1 Introduction to Systems of Linear Equations

In general, a **linear equation** in n variables is defined as follows:

A **linear equation in n variables** $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b. \tag{1}$$

- $a_1, a_2, \ldots, a_n$: **coefficients** (real nummbers)
- b : **constant term** (real number)

a_1 : the **leading coefficients**
x_1 : **leading variable**

Solution set : The set of *all solutions* of a linear equation.
When this set is found, the equation is said to be **solved**. To describe the entire solution set of a linear equation, a **parametric representation** is used.

Example 1. Solve the linear equation $x_1 + 2x_2 = 4$.

1. Solve x_1 in terms of x_2, obtain
$$x_1 = 4 - 2x_2$$
In this form, x_2 is **free -** it can take on any real value.
x_1 is not free since its value depends on the value assigned to x_2.

2. Represent the infinite number of solutions of this Eq, introduce a 3^{rd} var t called a **parameter**.
$\begin{cases} x_1 = 4 - 2t \\ x_2 = t \end{cases}$, t is any real numbers.

Parametric Representation of a Solution Set

Example 2.

$$3x + 2y - z = 3$$

Choosing y and z to be the free variables, obtain

$$x = 1 - \frac{2}{3}y + \frac{1}{3}z$$

Letting $y = s$ and $z = t$, obtain the parametric presentation

$$x = 1 - \frac{2}{3}s + \frac{1}{3}t, \quad y = s, \quad z = t$$

where s and t are any real numbers.

Systems of Linear Equations

A **system of m linear equation in n variables**: $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm}x_n = b_m \end{cases}$

A system of linear equations can have exactly one solution, an infinite number of solutions, or no solution.

- **consistent** : ≥ 1 solution
- **inconsistent** : no solution

Solving a System of Linear Equations

$$\begin{cases} x - 2y + 3z = 9 \\ \qquad y + 3z = 5 \\ \qquad\qquad z = 2 \end{cases}$$

This system is in **row-echelon form** (it follows a start-step pattern and has leading coefficients of 1). To solve such a system, use a procedure called **back-substitution** (work backward).

Gaussian Elimination

2 S.LN are called **equivalent**: have the same **solution set**.

Changing the initial S.LN into an equivalent S.LN that is in **row-echelon form**:

1. Interchange 2 equations
2. Multiply an Eq. by a nonzero constant
3. Add a multiple of an Eq. to another Eq.

2 Gaussian Elimination and Gauss-Jordan Elimination

This is a $m \times n$ matrix (m by n matrix) : $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

- **entry** : a_{ij}
- **row subscript** : i
- **column subscript** : j
- **main diagonal** : the line that contains the entries $a_{11}, a_{22}, \ldots$ (main diagonal entries)

System	Augmented Matrix	Coefficient Matrix
$\begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x \quad - 4z = 6 \end{cases}$	$\left[\begin{array}{ccc\|c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & 4 & 6 \end{array}\right]$	$\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & 4 \end{bmatrix}$

Elementary Row Operations

1. Interchange 2 Eq. ($R_1 \leftrightarrow R_2$)
2. Multiply an Eq. by a nonzero constant. ($(\frac{1}{2}R_2) \to R_2$)
3. Add a multiple of an Eq. to another Eq. ($R_3 + (-2)R_1 \to R_3$)

Row-equivalent matrices

2 matrices are said to be **row-equivalent** if one can be obtained from the other by a finite sequence of elementary row operations.

Row-echelon form & Reduce Row-echelon form

Matrices in row-echelon form

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrices in **reduced row-echelon form**

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Gauss-Jordan Elimination

Just the same but it continues the reduction until a *reduced row-echelon* form is obtained.

Example 1. $\begin{cases} x - 2y + 3z = 9 \\ -x + 3y \qquad = -4 \\ 2x - 5y + 5z = 17 \end{cases} \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

> **What is better?**
> S.Eq are usually solved by computer. Most computer programs use a form of Gaussian elimination, with special emphasis on ways to reduce rounding errors and minimize storage of data.

Homogeneous Systems of Linear Equations

S.Eq in which each of the **constant terms** is 0 - such systems are called **homogeneous**.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \vdots a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

$\implies$ Definitely **consistent** (has at least 1 solution), that is

trivial (obvious) solution : $x_1 = x_2 = \cdots = x_n = 0$

- fewer Eq. than variables $\rightarrow$ infinite number of solution
- consistent

3 Applications of Systems of Linear Equations

3.1 Polynomial Curve Fitting

Suppose a collection of data is represented by n points in the xy-plane,
$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Find a polynominal function of degree $n-1$

$p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$
whose graph pass through the specified points.

If all x-coordinates of the points are distinct, there is precisely 1 polynominal function of degree $n-1$ (or less) that fits the n points.

Let $a_0, a_1, \dots, a_{n-1}$ be the n **variables** and substitute each of the n points into the polynominal function

$$\begin{aligned}
a_0 + a_1 x_1 + \cdots + a_{n-1} x_1^{n-1} &= y_1 \\
a_0 + a_1 x_2 + \cdots + a_{n-1} x_2^{n-1} &= y_2 \\
&\vdots \\
a_0 + a_1 x_n + \cdots + a_{n-1} x_n^{n-1} &= y_n
\end{aligned}$$

What if the x-values are large ?

Translating Large x-Values Before Curve Fitting

$$\overbrace{(2006, 3)}^{(x_1, y_1)}, \quad \overbrace{(2007, 5)}^{(x_2, y_2)}, \quad \overbrace{(2008, 1)}^{(x_3, y_3)}, \quad \overbrace{(2009, 4)}^{(x_4, y_4)}, \quad \overbrace{(2010, 10)}^{(x_5, y_5)},$$

Translation $z = x - 2008$ to obtain

$$\overbrace{(-2, 3)}^{(z_1, y_1)}, \quad \overbrace{(-1, 5)}^{(z_2, y_2)}, \quad \overbrace{(0, 1)}^{(z_3, y_3)}, \quad \overbrace{(1, 4)}^{(z_4, y_4)}, \quad \overbrace{(2, 10)}^{(z_5, y_5)},$$

$$\implies p(z) = 1 - \frac{5}{4}z + \frac{101}{24}z^2 + \frac{3}{4}z^3 - \frac{17}{24}z^4.$$

$$\boxed{\implies p(= x) = 1 - \frac{5}{4}(x - 2008) + \frac{101}{24}(x - 2008)^2 + \frac{3}{4}(x - 2008)^3 - \frac{17}{24}(x - 2008)^4.}$$

Network Analysis

Network composed of **branches** and **junctions -** are used as models in economics, traffic analysis, and engineering.

Assume in each of the junctions:

$$\sum \text{flow}_{in} = \sum \text{flow}_{out}$$

Solve the linear equations for all junctions.

Kirchhoff's Laws

1. All the current flowing into a junction must flow out of it.
2. The sum of IR around a closed path is equal to the total voltage in the path.

PROJECT

3.2 Graphing Linear Equations

3.3 Underdetermined and Overdetermined Systems of Equations

- **underdetermined** : more **var** than **eq**
- **overdetermined : less var than eq**