Extension to Maximally Consistent Set with Henkin Property Theorem

kankanray

Theorem 1. Extension Theorem

Assume Γ is consistent, and there are infinitely many constant element symbols that occurs nowhere in Γ, then Γ must a subset of some maximally consistent set with Henkin property.

Proof. The main idea is to construct a proper sequence of all formulas in $\mathcal{L}$, go through every of them and add suitable ones into Γ, to get a maximally consistent set with Henkin property.

We first construct a sequence of formulas with properties we need.

Since there are countably infinitely many constant elements symbols that appears nowhere in Γ, then we use a sequence $(n_i)_{i\in\mathbb{N}}$ to enumerate them: $c_{n_0}, ..., c_{n_i}, ...$

Also, $\mathcal{L}$ is countably infinite, so we list formulas in it as $\varphi'_0, ..., \varphi'_i, ...$

Given $i \in \mathbb{N}$, consider formula φ'_i. Since there are only finite constant element symbols in it, there must be some integer j such that if c_{n_k} occurs in φ'_i, then actually $k < j$. We let $h(i)$ to be the smallest j(such j exists, by the basic property of natural numbers). So we've defined a function h from $\mathbb{N}$ to itself.

Now we use h to define another function recursively. Let $g(0) = h(0)$, and $g(n+1) = max(\{g(n)+1, h(n+1)\})$. Then we know that g is strictly increasing.

Then we define a new sequence of formulas. For every $i \in \mathbb{N}$, we define $\theta_{g(i)}$ as φ'. For arbitrary natural number m, iff $g(i) + 1 \leq m < g(i + 1)$ or $1 \leq (m + 1) \leq g(0)$, then we define θ_m as $(x_m \hat{=} x_m)$(harmless formula). We know all formulas in $\mathcal{L}$ appears in θ at least once.

What is new for θ? Note that given a natural number k, if $k \geq m$, then it occurs nowhere in θ_m. Why? The only case that c_k may occur in θ_m is when $m = g(i)$ for some natural number i. We know that $g(i) \geq h(i)$, so by definition of $h(i)$, we know that if c_{n_k} occurs in φ'_i, which is just θ_m, it must be $k < h(i)$. So for those $k \geq m = g(i) \geq h(i)$, c_{n_k} occurs nowhere in θ_m.

With θ, we define a new sequence of formulas φ, where φ_{2m} and φ_{2m+1} are both θ_m. Then we know φ has two good properties:

(1) Every formula in $\mathcal{L}$ appears in φ at least twice.

(2) For every integer $i \in \mathbb{N}$, and $k \geq i$, then c_{n_k} occurs nowhere in φ_i.

We discuss (2): The only possibility that c_{n_k} occurs in φ_i is when φ_i is just $\theta_{g(j)}$. In this case we know $i = 2g(j)$ or $i = 2g(j) + 1$, then we know that

$k \geq 2g(j) \geq g(j)$, so by properties of θ, we know c_{n_k} occurs nowhere in $\theta_{g(j)}$, which is just φ_i. So (2) follows.

Then we extend Γ with sequence φ.

We definite a sequence of sets recursively:

(1) $\Sigma_0 = \Gamma$.

...

(i+1) Assume Σ_i is already defined, then consider the following cases:

(a) If $\varphi_i \notin \Sigma_i$ and $\Sigma_i \cup \{\varphi_i\}$ is consistent, then let $\Sigma_{i+1} = \Sigma \cup \{\varphi_i\}$.

(b) If $\varphi_i \in \Sigma_i$ and φ_i is $(\exists x_j \psi)$. Then let $\Sigma_{i+1} = \Sigma_i \cup \{\psi_i(x_j; c_{n_i})\}$

(c) If it's not case (a) or (b), then let $\Sigma_{i+1} = \Sigma_i$.

After the process above is 'done', we define $\Sigma = \bigcup\{\Sigma_i | i \in \mathbb{N}\}$. We verify that Σ is maximally consistent, has Henkin property, and $\Gamma \subseteq \Sigma$.

Clearly $\Sigma_i \subseteq \Sigma_{i+1}$. Also, for every integer $i \in \mathbb{N}$, and $k \geq i$, then c_{n_k} occurs nowhere in Σ_i.

-Consistency: We prove by induction. $\Sigma_0 = \Gamma$ is consistent. Assume Γ_i is consistent. If Γ_{i+1} is inconsistent, then the only problem would be in case (b), when $\Sigma_{i+1} = \Sigma_i \cup \{\psi_i(x_j; c_{n_i})\}$ is inconsistent. So we have $\Sigma_i \vdash (\neg\psi_i(x_j; c_{n_i}))$. Since c_{n_i} appears nowhere in Σ_i, x_j has no free occurrence in $(\neg\psi(x_j; c_{n_i}))$ and x_j can substitute c_{n_i} in it so by Theorem on Constants, we know $\Sigma_i \vdash (\forall x_j(\neg\psi(x_j; c_{n_i})(c_{n_i}; x_j)))$ which is just $\Sigma_i \vdash (\forall x_j(\neg\psi))$.

But we know that $(\exists x_j \psi) \in \Sigma_i$, so $\Sigma_i \vdash (\exists x_j \psi)$, which means Σ_i is inconsistent, contradiction, we Σ_i must be consistent.

By induction, we know Σ_i are consistent for every i, then it's not hard to show that Σ is consistent also (by finiteness of proofs).

-Maximally consistency: For every formula φ, it must appear as φ_i for some i in the sequence. Then at step (i+1) in the recursive construction, if it's case (a) or (b), we know that $\varphi \in \Sigma_{i+1} \subseteq \Sigma$; if it's case (c), then we know either $\Sigma_i \cup \{\varphi\}$ is inconsistent, or $\varphi \in \Sigma_i$ but is not $(\exists x_j \psi)$, so we know either $\Sigma \cup \{\varphi\}$ is inconsistent, or $\varphi \in \Sigma$, so Σ is maximally consistent.

-Henkin property: For every $\varphi = (\exists x_j \psi) \in \Sigma$, it appears as φ_k and φ_l in the sequence with $k < l$. We know $\Sigma_k \cup \{\varphi\}$ is consistent, so after step (k+1) we know that $\varphi \in \Sigma_{k+1}$. So in step (l+1), it must be case (b) so by construction $\psi(x_j; c_{n_l}) \in \Sigma_{i+1} \subseteq \Sigma$. Thus Σ has Henkin property.

-$\Gamma \subseteq \Sigma_0 \subseteq \Sigma$.

As a conclusion, Σ is maximally consistent, with Henkin property and $\Gamma \subseteq \Sigma$. $\square$