Modal Logic

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1 Basic Concepts

1.1 Relational Structures

Definition 1.1. A relational structure $\mathfrak{F}$ is a tuple, where the first component is a non-empty set W called *universe*, and remaining components are relations on W. We assume there is at least one relation on W.

In the following 2 definitions, we assume W be a non-empty set and R a binary relation on W.

Definition 1.2. $R^+ := \bigcap\{R' | R'$ is a transitive binary relation on W and $R \subseteq R'\}$, is called the *transitive closure* of R.

Definition 1.3. $R^* := \bigcap\{R' | R'$ is a reflexive transitive binary relation on W and $R \subseteq R'\}$, is called the *reflexive transitive closure* of R.

Note that transitive closure of a binary relation has nice *finite steps* property, see Exercise 1.1.3.

Definition 1.4. A tree $\mathfrak{T}$ is a structural structure (T, S) where:
(i)T, the set of nodes, contains a unique $r \in T$ (root), such that $\forall t \in T S^* rt$.
(ii)For every $t \neq r$, there is a unique $t' \in T$, such that $St't$
(iii)$\forall t \neg S^+ tt$, so S is *acyclic*.

Question 1.1. Why we define tree like that?

1.2 Modal Languages

Definition 1.5. Basic modal language:

-A set of proposition letters(or proposition symbols or propositional variables) Φ, whose elements are usually denoted p, q, r, and so on.

-A unary modal operator $\Diamond$.

Then the well-formed *formulas* ϕ of the basic modal language are given by the rule:

$\phi ::= p|\bot|\neg\phi|\psi \vee \phi|\Diamond\phi$,

where p ranges over elements pf Φ.

There is also a dual operator $\Box$ which is defined by $\Box\phi := \neg\Diamond\neg\phi$.

Moreover, we can define conjunction, implication, bi-implication, and the constant true as usual:

$\phi \wedge \psi := \neg(\neg\phi \vee \neg\psi)$,

$\phi \to \psi := \neg\phi \vee \psi$,

$\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$, and

$\top := \neg\bot$.

The following 3 definitions generalize the concept of basic modal language.

Definition 1.6. A *modal similarity type* is a pair $\tau = (O, \rho)$ where O is a non-empty set, and ρ is a function $O \to \mathbb{N}$. The elements of O are called *modal operators*; we use $\triangle$, $\triangle_0$, $\triangle_1$,..., to denote elements of O. The function ρ assigns to each operator $\triangle \in O$ a finite *arity* indicating the number of arguments $\triangle$ can be applied to.

So we often refer to *unary* triangles as *diamonds*, and denote them by $\Diamond_a$ or $\langle a \rangle$, where a is taken from some index set.

Definition 1.7. A *modal language* $ML(\tau, \Phi)$, with modal similarity type $\tau = (O, \rho)$ and a set of proposition letters Φ. The well-formed formulas are given by the rule:

$\phi := p|\bot|\neg\phi|\phi_1 \vee \phi_2|\triangle(\phi_1, ..., \phi_{\rho(\triangle)})$,

where p ranges over elements of Φ.

Definition 1.8. Dual operators for non-nullary triangles. For each $\triangle \in O$ the dual $\triangledown$ is defined as $\triangledown(\phi_0, ..., \phi_n) = \neg\triangle(\neg\phi_0, ..., \neg\phi_n)$. The dual of a triangle of arity at least 2 is called a *nabla*. A *box*(unary triangle-down) is written $\Box_a$ or $[a]$.

Definition 1.9. A *substitution* is a map $\sigma : \Phi \to Form(\tau.\Phi)$(formulas).

Then a substitution σ induces a map $(\cdot)^\sigma : Form(\tau, \Phi) \to Form(\tau, \Phi)$, which can be recursively defined as follows:

$\bot^\sigma = \bot$,

$p^\sigma = \sigma(p)$,

$(\neg\psi)^\sigma = \neg\psi^\sigma$,

$(\psi \vee \theta)^\sigma = \psi^\sigma \vee \theta^\sigma$,

$(\triangle(\psi_1, ..., \psi_n))^\sigma = \triangle(\psi_1^\sigma, ..., \psi_n^\sigma)$