Exercises for Section 1.3

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Exercise 1.3.1

Consider basic modal language.

Let $\mathfrak{F}$ be a frame and V and V' are two valuations, and for every proposition letter p in ϕ, $V(p) = V'(p)$.We prove by induction that: for every $w \in W$, $(\mathfrak{F}, V), w \Vdash \phi$ iff $(\mathfrak{F}, V'), w \Vdash \phi$.

If ϕ is some proposition letter p, then for every $w \in W$, $(\mathfrak{F}, V), w \Vdash \phi$ iff $w \in V(p)$, iff $w \in V'(p)$, iff $(\mathfrak{F}, V'), w \Vdash \phi$.

If ϕ is $\perp$, then our claim holds naturally.

If ϕ is some $\neg\psi$, then $(\mathfrak{F}, V), w \Vdash \phi$ iff $(\mathfrak{F}, V), w \nVdash \psi$, by induction, iff $(\mathfrak{F}, V'), w \nVdash \psi$, iff $(\mathfrak{F}, V'), w \Vdash \phi$.

If ϕ is $\varphi \vee \psi$, then $(\mathfrak{F}, V), w \Vdash \phi$ iff $(\mathfrak{F}, V), w \Vdash \varphi$ or $(\mathfrak{F}, V), w \Vdash \psi$, by induction, iff $(\mathfrak{F}, V'), w \Vdash \varphi$ or $(\mathfrak{F}, V'), w \Vdash \psi$, iff $(\mathfrak{F}, V'), w \Vdash \phi$.

If ϕ is $\Diamond\psi$, then $(\mathfrak{F}, V), w \Vdash \phi$ iff there is some $v \in W$ with Rwv and $(\mathfrak{F}, V), v \Vdash \psi$, by induction, iff $(\mathfrak{F}, V'), v \Vdash \psi$, iff $(\mathfrak{F}, V'), w \Vdash \phi$.

So by induction, our claim is true, and our claim implies that $\mathfrak{F}, V \Vdash \phi$ iff $\mathfrak{F}, V' \Vdash \phi$.

Exercise 1.3.2 We only consider the case of $\mathfrak{N} = (\mathbb{N}, S_1, S_2)$, since I have no idea what $\mathbb{B}$ is.

A formula is valid in a frame iff it is valid in all state in the frame.

(a) Given a state $n \in \mathbb{N}$, and a valuation V. Assume $(\mathfrak{N}, V), n \Vdash (\Diamond_1 p \wedge \Diamond_1 q)$, we need to show that $(\mathfrak{N}, V), n \Vdash \Diamond_1(p \wedge q)$.

$(\mathfrak{N}, V), n \Vdash (\Diamond_1 p \wedge \Diamond_1 q)$ implies that $(\mathfrak{N}, V), n \Vdash \Diamond_1 p$ and $(\mathfrak{N}, V), n \Vdash \Diamond_1 q$. Since the only state m in $\mathbb{B}$ with $S_1 nm$ is $m = n+1$. So we have $(\mathfrak{N}, V), n+1 \Vdash p$ and $(\mathfrak{N}, V), n+1 \Vdash q$, which means $(\mathfrak{N}, V), n+1 \Vdash (p \wedge q)$, so by definition, we have $(\mathfrak{N}, V), n \Vdash \Diamond_1(p \wedge q)$.

This means that $\mathfrak{N}, n \Vdash (\Diamond_1 p \wedge \Diamond_1 q) \to \Diamond_1(p \wedge q)$, i.e. the formula is valid at state n. So (a) is valid in the frame.

(b) Assume we have countably infinitely many proposition letters, we enumerate them as $p_0, p_1, ..., p_n$. Consider a valuation $V(q_n) = \{n\}$.

Assume $p = p_k$ and $q = p_l$, and $k \neq l$, then consider a state $n \in \mathbb{N}$ with $n > max(k, l)$, then $(\mathfrak{N}, V), n \Vdash \Diamond_2 p_k$ and $(\mathfrak{N}, V), n \Vdash \Diamond_2 p_l$, since $(\mathfrak{N}, V), k \Vdash p_k$ and $(\mathfrak{N}, V), m \Vdash p_l$, so $(\mathfrak{N}, V), n \Vdash (\Diamond_2 p \wedge \Diamond_2 q)$.

However, in this case, no $m \in \mathbb{N}$ will satisfy that $(\mathfrak{N}, V), m \Vdash (p \wedge q)$, since this requires that $m \in V(p)$ and $m \in V(q)$, meaning that we need m to be k and l at the same time, which is impossible. So we have $(\mathfrak{N}, V), n \nVdash \Diamond_2(p \wedge q)$.

So $(\Diamond_2 p \wedge \Diamond_2 q) \rightarrow \Diamond_2(p \wedge q)$ fails to be valid at state n in general(when $p \neq q$). So (b) is not valid.

(c) Given a state $n \in \mathbb{N}$, and a valuation V. Assume $(\mathfrak{N}, V), n \Vdash (\Diamond_1 p \wedge \Diamond_1 q \wedge \Diamond_1 r)$, then $(\mathfrak{N}, V), n \Vdash \Diamond_1 p$, $(\mathfrak{N}, V), n \Vdash \Diamond_1 q$ and $(\mathfrak{N}, V), n \Vdash \Diamond_1 r$, so we have $(\mathfrak{N}, V), n+1 \Vdash p$, $(\mathfrak{N}, V), n+1 \Vdash q$ and $(\mathfrak{N}, V), n+1 \Vdash r$, so the part on the right follows trivially, as in (a).

Therefore, (c) is valid.

(d) Given a state $n \in \mathbb{N}$, and a valuation V. Assume $(\mathfrak{N}, V), n \Vdash p$.

Then $(\mathfrak{N}, V), n \Vdash \Diamond_1 \Box_2 p$ iff $(\mathfrak{N}, V), n+1 \Vdash \Box_2 p$, iff for every $m \leq n$, we have $(\mathfrak{N}, V), m \Vdash p$, which cannot hold in general, since we only know the case $m = n$.

Thus (d) is not valid.

(e) Given a state $n \in \mathbb{N}$, and a valuation V. Assume $(\mathfrak{N}, V), n \Vdash p$.

Then $(\mathfrak{N}, V), n \Vdash \Diamond_2 \Box_1 p$ iff there is a $m < n$ such that $(\mathfrak{N}, V), m \Vdash \Box_1 p$, which fails when $n = 0$ since no one is less than it.

Therefore (e) is not valid.

(f) Given a state $n \in \mathbb{N}$, and a valuation V. Assume $(\mathfrak{N}, V), n \Vdash p$.

Then $(\mathfrak{N}, V), n \Vdash \Box_1 \Diamond_2 p$ iff $(\mathfrak{N}, V), n+1 \Vdash \Diamond_2 p$, iff there is some $m < n$ such that $(\mathfrak{N}, V), m \Vdash p$, where we can simply take $m = n$.

Therefore (f) is valid.

(g) Given a state $n \in \mathbb{N}$, and a valuation V. Assume $(\mathfrak{N}, V), n \Vdash p$.

Then $(\mathfrak{N}, V), n \Vdash \Box_2 \Diamond_1 p$ iff for every $m < n$ we have $(\mathfrak{N}, V), m \Vdash \Diamond_1 p$, iff for every $m < n$ we have $(\mathfrak{N}, V), m+1 \Vdash p$, iff $\forall m$ with $0 < m \leq n$, we have $(\mathfrak{N}, V), m \Vdash p$. This cannot hold in general, since we know the case $m = n$ only.

Exercise 1.3.3 Note that G is dual of F, H is the dual of P.

In the following, I usually sloppily write $w \Vdash ...$, omit models in the background.

(a) First assume $w \Vdash GGp$, then we know, for every state v with $w < v$, we have $v \Vdash Gp$, then for every state v with $w < v$, for every state u with $v < u$, we have $u \Vdash p$.

Then if we simply take $V(p)$ is $\{x \in W | x \leq w\}$, then we know $w \Vdash GGp$ while $w \nVdash p$.

So (a) is not valid.

(b) Assume $w \Vdash (p \wedge Hp)$, then $w \Vdash p$ and $w \Vdash Hp$.

$w \Vdash Hp$ implies that for every $v < w$, we have $v \Vdash p$. As a conclusion, we have $u \Vdash p$ for every $u \leq w$.

On the right side, $w \Vdash FHp$ iff there is some $v > w$ with $v \Vdash Hp$, iff there is some $v > w$, such that for every $u < v$ we have $u \Vdash p$.

When $W = \mathbb{Z}$, we can simply take $v = w + 1$, so (b) is valid in $(\mathbb{Z}, <)$.

While when $W = \mathbb{Q}$ or $W = \mathbb{R}$, if we take $V(p) = \{x \in W | x \leq w\}$, then clearly $w \Vdash (p \wedge Hp)$, while $w \nVdash FHp$, so (b) is not valid in $(\mathbb{Q}, <)$ or $(\mathbb{R}, <)$.

(c) Assume $w \Vdash (Ep \wedge E\neg p \wedge A(p \rightarrow Hp) \wedge A(\neg p \rightarrow G\neg p))$.

-$w \Vdash Ep$, which implies that there is some $v_1 \in W$ such that $v_1 \Vdash p$.

-$w \Vdash E\neg p$, which implies that there is some $v_2 \in W$ such that $v_2 \Vdash \neg p$.

-$w \Vdash A(p \to Hp)$, which implies that for all $v_3 \in W$ we have $v_3 \Vdash p \to Hp$.

-$w \Vdash A(\neg p \to G\neg p)$, which implies that for all $v_4 \in W$ we have $v_4 \Vdash \neg p \to G\neg p$.

The above results says that, there is some v_1 such that $v_1 \Vdash p$, and for every v with $v \Vdash p$, we have $v \Vdash Hp$, which means that all $u \leq v$ satisfies $u \Vdash p$.

Also, there is some v_2 such that $v_2 \Vdash \neg p$, and for every v with $v \Vdash \neg p$, we have $v \Vdash G\neg p$, which means that all $u \geq v$ satisfies $u \Vdash \neg p$.

So at least we know $v_1 < v_2$.

What about those on the right hand side? We know $w \Vdash E(Hp \wedge G\neg p)$ iff there is some $v_5 \in W$ such that $v_5 \Vdash Hp \wedge G\neg p$, which means $v_5 \Vdash Hp$ and $v_5 \Vdash G\neg p$, which says that for every $u \in W$, if $u < v_5$, then $u \Vdash p$, and if $u > v_5$, then $u \Vdash \neg p$.

In $\mathbb{Z}$, we know there is a largest v_1 such that $v_1 \Vdash p$, and then $v_2 = v_1 + 1$ is the smallest element with $v_2 \Vdash \neg p$, in this case we take $v_5 = v_1$. So (c) is valid in $(\mathbb{Z}, <)$.

In $\mathbb{Q}$, assume r is an irrational number, and we define $V(p) = \{x \in \mathbb{Q} | x < r\}$, then we know under this valuation, $w \Vdash (Ep \wedge E\neg p \wedge A(p \to Hp) \wedge A(\neg p \to G\neg p))$ holds. However, now no $v_5 \in \mathbb{Q}$ helps. So (c) is not valid in $(\mathbb{Q}, <)$.

In $\mathbb{R}$, we know $\{x \in \mathbb{R} | x \Vdash p\}$ is non-empty and is upper-bounded, then it has a supreme s, and clearly we can take $v_5 = s$. So (c) is valid in $(\mathbb{R}, <)$.

Exercise 1.3.4 Clearly every propositional tautologies are valid.

To show that $\Box(p \to q) \to (\Box p \to \Box q)$ is valid, we first assume $w \Vdash \Box(p \to q)$, then for every v with Rwv, we must have $v \Vdash p \to q$.

Then we assume $w \Box p$, then for every v with Rwv, we must have $v \Vdash p$.

As a conclusion, for every v with Rwv, we must have $v \Vdash p$, which implies $w \Vdash \Box q$.

Therefore $\Box(p \to q) \to (\Box p \to \Box q)$ is valid.

Exercise 1.3.5

(a) Take $W = \{0, 1\}$, and $R = W \times W$.

(b) Take $W = \{0, 1\}$, and $R = W \times W$.

(c) Take $W = \mathbb{Z}$, and Rxy iff $y = x + 1$.

(d) Take $W = \mathbb{N}$, and Rxy iff $y < x$. Then consider $V(p) = \mathbb{N}$, so clearly $1 \Vdash \Diamond\Box p$, while $1 \nVdash \Box\Diamond p$.

Exercise 1.3.6

Assume $a \Vdash \phi \circ (\psi \circ \xi)$, then for some b, c with $Cabd$, we have $b \Vdash \phi$ and $c \Vdash (\psi \circ \xi)$, then for some x, y with $Ccxy$, we have $x \Vdash \psi$ and $y \Vdash \xi$. Then we can composite b after x, get a z, with $Czbx$, so $z \Vdash (\phi \circ \psi)$.Then we know that also $Cazy$, and we have $y \Vdash \xi$, so $a \Vdash (\phi \circ \psi) \circ \xi$. The other side follows similarly.

Assume $a \Vdash 1' \circ \phi$, then foe some b, c with $Cabc$, we have $b \Vdash 1'$ and $c \vdash \phi$, thus Ib, it's actually $a = c$, so $a \Vdash \phi$. The other side follows similarly.