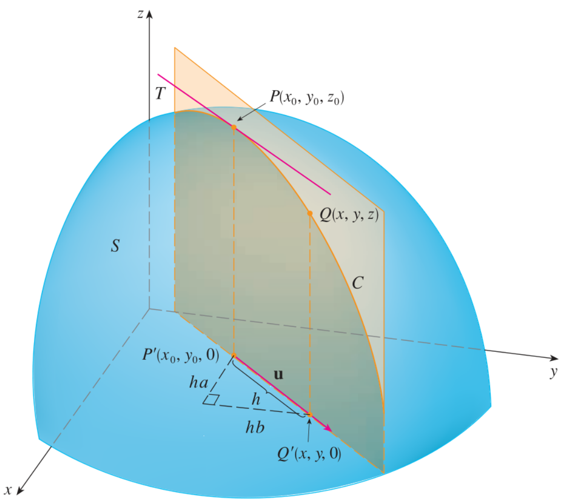
1 Directional Derivatives and the Gradient Vector



⚓ Directional Derivatives

We want the rate of change of z at (x_0, y_0) in the direction of an unit vector $\mathbf{u} = \langle a, b\rangle$.

- Consider the surface S of $z = f(x, y)$, the vertical plane that passes through $P(x_0, y_0, z_0)$ in the direction of $\mathbf{u}$ intersects S a curve C.
- The slope of tangent line T to C at P is what we need.

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to $\mathbf{u}$,

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb\rangle$$

Therefore $x - x_0 = ha$, $y - y_0 = hb$.

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of $\mathbf{u}$.

Definition : Directional Derivatives

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$\begin{aligned} D_u f(x_0, y_0) &= \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ &= f_x(x, y)a + f_y(x, y)b \\ &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \quad (\mathbf{u} \text{ makes an angle } \theta \text{ with the } x^+\text{-axis}) \end{aligned}$$

The directional derivative $D_{\mathbf{u}} f(1, 2)$ in Example 2 represents the rate of change of z in the direction of $\mathbf{u}$. This is the slope of the tangent line to the curve of intersection of the surface $z = x^3 - 3xy + 4y^2$ and the vertical plane through $(1, 2, 0)$ in the direction of $\mathbf{u}$ shown in Figure 5.

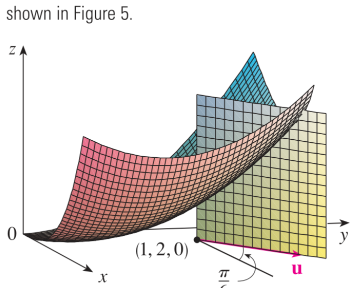


FIGURE 5

EXAMPLE. Find the directional derivative $D_u f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\mathbf{u}$ is given by $\theta = \pi/6$. What is $D_{\mathbf{u}} f(1, 2)$?

SOLUTION. $f_x(x, y) = 3x^2 - 3y \qquad f_y(x, y) = 8y - 3$

Therefore,

$$\begin{aligned} D_u f(x, y) &= \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(8y - 3) \\ &= \frac{3\sqrt{3}}{2}x^2 + \frac{4 - 3\sqrt{3}}{2}y - \frac{3}{2} \end{aligned}$$

Hence $D_u f(1, 2) = \dfrac{13 - 3\sqrt{3}}{2}$

The Gradient Vector

Notice that $D_{\mathbf{u}} = \langle f_x(x,y), f_y(x,y)\rangle \cdot \mathbf{u}$.

Definition : Gradient

The **gradient** of $f(x,y)$ is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

The directional derivative of $f(x,y)$ is $\quad D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$

EXAMPLE. If $f(x,y) = \sin x + e^{xy}$, then

$$\begin{aligned}
&\nabla f(x,y) = \langle f_x, f_y\rangle = \langle \cos x + ye^{xy}, xe^{xy}\rangle \\
&\nabla f(0,1) = \langle 2, 0\rangle
\end{aligned}$$

The gradient vector $\nabla f(2,-1)$ in Example 4 is shown in Figure 6 with initial point $(2,-1)$. Also shown is the vector $\mathbf{v}$ that gives the direction of the directional derivative. Both of these vectors are superimposed on a contour plot of the graph of f.

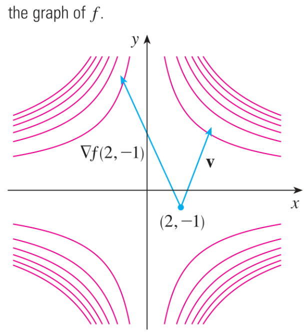


FIGURE 6

EXAMPLE. Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at $(2,-1)$ in the direction of $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

SOLUTION. We first compute the gradient vector at $(2,-1)$:

$$\begin{aligned}
\nabla f(x,y) &= 2xy^3\mathbf{i} + (3x^2y^2 - 4)\mathbf{i} \\
\nabla f(2,-1) &= -4\mathbf{i} + 8\mathbf{j}
\end{aligned}$$

The unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{2}{\sqrt{29}}\,\mathbf{i} + \dfrac{5}{\sqrt{29}}\,\mathbf{j}$

Therefore we have

$$\begin{aligned}
D_{\mathbf{u}} f(2,-1) &= \nabla f(2,-1) \cdot \mathbf{u} = (-4\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}}\,\mathbf{i} + \frac{5}{\sqrt{29}}\,\mathbf{j}\right) \\
&= \frac{-4\cdot 2 + 8\cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}
\end{aligned}$$

Functions of Three Variables

Definition : Directional Derivatives

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

The **gradient vector** is

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

And the directional derivative is $\quad D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$

EXAMPLE. If $f(x,y,z) = x \sin yz$, (a) find ∇f and (b) find $D_{\mathbf{u}} f(1,3,0)$ in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

SOLUTION.

$$\nabla f = \sin yz \cdot \mathbf{i} + xz \cos yz \cdot \mathbf{j} + xy \cos xz \cdot \mathbf{k}$$

The unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Therefore

$$\begin{aligned}
D_{\mathbf{u}} &= \nabla f(1,3,0) \cdot \mathbf{u} \\
&= 3\mathbf{k} \cdot \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \\
&= -\sqrt{\frac{3}{2}}
\end{aligned}$$

1.1 Maximizing the Directional Derivative

Definition : Maximum Value of the Directional Derivative

The maximum value of $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$, when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

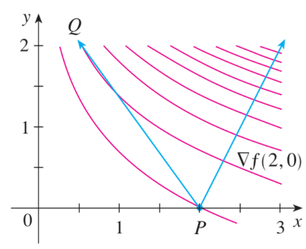


FIGURE 7

At (2, 0) the function in Example 6 increases fastest in the direction of the gradient vector $\nabla f(2,0) = \langle 1, 2\rangle$. Notice from Figure 7 that this vector appears to be perpendicular to the level curve through (2, 0). Figure 8 shows the graph of f and the gradient vector.

EXAMPLE.

(a) If $f(x,y) = xe^y$, find the rate of change of f at $P(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction, f has max $D_{\mathbf{u}}f$ and what's it?

SOLUTION.

(a)

$$\nabla f(x,y) = \langle f_x, f_y\rangle = \langle e^y, xe^y\rangle$$
$$\nabla f(2,0) = \langle 1, 2\rangle$$

The unit vector in the direction $\overrightarrow{PQ}$ is $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$, so we have

$$D_{\mathbf{u}}f(2,0) = \nabla f(2,0)\cdot\mathbf{u} = \langle 1,2\rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle$$
$$= 1\left(-\frac{3}{5}\right) + 2\left(\frac{4}{5}\right) = 1$$

(b) f increases fastest in the direction of $\nabla f(2,0) = \langle 1, 2\rangle$.

$$|\nabla f(2,0)| = |\langle 1,2\rangle| = \sqrt{5}$$

Tangent Planes to Level Surfaces

Suppose S of $F(x,y,z) = k$, and $P(x_0,y_0,z_0) \in S$. We can write $\nabla F \cdot \mathbf{r}'(t) = 0$

$$\nabla F(x_0,y_0,z_0)\cdot \mathbf{r}'(t_0) = 0$$

We see that the *gradient vector* $\nabla F(x_0,y_0,z_0)$ is **perpendicular** to the tangent vector to any curve C on S that pass through P.

Definition : Tangent plane

If $\nabla F(x_0,y_0,z_0) \neq \mathbf{0}$, there is a **tangent plane to the level surface** $F(x,y,z) = k$ **at** $P(x_0,y_0,z_0)$

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. The direction of it is given by $\nabla F(x_0,y_0,z_0)$ and its symmetric equation*s are

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

Special case. When $z = f(x,y)$, then $F(x,y,z) = f(x,y) - z = 0$, we have

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

EXAMPLE. Find the tangent plane and normal line at $(-2,1,-3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

SOLUTION. The ellipsoid is the level surface ($k = 3$) of the function

$$F(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Therefore we have

$$F_x(x,y,z) = \frac{x}{2} \qquad F_y(x,y,z) = 2y \qquad F_z(x,y,z) = \frac{2z}{9}$$

$$F_x(-2,1,-3) = -1 \qquad F_y(-2,1,-3) = 2 \qquad F_z(-2,1,-3) = -\frac{2}{3}$$

Figure 10 shows the ellipsoid, tangent plane, and normal line in Example 8.

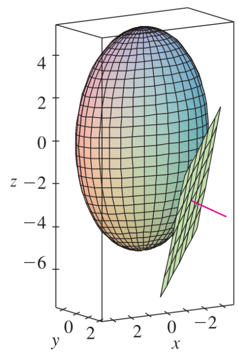


FIGURE 10

The equation of the tangent plane at (-2, 1, -3) is

$$-1(x+2)+2(y-1)-\frac{2}{3}(z+3)=0$$

The symmetric equations of the normal line are

$$\frac{x+2}{-1}=\frac{y-1}{2}=\frac{z+3}{-\frac{2}{3}}$$

Maximum and Minimum Values

Definition : Local extrema

Local maximum $f(a,b)$ if $f(x,y)\le f(a,b)$ when (x,y) is near (a,b).And the first-order partial derivatives of f exists there, then $f_x(a,b)=0$ and $f_y(a,b)=0$.

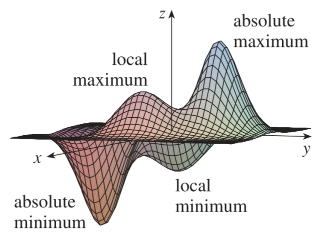


FIGURE 1

If we put $f_x(a,b)=0$ and $f_y(a,b)=0$ in the equation of a tangent plane, we get $z=z_0$. So the tangent plane at a local extrema must be *horizontal*. A point (a,b) is a **critical point** (or *stationary point*) of f if $f_x(a,b)=f_y(a,b)=0$, or if one of these does not exist.

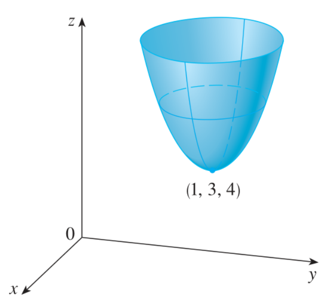


FIGURE 2
$z=x^2+y^2-2x-6y+14$

EXAMPLE. Let $f(x,y)=x^2+y^2-2x-6y+14$. Then

$$f_x(x,y)=2x-2 \qquad f_y(x,y)=2y-6$$

These derivatives are equal to 0 when $x=1, y=3$. So the only critical point is $(1,3)$.

$$f(x,y)=4+(x-1)^2+(y-3)^2$$

We have $f(x,y)\ge 4$. Therefore $f(1,3)=4$ is a local minimum, and in fact it is the **absolute minimum** of f.

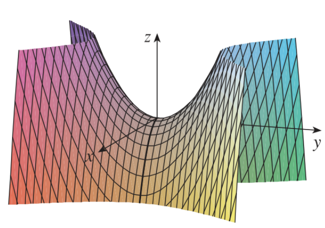


FIGURE 3
$z=y^2-x^2$

EXAMPLE. Find the extreme values of $f(x,y)=x^2+y^2$.
$f(x,y)$ is either maxima or minima depends on directions. So $(0,0)$ is a *saddle point* of f. Then how to determine?

Definition : Second Derivatives Test

Suppose $f_x(a,b)=f_y(a,b)=0$. Let

$$D=D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$$
$$=\begin{vmatrix} f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{vmatrix}=f_{xx}f_{yy}-(f_{xy})^2$$

(a) **Local minimum:** $D>0, f_{xx}(a,b)>0$.
(b) **Local maximum:** $D>0, f_{xx}(a,b)<0$.
(c) **Neither:** $D<0$.

Note. If $D=0$, we have no idea.

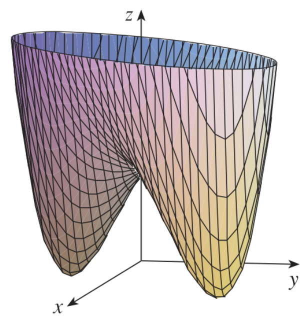


FIGURE 4
$z = x^4 + y^4 - 4xy + 1$

📍 **EXAMPLE.** Find the local maximum and minimum ad saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.
First we have

$$f_x = 4x^3 - 4y \qquad\qquad f_y = 4y^3 - 4x$$
$$x^3 - y = 0 \qquad\qquad y^3 - x = 0$$

which implies $0 = x^9 - x = x(x^2-1)(x^2+1)(x^4+1)$, so there're 3 roots: 0, 1, -1. The 3 critical points are $(0, 0), (1, 1), (-1, -1)$.
Next we calculate the second partial derivatives and $D(x, y)$

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_y y = 12y^2$$

$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0, 0) = -16 < 0$, it follows that (0,0) is a saddle point. And $D(1, 1) = 128 > 0, f_{xx}(1, 1) = 12 > 0$, so it's a local minimum. Similarly, $(-1, -1)$ is a local minimum.

📍 **EXAMPLE.** Find the shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

The distance from (x, y, z) to $(1, 0, -2)$ is

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$$

By solving the equation

$$f_x = 4x + 4y - 14 = 0$$
$$f_y = 4x + 10y - 24 = 0$$

we find that the only critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$. Since $f_{xx} = 4, f_{xy} = 4, f_{yy} = 10, D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 24 > 0$, so f has a local minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$. There must be a point on the given plane that is closest to (1,0,-2). We also find that $d = \frac{5}{6}\sqrt{6}$.

⚓ Absolute Maximum and Minimum Values

Definition : Extreme Value Theorem

If f is continuous on a closed, bounded set $D \in \mathbb{R}^2$ then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$. To find it,
1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. Determine the largest and smallest ones.

📍 **EXAMPLE.** Find the absolute maximum and minimum of $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) |\ 0 \leq x \leq 3, 0 \leq y \leq 2\}$.
Since f is a polynominal, it's continuous on D. First find the critical points

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

So the only critical point is (1,1), and $f(1, 1) = 1$.

Now we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4.

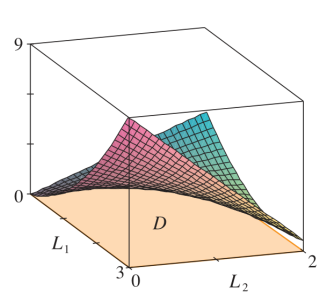


FIGURE 13
$f(x, y) = x^2 - 2xy + 2y$

- On L_1, we have $y = 0$ and

$$f(x,0) = x^2 \qquad 0 \leq x \leq 3$$

Its minimum value is $f(0,0) = 0$ and maximum value is $f(3,0) = 9$.

- On L_2, we have $x = 3$ and

$$f(3,y) = 9 - 4y \qquad 0 \leq y \leq 2$$

The maximum value is $f(3,0) = 9$ and the minimum value is $f(3,2) = 1$.

- On L_3 we have $y = 2$ and

$$f(x,2) = x^2 - 4x + 4 = (x-2)^2 \qquad 0 \leq x \leq 3$$

The minimum value is $f(2,2) = 0$ and the maximum value is $f(0,2) = 4$.

- On L_4 we have $x = 0$ and

$$f(0,y) = 2y \qquad 0 \leq y \leq 2$$

with maximum value $f(0,2) = 4$ and minimum valye $f(0,0) = 0$.

Thus, on the boundary, the minimum value is 0 and the maximum is 9.

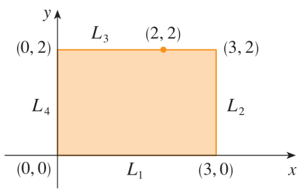


FIGURE 12

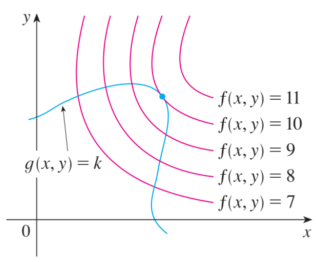


FIGURE 1

TEC Visual 14.8 animates Figure 1 for both level curves and level surfaces.

⚓ Lagrange Multipliers

We will discover Lagrange's methods for maximizing or minimizing a general function $f(x,y,z)$ to a constraint (or side contidition) of the form $g(x,y,z) = k$.

Definition : Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x,y,z)$ to the constraint $g(x,y,z) = k$ (assume they exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x,y,z) = k$):

(a) Find all x, y, z and λ (**Lagrange multiplier**) such that

$$\begin{aligned} \nabla f(x,y,z) &= \lambda \nabla g(x,y,z) \\ g(x,y,z) &= k \end{aligned}$$

(b) Evaluate f at all these points and find the largest and smallest ones.

Write (a) in terms of components

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x,y,z) = k$$

It's not necessary to find explicit values for λ.

EXAMPLE. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume.

SOLUTION. We wish to maximize $V = xyz$, where x, y, z are the length, width and height of the box, subject to the constraint

$$g(x,y,z) = 2xz + 2yz + xy = 12$$

We look for x, y, z, λ that $\nabla V = \lambda \nabla g$ and $g(x,y,z) = 12$.

$$\begin{aligned} V_x &= \lambda g_x \\ V_y &= \lambda g_y \\ V_z &= \lambda g_z \\ 2xz + 2yz + xy &= 12 \end{aligned}$$

which become

$$
\begin{aligned}
yz &= \lambda(2z+y)\\
xz &= \lambda(2z+x)\\
xy &= \lambda(2x+2y)\\
2xz+2yz+xy &= 12
\end{aligned}
$$

Observe that $\lambda \neq 0$, and we have $2xz+xy = 2yz+xy$ which gives $xz = yz$. But $z \neq 0$, or $V = 0$. So $x = y$. We also have $x = y = 2z$.

$$4z^2+4z^2+4z^2 = 12$$

Therefore we have $x = y = 2$, and $z = 1$.

In geometric terms, Example 2 asks for the highest and lowest points on the curve C in Figure 2 that lie on the paraboloid $z = x^2 + 2y^2$ and directly above the constraint circle $x^2 + y^2 = 1$.

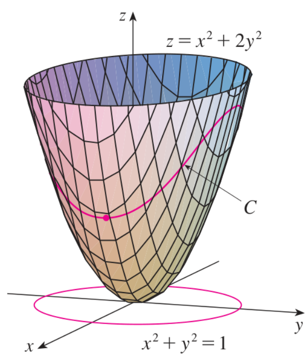


FIGURE 2

EXAMPLE. Find the extreme values of $f(x,y) = x^2+2y^2$ on the circle $x^2+y^2 = 1$.
Solve the equation

$$
\begin{aligned}
f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x,y) &= 1\\
2x &= 2x\lambda\\
4y &= 2y\lambda\\
x^2+y^2 &= 1
\end{aligned}
$$

- $x = 0$, then $y = \pm 1$.
- $\lambda = 1$, then $y = 0$, and $x = \pm 1$.

Evaluating f at these 4 points, we find that $f_{\max} = f(0,\pm 1) = 2$ and $f_{\min} = f(\pm 1, 0) = 1$.

The geometry behind the use of Lagrange multipliers in Example 2 is shown in Figure 3. The extreme values of $f(x, y) = x^2 + 2y^2$ correspond to the level curves that touch the circle $x^2 + y^2 = 1$.

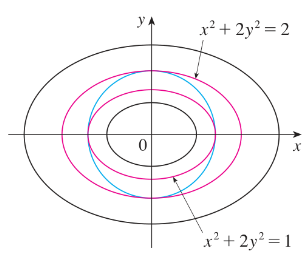


FIGURE 3

EXAMPLE. Find the points on the sphere $x^2+y^2+z^2 = 4$ that are closest and farthest from $(3, 1, -1)$.