Theorem on Constants

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Before the proof of *Theorem on Constants*, we prove two useful lemmas.

Lemma 1. Change Quantifiers

Given $\varphi \in \mathcal{L}$, x_i can substitute x_j in φ, and x_i has no free occurrence in $(\forall x_j \varphi)$, then $\vdash ((\forall x_j \varphi) \to (\forall x_i \varphi(x_j; x_i)))$.

Proof. Since x_i can substitute x_j in φ, so we have,

$\vdash ((\forall x_j \varphi) \to \varphi(x_j; x_i))$, then by generalization theorem:

$\vdash (\forall x_i ((\forall x_j \varphi) \to \varphi(x_j; x_i)))$, then by distribution law:

$\vdash ((\forall x_i ((\forall x_j \varphi) \to \varphi(x_j; x_i))) \to ((\forall x_i (\forall x_j \varphi)) \to (\forall x_i \varphi(x_j; x_i))))$, we get

$\vdash ((\forall x_i (\forall x_j \varphi)) \to (\forall x_i \varphi(x_j; x_i)))$.

Finally, since x_i is not a free variable of $(\forall x_j \varphi)$, we have $\vdash ((\forall x_j \varphi) \to (\forall x_i (\forall x_j \varphi)))$. By Deduction Theorem, we have $(\forall x_j \varphi) \vdash (\forall x_i (\forall x_j \varphi))$, combined with results above, we get $(\forall x_j \varphi) \vdash (\forall x_i \varphi(x_j; x_i))$, again by Deduction Theorem, we have

$\vdash ((\forall x_j \varphi) \to (\forall x_i \varphi(x_j; x_i)))$. □

With Generalization Theorem, we have an even shorter (and perfect) proof.

Proof. Since x_i can substitute x_j in φ, so we have,

$\vdash ((\forall x_j \varphi) \to \varphi(x_j; x_i))$, then by Deduction Theorem, we have $\{(\forall x_j \varphi)\} \vdash \varphi(x_j; x_i)$.

Since x_i is not a free variable of $(\forall x_j \varphi)$, by Generalization Theorem, we have $\{(\forall x_j \varphi)\} \vdash (\forall x_i \varphi(x_j; x_i))$.

Again, by Deduction Theorem, we have $\vdash ((\forall x_j \varphi) \to (\forall x_i \varphi(x_j; x_i)))$. □

Lemma 2. Lemma on Constants

Given $\Gamma \subseteq \mathcal{L}$, and c_i occurs nowhere in Γ. If $\langle B_1, ..., B_n \rangle$ is a Γ-proof, and x_j appears nowhere in Γ. Then $\langle B_1(c_i; x_j), ..., B_n(c_i; x_j) \rangle$ is still a Γ-proof.

Proof. (1) Note that $(A \to B)(c_i; x_j)$ is just $(A(c_i; x_j) \to B(c_i; x_j))$, $(\neg A)(c_i; x_j)$ is just $(\neg A(c_i; x_j))$; If $x_j \neq x_k$, then $(\forall x_k A)(c_i; x_j)$ is just $(\forall x_k A(c_i; x_j))$.

(2) If $\varphi \in \Gamma$, and c_j occurs nowhere in it, then $\varphi(c_i; x_j)$ is just φ, so $\varphi(c_i; x_j) \in \Gamma$.

(3) If $\psi \in \mathbb{L}$, and x_j occurs nowhere in it, then we have to show that $\psi(c_i; x_j)$ is still in $\mathbb{L}$.

To show this assumption, we have to use the minimal closure property of $\mathbb{L}$:

-If ψ is one of the tautologies, then by (1), $\psi(c_i; x_j)$ is also one of the tautologies, thus $\psi(c_i; x_j) \in \mathbb{L}$.

-If there is a formula φ, variable x_k, term τ, and τ can substitute x_k in φ, and ψ is just $((\forall x_k\varphi) \to \varphi(x_k;\tau))$. Then we know that $x_k \neq x_j$, x_j occurs nowhere in φ, thus for formula $\varphi(c_i;x_j)$, τ can still substitute x_k in it, then by (1), $\psi(c_i;x_j)$ is just $((\forall x_k\varphi(c_i;x_j)) \to \varphi(c_i;x_j)(x_k;\tau))$, so it's in $\mathbb{L}$.

-If A, B are two formulas, x_k is a variable, and ψ is $((\forall x_k(A \to B)) \to ((\forall x_k A) \to (\forall x_k B)))$, so $x_k \neq x_j$ and x_j occurs nowhere in A or B. Then by (1), $\psi(c_i;x_j)$ is just $((\forall x_k(A(c_i;x_j) \to B(c_i;x_j))) \to ((\forall x_k A(c_i;x_j)) \to (\forall x_k B(c_i;x_j))))$, so it's in $\mathbb{L}$.

-If there is formula φ, a variable x_k which is not a free variable of φ, and ψ is $(\varphi \to (\forall x_k\varphi))$. We know $x_k \neq x_j$, x_j appears nowhere in φ, then by (1), we know that $\psi(c_i;x_j)$ is just $(\varphi(c_i;x_j) \to (\forall x_k\varphi(c_i;x_j)))$, thus it's in $\mathbb{L}$.

-If ψ is $x_k \hat{=} x_k$, then $\psi(c_i;x_j)$ is still $x_k \hat{=} x_k$, thus in $\mathbb{L}$.

-If there are two formulas A and B, two variables x_k and x_l, and x_l can substitute x_k in both A and B, and after substitution, two formulas are identical, and our ψ is $((x_l \hat{=} x_k) \to (A \to B))$. Then we know $x_j \neq x_k$, $x_j \neq x_l$, x_j occurs nowhere in A or B, and x_l can substitute x_k in $A(c_i;x_j)$ and $B(c_i;x_j)$, and after substitution, two formulas are identical, then by (1) we know $\psi(c_i;x_j)$ is just $((x_l \hat{=} x_k) \to (A(c_i;x_j) \to B(c_i;x_j)))$, thus it's in $\mathbb{L}$.

We can see from above that all those 6 properties in the definition of $\mathbb{L}$ are constructive, telling us at least what formulas are in $\mathbb{L}$. With the results above, we can prove by induction on the length of formulas: If $\psi \in \mathbb{L}$ and x_j occurs nowhere in ψ, then $\psi(c_i;x_j) \in \mathbb{L}$:

Base step: The shortest formula satisfying all conditions is $(x_k \hat{=} x_k)$, and we've already verified this case.

Induction step: Assume all formulas with length less or equal than $n (n \geq 5)$ satisfies the proposition we want to prove, then for every formula ψ with length $(n+1)$, we need to verify all 7 cases in the definition of $\mathbb{L}$, and 6 of them are already verified above, so we only need to check the case of the generalization law:

Assume there is a formula $\varphi \in \mathbb{L}$ and ψ is just $(\forall x_k\varphi)$. Then the length of φ is less than n, thus by induction hypothesis, $\varphi(c_i;x_j) \in \mathbb{L}$, then $(\forall x_j\varphi(c_i;x_j)) \in \mathbb{L}$ also. Since $x_k \neq x_j$, $\psi(c_i;x_j)$ is just $(\forall x_j\varphi(c_i;x_j))$, thus it's in $\mathbb{L}$.

By induction, we finally proved that if $\psi \in \mathbb{L}$ and x_j occurs nowhere in ψ, then $\psi(c_i;x_j) \in \mathbb{L}$.

Now we go back to the proof of this lemma. For $1 \leq k \leq n$,

If $B_k \in \Gamma \cup \mathbb{L}$, then by (2) and (3), we know $B_k(c_i;x_j) \in \Gamma \cup \mathbb{L}$ also.

If there are $1 \leq s, t < k$ and B_t is just $(B_s \to B_k)$, then by (1), we know $B_t(c_i;x_j)$ is just $(B_s(c_i;x_j) \to B_k(c_i;x_j))$.

So $\langle B_1(c_i;x_j), ..., B_n(c_i;x_j)\rangle$ is still a Γ-proof. $\square$

So we can finally prove Theorem on Constants:

Theorem 1. Theorem on Constants

Assume (1) $\Gamma \vdash \varphi$, (2) c_i occurs nowhere in any Γ and (3) x_j has no free occurrence in φ and x_j can substitute c_i in φ.

Then $\Gamma \vdash (\forall x_j \varphi(c_i; x_j))$, and $(\forall x_j \varphi(c_i; x_j))$ is proved by a sequence $\langle B_1, ..., B_n \rangle$ satisfying

1. For $1 \le m \le n$, c_i doesn't occur in B_m.
2. For $1 \le m < n$, if c_k appears in B_m, then
(a) either c_k occurs in $(\forall x_j \varphi(c_i; x_j))$,
(b) or c_k occurs in some formula of Γ.

Proof. We first show $\Gamma \vdash (\forall x_j \varphi(c_i; x_j))$.

$\Gamma \vdash \varphi$ implies that there is a Γ-proof $\langle B_1, ..., B_n \rangle$. Since proof is finite, we take k large enough such that x_k occurs nowhere in the proof $\langle B_1, ..., B_n \rangle$.

Define $\Gamma_0 := \{B_1, ..., B_n\} \cap \Gamma$, then we know x_k appears nowhere in Γ_0. Also, $\langle B_1, ..., B_n \rangle$ is a Γ_0-proof, c_i occurs nowhere in Γ_0, then $\langle B_1(c_i; x_k), ..., B_n(c_i; x_k) \rangle$ is still a Γ_0-proof, thus $\Gamma_0 \vdash \varphi(c_i; x_k)$.

Since x_k occurs nowhere in Γ_0, by Generalization Theorem, we know $\Gamma_0 \vdash (\forall x_k \varphi(c_i; x_k))$.

Since every occurrence of x_k in $\varphi(c_i; x_k)$ happens at the same place of c_i's occurrence in φ, and x_j can substitute c_i means that every occurrence of c_i is not in range of any $\forall x_j$, so now x_j can substitute x_k in $\varphi(c_i; x_k)$. Also, since x_j has no free occurrence in φ, it also has no free occurrence in $(\forall x_k \varphi(c_i; x_k))$, so by Change Quantifiers Lemma, we know $\Gamma_0 \vdash ((\forall x_k \varphi(c_i; x_k)) \to (\forall x_j \varphi(c_i; x_k)(x_k; x_j)))$, and it's not hard to find out that $\varphi(c_i; x_k)(x_k; x_j)$ is just $\varphi(c_i; x_j)$, so combined with results above, we have $\Gamma_0 \vdash (\forall x_j \varphi(c_i; x_j))$. Therefore, $\Gamma \vdash (\forall x_j \varphi(c_i; x_j))$.

Now we prove the second result. $\Gamma \vdash (\forall x_j \varphi(c_i; x_j))$ implies that there is a Γ-proof $\langle C_1, ..., C_n \rangle$, with $C_n = \forall x_j \varphi(c_i; x_j)$.

First, we take k_0 large enough such that x_{k_0} occurs nowhere in the proof $\langle C_1, ..., C_n \rangle$. We know c_i occurs nowhere in Γ, x_{k_0} occurs nowhere in the proof, so by Lemma on Constants, $\langle C_1(c_i; x_k), ..., C_n(c_i; x_{k_0}) \rangle$ is also Γ-proof, it proves $(\forall x_j \varphi(c_i; x_j))(c_i; x_{k_0})$, which is just $(\forall x_j \varphi(c_i; x_j))$, since there is no c_i for substitution. So we find a proof of $(\forall x_j \varphi(c_i; x_j))$ with no c_i in it. We still call the proof $\langle C_1, ..., C_n \rangle$

Second, for every m, $1 \le m \le n$, if c_l occurs in C_m, and it occurs nowhere in $(\forall x_j \varphi(c_i; x_j))$ or some formula in Γ, then we do the 'large-enough-k_m' trick again, get a new proof of $(\forall x_j \varphi(c_i; x_j))(c_l; x_{k_m})$(we can do this since c_l occurs nowhere in Γ), since c_l occurs nowhere in $(\forall x_j \varphi(c_i; x_j))$, $(\forall x_j \varphi(c_i; x_j))(c_l; x_{k_m})$ is just $(\forall x_j \varphi(c_i; x_j))$. By recursive construction, we finally get the proof we want. $\square$