Equality Theorem

kankanray

Theorem 1. Equality Theorem

Assume $\varphi \in \mathcal{L}$ has no quantifiers, $\sigma_1, ..., \sigma_n$ and $\tau_1, ..., \tau_n$ are two collection of terms. Given variables $x_{m_1}, ..., x_{m_n}$, if each of them occurs nowhere in σ_i, τ_i for every $1 \leq i \leq n$, then we have

$\{\tau_i \hat{=} \sigma_i | 1 \leq i \leq n\} \cup \{\varphi(x_{m_1}, ..., x_{m_n}; \tau_1, ..., \tau_n)\} \vdash \varphi(x_{m_1}, ..., x_{m_n}; \sigma_1, ..., \sigma_n)$

Proof.

(1) Assume τ and σ are terms, variable x_m appears nowhere in τ or σ, and formula φ has no quantifiers, then we show that $\vdash ((\tau = \sigma) \to (\varphi(x_m; \tau) \to \varphi(x_m; \sigma)))$.

Take k large enough such that x_k appears nowhere in φ, then x_k can substitute x_m in φ. So consider two formulas φ and $\varphi(x_m; x_k)$. Note that x_k can substitute x_m in both two formulas, and after substitution, two formulas are identical. So we have:

$((x_k \hat{=} x_m) \to (\varphi(x_m; x_k) \to \varphi)) \in \mathbb{L}$.

Then use generalization law, we have:

$(\forall x_k (\forall x_m ((x_k \hat{=} x_m) \to (\varphi(x_m; x_k) \to \varphi)))) \in \mathbb{L}$.

Note that τ can substitute x_k in $(\forall x_m ((x_k \hat{=} x_m) \to (\varphi(x_m; x_k) \to \varphi)))$, so use specialization law, we get $\vdash (\forall x_m ((\tau \hat{=} x_m) \to (\varphi(x_m; \tau) \to \varphi)))$.

Again, since σ can substitute x_m in $((\tau \hat{=} x_m) \to (\varphi(x_m; \tau) \to \varphi))$, so use specialization law, we get $\vdash ((\tau \hat{=} \sigma) \to (\varphi(x_m; \tau) \to \varphi(x_m; \sigma)))$.

Now we use induction on the index i, $1 \leq i \leq n$.

Base step: By (1), we have $\vdash ((\tau_1 \hat{=} \sigma_1) \to (\varphi(x_{m_1}; \tau_1) \to \varphi(x_{m_1}; \sigma_1)))$. Use Deduction Theorem twice, we have $\{(\tau_1 \hat{=} \sigma_1)\} \cup \{\varphi(x_{m_1}; \tau_1)\} \vdash \varphi(x_{m_1}; \sigma_1)$.

Induction step: Assume $\{\tau_j \hat{=} \sigma_j | 1 \leq j \leq i\} \cup \{\varphi(x_{m_1}, ..., x_{m_i}; \tau_1, ..., \tau_i)\} \vdash \varphi(x_{m_1}, ..., x_{m_i}; \sigma_1, ..., \sigma_i)$, for $1 \leq i < n$. Then by (1) again, we have $\vdash ((\tau_{i+1} \hat{=} \sigma_{i+1}) \to (\varphi(x_{m_{i+1}}; \tau_{i+1}) \to \varphi(x_{m_{i+1}}; \sigma_{i+1})))$, combined with results above and Deduction Theorem, we have $\{\tau_j \hat{=} \sigma_j | 1 \leq j \leq (i+1)\} \cup \{\varphi(x_{m_1}, ..., x_{m_{i+1}}; \tau_1, ..., \tau_{i+1})\} \vdash \varphi(x_{m_1}, ..., x_{m_{i+1}}; \sigma_1, ..., \sigma_{i+1})$.

By induction, we proved our theorem. $\square$