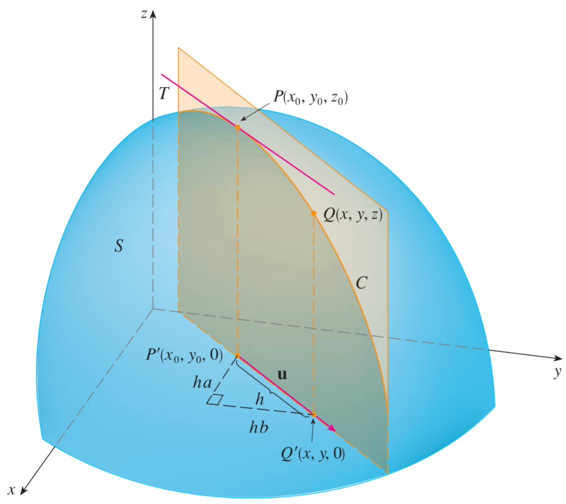
1 Directional Derivatives and the Gradient Vector



Directional Derivatives

We want the rate of change of z at (x_0, y_0) in the direction of an unit vector $\mathbf{u} = \langle a, b \rangle$.

- Consider the surface S of $z = f(x, y)$, the vertical plane that passes through $P(x_0, y_0, z_0)$ in the direction of $\mathbf{u}$ intersects S a curve C.
- The slope of tangent line T to C at P is what we need.

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to $\mathbf{u}$,

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

Therefore $x - x_0 = ha$, $y - y_0 = hb$.

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of $\mathbf{u}$.

Definition : Directional Derivatives

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$\begin{aligned} D_u f(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ &= f_x(x, y)a + f_y(x, y)b \\ &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \quad (\mathbf{u} \text{ makes an angle } \theta \text{ with the } x^{+}\text{-axis}) \end{aligned}$$

The directional derivative $D_{\mathbf{u}} f(1, 2)$ in Example 2 represents the rate of change of z in the direction of $\mathbf{u}$. This is the slope of the tangent line to the curve of intersection of the surface $z = x^3 - 3xy + 4y^2$ and the vertical plane through $(1, 2, 0)$ in the direction of $\mathbf{u}$ shown in Figure 5.

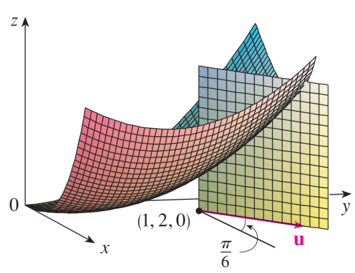


FIGURE 5

EXAMPLE. Find the directional derivative $D_u f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is given by $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?

SOLUTION. $f_x(x, y) = 3x^2 - 3y \qquad f_y(x, y) = 8y - 3$

Therefore,

$$\begin{aligned} D_u f(x, y) &= \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(8y - 3) \\ &= \frac{3\sqrt{3}}{2}x^2 + \frac{4 - 3\sqrt{3}}{2}y - \frac{3}{2} \end{aligned}$$

Hence $D_u f(1, 2) = \dfrac{13 - 3\sqrt{3}}{2}$

The Gradient Vector

Notice that $D_{\mathbf{u}} = \langle f_x(x,y), f_y(x,y)\rangle \cdot \mathbf{u}$.

Definition : Gradient

The **gradient** of $f(x,y)$ is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

The directional derivative of $f(x,y)$ is $\quad D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$

EXAMPLE. If $f(x,y) = \sin x + e^{xy}$, then

$$\begin{aligned} &\nabla f(x,y) = \langle f_x, f_y\rangle = \langle \cos x + ye^{xy}, xe^{xy}\rangle \\ &\nabla f(0,1) = \langle 2, 0\rangle \end{aligned}$$

The gradient vector $\nabla f(2,-1)$ in Example 4 is shown in Figure 6 with initial point $(2,-1)$. Also shown is the vector $\mathbf{v}$ that gives the direction of the directional derivative. Both of these vectors are superimposed on a contour plot of the graph of f.

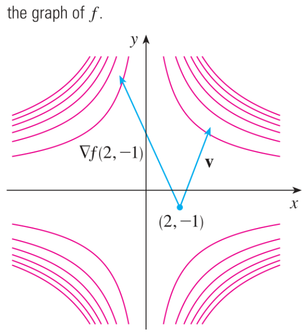


FIGURE 6

EXAMPLE. Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at $(2,-1)$ in the direction of $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

SOLUTION. We first compute the gradient vector at $(2,-1)$:

$$\begin{aligned} \nabla f(x,y) &= 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{i} \\ \nabla f(2,-1) &= -4\mathbf{i} + 8\mathbf{j} \end{aligned}$$

The unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{2}{\sqrt{29}}\,\mathbf{i} + \dfrac{5}{\sqrt{29}}\,\mathbf{j}$

Therefore we have

$$\begin{aligned} D_{\mathbf{u}} f(2,-1) = \nabla f(2,-1) \cdot \mathbf{u} &= (-4\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}\right) \\ &= \frac{-4\cdot 2 + 8\cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}} \end{aligned}$$

Functions of Three Variables

Definition : Directional Derivatives

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

The **gradient vector** is

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

And the directional derivative is $\quad D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$

EXAMPLE. If $f(x,y,z) = x \sin yz$, (a) find ∇f and (b) find $D_{\mathbf{u}} f(1,3,0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

SOLUTION.

$$\nabla f = \sin yz \cdot \mathbf{i} + xz \cos yz \cdot \mathbf{j} + xy \cos xz \cdot \mathbf{k}$$

The unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Therefore

$$\begin{aligned} D_{\mathbf{u}} &= \nabla f(1,3,0) \cdot \mathbf{u} \\ &= 3\mathbf{k} \cdot \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \\ &= -\sqrt{\frac{3}{2}} \end{aligned}$$