First order Logic

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1 Propositional Logic

1.1 Syntax

We have logic symbols $\neg$ and $\rightarrow$, (and).

We also have a set of proposition letters $A_0, A_1, \ldots$. We often assume we have countably many proposition letters.

We consider the set of all finite sequences of proposition letters and logic symbols, and select some suitable ones from them.

Definition 1.1. A a well-formed formula(or sentence) φ are defined by:

$\varphi ::= A_n|(\neg\psi)|(\psi \rightarrow \phi)$.

Denote $\mathcal{L}_0$ as the set of all well-formed formulas.

We know $\mathcal{L}$ is the smallest set that is closed under negation and implication, and contains all proposition letters.

We have several lemmas:

Lemma 1.1. For every $\varphi \in \mathcal{L}_0$, φ is exactly one of the three cases:

(1) φ is a proposition letter A_n,

(2) There is a $\psi \in \mathcal{L}_0$, such that $\varphi = (\neg\psi)$,

(3) There are $\psi \in \mathcal{L}_0$ and $\phi \in \mathcal{L}_0$ such that φ is $(\psi \rightarrow \phi)$

This one is obvious, just o through the definition of $\mathcal{L}_0$.

The following two lemmas strengthen our understanding of formulas in $\mathcal{L}_0$:

Lemma 1.2. For sentences $\varphi \in \mathcal{L}_0$, all real prefix of φ is not a well-formed formula.

Here, φ is a sequence of symbols and letters, essentially, so real prefix of φ is a sub-sequence, start from the head of φ, with no one is left in the middle, but shorter than φ.

This lemma can be proved by induction on complexity of formulas.

Then we obtain a useful theorem:

Theorem 1.1. For every $\varphi \in \mathcal{L}_0$, φ is exactly one of the three cases:

(1) φ is a unique proposition letter A_n,

(2) There is a unique $\psi \in \mathcal{L}_0$, such that $\varphi = (\neg\psi)$,

(3) There are unique $\psi \in \mathcal{L}_0$ and $\phi \in \mathcal{L}_0$ such that φ is $(\psi \to \phi)$

1.2 Semantics

After given the syntax of propositional logic, we need semantics.

We start from the valuation on proposition letters:

Definition 1.2. A *valuation* ν is a map from the set of proposition letters to the set $\{0, 1\}$. (1 means true, and 0 means false)

To extend the valuation to all formulas in $\mathcal{L}_0$, we need the following 2 functions:

Definition 1.3. $H_\neg : \{0, 1\} \to \{0, 1\}$, with $H_\neg(0) = 1$ and $H_\neg(1) = 0$.

$H_\to : \{0, 1\} \times \{0, 1\} \to \{0, 1\}$, with $H_\to(0, 0) = H_\to(0, 1) = H_\to(1, 1) = 1$, and $H_\to(1, 0) = 0$.

Then we have an obvious theorem:

Theorem 1.2. Given valuation ν on proposition letters, there is a unique extension $\bar{\nu}$ on $\mathcal{L}_0$, satisfying:

$\bar{\nu}(A_n) = \nu(A_n)$, for all natural number n,

$\bar{\nu}((\neg\psi)) = H_\neg(\bar{\nu}(\psi))$, and

$\bar{\nu}((\psi \to \phi)) = H_\to(\bar{\nu}(\psi), \bar{\nu}(\phi))$, for all formulas ψ and ϕ.

For some specific formulas, we need not know valuations on all proposition letters. We need only the letters that appear in those formulas. We have a theorem describing this:

Theorem 1.3. Given $\varphi \in \mathcal{L}_0$, and two valuation function ν and μ. Suppose for all proposition letters A_n that appears in φ, we have $\nu(A_n) = \mu(A_n)$. Then $\bar{\nu}(\varphi) = \bar{\mu}(\varphi)$.

We denote the relation $\Gamma \models \varphi$.

This can be by induction easily.

More corollaries on valuation and substitution are omitted, since they are quite obvious.

We have some basic concepts:

Definition 1.4. (1) A valuation ν *satisfies* a formula φ iff $\bar{\nu}(\varphi) = 1$.

(2) A valuation ν *satisfies* a set of formulas Γ iff for all formulas φ in Γ, we have $\bar{\nu}(\varphi) = 1$.

(3) A formula φ is said to be *satisfiable* iff there is a valuation satisfies it.

(4) A set of formulas Γ is said to be *satisfiable* iff there is a valuation satisfies it.

(5) A formula is called a *tautology* iff all valuation satisfies it.

(6) A formula is called a *contradiction* iff no valuation satisfies it.

Definition 1.5. Given a set of formulas Γ, and a formula φ, we say φ is a logical consequence iff for every valuation ν, if ν satisfies Γ, then ν satisfies ν also.

Definition 1.6. Two formulas φ and ψ are said to be equivalent iff $\{\varphi\} \models \psi$ and $\{\psi\} \models \varphi$.

Besides $\neg$ and $\rightarrow$, we often use *conjunction* $\wedge$ and *disjunction* $\vee$, and *equality* $\leftrightarrow$. They can all be defined with $\neg$ and $\rightarrow$:

Definition 1.7. $(\psi \wedge \phi) := \neg((\psi \rightarrow (\neg\phi))$,

$(\psi \vee \phi) := ((\neg\psi) \rightarrow \phi)$,

$(\psi \leftrightarrow \phi) : ((\psi \rightarrow \phi) \wedge (\phi \rightarrow \psi))$.

There valuation are quite obvious, thus omitted.