_R & H_{R_d} \\ 0 & 1 \end{bmatrix} \tag{2}$$

The matrix H can thus be split into the following matrices:

$$H_R = \begin{bmatrix} \cos(\theta_1) & 0 & -sin(\theta_1) & 0 \\ \sin(\theta_1) & 0 & cos(\theta_1) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_{R_d} = \begin{bmatrix} 0 & 0 & 0 & -sin(\theta_1)d_3 \\ 0 & 0 & 0 & cos(\theta_1)d_3 \\ 0 & 0 & 0 & d_1 - d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

Now, for H to be a valid homogenous transformation matrix, H_R must be a valid rotational matrix.
The necessary condition for H_R to be a valid rotation matrix is that $H_R \in SO(3)$. The following conditions must be satisfied:

- $H_R^{-1} = H_R^T$
- $det(H_R) = 1$
- $H_R^{-1} \in SO(3)$

Now, consider the condition $det(H_R) = 1$ mentioned above. From calculation of determinant, we get

$$det(H_R) = (-1)(sin^2(\theta_1) + cos^2(\theta_1)) = -1 \tag{4}$$

Now, for H_R to be a rotational matrix, $det(H_R) = 1$.
However, $det(H_R) = sin^2(\theta_1) + cos^2(\theta_1) \neq 1$ for any $\theta \in \mathbb{R}$.

This violates the condition needed for H_R to be a rotational matrix and since H_R is the rotational component of matrix H, this implies that <u>H is also not a valid homogenous transformation matrix.</u>

(b)

$$T = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1s_2d_3 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1s_2d_3 + c_1d_2 \\ s_2 & 0 & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Problem 1 [(b)] continued on next page...

The above transformation matrix T can be written in terms of rotational and translational components T_R and T_{R_d} respectively as:

$$T = \begin{bmatrix} T_R & T_{R_d} \\ 0 & 1 \end{bmatrix} \tag{6}$$

$$\implies T_R = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & 0 \\ s_1c_2 & c_1 & s_1s_2 & 0 \\ s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

For T to be a valid, homogenous transformation matrix, it is necessary for T_R to be a valid rotational matrix. Similar to (a), T_R must satisfy the following conditions for it to be a valid rotation matrix:

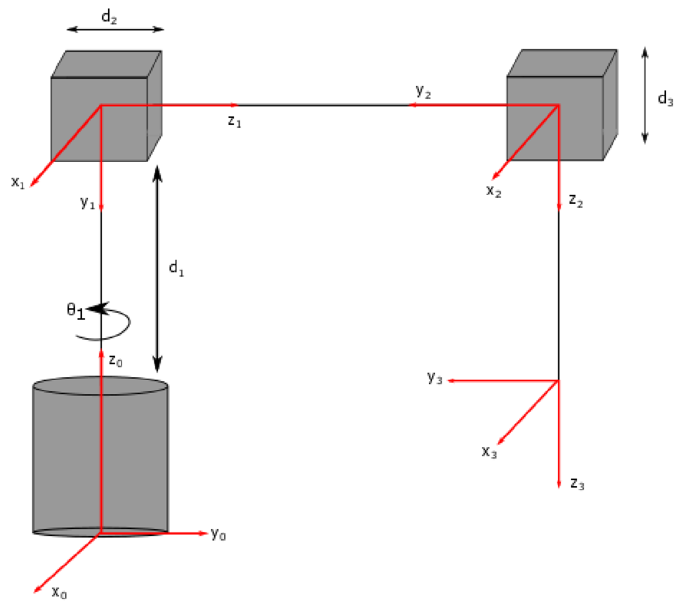
- $T_R^{-1} = T_R^T$
- $det(T_R) = 1$
- $T_R^{-1} \in SO(3)$

Considering the condition $det(T_R) = 1$, we have from calculation:

$$det(T_R) = cos^2(\theta) - sin^2(\theta) \tag{8}$$

Now, for T_R to be a rotation matrix, $det(T_R) = 1$.
But $det(T_R) = cos^2(\theta) - sin^2(\theta) \neq 1$ for any $\theta \in \mathbb{R}$.
Thus, the condition is needed for T_R to be a rotation matrix is not satisfied and since T_R is the rotational component of matrix T, this implies that $\underline{\underline{T \text{ is also not a valid homogenous transformation matrix.}}}$

Problem 2



Choice of Axes - DH Convention

From given figure, we form DH Parameters as:

Link (i)	a_i	α_i	d_i	θ_i
1	0	-90	d_1	$\boldsymbol{\theta_1}$
2	0	-90	$\mathbf{d_2}$	0
3	0	0	$\mathbf{d_3}$	0

where **bold** indicates a variable value.

Applying the DH convention, we get the following matrix:

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

(a)

We have

$$T_1^0 = A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

$$T_2^1 = A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$$\text{We have:} \quad T_2^0 = T_1^0 \times T_2^1 \tag{12}$$

$$\implies T_2^0 = \begin{bmatrix} c_1 & s_1 & 0 & -s_1 d_2 \\ s_1 & -c_1 & 0 & c_1 d_2 \\ 0 & 0 & -1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$T_3^2 = A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

$$\text{We have:} \quad T_3^0 = T_1^0 \times T_2^1 \times T_3^2 \tag{15}$$

$$\implies T_3^0 = \begin{bmatrix} c_1 & s_1 & 0 & -s_1 d_2 \\ s_1 & -c_1 & 0 & c_1 d_2 \\ 0 & 0 & -1 & d_1 - d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

Thus we have the matrices T_1^0**,** T_2^0 **and** T_3^0, which map the world frame to each joint origin.

(b)

From the figure above, we have a revolute joint at 1 and prismatic joints at 2 and 3.
Thus, we can write:

$$J_1 = \begin{bmatrix} z_0 \times (O_3 - O_0) \\ z_0 \end{bmatrix} \tag{17}$$

$$J_2 = \begin{bmatrix} z_1 \\ 0 \end{bmatrix} \tag{18}$$

$$J_3 = \begin{bmatrix} z_2 \\ 0 \end{bmatrix} \tag{19}$$

where

$$z_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \tag{20}$$

$$z_1 = \begin{bmatrix} -s_1 & c_1 & 0 \end{bmatrix}^T \tag{21}$$

$$z_2 = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T \tag{22}$$

$$z_3 = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T \tag{23}$$

and

$$O_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \tag{24}$$
$$O_1 = \begin{bmatrix} 0 & 0 & d_1 \end{bmatrix}^T \tag{25}$$
$$O_2 = \begin{bmatrix} -s_1 d_2 & c_1 d_2 & d_1 \end{bmatrix}^T \tag{26}$$
$$O_3 = \begin{bmatrix} -s_1 d_2 & c_1 d_2 & d_1 - d_3 \end{bmatrix}^T \tag{27}$$

This gives us the Jacobian matrix as

$$J = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix} \tag{28}$$

Now, to find the Jacobian at the end effector, we have:

$$z_0 \times (O_3 - O_0) \tag{29}$$
$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} -s_1 d_2 \\ c_1 d_2 \\ d_1 - d_3 \end{bmatrix} \tag{30}$$
$$= \begin{bmatrix} -c_1 d_2 \\ -s_1 d_2 \\ 0 \end{bmatrix} \tag{31}$$

So, we have

$$J_1 = \begin{bmatrix} -c_1 d_2 \\ -s_1 d_2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{32}$$

Similarly, we get

$$J_2 = \begin{bmatrix} z_1 \\ 0 \end{bmatrix} \tag{33}$$
$$= \begin{bmatrix} -s_1 \\ c_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{34}$$
$$J_3 = \begin{bmatrix} z_2 \\ 0 \end{bmatrix} \tag{35}$$
$$= \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{36}$$

Thus, from Eqn. 28, the final Jacobian matrix is:

$$J = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix} \tag{37}$$

$$= \begin{bmatrix} -c_1 d_2 & -s_1 & 0 \\ -s_1 d_2 & c_1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{38}$$

(c)

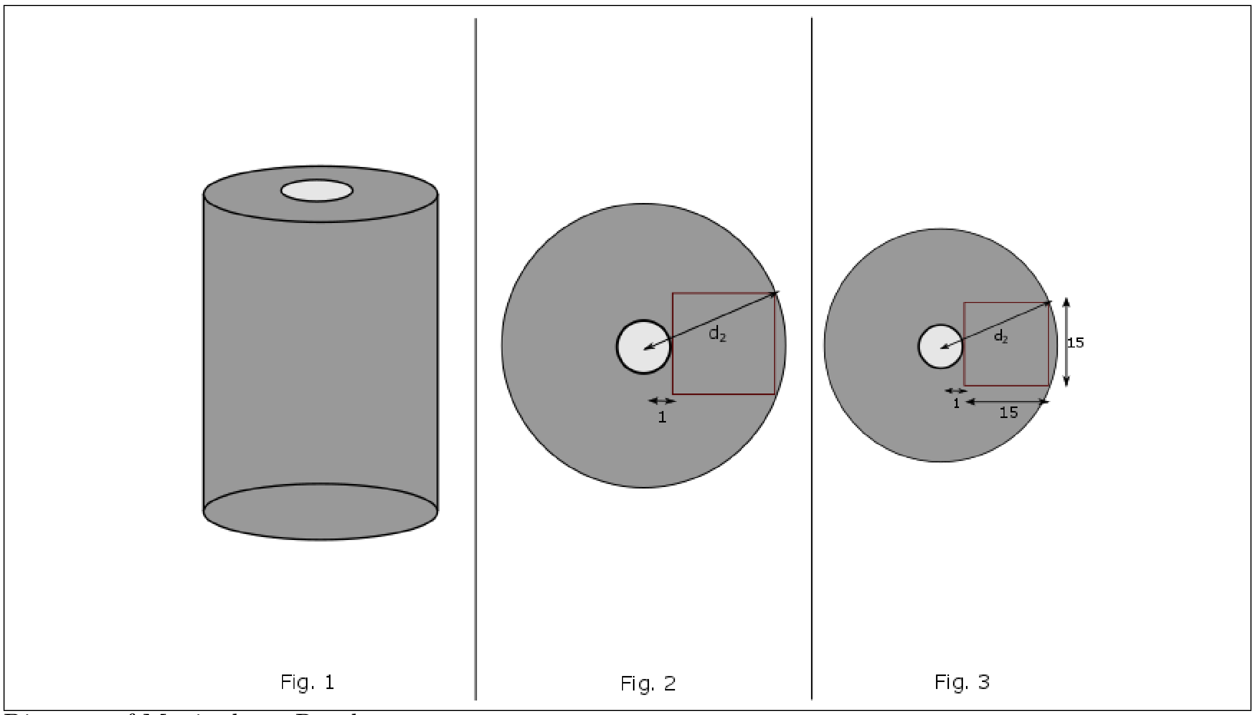


Diagram of Manipulator Reach

From the problem statement, we have $d_3 \geq 0$ and $1 \leq d_2 \leq d_{2_{max}}$.
Also from given problem statement, our work area will take the form of a cylinder with a 'hole' in its center, as shown in Fig. 1
We need to allow the manipulator to traverse a cube of dimensions $15 \times 15 \times 15$, which must fit into the cylinder work area depicted. This is shown is Fig. 2 and Fig. 3.

From these figures, we obtain:

$$(1 + 15)^2 + (15/2)^2 = (d_2)^2 \tag{39}$$

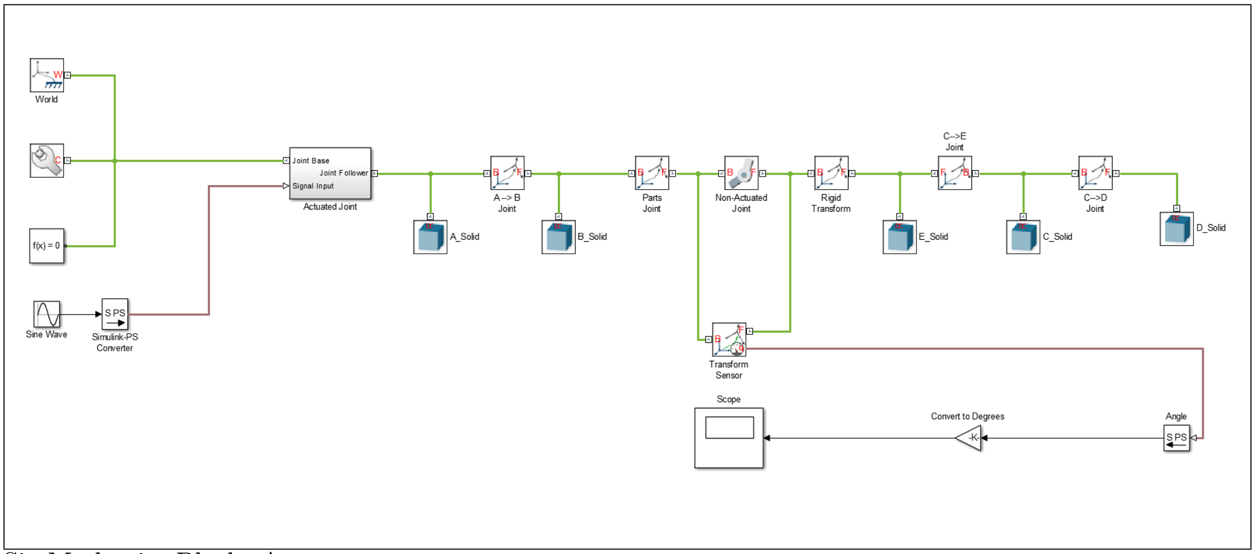
$$\implies d_2 = 17.6 \tag{40}$$

This gives us the lower limit for $d_{2_{max}} = 17.6$.
Now, d_3 must be a minimum of 15 units, since the the cube has a depth of 15 units.
Lastly, d_1 should be a minimum of 15 units so that joint 3 does not hit the base of the reachable area and allow it unrestricted movement.

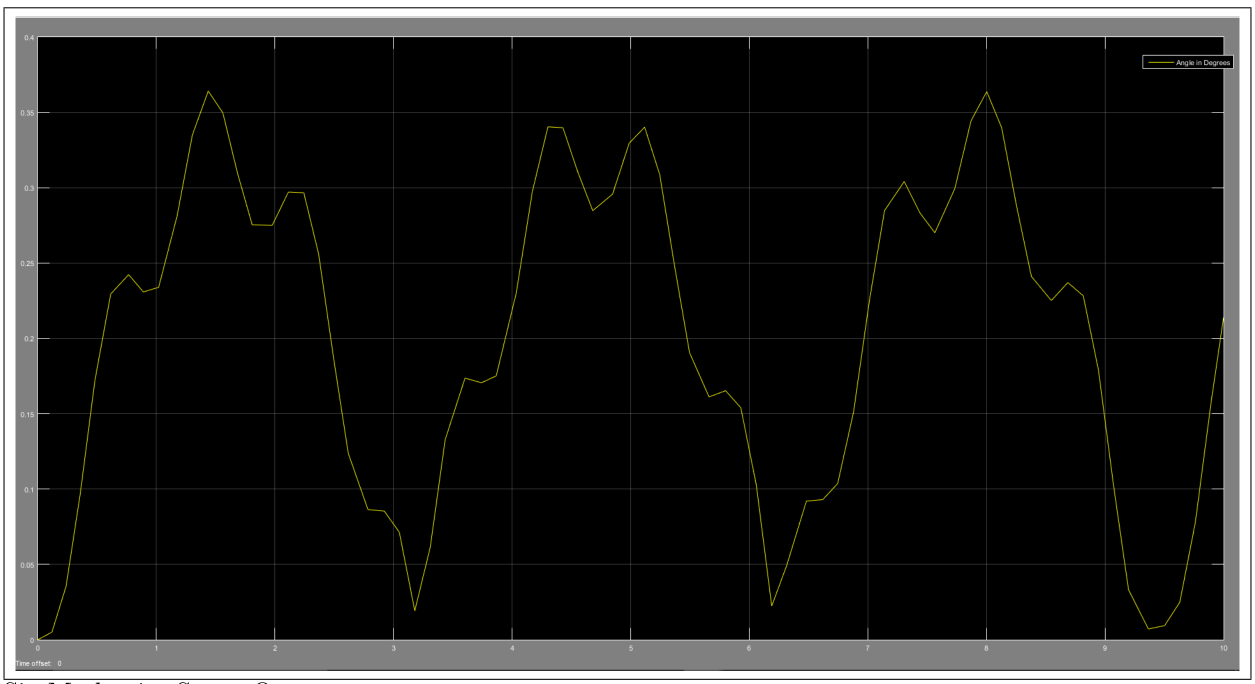
Problem 2 [(c)] continued on next page...

Thus, $\underline{d_1 \text{ must be a minimum of 15 units and } d_{2_{max}} \text{ must be at least 17.6 units}}$.

Problem 3

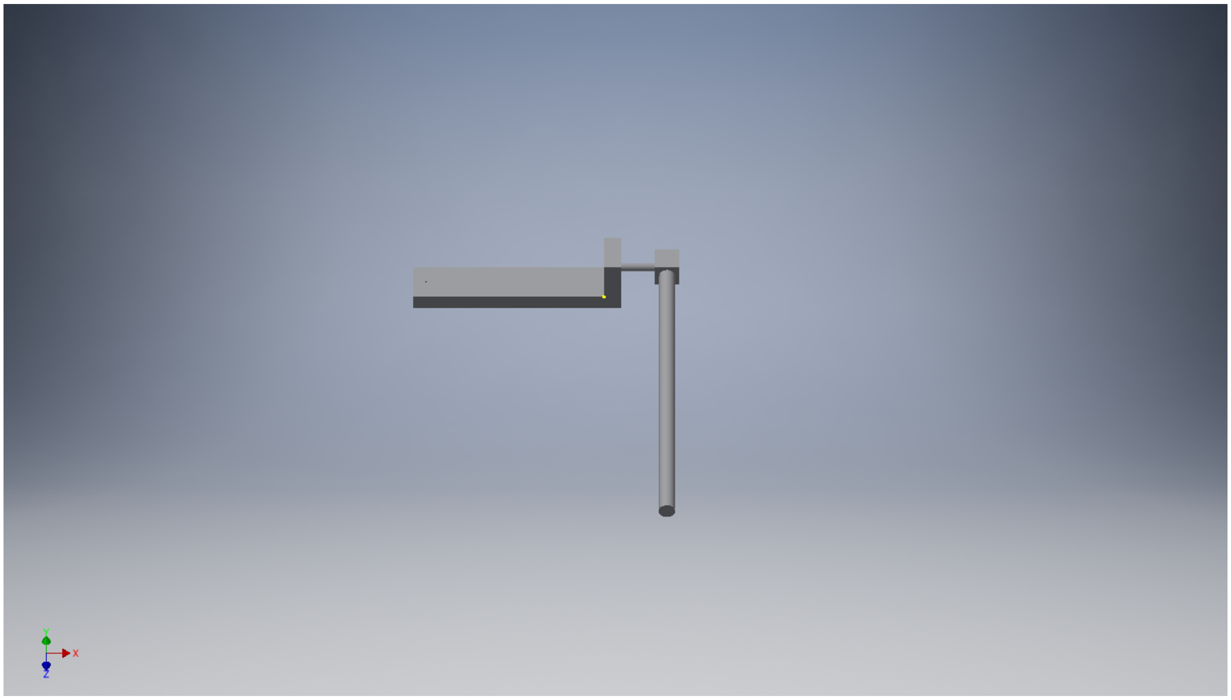


SimMechanics Blocks Arrangement

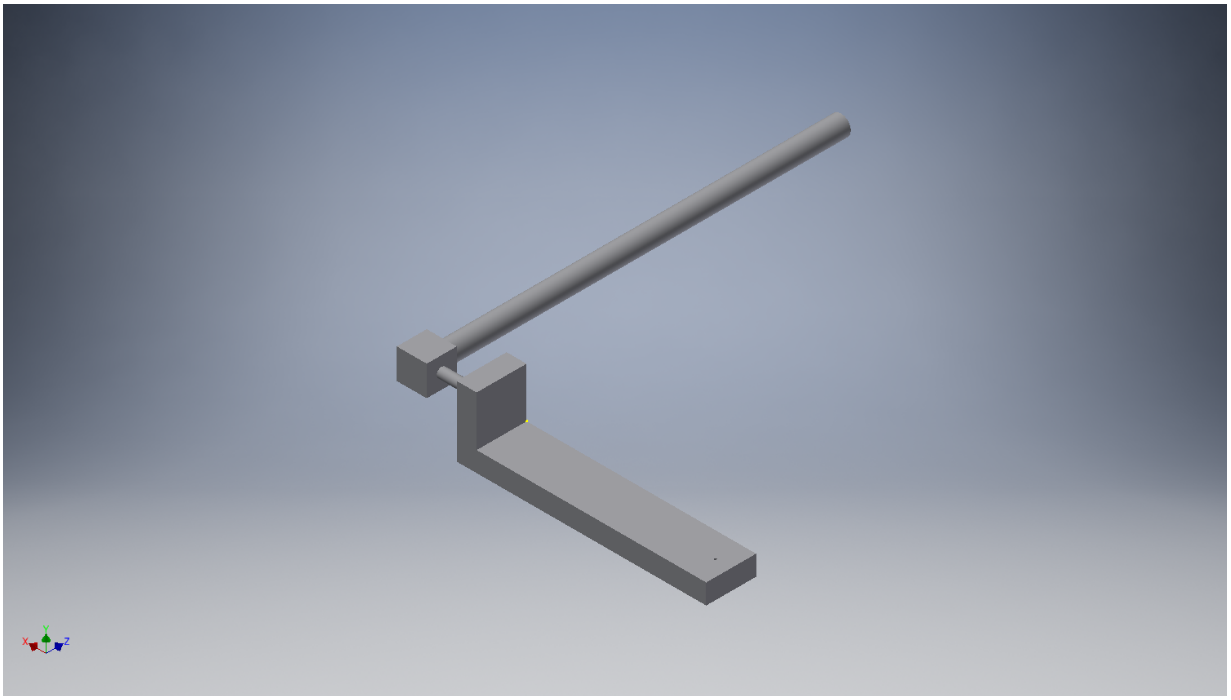


SimMechanics Scope Output

Problem 3 continued on next page. . .



CAD Model 1



CAD Model 2

Problem 4

(a)

It is given in problem statement that

$$J = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & (a_2c_2 + a_3c_{23}) & a_3c_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix} \tag{41}$$

Since precise manipulation is required, we need to find the best configuration of angles. To do so, we calculate manipulability μ.

For the first configuration given as $\theta_1 = \theta_2 = \theta_3 = 45°$, we have $a = 0.4m$.
By calculating JJ^T, we obtain 3 non-zero eigenvalues as

$$\lambda_1 = 2.64 \tag{42}$$
$$\lambda_2 = 1.08 \tag{43}$$
$$\lambda_1 = 0.06 \tag{44}$$

Now we know that J is a 6×3 matrix, which gives us a maximum rank of 3 for the same.

$$\mu_1 = K \times |\sqrt{\lambda_1\lambda_2\lambda_3}| \tag{45}$$
$$= 0.43K \tag{46}$$

For the second configuration given as $\theta_1 = 45°, \theta_2 = -45°, \theta_3 = 45°$, we have $a = 0.4m$.
By calculating JJ^T, we obtain 3 non-zero eigenvalues as

$$\lambda_1 = 2.64 \tag{47}$$
$$\lambda_2 = 1.46 \tag{48}$$
$$\lambda_1 = 0.06 \tag{49}$$

Now we know that J is a 6×3 matrix, which gives us a maximum rank of 3 for the same.

$$\mu_2 = K \times |\sqrt{\lambda_1\lambda_2\lambda_3}| \tag{50}$$
$$= 0.503K \tag{51}$$

For the third configuration given as $\theta_1 = 30°, \theta_2 = -30°, \theta_3 = 30°$, we have $a = 0.4m$.
By calculating JJ^T, we obtain 3 non-zero eigenvalues as

$$\lambda_1 = 2.69 \tag{52}$$
$$\lambda_2 = 1.55 \tag{53}$$
$$\lambda_1 = 0.06 \tag{54}$$

Now we know that J is a 6×3 matrix, which gives us a maximum rank of 3 for the same.

$$\mu_3 = K \times |\sqrt{\lambda_1\lambda_2\lambda_3}| \tag{55}$$
$$= 0.509K \tag{56}$$

From the values of μ_1, μ_2, μ_3, we see that μ_3 has the highest manipulability value.
Thus for the configuration given as $\theta_1 = 30°, \theta_2 = -30°, \theta_3 = 30°$, we have the best choice of joint angles for the best manipulability.

(b)

We have, from the problem statement,

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{57}$$

where $\theta_1 = 30°$, $\theta_2 = -30°$, $\theta_3 = 30°$
We find the torque τ (calculated in MATLAB) as

$$\tau = J^T \times F \tag{58}$$

$$= \begin{bmatrix} 1.2732 \\ 0.6536 \\ 0.034 \end{bmatrix} \tag{59}$$

Problem 5

(a)

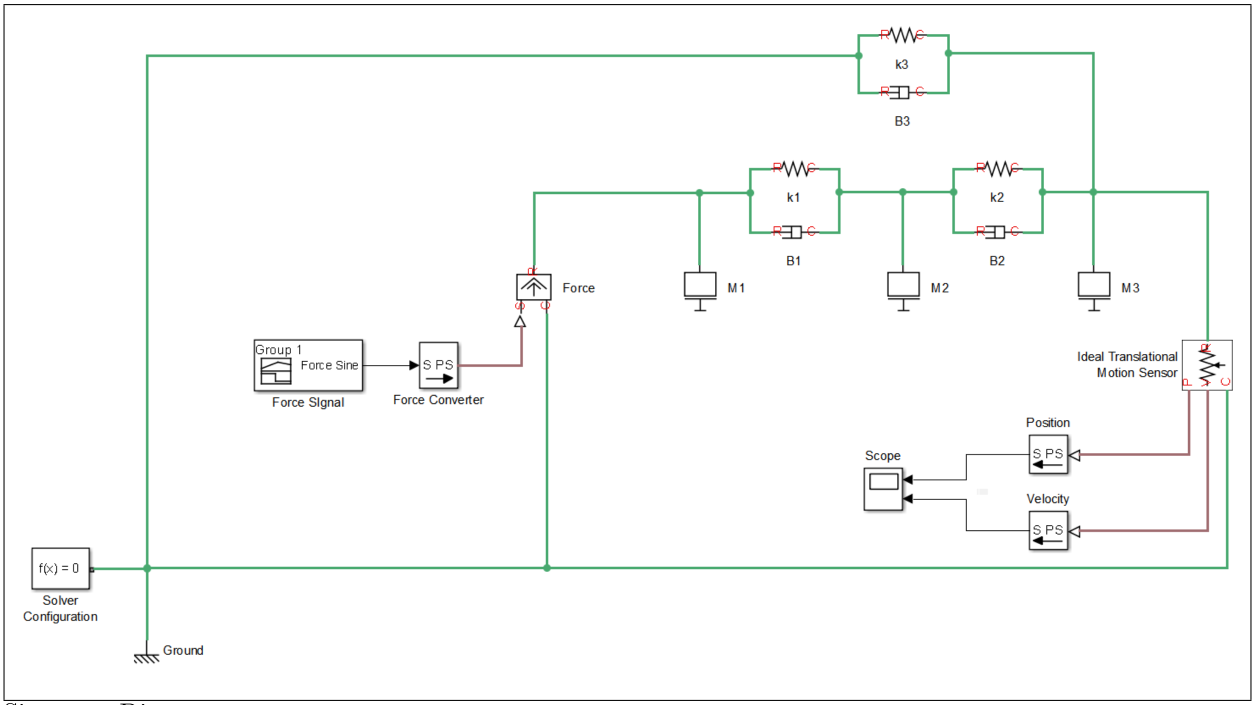
Initial conditions for the system is given as $x_1 = x_2 = x_3 = 0$. Also, the length of the springs at rest is also 0.
This implies that all the blocks at $t = 0$ are at same position. As we apply force $F(t)$, the mass M_1 starts moving. Since M_2 is connected to M_1 via the spring-damper system, and similarly since M_3 is connected to M_2 by the second spring-damper system, the motion of M_1 causes all three masses to move.
From this explanation, we have the following set of equations:

$$M_1\ddot{x} = F(t) - k_1(x_1 - x_2) - B_1(\dot{x}_1 - \dot{x}_2) \tag{60}$$

$$M_2\ddot{x} = k_1(x_1 - x_2) + B_1(\dot{x}_1 - \dot{x}_2) - k_2(x_2 - x_3) - B_2(\dot{x}_2 - \dot{x}_3) \tag{61}$$

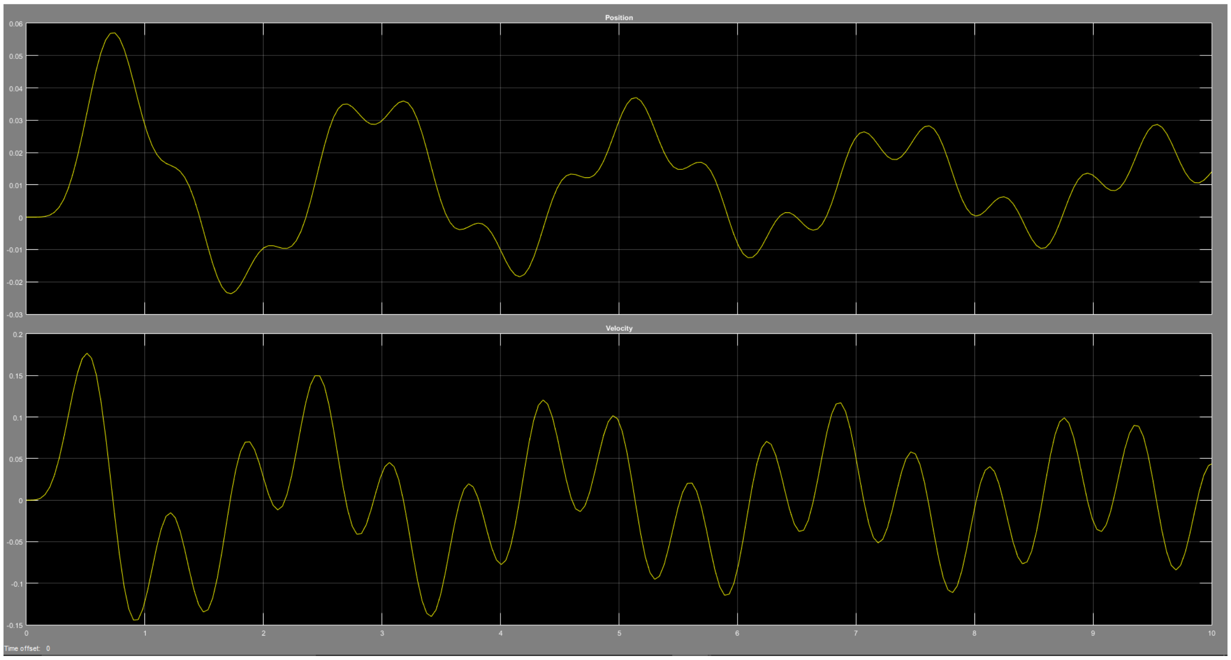
$$M_3\ddot{x} = k_2(x_2 - x_3) - B_2(\dot{x}_2 - \dot{x}_3) - k_3(x_3) - B_3\dot{x}_3 \tag{62}$$

(b)



Simscape Diagram

Problem 5 [(b)] continued on next page. . .



Simscape Scope Output