# Satisfiability Theorem

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**Lemma 1.** Assume $\Gamma$ is maximally consistent, then $(\tau \hat{=} \sigma) \in \Gamma$ iff $(\sigma \hat{=} \tau) \in \Gamma$.

*Proof.* By symmetry, we only need to prove one side. Assume $(\tau \hat{=} \sigma) \in \Gamma$. Consider $k$ large enough such that they appears nowhere in $\tau$ and $\sigma$. Define formula $\varphi := (x_k \hat{=} \tau)$, we have

$\{(\tau \hat{=} \sigma)\} \cup \{\varphi(x_k; \tau)\} \vdash \varphi(x_k; \sigma)$, which is just

$\{(\tau \hat{=} \sigma)\} \cup \{(\tau \hat{=} \tau)\} \vdash (\sigma \hat{=} \tau)$.

We have $(\tau \hat{=} \sigma) \in \Gamma$ by assumption, and $\Gamma \vdash (\tau \hat{=} \tau)$ combined with maximally consistency, we have $(\tau \hat{=} \tau) \in \Gamma$.

Thus we have $(\sigma \hat{=} \tau) \in \Gamma$. □

**Theorem 1.** Satisfiability Theorem

A maximally consistent set $\Gamma \subseteq \mathcal{L}$ with Henkin property is satisfiable.

*Proof.* The proof contains 4 parts: (1) Construct a domain $M$, (2) Give an interpretation $I$, (3) Give a valuation function $\nu$, and (4) Show that $((M, I), \nu) \models \Gamma$.

(1) Construct a domain $M$:

Consider the set of all constant element symbols: $C := \{c_i | i \in \mathbb{N}\}$. Consider a relation $\sim_\Gamma$ on $C$:

$c_i \sim_\Gamma c_j$ iff $(c_i \hat{=} c_j) \in \Gamma$.

We show that $\sim_\Gamma$ is an equivalent relation on C:

-Reflexivity: $(x_j \hat{=} x_j) \in \mathbb{L}$, then $(\forall x_j (x_j \hat{=} x_j)) \in \mathbb{L}$, and since $c_i$ can substitute $x_j$ in $(x_j \hat{=} x_j)$, we have $\vdash (c_i \hat{=} c_i)$, so $\Gamma \vdash (c_i \hat{=} c_i)$, and by maximally consistency, $(c_i \hat{=} c_i) \in \Gamma$, thus $c_i \sim_\Gamma c_i$.

-Symmetry: Assume $c_i \sim_\Gamma c_j$, then $(c_i \hat{=} c_j) \in \Gamma$. Then by lemma above, we know that $(c_j \hat{=} c_i) \in \Gamma$ also. So $c_j \sim_\Gamma c_i$.

-Transitivity: Assume $c_i \sim_\Gamma c_j$ and $c_j \sim_\Gamma c_k$, then $(c_i \hat{=} c_j) \in \Gamma$ and $(c_j \hat{=} c_k) \in \Gamma$. By symmetry, we know $(c_j \hat{=} c_i) \in \Gamma$. Then consider $\varphi := (x_0 \hat{=} c_k)$. Clearly, $\varphi$ has no quantifiers, $x_0$ appears nowhere in $c_j$ or $c_i$. Then by equality theorem, we have $\{(c_j \hat{=} c_i)\} \cup \{\varphi(x_0; c_j)\} \vdash \varphi(x_0; c_i)$, which is just $\{(c_j \hat{=} c_i)\} \cup \{(c_j \hat{=} c_k)\} \vdash (c_i \hat{=} c_k)$. Since $(c_j \hat{=} c_i) \in \Gamma$ and $(c_j \hat{=} c_k) \in \Gamma$, we have $\Gamma \vdash (c_i \hat{=} c_k)$. By maximally consistency, $(c_i \hat{=} c_k) \in \Gamma$, so $c_i \sim_\Gamma c_k$.

So we define our domain as $M := C/_{\sim_\Gamma}$, is the set of all equivalent classes under equivalent relation $\sim_\Gamma$.

(2) Give a interpretation $I$:

-For constant element symbol $c_i$, we define $I(c_i) = [c_i]$.

-For function symbol $F_n$ with $\pi(F_n) = k \geq 1$, we define $I(F_n)$ as the following:

If $(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} c_m) \in \Gamma$, then define $I(F_n)([c_{i_1}], [c_{i_k}]) = [c_m]$.

-For predicate symbol $P_n$ with $\pi(P_n) = k \geq 1$, we define $I(P_n)$ as the following:

$I(P_n) := \{([c_{i_1}], ..., [c_{i_k}]) \in M^k | P_n(c_{i_1}, ..., c_{i_k}) \in \Gamma\}$.

Note that there is no defining problem in constant element or predicate symbol. As for a function symbol $F_n$ with $\pi(F_n) = k \geq 1$, we have to verify that it is well-defined, and is really a function from $M^k$ to $M$.

To show $I(F_n)$ is well-defined, consider $(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} c_{m_1}) \in \Gamma$ and $(F_n(c_{j_1}, ..., c_{j_k}) \hat{=} c_{m_2}) \in \Gamma$, with $[c_{i_1}] = [c_{j_1}], ..., [c_{i_k}] = [c_{j_k}]$, we need to show that $[c_{m_1}] = [c_{m_2}]$. We use Equality Theorem: first consider $\varphi := (F_n(x_1, ..., x_k) \hat{=} c_{m_2})$:

$\{(c_{j_1} \hat{=} c_{i_1}), ..., (c_{j_k} \hat{=} c_{i_k})\} \cup \{\varphi(x_1, ..., x_k; c_{j_1}, ..., c_{j_k})\} \vdash \varphi(x_1, ..., x_k; c_{j_1}, ..., c_{j_k})$, which is just:

$\{(c_{j_1} \hat{=} c_{i_1}), ..., (c_{j_k} \hat{=} c_{i_k})\} \cup \{(F_n(c_{j_1}, ..., c_{j_k}) \hat{=} c_{m_2})\} \vdash (F_n(c_{i_1}, ..., c_{i_k}) \hat{=} c_{m_2})$.

So $(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} c_{m_2}) \in \Gamma$.

Then consider $\psi := (x_0 \hat{=} c_{m_2})$:

$\{(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} c_{m_1})\} \cup \{\psi(x_0; (F_n(c_{i_1}, ..., c_{i_k}))\} \vdash \psi(x_0; c_{m_1})$, which is just:

$\{(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} c_{m_1})\} \cup \{(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} c_{m_2})\} \vdash (c_{m_1} \hat{=} c_{m_2})$.

So $(c_{m_1} \hat{=} c_{m_2}) \in \Gamma$, thus $[c_{m_1}] = [c_{m_2}]$.

To show $I(F_n)$ is really a function from $M^k$ to $M$, consider arbitrary $k$ constant element symbols $c_{i_1}, ..., c_{i_k}$, we have to show that there exists another constant element symbol $c_m$ such that $(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} c_m) \in \Gamma$.

Consider formula $\theta := \forall x_0(\neg(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} x_0))$. If $\theta \in \Gamma$, then since $F_n(c_{i_1}, ..., c_{i_k})$ can substitute $x_0$ in $(\neg(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} x_0))$, then by specialization, we have $\Gamma \vdash (\neg(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} F_n(c_{i_1}, ..., c_{i_k})))$. However, it's not hard to show that $(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} F_n(c_{i_1}, ..., c_{i_k})) \in \Gamma$, so $\Gamma \vdash (F_n(c_{i_1}, ..., c_{i_k}) \hat{=} F_n(c_{i_1}, ..., c_{i_k}))$, contracts the fact that $\Gamma$ is consistent.

Thus we must have $(\neg\theta) \in \Gamma$, which is just $(\exists x_0(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} x_0))$. By Henkin property of $\Gamma$, we know that there must be a constant element symbol $c_m$ such that $(F_n(c_{i_1}, ..., c_{i_k}) \hat{=} c_m) \in \Gamma$, which completes our proof.

Note that we can also show $I(P_n)$ is 'well-defined', with just the same method. I don't think such proof is necessary here. However, we need such result, in part (4) of our whole proof.

(3) Give a valuation function $\nu$:

-For variable symbol $x_i$, there is a constant element symbol $c_j$ such that $(x_i \hat{=} c_j) \in \Gamma$, then define $\nu(x_i) = [c_j]$.

We have to show that $\nu$ is well-defined and is really a valuation function from $\mathcal{B}$ to $M$.

To show that $\nu$ is well-defined, consider we have $(x_i \hat{=} c_{j_1}) \in \Gamma$ and $(x_i \hat{=} c_{j_2}) \in \Gamma$, then we have to show that $(c_{j_1} \hat{=} c_{j_2}) \in \Gamma$. We use Equality Theorem. We take $k$ large enough such that $x_k \neq x_i$, then consider formula $\varphi := (x_k \hat{=} c_{j_2})$. Then we have:

$\{(x_i \hat{=} c_{j_1})\} \cup \{\varphi(x_k; x_i)\} \vdash \varphi(x_k; c_{j_1})$, which is just:

$\{(x_i \hat{=} c_{j_1})\} \cup \{(x_i \hat{=} c_{j_2})\} \vdash (c_{j_1} \hat{=} c_{j_2})$.

So we have $\Gamma \vdash (c_{j_1} \hat{=} c_{j_2})$, by maximally consistency, $(c_{j_1} \hat{=} c_{j_2}) \in \Gamma$.

To show that $\nu$ is really a valuation function from $\mathcal{B}$ to $M$, we have to show that given a variable symbol $x_i$, there must be a constant element symbol $c_j$ such that $(x_i \hat{=} c_j) \in \Gamma$.

Consider $k$ large enough such that $x_k \neq x_i$, then define a formula $\psi := (\forall x_k(\neg(x_i \hat{=} x_k)))$. If $\psi \in \Gamma$, then since $x_i$ can substitute $x_k$ in $(\neg(x_i \hat{=} x_k))$, by specialization law, we have $\Gamma \vdash (\neg(x_i \hat{=} x_i))$, while at the same time we have $\Gamma \vdash (x_i \hat{=} x_i)$, we get a contradiction. So $\psi \notin \Gamma$, by maximally consistency of $\Gamma$ we have $(\neg\psi) \in \Gamma$, which is just $(\exists x_k(x_i \hat{=} x_k))$. By Henkin property of $\Gamma$, we know there must be a constant element symbol $c_j$ such that $(x_i \hat{=} c_j) \in \Gamma$.

In the end, we consider valuation on terms. Given an arbitrary term $\tau$, we show that $\bar{\nu}(\tau) = [c_i]$ iff $(\tau \hat{=} c_i) \in \Gamma$.

We prove this by induction:

-Variable symbols: $\bar{\nu}(x_j) = \nu(x_j)$, then by results above, $\bar{\nu}(x_j) = [c_i]$ iff $(x_j \hat{=} c_i) \in \Gamma$.

-Constant element symbols: $\bar{\nu}(c_j) = I(c_j) = [c_j]$, by results in (1), $\bar{\nu}(c_j) = [c_i]$ iff $(c_j \hat{=} c_i) \in \Gamma$.

-Given function symbol $F_n$ with $\pi(F_n) = k \geq 1$, and terms $\tau_1, ..., \tau_k$. Assume that $\bar{\nu}(\tau_m) = [c_{i_m}]$ iff $(\tau_m \hat{=} c_{i_m}) \in \Gamma$, for $1 \leq m \leq k$.

Assume $\bar{\nu}(F_n(\tau_1, ..., \tau_k)) = [c_j]$. Then $I(F_n)(\bar{\nu}(\tau_1), ..., \bar{\nu}(\tau_k)) = [c_j]$. Assume $\bar{\nu}(\tau_m) = [c_{i_m}]$ for $1 \leq m \leq k$, then by results in (2), we know $(F_n(c_{i_1}, ..., c_{i_k}) = c_j) \in \Gamma$. Take $l_1, ..., l_k$ large enough such that they appears nowhere in $\tau_1, ..., \tau_k$, so we can use Equality Theorem. Consider $\theta := (F_n(x_{l_1}, ..., x_{l_k}) = c_j)$, then by Equality Theorem,

$\{(c_{i_1} \hat{=} \tau_1), ..., (c_{i_k} \hat{=} \tau_k)\} \cup \{\theta(x_{l_1}, ..., x_{l_k}; c_{i_i}, ..., c_{i_k})\} \vdash \theta(\theta(x_{l_1}, ..., x_{l_k}; \tau_1, ..., \tau_k)$, which is just

$\{(c_{i_1} \hat{=} \tau_1), ..., (c_{i_k} \hat{=} \tau_k)\} \cup \{(F_n(c_{i_1}, ..., c_{i_k}) = c_j)\} \vdash (F_n(\tau_1, ..., \tau_k) \hat{=} c_j)$.

By lemma above and induction hypothesis, we have $\{(c_{i_1} \hat{=} \tau_1), ..., (c_{i_k} \hat{=} \tau_k)\} \subseteq \Gamma$.

So we have $(F_n(\tau_1, ..., \tau_k) \hat{=} c_j) \in \Gamma$. This proves one side.

Assume $(F_n(\tau_1, ..., \tau_k) \hat{=} c_j) \in \Gamma$. Assume $\bar{\nu}(\tau_m) = [c_{i_m}]$ for $1 \leq m \leq k$. Use Equality Theorem similarly, we have $(F_n(c_{i_1}, ..., c_{i_k}) = c_j) \in \Gamma$, then $\bar{\nu}(F_n(\tau_1, ..., \tau_k)) = [c_j]$ follows.

(4) Show that $((M, I), \nu) \models \Gamma$:

We induct on the complexity of formulas in $\mathcal{L}$ to show that:

$\varphi \in \Gamma$ iff $((M, I), \nu) \models \varphi$.

-Base step:

—Consider $(\tau \hat{=} \sigma) \in \Gamma$, assume $\bar{\nu}(\tau) = [c_i]$ and $\bar{\nu}(\sigma) = [c_j]$. Then by (3), $(\tau \hat{=} c_i) \in \Gamma$, $(\sigma \hat{=} c_j) \in \Gamma$.

Take $k, l$ large enough such that $x_l, x_l$ occurs nowhere in $\tau, \sigma$. Then consider $A_1 := (x_k \hat{=} c_i)$, by Equality Theorem,

$\{(\tau \hat{=} \sigma)\} \cup \{A_1(x_k; \tau)\} \vdash A_1(x_k; \sigma)$, which is just

$\{(\tau \hat{=} \sigma)\} \cup \{(\tau \hat{=} c_i)\} \vdash (\sigma \hat{=} c_i)$

So we have $(\sigma \hat{=} c_i) \in \Gamma$.

With same method, using Equality Theorem with formula $(x_l\hat{=}c_j)$, and Lemma above, we have $(c_i\hat{=}c_j) \in \Gamma$, so $[c_i] = [c_j]$, which means $\bar{\nu}(\tau) = \bar{\nu}(\sigma)$. Thus $((M, I), \nu) \models (\tau\hat{=}\sigma)$.

The other hand, assume $\bar{\nu}(\tau) = \bar{\nu}(\sigma)$, then $\bar{\nu}(\tau) = [c_i]$ and $\bar{\nu}(\sigma) = [c_j]$ , then by (3), $(\tau\hat{=}c_i), (\sigma\hat{=}c_j), (c_i\hat{=}c_j) \in \Gamma$. A few 'Equality Theorems' will show that $(\tau\hat{=}\sigma) \in \Gamma$.

—Consider $P_n(\tau_1, ..., \tau_k) \in \Gamma$, with $\bar{\nu}(\tau_m) = [c_{i_m}]$, for $1 \leq m \leq k$. Consider $P_n(c_{i_1}, ..., c_{i_k})$, we take $s_1, ..., s_k$ large enough such that $x_{s_1}, ..., x_{s_k}$ appears nowhere $\tau_1, ..., \tau_k$, then use Equality Theorem with formula $P_n(x_{s_1}, ..., x_{s_k})$, and terms $\tau_1, ..., \tau_k$ and $c_{i_1}, ..., c_{i_k}$, we can prove that $P_n(c_{i_1}, ..., c_{i_k}) \in \Gamma$, thus $([c_{i_1}], ..., [c_{i_k}]) \in I(P_n)$, so $((M, I), \nu) \models P_n(\tau_1, ..., \tau_k) \in \Gamma$.

The other hand, assume $(\bar{\nu}(\tau_1), ..., \bar{\nu}(\tau_k)) \in I(P_n)$, then $P_n(c_{i_1}, ..., c_{i_k}) \in \Gamma$. Then Equality Theorem will tell us that $P_n(\tau_1, ..., \tau_k) \in \Gamma$.

-Induction step:

—Consider $(\neg\psi) \in \Gamma$, iff $\psi \notin \Gamma$, iff $((M, I), \nu) \not\models \psi$, iff $((M, I), \nu) \models (\neg\psi)$.

—Consider $(A \to B) \in \Gamma$, iff $A \notin \Gamma$ or $B \in \Gamma$, iff $((M, I), \nu) \not\models A$ or $((M, I), \nu) \models B$, iff $((M, I), \nu) \models (A \to B)$.

—Consider $(\forall x_i\theta)$. We need Substitution Theorem(so far away) to show that:

(a) Assume $\mu = \nu$ (mod $(\forall x_i\theta)$), and $\mu(x_i) = [c_j]$, then we consider the formula $\theta$, we know that $c_j$ can substitute $x_i$ in it, and at the same time, we have $\mu(x_i) = [c_j] = I(c_j) = \bar{\nu}(c_j)$, then by Substitution Theorem, we have $((M, I), \mu) \models \theta$ iff $((M, I), \nu) \models \theta(x_i; c_j)$.(a ends)

Then $((M, I), \nu) \models (\forall x_i\theta) \in \Gamma$ iff for every valuation function $\mu$ with $\mu = \nu$ (mod $(\forall x_i\theta)$)) we have $((M, I), \nu) \models \theta$,

by (a), iff for every valuation function $\mu$ with $\mu = \nu$ (mod $(\forall x_i\theta)$)) we have $((M, I), \nu) \models \theta(x_i; c_j)$,

then by induction hypothesis, iff for every valuation function $\mu$ with $\mu = \nu$ (mod $(\forall x_i\theta)$)) we have $\theta(x_i; c_j) \in \Gamma$. It seems that we almost got it. If we can prove the following result (b), then the bridge between two sides will be built.

(b) says that $(\forall x_i\theta) \in \Gamma$ iff for every constant element symbol $c_j$, we have $\theta(x_i; c_j) \in \Gamma$.

So iff (b) holds, then we continue our 'iff':

iff for every constant element symbol $c_j$, we have $\theta(x_i; c_j) \in \Gamma$, iff $(\forall x_i\theta) \in \Gamma$, completing our proof.

So now we prove (b):

On side is direct. Assume $(\forall x_i\theta) \in \Gamma$, since $c_j$ can substitute $x_i$ in $(\forall x_i\theta)$, by specialization law we have $\Gamma \vdash \theta(x_i; c_j)$, then by maximally consistency of $\Gamma$, we have $\theta(x_i; c_j) \in \Gamma$.

On the other hand, assume for $\theta(x_i; c_j) \in \Gamma$ for all $c_j$. Then if $(\forall x_i\theta) \notin \Gamma$, then $(\neg(\forall x_i\theta)) \in \Gamma$.

It not hard to show that, $(\neg(\forall x_i\theta)) \in \Gamma$ implies that $(\exists x_i(\neg\theta)) \in \Gamma$. By Henkin property of $\Gamma$, we know there is some $c_m$ such that $(\neg\theta(x_i; c_m)) \in \Gamma$, while this contradicts the assumption that $(\theta(x_i; c_m)) \in \Gamma$: Otherwise $\Gamma$ won't be consistent. So we proves (b).

Finally, we verified all cases. By induction, we proved that $((M, I), \nu) \models \varphi$ iff $\varphi \in \Gamma$ for every formula $\varphi$. This means that $((M, I), \nu) \models \Gamma$.

As a conclusion, we know that $\Gamma$ is satisfied by $((M, I), \nu)$, thus $\Gamma$ is satisfiable. $\square$