

# Equality Theorem

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## Theorem 1. Equality Theorem

Assume  $\varphi \in \mathcal{L}$  has no quantifiers,  $\sigma_1, \dots, \sigma_n$  and  $\tau_1, \dots, \tau_n$  are two collection of terms. Given variables  $x_{m_1}, \dots, x_{m_n}$ , if each of them occurs nowhere in  $\sigma_i, \tau_i$  for every  $1 \leq i \leq n$ , then we have

$$\{\tau_i \hat{=} \sigma_i | 1 \leq i \leq n\} \cup \{\varphi(x_{m_1}, \dots, x_{m_n}; \tau_1, \dots, \tau_n)\} \vdash \varphi(x_{m_1}, \dots, x_{m_n}; \sigma_1, \dots, \sigma_n)$$

*Proof.*

(1) Assume  $\tau$  and  $\sigma$  are terms, variable  $x_m$  appears nowhere in  $\tau$  or  $\sigma$ , and formula  $\varphi$  has no quantifiers, then we show that  $\vdash ((\tau = \sigma) \rightarrow (\varphi(x_m; \tau) \rightarrow \varphi(x_m; \sigma)))$ .

Take  $k$  large enough such that  $x_k$  appears nowhere in  $\varphi$ , then  $x_k$  can substitute  $x_m$  in  $\varphi$ . So consider two formulas  $\varphi$  and  $\varphi(x_m; x_k)$ . Note that  $x_k$  can substitute  $x_m$  in both two formulas, and after substitution, two formulas are identical. So we have:

$$((x_k \hat{=} x_m) \rightarrow (\varphi(x_m; x_k) \rightarrow \varphi)) \in \mathbb{L}.$$

Then use generalization law, we have:

$$(\forall x_k (\forall x_m ((x_k \hat{=} x_m) \rightarrow (\varphi(x_m; x_k) \rightarrow \varphi)))) \in \mathbb{L}.$$

Note that  $\tau$  can substitute  $x_k$  in  $(\forall x_m ((x_k \hat{=} x_m) \rightarrow (\varphi(x_m; x_k) \rightarrow \varphi)))$ , so use specialization law, we get  $\vdash (\forall x_m ((\tau \hat{=} x_m) \rightarrow (\varphi(x_m; \tau) \rightarrow \varphi)))$ .

Again, since  $\sigma$  can substitute  $x_m$  in  $((\tau \hat{=} x_m) \rightarrow (\varphi(x_m; \tau) \rightarrow \varphi))$ , so use specialization law, we get  $\vdash ((\tau \hat{=} \sigma) \rightarrow (\varphi(x_m; \tau) \rightarrow \varphi(x_m; \sigma)))$ .

Now we use induction on the index  $i$ ,  $1 \leq i \leq n$ .

Base step: By (1), we have  $\vdash ((\tau_1 \hat{=} \sigma_1) \rightarrow (\varphi(x_{m_1}; \tau_1) \rightarrow \varphi(x_{m_1}; \sigma_1)))$ . Use Deduction Theorem twice, we have  $\{(\tau_1 \hat{=} \sigma_1)\} \cup \{\varphi(x_{m_1}; \tau_1)\} \vdash \varphi(x_{m_1}; \sigma_1)$ .

Induction step: Assume  $\{\tau_j \hat{=} \sigma_j | 1 \leq j \leq i\} \cup \{\varphi(x_{m_1}, \dots, x_{m_i}; \tau_1, \dots, \tau_i)\} \vdash \varphi(x_{m_1}, \dots, x_{m_i}; \sigma_1, \dots, \sigma_i)$ , for  $1 \leq i < n$ . Then by (1) again, we have  $\vdash ((\tau_{i+1} \hat{=} \sigma_{i+1}) \rightarrow (\varphi(x_{m_1}, \dots, x_{m_i}, \tau_{i+1}) \rightarrow \varphi(x_{m_1}, \dots, x_{m_i}, \sigma_{i+1})))$ , combined with results above and Deduction Theorem, we have  $\{\tau_j \hat{=} \sigma_j | 1 \leq j \leq (i+1)\} \cup \{\varphi(x_{m_1}, \dots, x_{m_{i+1}}; \tau_1, \dots, \tau_{i+1})\} \vdash \varphi(x_{m_1}, \dots, x_{m_{i+1}}; \sigma_1, \dots, \sigma_{i+1})$ .

By induction, we proved our theorem.  $\square$