2-7 A **group** is a set X together with an operation $*$ defined on that set such that

- $x_1 * x_2 \in X$ for all $x_1, x_2 \in X$
- $(x_1 * x_2) * x_3 = x_1 * (x_2 * x_3)$
- There exists an element $I \in X$ such that $I * x = x * I = x$ for all $x \in X$
- For every $x \in X$, there exists some element $y \in X$ such that $x * y = y * x = I$

Show that SO(n) with the operation of matrix multiplication is a group.

2-8 Derive Equations (2.6) and (2.7).

2-9 Suppose A is a 2×2 rotation matrix. In other words $A^T A = I$ and $\det A = 1$. Show that there exists a unique θ such that A is of the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2-10 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the current z-axis.
3. Rotate by ψ about the world y-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-11 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the world z-axis.
3. Rotate by ψ about the current x-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-12 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.

2. Rotate by θ about the current z-axis.
3. Rotate by ψ about the current x-axis.
4. Rotate by α about the world z-axis.

for 2.12

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-13 Consider the following sequence of rotations:

1. Rotate by ϕ about the world x-axis.
2. Rotate by θ about the world z-axis.
3. Rotate by ψ about the current x-axis.
4. Rotate by α about the world z-axis.

Write the matrix product that will give the resulting rotation matrix (do not perform the matrix multiplication).

2-14 If the coordinate frame $o_1x_1y_1z_1$ is obtained from the coordinate frame $o_0x_0y_0z_0$ by a rotation of $\frac{\pi}{2}$ about the x-axis followed by a rotation of $\frac{\pi}{2}$ about the fixed y-axis, find the rotation matrix R representing the composite transformation. Sketch the initial and final frames.

2-15 Suppose that three coordinate frames $o_1x_1y_1z_1$, $o_2x_2y_2z_2$, and $o_3x_3y_3z_3$ are given, and suppose

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad R_3^1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the matrix R_3^2.

2-16 Derive equations for the roll, pitch, and yaw angles corresponding to the rotation matrix $R = (r_{ij})$.

2-17 Verify Equation (2.43).

2-18 Verify Equation (2.45).

2-19 If R is a rotation matrix show that $+1$ is an eigenvalue of R. Let k be a unit eigenvector corresponding to the eigenvalue $+1$. Give a physical interpretation of k.

2-20 Let $k = \frac{1}{\sqrt{3}}[1, 1, 1]^T$, $\theta = 90^\circ$. Find $R_{k,\theta}$.

2-21 Show by direct calculation that $R_{k,\theta}$ given by Equation (2.43) is equal to R given by Equation (2.47) if θ and k are given by Equations (2.48) and (2.49), respectively.

2-22 Compute the rotation matrix given by the product

$$R_{x,\theta}R_{y,\phi}R_{z,\pi}R_{y,-\phi}R_{x,-\theta}$$

2-23 Suppose R represents a rotation of 90° about y_0 followed by a rotation of 45° about z_1. Find the equivalent axis/angle to represent R. Sketch the initial and final frames and the equivalent axis vector k.

2-24 Find the rotation matrix corresponding to the Euler angles $\phi = \frac{\pi}{2}$, $\theta = 0$, and $\psi = \frac{\pi}{4}$. What is the direction of the x_1 axis relative to the base frame?

2-25 Section 2.5.1 described only the Z-Y-Z Euler angles. List all possible sets of Euler angles. Is it possible to have Z-Z-Y Euler angles? Why or why not?

2-26 Unit magnitude complex numbers $a + ib$ with $a^2 + b^2 = 1$ can be used to represent orientation in the plane. In particular, for the complex number $a + ib$, we can define the angle $\theta = \text{Atan2}(a, b)$. Show that multiplication of two complex numbers corresponds to addition of the corresponding angles.

2-27 Show that complex numbers together with the operation of complex multiplication define a group. What is the identity for the group? What is the inverse for $a + ib$?

2-28 Complex numbers can be generalized by defining three independent square roots for -1 that obey the multiplication rules

$$\begin{aligned} -1 &= i^2 = j^2 = k^2, \\ i &= jk = -kj, \\ j &= ki = -ik, \\ k &= ij = -ji \end{aligned}$$

Using these, we define a **quaternion** by $Q = q_0 + iq_1 + jq_2 + kq_3$, which is typically represented by the 4-tuple (q_0, q_1, q_2, q_3). A rotation by θ about the unit vector $n = [n_x, n_y, n_z]^T$ can be represented by the unit quaternion $Q = \left(\cos\frac{\theta}{2}, n_x \sin\frac{\theta}{2}, n_y \sin\frac{\theta}{2}, n_z \sin\frac{\theta}{2}\right)$. Show that such a quaternion has unit norm, that is, $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$.

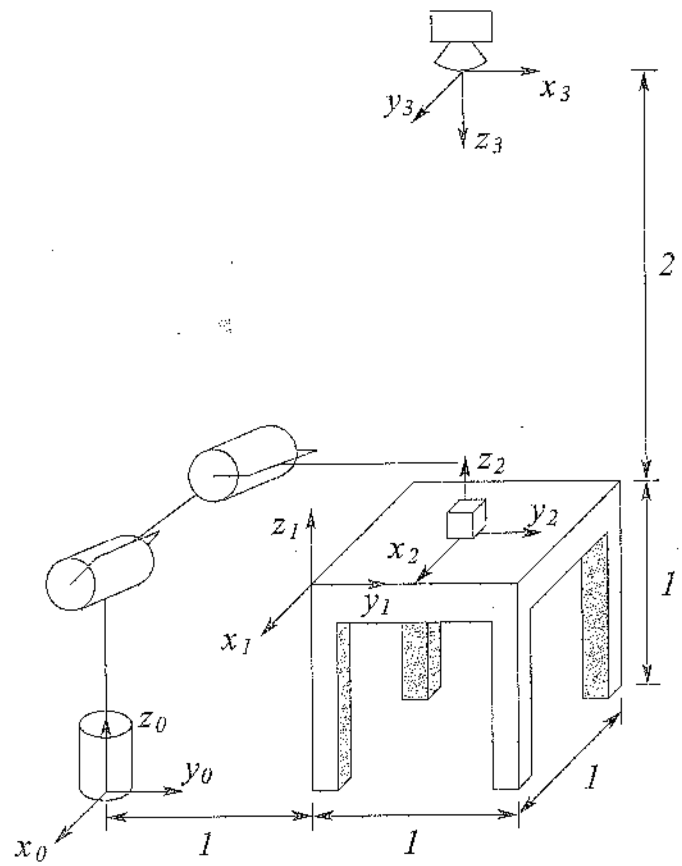

Figure 2.14: Diagram for Problem 2-39.

2-42 Consult an astronomy book to learn the basic details of the Earth's rotation about the sun and about its own axis. Define for the Earth a local coordinate frame whose z-axis is the Earth's axis of rotation. Define $t = 0$ to be the exact moment of the summer solstice, and the global reference frame to be coincident with the Earth's frame at time $t = 0$. Give an expression $R(t)$ for the rotation matrix that represents the instantaneous orientation of the earth at time t. Determine as a function of time the homogeneous transformation that specifies the Earth's frame with respect to the global reference frame.

2-43 In general, multiplication of homogeneous transformation matrices is not commutative. Consider the matrix product

$$H = \text{Rot}_{x,\alpha}\text{Trans}_{x,b}\text{Trans}_{z,d}\text{Rot}_{z,\theta}$$

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2.43

Determine which pairs of the four matrices on the right hand side commute. Explain why these pairs commute. Find all permutations of these four matrices that yield the same homogeneous transformation matrix, H.

NOTES AND REFERENCES

Rigid body motions and the groups $SO(n)$ and $SE(n)$ are often addressed in mathematics books on the topic of linear algebra. Standard texts for this material include [8], [23], and [40]. These topics are also often covered in applied mathematics texts for physics and engineering, such as [108], [119], and [139]. In addition to these, a detailed treatment of rigid body motion developed with the aid of exponential coordinates and Lie groups is given in [93].

3-5 Consider the three-link planar manipulator of Figure 3.26. Derive the forward kinematic equations using the DH convention.

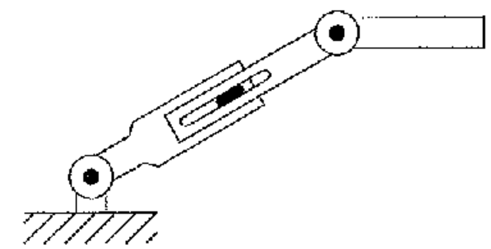

Figure 3.26: Three-link planar arm with prismatic joint of Problem 3-5.

3-6 Consider the three-link articulated robot of Figure 3.27. Derive the forward kinematic equations using the DH convention.

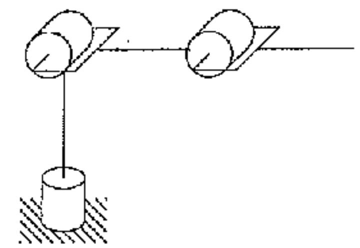

Figure 3.27: Three-link articulated robot.

3-7 Consider the three-link Cartesian manipulator of Figure 3.28. Derive the forward kinematic equations using the DH convention.

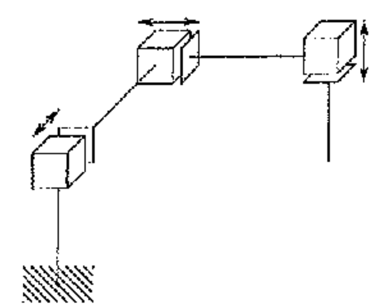

Figure 3.28: Three-link Cartesian robot.

3-8 Attach a spherical wrist to the three-link articulated manipulator of Problem 3-6 as shown in Figure 3.29. Derive the forward kinematic equations for this manipulator.

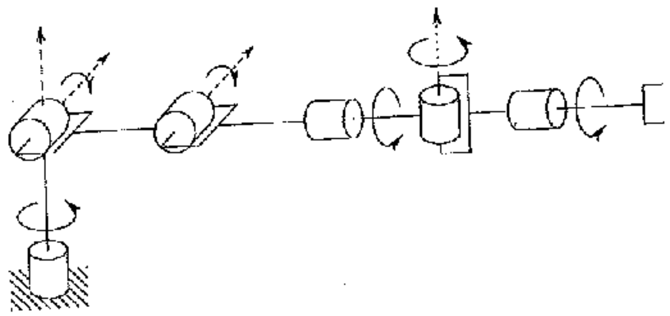

Figure 3.29: Elbow manipulator with spherical wrist.

3-9 Attach a spherical wrist to the three-link Cartesian manipulator of Problem 3-7 as shown in Figure 3.30. Derive the forward kinematic equations for this manipulator.

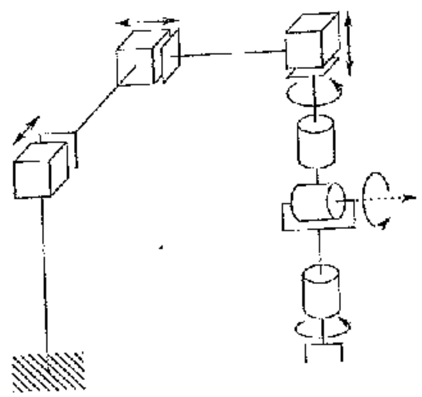

Figure 3.30: Cartesian manipulator with spherical wrist.