Lab 2 Mine Crafting

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I. Introduction

Accurately measuring the depth of deep mine shafts is critical for safety and operational efficiency. This report assesses the feasibility of determining the vertical depth of a 4-kilometer mine shaft at Earth's equator by analyzing the fall time of a dropped 1-kilogram test mass. Our analysis incrementally incorporates realistic conditions—constant gravity, height-dependent gravity, aerodynamic drag, and the Coriolis effect—to evaluate their impact on the accuracy and practicality of this measurement method. We also explore homogeneous vs inhomogeneous densities as well as lunar mine shaft possibilities by testing fall times and finding relation between crossing times and orbital periods and densities.

II. Calculation of Fall Times

First, we explored the fall times of our test particle in a drag free simulation. We used the 2nd order differential equation given below to calculate the time the test mass reaches the bottom of our mine shaft.

$$\frac{d^2y}{dt^2} = -g + \alpha \left| \frac{dy}{dt} \right|^{\gamma}$$

By reducing the above into a system of ordinary differential equations which are easier to understand and solve using numerical computational methods such as scipy.integrates.solve_ivp.

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -g + \alpha \mid v \mid^{\gamma}$$

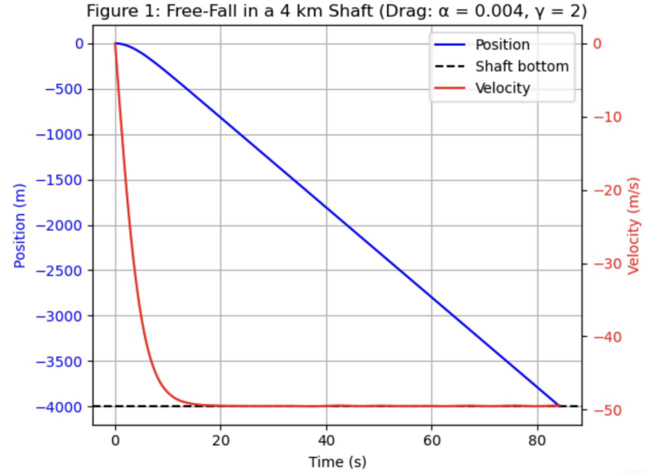
The solve_ivp method uses the popular Runge-Kutta method in the background and is very useful in producing correct numerical approximations. Initially, without the use of such complex ODE solvers, the fall time of the test mass under ideal conditions—assuming constant gravity ($g = 9.81$ m/s²) and no drag—was calculated to be 28.6 seconds algebraically. Our numerical approximation was also very close to the theoretical time calculation, only differing by 1.4E-14 of a second. This shows and proves the usefulness of numerical solvers methods such as solve_ivp.

In practice, however, we cannot ignore drag and a constant gravitational force. Therefore, we next introduced the height-dependent gravity.

$$g(r) = g_o \left(\frac{r}{R_{\oplus}} \right)$$

The results for fall time did not yield any significant difference, maintaining a fall time of approximately 28.6 seconds due to the shallow depth relative to Earth's radius. However, there was a slight difference present in the later digits of the decimal number which accounts for the decrease in

Figure 1: Free-Fall in a 4 km Shaft (Drag: α = 0.004, γ = 2)



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acceleration as the particle approaches the center of the Earth.
However, incorporating drag (with drag coefficient α calibrated to 0.004 for a terminal velocity of 50 m/s and gamma set to 2) significantly increased the fall time to approximately 84.2 seconds. This highlights the substantial effect of aerodynamic resistance, nearly quadrupling the fall duration.

III. Feasibility Considering Coriolis Effects

The Coriolis force, arising from Earth's rotation, introduces a substantial transverse displacement. Our new force equations all have a new term which accounts for the Coriolis force. Force of Coriolis on y is 2m*omega*velocity in y direction and similarly, the force of Coriolis on x is 2m*omega*velocity in x direction (please refer to the constants charts for values on omega and m).

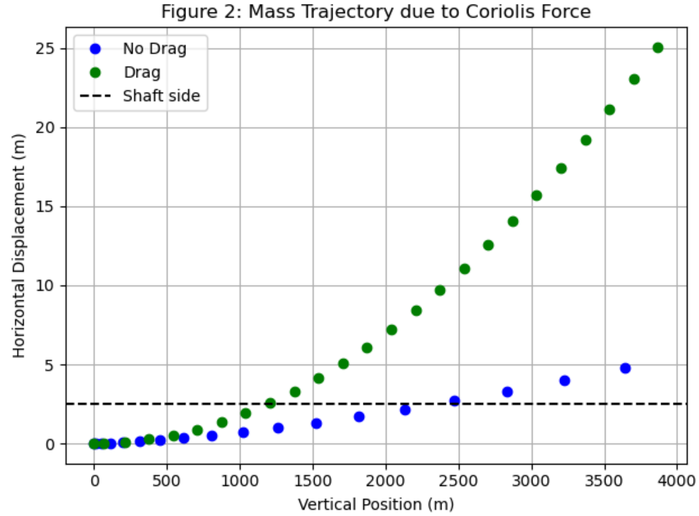
$$\frac{dy_y}{dt} = v_y \qquad \frac{dy_x}{dt} = v_x$$

$$\frac{dv_y}{dt} = -g + \alpha \mid v \mid^{\gamma} + \frac{F_{c,y}}{m} \qquad \frac{dv_x}{dt} = \frac{F_{c,x}}{m}$$

We use the previously used solve_ivp method to calculate the trajectory of the particle. Additionally, we keep track of the transverse position (side to side movement in the shaft) of the particle to check if it hit a shaft wall. To do so, we use an event marker to exit when the transverse position exceeds 2.5 meters (starting from the center of a 5 meter diameter shaft).

Without drag, transverse displacement reached 5.5 meters, exceeding the 2.5 meters (shaft radius), indicating the mass would collide with the shaft wall at approximately 21.9 seconds and a depth of about 2,353 meters—well before reaching the bottom. With drag included, the transverse displacement dramatically increased to approximately 26.6 meters, intensifying the impracticality of this measurement method. Thus, due to significant lateral deviations caused by the Coriolis force, this method is deemed unfeasible for accurately measuring shaft depth.

Figure 2: Mass Trajectory due to Coriolis Force



IV. Trans-planetary and Trans-lunar Tunnel Analysis

Exploring theoretical scenarios, the fall time for a test mass traversing an idealized shaft through Earth depends critically on internal density distributions.

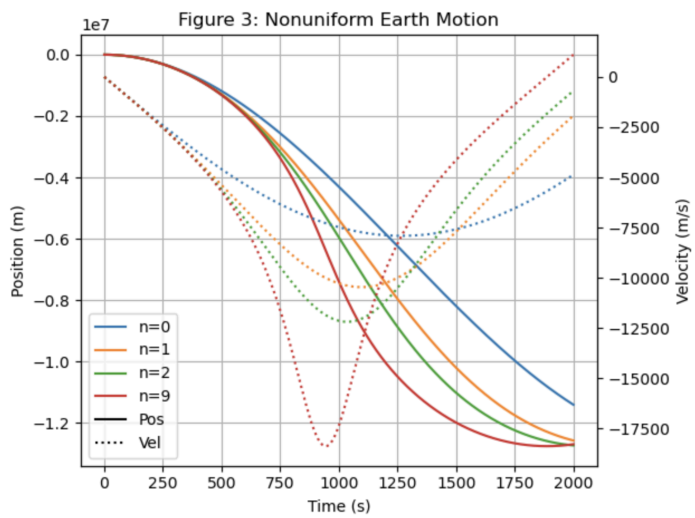
Exploring only the homogeneous Earth with constant density we solve the free fall ODE and calculate the full Earth transit to be 2419.0 seconds. We then compare it with the orbital period (the time it takes for an object going around the Earth) which equals (2 * pi * Radius of Earth) / sqrt(G * Mass of Earth / Radius of Earth) or 5069.1 seconds. This is approximately 2x the full Earth transit or a particle will travel back and forth one time in the orbital period.

However, it is unrealistic to assume constant mass density. To account for this we follow the following equations with the final goal of calculating the gravitational acceleration. We first use the first equation to calculate our normalization constant (rho_n) for each n as the density changes. We do so by using the quad function of scipy (another very useful method for solving integrals). We use our calculated rho_n which is the normalization constant to calculate rho as a function of r (r changes as the particle travels). Then we calculate the force profile for each n which sets rho_n as well. This is calculated by F(r) = G*M(r)/r^2.

$$M = \int_V \rho(r) dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{R_\oplus} \rho(r) r^2 \sin\phi \, dr \, d\phi \, d\theta = 4\pi \int_0^{R_\oplus} \rho(r) r^2 \, dr \qquad \rho(r) = \rho_n \left(1 - \frac{r^2}{R_\oplus^2}\right)^n$$

By using the force profiles, we solve the free fall equations described in the first part for n = 0,1,2,9. Using solve_ivp and the event marker we obtain the trajectory and crossing time for each n value.

For a homogeneous Earth (n=0), the crossing time was approximately 2032.2 seconds, whereas a highly non-homogeneous Earth (n=9) reduced crossing time to about 1726.7 seconds. We also see from the graph that the red marker (n = 9) is shifted more to left indicating a faster turnaround time whereas the blue (n = 0) is more to the right. Increased central density significantly enhances gravitational attraction, reducing travel times and increasing terminal velocities.



Similarly, we calculated the travel time for a pole to pole moon shaft for constant density (n=0) to be 3042.9 seconds. We also calculated the density of the Moon and Earth by density = M / ((4/3) * pi * R**3).
Earth's density was calculated to be 3,341.8 kg/m^3 and the Moon's density was calculated to be 5494.9 kg/m^3 . The inverse square of their density ratio was calculated to be 0.7798, confirming the derived relationship between density and crossing time. We then found a relation between density and crossing time/orbital period by exploring the T = SQRT(3*pi / G * Rho) equation. We simplified and proved that the ratio between times is equal to the square root of the ratio between the densities (both of the ratios ~ 0.78).

V. Discussion and Future Work

Our findings clearly indicate that the method of measuring mine shaft depth via free fall timing, while theoretically viable, is practically limited due to aerodynamic drag and significant Coriolis-induced lateral displacement. Additionally, realistic planetary density distributions markedly influence fall dynamics, crucial for deeper hypothetical scenarios.

Future studies should address the oversimplifications made, such as assuming spherical Earth geometry, constant drag coefficients, and uniform air density. Investigating non-spherical geometries, varying density layers within Earth, and altitude-dependent drag coefficients would refine these models, enhancing their applicability and accuracy for practical depth measurements.