1 Partial Derivatives

Suppose we let x vary while keeping y fixed ($y = b$) in $f(x, y)$, we got a function $g(x) = f(x, b)$. If g has a derivative at a, we call it the **partial derivative of f with respect to x at (a,b)**. We have

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

and so it become

$$\boxed{\begin{aligned} f_x(x,y) &= \lim_{h \to 0} \frac{f(x+h,b) - f(x,y)}{h} \\ f_y(x,y) &= \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} \end{aligned}}$$

To compute partial derivatives, we have the following rule.

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

EXAMPLE. If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$. Holding y constant and differentiating with respect to x, we get

$$\begin{aligned} f_x(x,y) &= 3x^2 + 2xy^3 \\ f_x(2,1) &= 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16 \end{aligned}$$

Do the same with y

$$\begin{aligned} f_y(x,y) &= 3x^2y^2 - 4y \\ f_y(2,1) &= 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8 \end{aligned}$$

Interpretations of Partial Derivatives

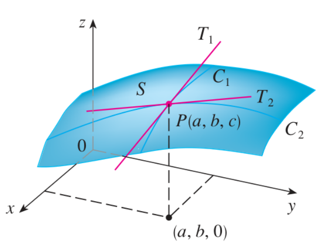


FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

- The equation $f(x, y)$ represent a surface S. By fixing $y = b$, we got the curve C_1 (the trace of S in the pane $y = b$).
- Notice that C_1 is the graph of $g(x) = f(x, b)$, so the slope of its tangent T_1 is $g'(a) = f_x(a, b)$.

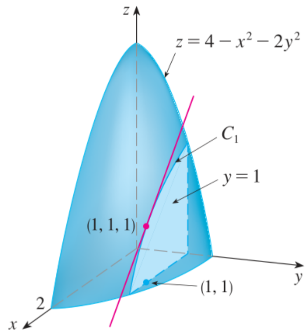


FIGURE 2

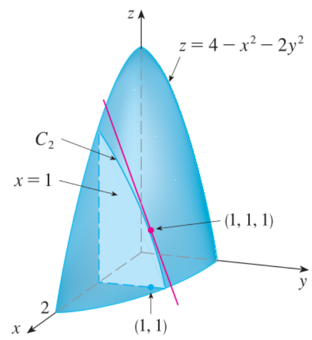


FIGURE 3

EXAMPLE. If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.
We have

$$f_x(x, y) = -2x \qquad f_y(x.y) = -4y$$
$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

The vertical plane $y = 1$ intersects $f(x, y)$ in the parabola $z = 2 - x^2$, $y = 1$ (C_1). The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$.

EXAMPLE. If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\delta f}{\partial x}$ and $\frac{\delta f}{\partial y}$.

Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$
$$\frac{\delta f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

EXAMPLE. Find $\partial z/\partial x$ and $\partial z/\partial y$ of z is defined as follow

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Some computer algebra systems can plot surfaces defined by implicit equations in three variables. Figure 6 shows such a plot of the surface defined by the equation in Example 4.


FIGURE 6

First differentiate implicitly with respect to x, treat y as a constant.

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Solving this for $\partial z/\partial x$, we obtain

$$\frac{\partial z}{\partial x} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Functions of More Than Two Variables

Regarding y and z as constants and differentiating with respect to x.

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

EXAMPLE. Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

Holding y and z constant and differentiating with respect to x, we have

$$f_x = ye^{xy} \ln z$$

Similarly,

$$f_y = xe^{xy} \ln z \quad \text{and} \quad f_z = \frac{e^{xy}}{z}$$

Higher Derivatives

If f is a function of 2 variables, then f_x and f_y are also functions of 2 variables. So we can consider the **second partial derivatives** of f, that is $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$.

$$\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\
(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x} \\
(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y} \\
(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}
\end{aligned}$$

Thus f_{xy} (or $\partial^2 f/\partial y\,\partial x$) means that we first differentiate with respect to x and then with respect to y.

EXAMPLE. Find the second partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

SOLUTION We find that

$$f_x(x,y) = 3x^2 + 2xy^3 \qquad f_y(x,y) = 3x^2y^2 - 4y$$

Therefore

$$\begin{aligned}
f_{xx} &= \frac{\partial}{\partial x}(3x^2+2xy^3) = 6x + 2y^3 \qquad & f_{xy} &= \frac{\partial}{\partial y}(3x^2+2xy^3) = 6xy^2 \\
f_{yx} &= \frac{\partial}{\partial x}(3x^2y^2-4y) = 6xy^2 \qquad & f_{yy} &= \frac{\partial}{\partial y}(3x^2y^2-4y) = 6x^2y - 4
\end{aligned}$$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous in D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Partial derivatives of order 3 or higher can also be defined

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x\,\partial y}\right) = \frac{\partial^3 f}{\partial y^2\,\partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$.

EXAMPLE. Calculate f_{xxyz} if $f(x,y,z) = \sin 3x + yz$.

SOLUTION.

$$\begin{aligned}
f_x &= 3\cos(3x+yz) \\
f_{xx} &= -9\sin(3x+yz) \\
f_{xxy} &= -9z\cos(3x+yz) \\
f_{xxyz} &= -9\cos(3x+yz) + 9yz\sin(3x+yz)
\end{aligned}$$

Partial Differential Equations

Laplace's equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2 Tangent Planes and Linear Approximations

As we zoom in toward a point on a surface of a differentiable function, the surface looks more and more like a plane (its tangent plane) and we can approximate it by a linear function of 2 variables.

Tangent Planes

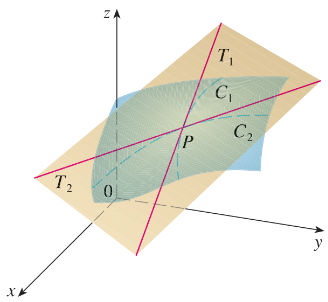


FIGURE 1
The tangent plane contains the tangent lines T_1 and T_2.

Suppose surface S of $z = f(x, y)$ has continuous first partial derivatives, let $P(x_0, y_0, z_0) \in S$.

- C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x0$ with S.
- Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at P. Then the **tangent plane** to the surface S at the point P contains T_1 and T_2. In fact, it consists of *all possible* tangent lines at P.
- We know the plane has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

By dividing this by C and letting $a = -A/C$ and $b = -B/C$,

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

The tangent plane's intersection with the plane $y = y_0$ must be the tangent line T_1.

$$z - z_0 = a(x - x_0) \quad \text{where } y = y_0$$

This is a line with slope $a = f_x(x_0, y_0)$. Similarly, $z - z_0 = b(y - y_0)$, and $b = f_y(x_0, y_0)$.

Definition : Equation of Tangent Plane

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

EXAMPLE. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$
$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then the equation of the tangent plane at (1, 1, 3) is

$$z - 3 = 4(x - 1) + 2(y - 1)$$
$$z = 4x + 2y - 3$$

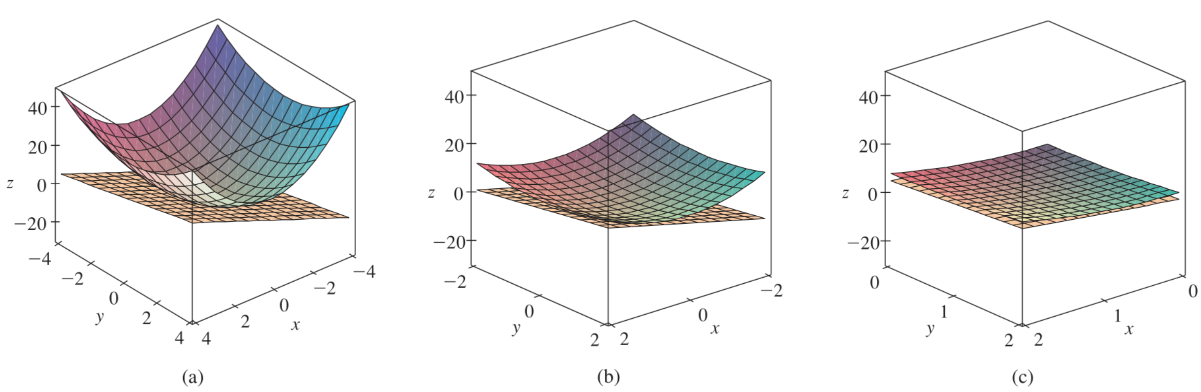


FIGURE 2 The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).

By zooming toward the point $(1, 1)$ on a contour map, we see that the more we zoom in, the more the level curves look like equally spaced parallel lines.

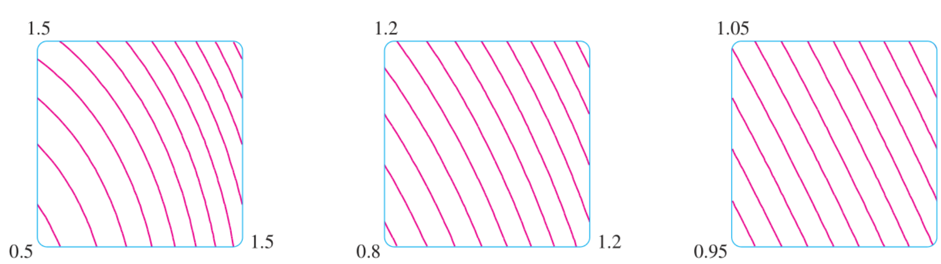


FIGURE 3
Zooming in toward (1, 1) on a contour map of $f(x, y) = 2x^2 + y^2$

Linear Approximations

The equation of the tangent plane of $f(x, y) = 2x^2 + y^2$ at the point (1, 1, 3) is $z = 4x + 2y - 3$. Therefore, the linear function of 2 variables

$$L(x, y) = 4x + 2y - 3$$

is the *linearization of* f at (1, 1) and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is the *linear approximation* or *tangent plane approximation* of f at (1, 1).

Eg: At the point $(1.1, 0.95)$, the linear approximation gives

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$

True value: $f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225$.

Definition

The linearization of f at (a, b). $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$
The linear approximation of f at (a, b). $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

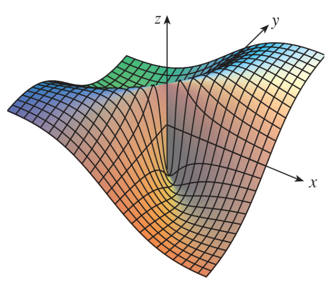


FIGURE 4

$f(x, y) = \dfrac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$,
$f(0, 0) = 0$

What if f_x and f_y are not continuous?

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Even though $f_x(0, 0) = f_y(0, 0) = 0$, but they are not continuous. The linear approximation would be $f(x, y) \approx 0$, but $f(x, y) = \frac{1}{2}$ at all points on the line $y = x$. So we define it as follow.

Definition : Differentiable

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed as

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Pretty ..dumb.

Theorem. If f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Figure 5 shows the graphs of the function f and its linearization L in Example 2.

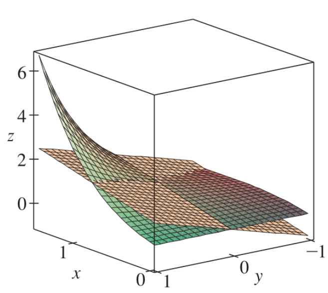


FIGURE 5

EXAMPLE. Show that $f(x, y) = xe^{xy}$ is differentiable at (1,0) and find its linearization there. Approximate f(1.1, -0.1).
The partial derivatives are

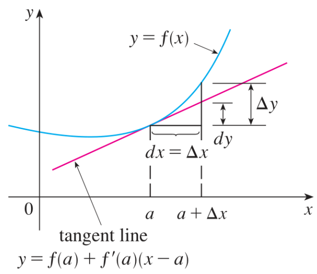
$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_x(x, y) = x^2 e^{xy}$$
$$f_x(1, 0) = 1 \qquad f_y(1, 0) = 1$$

Both f_x and f_y are continuous, so by the above Theorem, we got f differentiable. The linearization is

$$\begin{aligned} L(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \\ &= x + y \end{aligned}$$

So $f(1.1, -0, 1) \approx 1.1 - 0.1 = 1$, actual value: 0.98542.

Differentials



■ For $y = f(x)$, we define dx an independent variable. And $dy = f'(x)\,dx$, represents the change in height when x changes dx.
■ For a differentiable $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables.

Definition : Total differential

Then the **differential** dz (the **total differential**), is defined as follow.

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$, then $f(x, y) \approx f(a, b) + dz$.

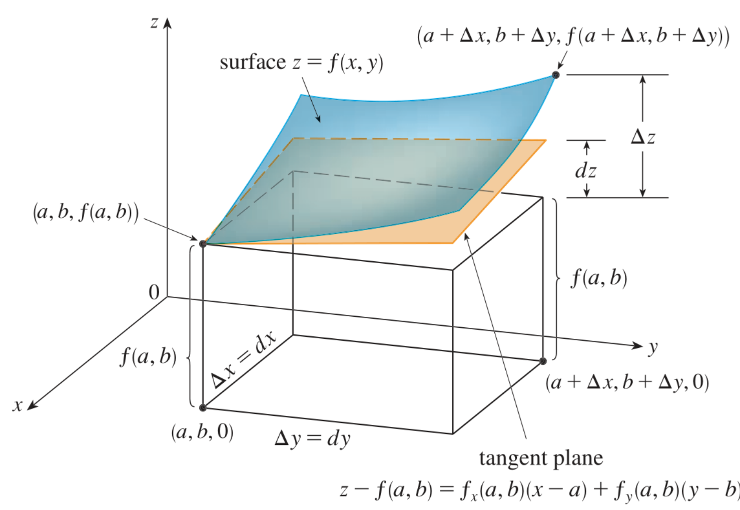


FIGURE 7

EXAMPLE.

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.
(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, comppare the values of Δz and dz.

SOLUTION

(a) Applying the formula,

$$dz = dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x + 3y)\,dx + (3x - 2y)\,dy$$

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, $dy = \Delta y = -0.04$, we get

$$dz = [2(2) + 3(3)]0.05 + [3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3)\\ &= [(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2]\\ &= 0.6449\end{aligned}$$

> Notice that $\Delta z \approx dz$ but dz is easier to compute.

In Example 4, dz is close to Δz because the tangent plane is a good approximation to the surface $z = x^2 + 3xy - y^2$ near $(2, 3, 13)$. (See Figure 8.)

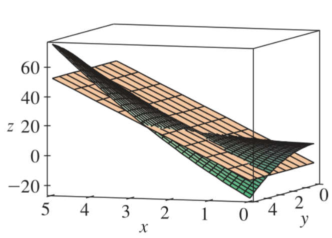


FIGURE 8

Functions of Three or More Variables

Linear approximation. $f(x, y, z) \approx f(a, b, c) + \Sigma f_x(a, b, c)(x - a)$

Total differential. $dw = \dfrac{\partial w}{\partial x}dx + \dfrac{\partial w}{\partial y}dy + \dfrac{\partial w}{\partial z}dz$

3 The Chain Rule

Definition : The Chain Rule (Case 1)

Suppose that $z = f(x, y)$ is differentiable, where $x = g(t)$ and $y = h(t)$ are both differentiable. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

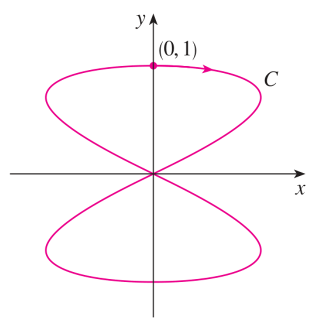


FIGURE 1
The curve $x = \sin 2t$, $y = \cos t$

EXAMPLE. If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.
The Chain Rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\\ &= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)\end{aligned}$$

When $t = 0$, $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2\cos 0) + (0+0)(-0) = 6$$

Definition : The Chain Rule (Case 2)

Suppose that $z = f(x, y)$ is differentiable, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable. We can hold the the other variable fixed.

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} \qquad \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

EXAMPLE. If $z = e^x \sin y$, where $x = st^2$, and $y = s^2t$, find $\partial z/\partial s$ and $\partial z/\partial t$.

Applying Case 2 of the Chain Rule, we get

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)\\ &= t^2 e^{st^2} \sin(s^2t) + 2ste^{st^2} \cos(s^2t)\\ \frac{\partial z}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2)\\ &= 2ste^{st^2} + s^2e^{st^2} \cos s^2 t\end{aligned}$$

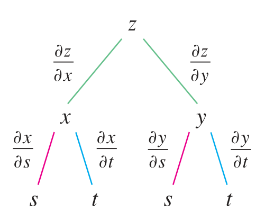


FIGURE 2

Note. s, t are **independent** variables, x, y are **intermediate** variables, and z is the **dependent** variable.
For the gemeral version of n variables, it's similar.

EXAMPLE. If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

EXAMPLE. If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find
a. $\partial z/\partial r$
b. $\partial^2 z/\partial r^2$
SOLUTION.
a. The Chain Rule gives

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

b. Applying the Product Rule, we get

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)$$
$$= 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

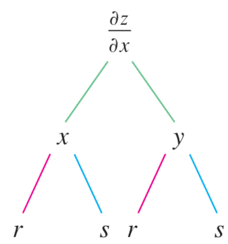


FIGURE 5

Using the Chain Rule again (Figure 5), we have

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y \partial x}(2s)$$
$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}(2r) + \frac{\partial^2 z}{\partial y^2}(2s)$$

Putting these into the previous equation,

$$\frac{\partial^2 z}{\partial r^2} = 2\frac{\partial z}{\partial x} + 4r^2\frac{dd^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}$$

Implicit Differentiation

Suppose $F(x, y) = 0$, $y = f(x)$ is differentiable. If F is differentiable, apply Case 1 of the Chain Rule to differentiate both sides with respect to x.

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

EXAMPLE. Find y' if $x^3 + y^3 = 6xy$.

SOLUTION. $F(x, y) = x^3 + y^3 - 6xy = 0$ which gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Definition : Implicit Function Theorem

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

EXAMPLE. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

SOLUTION. Let $f(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$, then we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$