Divergence and curl of a vector field —

Divergence of vector field — Divergence of a vector field at a given point in a vector field is a scalar and is defined as the amount of flux diverging from a unit volume element, per second around that point.

If $\vec{F}$ represents a vector field i.e $\vec{F}$ be a continuously differentiable vector point function, the function

$$\hat{a}_x \cdot \frac{\partial \vec{F}}{\partial x} + \hat{a}_y \cdot \frac{\partial \vec{F}}{\partial y} + \hat{a}_z \cdot \frac{\partial \vec{F}}{\partial z}$$ is a scalar and it is called divergence of $\vec{F}$.

Now $$\operatorname{div} \vec{F} = \left(\hat{a}_x \cdot \frac{\partial \vec{F}}{\partial x} + \hat{a}_y \cdot \frac{\partial \vec{F}}{\partial y} + \hat{a}_z \cdot \frac{\partial \vec{F}}{\partial z}\right) = \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right) \cdot \vec{F}$$

$$\boxed{\operatorname{div} \vec{F} = \vec{\nabla} \cdot F}$$

Therefore the divergence of a vector point function $\vec{F}$ is a scalar product of the Del operator with $\vec{F}$.

Curl of a vector field — The curl of a vector at any point in a vector field is a vector whose magnitude is equal to the line integral per unit area along the boundary of an infinitesimal area drawn around that point and whose direction is along the normal to this area.

The function $$\hat{a}_x \times \frac{\partial \vec{F}}{\partial x} + \hat{a}_y \times \frac{\partial \vec{F}}{\partial y} + \hat{a}_z \times \frac{\partial \vec{F}}{\partial z}$$ is a vector and it is called curl of $\vec{F}$.

Now $$\operatorname{curl} \vec{F} = \left(\hat{a}_x \times \frac{\partial \vec{F}}{\partial x} + \hat{a}_y \times \frac{\partial \vec{F}}{\partial y} + \hat{a}_z \times \frac{\partial \vec{F}}{\partial z}\right) = \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right) \times \vec{F}$$

$$\boxed{\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}}$$

Therefore the curl of a vector point function $\vec{F}$ is the vector product of the Del operator with $\vec{F}$.

If $\vec{F} = \hat{a}_x F_x + \hat{a}_y F_y + \hat{a}_z F_z$ then $\operatorname{div} \vec{F} = \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right) \cdot \left(\hat{a}_x F_x + \hat{a}_y F_y + \hat{a}_z F_z\right)$

or $$\operatorname{div} \vec{F} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)$$

and $$\operatorname{curl} \vec{F} = \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right) \times \left(\hat{a}_x F_x + \hat{a}_y F_y + \hat{a}_z F_z\right) = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

Introduction — Electricity and magnetism ... in an isolated physical phenomena i.e. electricity nothing to do with magnetism and vice-versa. were considered to be separate ①

In 1820 Scientist Hans Oersted observed that if electric current is passed through a wire in the vicinity (around) of the wire a ~~magd~~ magnetic field is produced. Later Faraday experimentally proved that a time varying magnetic field produces electric field.

Electromagnetic field — In electromagnetics time varying electric or time varying magnetic field is considered. If we vary electric field then magnetic field is produced and if we vary magnetic field then electric field is produced. These time varying fields are mutualy (पारस्परिक) dependent on each other and can-not exist without the other. (आत्मनिर्भर) So they called electromagnetic fields.

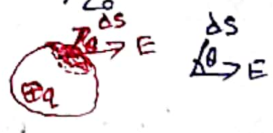
Maxwell's Equations — A complete set of relations giving the connection between charges at rest (electrostatics), charges in motion (current electricity), varying electric fields and magnetic fields (electromagnetism) were derived theoratically and summarized in four (4) equations known as Maxwell's Equations.

Maxwell brought together and extended four basic laws in electromagnetism such as —

(I) Gauss's law in electrostatics (II) Gauss's law in magnetism
(III) Faraday's law (IV) Ampere's law.

(I) Gauss's law of Electrostatics — Gauss's law states that the total electric flux ϕ_E enclosed by a closed surface is $1/\varepsilon_0$ times the total charge q enclosed by that surface.

$$\phi_E = \frac{1}{\varepsilon_0} q \quad ①$$



where $\phi_E = \oint_S E \cdot ds$ ②

($\oint_S \Rightarrow$ Surface integral of closed surface)

from ① and ②

$$\oint_S E \cdot ds = \frac{q}{\varepsilon_0}$$

[If closed surface does not enclose any charge then $\phi_E = 0$]

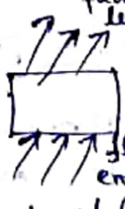
where ε_0 is the permitivity of free space. [permitivity of free space (~~...~~ a vacuum) is a physical constant and equal to 8.85×10^{-12} farad per meter]

(II) Gauss's law in magnetostatics — The magnetic flux entering a closed surface is always equal to the magnetic flux leaving the surface of the same volume because magnetic line of forces are continuous by nature. Therefore the net flux through a closed surface must be zero.

i.e.

$$\phi_B = \oint_S B \cdot ds = 0$$

flux leaving



flux entering

(III) Faraday's law of Electromagnetic induction — There are two laws such as bellow

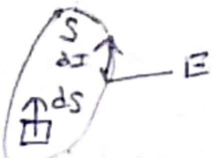
(A) Whenever the magnetic flux linked with a circuit is changed an electro motive force (e.m.f.) is induced in the circuit.

~~If ϕ_B be the magnetic flux~~

(B) The magnitude of induced e.m.f. is directly proportional to the negative rate of vibration of magnetic flux linked with the circuit.

If ϕ_B be the magnetic flux linked with circuit at any instant and e be the induced e.m.f. then

$$e = -\left(\frac{d\phi_B}{dt}\right) \quad (I)$$



The line integral of the electric field gives the induced e.m.f. in the closed circuit i.e.

$$\oint E \cdot dl = -\frac{d\phi_B}{dl} \quad (II)$$

thus

$$e = -\oint E \cdot dl \quad (III)$$

The magnetic flux through a small area ds will be B·ds. Thus the flux through the entire circuit is

$$\phi_B = \int_S B \cdot ds \quad (IV)$$

now from eqn (II) and (IV) we have

$$\oint E \cdot dl = -\frac{d}{dt}\int_S B \cdot ds \Rightarrow \oint E \cdot dl = -\frac{d}{dt}\int_S B \cdot ds \quad (V)$$

This is the integral form of Faraday's law

Therefore the line integral of the electric field around any closed circuit is equal to the negative rate of change of magnetic flux through the circuit.

By Stokes theorem we know that

$$\oint E \cdot dl = \int_S (\vec{\nabla} \times E) \cdot ds \quad (VI)$$

now from eqn (V) and (VI)

Since $\int_S \frac{\partial B}{\partial t} = \frac{d}{dt}\int_S B$

$$\int_S (\vec{\nabla} \times E) \cdot ds = -\int_S \frac{\partial B}{\partial t} \cdot ds$$

therefore

$$\boxed{(\vec{\nabla} \times E) = -\left(\frac{\partial B}{\partial t}\right)}$$

This is the differential form of Faraday's law.

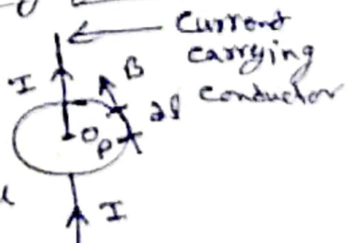
(IV) Ampere's circuital law (Amphere's law)

According to Ampere's law, the line integral of magnetic field B along a closed curve is equal to μ_0 times the net current through the area bounded by the curve.

$$\boxed{\oint B \cdot dl = \mu_0 I}$$

where μ_0 is the permeability of free space or air.

or $\boxed{\oint H \cdot dl = I}$



Vector product - A vector is represented by a straight line bearing an arrow. The length of line measures the magnitude of vector on a convinient scale and arrow gives direction, and it is represented as $\vec{A}$ and magnitude is written as $|A|$.

A vector $\vec{A}$ with its component A_x, A_y, A_z along X, Y, Z direction is expressed as $\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$ and the magnitude of $\vec{A}$ is $|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

(I) $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ (II) $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

scalar (or dot) product - (I) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ (II) $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

Displacement current — Maxwell postulated or proved that, it is not only the current in the conductor that produce magnetic field, but a changing electric field also produce a magnetic field. It means that a changing electric field is equivalent to a current which flows as the electric field is changing and produces the same magnetic effect as an ordinary conduction current. This is known as displacement current.

Let q be the charge on a capacitor plate at any particular time. Then the conduction current (i_c) is defined as time rate of flow of charge i.e.

$$i_c = \frac{dq}{dt} \quad (1)$$

Let D be the electric displacement with di-electric region AB. then

$$D = q/A \quad (ii) \quad \text{(where A is the cross-section area of each plate)}$$

Now from eqn (i) and (ii) we have

$$i_c = \frac{d}{dt}(DA) \Rightarrow$$

$$i_c = A\left(\frac{\partial D}{\partial t}\right) \quad (iii)$$

Maxwell suggested that the term $A\left(\frac{\partial D}{\partial t}\right)$ should be considered as the current inside the di-electric and named this as Displacement current denoted by i_d.

Thus

$$i_d = A\left(\frac{dD}{dt}\right)$$

Thus it is clear that the displacement current is finite when applied voltage is changing and for constant applied voltage $i_d = 0$

Now the displacement current density J_d is given by

$$J_d = \frac{i_d}{A} = \frac{dD}{dt}$$

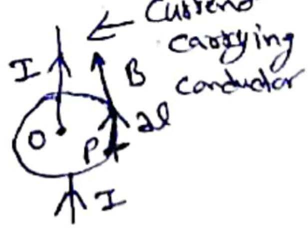
The vector D may vary with space, hence

$$\boxed{J_d = \frac{\partial D}{\partial t}}$$

Ampere's circuital law (Ampere's law) — According to Ampere's law, the line integral of magnetic field B along a closed curve is equal to μ_0 times the net current through the area bounded by the curve.

i.e

$$\oint B \cdot dl = \mu_0 I \quad \text{or} \quad \oint H \cdot dl = I$$



where μ_0 is the permeability of free space or air. and H is the magnetic field intensity.

Equation of Continuity — The net amount of charge in an isolated system remains constant. (3)

As we know that rate of change of charge constitute the current, hence

$$I = -\frac{dq}{dt} \quad (I)$$ {-ve sign indicates that the charge contained in a certain volume decreases with time}

The charge q in terms of charge density ρ can be written as

$$q = \int_V \rho \cdot dv \quad (II)$$ then $$I = -\frac{d}{dt}\int_V \rho \cdot dv \Rightarrow I = -\int_V \frac{\partial \rho}{\partial t} \cdot dv \quad (III)$$

The current I in terms of current density $\vec{J}$ can be written as

$$I = \int_S \vec{J} \cdot d\vec{S} \quad (IV)$$

Therefore from eqn (III) and (IV) we have

$$\int_S \vec{J} \cdot d\vec{S} = -\int_V \frac{\partial \rho}{\partial t} \cdot dV \quad (V)$$

From Gauss divergence theorem, we know that

$$\boxed{\int_S \vec{J} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{J}\, dv}$$

then we have

$$\int_V \vec{\nabla} \cdot \vec{J}\, dV = -\int \frac{\partial \rho}{\partial t} \cdot dV$$

or $$\int_V \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}\right) \cdot dV = 0 \Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

This is known as equation of continuity.

NOTE ⇒ In case of stationary current, charge density at any point within the region should remain constant.

i.e $\frac{\partial \rho}{\partial t} = 0$. Then we have $\boxed{\vec{\nabla} \cdot \vec{J} = 0}$

Which indicates that there is not net outward flux of current density $\vec{J}$.