1 Surface Area

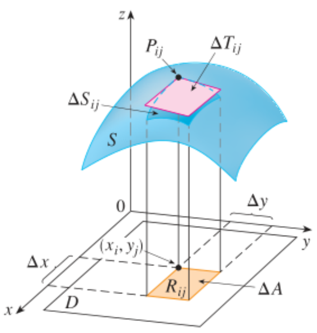


FIGURE 1

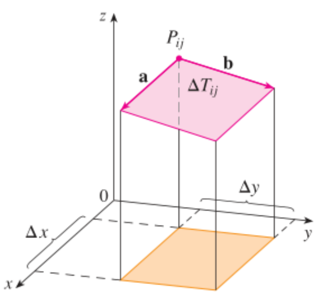


FIGURE 2

Divide into $m \times n$ square. Then $A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$. Since $\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}\,|$. Recall that $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij}.

$$\mathbf{a} = \Delta x\mathbf{i} + f_x(x_i, y_j)\Delta x\mathbf{k}$$
$$\mathbf{b} = \Delta y\ \mathbf{j} + f_y(x_i, y_j)\Delta y\mathbf{k}$$

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\Delta y \end{vmatrix} \\
&= [-f_x(x_i, y_j)\ \mathbf{i} - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}\ ]\Delta A \\
\Delta T_{ij} = |\ \mathbf{a} \times \mathbf{b}\ | &= \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\,\Delta A
\end{aligned}$$

Hence we have

Definition : The Area of the Surface

If f_x, f_y are continuous.

$$\begin{aligned}
A(S) &= \iint\limits_D \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\,dA \\
&= \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA
\end{aligned}$$

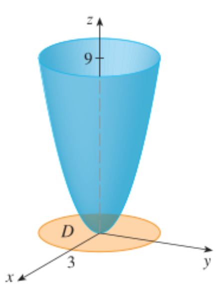


FIGURE 5

EXAMPLE. Area of $z = x^2 + y^2$ that lies under $z = 9$.

$$\begin{aligned}
A &= \iint\limits_D \sqrt{1 + (2x)^2 + (2y)^2}\,dA \\
&= \iint\limits_D \sqrt{1 + 4(x^2 + y^2)}\,dA
\end{aligned}$$

Converting to polar coordinates, we obtain

$$\begin{aligned}
A &= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\,r\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^3 \frac{1}{8}\sqrt{1 + 4r^2}(8r)\,dr \\
&= 2\pi\left(\tfrac{1}{8}\right)\tfrac{2}{3}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}(37\sqrt{37} - 1)
\end{aligned}$$

2 Triple Integrals

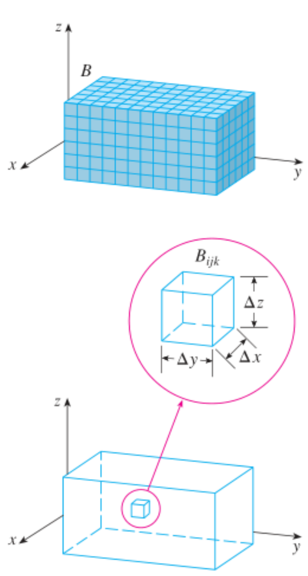


FIGURE 1

Divide into subboxes.

Definition : Triple Integrals

Let $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$, then

$$\iiint\limits_B \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}*, y_{ijk}*, z_{ijk}*)\Delta V$$

Fubini's Theorem. $\iiint\limits_B f(x, y, z)\,dV = \int_a^b \int_r^s \int_c^d f(x, y, z)\,dy\,dz\,dx$

Just the same, wrap E inside a box, and we got $\iiint\limits_B F(x,y,z)\,dV$.

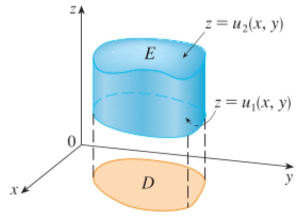


FIGURE 2
A type 1 solid region

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$$

$$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

Definition : 3 Types of Triple Integrals

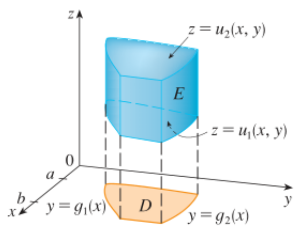


FIGURE 3
A type 1 solid region where the projection D is a type I plane region

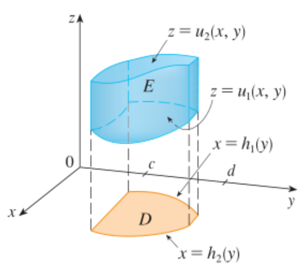


FIGURE 4
A type 1 solid region with a type II projection

- **Type I.** D is the projection on the xy-plane.

$$E = \{(x,y,z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\}$$

$$\iint\limits_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

- **Type II.** D is the projection on the yz-plane.
- **Type III.** D is the projection on the xz-plane.

EXAMPLE. $\iiint\limits_E \sqrt{x^2+z^2}\,dV$, where E bounded by $y = x^2+z^2$ and $y = 4$.

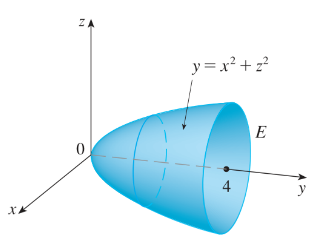


FIGURE 9
Region of integration

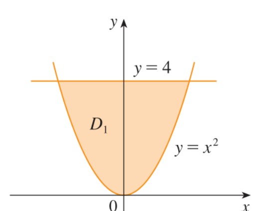


FIGURE 10
Projection onto xy-plane

$$\iint\limits_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right] dA = \iint\limits_{D_3} (4-x^2-z^2)\sqrt{x^2+z^2}\,dA$$

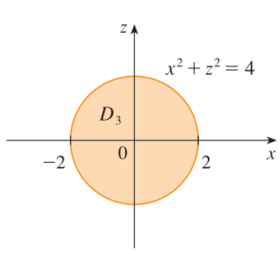


FIGURE 11
Projection onto xz-plane

Convert to polar coordinate in the xz-plane: $x = r\cos\theta, z = r\sin\theta$, which gives

$$\begin{aligned}
\iiint\limits_E \sqrt{x^2+z^2}\,dV &= \iint\limits_{D_3} (4-x^2-z^2)\sqrt{x^2+z^2}\,dA \\
&= \int_0^{2\pi}\int_0^2 (4-r^2)r\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r^2-r^4)\,dr \\
&= 2\pi\left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 = \frac{128\pi}{15}
\end{aligned}$$

3 Applications of Triple Integrals

First, begin with the special case where $f(x,y,z) = 1$ for all points in E. That would be the volume of the shape.

4 Triple Integrals in Cylindrical Coordinates

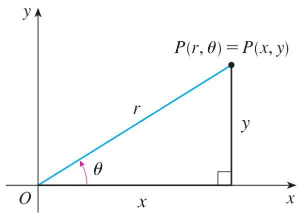


FIGURE 1

Recall the connection between polar and Cartesian coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

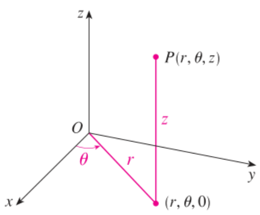


FIGURE 2
The cylindrical coordinates of a point

Cylindrical Coordinates

Represented by (r, θ, z),

- r, θ: polar coordinates of the **projection** of P onto the xy-plane.
- z: the directed distance from the xy-plane to P.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

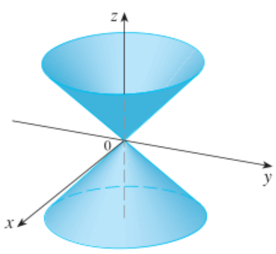
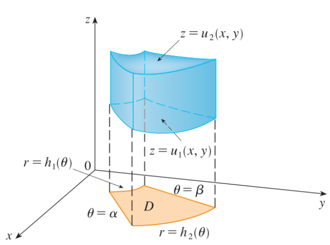


FIGURE 5
$z = r$, a cone

This is the surface of $z = r$.

Evaluating Triple Integrals with Cylindrical Coordinates



Suppose $E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$, and

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

Definition : Triple Integrals with Cylindrical Coordinates

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta}^{u_2(r\cos\theta, r\sin\theta))} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

5 Triple Integrals in Spherical Coordinates

Definition : Spherical Coordinates

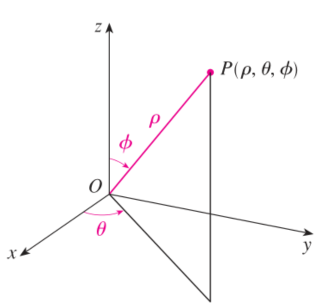


FIGURE 1
The spherical coordinates of a point

The **spherical coordinates** (ρ, θ, ϕ) of a point P:

- $\rho = |OP| \ge 0$: the distance from O to P.
- θ: the same angle as in cylindrical coordinates.
- $0 \le \phi \le \pi$: the angle between the positive z and OP.

Useful when there is symmetry about a point.

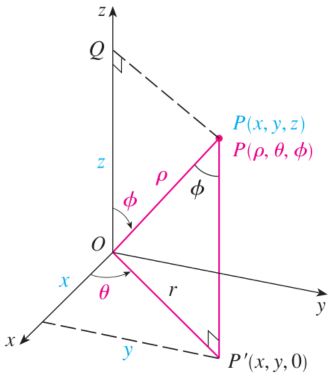


FIGURE 5

We have $z = \rho\cos\phi$ and $r = \rho\sin\phi$.

$$\boxed{x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi}$$

The distance formula

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

Definition : Evaluating Triple Integrals with Spherical Coordinates

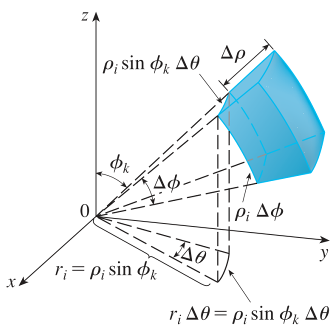


FIGURE 7

$$\iiint_E f(x,y,z)\,dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

where E is a **spherical wedge**

$$E = \{(\rho,\theta,\phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$$

where $a \ge 0,\ \beta - \alpha \le 2\pi,\ d - c \le \pi$.
The formula can be extended as $g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi)$.

6 Change of Variables in Multiple Integrals

Consider a **transformation** T from the uv-plane to the xy-plane: $x = g(u,v)$, $y = h(u,v)$.
Assume T is a C^1 **transformation** (g, h have continuous 1st-order partial derivatives).

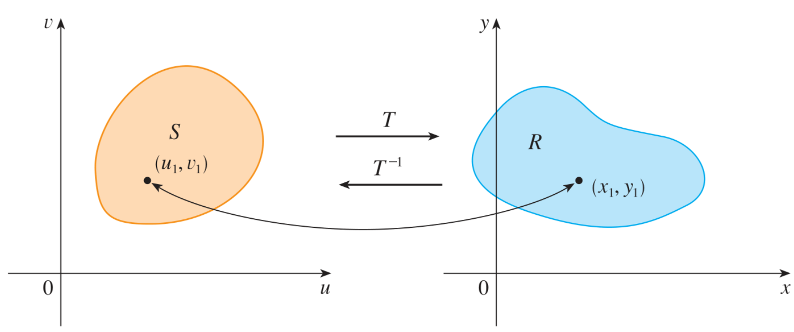


FIGURE 1

If T is **one-to-one**, the the **inverse transformation** T^{-1} exist: $u = G(x,y),\ v = H(x,y)$.

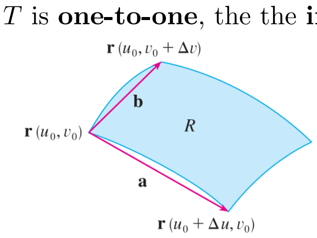


FIGURE 4

We approximate the image R by a parallelogram determined by

$$\begin{cases} \mathbf{a} &= \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \mathbf{r}_u \\ \mathbf{b} &= \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \mathbf{r}_v \\ S_R &= |(\Delta u \mathbf{r}_u) \times (\Delta v \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u\,\Delta v \end{cases}$$

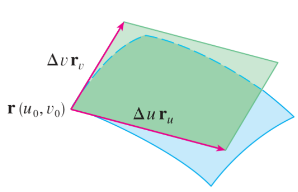


FIGURE 5

Computing the cross product

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

Definition : The Jacobian

The **Jacobian** of the transformation T: $x = g(u,v)$, $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Hence we got $\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\Delta u \Delta v\right|$ where the Jacobian is evaluated at (u_0, v_0).

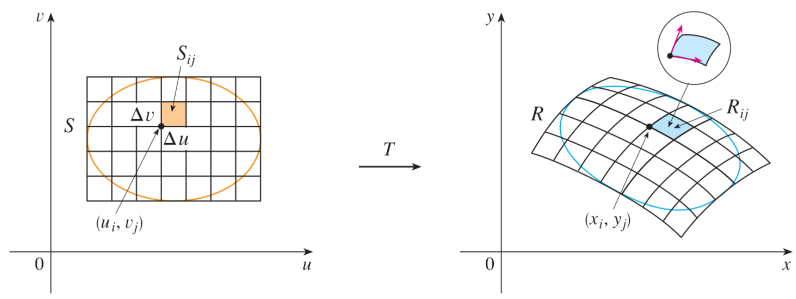


FIGURE 6

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\,\Delta A$$
$$\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i,v_i), h(u_i,v_i)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$$

Definition : Change of Variables in a Double Integral

Suppose T is a C^1 transformation whose Jacobian is nonzero, map from uv to xy. R, S are type I, II, f is continuous.

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v$$

Triple Integral

The **Jacobian** of T is the determinant

$$\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases} \qquad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Definition : Triple Integration in Spherical Coordinates

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$