Set Theory

kankanray

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1 Set Theory Language

We sloppily introduce the following. For more details you may refer to first order logic.

Definition 1.1.
We have logical symbols: $\neg$, $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$, $\exists$, and two brackets (,).
Equation symbol $=$.
Variable symbols: x, y, z,..., x_0, x_1, ... and so on.
Binary predicate symbol $\in$.

Then the well-formed formulas in set theory:

Definition 1.2.
(1) Basic formulas : $x \in y$.
(2) Formulas:
-All basic formulas,
-If φ is a formula, then $(\neg\varphi)$ is a formula,
-If ψ and ϕ are formulas, then $(\psi \rightarrow \phi)$ is a formula,
-$\wedge$, $\vee$ and $\leftrightarrow$ are defined similarly,
-If φ is a formula, x is a variable symbol, then $\forall x\varphi$ is a formula; $\exists$ is defined similarly.
-All formulas can only be constructed by the above methods.

2 Zermelo Set Theory

In this part we consider seven axioms, and some basic properties.

2.1 First 6 axioms

Axiom 1. Axiom of extensionality
$((x = y) \leftrightarrow (\forall z(z \in x \leftrightarrow z \in y)))$

Actually, the axiom above tells us what does 'equal' means in set theory.
Then it's easy to verify the following proposition:

Proposition 2.1.

(1) $x = x$,

(2) If $x = y$, then $y = x$,

(3) If $x = y$ and $y = z$, then $x = z$.

Axiom 2. Existence of sets

$(\exists x(\exists y(x \in y)))$

We need Axiom 2, since we are not going to play with nothing.

We also need more axioms to get various sets:

Axiom 3. Axiom of pair

$\forall x \forall y \exists u(\forall z(z \in u \leftrightarrow (z = x \vee z = y)))$

We often write such u as $\{x, y\}$.

Axiom 4. Axiom of union

$\forall x \exists y \forall z(z \in y \leftrightarrow \exists u(u \in x \wedge z \in u))$

We often write such y as $\bigcup x$.

We define the concept of 'subset':

$x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y)$

We say $x \subset y$ when $x \subseteq y$ and $x \neq y$.

Axiom 5. Axiom of the power set

$\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$

We often write such y as $\mathfrak{P}(x) = \{z | z \subseteq x\}$

Axiom 6. Axiom schema of separation

Given an arbitrary sentence with one free variable $\varphi(x)$, describing some separation properties, we have

$\forall x \exists y \forall z(z \in y \leftrightarrow (z \in x \wedge \varphi(z)))$

We often write such y as $\{z \in x | \varphi(z)\}$

Note that, Axiom 3 to 6 gives us existences of some specific kinds of sets. It can be easily verified that all those sets are unique in '=' sense.

With those axioms we have more concepts and propositions:

Definition 2.1.

Intersection: Assume A is non-empty, then the intersection is defined by $\bigcap A := \{a \in \bigcup A | \forall x(x \in A \rightarrow a \in x)\}$. Note that we used Axiom 4 and 6 here.

Difference between sets is defined by: $x - y := \{z \in x \cup y | z \in x \wedge z \notin y\}$. Note that we used Axiom 3, 4 and 6 here.

Symmetry difference of sets is defined by: $x \Delta y := (x - y) \cup (y - x)$. Note that we used Axiom 3, 4 and 6 here.

Proposition 2.2.

$(x = y) \leftrightarrow (x \subseteq y \wedge y \subseteq x)$

Proposition 2.3. Existence of empty set

$\exists x \forall y(y \notin x)$

Proof. By Axiom 2, we have a set A. Then consider sentence $\varphi(x) := \forall y(y \neq x)$.

Then use Axiom 6, we have $B = \{z \in A | \varphi(z)\}$.

We verify that B is an empty set:

For all y, $y \in B$ iff $\varphi(y)$, iff $\forall x(x \neq y)$, while we have $y = y$, so $y \notin B$, which means B is empty. □

We can also show that the empty set is unique in '=' sense. So we will denote it as $\emptyset$. Note also that $\emptyset$ is subset of every set.

Now we consider an interesting 'paradox', or a theorem now. It says that there exists no such set, which contains all sets that are not element of themselves.

Theorem 2.1. (Russell)

$\neg(\exists x \forall y(y \in x \leftrightarrow y \notin y))$

Proof. Otherwise, we assume such set exists, call it A.

Then if $A \in A$, then by its definition, $A \notin A$, contradicts.

If $A \notin A$, then by its definition again, $A \in A$, contradicts.

So such A cannot exist. □

More definitions:

Definition 2.2. Ordered pair

$(x, y) := \{\{x\}, \{x, y\}\}$. Note that we used Axiom 3 twice.

Proposition 2.4.

$(a, b) = (c, d)$ iff $a = b$ and $c = d$.

Definition 2.3. Cartesian product

Given 2 sets A and B, $A \times B := \{(a, b) | a \in A \wedge b \in B\}$.

We used Axiom 5(power set) twice, and then use Axiom 6(separation schema).

The definition of relations(subsets of $A \times B$), functions(a special kind of relations) and more concepts are omitted here. They can be constructed easily in common sense.

2.2 Axiom of Infinity and Natural Numbers

We introduce the 7th axiom, also the last axiom in Zermelo's set theory.

First consider a legitimate operation:

Definition 2.4. Successor

Given a set x, its *successor* is defined by:

$S(x) := x \cup \{x\}$

Axiom 7. Axiom of Infinity

$\exists x((\emptyset \in x) \wedge (\forall y(y \in x \rightarrow S(y) \in x)))$

For simplicity, we define $Inf(x) := ((\emptyset \in x) \wedge (\forall y(y \in x \rightarrow S(y) \in x)))$

Clear, a set x with $Inf(x)$ may be too large, while the set of natural numbers in our imagination is relatively small. We must do some proper 'separations' to reduce a infinity set.

Definition 2.5. Given a set u satisfying $Inf(u)$, we define a subset of u:
$W(u) := \{a \in u | \forall v(Inf(v) \to a \in v)\}$

$W(u)$ has good properties:

Theorem 2.2.
(1) $Inf(u) \to Inf(W(u))$
(2) If $Inf(u_1)$ and $Inf(u_2)$, then $W(u_1) = W(u_2)$.
(3) There exists a unique set satisfying: $Inf(u)$ and $W(u) = u$.

Proof. Use definitions and axioms. □

Definition 2.6. Natural number
We define that unique set as $\mathbb{N}$ or ω, satisfying $Inf(\omega)$ and $W(\omega) = \omega$.

Natural number set has many properties:

Proposition 2.5. If $Inf(u)$, then $\omega \subseteq u$.

Theorem 2.3.
1. $\forall a \in \omega(a = \emptyset \vee \emptyset \in a)$.
2. $\forall a \in \omega \forall b \in \omega(a \in S(b) \leftrightarrow (a = b \vee a \in b))$.
3. $\forall a \in \omega(a \subseteq \omega)$.
4. $\forall a \in \omega \forall b \in \omega \forall c \in \omega((a \in b \wedge b \in c) \to a \in c)$.

Definition 2.7. We say a subset x is transitive, iff $\forall y \in x$, we have $y \subseteq x$.

So ω itself is a transitive set. Actually all elements of ω are transitive also (use induction to prove it).

Theorem 2.4.
5. $\forall x \in \omega(x \notin x)$.
6. $\forall x \in \omega \forall y \in \omega(x \in y \to y \notin x)$.
7. $\forall x \in \omega \forall y \in \omega(x \in y \to (S(x) = y \vee S(x) \in y))$.
8. $\forall x \in \omega \forall y \in \omega(x \in y \vee x = y \vee y \in x)$.
9. $\forall x \in \omega(x \neq \emptyset \to \exists y(y \in \omega \wedge x = S(y)))$

We notice that $\in$ gives an order on ω.

We have definition of linear ordered set, well-ordered set. Linear order says that every two elements in the set can be compared, and well-ordered set is a linear ordered set, and for arbitrary non-empty subset, it has a minimal element.

We find that $(\omega, \in)$ is a well-ordered set:

Theorem 2.5.
10. $\forall a \in \omega \forall x((\emptyset \neq x \subseteq a) \to \exists b(b \in x \wedge b \cap x = \emptyset))$.
11. If $A \neq \emptyset$ and $A \subseteq \omega$, then $\exists a(a \in A \wedge \forall x \in A(x = a \vee a \in x))$.
...

We have more properties, like a bounded subset has maximal. We left those verification to readers.

We also have mathematical induction, essentially, just follows the fact that $(\omega, \in)$ is a well-ordered set.

2.3 Cardinality: First Glance

We omit basic definitions. Consider:

Theorem 2.6. There is no surjection from A to $\mathfrak{P}(A)$. Especially, $|\omega| < |\mathfrak{P}(\omega)|$.

Proof. Given $f : A \to \mathfrak{P}(A)$, we define a set $X = \{a \in A | a \notin f(a)\}$.

Now we ask whether X is in the range of f or not. If there is some $x \in A$ such that $f(x) = X$, then:

If $x \in f(x)$, then $x \notin X = f(x)$, contradiction.

If $x \notin f(x)$, then $x \in X = f(x)$, contradiction.

So we know that f cannot be surjective, so it must be $|A| < |\mathfrak{P}(A)|$. $\square$

3 Ordinal Number Basics

Definition 3.1. A set x is said to be an ordinal number iff it satisfies the following properties:

(a) x is a transitive set: $\forall y(y \in x \to y \subseteq x)$.

(b) Any two of x's elements can compare: $\forall u \in x \forall v \in x(u \in v \vee u = v \vee v \in u)$.

(c) Every non-empty set of x has a $\in$-minimal element:
$\forall y((y \subseteq x \wedge y \neq \emptyset) \to \exists z(z \in y \wedge z \cap y = \emptyset))$.

We often use Greek alphabet to denote ordinal numbers.

Definition 3.2. We use Ord to denote the collection of all ordinal numbers, and use $\alpha \in Ord$ to express that α is an ordinal number.

Note that we say Ord is a class, not a set. We'll show that Ord cannot be a set.

We start from some basic properties of ordinal numbers:

Proposition 3.1. Assume α is an ordinal number, and $\emptyset \neq A \subseteq \alpha$.

(1) If $\beta \in A$ and $\gamma \in A$ are both $\in$-least element of A, i.e. $\beta \cap A = \gamma \cap A = \emptyset$, then $\beta = \gamma$.

(2) By (1), we use $min(A)$ to denote A's $\in$-;east element. For every $\gamma \in A$, either $\gamma = min(A)$ or $min(A) \in \gamma$.

By that result, we can formally define:

Definition 3.3. Assume $\alpha \in Ord$ and $\emptyset \neq A \subseteq \alpha$, then we use $min(A)$ to denote the unique $\in$-minimal element of A.

Lemma 3.1. $\alpha \in Ord$, then we have:

(1) $\alpha \notin \alpha$.

(2) $\forall \gamma(\gamma \in \alpha \to \gamma \notin \gamma)$.

(3) $\beta = \alpha \cup \{\alpha\} \in Ord$ also.

Note that every ordinal number is a ordered set naturally:

Theorem 3.1. $\alpha \in Ord$, then define a relation $<$ on α:

$x < y$ iff $x \in y$, for every $x, y \in \alpha$.

Then $(\alpha, <)$ is a (well-)ordered set.

Note that ordinal numbers have a lot of properties that are similar to ω:

Lemma 3.2. $\alpha \in Ord$ and $x \in \alpha$, then x is transitive also.

Proposition 3.2. $\alpha \in Ord$ and $\beta \in \alpha$, then $\beta \in Ord$.

Lemma 3.3. $\alpha \in Ord$, with $\alpha = \beta \cup \{\beta\}$ and $\alpha = \gamma \cup \{\gamma\}$, then $\beta = \gamma$.

There are two different kinds of ordinal numbers, which are discussed in the following definition:

Definition 3.4.

(1) We say a ordinal number α is *successor ordinal*, iff $\exists \beta(\beta \in \alpha \wedge \alpha = \beta \cup \{\beta\})$.

(2) We say a ordinal number is a *limit ordinal* iff it's not a successor ordinal.

Clearly, $\emptyset$ and ω are limit ordinal, while every non-zero natural numbers are successor ordinals.

Lemma 3.4. $\alpha \in Ord$ and $A \subseteq \alpha$. If A is an ordinal number, then either $A = \alpha$ or $A \in \alpha$.

With this lemma, it's not hard to show that every two ordinals can compare with $\in$:

Theorem 3.2. $\alpha, \beta \in Ord$, then it must be one of the three cases: $\alpha \in \beta$, $\alpha = \beta$, or $\beta \in \alpha$.

So with this theorem we know the structure of ordinal number:

Corollary 3.1. $\alpha \in Ord$, then $\alpha = \{\beta \in Ord | \beta \in \alpha\}$

Proposition 3.3. Ord is not a set.

Proof. Otherwise assume Ord is a set, then it is not hard to show that Ord is a ordinal by definition, then by definition again we have $Ord \in Ord$, which is a contradiction. □

Although Ord is not a set, it still has some good properties. We first talk about $\in$-minimal element for arbitrary set.

Definition 3.5. For set x, we say another set a is $\in$-minimal element of x iff $a \in x$ and $a \cap x = \emptyset$.

We say a is $\in$-minimal element(same word, different meaning) of x iff $a \in x$, and $\forall y((y \in x \wedge y \neq a) \to a \in y)$.

Theorem 3.3. For every non-empty set of ordinals X, X has a $\in$-minimal element, which we denote as $min(X)$.

Theorem 3.4. Order completeness

Assume x is a non-empty set of ordinals.

(1) x is an ordinal iff x is transitive.

(2) $\bigcup x$ is an ordinal.

(3) $\forall \alpha \in x$, we have $\alpha \subseteq \bigcup x$; If $\gamma \in Ord$ and $\forall \alpha \in x$ we have $\alpha \subseteq \gamma$, then $\bigcup x \subseteq \gamma$.

Definition 3.6. Assume x is a non-empty set of ordinals. Let $sup(x) = \bigcup x$, we call the the *supreme* of x. If x has a maximal, then use $max(x)$ to denote it.

In the end, we discuss transfinite induction, which is a kind of induction on Ord:

Theorem 3.5. Minimal Ordinal

Let $\phi(v)$ be a formula. If $\exists \alpha(\phi(\alpha) \wedge \alpha \in Ord)$, then $\exists \alpha(\phi(\alpha) \wedge \alpha \in Ord \wedge \forall \beta(\beta \in \alpha \to \neg\phi(\beta)))$.

Then we have two versions of transfinite induction theorem:

Theorem 3.6. Transfinite Induction: version 1

Assume $\phi(u)$ is a formula. If for every $\alpha \in Ord$ we have $(\forall \beta(\beta \in \alpha \to \phi(\beta))) \to \phi(\alpha)$, then $\phi(\gamma)$ for every $\gamma \in Ord$.

Theorem 3.7. Transfinite Induction: version 2

Assume $\phi(u)$ is a formula. If

(a) $\phi(\emptyset)$ holds.

(b) $\forall \beta((\beta \in Ord \wedge \phi(\beta)) \to \phi(\beta + 1))$.

(c) For every limit ordinal α, we have $(\forall \beta(\beta \in \alpha \to \phi(\beta))) \to \phi(\alpha)$.

Then $\phi(\gamma)$ for every $\gamma \in Ord$.

4 Order

In this section we discuss (well-)ordered set, mainly on the *rigidity* of it.

The a ordered set is a linear ordered set, and every non-empty subset of it has a minimal element. Note that every ordinal number is a ordered set naturally.

We start from natural number first:

Definition 4.1. Strictly Increasing Function

Assume $X \subseteq \omega$ is non-empty, and $f : X \to \omega$. We say f is *strictly increasing* iff for every $x, y \in X$, $x < y$ then $f(x) < f(y)$.

Definition 4.2. Order Isomorphism

Assume $X, Y \subseteq \omega$ are both non-empty, and $f : X \to Y$

(1) f is said to be an *order isomorphism*, iff it is a strictly increasing surjection.

(2) X and Y are isomorphic iff f is an order isomorphism.

Then we have *rigidity* of natural number orders.

Theorem 4.1. Rigidity Theorem

(1) Assume $f : n \to n$ is a strictly increasing function, then it's actually an identity map.

(2) Assume $f : \omega \to \omega$ is an order isomorphism, then it is actually identical map.

(3) Assume $X, Y \subseteq \omega$ are finite subset and $|X| = |Y|$, then there is a unique order isomorphism between the two sets.

(4) If $f : \omega \to \omega$ is a strictly increasing function, then for every n, we have $f(n) \geq n$.

(5) If $X \subseteq \omega$ and there is an order isomorphism between them, then such order isomorphism is unique.

The question is, what kind of subset of ω has order isomorphism with ω. We have the representation as the following:

Theorem 4.2. Representation Theorem

If $X \subseteq \omega$, then

(1) either $\exists n \in \omega((X, <) \cong (n, <))$,

(2) or $(X, <) \cong (\omega, <)$.

We want to know more about representing subsets of ω: we want to know that the order isomorphism map is.

Definition 4.3. (1) $X \subseteq \omega$ is said to be unbounded iff $\forall n \in \omega \exists x(x \in X \wedge n < n)$.

(2) $X \subseteq \omega$ is said to be bounded iff it's not unbounded.

Lemma 4.1. $m \in \omega$, $f : m \to \omega$, then $f[m]$ is bounded in ω.

Theorem 4.3. (1) If $X \in \omega$ is bounded, then there exists a unique pair (m_X, π_X):

(i) $m_X \in \omega$ and $\pi_X : X \to m_X$ is a order isomorphism. We sometimes call this map *avalanche mapping*.

(ii) $\forall i \in X(\pi_X(i) = \pi[i \cap X]) = \{\pi_X(j) | j \in i \cap X\}$. Which means π_X is continuous.

(2) If $X \subseteq \omega$ is unbounded, then there is a unique order isomorphism from X to ω, called π_X, satisfying:

$\forall k \in X(\pi_X(k) = \pi_X[k \cup X])$.

We sometimes can such map *avalanche mapping*. And call π_X^{-1} the *natural list* of X.

Now we've showed the rigidity of order on natural numbers. We want to know how much such rigidity can be extended. We first consider our familiar ordinal numbers.

Note that the increasing functions and some other concepts are defined in the similar way for ordinal numbers.

Lemma 4.2. $\alpha \in Ord$, and $f : (\alpha, <) \to (\alpha, <)$ is a increasing function, then $\forall \gamma \in \alpha(\gamma < f(\gamma))$.

Theorem 4.4. Rigidity for Ordinals

$\alpha \in Ord$

(1) $f : \alpha \to \alpha$ is a order isomorphism, them it must be identical map.

(2) $\beta \in Ord$, and $f : \alpha \to \beta$ is an order isomorphism, then $\alpha = \beta$ and f is identical map.

Great! Orders on ordinal numbers have rigidity too. Now, we consider a more general case: well-ordered set. Note that all ordinals are well-ordered set naturally.

Definition 4.4. Given ordered set $(W, <)$, a subset x of W is said to be a prefix iff

$\exists u \in W(\forall y(y \in x \leftrightarrow (y \in W \wedge y < u)))$.

So we often use $W[u] = \{y \in W | y < u\}$ to denote such prefix.

Lemma 4.3. (1) Assume X is a prefix of W, then there exists a $u \in W$, such that $X = W[u]$.

(2) For $u, v \in W$, iff $W[u] = W[v]$, then $u = v$.

Theorem 4.5. (1) Every ordered set has unique self order isomorphism, which is the identical map.

(2) If two ordered set are order isomorphic, then the isomorphism is unique.

(3) No ordered set will be order isomorphic to any prefix of it.

Theorem 4.6. Comparable Theorem for Ordered Set

Assume $(W_1, <_1)$ and $(W_2, <)$ are two ordered sets. Then it must one of the following three cases:

(1) $(W_1, <_1) \cong (W_2, <_2)$.

(2) $\exists u \in W_1((W_1[u], <_1) \colon (W_2, <_2))$.

(3) $\exists v \in W_2((W_1, <_1) \cong (W_2[v], <_2))$

5 Recursive Definition

Note that in mathematics we often define something recursively, that is first define $f(0)$, then assume $f(n)$ is defined, and use it to define $f(n+1)$. But why we can do that? We will answer such question in this section.

Definition 5.1. (1) Given $n \in \omega$, A is a non-empty set. Then any element of A^n is said to be a sequence with length n in A. We think A^1 is the same as A. For A^n, we also equating it with n-times Cartesian product.

(2) Let $A^{<\omega} = \bigcup\{A_n | n \in \omega\}$, we say it's the set of all finite sequences on A. A^ω is said to be the set of all infinite sequences on A.

(3) $[A]^n = \{x \in \mathfrak{P}(A) | |x| = |n|\}$. $[A]^{<\omega} = \{x \in \mathfrak{P}(A) | \exists n \in \omega(|x| = |n|)\}$.

Note that in general X^Y is defined to be the set of all functions from Y to X.

To accurately discuss recursive definition, we need more definitions:

Definition 5.2. (1) Given two functions f and g, we say they are *harmonious* iff $\forall x(x \in dom(f) \cap dom(g) \to f(x) = g(x))$.

(2) Assume F is a set of functions. We say F is a *harmonious function system* iff every two functions $f, g \in F$ are harmonious.

Harmonious functions have good properties:

Proposition 5.1. (1) Two functions f and g are harmonious iff $f \cup g$ is a function.

(2) Assume F is a harmonious function system. Let $H = \bigcup F$, then
(a) H is a function.
(b) $dom(H) = \bigcup\{dom(f) | f \in F\}$.
(c) $\forall f(f \in F \to f \subseteq H)$.

Then we can discuss recursive definitions, start from two lemmas, then the theorem. We assume A is non-empty, $a \in A$, $g : A \times \omega \to A$.

Lemma 5.1. Existence of Prefix

For every $m \in \omega$, there exists a function $f_m : m + 1 \to A$ satisfying:
(1) $f_m(0) = a$.
(2) $\forall(n \in m \to f_m(n + 1) = g(f_m(n), n))$.

Proof. Induct on m. □

Lemma 5.2. Harmoniousness of Prefixes

Assume $m, n \in \omega$, and f_m, f_n are as in the above lemma, and assume $n \in m$ or $n = m$, then for every $i \in n + 1$, we have $f_n(i) = f_m(i)$.

Theorem 5.1. First Recursive Definition Theorem

Assume A is non-empty, $a \in A$, $g : A \times \omega \to A$. Then there exists a unique function $f : \omega \to A$ satisfying:
(1) $f(0) = a$.
(2) $\forall n \in \omega(f(n + 1) = g(f(n), n))$.

Proof. Collect all such functions with domain $m+1$, and prove that this collection is a harmonious system, and check that its union is what we want. □

We have some corollaries:

Corollary 5.1. Assume B is non-empty, $g : B^{<\omega} \to B$. Then there exists a unique function $f : \omega \to \omega$, satisfying:

$\forall n \in \omega(f(n) = g(f \upharpoonright n))$

Here we arranges that $f \upharpoonright 0 = \emptyset$

I guess we more commonly use this corollary.

Actually we can generalize our recursive definition a little bit on the given function g as the following:

Corollary 5.2. Assume A, P are non-empty. Assume $a : P \to A$ and $g : P \times A \times \omega$. Then there exists a unique function $f : P \times \omega \to A$ satisfying the two following properties:

(1) $\forall p \in P(f(p, 0) = a(p))$.

(2) $\forall p \in P \forall n \in \omega(f(p, n+1) = g(p, f(p, n), n))$.

With such recursive definitions, we can construct $+$-addition, $\cdot$-multiplication, and m^n-exponential map. Details are omitted here.

In the end we discuss two additional parts.

5.1 Cantor-Bernstein Theorem

It's mainly about a seemingly obvious results:

Lemma 5.3. Sandwich Lemma

Assume $C \subseteq B \subseteq A$. If $|A| = |C|$, then $|B| = |A|$

Theorem 5.2. Cantor-Bernstein Theorem

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

5.2 Typical Order on $\omega \times \omega$

Definition 5.3. We define an typical order on the set $\omega \times \omega$:

We say $(a, b) < (c, d)$, iff $(max(a, b) < max(c, d)) \vee (max(a, b) = max(c, d) \wedge a < c) \vee (max(a, b) = max(c, d) \wedge a = c \wedge b < b < d)$.

It has obvious geometric interpretations.

Theorem 5.3. (1) $<$ is a linear order on $\omega \times \omega$.

(2) $<$ is an order on $\omega \times \omega$.

(3) $(m, n) \in \omega \times \omega$, then $\{(a, b) \in \omega \times \omega | (a, b) < (m, n)\}$ is finite.

(4) Given $(m, n) \in \omega \times \omega$, define $g(m, n) = k$ iff $|k| = \{(a, b) \in \omega \times \omega | (a, b) < (m, n)\}$, then g is a order isomorphism from $\omega \times \omega$ to ω.

Proposition 5.2. $|\omega \times \omega| = |\omega|$.

Proposition 5.3. (1) $|\omega^{n+2}| = \omega$, for $n \in \omega$.

(2) $|\omega^{<\omega}| = |\omega|$.

(3) $|[\omega]^{<\omega}| = |\omega|$.

Theorem 5.4. Assume A and B are countable, then $A \cup B$ and $A \times B$ are countable.

Theorem 5.5. A set X is infinite iff $\mathfrak{P}(\mathfrak{P}(X))$ contains a countably infinite subset.

6 Transitive Sets and ZF

In this section we add two more axioms, extent our set theory to ZF system, then we have more properties about sets.

Axiom 8. Axiom Schema of Replacement
$(\forall u \forall v \forall w(\phi(u,v) \wedge \phi(u,w) \rightarrow v = w)) \rightarrow (\forall z \exists y \forall v(v \in y \leftrightarrow \exists u(u \in z \wedge \phi(u,v))))$

In this axiom we can think ϕ as a 'map' from some collection of sets to the collection of all sets. And this axiom tells us that given a set in range of ϕ, its image is also a set.

Definition 6.1. We say a formula $\varphi(u,v)$ is called *functional defining formula* iff
$\forall u \forall v \forall w(\phi(u,v) \wedge \phi(u,w) \rightarrow v = w)$.

With the new axiom, we have a new version of recursive definitions.

Theorem 6.1. Second Recursive Definition Theorem
Assume $\varphi(x_1, x_2, x_3)$ is a formula with three free variables, and
$\forall n \in \omega \forall x \exists y(\varphi(n,x,y))$, and
$\forall n \in \omega \forall u \forall v \forall w(\varphi(n,u,v) \wedge \varphi(n,u,w) \rightarrow v = w)$.
We use this φ to define a *functional* G:
$G(n,x) = y \leftrightarrow \varphi(n,x,y)$.
Let a be a set. Then there exists a unique function satisfying:
1. $dom(f) = \omega$.
2. $f(0) = a$.
3. $\forall n \in \omega(f(n+1) = G(n, f(n)))$.

Corollary 6.1. Given a functional $\varphi(x,y)$ and a set a. Then there exists a unique function f with domain ω satisfying:
(1) $f(0) = a$.
(2) $\forall n \in \omega(\varphi(f(n), f(n+1)))$.

We use results above to get two theorems:

Theorem 6.2.
Generalized Successor Closure.
$\forall u \exists w(u \in w \wedge \forall a \in w \forall b \in w(a \cup \{b\} \in w))$

We use this theorem to prove:

Theorem 6.3. Assume $\alpha \in Ord$, then there exists a limit ordinal γ such that $\alpha \in \gamma$.

Now we show that every set is a subset of some transitive set, and among such sets, there is a minimal one.

Definition 6.2. For every set X, there is a unique set Y satisfying:

1. Y is transitive.
2. $X \subseteq Y$.
3. For every transitive set Z with $X \subseteq Z$, we have $Y \subseteq Z$.

Definition 6.3. Given arbitrary set X, we say Y is a transitive closure of X, denote as $\mathcal{TC}(X)$.

To proceed our discussion, we need last axiom in ZF system.

Axiom 9. Axiom Schema of Foundation
$(\exists x \phi(x, x_1, ..., x_n)) \rightarrow (\exists x(\phi(x, x_1 ..., x_n) \wedge \forall y \in x(\neg\phi(x, x_1, ..., x_n))))$.
Where y is not a free variable of $\phi(x, x_1, ..., x_n)$.

Use this axiom, we have several results.

Proposition 6.1. $\forall x(x \notin x)$.

Proposition 6.2. If A is non-empty, then $\exists x(x \in A \wedge x \cap A = \emptyset)$.

Theorem 6.4. $\in$-induction theorem
Assume $\varphi(u)$ is a formula, then if
(1) $\varphi(\emptyset)$ holds.
(2) For every set x, we have $(\forall y(y \in x \rightarrow \varphi(y)) \rightarrow \varphi(x))$.
Then $\varphi(x)$ holds for every set x.

Theorem 6.5. Rigidity for Transitive Sets
Assume M and N are two transitive sets, and $f : M \rightarrow N$ is a isomorphism, i.e. f is bijective and,
$\forall x \in M \forall y \in M(x \in y \rightarrow f(x) = f(y))$.
Then $M = N$ and f is identical map.

Now we use Axiom 9 to discuss a more general recursive definition.

Definition 6.4. Given functional $\varphi(x, y)$, we can define a *class function* G:
$G(x) = y \leftrightarrow \varphi(x, y)$.

Theorem 6.6. $\in$-Recursive Definition
Given a class function G, there is a unique class function satisfying all the following properties:
For every x, $F(x) = G(F \restriction x)$.

Theorem 6.7. Given a set of ordinals X, there is unique ordinal α and a avalanche map $\pi_X : X \rightarrow \alpha$, which is order isomorphism.

With this theorem, we define:

Definition 6.5. We say α in the above theorem is the *ordinal type* $ot(X)$.

Again and again, we generalize the concept of transitivity.

Definition 6.6. Given a non-empty set A, we say that A is *extensional* iff
$\forall x \in A \forall y \in A(\forall z \in A(z \in x \leftrightarrow z \in y) \to x = y)$.
Or we can write it more simply, $\forall x \in A \forall y \in A(x \cap A = y \cap A \to x = y)$.

Theorem 6.8. Avalanche Theorem
Given A is non-empty and extensional, then exists a unique avalanche map π with a unique transitive set M satisfying:
(i) $\pi : (A, \in) \to (M, \in)$ is a $\in$-isomorphism.
(ii) If $B \subseteq A$ is a transitive set, then $\pi \restriction B$ is an identical map.

7 Transfinite Recursive Definition

Note that we've discussed transfinite induction theorem in Section 5. Now we consider transfinite recursive definition.

Definition 7.1. Given $\alpha \in Ord$ and a set A, we call a map from α to A a α-sequence on A, sometimes denote as $\langle a_\xi : \xi \in \alpha \rangle$.

Definition 7.2. We say G is a class function iff there is a defining formula $\phi(u, v)$ satisfying:
(1) $\forall x \exists y \phi(x, y)$.
(2) $\forall x \forall y \forall z((\phi(x, y) \wedge \phi(x, z)) \to y = z)$.
(3) $\forall x \forall y(G(x) = y \leftrightarrow \phi(x, y))$.

Definition 7.3. We say G is a class sequence iff there is a defining formula $\phi(u, v)$ satisfying:
(1) $\forall x \in Ord \exists y \phi(x, y)$.
(2) $\forall x \in Ord \forall y \forall z((\phi(x, y) \wedge \phi(x, z)) \to y = z)$.
(3) $\forall x \in Ord \forall y(G(x) = y \leftrightarrow \phi(x, y))$.
We sometimes denote as $\langle a_\xi : \xi \in Ord \rangle$

Theorem 7.1. Transfinite Recursive Definition
Given a class function G, then the following formula $\phi_G(u, v)$ is a functional:
$\phi_G(\alpha, x) \leftrightarrow (\alpha \in Ord \wedge \exists f(\sigma(f, \alpha) \wedge \eta_0(f, \alpha) \wedge \eta_1(f, x)))$
Where
(i) $\sigma(f, \alpha) \leftrightarrow f$ is a sequence with length α.
(ii) $\eta_0(f, \alpha) \leftrightarrow \sigma(f, \alpha) \wedge \forall \xi(\xi \in \alpha \to f(\xi) = G(f \restriction \xi))$.
(iii) $\eta_1(f, x) \leftrightarrow x = G(f)$.
And it defines a unique class sequence F satisfying
$\forall \alpha \in Ord(F(\alpha) = G(F \restriction \alpha))$.

As a special case of the above theorem, and a generalization of recursive definition on ω also, we have the following theorem.

Theorem 7.2. Assume θ is a non-zero limit ordinal, and A is a non-empty set. Let $S = A^{<\theta} = \bigcup\{A^\alpha | \alpha \in \theta\}$. Assume $g : S \to A$, then there exists a unique sequence f with length θ, satisfying
$\forall \alpha \in \theta(f(\alpha) = g(f \upharpoonright \alpha))$.

We also have generalized recursive definition with parameters.

Theorem 7.3. Given a binary class function G. Let $\varphi_G(z, x, y)$ iff
(1) Either $x \in Ord$, and there is a sequence t defined on $x + 1$ satisfying:
$(\forall \alpha \in (x + 1)(t(\alpha) = G(z, t \upharpoonright \alpha))) \wedge y = t(x)$.
(2) Or $x \notin Ord$ and $y = \emptyset$.
Then $\varphi_G(z, x, y)$ defines a unique binary class function F satisfying:
$\forall z \forall \alpha \in Ord(F(z, \alpha) = G(z, F_z \upharpoonright \alpha))$.
Note that F_z is define by $F(z, \cdot)$ in the obvious way.

We will consider a complicated case of class functions:

Definition 7.4. Assume G_1, G_2, G_3 are three class functions, then we define another class function G as the following:

$$G(x) = \begin{cases} G_1(\emptyset) & \text{if } x = \emptyset \\ G_2(x(\alpha)) & \text{if } x \text{ is a function defined on } \alpha + 1 \\ G_3(x) & \text{if } x \text{ is a function defined on a non-zero limit ordinal} \\ \emptyset & \text{other cases} \end{cases} \tag{1}$$

We define a formula $\varphi_G(x, y)$ as the following, which is indeed a functional:
$\varphi_G(x, y) \leftrightarrow$ either $x \in Ord$, and there is a sequence t defined on $x + 1$ satisfying: $(\forall \alpha \in (x + 1)(t(\alpha) = G(t \upharpoonright \alpha))) \wedge y = t(x)$, or $x \notin Ord$ and $y = \emptyset$.
Then such functional φ_G defines a class sequence F satisfying:
(1) $F(0) = G_1(\emptyset)$.
(2) $\forall \alpha \in Ord(F(\alpha + 1) = G_2(F(\alpha)))$.
(3) $\forall \alpha \in Ord((0 \in \alpha \wedge \alpha = \bigcup \alpha) \to F(\alpha) = G_3(F \upharpoonright \alpha))$.

Note that in the case (3) in the above theorem, $\alpha = \bigcup \alpha$ says that α is actually a limit ordinal.
Use the theorem above, we define a special kind of sets:

Definition 7.5. We have the following class sequence $\langle V_\alpha : \alpha \in Ord \rangle$:
$V_0 = \emptyset$,
$V_{\alpha+1} = \mathfrak{P}(V_\alpha)$,
$V_\lambda = \bigcup\{V_\alpha | \alpha \in \lambda\}$, where λ is a limit ordinal.
And finally we have a class: $V^* = \bigcup\{V_\alpha | \alpha \in Ord\}$.

Definition 7.6. We define another class $V = \{x | x = x\}$, which is the collection of all sets.

Theorem 7.4. $V^* = V$.

Definition 7.7. Given $x \in V$, we define the rank of x, as the the minimal $\alpha \in Ord$ satisfying $x \in V_{\alpha+1}$.

By the way, the with two more axioms we can answer the question about the ordered set and ordinal numbers:

Theorem 7.5. Representation Theorem
Every Ordered set is order isomorphic to some ordinal.

To prove the theorem above, we need a lemma:

Lemma 7.1. Assume $(W, <)$ is a ordered set. Then we have
$\forall x \in W \exists \alpha \in Ord((W[x], <) \cong (\alpha, \in))$.

Definition 7.8. Then given an ordered set $(W, <)$, we say that α in the above theorem is the *order type* of W, denote as $ot(W, <)$. If W is a set if ordinals, then we simply write as $ot(W)$.

Given an arbitrary set, we are curious about whether it can be ordered or not. The following theorem gives us a 'iff' condition:

Theorem 7.6. Given a set X, then X can be ordered iff it has a bijective map between an ordinal.

7.1 Cardinal Numbers

To consider the cardinal numbers, we start from a special case: natural numbers.

Definition 7.9. Define $Wo(\omega) = \{R \subset \omega \times \omega | (dom(R) \cup rng(R), R)$ is an ordered set$\}$.

By results before, we know that given $R \in Wo(\omega)$, there is a unique ordinal β such that $(dom(R) \cup rng(R), R) \cong (\beta, \in)$.

The next theorem says that there is a large enough ordinal number for $Wo(\omega)$:

Theorem 7.7. There exists a ordinal α such that $\forall R \in Wo(R) \exists \beta \in \alpha(\beta = ot(R))$.

Then by the minimal property on the class of ordinals, we define:

Definition 7.10. $\omega_1 = \omega^+ := min(\{\alpha \in Ord | \forall R \in Wo(R) \exists \beta \in \alpha(\beta = ot(R))\})$

We'll find that ω_1 is truly different from those ordinals smaller than it:

Theorem 7.8. (1) $|\omega| < |\omega_1|$.
(2) If $\omega \in \alpha \in Ord$, and $|\omega| < |\alpha|$, then $\omega_1 \leq \alpha$.
(3) If $\omega \in \alpha \in \omega_1$, then $|\omega| = |\alpha|$, so $\forall \beta \in \omega_1(|\beta| < |\omega_1|)$.

We can generalize those results above.

Definition 7.11. Given ordinal $\alpha \geq \omega$, there exists a ordinal λ:
$\forall R \in Wo(\alpha)\exists\beta \in \lambda(\beta = ot(R))$.

Definition 7.12. Given ordinal $\alpha \geq \omega$, we define $\alpha^+ = min(\{\lambda \in Ord | \forall R \in Wo(\alpha)\exists\beta \in \lambda(\beta = ot(R))\})$.

Theorem 7.9. Given ordinal $\alpha \geq \omega$, then
(1) $|\alpha| < |\alpha^+|$.
(2) If $\alpha \in \gamma \in Ord$, and $|\alpha| < |\gamma|$, then $\alpha^+ \leq \gamma$.
(3) If $\alpha \in \beta \in \alpha^+$, then $|\beta| = |\alpha|$, and $\forall\beta \in \alpha^+(|\beta| \in |\alpha^+|)$.

Then we have a more concrete concept:

Definition 7.13. (1) An ordinal α is said to be *cardinal number*, iff $\forall\beta \in \alpha(|\beta| < |\alpha|)$.
(2) Given arbitrary ordinal α, we define $\mathbf{Card}(\alpha) = min(\{\beta \in Ord||\beta| = |\alpha|\})$, we call it the *cardinality* of α.

What about cardinality of arbitrary set?

Definition 7.14. Given a set X, iff X can be ordered, then we define $\mathbf{Card}(X)$ to be the unique cardinal number that has a bijective map with X.
If X cannot be ordered, then we define its cardinality as the following:
$\mathbf{Card}(X) = \{y \in V_{rank(X)+1}||y| = |X| \wedge \forall z(|z| = |X| \to rank(y) < rank(z))\}$.

The definition for those sets who cannot be ordered are quite strange. In the future, with Axiom of Choice, we know that every set can be ordered, then everything will be fine.
Now we discuss some properties of Cardinals:

Theorem 7.10. (1) If α is an ordinal, then $\mathbf{Card}(\alpha) \in (\alpha + 1)$, and α is a cardinal number, so $\mathbf{Card}(\mathbf{Card}(\alpha)) = \mathbf{Card}(\alpha)$.
(2) An ordinal α is a cardinal iff $\mathbf{Card}(\alpha) = \alpha$.
(3) Every ordinal in $\omega + 1$ is a cardinal.
(4) If α is an infinite cardinal, then it must be a limit ordinal.
(5) If α is an infinite ordinal, then α^+ is a cardinal.

We have some familiar concepts:

Definition 7.15. (1) A cardinal λ is said to be a *successor cardinal* iff $\exists\beta \in \lambda(\beta^+ = \lambda)$.
(2) A cardinal λ is said to be *limit cardinal* iff $\forall\beta \in \lambda(\beta^+ \in \lambda)$.

Proposition 7.1. A ordinal α is a cardinal iff
$\forall R \in Wo(\alpha)((dom(R) \cup rng(R) = \alpha) \to \alpha \leq ot(R))$.

Theorem 7.11. Continuity of Cardinal
Assume X is a non-empty set of cardinals, then $\bigcup X$ is a cardinal. Moreover, iff X has no maximal, then every cardinal in X is less than $\bigcup X$.

Then we define a class sequence of cardinals:

Definition 7.16. (1) $\aleph_0 = \omega_0 = \omega$.
(2) Given $\alpha \in Ord$, $\aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+$.
(3) Given a non-zero limit ordinal λ, $\aleph_\lambda = \omega_\lambda = \bigcup\{\omega_\alpha | \alpha < \lambda\}$.

Lemma 7.2. $\forall\alpha(\alpha \leq \omega_\alpha)$

We are curious about whether $\aleph$ shows all the cardinal, the following theorem and proposition answers such question:

Theorem 7.12. $\aleph_\alpha$ is a cardinal; If λ is a cardinal, then it must be some $\aleph_\alpha$.

Proposition 7.2. Given $\alpha \in Ord$, for every infinite cardinal $\lambda < \omega_\alpha$, there must be an ordinal $\gamma < \alpha$ such that $\lambda = \omega_\gamma$.

Corollary 7.1. A set X can be ordered iff $|X| = |\lambda|$ for some cardinal λ.

In the end, we discuss fixed point of $\aleph$:

Definition 7.17. We say $\gamma \in Ord$ is a fixed point of $\aleph$ iff $\gamma = \aleph_\gamma$.

Theorem 7.13. (1) If $\alpha < \beta \in Ord$, then $\aleph_\alpha < \aleph_\beta$.
(2) Assume $0 < \gamma \in Ord$ is a limit ordinal, then $\forall\beta \in \gamma(\aleph_\beta < \aleph_\gamma)$.
(3) $\forall\alpha \in Ord \exists\gamma \in Ord(\alpha < \gamma \wedge \gamma = \aleph_\gamma)$.

8 Regular Ordinal and Ordinal Arithmetic

In this section we mainly focus on non-zero limit ordinals.

Definition 8.1. Assume $\alpha > 0$ is a limit ordinal.
(1) We say $A \subseteq \alpha$ is *unbounded* iff $\forall\beta \in \alpha\exists\gamma \in A(\beta < \gamma)$.
We say A is *bounded* iff A is not unbounded.
(2) We say A is a *unbounded closed subset* of α iff A is unbounded and $\forall\gamma \in \alpha((\gamma \text{ is a limit ordinal} \wedge A \cap \gamma \text{ is unbounded in } \gamma) \to \gamma \in A)$.
(3) $f : \beta \to \alpha$ is said to be bounded iff $f[\beta]$ is bounded in α.
f is said to be unbounded iff $f[\beta]$ is unbounded in α.
(4) Assume f is bounded in α now, the *supreme* of f is defined by:
$sup(f) = min(\{\gamma \in \alpha | f[\beta] \subseteq \gamma\}) = sup\{f(\gamma) + 1 | \gamma \in \beta\}$.
(5) We say f is (strictly) increasing iff $\forall\delta < \eta < \beta(f(\delta) < f(\eta))$
(6) We say f is increasing and continuous iff f is increasing and for all $\delta \in \beta$, if δ is a limit ordinal, then
$f(\delta) = \bigcup\{f(\eta) | \eta \in \delta\}$.

Definition 8.2. Cofinality
Given a non-zero limit ordinal α, we define its *cofinality* as the following:
$cf(\alpha) = min(ot(A) | A \text{ is an unbounded subset of } \alpha)$

The next theorem tells us that we need consider unbounded and closed subsets only:

Theorem 8.1. $cf(\alpha) = min(ot(A)|A$ is an unbounded and closed subset of $\alpha)$

Theorem 8.2. Given two limit ordinals $\alpha \geq \gamma \geq \omega$, the following two are equivalent:
(1) $\gamma = cf(\alpha)$.
(2) There exists a function from γ to α and for every $\eta < \gamma$, any function from η to α must be bounded.

Theorem 8.3. $cf(cf(\alpha)) = cf(\alpha)$.

Corollary 8.1. Given a non-zero limit ordinal α.
(1) $A \subseteq cf(\alpha)$, then A is unbounded in $cf(\alpha)$ iff $ot(A) = cf(\alpha)$.
(2) $\beta < cf(\alpha)$, and $f : \beta \to cf(\alpha)$, then f is bounded in $cf(\alpha)$.

Theorem 8.4. (1) Assume α is a non-zero limit ordinal, $A \subseteq \alpha$ is unbounded, then there must be an unbounded subset of A with order type $cf(\alpha)$.
(2) Assume α, γ are a non-zero limit ordinals. Assume $f : \gamma \to \alpha$ is non-decreasing, and α is the supreme of f, then $cf(\alpha) = cf(\gamma)$.

Now we discuss some special kinds of ordinals:

Definition 8.3. An limit ordinal $\alpha > 0$ is said to be a *regular ordinal* iff $\alpha = cf(\alpha)$.

Regular ordinals a so special, that they are all cardinals:

Theorem 8.5. All regular ordinals are cardinals.

However, not all cardinals are regular ordinals:

Definition 8.4. We say a cardinal κ a singular cardinal iff $cf(\kappa) < \kappa$. We say a cardinal α a regular cardinal iff $cf(\alpha) = \alpha$.

Proposition 8.1. Every regular ordinal is a regular cardinal.
Assume $\kappa > \omega$ is a regular limit cardinal, then $\kappa = \aleph_\kappa$.

Theorem 8.6. $\kappa \geq \omega$ is a regular cardinal.
(1) If $X \subset \kappa$, and $\mathbf{Card}(X) < \kappa$, then X is bounded in κ.
(2) If $\lambda < \kappa$, $f : \lambda \to \kappa$, then $f[\lambda]$ is bounded in κ.

Theorem 8.7. Assume $\alpha > 0$ is a limit ordinal, then the following are equivalent:
(1) α is a regular cardinal.
(2) Every unbounded subset of α are isomorphic under $\in$, i.e. every unbounded subset of α has order type α.

8.1 Ordinal Arithmetic

We start from some definitions:

Definition 8.5. Given two disjoint linearly ordered sets $(A, <_A)$ and $(B, <_B)$, we define their direct product sum $< = <_A \cup <_B \cup \{(a,b) | a \in A \wedge b \in B\}$

Definition 8.6. Given two linearly ordered sets $(A, <_A)$ and $(B, <_B)$, we define their product $(A \times B, <)$ as:

$(a,b) < (x,y)$ iff $a <_A x \vee (a = x \wedge b <_B y)$.

This is called *vertical dictionary order.*

Definition 8.7. Given two linearly ordered sets $(A, <_A)$ and $(B, <_B)$, we define their product $(A \times B, <)$ as:

$(a,b) < (x,y)$ iff $b <_B y \vee (b = y \wedge a <_A x)$.

This is called *horizontal dictionary order.*

Then we try to use the definitions above to construct addition and multiplication on Ord.

Proposition 8.2. Given two ordinal α, β.

(1) Let $(\alpha \times \{0\}, <_0)$ and $(\beta \times \{1\}, <_2)$ be their products, then the direct sum of those two sets is an ordered set.

(2) Under vertical or horizontal dictionary order, $(\alpha \times \beta, <)$ is an ordered set.

Not, addition:

Definition 8.8. Addition

Given two ordinal α, β we define $\alpha + \beta$ as the unique ordinal that is order isomorphism with $(\alpha \times \{0\} + \beta \times \{1\}, <)$.

Proposition 8.3. We can also 'define' addition recursively:

(1) $\alpha + 0 = \alpha$.

(2) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$

(3) For non-zero limit ordinal β, define $\alpha + \beta = \bigcup\{\alpha + \xi | \xi \in \beta\}$.

Note that addition is 'continuous':

Lemma 8.1. Assume $\gamma > 0$ is a limit ordinal, $\langle \beta_\xi : \xi \in \gamma$ is an increasing sequence of ordinals.

Consider $\beta = \bigcup\{\beta_\xi | \xi \in \gamma\} = \sup\limits_{\xi < \gamma} \beta_\xi$.

Then for every ordinal α, we have $\alpha + \beta = \sup\limits_{\xi < \gamma}(\alpha + \beta_\xi)$.

Addition also have some common properties:

Theorem 8.8. Basic properties of addition:

(1) Associativity: $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(2) Left cancellation law:

(a) $\forall\alpha\forall\beta\forall\gamma(\alpha < \beta \leftrightarrow \gamma + \alpha < \gamma + \beta)$
(b) $\forall\alpha\forall\beta\forall\gamma(\alpha = \beta \leftrightarrow \gamma + \alpha = \gamma + \beta)$.
(3) Right weak order-preserving:
$\forall\alpha\forall\beta\forall\gamma(\alpha < \gamma \to \alpha + \beta \leq \gamma + \beta)$.
(4) Difference Elimination law:
$\forall\alpha\forall\beta(\alpha \leq \beta \to \exists\gamma(\alpha + \gamma = \beta))$, and such γ is unique.

Now we consider multiplication:

Definition 8.9. Given two ordinal α, β we define $\alpha \cdot \beta$ as the unique ordinal that is order isomorphism with the horizontal dictionary order of $(\alpha \times \beta, <)$.

At the same time $\beta \cdot \alpha$ is define with the vertical dictionary order of $(\alpha \times \beta, <)$.

Like addition, we can also define multiplication recursively

Proposition 8.4. Order multiplication can be defined recursively:
(1) $\alpha \cdot 0 = 0$.
(2) $\forall\beta \in Ord(\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha)$
(3) For non-zero limit ordinal β, we have $\alpha \cdot \beta = \bigcup\{\alpha \cdot \xi | \xi \in \beta\}$.

Multiplication has continuity also:

Lemma 8.2. Assume $\gamma > 0$ is a limit ordinal, $\langle\beta_\xi : \xi \in \gamma$ is an increasing sequence of ordinals.

Consider $\beta = \bigcup\{\beta_\xi | \xi \in \gamma\} = \underset{\xi<\gamma}{sup}\beta_\xi$.

Then for every ordinal α, we have
$\alpha \cdot \beta = \underset{\xi<\beta}{sup}(\alpha \cdot \beta_\xi)$

Multiplication has some common properties:

Theorem 8.9. (1) Distributive law: $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.
(2) Associative law: $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.
(3) Left cancellation law:
(a) $\forall\alpha\forall\beta\forall\gamma(\beta \neq 0 \to (\alpha < \gamma \leftrightarrow \beta \cdot \alpha < \beta \cdot \gamma))$.
(b) $\forall\alpha\forall\beta\forall\gamma(\beta \neq 0 \to (\alpha = \gamma \leftrightarrow \beta \cdot \alpha = \beta \cdot \gamma))$.
(4) Right order conservation: $\forall\alpha\forall\beta\forall\gamma(\alpha < \gamma \to \alpha \cdot \beta \leq \gamma \cdot \beta)$.

As in the natural numbers, we can define exponential map recursively also.

Definition 8.10. Exponential map
(1) $\alpha^0 = 1$.
(2) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$.
(3) For non-zero limit ordinal β, $\alpha^\beta = \bigcup\{\alpha^\xi | \xi < \beta\}$.

Basics properties:

Lemma 8.3. (1) $\alpha \leq \beta \to \alpha^\gamma \leq \beta^\gamma$.
(2) $(1 < \gamma \wedge \beta < \gamma) \to \alpha^\gamma < \alpha^\gamma$.
(3) $\beta < \gamma \to \forall k \in \omega(\omega^\beta \cdot k < \omega^\gamma)$.

Exponential map is continuous also:

Lemma 8.4. Assume $\gamma > 0$ is a limit ordinal, $\langle \beta_\xi : \xi \in \gamma$ is an increasing sequence of ordinals.

Consider $\beta = \bigcup\{\beta_\xi | \xi \in \gamma\} = \sup_{\xi<\gamma} \beta_\xi$.

Then for every ordinal α, we have
$\alpha^\beta = \sup_{\xi<\beta}(\alpha^{\beta_\xi})$

Corollary 8.2. (1) $\alpha \mapsto \omega \cdot \alpha$ is a continuous functional.
(2) $\alpha \mapsto \omega^\alpha$ is a continuous functional.

Theorem 8.10. Assume δ is a regular ordinal. Then
$\forall \alpha < \delta \forall \beta < \delta(\alpha + \beta < \delta \wedge \alpha \cdot \beta < \delta \wedge \alpha^\beta < \delta)$.
So when $\delta > \omega$, the following three sets are unbounded closed sets:
$\{\gamma < \delta | \omega + \gamma = \gamma\}$,
$\{\gamma < \delta | \omega \cdot \gamma = \gamma\}$,
$\{\gamma < \delta | \omega^\gamma = \gamma\}$.

Lemma 8.5. (1) If $0 < \alpha \leq \gamma$, then the set $\{\beta | \alpha \cdot \beta \leq \gamma\}$ has maximal.
(2) If $1 < \alpha \leq \gamma$, then the set $\{\beta | \alpha^\beta \leq \gamma\}$ has maximal.

There are more to discuss, but I think we'd better stop here.

9 Uncountable Regular Cardinals

9.1 Filters and Ideals

Definition 9.1. Assume X is a set, a subset of $\mathfrak{P}(X)$ $\mathcal{F}$ is said to be a *filter* on X iff
(1) $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
(2) If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
(3) If $B \subseteq A \subseteq X$ and $B \in \mathcal{F}$, then $A \in \mathcal{F}$.
Moreover $\mathcal{F}$ is said to be a *ultra filter* iff,
(4) If $A \subset X$, then either $X \in \mathcal{F}$ or $(X - A) \in \mathcal{F}$.

Definition 9.2. We have trivial ultra-filter, which is generated by a single element of X:
Assume $a \in X$, then we define a ultra filer:
$\mathcal{F}_a = \{A \subseteq X | a \in A\}$

Now we consider a rather dual concept of filter:

Definition 9.3. Assume X is a set, a subset of $\mathfrak{P}(X)$ $\mathcal{I}$ is said to be a *ideal* on X iff
(1) $X \notin \mathcal{I}$ and $\emptyset \in \mathcal{F}$.
(2) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.
(3) If $B \subseteq A \subseteq X$ and $A \in \mathcal{I}$, then $B \in \mathcal{I}$.
Moreover $\mathcal{I}$ is said to be a *ultra filter* iff,
(4) If $A \subset X$, then either $X \in \mathcal{I}$ or $(X - A) \in \mathcal{I}$.

Definition 9.4. Given a non-empty set X and ideal I on it. A subset A of X is said to be a $\mathcal{I}$-positive measure set iff $A \notin \mathcal{I}$.

We define $\mathcal{I}^+ = \{A \subseteq X | A \notin \mathcal{I}\}$.

Filters and ideals have some familiar examples, but we omit them here.

We have a basic dual property between the two concepts:

Proposition 9.1. Assume X is a set.

(1) If $\mathcal{F}$ is a(n) (ultra) filter on X, then

$\mathcal{I} = \{A \subseteq X | (X - A) \in \mathcal{F}\}$ is an(a) (prime) ideal on X.

(2) If $\mathcal{I}$ is an(a) (prime) ideal on X, then

$\mathcal{F} = \{A \subseteq X | (X - A) \in \mathcal{I}\}$ is a(n) (ultra) filter on X.

9.2 Ordinary Filters and Ideals on Uncountable Regular Cardinals

We restate some familiar concepts here:

Definition 9.5. (1) Given a non-zero limit ordinal α and its subset A, we say A is unbounded in α iff

$\forall \beta \in \alpha \exists \gamma \in A(\beta < \gamma)$.

(2) Given a non-zero limit ordinal α and a non-empty set of ordinals A, we say α is a *limit point* of A iff $A \cap \alpha$ is unbounded in α.

We use A' to denote the set of all limit points of A.

(3) Given a set of ordinals C, we say C is closed iff every limit point of C is in C.

(4) Given a non-zero limit ordinal α and its subset C, we say C is a closed subset of α iff $C \cup \{\alpha\}$ is a closed set. Or equivalently, $\forall \beta \in \alpha$, if β is a limit point of C, then $\beta \in C$.

(5) Given a non-zero limit ordinal α and its subset C, we say C is an unbounded closed subset of α iff it's unbounded in C and closed in C.

The set A' has some good properties(as in topology?):

Proposition 9.2. (1) If α is a limit point of A', then there exists a strictly increasing function from $\alpha \cap A'$ to $\alpha \cap A$.

(2) If α is a limit point of A', then the length of $\alpha \cap A'$ is less or equal than than of $\alpha \cap A$.

(3) A' is a closed set.

The following theorem describes some properties between closed sets and continuous maps:

Theorem 9.1. (1) A set of ordinals A is a closed set iff its natural list is an increasing continuous function defined on a successor ordinal.

(2) A strictly increasing function defined on a non-zero limit ordinal is continuous iff its range is a closed subset of its supreme.

(3) Assume α is non-zero limit ordinal, then A is an unbounded closed subset of α iff the natural list from $ot(A)$ to α is a unbounded continuous function.

Theorem 9.2. Assume $\alpha > 0$ is a limit ordinal, then $cf(\alpha)$ is the length of the shortest unbounded closed subset.

Theorem 9.3. Assume $\delta > \omega$ is a regular ordinal and $C \subseteq \delta$, then the following three are equivalent:

(1) C is a unbounded closed subset of δ.

(2) The natural list of C is an increasing continuous function on δ.

(3) C is range of some increasing and continuous function from δ to δ.

Theorem 9.4. Assume $\alpha > 0$ is a regular ordinal, $f : \alpha \to \alpha$ is a continuous embedding function, then the set $\{\gamma \in \alpha | f(\gamma) = \gamma\}$ is an unbounded closed subset of α.

Proof. Note that by continuity of f, it's not hard to show that the set is closed in α.

Unboundedness is trivial also: $f(\beta) \geq \beta$ for every $\beta < \alpha$. □

Now consider sequence of unbounded closed subsets on an uncountable regular cardinal:

Theorem 9.5. (κ-completeness)

Assume κ is an uncountable regular cardinal, and $\lambda < \kappa$, and $\langle C_\alpha : \alpha < \gamma$ is a sequence of unbounded closed subsets of κ.

Then $\bigcap \{C_\alpha | \alpha < \gamma\}$ is an unbounded closed subset of κ.

Proof. The ideas are the following:

(1) The general idea is to use transfinite induction.

(2) Prove that the intersection of two unbounded closed sets are still closed and unbounded, the harder part is about the unboundedness, which can be shown by constructing a sequence with element one from C_0 and another one from C_1, then C_0 and so on, will this we climb up to the top.

(3) About the limit cardinal in induction process. The harder part is still the unboundedness. We first reconsider another decreasing sets D, by induction hypothesis, they are unbounded and closed, then use transfinite recursive definition, and verify carefully.

Note that we need regularity of κ in (3). □

Note that γ must be strictly smaller than κ, otherwise C may not be unbounded and closed anymore(we may add an example here). Anyway, we can solve such problem by defining another king of 'intersection':

Definition 9.6. Assume κ is an uncountable regular cardinal, and $\langle C_\alpha : \alpha < \kappa$ is a sequence of unbounded closed subsets of κ with length κ.

Then the *diagonal intersection* is defined by:

$\Delta_{\alpha<\kappa} C_\alpha = \{\gamma \in \kappa | \forall \alpha \in \gamma (\gamma \in C_\alpha)\}$.

Theorem 9.6. The diagonal intersection defined above is a unbounded closed subset of κ.

We have a example of *ordinary unbounded closed subset* on a uncountable regular cardinal, but I decided to omit it here.

Definition 9.7. Assume κ is a uncountable regular cardinal, then define
$\mathcal{G}_\kappa = \{X \subseteq \kappa | \exists C \subseteq X(C \text{ is an unbounded closed subset of } \kappa)\}$.
We say $\mathcal{G}_\kappa$ is a ordinary filter on κ.

It has good properties:

Theorem 9.7. $\mathcal{G}_\kappa$ is a κ-complete and regular filter, i.e.
(1) $\mathcal{G}_\kappa$ is a filter.
(2) $\mathcal{G}_\kappa$ is κ-complete, i.e for $\lambda < \kappa$, and $\langle X_\alpha : \alpha < \gamma$ is a sequence of subsets in $\mathcal{G}_\kappa$, we have $\bigcap\{X_\alpha | \alpha < \gamma\} \in \mathcal{G}_\kappa$
(3) $\mathcal{G}_\kappa$ is regular, i.e. closed under diagonal intersection.

Then we use $\mathcal{G}_\kappa$ to construct an ideal on κ:

Definition 9.8. We define $NS_\kappa = \{X \subset \kappa | (\kappa - X) \in \mathcal{G}_\kappa\}$.
We call it the ordinary ideal on κ, or *non-stationary* ideal.

Similar to diagonal intersection, we define diagonal union as the following:

Definition 9.9. Assume κ is a uncountable cardinal, and $\langle X_\alpha : \alpha < \kappa\rangle$ is a sequence of non-stationary sets(complement of a unbounded closed subset). The diagonal union of this sequence is defined as:
$\nabla_{\alpha<\kappa} X_\alpha = \{\gamma \in \kappa | \exists \alpha < \gamma(\gamma \in X_\alpha)\}$.

Corollary 9.1. (1) Assume $\langle X_\alpha : \alpha < \gamma\rangle$, with $\gamma < \kappa$, and X_α are non-stationary, then $\bigcup\limits_{\alpha\in\gamma} X_\alpha$ is a non-stationary set also.
(2) Diagonal union is also non-stationary.

So what about stationary sets? Here is the definition:

Definition 9.10. Assume κ is a uncountable cardinal, we say a subset $X \subseteq \kappa$ is *stationary* iff
$\forall A \in \mathcal{G}_\kappa(X \cap A \neq \emptyset)$.

Theorem 9.8. Choice Function
Assume κ is a uncountable cardinal, and a stationary subset $X \subseteq \kappa$. If f is a choice function on X, then f must be constant on some stationary subset of κ.

Proof. Otherwise for every $\alpha < \kappa$, we have:
$f^{-1}(\alpha) = \{\beta \in X | f(\beta) = \alpha\}$ is NOT a stationary subset. Then there must be some C_α which is unbounded and closed, and $f^{-1}(\alpha) \cap C_\alpha = \emptyset$. Then consider the diagonal intersection C, which is also a unbounded and closed subset.

Since X is a stationary set, $X \cap C \neq$, so take $\beta \in X \cap C \neq \emptyset$, and we know f is a choice function on X, thus $\alpha := f(\beta) \in \beta$. By the definition of diagonal intersection, $\beta \in C_\alpha$, which means that $\beta \in f^{-1}(\alpha) \bigcap C_\alpha$, which is a contradiction.

So there must be some $\alpha \in \kappa$, such that $f^{-1}(\alpha)$ is a stationary subset. $\square$

10 Combination Principle of Regular Cardinals

A special kind of partial order sets are extremely important, they are *trees*.

Definition 10.1. Partially ordered set.

Definition 10.2. Tree

(1) A PO set $(T, <)$ is a *tree* iff

(i) T has a $<$-minimal element, called the *root*.

(ii) If $x \in T$, then $(\{y \in T | y < x\}, <)$ is an ordered set.

We have more definitions as the following:

(2) Elements in T are called *nodes*. If $x < y$, then x is called the *predecessor* of y, y is called the *successor* of x. If $x < y$ and $(x, y) = \{z \in T | x < z < y\}$ is empty, then we say x is *direct predecessor* of y, y is *direct successor* of x. If x has no successors, then we say x is an *ending node*. Two nodes x, y are incomparable iff $(\neg x < y) \vee (\neg y < x)$.

(3) Height of a node x: $h_T(x) = ot(\{y \in T | y < x\}, <)$.

If $h_T(x)$ is a successor ordinal, then we say x is a *successor node*. If $h_T(x)$ is a limit ordinal, then we say x is a *limit ordinal*.

(4) The α-th level of the tree T: $T_\alpha = \{x \in T | h_T(x) = \alpha\}$. The height of T, $ht(T) = min(\{\alpha | T_\alpha = \emptyset\})$.

(5) A branch b of T is a maximum ordered subset of T. The length of b is $l(b) = ot(b, <)$. If $l(b) = ht(T)$, then we say b is a equal height branch.

(6) Given $\alpha \leq ht(T)$, we define:

$T \restriction_\alpha = \bigcup\limits_{\beta < \alpha} T_{beta}$

We say $T \restriction_\alpha$ is a truncated sub-tree with height α of T.

(7) $T' \subset T$ is said to be a sub-tree iff

$\forall x \in T' \forall y \in T (y < x \rightarrow y \in T')$.

We have good properties for some special kinds of trees:

Theorem 10.1. König Lemma

Assume $(T, <)$ is a tree with height ω, and for every $x \in T$, the set of all direct successors of x is finite, then T must have a equal height branch.

We dare ask is such property holds for every regular cardinal? Anyway, we define such property as the following:

Definition 10.3. We say a regular cardinal κ has tree-property iff for every tree $(T, <)$, if its height is κ, and $|T_\alpha| < |\kappa|$ for every α-level, then it has a equal height branch.

Definition 10.4. We say a tree $(T, <)$ is a *Aronszajn* tree iff $ht(T) = \omega_1$, $\forall \alpha < \omega_1 (|T_\alpha| \leq |\omega|)$, however, T has no equal height branch.

Then we know ω_1 is strange enough:

Theorem 10.2. ω has no tree property!(By the way, is ω_1 a regular cardinal?)

10.1 Some Combinatorics Problems

Definition 10.5. Assume r, s, κ, λ are cardinals. We use $k \to (\lambda)^r_s$ to denote the following propositions:

If X is a set with $|X| = \kappa$, $[X]^r = \{A \subseteq X||A| = r\}$, $f : [X]^r \to s$, then there must exists a subset $H \subseteq X$ satisfying the following:

(1) $|H| = \lambda$.

(2) $\exists i \in s(f[[H]^r]) = \{i\}$.

We have the following results:

Theorem 10.3. $\omega \to (\omega)^2_{n+2}$

And:

Theorem 10.4. Ramsey's Theorem

$\omega \to (\omega)^{k+2}_{n+2}$.

With results above, we can prove similar results for finite cardinals, but I decide to omit them all.

The following example says that Ramsay's Theorem cannot be extended to ω_1:

Example 10.1. (1) Assume $X \subset \mathbb{R}$ is a set with $|X| = \aleph_1$. Let $h : X \to \omega_1$ be a bijection. We define $f : [X]^2 \to 2$ as the following:

Given $\{a, b\} \in [X]^2$, we define:

$f(\{a,b\}) = \begin{cases} 0 & \text{if } a < b \wedge h(a) < h(b) \\ 1 & \text{if } a < b \wedge h(a) > h(b) \end{cases}$ Then no subset of X with cardinality $\aleph_1$ which is also 'uniform'.

(2) Actually $(2^{\aleph_0}) \not\to (\omega_1)^2_2$.

The above example is actually a special case of a more general result.

Lemma 10.1. Assume $\lambda \geq \omega$ is a cardinal. Let $A = \{0,1\}^\lambda = 2^\lambda$. Let $f, g \in A$ and $f \neq g$. Then define:

$\delta(f, g) = min(\{\gamma < \lambda | f(\gamma) \neq g(\gamma)\})$, and

$f < g$ iff $f(\delta(f, g)) < g(\delta(f, g))$.

Then this is a linear order on A. And there will not be any strictly increasing or decreasing sequence with length λ^+ under this linear order.

Example 10.2. $2^\lambda \not\to (\lambda^+)^2_2$.

However we have some seemingly quite different theorems:

Theorem 10.5. Assume $\kappa \geq \omega$ is a cardinal, then $(2^\kappa)^+ \to (\kappa^+)^2_2$.

Definition 10.6. $\kappa \to (\kappa, \omega + 1)^2_2$ is defined as the following:

If $f : [\kappa]^2 \to 2$, then either there exists an uniform subset with cardinality κ, or there exists an uniform subset with order type $\omega + 1$.

Theorem 10.6. Assume $\kappa \geq \omega_1$ is regular, then $\kappa \to (\kappa, \omega + 1)^2_2$

More theorems based on the example above are omitted for now.

11 Cardinal Arithmetic and Axiom of Choice

Definition 11.1. Assume κ and λ are two cardinals.

(1) Addition: $\kappa + \lambda = min(\{\gamma||\gamma| = |\kappa \times \{0\} \cup \lambda \times \{1\}|\})$.

(2) Multiplication: $\kappa \cdot \lambda = min(\{\gamma|\exists R \in Wo(\kappa \times \lambda)(dom(R) \cup rng(R) = \kappa \times \lambda \wedge (\kappa \times \lambda, R) \cong (\gamma, \in))\})$

Compared with arithmetic on ordinals, here in cardinals we have better properties:

Lemma 11.1. Assume κ and λ are two cardinals.

(1) $\kappa + \lambda$ and $\kappa \cdot \lambda$ are both cardinals.

(2) Assume X and Y are two sets, $|X| = \kappa$ and $|Y| = \lambda$, and $X \cap Y = \emptyset$, then we have $|\kappa + \lambda| = |X \cup Y|$.

(3) Assume X and Y are two sets, $|X| = \kappa$ and $|Y| = \lambda$, then $|\kappa \cdot \lambda| = |X \times Y|$

Lemma 11.2. Assume κ, λ and μ are three cardinals.

(1) $\kappa + \lambda = \lambda + \kappa$; $\kappa \cdot \lambda = \lambda \cdot \kappa$.

(2) $\kappa + (\lambda + \delta) = (\kappa + \lambda) + \delta$; $\kappa \cdot (\lambda \cdot \delta) = (\kappa \cdot \lambda) \cdot \delta$.

(3) $\kappa \cdot (\lambda + \delta) = \kappa \cdot \lambda + \kappa \cdot \delta$.

To discover more properties of cardinal arithmetic, we consider the order on $Ord \times Ord$.

Definition 11.2. Ordinary Order on $Ord \times Ord$:$<$

Theorem 11.1. (1) $<$ is a linear order.

(2) Every non-empty set of elements in $Ord \times Ord$, it has a minimal element.

(3) $\alpha \times \alpha = \{(\gamma, \delta) \in Ord \times Ord | (\gamma, \delta) < (0, \alpha)\}$.

Definition 11.3. Gödel coding of (α, β): the unique ordinal that is order isomorphic with $\{(\gamma, \delta) \in Ord \times Ord | (\gamma, \delta) < (\alpha, \beta)\}$.

Most properties just follows those in $\omega \times \omega$. We omit them here.

Theorem 11.2. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

Corollary 11.1. $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = max\{\aleph_\alpha, \aleph_\beta\}$.

11.1 Examples of Cardinality of Some Sets

Theorem 11.3. (1) $|2^\omega| = |2^{\omega \times \omega}| = |2^\omega \times 2^\omega|$.

(2) $|\omega^\omega| = |\omega^{\omega \times \omega}| = |\omega^\omega \times \omega^\omega|$

(3) $|(\omega^\omega)^\omega| = |\omega^\omega|$.

Theorem 11.4. $|2^\omega| = |\omega^\omega|$.

Theorem 11.5. (1) If $A \subset 2^\omega$ is countable, then $|2^\omega - A| = |2^\omega|$.

(1) If $A \subset \omega^\omega$ is countable, then $|\omega^\omega - A| = |\omega^\omega|$.

11.2 Axiom of Choice and Some Equivalent Propositions

Definition 11.4. Given a non-empty set S, a *choice function* c is a function from S to $\{\emptyset\} \cup \bigcup S$, satisfying:
-$\forall x \in S(x \neq \emptyset \to c(x) \in x)$.
-If $\emptyset \in S$, then $c(\emptyset) = \emptyset$.

Here we omit some seemingly non-important examples and theorems, except this one:

Theorem 11.6. Given a set X, X can be ordered iff $\mathfrak{P}(X)$ has a choice function.

It shows that choice function is strongly related to ordered set. Now we introduce the axiom of choice:

Axiom 10. Axiom of Choice
Every non-empty set has a choice function

Axiom of Choice has many equivalent propositions:

Theorem 11.7. The following four propositions are equivalent:
(1) Axiom of choice.
(2) Every set can be ordered
(3) Every set has a bijection to some cardinals.
(4) $\forall X \forall Y(|X| \leq |Y| \vee |Y| \leq |X|)$.

Moreover, we have:

Theorem 11.8. The following propositions are equivalent:
(1) Axiom of Choice.
(5) Assume X is a non-empty set, and E is a equivalent relation on X, then there is a choice function on X/E.
(6) Assume I is a non-empty set, and for every $i \in I$, X_i is a non-empty set, then there exists a function defined on I satisfying:
$\forall i \in I(f(i) \in X_i)$,
i.e. the set $\prod\limits_{i \in I} X_i = \{f | f : I \to \bigcup\{X_i | i \in I\} \wedge \forall i \in I(f(i) \in X_i)\}$ is non-empty.

With results above, we can define the product space:

Definition 11.5. Given S non-empty, and every element of S is non-empty, then we can define ΠS as the set of all choice functions on S.

The Axiom of Choice tells us that the above ΠS is non-empty.
In the end, we discuss some direct and also important applications of AC(Axiom of Choice).

Theorem 11.9. AC
(1) Assume $B = \{A_i | i \in \omega\}$ and every A_i is countable, then $\bigcup B$ is countable also.

(2) Assume $\omega \leq \kappa < \aleph_{\alpha+1}$, and $f : \kappa \to \aleph_{\alpha+1}$, then f is bounded in $\aleph_{\alpha+1}$. Thus $\aleph_{\alpha+1}$ is a regular ordinal.

(3) If X is infinite, then it must contains a countably infinite subset.

(4) Given set A, and f with domain A, then $|f[A]| \leq |A|$.

(5) Assume $|S| \leq \aleph_\alpha$, and $\forall A \in S(|A| \leq \aleph)$, then $|\bigcup S| \leq \aleph_\alpha$.

Theorem 11.10. AC

Assume $(A, <)$ is a linear order. Assume $\forall x \in A(|\{y \in A|y \leq x\}| < \aleph_\lambda)$, then $|A| \leq \aleph_\lambda$.

12 Transfinite Cardinal Arithmetic

With Axiom of Choice, we can define addition and multiplication with extremely large number of cardinals.

We start from some lemma:

Lemma 12.1. Assume $\langle A_i : i \in I\rangle$ and $\langle A'_i : i \in I\rangle$ are two groups of pairwise disjoint sequence of sets.

(1) If $|A_i| = |A'_i|$, then $|\bigcup_{i\in I} A_i| = |\bigcup_{i\in I} A'_i|$.

(2) If $|A_i| \leq |A'_i|$, then $|\bigcup_{i\in I} A_i| \leq |\bigcup_{i\in I} A'_i|$.

With this lemma, we define transfinite cardinal addition:

Definition 12.1. Addition

Given a non-empty index set I, and for every $i \in I$ A_i is non-empty, and $|A_i| = \kappa_i$, and we assume $A_i \cap A_j = \emptyset$ for $i \neq j$, then we define:

$\sum\limits_{i\in I} \kappa_i = |\bigcup_{i\in I} A_i|$.

Clearly, the definition does not depend on the choice of A_i's.

We then have a direct way to 'calculate' the sum:

Lemma 12.2. $\sum\limits_{i\in I} \kappa_i = |\bigcup_{i\in I} \kappa_i \times \{i\}|$.

The next theorem tells us that addition is actually quite simple!

Theorem 12.1. Assume λ is a infinite cardinal. $\langle \kappa_\alpha : \alpha < \lambda\rangle$ is a sequence of non-zero cardinals.

Let $\kappa = sup\{\kappa_\alpha|\alpha < \lambda\} = \bigcup\{\kappa_\alpha : \alpha < \lambda\} = min(\{\gamma \in Ord|\forall\alpha \in \lambda(\kappa_\alpha \leq \gamma)\})$.

Then $\sum\limits_{\alpha\in\lambda} \kappa_\alpha = \lambda \cdot \kappa$.

Then we have some quick results, which are omitted here.

We also use transfinite cardinal addition to give a new describe of singular cardinals:

Theorem 12.2. A infinite cardinal κ is a singular cardinal iff there exists a sequence of cardinals $\langle \kappa_\gamma|\gamma < \lambda\rangle$ satisfying:

$\lambda < \kappa \wedge (\forall\gamma < \lambda(\kappa_\gamma < \kappa)) \wedge \kappa = \sum\limits_{\gamma<\lambda} \kappa_\gamma$.

Now, what about multiplications? Exponential? We start from two lemmas:

Lemma 12.3. Assume $\langle A_i : i \in I \rangle$ and $\langle A'_i : i \in I \rangle$ are two sequence of non-empty sets.

(1) If $|A_i| = |A'_i|$, then $|\prod\limits_{i\in I} A_i| = |\prod\limits_{i\in I} A'_i|$.

(2) If $|A_i| \leq |A'_i|$, then $|\prod\limits_{i\in I} A_i| \leq |\prod\limits_{i\in I} A'_i|$.

Lemma 12.4. (1) If $|A| = |C|$ and $|B| = |D|$, then $|A^B| = |C^D|$.

(2) Assume κ, λ are two cardinals, A, B are two sets with $|A| = \kappa$, $|B| = \lambda$, then $\kappa^\lambda = |A^B|$.

Then we define multiplication and exponential:

Definition 12.2. Given a non-empty index set I, and for every $i \in I$ A_i is non-empty, and $|A_i| = \kappa_i$, then we define:
$\prod\limits_{i\in I} \kappa_i = |\prod\limits_{i\in I} A_i|$

Definition 12.3. Exponential

Assume λ and κ are two cardinals. For $\alpha < \lambda$, define κ_α. Then we define:
$\kappa^\lambda = \prod\limits_{\alpha<\lambda} \kappa_\alpha$.

Note that here 2^κ means exponential map, but not the set of all function from κ to 2.

So, we also some quick lemma:

Lemma 12.5. $\prod\limits_{i\in I} \kappa_i = |\prod\limits_{i\in I} \kappa_i \times \{i\}|$

And some examples, we also omit them now.

12.1 Some Inequalities

Theorem 12.3. König Theorem

Assume $\langle \kappa_i, \lambda_i : i \in I \rangle$ are two sequences of cardinals. Assume $\kappa_i < \lambda_i$ for every $i \in I$, then we have inequality:
$\sum\limits_{i\in} \kappa_i < \prod\limits_{i\in I} \lambda_i$.

Then we have an important property about infinite cardinals:

Corollary 12.1. (1) $2^\kappa > \kappa$.

(2) If $\alpha \leq \beta$, then $2^{\aleph_\alpha} \leq 2^{\aleph_\beta}$.

(3) $\alpha \in Ord$, then $cf(2^{\aleph_\alpha}) > \aleph_\alpha$.

Lemma 12.6. If $\alpha \leq \beta$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$

Theorem 12.4. (Hausdorff)
$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$.

12.2 Existence of Non-Trivial Ultra Filter on Infinite Regular Cardinal

We use Zorn's Lemma show answer the question.

Definition 12.4. Assume X is a non-empty set. We say a subset $\mathcal{E}$ of $\mathfrak{P}(X)$ has the property of *finite intersection* iff:

(1) $\emptyset \notin \mathcal{E} \neq \emptyset$.

(2) If $A \in [\mathcal{E}]^{<\omega}$ is non-empty, then $\bigcap A \in \mathcal{E}$.(closed under finite intersections)

Theorem 12.5. (Tarski)

(1) Assume $\mathcal{E}$ is a subset of $\mathfrak{P}(X)$ with finite intersection property, then it can be extended to a ultra filter $\mathcal{F}$ on X.

(2) If $\mathcal{I}$ is an ideal on X, then it can be extended to a prime ideal on X.

Corollary 12.2. Assume κ is a infinite cardinal, then there exists an ultra filter on κ which is disjoint with ideal $[\kappa]^{<\kappa}$.

13 Large Cardinals

Theorem 13.1. (1) $\forall n \in \omega(|V_n| < \omega)$

(2) $|V_\omega| = \aleph_0$

Then we start with some definitions:

Definition 13.1. (1) A infinite cardinal $\aleph_\alpha$ is said to be *strong limit cardinal* iff $\alpha = 0$ or $\forall \beta < \alpha(2^{\aleph_\beta} < \aleph_\alpha)$.

(2) A infinite cardinal κ is said to be *unreachable cardinal* iff κ is a regular strong limit cardinal.

(3) A infinite cardinal κ is said to be *weak unreachable cardinal* iff κ is a regular limit cardinal

We discuss their properties.

The next theorem tells us that strong limit cardinal can be really large:

Theorem 13.2. Assume $\aleph_\alpha$ is a strong limit cardinal, κ and λ are two infinite cardinals, then $\kappa^\lambda < \aleph_\alpha$.

Corollary 13.1. Assume κ is an unreachable cardinal, then the following two sets are unbounded closed subset of κ:

$\{\alpha < \kappa | \alpha = |V_\alpha| = \aleph_\alpha\}$ and $\{\alpha < \kappa | \alpha = \aleph_\alpha\}$

Now recall the class sequence V_α, we then consider another class sequence as the following:

Theorem 13.3. Given a $\alpha \in Ord$, we define $f(\alpha) = |V_\alpha|$. Then

(1) $\forall \alpha (f(\alpha) < f(\alpha + 1))$.

(2) $\forall \alpha < \gamma (f(\alpha) < f(\gamma))$.

(3) Assume $\alpha > 0$ is a limit ordinal, then $f(\alpha) = sup(\{f(\gamma)|\gamma < \alpha\})$. (continuity)

(4) $\forall\alpha \in Ord\exists\gamma \in Ord(\alpha < \gamma \wedge f(\gamma) = |V_\gamma| = \gamma)$.

(5) If $\alpha \in Ord$ and $f(\alpha) = \alpha$, then α is a strong limit ordinal.

Theorem 13.4. Assume κ is an unreachable cardinal, then

(1) If $|X| < \kappa$, then $|\mathfrak{P}(X)| < \kappa$.

(2) If $|S| < \kappa$, and $\forall X \in S(|X| < \kappa)$, then $|\bigcup S| < \kappa$.

(3) If $|X| < \kappa$ and $f : X \to \kappa$, then $sup(f[X]) < \kappa$.

(4) $\forall\alpha < \kappa(\aleph_\alpha < \kappa)$, so $\kappa = \aleph_\kappa$.

(5) $|V_\kappa| = \kappa$.

Note till now we have no idea on the existence of unreachable cardinals. Anyway it won't keep us from discussing them.

As a conclusion, we have:

Proposition 13.1. Assume κ is an unreachable cardinal, then we take κ as our domain, and consider two functions:

$\alpha(\in \kappa) \mapsto |V_\alpha|(\in \kappa)$ and

$\alpha(\in \kappa) \mapsto \aleph_\alpha(\in \kappa)$.

The two functions are both strictly increasing and continuous, has so many fixed point that those fixed points can be collect as a unbounded closed subset of κ.

Then we consider extension of Ramsay's Theorem to large cardinals.

Definition 13.2. We say a uncountable cardinal κ a *weak compact cardinal* iff $\kappa \to (\kappa)^2_2$

Then we have a theorem describing this:

Theorem 13.5. Assume $\kappa > \omega$ is a cardinal, then the following three are equivalent:

(1) κ is a weak compact cardinal.

(2) κ is an unreachable cardinal, and it has tree property.

(3) $\forall 1 < \eta < \kappa(\kappa \to (\kappa)^2_\eta)$

(what a difficult theorem)

Theorem 13.6. Ramsay's Theorem

Given a cardinal $\kappa > \omega$, the formula $\kappa \to (\kappa)^{<\omega}_2$ is equivalent to:

$\forall f : [\kappa]^{<\omega} \to 2\exists H \in [\kappa]^\kappa\forall n \in \omega|f[[H]^{n+1}]| = 1$.

We call a uncountable cardinal κ a *Ramsay cardinal* iff $\kappa \to (\kappa)^{<\omega}_2$.

Then we discuss *measurable cardinals*:

Definition 13.3. We call a cardinal κ a *measurable cardinal* iff there exists a regular ultra filter $\mathcal{U}$ on κ satisfying the following:

(1) $\mathcal{U}$ is non-trivial, i.e. $[\kappa]^{<\kappa} \cap \mathcal{U} = \emptyset$.

(2) $\mathcal{U}$ is κ-complete, i.e. if $\gamma < \kappa$, $\langle A_\alpha : \alpha < \gamma\rangle \in \mathcal{U}^\lambda$, then $\bigcap\limits_{\alpha<\gamma} A_\alpha \in \mathcal{U}$.

(3) $\mathcal{U}$ is regular, i.e. if $A \in \mathcal{U}$, f is a choice function on A, then f must be constant on a $B \in \mathcal{U}$, with $B \subseteq A$.

Then we have three theorems:

Theorem 13.7. (1) If κ is a measurable cardinal, then it's a unreachable cardinal.

(2) If κ is a measurable cardinal, $\mathcal{U}$ is a regular ultra cardinal on κ, then $\{\lambda \in \kappa | \lambda \text{ is an unreachable cardinal}\} \in \mathcal{U}$.

Theorem 13.8. Assume κ is a measurable cardinal and $\mathcal{U}$ is a non-trivial regular and κ-complete ultra filter. If $C \subseteq \kappa$ is an unbounded closed subset of κ, then $C \in \mathcal{U}$.

Theorem 13.9. Assume κ is a measurable cardinal, then $\kappa \to (\kappa)_2^{<\omega}$. Moreover, assume $\mathcal{U}$ is a non-trivial regular and κ-complete ultra filter, $f : [\kappa]^{<\omega} \to 2$, then

$\exists H \in \mathcal{U} \forall n \in \omega(|f[[H]^n]| = 1)$

14 Domain of Set Theory

This time we discuss some huge things, on the domain of set theory.

Definition 14.1. Well-founded relation

Given a set A and relation R on it, we say R is a *well-founded relation* on A iff for every non-empty subset X of A, it has a R-minimal a, satisfying if $b \in X$, then either $\neg bRa$ or $(b, a) \notin R$.

Actually, Axiom of Foundation says that $\in$ is a well-founded relation.

Definition 14.2. Assume A is non-empty and R is a binary relation on A. We say $f : A \to \alpha$ is map to an ordinal α. We say f is a R-order function iff:

$\forall x \in A \forall y \in A((x, y) inR \to f(x) < f(y))$

Theorem 14.1. Assume R is a well-founded relation on non-empty set A, then there must be an ordinal α and a (typical) R-order function $\rho : A \to \alpha$ satisfying:

if $x \in A$, then $\rho(x) = sup(\{\rho(y) + 1 | y \in R \wedge (y, x) \in R\})$(what about there is no y such that $(y, x) \in R$?)

On the other hand, assume R is a binary relation, and there is R-order function, then R is actually well-founded.

Definition 14.3. Given a non-empty set A and a well-founded relation E on A, we say E is *extensional* iff

$\forall x \in A \forall y \in A((\forall z \in A(zEx \leftrightarrow zEy)) \to x = y)$.

We define $ext_E(x) = \{y \in A | (y, x) \in E\}$. We know if E is extensional, then $ext_E(x) = ext_R(y)$ iff $x = y$.

Theorem 14.2. Assume G is a binary class function, and (A, E) is extensional and well-founded, then there is a unique recursive function F on A satisfying: for every $x \in A$

$F(x) = G(x, F \restriction ext_E(x))$.

Theorem 14.3. Representation Theorem

Assume A is non-empty, E on A is extensional and well-founded. Then there must be a map $\pi : (A, E) \to (M, \in)$, where M is transitive, satisfying:

$\forall x \in A \forall y \in A((x, y) \in E \leftrightarrow \pi(x) \in \pi(y))$.

The map π and transitive set M are unique. We call π *avalanche map* of (A, E).

Definition 14.4. Assume κ is a measurable cardinal, and $\mathcal{V}$ is a non-trivial and κ-complete ultra filter on κ. Assume M is a transitive set. For $f, g \in M^\kappa$, we define:

$f =^* g$ iff $\{\alpha \in \kappa | f(\alpha) = g(\alpha)\} \in \mathcal{V}$, and

$[f] \in^* [g]$ if $\{\alpha \in \kappa | f(\alpha) \in g(\alpha)\} \in \mathcal{V}$.

Define $ult(M, \mathcal{V}) = (M/\mathcal{V}, \in^*)$, and call it *ultra power* on ultra filter $\mathcal{V}$

Lemma 14.1. (1) $=^*$ is a equivalent relation.

(2) $\in^*$ is well-defined.

Ah, oh, too hard for me now, and I give up!