Applications of Vector Spaces

1 Linear Differential Equations *(Calculus)*

A **linear differential equation of order n** is of the form

$$y^{(n)} + g_{n-1}(x)y^{(n-1)} + \cdots + g_1(x)y' + g_0(x)y = f(x)$$

If $f(x) = 0$, the function is **homogeneous**, otherwise, **nonhomogeneous**. A function y is called a solution of the linear differential equation if the equation satisfied when y and its first n derivatives are substituted into the equation.

EXAMPLE 1. **A Second-Order Linear Differential Equation** Show that both $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions for the second-order linear differential equation

$$y'' - y = 0$$

There are 2 observations about this problem.

1. In the vector space $C''(-\infty, \infty)$ of all twice differentable functions defined on the entire real line, the 2 solutions $y_1 = e^x$ and $y_2 = e^{-x}$ are *linearly independent*. This means that the only solution of

$$C_1y_1 + C_2y_2 = 0$$

   that is valid for all x is $C_1 = C_2 = 0$.

2. Every *linear combination* of y_1 and y_2 is also a solution of the linear differential equation.

Solutions of a Linear Homogeneous Differential Equation.
Every nth-order linear homogeneous differential equation

$$y^{(n)} + g_{n-1}(x)y^{(n-1)} + \cdots + g_1(x)y' + g_0(x)y = 0$$

has n linear independent solutions. Moreover, if $\{y_1, y_2, \ldots, y_n\}$ is a set of linearly independent solutions, then every solution is in the form

$$y = C_1y_1 + C_2y_2 + \cdots + C_ny_n \quad (C_1, C_2, \ldots \text{ are real numbers})$$

We can see the importance of being able to determine whether a set of solutions is linearly independent. Let's get started with a preliminary definition.

Definition of the Wronskian of a Set of Functions.
Let $y = \{y_1, y_2, \ldots, y_n\}$ be a set of solutions, each of which has $n - 1$ derivatives on an interval I. The determinant

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of the given set of functions.

Example. Finding the Wronskian of a Set of Functions.

(a) $\{1-x, 1+x, 2-x\}$

$$W = \begin{vmatrix} 1-x & 1+x & 2-x \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

The Wronskian of this set is **identically equal to zero**, since it is zero for *any* value of x.

(b) $\{x, x^2, x^3\}$

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$$

Wronskian Test for Linear Independence.
Let $y = \{y_1, y_2, \ldots, y_n\}$ be a set of n solutions of an nth-order linear homogeneous differential equation. This set is **linearly independent** if and only if the Wronskian is not *identically equal to zero.*

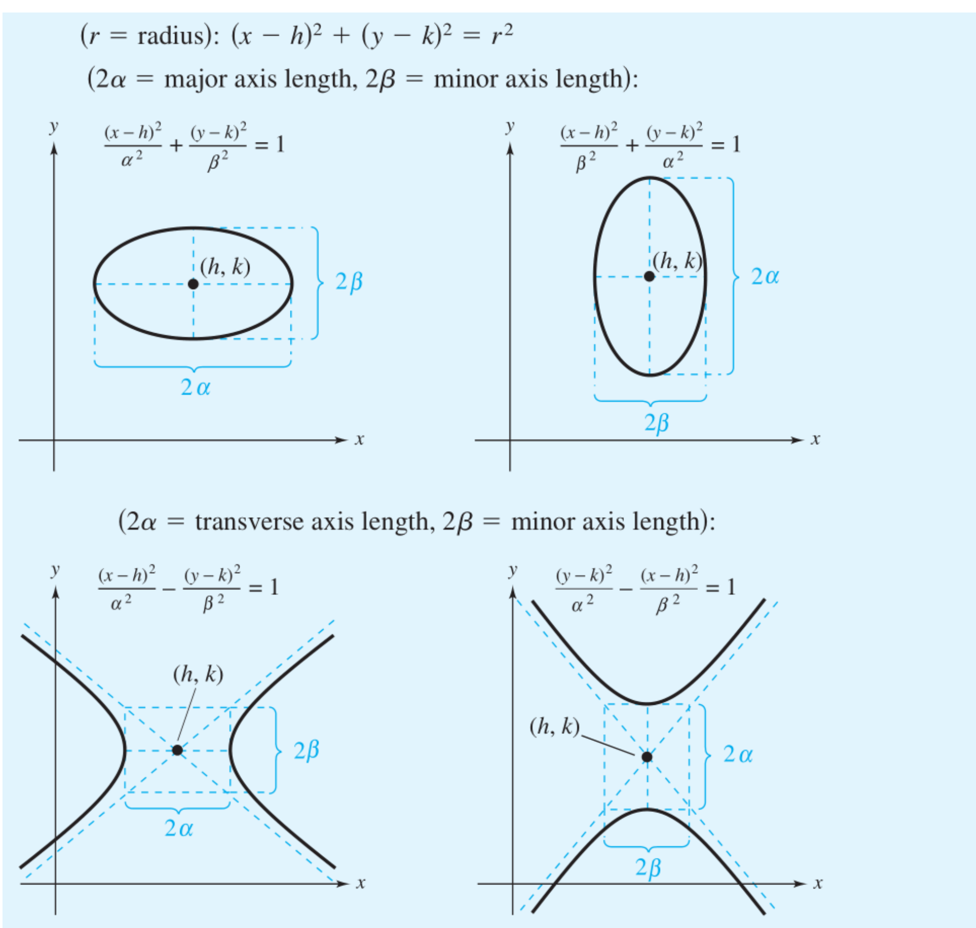
2 Conic Sections and Rotation

Every conic section in the xy-plane has an equation of the form

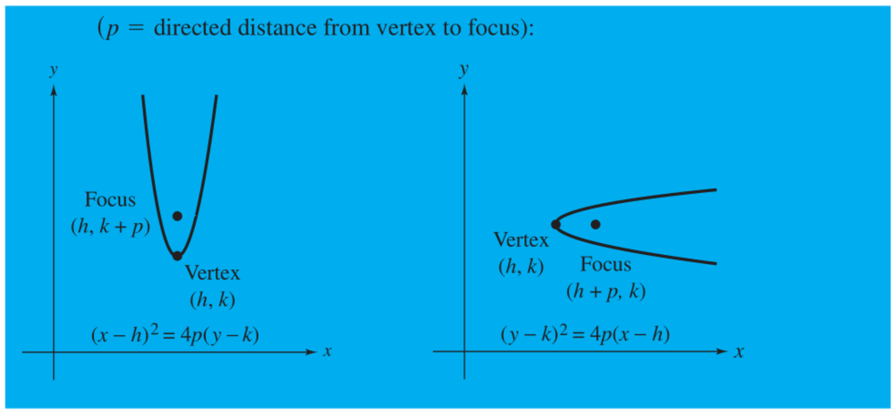
$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Identifying the graph is simple if $b = 0$. In such cases, the conic axes are parallel to the coordinate axes.

Standard Forms of Equations of Conics



Standard Forms of Equations of Conics (cont.)



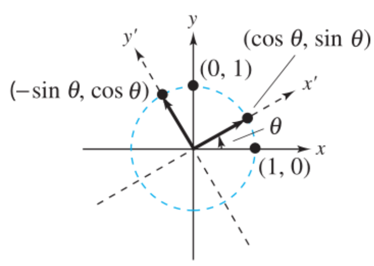
For a second-degree polynomial equations that have an xy-term, the axes are not *parallel* to the coordinate axes. In such cases, it is helpful to *rotate* the standard axes to form the new x'-axis and y'-axis.

The required rotation angle θ (measure counterclockwise) is $\cot 2\theta = (a-c)/b$. With this rotation, the standard basis in the plane

$$B = \{(1,0), (0,1)\}$$

is rotated to form the new basis

$$B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)$$



To find the coordinates of (x, y) relative to this new basis, use a *transition matrix*.

2.1 A Transition Matrix for Rotation in the Plane

Find the coordinate of a point (x, y) in R^2 relative to the basis

$$B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$$

SOLUTION. By Theorem 4.21, you have

$$[B' \vdots B] = \begin{bmatrix} \cos\theta & -\sin\theta & \vdots & 1 & 0 \\ \sin\theta & \cos\theta & \vdots & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I & \vdots & P^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \vdots & \cos\theta & \sin\theta \\ 0 & 1 & \vdots & -\sin\theta & \cos\theta \end{bmatrix}$$

The x'- and y'- coordinates are

$$\begin{cases} x' = x\cos\theta + y\cos\theta \\ y' = -x\sin\theta + y\cos\theta \end{cases}$$

It is also important to express the xy-coordinates in terms of the $x'y'$-coordinates.

$$x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta$$

Substituting these expressions for x and y into the given second-degree equation produces a second-degree polynomial equation in x' and y' that has no $x'y'$-term.

Rotation of Axes. The second-degree equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can be written in the form

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$$

by rotating the coordinates axes counterclockwise through the angle θ, where θ is defined by $\cot 2\theta = \frac{a-c}{b}$. The coefficients of the new equation are obtained from the substitutions

$$\begin{cases} x = x' \cos\theta - y' \sin\theta \\ y = x' \sin\theta + y' \cos\theta \end{cases}$$