Exercises for Section 1.2

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Exercise 1.2.1

(a) $\phi \to MK\phi$.
(b) $(M\phi \wedge M\psi) \to M(\phi \wedge \psi)$.
(c) $K\phi \to M\phi$.
(d) $MM\phi \to M\phi$.
Only (b) seems clear.
In my opinion, (b) (c) are plausible; (a) (d) are not.

Exercise 1.2.2

$\Box\phi$ can be interpreted as 'it's not the case that the negation of ϕ is permissible', or equivalently 'the negation of ϕ is not permissible'.

Some formulas which seem plausible under this interpretation:
$(\Diamond\phi \wedge \Diamond\psi) \leftrightarrow \Diamond(\phi \wedge \psi)$,
$\Diamond(\phi \vee \psi) \to (\Diamond\phi \vee \Diamond\psi)$.

Finally, consider the Löb formula: $\Box(\Box p \to p) \to \Box p$. Assume $\Box(\Box p \to p)$ is true, then we know '$\neg(\Box p \to p)$ is not permissible', which is '$\Box p \wedge \neg p$ is not permissible'. Then we conclude that '$\neg p$ is not permissible', which is just described by $\Box p$.

So the Löb formula may be plausible.

Exercise 1.2.3

'while ϕ do π': $(\phi?;\pi)^*;\neg\phi?$.
'repeat π until ϕ': $\pi;(\neg\phi?;\pi)^*;\phi?$.

Note that the 'programs' given above are only in an intuitive sense, since we have no semantics here.

Exercise 1.2.4

$1'\circ p \leftrightarrow p$ should be always true.
$\otimes(p \circ q) \leftrightarrow \otimes q \circ \otimes p$ should be always true.
$p \circ (q \circ r) \leftrightarrow (p \circ q) \circ r$ should be always true.

Exercise 1.2.5

Assume that χ is a substitution instance of ψ, then there is a substitution σ such that $\psi^\sigma = \chi$.

Assume also that ψ is a substitution instance of ϕ, then there is a substitution τ such that $\phi^\tau = \psi$.

Then consider $\sigma \circ \tau$, is also a substitution, and we have
$\perp^{\sigma\circ\tau} = \perp$,
$p^{\sigma\circ\tau} = \sigma \circ \tau(p) = \sigma(\tau(p)) = (p^\tau)^\sigma$,
$(\neg\psi)^{\sigma\circ\tau} = \neg\psi^{\sigma\circ\tau}$, by induction hypothesis, $= \neg(\psi^\tau)^\sigma$,
$(\psi \vee \theta)^{\sigma\circ\tau} = \psi^{\sigma\circ\tau} \vee \theta^{\sigma\circ\tau}$, by induction hypothesis, $= (\psi^\tau)^\sigma \vee (\theta^\tau)^\sigma$,
$(\triangle(\psi_1, ..., \psi_n))^{\sigma\circ\tau} = \triangle(\psi_1^{\sigma\circ\tau}, ..., \psi_n^{\sigma\circ\tau})$, by induction hypothesis, $= \triangle((\psi_1^\tau)^\sigma, ..., (\psi_n^\tau)^\sigma)$.

As a conclusion, we have $\phi^{\sigma\circ\tau} = (\phi^\tau)^\sigma = \psi^\sigma = \chi$, so χ is a substitution instance of ϕ.