1 Double Integrals over Rectangles

The Riemann sum

$$\int_a^b f(x)\,dx = \lim_{n\to\inf} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

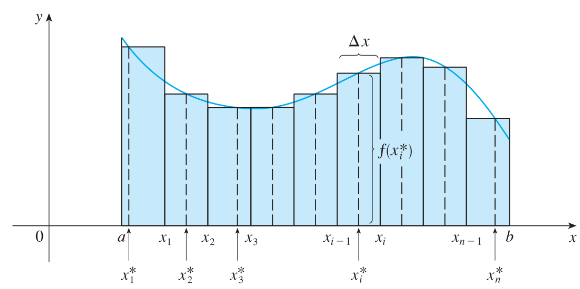

FIGURE 1

Volumes and Double Integrals

Form the subrectangles

$$F_{ij} = [x_{i-1}, x_y] \times [y_{i-1}, y_i] = \{(x,y) \mid x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$$

each with area $\Delta A = \Delta x \Delta y$.

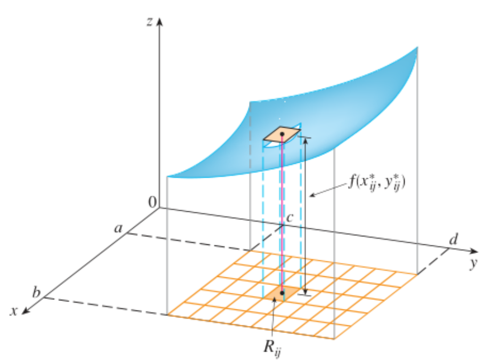

FIGURE 4

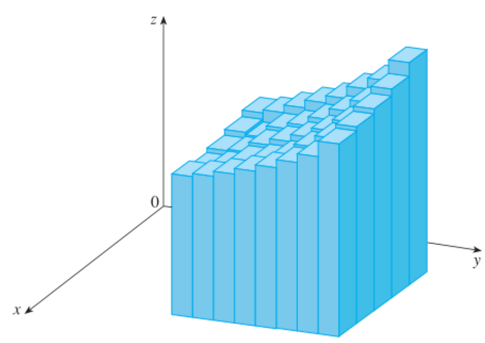

FIGURE 5

Definition : Double Integral

The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)\,dA = V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

EXAMPLE 1. Estimate the volume

$$R = [0,2] \times [0,2], \quad z = 16 - x^2 - 2y^2$$

Divide R into 4 squares and choose the sample point to be the upper right corner of each square R_{ij}.

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\Delta A \\ &= f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A \\ &= 13(1) + 7(1) + 10(1) + 4(1) = 34 \end{aligned}$$

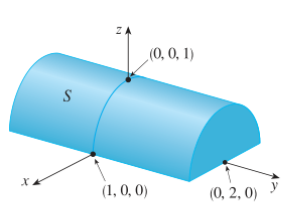

FIGURE 9

EXAMPLE. If $R = \{(x,y) | -1 \le x \le 1, -2 \le y \le 2\}$, evaluate

$$\iint_R \sqrt{1-x^2}\,dA$$

Since $\sqrt{1-x^2} \ge 0$, we can interpreting it as a volume. $x^2 + z^2 = 1$ and $z \ge 0$.

$$\iint_R \sqrt{1-x^2}\,dA = \frac{1}{2}\pi(1)^2 \times 4 = 2\pi$$

⚓ The Midpoint Rule

Take $(x_i*, y_i*) = (\overline{x_i}, \overline{y_i})$ (the middle point between x_i, x_{i-1}).

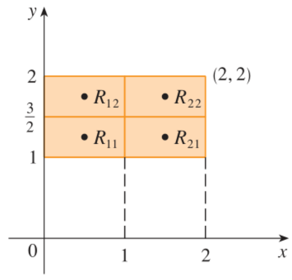

FIGURE 10

EXAMPLE. $m = n = 2,\ R = \{(x,y)|0 \le x \le 2, 1 \le y \le 2\}$

$$\begin{aligned}\iint_R (x - 3y^2)\, dA &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x_i}, \overline{y_i})\Delta A \\ &= f(\overline{x_1}, \overline{y_1})\Delta A + \cdots + f(\overline{x_2}, \overline{y_2})\Delta A \\ &= f(\frac{1}{2}, \frac{5}{4})\Delta A + f(\frac{1}{2}, \frac{7}{4})\Delta A + f(\frac{3}{2}, \frac{5}{4})\Delta A + f(\frac{3}{2}, \frac{7}{4})\Delta A \\ &= -11.875\end{aligned}$$

Note. Double integral as a bolume is valid only when f is a *positive* function. So in the previous example, the integral is not a volume.

⚓ Average Value

The average value of $f(x)$ on (a,b) is $f_{ave} = \dfrac{1}{b-a}\int_a^b f(x)\, dx$.

Definition : Average Value

The **average value** of $f(x,y)$ on a rectangle R is

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\, dA$$

If $f(x,y) \ge 0$, the equation $A(R) \times f_{\text{ave}} = \iint_R f(x,y)\, dA$ says that it has the same V as a box with base R and height f_{ave}.

⚓ Properties of Double Integrals

The *linearity* of the integral $(+, c\times)$.

If $f(x,y) \ge g(x,y)$ for all $(x,y) \in R$, then

$$\iint_R f(x,y)\, dA \ge \iint_R g(x,y)\, dA$$

⚓ Iterated Integrals

$\int_c^d f(x,y)\, dy$ means that x is fixed and $f(x,y)$ is integrated with respecto y from $c \to d$. (*partial integration with respect to* y).

Definition : Iterated Integral

$$\int_c^d \int_a^b f(x,y)\, dx\, dy = \int_c^d \left[\int_a^b f(x,y)\, dx\right] dy$$

work from the inside out.

EXAMPLE.

(a)

$$\begin{aligned}\int_0^3 \int_1^2 x^2 y\, dy\, dx &= \int_0^3 \left[\int_1^2 x^2 y\, dy\right] dx \\ &= \int_0^3 \frac{3}{2}x^2\, dx = \frac{x^3}{2} = \frac{27}{2}\end{aligned}$$

(b)

$$\begin{aligned}\int_1^2 \int_0^3 x^2 y\, dx\, dy &= \int_1^2 \left[\int_0^3 x^2 y\, dx\right] dy \\ &= \int_1^2 9y\, dy = 9\frac{y^2}{2}\Bigg]_1^2 = \frac{27}{2}\end{aligned}$$

Definition : Fubini's Theorem

If f is continuous on $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$, then

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

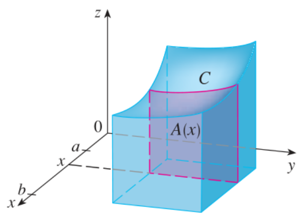

FIGURE 1

TEC Visual 15.2 illustrates Fubini's Theorem by showing an animation of Figures 1 and 2.

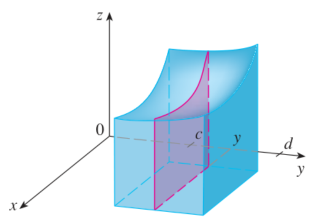

FIGURE 2

$$V = \int_a^b A(x)\,dx$$

where $A(x)$ is the area of the surface that is perpendicular to the $x-$ axis.

$$A(x) = \int_c^d f(x,y)\,dy$$

Definition : Special case

In case $f(x,y) = g(x)\,h(y)$,

$$\iint\limits_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

EXAMPLE. $R = [0, \pi/2] \times [0, \pi/2]$, then

$$\iint\limits_R \sin x \cos y\,dA = \int_0^{\pi/2} \sin x \int_0^{\pi/2} \cos y\,dy$$

$$= [-\cos x]_0^{\pi/2}\,[\sin y]_0^{\pi/2} = 1 \cdot 1 = 1$$

Double Integrals over General Regions

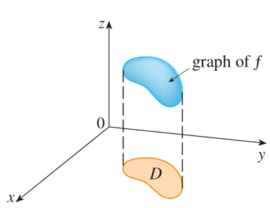

FIGURE 3

The **double integral of f over D** is

$$\iint\limits_D f(x,y)\,dA = \iint\limits_R F(x,y)\,dA$$

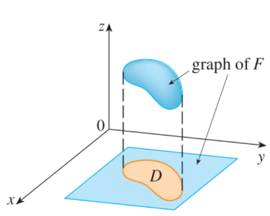

FIGURE 4

where $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$

Type I: D lies between 2 continuous function of x

$$D = \{(x,y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

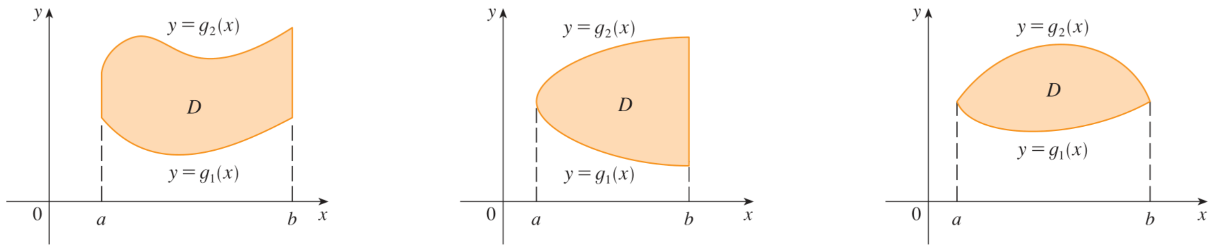

FIGURE 5 Some type I regions

> **Definition : Type I**
>
> If f is continuous on a type I region D such that
>
> $$D = \{(x,y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$$
>
> then
>
> $$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

which leads to the definition for **Type II**,

$$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

EXAMPLE. $y = 2x^2, y = 1 + x^2$, evaluate $\iint_D (x+2y)\, dA$.

$$\begin{aligned}\int_D (x+2y)\, dA &= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\, dy\, dx \\ &= \int_{-1}^{1} \left[xy + y^2\right]_{y-2x^2}^{y=1+x^2} dx \\ &= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx \\ &= \frac{32}{15}\end{aligned}$$

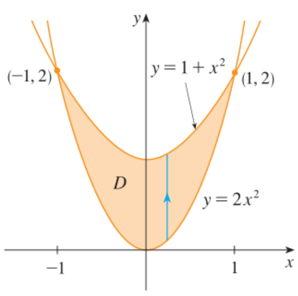

FIGURE 8

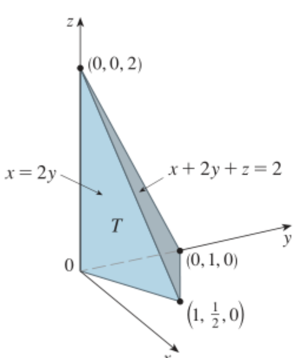

FIGURE 13

EXAMPLE. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2, x = 2y.x = 0, z = 0$.

$$D = \left\{(x,y) \mid 0 \le x \le 1, x/2 \le y \le 1 - x/2\right\}$$

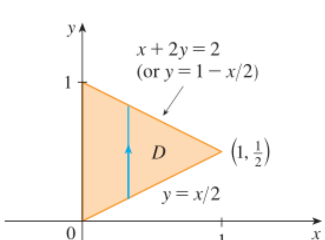

FIGURE 14

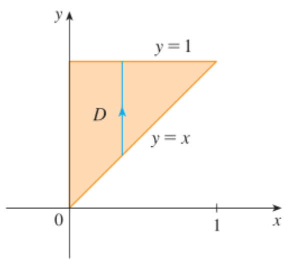

FIGURE 15
D as a type I region

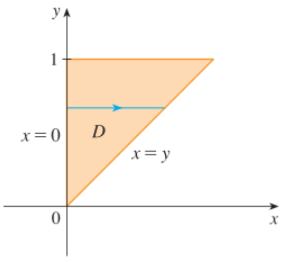

FIGURE 16
D as a type II region

EXAMPLE.

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA$$

$$D = \{(x, y) \mid 0 \le x \le 1, x \le y \le 1\}$$

can be transformed to

$$D = \{(x, y) \mid 0 \le y \le 1, 0 \le x \le y\}$$

$$\begin{aligned}\int_0^1 \int_0^y \sin(y^2)\,dx\,dy &= \int_0^1 \left[x \sin(y^2)\right]_{x=0}^{x=y} dy \\ &= \int_0^1 y \sin(y^2)\,dy \\ &= -\frac{1}{2}\cos(y^2)\Big]_0^1 = \frac{1}{2}(1 - \cos 1)\end{aligned}$$

Properties 1: Double Integrals

Beside sum and constant multiplier.

- If $f(x, y) \ge g(x, y)$ for all $(x, y) \in D$.

$$\iint_D f(x, y)\,dA \ge \iint_D g(x, y)\,dA$$

- If $D = D_1 \cup D_2$, and they don't overlap except perhaps on their bound daries

$$\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA$$

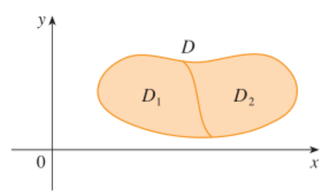

FIGURE 17

- Since $\iint_D 1\,dA = A(D)$, so if $m \le f(x, y) \le M$ for all $(x, y) \in D$.

$$mA(D) \le \iint_D f(x, y)\,DA \le MA(D)$$

EXAMPLE. Estimate $\iint_D e^{\sin x \cos y}\,dA$, where D is the disk with center the origin and $r = 2$. Since $-1 \le \sin x \le 1$ and $-1 \le \cos y \le 1$, we have $-1 \le \sin x \cos y \le 1$. Therefore

$$e^{-1} \le e^{\sin x \cos y} \le e^1$$

$$\frac{4\pi}{e} \le \iint_D e^{\sin x \cos y}\,dA \le 4\pi e$$

2 Double Integrals in Polar Coordinate

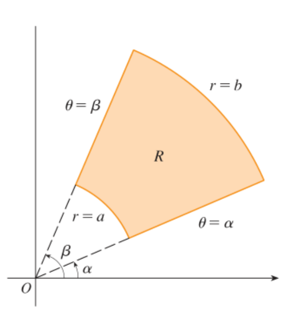

FIGURE 3 Polar rectangle

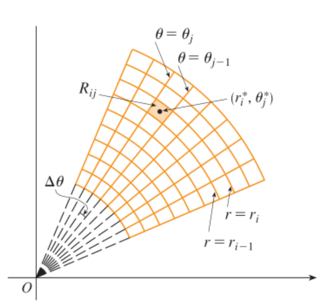

FIGURE 4 Dividing *R* into polar subrectangles

Divide into m subinterval $[r_{i-1}, r_i]$ of $\Delta r = (b - a)/m$ and n subinterval of $(\beta - \alpha)/n$.

- Then the center of the polar subrectangles has polar coordinate

$$r_i* = \frac{1}{2}(r_{i-1} + r_i), \quad \theta_j* = \frac{1}{2}(\theta_{j-1} + \theta_j)$$

- And the area

$$\begin{aligned}\Delta A_i &= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta \\ &= r_i^* \Delta r \Delta\theta\end{aligned}$$

Definition : Change to Polar Coordinates in a Double Integral

If f is continuous on a polar rectangle R $(0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi)$, then

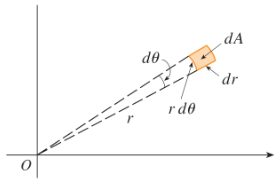

FIGURE 5

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

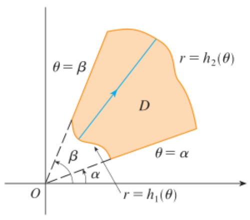

FIGURE 7
$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$

$$\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

EXAMPLE. Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

$$D = \{(r,\theta) \mid -\pi/4 \le \theta \le \pi/4, 0 \le r \le \cos 2\theta\}$$

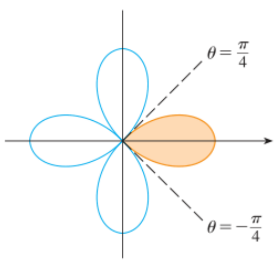

FIGURE 8

So the area is

$$A(D) = \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\, dr\, d\theta$$
$$= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2} r^2\right]_0^{\cos 2\theta} = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos 2\theta^2\, d\theta$$
$$= \frac{1}{4} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta)\, d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$$

EXAMPLE. $z = x^2 + y^2$, $x^2 + y^2 = 2x$.

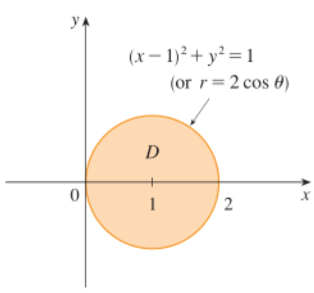

FIGURE 9

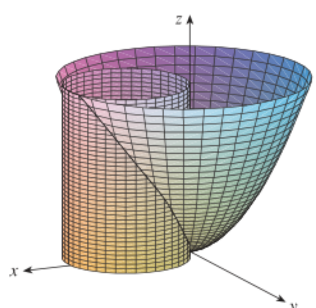

FIGURE 10

$$D = \{(r,\theta) \mid -\pi/2 \le \theta \le \pi/2, 0 \le r \le 2\cos\theta\}$$

$$V \iint_D (x^2 + y^2)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos theta} r^2 r\, dr\, d\theta$$
$$= 4\int_{-\pi/2}^{\pi/2} \cos\theta^4\, d\theta = 8\int_0^{\pi/2} \cos\theta^4\, d\theta = 8\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2}\right)^2 d\theta$$
$$= 2\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

Why split into 2 parts?