Exercises for Section 1.1

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Exercise 1.1.1

(a) Rss and Rss implies $s \sim s$(reflexivity).

$s \sim t$ iff Rst and Rts, iff Rts and Rst, iff $t \sim s$(symmetry).

Assume $r \sim s$ and $s \sim t$, then Rrs and Rsr,Rst and Rts. Since R is transitive, we have Rrt and Rtr, so $r \sim t$(transitivity).

(b) Assume $s, s' \in [s]$ and $t, t' \in [t]$, and Rst. To show the relation is well-defined, it suffices to show $Rs't'$.

$s \sim s'$ implies $Rs's$, combining with Rst, we have $Rs't$. Also, $t \sim t'$ implies Rtt', so $Rs't'$.

(c) Rss implies $[s] \leq [s]$(reflexivity).

Assume $[s] \leq [t]$ and $[t] \leq [s]$, then Rst and Rts, so $[s] = [t]$(anti-symmetry).

Assume $[r] \leq [s]$ and $[s] \leq [t]$, then Rrs and Rst, so Rrt, then $[r] \leq [t]$(transitivity).

Exercise 1.1.2

$\Rightarrow$: Assume R is *well-founded*, i.e. there are no infinite paths...$Rs_2Rs_1Rs_0$. Then for an element $x \in W$, Rxx implies a infinite path ...$RxRxRx$, which is a contradiction. So $\forall x \neg Rxx$.

$\Leftarrow$: Assume R is irreflexive. If there is a infinite path ...$Rs_2Rs_1Rs_0$. Since W is finite, there is an element $x \in W$ that occurs in the path at least once. So there is a finite sub-path $xRs_i...s_jRx$, combining with transitivity of R, we have Rxx, which contradicts irreflexivity. Thus R must be well-founded.

Exercise 1.1.3

R^+, R^* and $R^{\mathbf{r}}$ is well-defined, since 'intersection of two transitive/reflexive binary relations is still a transitive/reflexive binary relations', which is easy to verify.

The equivalence of the two definitions of reflexive closure is trivial: clearly $R \cup \{(u, u) | u \in W\}$ is a subset of every reflexive binary relation containing R, so $R \cup \{(u, u) | u \in W\} \subseteq R^{\mathbf{r}}$. On the other hand, $R^{\mathbf{r}} \subseteq R \cup \{(u, u) | u \in W\}$, since $R \cup \{(u, u) | u \in W\}$ itself is a reflexive binary relation.

Finally we show R^+uv iff there is a sequence of elements $u = w_0, w_1, ..., w_n = v$ from W such that for each $i < n$ we have Rw_iw_{i+1}. We prove this by giving a sequence-based definition of transitive closure.

Define $R^{\mathbf{t}} = \{(u, v) \in W \times W |$ There is a sequence of elements $u = w_0, w_1, ..., w_n = v$ from W such that for each $i < n$ we have Rw_iw_{i+1} $\}$. We show that $R^{\mathbf{t}} = R^+$.

$R^{\mathbf{t}}$ is transitive: assume $R^{\mathbf{t}}xy$ and $R^{\mathbf{t}}yz$, then the corresponding sequences can be joined together to form a new sequence, implying $R^{\mathbf{t}}xz$.

$R \subseteq R^{\mathbf{t}}$: For $(u, v) \in R$, $u = w_0, w_1 = v$ is a sequence with Ruv, so $(u, v) \in R^{\mathbf{t}}$.

By definition of R^+, $R^+ \subseteq R^{\mathbf{t}}$.

On the other hand, given $(u, v) \in R^{\mathbf{t}}$, and corresponding sequence $u = w_0, w_1, ..., w_n = v$. For arbitrary transitive binary relation R' with $R \subseteq R'$, $Rw_0w_1, ..., Rw_{n-1}w_n$, implies $R'w_0w_1, ..., R'w_{n-1}w_n$, implies $R'w_0w_n$, so $R^{\mathbf{t}} \subseteq R'$. Therefore, $R^{\mathbf{t}} \subseteq R^+$.

As a conclusion, $R^+ = R^{\mathbf{t}}$.

In the end, we give a sequence-based definition of reflexive transitive closure.

Define $R^{\mathbf{rt}} = \{(u, v) \in W \times W |$ There is a sequence of elements $u = w_0, w_1, ..., w_n = v$ from W such that for each $i < n$ we have Rw_iw_{i+1} $\} \cup \{(u, u) | u \in W\}$. We show that $R^* = R^{\mathbf{rt}}$.

$R^{\mathbf{rt}}$ is reflexive: Trivial.

$R^{\mathbf{t}}$ is transitive: assume $R^{\mathbf{rt}}xy$ and $R^{\mathbf{rt}}yz$. Then there are 4 cases: $x = y$ and $y = z$, $x = y$ and $y \neq z$, $x \neq y$ and $y = z$, $x \neq y$ and $y \neq z$. There first 3 cases are trivial, and the last case is similar to *transitive closure* case.

$R \subseteq R^{\mathbf{rt}}$: For $(u, v) \in R$, $u = w_0, w_1 = v$ is a sequence with Ruv, so $(u, v) \in R^{\mathbf{rt}}$.

By definition of R^*, $R^* \subseteq R^{\mathbf{rt}}$.

On the other hand, given $(u, v) \in R^{\mathbf{rt}}$. For arbitrary reflexive transitive binary relation R' with $R \subseteq R'$, if (u, v) has its defining sequence $u = w_0, w_1, ..., w_n = v$, then $Rw_0w_1, ..., Rw_{n-1}w_n$, implies $R'w_0w_1, ..., R'w_{n-1}w_n$, implies $R'w_0w_n$, so $(u, v) \in R'$. If $u = v$, then $(u, v) \in R'$ also. Thus $R^{\mathbf{rt}} \subseteq R'$, for all such R'. Therefore $R^{\mathbf{rt}} \subseteq R^*$.

As a conclusion, $R^* = R^{\mathbf{rt}}$.

Exercise 1.1.4

(a) (T, S^+) is a SPO:

Irreflexivity: Follows the definition of tree.

Transitivity: Follows the definition of transitive closure.

The root $r \in T$: for $t \neq r$, by equivalent descriptions of R^* and R^+ in Exercise 1.1.3, $(r, t) \in S^+$, and this satisfies (i).

Since S^+rt, there is a sequence $r = w_0, w_1, ..., w_n = t$ such that Sw_iw_{i+1} for $i < n$. Clearly $\{w_0, ..., w_n\} \subseteq \{s \in T | S^+st\}$.

On the other hand, for every $s \in T$ with S^+st, there is a sequence $s = u_0, u_1, ..., u_m = t$ such that Su_iu_{i+1} for $i < m$. Then we can prove by induction that $u_{m-i} = w_{n-i}$ for all $0 \leq i \leq m$:

Base step: $i = 0$, $u_m = t = w_n$.

Induction step: Assume for $0 \geq i < m$, we have $u_{m-i} = w_{n-i}$. Then, since $Su_{m-(i+1)u_{m-i}}$, so $w_{n-i} \neq r$, thus $Sw_{n-(i+1)}w_{n-i}$. Since the predecessor is unique, we have $u_{m-(i+1)} = w_{n-(i+1)}$.

So $s = u_0 = w_{n-m}$. Therefore $\{s \in T | S^+st\} \subseteq \{w_0, w_1, ..., w_n\}$.

As a conclusion, $\{s \in T|S^+st\} = \{w_0, w_1, ..., w_n\}$, thus finite and linear ordered. This is (ii).

Now we conclude that (T, S^+) is a transitive tree.

(b)$\Rightarrow$: Assume a SPO $(T, <)$ is a transitive tree.

It's not hard to verify that $S^*_< =< \cup\{(s,s)|s \in T\}$, so r is root of $(T, S_<)$.

The uniqueness of predecessor: Suppose $t_1 S_< t$ and $t_1 S_< t$. Since $\{s \in T|s < t\}$ is linearly-ordered, if $t_1 \neq t_2$, then without loss of generality, assume $t_1 < t_2$, then there cannot be $S_< t_1 t$. So we must have $t_1 = t_2$.

Acyclic is obvious: $<$ itself is acyclic.

Therefore $(T, S_<)$ is a tree.

$\Leftarrow$: Assume $(T, S_<)$ is a tree.

It' not hard to verify that $S^+_< =<$, so by (a), $(T, <)$ is a transitive tree.

(c) We need the assumption of 'unique predecessor', then

Root r: $S^* = S^+ \cup \{(s,s)|s \in T\}$, so r is also root element of (T, S).

Uniqueness of predecessor: Follows assumption above.

$\forall t \neg S^+ tt$, since S^+ is a strict partial order.

Exercise 1.1.5

Define a *reflexive and transitive tree* is a PO $(T, \leq)$ such that (i)there is a *root* $r \in T$ satisfying $r \leq t$ for all $t \in T$ and (ii) for each $t \in T$, the set $\{s \in T|s \leq t\}$ is finite and totally ordered by $\leq$.

Now assume (T, S) is a tree, consider (T, S^*).

Reflexivity: Follows definition of reflexive and transitive closure.

Transitivity: Follows definition of reflexive and transitive closure.

Anti-symmetry: $S^* = S\mathbf{rt}$ in Exercise 1.1.3, so if S^*xy and S^*yx, then it must be $x = y$.

Thus S^* is a partial order.

r is also a root of (T, S^*), by the defining property of the tree (T, S).

For all t, $\{s \in T|S^*st\} = \{s \in T|S^+st\} \cup \{(t,t)\}$, so by Exercise 1.1.4(a), it's finite, and totally ordered by S^*.

As a conclusion, (T, S^*) is a reflexive and transitive tree.

Exercise 1.1.6

(a) Rxy iff $x_0 = y_1$ and $x_1 = y_0$, iff $y_0 = x_1$ and $y_1 = x_0$, iff Ryx.

(b) I think the correct sentence should be $\forall xy(\exists z(Cxyz \wedge Iz) \leftrightarrow x = y)$

Assume $Cxyz$ and Iz, then $x_0 = y_0$, $x_1 = z_1$, $y_1 = z_0$ and $z_0 = z_1$, so $x_0 = y_0$ and $x_1 = z_1 = z_0 = y_1$, thus $x = y$.

On the other hand, assume $x = y$, then $x_0 = y_0$, $x_1 = y_1$, then $z := (x_1, x_1)$ satisfies $Cxyz$ and Iz.

(c) I shall rewrite the sentence: $\forall xabc(\exists y(Cxay \wedge Cybc) \leftrightarrow \exists z(Cxzc \wedge Czab))$.

Assume $Cxay$ and $Cybc$, then $x_0 = a_0$, $x_1 = y_1$, $a_1 = y_0$ and $y_0 = b_0$, $y_1 = c_1$, $b_1 = c_0$. Then it $z := (a_0, b_1)$ satisfies $Cxzc$ and $Czab$.

Assume $Cxzc$ and $Czab$, then $x_0 = z_0$, $x_1 = c_1$, $z_1 = c_0$ and $z_0 = a_0$, $z_1 = b_1$, $a_1 = b_0$. Then $y := (b_0, c_1)$ satisfies $Cxay$ and $Cybc$.

(If you draw a diagram then everything will be obvious)