# ENPM 808M: Homework #3

Due on Wednesday, October 14, 2015

*Dr. William Levine 4:00 PM*

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# Contents

# Problem 1

By definition, skew matrices have the following property:

$$S + S^T = 0 \tag{1}$$

We have

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots \tag{2}$$

$$\implies e^S = I + S + \frac{1}{2}S^2 + \frac{1}{3!}S^3 + \dots \tag{3}$$

$$\implies \left(e^S\right)^T = \left(I + S + \frac{1}{2}S^2 + \frac{1}{3!}S^3 + \dots\right)^T \tag{4}$$

$$\implies \left(e^S\right)^T = e^{S^T} \tag{5}$$

Now, given $S \in so(3)$, we get from above

$$(e^S)^T = e^{S^T} \tag{6}$$

Thus, we have

$$e^S \times e^{S^T} = e^{S+S^T} = e^0 = I \tag{7}$$

Now, let $e^S = R$ which gives us

$$RR^T = I \tag{8}$$

which is possible only when sum of the squares of each row $= 1$, which is a property of $R \in SO(3)$. This however gives us two possible determinant values for $R$, namely $+1$ or $-1$.

We also have

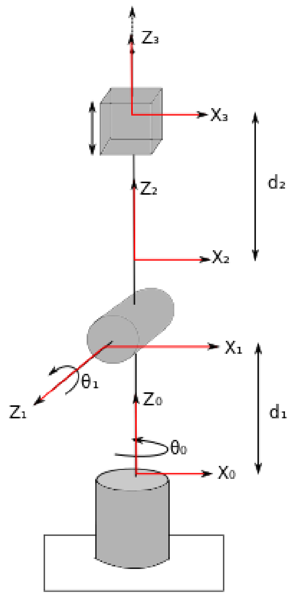
$$det(e^S) = e^{Tr(S)} = e^0 = 1 \tag{9}$$

Thus, both conditions for $R \in SO(3)$ are met, namely

1. $RR^T = 1$
2. $det(R) = 1$

Thus, $R = e^S \in SO(3)$.

## Problem 2




Choice of Axes - DH Convention

From given figure, we form DH Parameters as:

| Link ($i$) | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 90 | $d_1$ | $\boldsymbol{\theta_1}$ |
| 2 | 0 | 90 | 0 | $\boldsymbol{\theta_2}$ |
| 3 | 0 | 0 | $\mathbf{d_2}$ | 0 |

where **bold** indicates a variable parameter value.

We now have the following transformation matrices:

$$T_1 = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

$$T_2 = \begin{bmatrix} c_2 & 0 & -s_2 & 0 \\ s_2 & 0 & c_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

Taking products, we get:

$$T_1T_2 = \begin{bmatrix} c_1c_2 & -s_1 & -s_2c_1 & 0 \\ s_1c_2 & c_1 & -s_2s_1 & 0 \\ s_2 & 0 & c_2 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$T_1T_2T_3 = \begin{bmatrix} c_1c_2 & -s_1 & -s_2c_1 & -s_2c_1d_2 \\ s_1c_2 & c_1 & -s_2s_1 & -s_2s_1d_2 \\ s_2 & 0 & c_2 & c_2d_2+d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

From above matrices, we get:

$$O_0 = [0 \quad 0 \quad 0]^T \tag{15}$$

$$O_1 = [0 \quad 0 \quad d_1]^T \tag{16}$$

$$O_2 = [0 \quad 0 \quad d_1]^T \tag{17}$$

$$O_3 = [-s_2c_1d_2 \quad -s_2s_1d_2 \quad c_2d_2+d_1]^T \tag{18}$$

Also, we have:

$$Z_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \tag{19}$$

$$Z_1 = \begin{bmatrix} s_1 & -c_1 & 0 \end{bmatrix}^T \tag{20}$$

$$Z_2 = \begin{bmatrix} -s_2c_1 & -s_2s_1 & c_2 \end{bmatrix}^T \tag{21}$$

$$Z_3 = \begin{bmatrix} -s_2c_1 & -s_2s_1 & c_2 \end{bmatrix}^T \tag{22}$$

From the figure, we have revolute joints at joints 1 and 2 and joint 3 as prismatic joint. This gives us:

$$J_1 = \begin{bmatrix} Z_0 \times (O_3 - O_0) \\ Z_0 \end{bmatrix} \tag{23}$$

$$J_2 = \begin{bmatrix} Z_1 \times (O_3 - O_1) \\ Z_1 \end{bmatrix} \tag{24}$$

$$J_3 = \begin{bmatrix} Z_2 \\ 0 \end{bmatrix} \tag{25}$$

$$J = [J_1 \quad J_2 \quad J_3] \tag{26}$$

We need to calculate $J_{11}$, which is the upper half of the matrix $J$. Thus,

$$J_{11} = \begin{bmatrix} \underline{Z_0 \times (O_3 - O_0)} & \underline{\underline{Z_1 \times (O_3 - O_1)}} & Z_2 \end{bmatrix} \tag{27}$$

For $\underline{Z_0 \times (O_3 - O_0)}$, we have

$$(O_3 - O_0) = \begin{bmatrix} -s_2c_1d_2 \\ -s_2s_1d_2 \\ c_2d_2+d_1 \end{bmatrix} \tag{28}$$

$$Z_0 \times (O_3 - O_0) = S(Z_0)(O_3 - O_0) \tag{29}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} -s_2c_1d_2 \\ -s_2s_1d_2 \\ c_2d_2+d_1 \end{bmatrix} \tag{30}$$

$$\implies \begin{bmatrix} s_2s_1d_2 \\ -s2c_1d_2 \\ 0 \end{bmatrix} \tag{31}$$

For $\underline{\underline{Z_1 \times (O_3 - O_1)}}$, we have

$$(O_3 - O_1) = \begin{bmatrix} -s_2 c_1 d_2 \\ -s_2 s_1 d_2 \\ c_2 d_2 \end{bmatrix} \tag{32}$$

$$Z_1 \times (O_3 - O_1) = S(Z_1)(O_3 - O_1) \tag{33}$$

$$= \begin{bmatrix} 0 & 0 & -c_1 \\ 0 & 0 & -s_1 \\ c_1 & s_1 & 0 \end{bmatrix} \times \begin{bmatrix} -s_2 c_1 d_2 \\ -s_2 s_1 d_2 \\ c_2 d_2 \end{bmatrix} \tag{34}$$

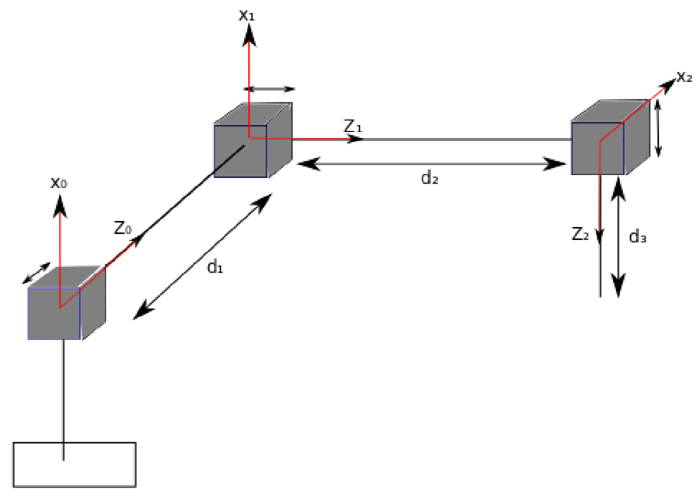
$$\implies \begin{bmatrix} -c_1 c_2 d_2 \\ -s1 c_2 d_2 \\ -s_2 d_2 \end{bmatrix} \tag{35}$$

Combining results above, we get

$$J_{11} = \begin{bmatrix} s_2 s_1 d_2 & -c_1 c_2 d_2 & -s_2 c_1 \\ -s_2 c_1 d_2 & -s_1 c_2 d_2 & -s_2 s_1 \\ 0 & -s_2 d_2 & c_2 \end{bmatrix} \tag{36}$$

Problem 2 continued on next page... 

## Problem 3




Choice of Axes - DH Convention

From given figure, we form DH Parameters as:

| Link ($i$) | $a_i$ | $\alpha_i$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | $-90$ | $\mathbf{d_1}$ | 0 |
| 2 | 0 | $-90$ | $\mathbf{d_2}$ | 90 |
| 3 | 0 | 0 | $\mathbf{d_3}$ | 0 |

where **bold** indicates a variable parameter value.
We now have the following transformation matrices:

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{37}$$

$$T_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{38}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{39}$$

Taking products, we get:

$$T_1T_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & d_2 \\ -1 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{40}$$

$$T_1T_2T_3 = \begin{bmatrix} 0 & 0 & -1 & d_3 \\ 0 & -1 & 0 & d_2 \\ -1 & 0 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{41}$$

For a prismatic joint, we have

$$J_i = \begin{bmatrix} Z_{i-1} \\ 0 \end{bmatrix} \tag{42}$$

$$J = [J_1 \quad J_2 \quad J_3 \quad \dots] \tag{43}$$

We thus have:

$$Z_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \tag{44}$$

$$Z_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \tag{45}$$

$$Z_2 = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^T \tag{46}$$

$$\therefore J = \begin{bmatrix} Z_0 & Z_1 & Z_2 \\ 0 & 0 & 0 \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{48}$$

We can see that $J$ is a $6 \times 3$ matrix. So, the maximum rank possible is 3.
Since, the actual rank of $J$ is the maximum possible rank, we can say that <u>there are no singular configurations.</u>

# Problem 4

From Equation 4.87 in the textbook, we have the following:

$$B(\alpha) = \begin{bmatrix} c_\phi s_\theta & -s_\phi & 0 \\ s_\phi s_\theta & c_\phi & 0 \\ c_\theta & 0 & 1 \end{bmatrix} \tag{49}$$

Taking the determinant of $B(\alpha)$ we get

$$det(B(\alpha)) = (c_\phi s_\theta)(c_\phi)(1) + (-s_\phi)(0)(c_\theta) + (0)(s_\phi s_\theta)(0) - (0)(c_\phi)(c_\theta) - (-s_\phi)(s_\phi s_\theta)(1) - (c_\phi s_\theta)(0)(0) \tag{50}$$

$$\implies det(B(\alpha)) = (c_\phi s_\theta)(c_\phi) - (-s_\phi)(s_\phi s_\theta) \tag{51}$$

$$\implies det(B(\alpha)) = c_\phi^2 s_\theta + s_\phi^2 s_\theta \tag{52}$$

$$\implies det(B(\alpha)) = s_\theta \tag{53}$$

Now, given that $s_\theta \neq 0$, $det(B(\alpha)) = k$ where $k$ is some non-zero value.
Thus $B(\alpha)$ is invertible if $s_\theta \neq 0$