Triple Integral

EXAMPLE 5. Find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2, x = 2y, x = 0, z = 0$.

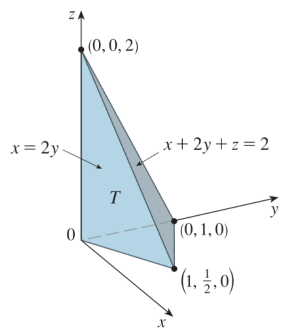


FIGURE 14

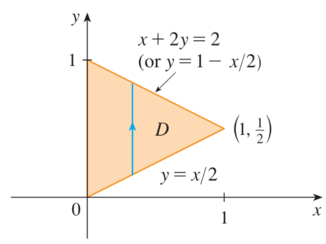


FIGURE 15

$$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\,dy\,dx$$
$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\,dy\,dx = \frac{1}{3}$$

Or we can use the formula

$$V = hS = \left(\frac{1}{3}\right)(2.\frac{1}{2}) = \frac{1}{3}$$

1 Triple Integrals in Cylindrical Coordinates

Evaluating Triple Integrals with Cylindrical Coordinates

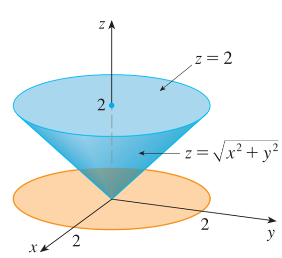


FIGURE 9

EXAMPLE 4. Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx = \iiint_E (x^2+y^2)\,dV$$
$$= \int_0^{2\pi}\int_0^2\int_r^2 r^2\,r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta \int_0^2 r^3(2-r)dr$$
$$= 2\pi\left[\frac{1}{2}r^4 - \frac{1}{5}r^5\right]_0^2 = \frac{16}{5}\pi$$

2 Triple Integrals in Spherical Coordinates

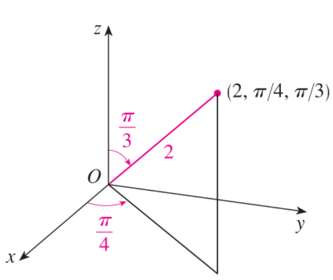


FIGURE 6

EXAMPLE 1. Find the rectangular coordinates of $(2, \pi/4.\pi/3)$.

$$x = \rho \sin\phi \cos\theta = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin\phi \sin\theta = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho\cos\phi = 2\cos\frac{\pi}{3} = 2(\tfrac{1}{2}) = 1$$

EXAMPLE 2. Find spherical coordinates of $(0, 2\sqrt{3}, -2)$.

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 12 + 4} = 4$$

So we have

$$\begin{aligned} \cos\phi &= \frac{z}{p} = -\frac{2}{4} = -\frac{1}{2} \qquad \frac{2\pi}{3} \\ \cos\theta &= \frac{x}{\rho\sin\phi} = 0 \qquad \theta = \frac{\pi}{2} \end{aligned}$$

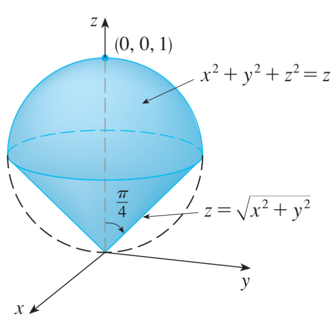
Note. $\theta \neq 3\pi/2$ since $y > 0$.

Evaluating Triple Integrals with Spherical Coordinates

EXAMPLE 3. Evaluate $\iiint\limits_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$, where $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$.

We use spherical coordinates: $B = \{(\rho, \theta, \phi), \mid 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$.

$$\begin{aligned} \iiint\limits_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV &= \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{(p^2)^{\frac{3}{2}}} \rho^2 \sin\theta \, d\rho \, d\theta \, d\phi \\ &= \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^1 \rho^2 e^{p^3} \, d\rho \\ &= [-\cos\phi]_0^{\pi} (2\pi) \left[\frac{1}{3} e^{p^3}\right]_0^1 = \frac{4}{3}\pi(e - 1) \end{aligned}$$



EXAMPLE 4. Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

■ The sphere has center $\left(0, 0, \frac{1}{2}\right)$. Hence $\rho^2 = \rho\cos\phi$ or $\rho = \cos\phi$. Theequation of the cone can be written as

$$\rho\cos\phi = \sqrt{\rho^2 \sin\phi^2 \cos\theta^2 + \rho^2 \sin\phi^2 \sin\theta^2} = \rho\sin\phi$$

This gives $\sin\phi = \cos\phi$, or $\phi = \pi/4$. Therefore

$$E = \{(\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi/4,\ 0 \leq \rho \leq \cos\phi\}$$

$$\begin{aligned} V(E) &= \iiint\limits_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &= \int_0^{2\pi} d\,\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\cos\phi} d\phi \\ &= \frac{2\pi}{3}\int_0^{\pi/4} \sin\phi\cos\phi^3 \, d\phi = \frac{2\pi}{3}\left[-\frac{\cos\phi^4}{4}\right]_0^{\pi/4} = \frac{\pi}{8} \end{aligned}$$

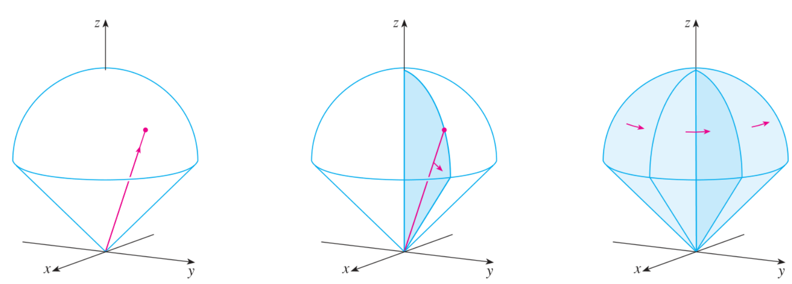


FIGURE 11

ρ varies from 0 to $\cos\phi$ while ϕ and θ arec onstant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π.

3 Change of Variables in Multiple Integrals

EXAMPLE 1. A transformation is defined by the equations

$$x = u^2 - v^2 \qquad\qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$.
Finding the images of the sides (boundary) of S.

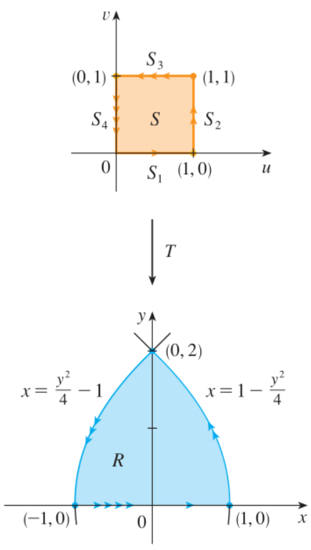


FIGURE 2

EXAMPLE 2. $\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$, and R bounded by x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x, y \geq 0$.
Evaluate $\iint_R y\, dA$.

SOLUTION.

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Therefore,

$$\iint\limits_R y\, dA = \iint\limits_S 2uv \frac{\partial(x, y)}{\partial(u, v)} = \int_0^1 \int_0^1 (2uv)4(u^2 + v^2)\, du\, dv$$
$$= 2$$

EXAMPLE 3. Evaluate $\iint\limits_R e^{(x+y)/(x-y)}\, dA$, where R is the trapezoidal region with vertices (1,0), (2,0), (0,-2) and (0,-1).

SOLUTION. Let $u = x + y,\ v = x - y$. Then $x = \frac{1}{2}(u + v),\ y = \frac{1}{2}(u - v)$.
The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

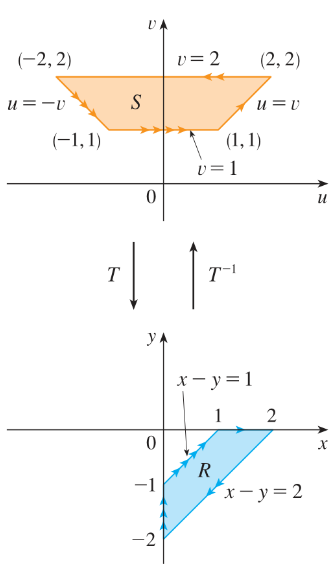


FIGURE 8

The sides of R are

$$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

The sides of S are

$$u = v \qquad v = 2 \qquad u = -v \qquad v = 1$$

$$S = \{(u, v) \mid 1 \leq v \leq 2, -v \leq u \leq v\}$$

Therefore we have

$$\iint\limits_R e^{(x+y)/(x-y)}\, dA = \iint\limits_S e^{u/v} \frac{\partial(x, y)}{\partial(u, v)} du\, dv$$
$$= \int_1^2 \int_{-v}^{v} e^{u/v} \left(\frac{1}{2}\right) du\, dv = \frac{1}{2} \int_1^2 \left[ve^{u/v}\right]_{u=-v}^{u=v} dv$$
$$= \frac{1}{2} \int_1^2 (e - e^{-1})v\, dv = \frac{3}{4}(e - e^{-1})$$