Set Theory

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1 Set Theory Language

We sloppily introduce the following. For more details you may refer to first order logic.

Definition 1.1.
We have logical symbols: $\neg$, $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$, $\exists$, and two brackets (,).
Equation symbol $=$.
Variable symbols: x, y, z,..., x_0, x_1, ... and so on.
Binary predicate symbol $\in$.

Then the well-formed formulas in set theory:

Definition 1.2.
(1) Basic formulas : $x \in y$.
(2) Formulas:
-All basic formulas,
-If φ is a formula, then $(\neg\varphi)$ is a formula,
-If ψ and ϕ are formulas, then $(\psi \rightarrow \phi)$ is a formula,
-$\wedge$, $\vee$ and $\leftrightarrow$ are defined similarly,
-If φ is a formula, x is a variable symbol, then $\forall x\varphi$ is a formula; $\exists$ is defined similarly.
-All formulas can only be constructed by the above methods.

2 Zermelo Set Theory

In this part we consider seven axioms, and some basic properties.

2.1 First 6 axioms

Axiom 1. Axiom of extensionality
$((x = y) \leftrightarrow (\forall z(z \in x \leftrightarrow z \in y)))$

Actually, the axiom above tells us what does 'equal' means in set theory.
Then it's easy to verify the following proposition:

Proposition 2.1.
(1) $x = x$,
(2) If $x = y$, then $y = x$,
(3) If $x = y$ and $y = z$, then $x = z$.

Axiom 2. Existence of sets
$(\exists x(\exists y(x \in y)))$

We need Axiom 2, since we are not going to play with nothing.
We also need more axioms to get various sets:

Axiom 3. Axiom of pair
$\forall x \forall y \exists u(\forall z(z \in u \leftrightarrow (z = x \vee z = y)))$
We often write such u as $\{x, y\}$.

Axiom 4. Axiom of union
$\forall x \exists y \forall z(z \in y \leftrightarrow \exists u(u \in x \wedge z \in u))$
We often write such y as $\bigcup x$.

We define the concept of 'subset':
$x \subseteq y \leftrightarrow \forall z(z \in x \to z \in y)$
We say $x \subset y$ when $x \subseteq y$ and $x \neq y$.

Axiom 5. Axiom of the power set
$\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$
We often write such y as $\mathfrak{P}(x) = \{z | z \subseteq x\}$

Axiom 6. Axiom schema of separation
Given an arbitrary sentence with one free variable $\varphi(x)$, describing some separation properties, we have
$\forall x \exists y \forall z(z \in y \leftrightarrow (z \in x \wedge \varphi(z)))$
We often write such y as $\{z \in x | \varphi(z)\}$

Note that, Axiom 3 to 6 gives us existences of some specific kinds of sets. It can be easily verified that all those sets are unique in '=' sense.
With those axioms we have more concepts and propositions:

Definition 2.1.
Intersection: Assume A is non-empty, then the intersection is defined by $\bigcap A := \{a \in \bigcup A | \forall x(x \in A \to a \in x)\}$. Note that we used Axiom 4 and 6 here.
Difference between sets is defined by: $x - y := \{z \in x \cup y | z \in x \wedge z \notin y\}$. Note that we used Axiom 3, 4 and 6 here.
Symmetry difference of sets is defined by: $x \Delta y := (x - y) \cup (y - x)$. Note that we used Axiom 3, 4 and 6 here.

Proposition 2.2.
$(x = y) \leftrightarrow (x \subseteq y \wedge y \subseteq x)$

Proposition 2.3. Existence of empty set
$\exists x \forall y (y \notin x)$

Proof. By Axiom 2, we have a set A. Then consider sentence $\varphi(x) := \forall y (y \neq x)$.
Then use Axiom 6, we have $B = \{z \in A | \varphi(z)\}$.
We verify that B is an empty set:
For all y, $y \in B$ iff $\varphi(y)$, iff $\forall x (x \neq y)$, while we have $y = y$, so $y \notin B$, which means B is empty. □

We can also show that the empty set is unique in '=' sense. So we will denote it as $\emptyset$. Note also that $\emptyset$ is subset of every set.
Now we consider an interesting 'paradox', or a theorem now. It says that there exists no such set, which contains all sets that are not element of themselves.

Theorem 2.1. (Russell)
$\neg (\exists y \forall y (y \in x \leftrightarrow y \notin y))$

Proof. Otherwise, we assume such set exists, call it A.
Then if $A \in A$, then by its definition, $A \notin A$, contradicts.
If $A \notin A$, then by its definition again, $A \in A$, contradicts.
So such A cannot exist. □

More definitions:

Definition 2.2. Ordered pair
$(x, y) := \{\{x\}, \{x, y\}\}$. Note that we used Axiom 3 twice.

Proposition 2.4.
$(a, b) = (c, d)$ iff $a = b$ and $c = d$.

Definition 2.3. Cartesian product
Given 2 sets A and B, $A \times B := \{(a, b) | a \in A \wedge b \in B\}$.
We used Axiom 5(power set) twice, and then use Axiom 6(separation schema).

The definition of relations(subsets of $A \times B$), functions(a special kind of relations) and more concepts are omitted here. They can be constructed easily in common sense.

2.2 Axiom of Infinity and Natural Numbers

We introduce the 7th axiom, also the last axiom in Zermelo's set theory.
First consider a legitimate operation:

Definition 2.4. Successor
Given a set x, its *successor* is defined by:
$S(x) := x \cup \{x\}$

Axiom 7. Axiom of Infinity

$\exists x((\emptyset \in x) \wedge (\forall y(y \in x \rightarrow S(y) \in x)))$

For simplicity, we define $Inf(x) := ((\emptyset \in x) \wedge (\forall y(y \in x \rightarrow S(y) \in x)))$

Clear, a set x with $Inf(x)$ may be too large, while the set of natural numbers in our imagination is relatively small. We must do some proper 'separations' to reduce a infinity set.

Definition 2.5. Given a set u satisfying $Inf(u)$, we define a subset of u:

$W(u) := \{a \in |\forall v(Inf(v) \rightarrow a \in v)\}$

$W(u)$ has good properties:

Theorem 2.2.

(1) $Inf(u) \rightarrow Inf(W(u))$

(2) If $Inf(u_1)$ and $Inf(u_2)$, then $W(u_1) = W(u_2)$.

(3) There exists a unique set satisfying: $Inf(u)$ and $W(u) = u$.

Proof. Use definitions and axioms. □

Definition 2.6. Natural number

We define that unique set as $\mathbb{N}$ or ω, satisfying $Inf(\omega)$ and $W(\omega) = \omega$.

Natural number set has many properties:

Proposition 2.5. If $Inf(u)$, then $\omega \subseteq u$.

Theorem 2.3.

1. $\forall a \in \omega(a = \emptyset \vee \emptyset \in a)$.
2. $\forall a \in \omega \forall b \in \omega(a \in S(b) \leftrightarrow (a = b \vee a \in b))$.
3. $\forall a \in \omega(a \subseteq \omega)$.
4. $\forall a \in \omega \forall b \in \omega \forall c \in \omega((a \in b \wedge b \in c) \rightarrow a \in c)$.

Definition 2.7. We say a subset x is transitive, iff $\forall y \in x$, we have $y \subseteq x$.

So ω itself is a transitive set. Actually all elements of ω are transitive also (use induction to prove it).

Theorem 2.4.

5. $\forall x \in \omega(x \notin x)$.
6. $\forall x \in \omega \forall y \in \omega(x \in y \rightarrow y \notin x)$.
7. $\forall x \in \omega \forall y \in \omega(x \in y \rightarrow (S(x) = y \vee S(x) \in y))$.
8. $\forall x \in \omega \forall y \in \omega(x \in y \vee x = y \vee y \in x)$.
9. $\forall x \in \omega(x \neq \emptyset \rightarrow \exists y(y \in \omega \wedge x = S(y)))$

We notice that $\in$ gives an order on ω.

We have definition of linear ordered set, well-ordered set. Linear order says that every two elements in the set can compare, and well-ordered set is a linear ordered set, and for arbitrary non-empty subset, it has a least element.

We find that $(\omega, \in)$ is a well-ordered set:

Theorem 2.5.

10. $\forall a \in \omega \forall x((\emptyset \neq x \subseteq a) \to \exists b(b \in x \wedge b \cap x = \emptyset))$.
11. If $A \neq \emptyset$ and $A \subseteq \omega$, then $\exists a(a \in A \wedge \forall x \in A(x = a \vee a \in x))$.

...

We have more properties, like a bounded subset has maximal. We left those verification to readers.

We also have mathematical induction, essentially, just follows the fact that $(\omega, \in)$ is a well-ordered set.

2.3 Cardinality: First Glance

We omit basic definitions. Consider:

Theorem 2.6. There is no surjection from A to $\mathfrak{P}(A)$. Especially, $|\omega| < |\mathfrak{P}(\omega)|$.