Generalization Theorem

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Theorem 1. Generalization Theorem

Given $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$. Assume $\Gamma \vdash \varphi$, and x_i is not a free variable of any formula in Γ, then $\Gamma \vdash (\forall x_i \varphi)$.

Proof. $\Gamma \vdash \varphi$ implies that there is a proof of φ in Γ: $\langle B_1, ..., B_n \rangle$ with $B_n = \varphi$. We use induction on index j, $1 \leq j \leq n$, to show that $\Gamma \vdash (\forall x_i B_j)$.

Base step: B_1 must be in $\Gamma \cup \mathbb{L}$. If $B_1 \in \Gamma$, then $\Gamma \vdash B_1$. Since x_i is not a free variable of B_1, we have $\Gamma \vdash (\varphi \to (\forall x_i \varphi))$, thus we have $\Gamma \vdash (\forall x_i B_1)$. If $B_1 \in \mathbb{L}$, then we have $(\forall x_i B_1) \in \mathbb{L}$ also, thus $\Gamma \vdash (\forall x_i B_1)$. In both case, we have $\Gamma \vdash (\forall x_i B_1)$.

Induction step: Assume for all j, $1 \leq j \leq k < n$, we have $\Gamma \vdash (\forall x_i B_j)$. Consider B_{k+1}. If $B_{k+1} \in \Gamma \cup \mathbb{L}$, then repeat the process in base step, we know $\Gamma \vdash (\forall x_i B_{k+1})$. If there exists $1 \leq s, t < (k+1)$, and B_t is just $(B_s \to B_{k+1})$. Then by induction hypothesis, we have:

$\Gamma \vdash (\forall x_i B_s)$, and

$\Gamma \vdash (\forall x_i (B_s \to B_{k+1}))$.

We also have $\Gamma \vdash ((\forall x_i (B_s \to B_{k+1})) \to ((\forall x_i B_s) \to (\forall x_i B_{k+1})))$, so we then have $\Gamma \vdash (\forall x_i B_{k+1})$.

By induction, we have $\Gamma \vdash (\forall x_i B_n)$, i.e. $\Gamma \vdash (\forall x_i \varphi)$ $\square$