# Part I

# Introduction to Linear Transformations

In this chapter, we will learn about functions that **map** a vector space $V$ into a vector space $W$. This type of function is denoted by

$$T : V \to W$$

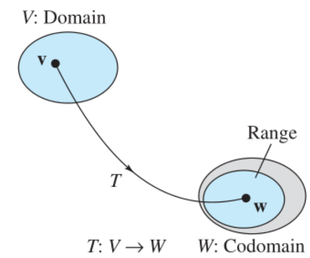
For instance, $V$ is called the **domain** of $T$, and $W$ is called the **codomain** of $T$. If $\mathbf{v} \in V, \mathbf{w} \in W$ such that

$$T(\mathbf{v}) = \mathbf{w}$$

then $\mathbf{w}$ is called the **image** of $v$ under $T$. The set of all images of vectors in $V$ is called the **range** of $T$, and the set of all $\mathbf{v}$ in $V$ such that $T(\mathbf{v}) = \mathbf{w}$ is called **preimage** of $\mathbf{w}$.

**REMARK.** For a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$, it would be technically correct to use double parentheses to denote $T(\mathbf{v})$. Howerver, for convenient, one parentheses is dropped

$$T(\mathbf{v}) = T(v_1, v_2, \ldots, v_n)$$




**Example 1.** **A Function from $\mathbb{R}^2$ into $\mathbb{R}^2$**

For any vector $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(v_1, v_2), (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of $\mathbf{v} = (-1, 2)$.
(b) Find the preimage of $\mathbf{w} = (-1, 11)$.

*SOLUTION.* (a) For $\mathbf{v} = (-1, 2)$ you have

$$T(-1, 2) = (-1 - 2, -1 + 2(2)) = (-3, 3)$$

(b) If $T(\mathbf{v}) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$, then $\begin{cases} v_1 - \phantom{2}v_2 = -1 \\ v_1 + 2v_2 = 11 \end{cases}$

This system of equations has the unique solution $v_1 = 3$ and $v_2 = 4$. So, the preimage of $(-1, 11)$ is $(3, 4)$.

**Definition of Linear Transformations.** Let $V$ and $W$ be vector spaces. The function $T : V \to W$ is called a **linear transformation** of $V$ into $W$ if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$

for all $\mathbf{u}$ and $\mathbf{v}$ in $V$.

A linear transformation is said to be *operation preserving*, because the same result occurs whether the operations of addition and scalar multiplication are performed *before* or *after* the linear transformation is applied.


**REMARK.** A linear transformation $T : V \to V$ from a vector space **itself** is called a **linear operator**.

**Example 3.** **Some functions that are not Linear Transformations**
(a) $f(x) = \sin x$ is not a linear transformation from $\mathbb{R}$ to $\mathbb{R}$ because, in general,

$$\sin x_1 + x_2 \neq \sin x_1 + \sin x_2$$

(b) $f(x) = x^2$is not a linear transformation from $\mathbb{R}$ to $\mathbb{R}$ because, in general,

$$(x_1 + x_2)^2 \neq x_1^2 + x_2^2$$

(c) $f(x) = x + 1$ is not a linear transformation from $\mathbb{R}$ to $\mathbb{R}$ because

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

whereas $f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$.

Two simple linear transformations are

1. **Zero transformation** : $T(\mathbf{v}) = \mathbf{0}, \quad \forall \mathbf{v}$.
2. **Identity transformation** : $T(\mathbf{v}) = \mathbf{v}, \quad \forall \mathbf{v}$.

*THEOREM 6.1* **Properties of Linear Transformations**

Let $T$ be a linear transformation from $V$ to $W$, where $\mathbf{u}$ and $\mathbf{v}$ are in $V$.

1. $T(\mathbf{0}) = \mathbf{0}$
2. $T(-\mathbf{v}) = -T(\mathbf{v})$
3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$
4. If $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n(v)_n$, then

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n(v)_n) \\ &= c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n) \end{aligned}$$

*Property 4.* tells you that a linear transformation $T : V \to W$ is determined completely by its action on a basis of $V$. In other words, if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis for $V$, and if $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)$ is given, then $T(\mathbf{v})$ is determined for *any* $\mathbf{v}$ in $V$.

**Example 4.** Linear Transformations and Bases

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$\begin{aligned} T(1, 0, 0) &= (2, -1, 4) \\ T(0, 1, 0) &= (1, 5, -2) \\ T(0, 0, 1) &= (0, 3, 1) \end{aligned}$$

Find $T(2, 3, -2)$.

*SOLUTION.*

$$\begin{aligned} T(2, 3, -2) &= 2T(1, 0, 0) + 3T(0, 0, 0) - 2T(0, 0, 1) \\ &= (7, 7, 0) \end{aligned}$$

**Note.** The properties in *THEOREM 6.1* provide a quick way to spot functions that are not linear transformations.

**Example 5.** A Linear Transformation Defined by a Matrix

The function $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined as follows.

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

We can use the properties of *matrix multiplication* to show that $T$ is a linear transformation

$$T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$

$$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$$

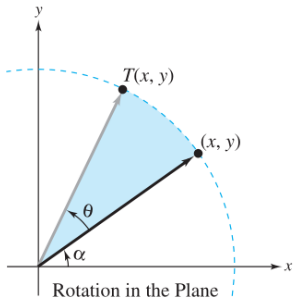
*THEOREM 6.2* **The Linear Transformation Given by a Matrix**

Let $A$ be an $m \times n$ matrix. The function $T$ defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

**Example 7.** **Rotation in the Plane**



Rotation in the Plane

Show that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has the property that it rotate every vector in $\mathbb{R}^2$ counterclockwise about the origin through the angle $\theta$.

*SOLUTION.* Let $\mathbf{v} = (x, y)$be a vector in $\mathbb{R}^2$. Using polar coordinates, we can write $\mathbf{v}$ as

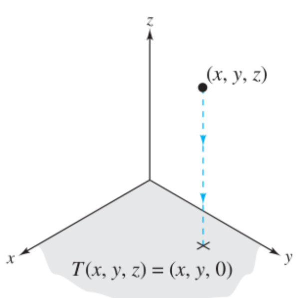
$$\mathbf{v} = (x, y) = (r\cos\alpha, r\sin\alpha)$$

Now applying the linear transformation $T$ to $\mathbf{v}$ produces

$$\begin{aligned} T(\mathbf{v}) = A\mathbf{v} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix} \\ &= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix} \\ &= \begin{bmatrix} r\cos\theta + \alpha \\ r\sin\theta + \alpha \end{bmatrix} \end{aligned}$$

**REMARK.** The above linear transformation is called a **rotation** in $\mathbb{R}^2$. Rotations in $\mathbb{R}^2$ preserve both vector *length* and the *angle* between 2 vectors.

**Example 8.** **A Projection in $\mathbb{R}^3$**



Projection onto $xy$-plane

A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ represented by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is called a **projection** in $\mathbb{R}^3$.
$T(x, y, z) = (x, y, 0)$, in other words, $T$ maps every vector in $\mathbb{R}^3$ to its orthogonal projection in the $xy$-plane.

**Example 9.** **A Linear Transformation from $M_{m\times n}$ into $M_{n\times m}$**

Let $T : M_{m,n} \to M_{n,m}$ be the function that maps an $m \times n$ matrix $A$ to its transpose.

$$T(A) = A^T$$

Show that $T$ is a linear transformation.

*SOLUTION.* Let $A$ and $B$ be $m \times n$ matrices. Then,

$$\begin{aligned} T(A + B) &= (A + B)^T \\ &= A^T + B^T \\ &= T(A) + T(B) \end{aligned}$$

and

$$\begin{aligned} T(cA) &= (cA)^T \\ &= c(A^T) \\ &= cT(A) \end{aligned}$$

**Example 10.** **The Differential Operator (Calculus)**

Let $C'[a, b]$ be the set of all functions whose derivatives are continuous on$[a, b]$. Show that the differential operator $D_x$ defines a linear transformation from $C'[a, b]$ into $C[a, b]$.

*SOLUTION.* Using operator notation, we can write

$$D_x(f) = \frac{d}{dx}(f)$$

where $f \in C'[a,b]$.

We have

$$\begin{aligned} D_x(f+g) = \frac{d}{dx}[f+g] &= \frac{d}{dx}[f] + \frac{d}{dx}[g] \\ &= D_x(f) + D_x(g) \end{aligned}$$

Similarly,

$$\begin{aligned} D_x(cf) = \frac{d}{dx}[cf] &= c\frac{d}{dx}[f] \\ &= cD_x(f) \end{aligned}$$

So, $D_x$ is a linear transformation from $C'[a,b]$ to $C[a,b]$. It's called **differential operator**.

For polynomials, the differential operator is $D_x : P_n \to P_{n-1}$ because the derivative of a polynomial function of degree $n$ is a polynomial function of degree $n-1$ or less.

$$D_x(a_n x^n + \cdots + a_1 x + a_0) = n a_n x^{x-1} + \cdots + a_1$$

**Example 11.** **The Definite Integral as a Linear Transformation (Calculus)**

Let $T : P \to \mathbb{R}$ be defined by

$$T(p) = \int_a^b p(x)\,dx$$

Show that $T$ is a linear transformation from $P$, the vector space of polynomial functions, into $\mathbb{R}$, the vector space of real numbers.

*SOLUTION.*

Using the properties of definite integrals, we can write

$$\begin{aligned} T(p+q) &= \int_a^b [p(x)+q(x)]\,dx \\ &= \int_a^b p(x)\,dx + \int_a^b q(x)\,dx \\ &= T(p) + T(q) \end{aligned}$$

and

$$T(cp) = \int_a^b c[p(x)]\,dx = c\int_a^b p(x)\,dx = cT(p)$$

So, $T$ is a linear transformation.

# Part II

# The Kernel and Range of a Linear Transformation

**Definition of Kernel of a Linear Transformation**
Let $T : V \to W$ be a linear transformation. Then the set of all vectors $\mathbf{v}$ in $V$ that satisfy $T(\mathbf{v}) = \mathbf{0}$ is called the **kernel** of $T$ and is denoted by $\ker(T)$.

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$$

Sometimes the kernel of a transformation is obvious.

**Example 1.** **Finding the Kernel of a Linear Transformation**

Let $T : M_{3,2} \to M_{2,3}$ be the linear transformation that maps a $3 \times 2$ matrix $A$ to its transpose. The kernel of $T$ consists of a single element: the zero matrix in $M_{3,2}$.

**Example 2.** **The Kernels of the Zero and Identity Transformations**
(a) The zero transformation: $\ker(T) = V$.
(b) The identity transformation: $\ker(T) = \{\mathbf{0}\}$.

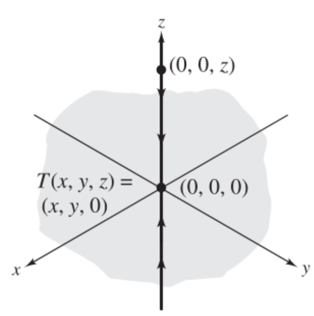
**Example 3.** **Finding the Kernel of a Linear Transformation**

Find the kernel of the projection $T : \mathbb{R}^3 \to \mathbb{R}^3$ represented by

$$T(x, y, z) = (x, y, 0)$$

*SOLUTION.* The kernel consists of all vectors lying on the $z$-axis.

$$\ker(T) = \{(0, 0, z) : z \in \mathbb{R}\}$$



The kernel of $T$ is the set of all vectors on the $z$-axis.

These above kernels were fairly easy to find. Generally, the kernel of a linear transformation is not so obvious.

**Example 4.** **Finding the Kernel of a Linear Transformation**

Find the kernel of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ represented by

$$T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1)$$

*SOLUTION.* To find $\ker(T)$, we need to find all $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ such that

$$T(x_1, x_2) = (x_1 - 2x_2, 0, -x_1) = (0, 0, 0)$$

This leads to the homogeneous system $\begin{cases} x_1 - 2x_2 &= 0 \\ \quad\quad 0 &= 0 \\ -x_1 &= 0 \end{cases}$, which has only the trivial solution.

So, we have $\ker(T) = \{(0, 0)\} = \{\mathbf{0}\}$.

**Example 5.** **Finding the Kernel of a Linear Transformation**

Find the kernel of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}$$

*SOLUTION.* The kernel of $T$ is the set of all $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ such that

$$T(x_1, x_2, x_3) = (0, 0)$$

From this equation, we can write the homogeneous system

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{cases} x_1 - \phantom{2}x_2 - 2x_3 = 0 \\ -x_1 + 2x_2 + 3x_3 = 0 \end{cases}$$

Writing the augmented matrix of this system in reduced row-echelon form produces

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \implies \begin{cases} x_1 = x_3 \\ x_2 = -x_3 \end{cases}$$

Using the parameter $t = x_3$ produces the family of solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

So, the kernel of $T$ is represented by

$$\begin{aligned} \ker(T) &= \{t(1, -1, 1) : t \in \mathbb{R}\} \\ &= \operatorname{span}(1, -1, 1) \end{aligned}$$

> *THEOREM 6.3* **The Kernel Is a Subspace of $V$**
> The kernel of a linear transformation $T : V \to W$ is a subspace of the domain $V$.

**REMARK.** As a result of Theorem 6.3, the kernel $T$ is sometimes called the **nullspace** of $T$.

**Example 6. Finding a Basis for the Kernel**

Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is in $\mathbb{R}^5$ and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for $\ker(T)$ as a subspace of $\mathbb{R}^5$.

*SOLUTION.* Reduce the augmented matrix $[A \vdots \mathbf{0}]$ to echelon form as follows.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{cases} x_1 \qquad\qquad = -2x_3 + x_5 \\ \qquad x_2 \qquad = x_3 + 2x_5 \\ \qquad\qquad x_4 = -4x_5 \end{cases}$$

Letting $x_3 = s$ and $x_5 = t$, we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s + t \\ s + 2t \\ s + 0t \\ 0s - 4t \\ 0s + t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

So one basis for the kernel $T$ is $B = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}$.

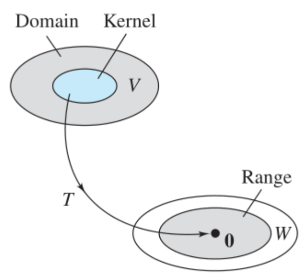
> ***THEOREM : COROLLARY***
>
> Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then the kernel of $T$ is equal to the solution space of $A\mathbf{x} = \mathbf{0}$.
> $$\ker(T) = N(A)$$

# 1 The Range of a Linear Transformation

Two critical subspaces associated with a linear transformation are the **kernel** and the **range** of $T$.

$$\text{range}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$$

> *THEOREM 6.4* **The Range of $T$ Is a Subspace of $W$**
> The range of a linear transformation $T : V \to W$ is a subspace of $W$.




**Note.** The kernel and range of $T : V \to W$ are subspaces of $V$ and $W$, respectively.

To find a basis for the **range** of $T(\mathbf{x}) = A\mathbf{x}$, observe that the range consists of all vectors $\mathbf{b}$ such that the system $A\mathbf{x} = \mathbf{b}$ is *consistent.* By writing the system

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in the form

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \mathbf{b}$$

You can see that $\mathbf{b}$ is in the range of $T$ if and only if $\mathbf{b}$ is a linear combination of the column vectors of $A$. So *the column space of $A$ is the same as the range of $T$.*

> ***THEOREM :*** ***COROLLARY***
>
> Let $T : \mathbb{R}^n \to \mathbb{R}^m$ to be the linear transformation given by $T(\mathbf{x} = A\mathbf{x}.)$ Then the column space of $A$ is equal to the range of $T$.
>
> $$R(A) = \text{range}(T)$$

**Example 7.** **Finding a Basis for the Range of a Linear Transformation**

For the linear transformation $\mathbb{R}^5 \to \mathbb{R}^4$ from the Example 6, find a basis for the range of $T$.

*SOLUTION.* The echelon from of $A$ was calculated in Example 6.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Because the leading 1's appear in columns 1, 2, and 4 of the reduced matrix on the right, the corresponding column vectors of $A$ form a basis for $R(A)$. One basis for range$(T)$ is

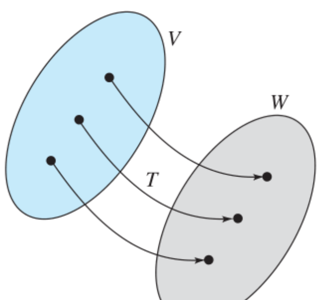
$$B = \{(1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2)\}$$

> **Definition of Rank and Nullity of a Linear Transformation**
> Let $T : V \to W$ be a linear transformation. The dimension of the kernel of $T$ is called the **nullity** of $T$. The dimension of the range of $T$ is called the **rank** of $T$.
>
> $$\textbf{nullity}(T) = \dim(\ker(T))$$
>
> $$\textbf{rank}(T) = \dim(\text{range}(T))$$

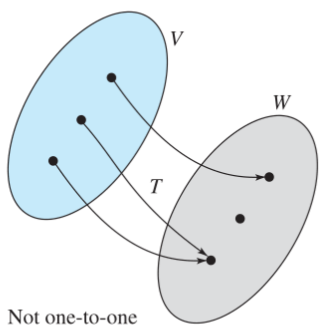
**REMARK.** If $T$ is provided by a matrix $A$, then $\text{rank}(T) = \text{rank}(A)$.

> *THEOREM 6.5* **Sum of Rank and Nullity**
> Let $T : V \to W$ be a linear transformation from an $n$-dimensional vector space $V$ into a vector space $W$. Then the sum of the dimensions of the range and the kernel is equal to the dimension of the **domain**.
> $$\text{rank}(T) + \text{nullity}(T) = n$$
> $$\text{dim(range)]} + \text{dim(kernel)} = \text{dim(domain)}$$

## 2 One-to-One and Onto Linear Transformation




One-to-one




Not one-to-one

A function $T : V \to W$ is called **one-to-one** if the preimage of every $\mathbf{w}$ in the range consists of a single vector.

$$T(\mathbf{u}) = T(\mathbf{v}) \implies \mathbf{u} = \mathbf{v}$$

for all $\mathbf{u}$ and $\mathbf{v}$ in $V$.

> *THEOREM 6.6* **One-to-One Linear Transformations**
> Let $T : V \to W$ be a linear transformation. Then $T$ is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

A function $T : V \to W$ is called **onto** if every element in $W$ has a preimage in $V$. In other words, $T$ is onto $W$ when $W$ is equal to the range of $T$.

> *THEOREM 6.7* **Onto Linear Transformations**
> Let $T : V \to W$ be a linear transformation, where $W$ is finite dimensional. Then $T$ is onto $W$ if and only if the rank of $T$ is equal to the dimension of $W$.

For vector spaces of equal dimensions, you can combine the results of Theorems 6.5, 6.6 and 6.7 to obtain the next theorem relating to the concepts of one-to-one and onto.

> *THEOREM 6.8* **One-to-One and Onto Linear Transformations**
> Let $T : V \to W$ be a linear transformation with vector spaces $V$ and $W$ *both* of dimension $n$. Then $T$ is one-to-one if and only if it is onto.

**Example 11.** **Summarizing Several Results**

The linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is represented by $T(\mathbf{x}) = A\mathbf{x}$. Find the nullity and rank of $T$, then determine whether $T$ is one-to-one, onto, or neither.

$$\text{(a)}A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(b)}A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{(c)}A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{(d)}A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

| $T: R^n \to R^m$ | Dim(domain) | Dim(range) Rank($T$) | Dim(kernel) Nullity($T$) | One-to-One | Onto |
|---|---|---|---|---|---|
| (a) $T: R^3 \to R^3$ | 3 | 3 | 0 | Yes | Yes |
| (b) $T: R^2 \to R^3$ | 2 | 2 | 0 | Yes | No |
| (c) $T: R^3 \to R^2$ | 3 | 2 | 1 | No | Yes |
| (d) $T: R^3 \to R^3$ | 3 | 2 | 1 | No | No |

# 3 Isomorphisms of Vector Spaces

This is a very important concept that can be a great aid in your understanding of vector spaces. The concept provides a way to think of distinct vector spaces as being ”essentially the same” - at least with respect to the operations of vector addition and scalar multiplication.

**Definition : Isomorphism**

A linear transformation $T : V \to W$ that is one-to-one and onto are called an **isomorphism**.

If $V$ and $W$ are vector spaces such that there exists an isomorphism from $V$ to $W$, then $V$ and $W$ are **isomorphic** to each other.

*THEOREM 6.9* **Isomorphic Spaces and Dimension**
Two finite-dimensional vector spaces $V$ and $W$ are isomorphic if and only if they are of the same dimension.

**Example 12. Isomorphic Vector Spaces**

The vectors listed below are mutually isomorphic.
(a) $\mathbb{R}^4$ = 4-space
(b) $M_{4,1}$ = space of all $4 \times 1$ matrices
(c) $M_{2,2}$ = space of all $2 \times 2$ matrices
(d) $P_3$ = space of all polynomials of degree 3 or less
(e) $V = \{(x_1, x_2, x_3, x_4, 0) : x_i \in \mathbb{R}\}$ (subspace of $\mathbb{R}^5$)

# Part III

# Matrices for Linear Transformations

Which representation of $T\mathbb{R}^3 \to \mathbb{R}^3$ is better,

$$T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

or

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}?$$

***ANSWER***

The second one is better for 3 reasons: it is simpler to write, simpler to read, and more easily adapted for computer use.

The key to representing $T : V \to W$ by a matrix is to determine how it acts on a **basis** of $V$. Once you know the image of every vector in the basis, you can determine $T(\mathbf{v})$ for any $\mathbf{v} \in V$.

*THEOREM 6.10* **Standard Matrix for a Linear Transformation**
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, T(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Then the $m \times n$ matrix whose $n$ columns correspond to $T(\mathbf{e}_i)$,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in $\mathbb{R}^n$. A is called the **standard matrix** for $T$.

**Example 1. Finding the Standard Matrix for a Linear Transformation**

Find the standard matrix for $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (x - 2y, 2x + y)$$

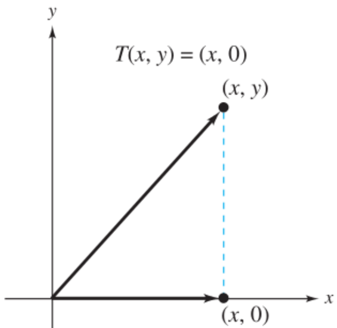
*SOLUTION.* Begin by finding the images of $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$.

| *Vector Notation* | *Matrix Notation* |
|---|---|
| $T(\mathbf{e}_1) = T(1, 0, 0) = (1, 2)$ | $T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ |
| $T(\mathbf{e}_2) = T(0, 1, 0) = (-2, 1)$ | $T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ |
| $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0)$ | $T(\mathbf{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ |

By Theorem 6.10, we have

$$A = [T(\mathbf{e}_1) \vdots T(\mathbf{e}_2) \vdots T(\mathbf{e}_3)] = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

**Example 2. Finding the Standard Matrix for a Linear Transformation**



Projection onto the $x$-axis

The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by projecting each point in $\mathbb{R}^2$ onto the $x$-axis.

$$T(x, y) = T(x, 0)$$

The standard matrix for $T$ is

$$A = \begin{bmatrix} T(1,0) & \vdots & T(0,1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

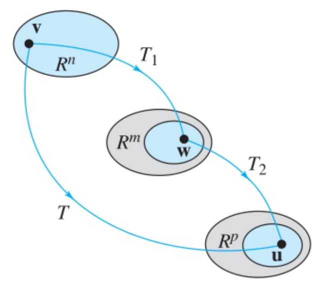
# 4 Composition of Linear Transformations

The **composition** , $T$, of $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ with $T_2 : \mathbb{R}^m to \mathbb{R}^p$ is defined by

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$$

where $\mathbf{v}$ is a vector $\mathbb{R}^n$. This composition is denoted by

$$T = T_2 \circ T_1$$



Composition of Transformations

> *THEOREM 6.11* **Composition of Linear Transformations**
> Let $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformation with standard matrices $A_1$ and $A_2$. The **composition** $T : \mathbb{R}^n \to \mathbb{R}^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a linear transformation, which standard matrix $A$ is given by
> $$A = A_2 A_1$$

**REMARK.** Theorem 6.11 can be generalized to cover the composition of $n$ linear transformation. That is, if the standard matrices of $T_1, T_2, \ldots, T_n$ are $A_1, A_2, \ldots, A_n$, then

$$A = A_n A_{n-1} \cdots A_2 A_1$$

**Example 3.** **The Standard Matrix for a Composition**

Let $T_1$ and $T_2$ be linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T_1(x, y, z) = (2x + y, 0, x + z) \quad \text{and} \quad T_2(x, y, z) = (x - y, z, y)$$

Find the standard matrices for the compositions $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.

*SOLUTION.* The standard matrices for $T_1$ and $T_2$ are

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

By Theorem 6.11, the standard matrix for $T$ is

$$A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the standard matrix for $T'$ is

$$A' = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Another benefit of matrix representation is that it can represent the **inverse** of a linear transformation.

**Definition : Inverse of Linear Transformation**

If $T_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $T_2 : \mathbb{R}^n \to \mathbb{R}^n$ are linear transformations such that for every $\mathbf{v} \in \mathbb{R}^n$

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v},$$

then $T_2$ is called the **inverse** of $T_1$, and $T_1$ is said to be **invertible**.

If $T_1$ is invertible, the inverse is **unique** and is denoted by $T_1^{-1}$. The inverse of $T$ can be thought as *undoing the mapping* done by $T$ - in other words, mapping back to the preimage under $T$.

*THEOREM 6.12* **Existence of an Inverse Transformation**
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix $A$. Then the following conditions are equivalent.
1. $T$ is invertible.
2. $T$ is an isomorphism.
3. $A$ is invertible.
And, the standard matrix for $T^{-1}$ is $A^{-1}$.

# 5 Nonstandard Bases and General Vector Spaces

We will now consider the problem of finding a matrix for $T : V \to W$, where $B$ and $B'$ are ordered bases for $V$ and $W$. In order to represent $T$, $A$ must be multipled by a *coordinate matrix relative to* $B$. The result will be a *coordinate matrix relative to* $B'$. That is,

$$[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B$$

$A$ is called the **matrix of** $T$ **relative to the bases** $B$ **and** $B'$.

**Transformation Matrix for Nonstandard Bases.**
Let $V$ and $W$ be finite-dimensional vector spaces with bases $B$ and $B'$, where

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

If $T : V \to W$ is a linear transformation such that

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, [T(\mathbf{v}_n)]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m \times n$ matrix whose $n$ columns correspond to $[T(\mathbf{v}_i)]_{B'}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B$ for $\forall \mathbf{v} \in V$.

**Example 5. Finding a Matrix Relative to Nonstandard Matrix**

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by

$$T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

Find the matrix for $T$ relative to the bases

$$B = \{\overset{\mathbf{v}_1}{(1,2)}, \overset{\mathbf{v}_2}{(-1,1)}\} \quad \text{and} \quad B' = \{\overset{\mathbf{w}_1}{(1,0)}, \overset{\mathbf{w}_2}{(0,1)}\}$$

*SOLUTION.* By the definition of $T$, we have

$$T(\mathbf{v}_1) = T(1,2) = (3,0) = 3\mathbf{w}_1 + 0\mathbf{w}_2$$
$$T(\mathbf{v}_2) = T(-1,1) = (0,-3) = 0\mathbf{w}_1 - 3\mathbf{w}_2$$

The coordinate matrices for $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ relative to $B'$ are

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

The matrix for $T$ relative to $B$ and $B'$ is $A = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$.

**Example 6.** **Using a Matrix to Represent a Linear Transformation**

Using $T : \mathbb{R}^2 \to \mathbb{R}^2$ from Example 5, find $T(\mathbf{v})$, where $\mathbf{v} = (2, 1)$.

*SOLUTION.* Using the basis $B$, we find $[\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So,$[T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.
It follows that $T(\mathbf{v}) = 3(1, 0) + 3(0, 1) = (3, 3)$.

In the special case, where $V = W$ and $B = B'$, then $A$ is called the **matrix of $T$ relative to basis $B$**. The matrix of the identity transformation is just $I_n$.

**Example 7.** **A Matrix for the Differential Operator (Calculus)**

Let $D_x : P_2 \to P_1$ be the differential operator that maps a quadratic polynomial $p$ onto its derivative $p'$. Find the matrix for $D_x$ using the bases

$$B = \{1, x, x^2\} \quad \text{and} \quad B' = \{1, x\}$$

*SOLUTION.* The derivatives of the basis vectors are

$$\begin{aligned} D_x(1) &= 0 = 0(1) + 0(x) \\ D_x(x) &= 1 = 1(1) + 0(x) \\ D_x(x^2) &= 2x = 0(1) + 2(x) \end{aligned}$$

So, the matrix for $D_x$ is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Note that this matrix *does* produce the derivative of a quadratic polynomial $p(x) = a + bx + cx^2$.

$$Ap = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ 2c \end{bmatrix}$$

# Part IV

# Transition Matrices and Similarity

The matrix for $T : V \to V$ depends on the *basis* of $V$. In other words, the matrix for $T$ relative to a basis $B$ is different from the one for $B'$.

**A classic problem**

Is it possible to find a basis $B$ such that the matrix for $T$ relative to $B$ is diagonal? The solution will be discussed in Chapter 7. We will see how matrices for $T$ relative to 2 different bases are related.

In this section, $A, A', P$ and $P^{-1}$ represent the 4 square matrices listed below.

1. Matrix for $T$ relative to $B$: $A$
2. Matrix for $T$ relative to $B'$: $A'$
3. Transition matrix from $B'$ to $B$: $P$
4. Transition matrix from $B$ to $B'$: $P^{-1}$

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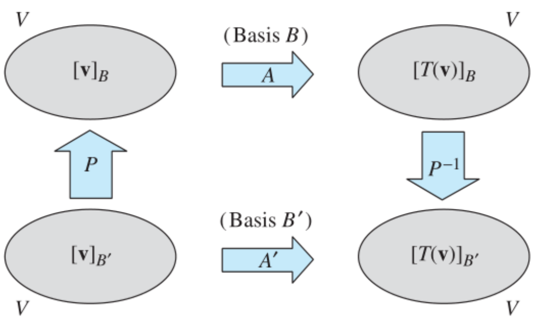
There are 2 ways to get from the coordinate matrix $[\mathbf{v}]_{B'}$ to $[T(\mathbf{v})]_{B'}$.




One way is direct, using $A'$ to obtain:

$$A'[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

The other way is indirect, using $P, A$ and $P^{-1}$ to obtain

$$P^{-1}AP[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

**Conclude**

By the definition of the matrix of a linear transformation relative to a basis, this implies that

$$A' = P^{-1}AP$$

**Example 1. Finding a Matrix of a Linear Transformation**

Find the matrix $A$ for $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$$

*SOLUTION.* The standard matrix for $T$ is $A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$.

The transition matrix from $B'$ to $B$ is $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

The inverse of this matrix is the transition matrix from $B$ to $B'$: $P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

The matrix for $T$ relative to $B'$ is

$$A' = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

**Example 2. Finding a Matrix for a Linear Transformation**

Let

$$B = \{(-3, 2), (4, -2)\} \quad \text{and} \quad B' = \{(-1, 2), (2, -2)\}$$

be bases for $\mathbb{R}^2$, and let

$$A = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}$$

be the matrix for $T : \mathbb{R}^2 \to \mathbb{R}^2$ relative to $B$. Find $A'$, the matrix of $T$ relative to $B'$.

*SOLUTION.* We can easily obtain

$$P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$

So the matrix of $T$ relative to $B'$ is

$$A' = P^{-1}AP = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

**REMARK.** It is instructive to note that $T$ in Example 2 is represented by the rule $T(x, y) = (x - \frac{3}{2}y, 2x + 4y)$.

## 6 Similar Matrices

**Definition : Similar Matrices**

For square matrices $A$ and $A'$ of order $n$, $A'$ is said to be **similar** to $A$ if there exist an invertible matrix $P$ such that $A' = P^{-1}AP$.

If $A'$ is similar to $A$, then it is also true that $A$ is similar to $A'$. Just simply say that ***A* and *A'* are similar**.

*THEOREM 6.13* **Properties of Similar Matrices**
Let $A, B$ and $C$ be square matrices of order $n$. Then the following properties are true.
1. $A$ is similar to $A$.
2. If $A$ is similar to $B$, then $B$ is similar to $A$.
3. If $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$.

It follows that any 2 matrices that represent the same $T : V \to V$ with respect to different bases must be similar.

**Example 4.** **Similar Matrices**

From Example 1, the matrices $A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$ and $A' = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$ are similar because $A' = P^{-1}AP$, where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

---

You have seen that the matrix for $T : V \to V$ depends on the basis used for $V$.

What choice of basis will make the matrix for $T$ as *simple* as possible? Is it always the *standard* basis? Not necessarily, as the next example will demonstrate.

**Example 5.** **A Comparision of Two Matrices for a Linear Transformation**

Suppose

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is the matrix for $T : \mathbb{R}^3 \to \mathbb{R}^3$ relative to the standard basis. Find the matrix for $T$ relative to the basis

$$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$$

*SOLUTION.* We have

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, the matrix for $T$ relative to $B'$ is

$$\begin{aligned}
A' &= P^{-1}AP \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}
\end{aligned}$$

Note that $A'$ is diagonal.

## Diagonal Matrices

Diagonal matrices have many computational advantages over nondiagonal ones. For instance, for the diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

the $k$th power is represented as follows.

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

A diagonal matrix is its own *transpose*. Moreover, if all the diagonal elements are nonzero, then the inverse of a diagonal matrix is the matrix whose main diagonal elements are reciprocals of corresponding elements in the original one.

**Conclude**

With such computational advantages, it's important to find ways (if possible) to choose a basis for $V$ such that the transformation matrix is diagonal.

## Part V

# Applications of Linear Transformations

## 7 The Geometry of Linear Transformations in the Plane

**Definition : Elementary Matrices for Linear Transformations in the Plane**

*Reflection in $y$-Axis*
$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

*Reflection in $x$-Axis*
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

*Reflection in Line $y = x$*
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

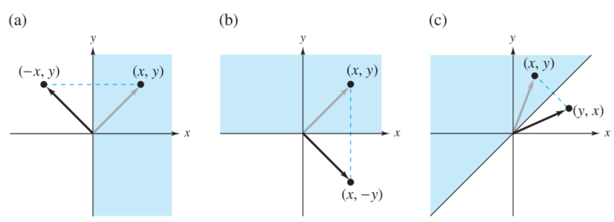
*Horizontal Expansion $(k > 1)$ or Contraction $(0 < k < 1)$*
$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

*Vertical Expansion $(k > 1)$ or Contraction $(0 < k < 1)$*
$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

*Horizontal Shear*
$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

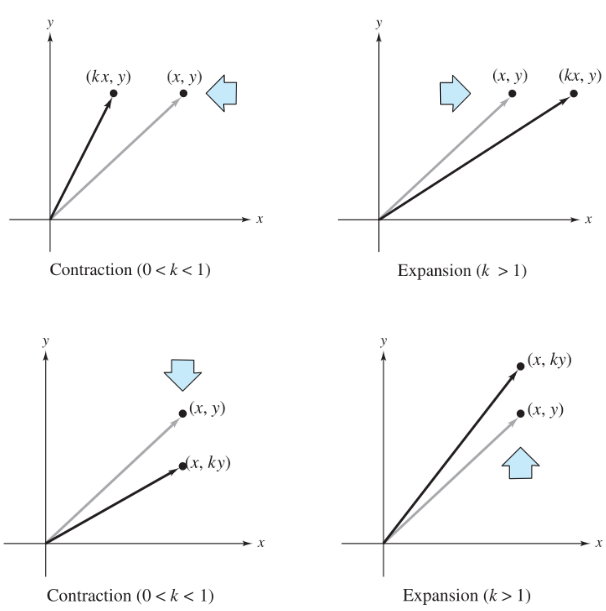
*Vertical Shear*
$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

**Example 1.** Reflections in the Plane



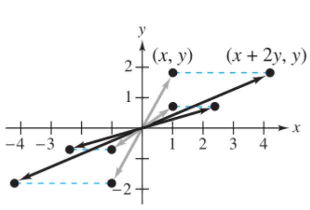
Reflections in the Plane

**Example 2.** Expansions and Contractions in the Plane



Contraction (0 < k < 1)
Expansion (k > 1)
Contraction (0 < k < 1)
Expansion (k > 1)

**Example 3.** Shears in the Plane

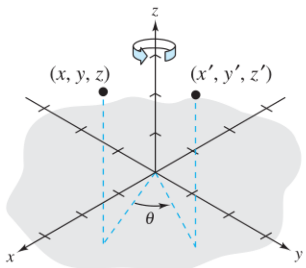



The transformations defined by the following matrices are shears.

$$T(x, y) = (x + ky, y)$$
$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

$$T(x, y) = (x, y + kx)$$
$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$

# 8 Computer Graphics




Suppose we want to rotate the point $(x, y, z)$ counterclockwise about the $z$-axis through an angle $\theta$.

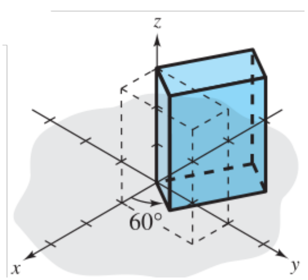
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}$$

**Example 4. Rotation About the $z$-Axis**

The 8 vertices of a rectangular box having sides of lengths 1, 2, and 3 are as follows.

$$V_1 = (0,0,0), \quad V_2 = (1,0,0), \quad V_3 = (1,2,0), \quad V_4 = (0,2,0)$$
$$V_5 = (0,0,3), \quad V_6 = (1,0,3), \quad V_7 = (1,2,3), \quad V_8 = (0,2,3)$$

Find the coordinates of the box when it is rotated $60^o$ counterclockwise about the $z$-axis.




The matrix that yields a rotation of $60^o$ is

$$A = \begin{bmatrix} \cos 60^o & -\sin 60^o & 0 \\ \sin 60^o & \cos 60^o & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And multiplying this matrix by each of the vertices.

**Definition**

All three types of rotations are summarized as follows.

*Rotation About the $x$-Axis*

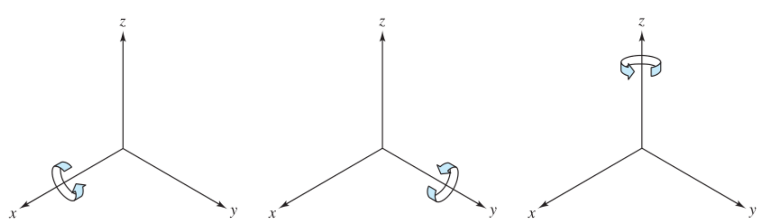
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

*Rotation About the $y$-Axis*

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

*Rotation About the $z$-Axis*

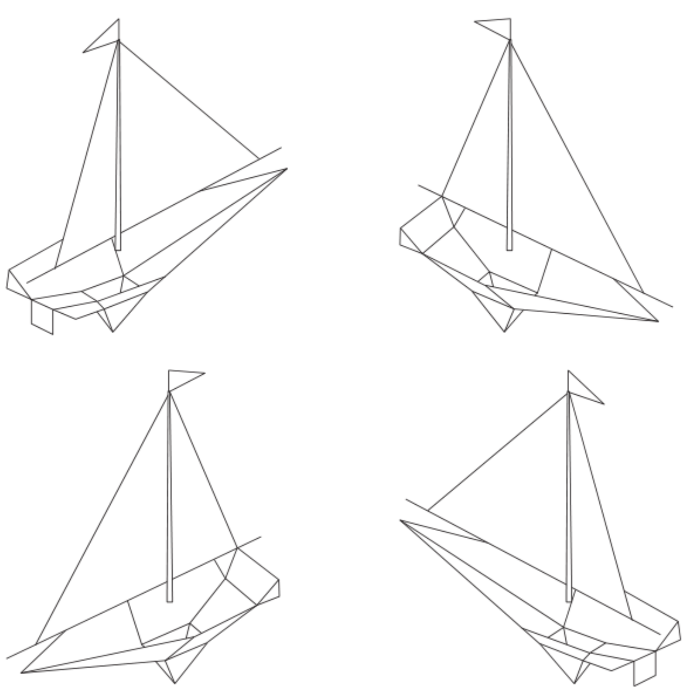
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$




Rotation about the $x$-axis

Rotation about the $y$-axis

Rotation about the $z$-axis



The use of computer graphics has become common among designers in many fields. By simply entering the **coordinates** that form the outline of an object into a computer, a designer can see the object before it is created.