1 The Cross Product of Two Vectors in Space

Here we will look at a vector product that yields a vector in $\mathbb{R}^3$ orthogonal to 2 vectors. This vector product is called the **cross product**, and it is defined and calculated with standard unit vectors

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Definition of Cross Product of Two Vectors.
Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in $\mathbb{R}^3$. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Alternative form of the Cross Product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example. **Finding the Cross Product of Two Vectors**

Provided that $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

(a)

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$
$$= 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

(b) $\mathbf{v} \times \mathbf{u} = -3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$

(c) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

Those results suggest some *algebraic* properties of the cross product.

THEOREM 5.17 **Algebraic Properties of the Cross Product**
If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, and c is a scalar, then the following properties are true.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w} = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = c\mathbf{u} \times \mathbf{v} = \mathbf{u} \times c\mathbf{v}$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Note. $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have equal lengths but *opposite direction.*

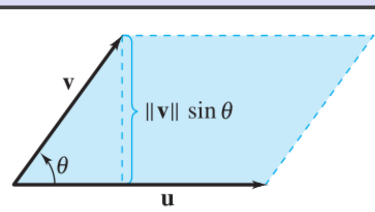
THEOREM 5.18 **Geometric Properties of the Cross Product**
If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, then the following properties are true.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is given by

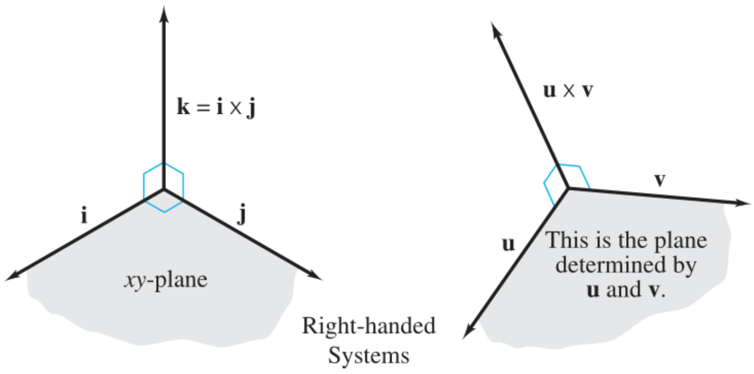
$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$$

3. $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = 0$.
4. The *parallelogram* having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides has an area of $\|\mathbf{u} \times \mathbf{v}\|$.

PROOF.



Note. The 3 vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a *right-handed system*, whereas the 3 vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ form a *left-handed system*.



2 Least Square Approximations (Calculus)

Many problems in the physical sciences and engineering involve an approximation of a function f by another function g. If f is in $C[a, b]$ (the inner product space of all continuous funtions on $[a, b]$), then g is usually chosen from a subspace W of $C[a, b]$.

For instance, to approximate the function

$$f(x) = e^x, \quad 0 \leq x \leq 1,$$

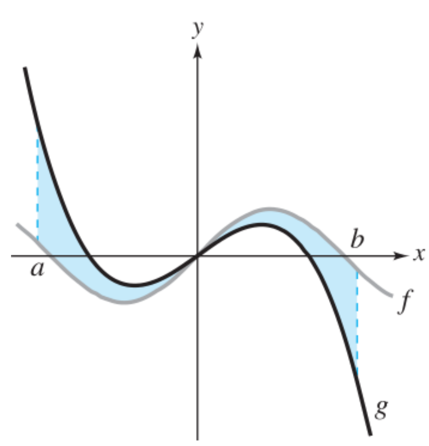
you could choose one of the following forms of g.

1. $g(x) = a_0 + a_1 x, \quad 0 \leq x \leq 1$ — Linear
2. $g(x) = a_0 + a_1 x + a_2 x^3, \quad 0 \leq x \leq 1$ — Quadratic
3. $g(x) = a_0 + a_1 \cos x + a_2 \sin x, \quad 0 \leq x \leq 1$ — Trigonometric

Before discussing ways of finding g, we must define how 1 function can "best" approximate another. One natural way would require the area bounded by the graphs of f and g on the interval $[a, b]$.

$$\text{Area} = \int_a^b |f(x) - g(x)|\, dx,$$

to be a minimum with respect to other functions in the subspace W



Integrands involving absolute value are difficult to evaluate, so it is more common to **square** the integrand.

Definition of Least Squares Approximation
Let f be continuous on $[a, b]$, and W be a subspace of $C[a, b]$. A function g in W is called a **least squares approximation** of f with respect to W if the value of

$$I = \int_a^b [f(x) - g(x)]^2\, dx$$

is a minimum with respect to all other functions in W.

REMARK. If the subspace W is the entire space $C[a, b]$, then $g(x) = f(x)$, which implies that $I = 0$.

Example 4. Finding a Least Squares Approximation

Find the least squares approximation $f(x) = a_0 + a_1 x$ for

$$f(x) = e^x, \quad 0 \leq x \leq 1$$

SOLUTION. For this approximation, we need to find the constant a_0 and a_1 that minimize the value of

$$\begin{aligned} I &= \int_0^1 [f(x) - g(x)]^2 \, dx \\ &= \int_0^1 (e^x - a_0 - a_1 x)^2 \, dx \end{aligned}$$

Evaluating this integral, you have

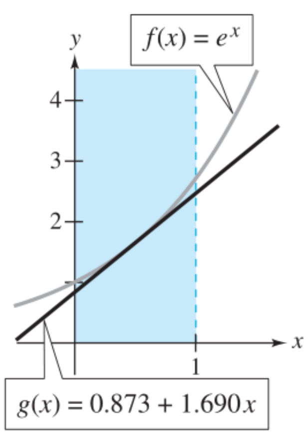
$$\begin{aligned} I &= \int_0^1 (e^x - a_0 - a_1 x)^2 \, dx \\ &= \int_0^1 (e^{2x} - 2a_0 e^x - 2a_1 x e^x + a_0^2 + 2a_0 a_1 x + a_1 x^2) \, dx \\ &= \left[\frac{1}{2} e^{2x} - 2a_0 e^x - 2a_1 e^x (x-1) + a_0^2 x + a_0 a_1 x^2 + a_1^2 \frac{x^3}{3} \right]_0^1 \\ &= \frac{1}{2}(e^2 - 1) - 2a_0(e-1) - 2a_1 + a_0^2 + a_0 a_1 + \frac{1}{3} a_1^2 \end{aligned}$$

Now, considering I to be a function of the variables a_0 and a_1, use calculus to determine their values to minimize I. Specifically, by setting the partial derivatives

$$\begin{aligned} \frac{\partial I}{\partial a_0} &= 2a_0 - 2e + 2 + a_1 \\ \frac{\partial I}{\partial a_1} &= a_0 + \frac{2}{3} a_1 - 2 \end{aligned}$$

equal to zero, we obtain the following 2 linear equations in a_0 and a_1

$$\begin{cases} 2a_0 + a_1 = 2(e-1) \\ 3a_0 + 2a_1 = 6 \end{cases}, \text{which solution is} \begin{cases} a_0 = 4e - 10 \approx 0.873 \\ a_1 = 18 - 6e \approx 1.690 \end{cases}$$



So, the best *linear approximation* of $f(x) = e^x$ on the interval $[0, 1]$ is

$$\begin{aligned} g(x) &= 4e - 10 + (18 - 6e)x \\ &\approx 0.873 + 1.690x \end{aligned}$$

Of course, the approximation obtained depends on the *definition of the best approximation.* If that definition of "best" had been the *Taylor polynomial of degree 1* centered at 0.5, g would have been

$$\begin{aligned} g(x) &= f(0.5) + f'(0.5)(x - 0.5) \\ &= e^{0.5} + e^{0.5}(x - 0.5) \\ &\approx 0.824 + 1.649x \end{aligned}$$

Example 5. Finding a Least Squares Approximation

Find the least squares approximation $f(x) = a_0 + a_1 x + a_2 x^2$ for

$$f(x) = e^x, \quad 0 \le x \le 1$$

SOLUTION. We need to find the values of a_0, a_1 and a_2 that minimize the value of

$$\begin{aligned} I &= \int_0^1 [f(x) - g(x)]^2 \, dx \\ &= \int_0^1 (e^x - a_0 - a_1 x - a_2 x^2)^2 \, dx \\ &= \frac{1}{2}(e^2 - 1) + 2a_0(1 - e) + 2a_2(2 - e) \\ &\quad + a_0^2 + a_0 a_1 + \frac{2}{3} a_0 a_2 + \frac{1}{2} a_1 a_2 + \frac{1}{3} a_1^2 + \frac{1}{5} a_2^2 - 2a_1 \end{aligned}$$

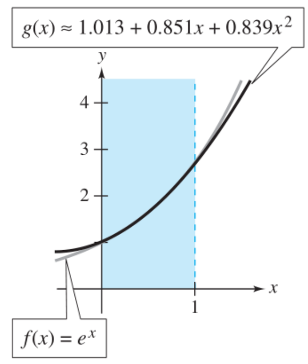
Integrating and then setting the partial derivatives of I (with respect to a_0, a_1 and a_2) equal to zero produces the following system of linear equations.

$$\begin{cases} 6a_0 + 3a_1 + 2a_2 = 6(e-1) \\ 6a_0 + 4a_1 + 3a_2 = 12 \\ 20a_0 + 15a_1 + 12a_2 = 60(e-2) \end{cases}$$

The solution of this system is $\begin{cases} a_0 = -105 + 39e \approx 1.013 \\ a_1 = 588 - 216e \approx 0.851 \\ a_2 = -570 + 210e \approx 0.839 \end{cases}$

So, the approximating function g is

$$g(x) \approx 1.013 + 0.851x + 0.839x^2$$



The integral I can be expressed in vector form. First, use the **inner product**

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$$

With this, we have: $$I = \int_a^b [f(x) - g(x)]^2\,dx = \langle f-g, f-g\rangle = \|f-g\|^2$$

In other words, $g \in W$ is *closest* to f in term of the inner product $\langle f, g\rangle$.

> *THEOREM 5.19* **Least Squares Approximation**
> Let f be continuous on $[a,b]$, W be a finite-dimensional subspace of $C[a,b]$.
> The least squares approximating function of f with respect to W is given by
>
> $$g = \langle f, \mathbf{w}_1\rangle \mathbf{w}_1 + \langle f, \mathbf{w}_2\rangle \mathbf{w}_2 + \cdots + \langle f, \mathbf{w}_n\rangle \mathbf{w}_n,$$
>
> where $B = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is an *orthornormal* basis for W.

Now observe how this can be used to produces least squares approximation obtained in **Example 4.**

First, apply the Gram-Schmidt orthornormalization process to the standard basis $B = \{1, x^2\}$ of W to obtain the orthornormal basis $B = \{1, \sqrt{3}(2x-1)\}$. Then we got

$$\begin{aligned} g(x) &= \langle e^x, 1\rangle(1) + \langle e^x, \sqrt{3}(2x-1)\rangle\sqrt{3}(2x-1) \\ &= \int_0^1 e^x\,dx + \sqrt{3}(2x-1)\int_0^1 \sqrt{3}e^x(2x-1)\,dx \\ &= \int_0^1 e^x\,dx + 3(2x-1)\int_0^1 e^x(2x-1)\,dx \\ &= (e-1) + 3(2x-1)(3-e) \\ &= 4e - 10 + (18-6e)x \end{aligned}$$

which agrees with the result obtained in **Example 4**.

Example 6. Finding a Least Squares Approximation

Find the least squares approximation for $f(x) = \sin x, \quad 0 \le x \le \pi$, with respect to the subspace W of quadratic functions.

SOLUTION. Applying the Gram-Schmidt orthonormalization process to the standard basis for W, $\{1, x, x^2\}$, obtaining the orthornormal basis

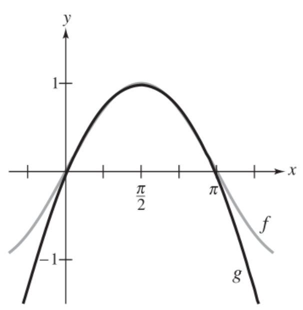
$$\begin{aligned} B &= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} \\ &= \left\{\frac{1}{\sqrt{\pi}}, \frac{\sqrt{3}}{\pi\sqrt{\pi}}(2x-\pi), \frac{\sqrt{5}}{\pi^2\sqrt{\pi}}(6x^2 - 6\pi x + \pi^2)\right\} \end{aligned}$$

The least squares approximating function g is

$$g(x) = \langle f, \mathbf{w}_1 \rangle \mathbf{w}_1 + \langle f, \mathbf{w}_2 \rangle \mathbf{w}_2 + \langle f, \mathbf{w}_3 \rangle \mathbf{w}_3$$

and we have

$$\begin{aligned}
\langle f, \mathbf{w}_1 \rangle &= \frac{1}{\sqrt{\pi}} \int_0^{\pi} \sin x \, dx = \frac{2}{\sqrt{\pi}} \\
\langle f, \mathbf{w}_2 \rangle &= \frac{\sqrt{3}}{\pi\sqrt{\pi}} \int_0^{\pi} \sin x (2x - \pi) \, dx = 0 \\
\langle f, \mathbf{w}_3 \rangle &= \frac{\sqrt{5}}{\pi^2\sqrt{\pi}} \int_0^{\pi} \sin x (6x^2 - 6\pi x + \pi^2) \, dx \\
&= \frac{2\sqrt{5}}{\pi^2\sqrt{\pi}} (\pi^2 - 12)
\end{aligned}$$



So, g is

$$\begin{aligned}
g(x) &= \frac{2}{\pi} + \frac{10(\pi^2 - 12)}{\pi^5} (6x^2 - 6\pi x + \pi^2) \\
&\approx -0.4177x^2 + 1.3122x - 0.0505
\end{aligned}$$

3 Fourier Approximations (Calculus)

We will now look at a special type of least squares approximation called a **Fourier approximation**. For this approximation, consider functions of the form

$$g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$$

in the subspace W of $C[0, 2\pi]$ spanned by the basis

$$S = \{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx\}$$

These $2n + 1$ vectors are *orthogonal* in the inner product space $C[0, 2\pi]$ because

$$\langle f, g \rangle = \int_0^{2\pi} f(x) g(x) \, dx = 0, f \neq g,$$

Moreover, by normalizing each function in this basis, we obtain the orthonormal basis

$$B = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \ldots, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin x, \ldots, \frac{1}{\sqrt{\pi}} \sin nx \right\}$$

With this orthonormal basis, we can apply Theorem 5.19 to write

$$g(x) = \langle f, \mathbf{w}_0 \rangle \mathbf{w}_0 + \langle f, \mathbf{w}_1 \rangle \mathbf{w}_1 + \cdots + \langle f, \mathbf{w}_{2n} \rangle \mathbf{w}_{2n},$$

It will now called the n**th-order Fourier approximation** of f on the interval $[0, 2\pi]$.

THEOREM 5.20 **Fourier Approximation**
On the interval $[0, 2\pi]$, the least squares approximation of a continuous function f with respect to the vector space spanned by $\{1, \cos x, \cos 2x, \ldots, \cos nx, \sin x, \sin 2x, \ldots, \sin nx\}$ is given by

$$g(x) = \frac{a_0}{2} + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$$

where the **Fourier coefficients** $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ are

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx \\
a_j &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos jx, \quad j = 1, 2, \ldots, n \\
b_j &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin jx, \quad j = 1, 2, \ldots, n
\end{aligned}$$

Example 7. **Finding a Fourier Approximation**

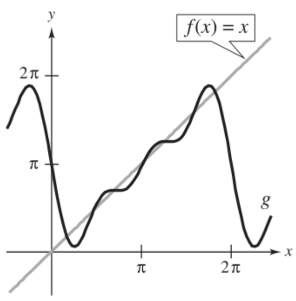
Find the third-order approximation of $f(x) = x, 0 \le x \le 2\pi$.

SOLUTION. Using Theorem 5.20, we have

$$g(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}2\pi^2 = 2\pi \\ a_j &= \frac{1}{\pi}\int_0^{2\pi} x\cos jx = \left[\frac{1}{\pi j^2}\cos jx + \frac{x}{\pi j}\sin jx\right]_0^{2\pi} = 0 \\ b_j &= \frac{1}{\pi}\int_0^{2\pi} x\sin jx = \left[\frac{1}{\pi j^2}\sin jx + \frac{x}{\pi j}\cos jx\right]_0^{2\pi} = -\frac{2}{j} \end{aligned}$$



Third-Order Fourier Approximation

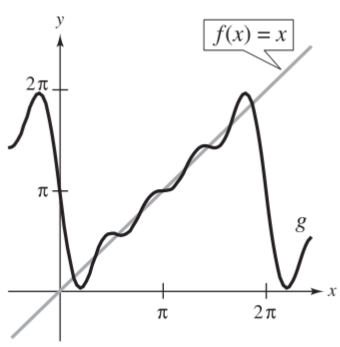
So, we have

$$\begin{aligned} g(x) &= \frac{2\pi}{2} - 2\sin x - \sin 2x - \frac{2}{3}\sin 3x \\ &= \pi - 2\sin x - \sin 2x - \frac{2}{3}\sin 3x \end{aligned}$$

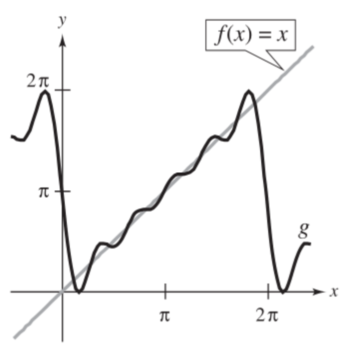
We've got a general pattern for the Fourier coefficients here. The nth-order Fourier approximation is

$$g(x) = \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots + \frac{1}{n}\sin nx\right)$$

As n increases, the Fourier approximation improves.



Fourth-Order Fourier Approximation



Fifth-Order Fourier Approximation

In advanced courses, it is shown that as $n \to \infty$, the approximation error $\|f - g\|$ approaches zero for all $x \in [0, 2\pi]$. The infinite *series* for $g(x)$ is called a **Fourier series**.

Example 8. **Finding a Fourier Approximation**

Find the fourth-order Fourier approximation of $f(x) = |x - \pi|, 0 \le x \le 2\pi$.

SOLUTION. Applying Theorem 5.20, find the Fourier coefficients and obtaining

$$g(x) = \frac{\pi}{2} + \frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x$$