Deduction Theorem

kankanray

Theorem 1. Deduction Theorem
Given $\Gamma \subseteq \mathcal{L}$, $\varphi, \psi \in \mathcal{L}$. Then $\Gamma \cup \{\varphi\} \vdash \psi$ iff $\Gamma \vdash (\varphi \to \psi)$.

Proof. $\Rightarrow$: Assume $\Gamma \cup \{\varphi\} \vdash \psi$.

Then there is a proof of ψ in $\Gamma \cup \{\varphi\}$: $\langle B_1, ..., B_n \rangle$, with $B_n = \psi$. We induction on the index of B_i's, show that $\Gamma \vdash (\varphi \to B_i)$

Base step: B_1 must be a formula in $\Gamma \cup \mathbb{L}$. If $B_1 \in \Gamma$, then $\Gamma \vdash B_1$. We also have $\Gamma_1 \vdash (B_1 \to (\varphi \to B_1))$, thus we have $\Gamma \vdash (\varphi \to B_1)$.

Induction step: Assume for every k, $1 \leq k \leq i < n$, we have $\Gamma \vdash (\varphi \to B_k)$. Then if $B_{i+1} \in \Gamma \cup \mathbb{L}$, then repeat the process above, we know that $\Gamma \vdash (\varphi \to B_{i+1})$. If there is $1 \leq s, t < (i+1)$, with B_t is just $(B_s \to B_{i+1})$, then by induction hypothesis, $\Gamma \vdash (\varphi \to B_s)$ and $\Gamma \vdash (\varphi \to (B_s \to B_{i+1}))$. We also have $\Gamma \vdash ((\varphi \to (B_s \to B_{i+1})) \to ((\varphi \to B_s) \to (\varphi \to B_{i+1})))$, so then we get $\Gamma \vdash (\varphi \to B_{i+1})$.

So by induction, we have $\Gamma \vdash (\varphi \to B_n)$, i.e. $\Gamma \vdash (\varphi \to \psi)$.

$\Leftarrow$: Assume $\Gamma \vdash (\varphi \to \psi)$.

Then $\Gamma \cup \{\varphi\} \vdash (\varphi \to \psi)$ and $\Gamma \cup \{\varphi\} \vdash \varphi$, so then we have $\Gamma \cup \{\varphi\} \vdash \psi$. $\square$