1 Determinant of a matrix

Historically, the use of determinants arose from the recognition of **special patterns** that occur in the **solutions** of systems of linear equations. For instance, the general solution of the system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 &= b_1 \\
a_{21}x_1 + a_{22}x_2 &= b_2
\end{aligned}
$$

can be shown to be

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \text{ and } x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}},$$

provided that $a_{11}a_{22} - a_{21}a_{12} \neq 0$. Note that both fractions have the same **denominator**. This quantity is called the determinant of the coefficient matrix A.

The **determinant** of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$det(A) = |A| = a_{11}a_{22} - a_{21}a_{21}$$

For matrix of order higher than 2

The **minor** M_{ij} of the element a_{ij} is the *determinant* of the matrix obtained by *deleting* the ith row and the jth column. The **cofactor** C_{ij} is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

To obtain the cofactors of a matrix, first find the minors then apply the sign $+$ or $-$.

DETERMINANT OF A MATRIX

The next definition is called **inductive** since it uses determinants of matrices of order $n-1$ to define the determinant of a matrix of order n.

Laplace's Expansion of a Determinant - *expansion by Cofactors*

$|A|$ is the sum of the entries in **any row or column** of A multiplied by their cofactors

$$det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{n1}C_{n1}$$

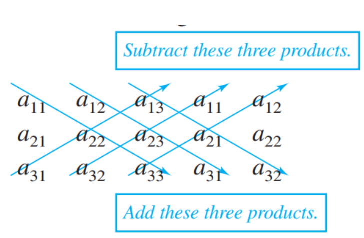
(expanding by cofactors in the first row)

$$|A| = \sum_{i=1}^{n} a_{1i}C_{1i} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

(expanding by cofactors in the first column)

The row or column that containing the most zeros is usually the best choice for expansion by cofactors.

Determinant of a 3×3 matrix



Triangular Matrices

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Determinant of a triangular matrix is the product of its entries on the main diagonal

$$det(A) = |A| = a_{11}a_{22}\ldots a_{nn}$$

1.1 The Effects of Elementary Row Operations on a Determinant

1. Interchanging 2 rows of A, then
$$det(B) = -det(A)$$
2. Adding a multiple of a row to **another row**
$$det(B) = det(A)$$
3. Multiplying a row with $c \neq 0$
$$det(B) = c.det(A)$$

To prove **Property 1**, use mathematical induction. First, it can be easily shown that this property is true for 2×2 matrices. Now assume that it's true for matrices of order $n-1$. Interchanging 2 rows of A, we obtain B. Then, to find $|A|$ and $|B|$, expand along a row other than the 2 interchanged rows. By the induction assumption, the cofactors of B will be the negatives of the cofactors of A since the corresponding $(n-1)\times(n-1)$ matrices have 2 rows interchanged. Finally $|B| = -|A|$, and the proof complete.

REMARK. *Property 3* enables us to divide a row by the common factor.

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

(Factor 2 out of the first row)

1.2 Conditions that Yield a Zero Determinant

If one of following conditions is true, then $det(A) = 0$

1. An entire row (column) consists of zeros.
2. 2 rows (columns) are equal.
3. 1 row (column) is a multiple of another row (column).

REMARK. $det(A) = 0$ if and only if it is **row- (column-) equivalent** to a matrix that has at least 1 row (column) consisting entirely of zeros. This will be proved in the next section.

What is better for computing?

TABLE 3.1

Order n	*Cofactor Expansion*		*Row Reduction*	
	Additions	*Multiplications*	*Additions*	*Multiplications*
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

For this reasons, most computer and calculator algorithm use the method involving elementary row operations.

In fact, the number of operations for the cofactor expansion of an $n \times n$ matrix grows like $n!$. Because $30! \approx 2.65 \times 10^{32}$, even a relatively small 30×30 matrix would require more than 10^{32} operations.

Techniques

Create a row or column having all zeros in all but one position - then using cofactor expansion to reduce the order of the matrix by 1.

$$A = \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ -3 & 0 & 0 \end{vmatrix} = (-3)(-1)^4 \begin{vmatrix} 5 & -4 \\ -4 & 3 \end{vmatrix} = 3$$

2 Properties of Determinants

Determinant of a Product. $|A_1 A_2 \cdots A_k| = |A_1||A_2| \cdots |A_k|$

Proof. If E is an elementary matrix,

1. Interchanging 2 rows of I : $|E| = -1$
2. Multiplying a row of I by c : $|E| = c$
3. Adding a multiple of 1 row of I to another row of I : $|E| = 1$

It follows that, $|EB| = |E|.|B|$. This can be generalized to conclude that

$$|E_k \dots E_2 E_1 B| = |E_k| \cdots |E_2||E_1|.|B|$$

Now consider the matrix AB. If A is **nonsingular**, then it can be written as $E_k \dots E_2 E_1$ and we can write

$$|AB| = |E_k| \cdots |E_2||E_1||B| = |E_k \cdots E_2 E_1||B| = |A||B|$$

If A is **singular**, then it is row-equivalent to a matrix with an entire row of zeros. Moreover, we can conclude that AB is also **singular**.

Determinant of a Scalar Multiple of a Matrix. $|cA| = c^n|A|$

Proof. Factor the scalar c out of each rows of $|cA|$, proof complete.

2.1 Determinants and the Inverse of a Matrix

Determinant of an invertible matrix. $det(A) \neq 0$

Proof.

1. Assume A is invertible $\Leftrightarrow AA^{-1} = I$.

$\Leftrightarrow |A||A^{-1}| = |I| = 1$. Then neither $|A|$ nor $|A^{-1}|$ is 0.

2. Assume $|A| \neq 0$.

Using Gauss-Jordan elimination, find a matrix B in row-echelon form that is row-equivalent to A. This implies, $B = I_n$ or B has at least 1 row that consists entirely of 0 - $|B| = 0$ which leads to $|A| = 0$. So B must be I_n.

A is, therefore, row-equivalent to the identity matrix - in other words, it is **invertible**.

Determinant of an Inverse Matrix. $|A^{-1}| = \frac{1}{|A|}$

Proof. Since $|A|.|A^{-1}| = 1$, proof complete.

2.2 Equivalent Conditions for a Nonsingular Matrix

If A is an $n \times n$ matrix, the following statements are equivalent.

1. A is **invertible**.
2. $A\mathbf{x} = \mathbf{b}$ has a **unique solution** for EVERY $n \times 1$ column matrix $\mathbf{b}$
3. $A\mathbf{x} = P$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. A can be written as product of elementary matrices.
6. $det(A) \neq 0$

REMARK. A can have a determinant of zaro if A is row-equivalent to a matrix that has at least one row consisting entirely of zeros. (Properties 4 and 6)

2.3 Determinants and the Transpose of a Matrix

Determinant of a Transpose. $det(A) = det(A^T)$

3 Introduction to Eigenvalues

This is one of the most important topics of linear algebra - **eigenvalues**. One applicaion of eigenvalues involves the study of population growth.

Example. Suppose that half of a population of rabbits raised in a laboratory survive their first year. Of those, half survive their second year. Their maximum life span is 3 years. Furthermore, during the first year the rabbits produce no offspring, whereas the average number of offspring is 6 during the second year and 8 during the third year. If there are 24 rabbits in each age class now, what will the distribution be in 1 year? In 20 years?

Eigenvalue problem. If A is an $n \times n$ matrix, do there exist $n \times 1$ nonzero matrices x such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?

- **eigenvalue** of A: λ *(the scalar)*
- **eigenvector** of A: $\mathbf{x}$ *(corresponding to λ)*

The fundamental equation for the eigenvalue problem is

$$Ax = \lambda x$$

Example 1. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

That is, $\lambda_1 = 5$ is an **eigenvalue** of A corresponding to x_1, and $\lambda_2 = -1$ is an **eigenvalue** of A corresponding to x_2.

Notice. If x is an eigenvector corresponding to λ, then so is **any nonzero multiple of x**.

3.1 Finding Eigenvalues and Eigenvectors

Provided with an $n \times n$ matrix, how to find the **eigenvalues** and the corresponding **eigenvector**? The key is to write the equation $A\mathbf{x} = \lambda\mathbf{x}$ in the equivalent form

$$(\lambda I - A)\mathbf{x} = 0 \textit{(characteristic equation)}$$

This **homogeneous** system of equations has nonzero solutions if and only if the coefficient matrix $(\lambda I - A)$ is **singular** - that is,

$$det(\lambda I - A) = 0$$

That equation is called the **characteristic equation** of A - and is a polynomial equation of degree n in the variable λ. Once we have found the **eigenvalues** of A, you can use Gaussion elimination to find the corresponding eigenvectors.

Example 2. Find eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

SOLUTION. The characteristic equation of A is

$$\begin{aligned} |\lambda I - A| &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \right| \\ &= \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda - 5)(\lambda + 1) = 0 \end{aligned}$$

For $\lambda_1 = 5$, the coefficient matrix is $5I - A = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}$, which row reduced to $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

The solutions of the homogeneous system having this coefficient matrix are all of the form $\begin{bmatrix} t \\ t \end{bmatrix}$. So, the **eigenvectors** corresponding to the **eigenvalue** $\lambda_1 = 5$ are the nonzero scalar multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

And the eigenvectors corresponding to the eigenvalue $\lambda_2 = -1$ are the nonzero scalar multiples of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Example 3. Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

SOLUTION. The characteristic equation of A is

$$
\begin{aligned}
|\lambda I - A| &= \left| \begin{bmatrix} \lambda - 1 & -2 & 2 \\ -1 & \lambda - 2 & -1 \\ 1 & 1 & \lambda \end{bmatrix} \right| \\
&= (\lambda - 1) \begin{vmatrix} \lambda - 2 & -1 \\ 1 & \lambda \end{vmatrix} - (-2) \begin{vmatrix} -1 & -1 \\ 1 & \lambda \end{vmatrix} + 2 \begin{vmatrix} -1 & \lambda - 2 \\ 1 & 1 \end{vmatrix} \\
&= \lambda^3 - 3\lambda^2 - \lambda + 3 \\
&= (\lambda^2 - 1)(\lambda - 3) = 0
\end{aligned}
$$

For $\lambda_1 = 1$, the coefficient matrix is

$$
I - A = \begin{bmatrix} 0 & -2 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}
$$

which row reduces to $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

The solutions of the homogeneous system having this coefficient matrix are all of the form $\begin{bmatrix} -2t \\ t \\ t \end{bmatrix}$. where t is a real number.

So, the eigenvectors corresponding to the eigenvalue $\lambda_1 = 1$ are the nonzero scalar multiples of $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

Similarly, solve for $\lambda_2 = -1$ and $\lambda_3 = 3$.

In **Chapter 7**, we will study eigenvalues and their applicaions in more detail.

4 Applications of Determinants

4.1 The Adjoint of a Matrix

If A is a square matrix, then the **matrix of cofactors** of A has the form

$$
\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{21} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}
$$

The transpose of this matrix is called the **adjoint** of A and is denoted by

$$
\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}
$$

The Inverse of a Matrix Given by Its Adjoint. If A is an $n \times n$ invertible matrix, then

$$
A^{-1} = \frac{1}{det(A)} \text{adj}(A)
$$

Proof. Consider the product

$$
\begin{aligned}
A[\text{adj}(A)] &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \\
&= det(A) I
\end{aligned}
$$

REMARK. Using Gauss-Jordan elimination is much better. This theorem just provides a concise formula.

4.2 Cramer's Rule

If a system of n linear equatios in n variables has a coefficient matrix with a nonzero determinant $|A|$, then the solution of the system is given by

$$x_1 = \frac{det(A_1)}{det(A)}, \quad x_2 = \frac{det(A_2)}{det(A)}, \quad \ldots, \quad x_n = \frac{det(A_n)}{det(A)}$$

where the ith column of A_i is the column of constants in the system of equations.

Proof. Let the system be represented by $AX = B$. Since $|A| \neq 0$, you can write

$$X = A^{-1}B = \frac{1}{|A|} adj(A)B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

If the entries of B are $b_1, b_2, \ldots, b_n$ then

$$x_i = \frac{1}{|A|} \sum_{j=1}^{n} b_j C_{ji} = \frac{|A_i|}{|A|}$$

4.3 Area, Volume, and Equation of Lines and Planes