1 Vectors in $\mathbb{R}^n$

A vector is characterized by 2 quantities: ***length*** and ***direction***, and is represented by a directed line segment. But they are just 2 special types of vectors.

1.1 Vectors in the Plane

A **vector in the plane** is represented geometrically by a **directed line segment** whose **initial point** is the origin and whose **terminal point** is the point (x_1, x_2). This vector is represented by the same **ordered pair** used to represent its terminal point

$$\mathbf{x} = (x_1, x_2)$$

- x_1, x_2 : the **components** of the vector $\mathbf{x}$
- $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1$ and $u_2 = v_2$

1.2 Vectors in $\mathbb{R}^n$

A vector in n-space is represented by an **ordered n-tuple**. The set of all n-tuple is called **n-space** and is denoted by $\mathbb{R}^n$.

Properties of Vectors Addition and Scalar Multiplication in $\mathbb{R}^n$

1. $\mathbf{u} + \mathbf{v}$ is a vector in R^n.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = (\mathbf{u} + (\mathbf{v} + \mathbf{w}))$
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. $\mathbf{u} + (\mathbf{-u}) = \mathbf{0}$
6. $c\mathbf{u}$ is a vector in R^n.
7. $c\mathbf{u} + \mathbf{v} = \mathrm{c}\mathbf{u} + \mathrm{c}\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{v}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

THEOREM 4.3 *Properties of Additive Identity and Additive Inverse*

1. $\mathbf{v} + \mathbf{u} = \mathbf{v}$, then $\mathbf{u}$ =0.
2. $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = \mathbf{v}$.
3. $0\mathrm{v} = 0$ (scalar)
4. $\mathrm{c}0 = 0$ (vector 0)
5. $\mathrm{cv} = 0$, then $c = 0$ or $v = 0$.
6. $-(-v) = v$

Writing a Vector as a Linear Combination of Other Vectors

Vector $\mathbf{x}$ can be written as the sum of scalar multiples of n other vectors $v_1, v_2, \cdots, v_n$

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

then the vector $\mathbf{x}$ is called a **linear combination** of the vectors $v_1, v_2, \cdots, v_n$

2 Vector Spaces

Any set that satisfies these aforementioned properties (or **axioms**) is called **vector space**, and the objects in the set are called **vectors**.

Definition of Vector Space. Let V be a set on which 2 operations (**vector addtition** and **scalar multiplication**) are defined. **u, v, w** $\in V$.
V is called a **vector space** if the listed axioms are satisfied for every **u, v, w** and real number c, d :

1. $\mathbf{u} + \mathbf{v}$ in V
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = (\mathbf{u} + (\mathbf{v} + \mathbf{w}))$
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$: V has a **zero vector 0**
5. **u + (-u) = 0** : For every **u** in V, there exists $-u$.
6. $c\mathbf{u}$ in V
7. $c\mathbf{u} + \mathbf{v} = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{v}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

REMARK. A vector space consists of 4 entities:

1. a set of vectors
2. a set of scalars
3. 2 operations

Example 3.
1. The set of all 2×3 matrices with the operations of matrix addtition and scalar multiplication is a vector space.
2. The set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ with the operations of matrix addition and scalar multiplication is a vector space.

Example. The Vector Space of All Polynomials of Degree 2 or Less

Let P_2 be the set of all polynomials of the form

$$p(x) = a_2x^2 + a_1x + a_0$$

with the usual operations of polynomial addition and scalar multiplication. P_2 is a vector space. This can be extended to P_n.

Summary of Important Vector Spaces.

$$\begin{aligned}
R &= \text{set of all real numbers} \\
R^2 &= \text{set of all ordered pairs} \\
R^3 &= \text{set of all ordered triples} \\
R^n &= \text{set of all n-tuples} \\
C(-\infty, \infty) &= \text{set of all continuous functions defined on the real number line} \\
C[a, b] &= \text{set of all continuous functions defined on a closed interval}[a, b] \\
P &= \text{set of all polynomials} \\
P_n &= \text{set of all polynomials of degree} \leq n \\
M_{m,n} &= \text{set of all } m \times n \text{matrices} \\
M_{n,n} &= \text{set of all square matrices}
\end{aligned}$$

THEOREM 4.4 *Properties of Scalar Multiplication*

1. $0\mathbf{v} = \mathbf{0}$
2. $c\mathbf{0} = \mathbf{0}$
3. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$
4. $(-1)\mathbf{v} = -\mathbf{v}$

Example.
The Set of Integer is Not a Vector Space.
The Set of Second-Degree Polynomials Is Not a Vector Space.

$$p(x) = x^2, \quad q(x) = -x^2 + x + 1 \implies p(x) + q(x) \notin V$$

3 Subspaces of Vector Spaces

Definition of Subspace of a Vector Space.
A non-empty subset W of a vector space V is called a **subspace** of V if W is a vector space under the operations of addition and scalar multiplication defined in V.

Example 1. The set $W = (x_1, 0, x_3) : x_1$ and $x_3 \in \mathbb{R}$ is a subspace of $\mathbb{R}^e$ with the standard operations. Graphically, it's the xz-plane.

Test for a subspace. If W is a non-empty subset of V, then W is a subspace of V if and only if

1. $\mathbf{u} + \mathbf{v} \in W$
2. $c\mathbf{u} \in W$

The 2 trivial subspaces are the **zero subspace** (contain only the zero vector), and the space itself. The subspaces other than those 2 are **proper** (or nontrivial) subspaces.
Example. The Set of Singular Matrices Is Not a Subspace of $M_{n,n}$.

To prove, we need to show that W is empty, W is not closed under addtion, or W is not closed under scalar multiplication. For this particular set, W is not closed under addtion

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \implies A + B \notin W$$

Note. Sequences of subspaces nested within each other

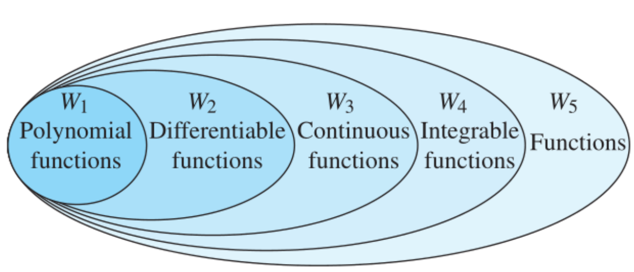
$$P_o \subset P_1 \subset \cdots \subset P_n$$

Example 5. Subspaces of Functions (Calculus)

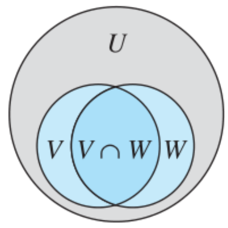
Let W_5 be the *vector space* of all functions defined n $[0, 1]$, and let

- W_1 = set of all polynominal functions defined on the interval $[0, 1]$
- W_2 = set of all functions that are differentable on $[0, 1]$
- W_3 = set of all functions that are continuous on $[0, 1]$
- W_4 = set of all functions that are integrable on $[0, 1]$

Show that $W_1 \subset W_2 \subset \cdots \subset W_5$, and $W_i \subset W_j$ for $i \leq j$.



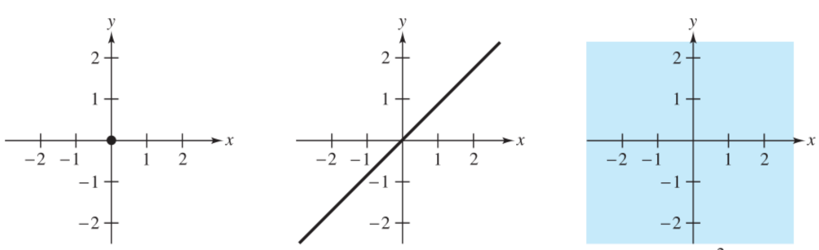
The Intersection of 2 Subspaces Is a Subspace. If V and W are both subspaces of a vector space U, then $V \cap W$ is also a subspace of U.



3.1 Subspace of $\mathbb{R}^n$

A characteristic of $\mathbb{R}^2$. If $W \subset R^2$, then it is a subspace if and only if **one** of the three possibilities listed below is true.

1. W consists of the *single point* $(0,0)$.
2. W consists of all points on a *line* that pass through the origin.
3. W consists of all of R^4.



$W = \{(0, 0)\}$ W = all points on a line passing through the origin $W = R^2$

Example 7. The Set of all points on the circle $x^2 + y^2 = 1$ is not a subspace.
Take the sum $a = (0, 1)$ and $b(1, 0)$, then the Set is not closed under addtion.

REMARK. Another way: a subspace of $\mathbb{R}^2$ must contain $(0, 0)$.

3.2 Subspaces of $\mathbb{R}^3$

Subspaces of $\mathbb{R}^3$. $W \subset \mathbb{R}^3$ is a subspace of $\mathbb{R}^3$ (with the standard operations) if and only if it has **one** of the forms listed below.

1. W consists of the *single point* $(0,0)$.
2. W consists of all points on a *line* that pass through the origin.
3. W consists of all points on a *plane* that pass through the origin.
4. W consists of all of R^3.

4 Spanning Sets and Linear Independence

Def. Linear Combination of Vectors.
A vector $\mathbf{v}$ in a vector space V is called a **linear combination** of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k,$$

where $c_1, c_2, \ldots, c_k$ are scalars.

Example 1.

(a) For the set of vectors in R^3,

$$S = \overbrace{(1,3,1)}^{v_1}, \overbrace{(0,1,2)}^{v_2}, \overbrace{(1,0,-5)}^{v_3},$$

$\mathbf{v}_1$ is a **linear combination** of $\mathbf{v}_2$ and $\mathbf{v}_3$ since $\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3$.

(b) For the set of vectors in $M_{2,2}$, (page 226)

4.1 Finding a Linear Combination

Def. Spanning Set of a Vector Space.
Let $S = \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$. S is called a **spanning set** of V if *every* vector in V can be written as a linear combination of vectors in S. (S **spans** V)

Example. $S = (1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{R}^3$ because any vector $\mathbf{u} = (u_1, u_2, u_3)$ can be written as

$$\mathbf{u} = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$$

Example. $S = 1, x, x^2$ spans P_2 because any polynominal function $p(x) = a + bx + cx^2$ in P_2 can be written as

$$p(x) = a(1) + b(x) + c(x^2)$$

These above 2 spanning sets are called **standard spanning sets** of R^3 and P_2, respectively.

A nonstandard Spanning Set of R^3.

$$S_1 = (1,2,3), (0,1,2), (-2,0,1)$$

$$\begin{cases} c_1 \qquad\quad - 2c_3 = u_1 \\ 2c_1 + \ \ c_2 \qquad\ = u_2 \\ 3c_1 + 2c_2 + \ \ c_3 = u_3 \end{cases}$$

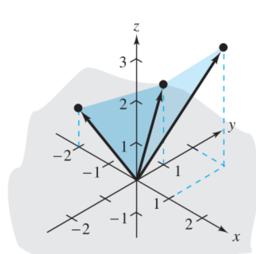
Since the coefficient matrix of the system has a nonzero determinant, this system has a unique solution.

A Set that Does Not Span R^3

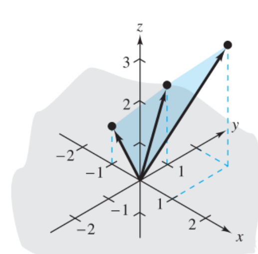
$$S_2 = (1,2,3), (0,1,2), (-1,0,1)$$

does not span R^3 since $\mathbf{w} = (1,-2,2)$ can not be expressed as a linear combination of these vectors.

Consider these 2 aforementioned sets S_1, S_2. The vectors in S_2 lie in a common plane while those in S_1 do not.



$S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$
The vectors in S_1 do not lie in a common plane.



$S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$
The vectors in S_2 lie in a common plane.

Although S_2 does not span R^3, it does span a subspace of R^3 - namely, the plane in which the 3 vectors of S_2 lie. This subspace is called the **span of** S_2.

Def. The Span of a Set. If $S = \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is a set of vectors in V, then the **span of *S*** is the set of all linear combinations of the vectors in S.

$$span(S) = span(v_1, v_2, \ldots, v_k) = \{c_1v_1 + c_2v_2 + \cdots + c_kv_k : c_1, c_2, \ldots, c_k \in \mathbb{R}\}$$

If $span(S) = V$: V is **spanned** by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, or S **spans** V.

THEOREM 4.7 **Span(S) Is a Subspace of V**
If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in V, then span(S) is a subspace of V. Moreover, span(S) is the smallest subspace of V that contains S, in the sense that every other subspace of V that contains S must contain span(S).

4.2 Linear Dependence and Linear Independence

The set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in V is called **linearly independent** if the vector equation

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

has only the trivial solutions. If there are also nontrivial solutions, then S is **linearly dependent**.

4.2.1 Testing for Linear Independence and Linear Dependence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors n a vector space V. To determine whether S is linearly independent or not:

1. Write a homogeneous system of equations in the variables $c_1, c_2, \ldots, c_k$

$$\{c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

2. Determine whether the matrix of k column vectors $v_1, v_2, \ldots, v_k$ has a nonzero determinant or not.
3. If yes, it is linearly independent. Otherwise, linearly dependent.

Example.(page 233)

THEOREM 4.8 **A Property of Linearly Dependent Sets** A set S containing ≥ 2 vectors is linearly dependent if and only if at least 1 vector v_j can be written as a linear combination of the other ones.

If S is linearly dependent, assuming $c_1 \neq 0$, then you can solve the equation for v_1 and write v_1 as a linear combination of the remaining other ones. In other words, v_1 *depends* on the other vectors on the set.

$$v_1 = -\frac{c_2}{c_1}v_2 - \frac{c_3}{c_1}v_3 - \cdots - \frac{c_k}{c_1}v_k$$

Corolarry. 2 vectors $\mathbf{u}$ and $\mathbf{v}$ in V are linearly dependent if and only if one is a scalar multiple of the other.

5 Basis and Dimension

Def. Basis
A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in V is called a **basis** for V if

1. S spans V.
2. S is linearly independent.

REMARK. A basis has 2 features. It must have *enough vectors* to span V, but *not so many vectors* that 1 one them could even be written as a linear combination of the others in S.

If V has a basis consisting of a finite number of vectors, then V is **finite dimesional**. Otherwise, like the vector space P of all polynominals, is **infinite dimesional**.

Example 1. The Standard Basis for $\mathbb{R}^3$

$$S = \{(1,0,0), (0,1,0), (0,0,1)\}$$

- S span $\mathbb{R}^3$.
- S is linear independent because the vector equation $c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = 0$ has only the trivial solution. Thus, S is a basis for $\mathbb{R}^3$.

That basis is called the **standard basis** for $\mathbb{R}^3$. That is, the vectors

$$\begin{aligned}\mathbf{e}_1 &= (1,0,\ldots,0)\\ \mathbf{e}_2 &= (0,1,\ldots,0)\\ &\vdots\\ \mathbf{e}_n &= (0,0,\ldots,1)\end{aligned}$$

form the **standard basis** for $\mathbb{R}^n$.

Standard Basis for P_n. $S = \{1, x, x^2, \ldots, x^n\}$

Standart Basis for $M_{2,2}$. $S = \left\{ \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} \right\}$

THEOREM 4.9 **Uniqueness of Basis Representation** If S is a basis for V, *every* vectors in V can be written in **only one** way as a linear combination of vectors of S.

Now, these are 2 important theorems concerning bases.

THEOREM 4.10 **Bases and Linear Dependence**
If S is a set of n vectors and is a basis for V, then every set contaning $\geq n$ vectors in V is **linearly dependent**.
THEOREM 4.11 **Number of Vectors in a Basis** If V has one basis with n vectors, then every basis for V has n vectors.

5.1 The Dimension of a Vector Space

Def. Dimension of a Vector Space.
If V has a basis with n vectors, then n is the **dimension** of V, denoted by $\dim(V) = n$.

If $V = \{0\}$, $\dim(V) = 0$.

1. $\dim(R^n) = n$.
2. $\dim(P_n) = n + 1$.
3. $\dim(M_{m,n}) = mn$.

If W is V's subspace, W is finite dimensional and $\dim(W) \leq n$.

Example. 247

Basis Tests in an n-Dimensional Space. Let V be a vector space of dimension n.

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in $V \implies S$ is a basis for V.
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans $V \implies S$ is a basis for V.

6 Rank of a Matrix and System of Linear of Equations

***Def.*Row Space and Column Space of a Matrix.** Let A be a $m \times n$ matrix.

1. The **row space** of A is the subspace of R^n spanned by the **row vectors** of A.
2. The **column space** of A is the subspace of R^m spanned by the **column vectors** of A.

Recall that 2 matrices are **row-equivalent** if one can be obtained from the other by elementary row operations.

THEOREM 4.13 **Row-Equivalent Matrices Have the Same Row Space.**
If $A_{m\times n}$ is row-equivalent to $B_{m\times n}$, then they have the *same* **row space**.

REMARK. The row space of a matrix does not change by elementary row operations. However, it can **change** the *column space.*

Basis for the Row Space of a Matrix. If A is row-equivalent to a matrix B in *row-echelon* form, then the nonzero row vectors of B form a basis for A's **row space**.

Suppose you are ask to find a **basis** for the subspace spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

- Form a matrix A consisting B's vectors as **row vectors**.
- Rewrite A in **row-echelon** form.

Find a Basis for the Column Space. There are 2 options. On the one hand, the column space of A is the row space of A^T. On the other hand, observe that:

REMARK. Although row operations can *change* the column space, they *do not change* the **dependency relationships** between columns. *(Notice that, the determinant remains unchanged.)*

THEOREM 4.15 **Row and Column Spaces Have Equal Dimension.**

Proof. (p.256) Suppose the basis S of A's row space has r vectors. You can rewrite the matrix and observe that the column vectors of A are all linear combination of the vectors S.

Thus, dim(col space) $\leq$ dim(row space). Doing the same, we have dim(row space) $\leq$ dim(col space). Proof complete.

Def. **Rank of a Matrix.** The dimension of the row (or col) space of A is called the **rank** of A, and denoted by $\text{rank}(A)$.

6.1 The Nullspace of a Matrix

Solutions of a Homogeneous System.
Consider the homogeneous linear system $A_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{0}$
The set of all solutions of this is a subspace of R^n called the **nullspace** :

$$N(A) = \{\mathbf{x} \in R^n : A\mathbf{x} = \mathbf{0}\}$$

The dimension of A's nullspace is the **nullity** of A.

REMARK. The *nullspace* of A is also the **solution space** of the system $A\mathbf{x} = \mathbf{0}$.

Dimension of the Solution Space. If A is an $m \times n$ matrix of rank r, then the dimension of the nullspace is $n - r$. That is,

$$n = \text{rank}(A) + \text{nullity}(A)$$

Example. 261

6.2 Solutions of Systems of Linear Equations

The set of all solutions vectors of the *homogeneous* linear system $A\mathbf{x} = \mathbf{0}$ is a **subspace**. However, for $A\mathbf{x} = \mathbf{b} \neq 0$, it's not a subspace - since $\mathbf{0}$ is never a solution of a nonhomogeneous system. However, there is a *relationship* between them.

THEOREM 4.18 **Solutions of a Nonhomogeneous Linear System.**
If $\mathbf{x}_p$ is a *particular* solution for $A\mathbf{x} = \mathbf{b}$, then *every* solution of this system can be written in the form

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$$

where $\mathbf{x}_h$ is a solution of $A\mathbf{x} = \mathbf{0}$.

Example 8. Finding the Solution Set of a Nonhomogeneous System.

Find the set of all solution vectors of the system of linear equations

$$\begin{cases} x_1 \quad\quad - 2x_3 + \ x_4 = 5 \\ 3x_1 + \ x_2 - 5x_3 \quad\quad = 8 \\ x_1 + 2x_2 \quad\quad - 5x_4 = -9 \end{cases}$$

SOLUTION. The augmented matrix for the system $A\mathbf{x} = \mathbf{b}$ reduces as follows.

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system of linear equations corresponding to the reduced row-echelon matrix is

$$\begin{cases} x_1 \quad\quad - 2x_3 + \ x_4 = 5 \\ \quad\quad x_2 + \ x_3 - 3x_4 = -7 \end{cases}$$

Letting $x_3 = s$ and $x_4 = t$, you can write a representative solution of $A\mathbf{x} = \mathbf{b}$ as follows.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s - t + 5 \\ -s + 3t - 7 \\ s + 0t + 0 \\ 0s + t + 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$
$$= s\mathbf{u}_1 + t\mathbf{u}_2 + \mathbf{x}_p$$

THEOREM 4.19 **Solutions of a System of Linear Equations**
The system of linear equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

6.3 Sys. Linear Equations with Square Coefficient Matrices

SUMMARY. **Equivalent Conditions for Square Matrices.**
If A is an $n \times n$ matrix, then the following conditions are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for *any* $n \times 1$ matrix $\mathbf{b}$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. $det(A) = |A| \neq 0$
6. Rank$(A) = n$
7. The n row vectors of A are linearly independent.
8. The n col vectors of A are linearly independent.

7 Coordinates and Change of Basis

If B is a basis for V, then every vector $\mathbf{x} \in V$ can be expressed in one and only one way as a linear combination of B's vectors. The *coefficients* in that linear combination are the **coordinates of x relative**

to B . In the context of *coordinate*, the order of vectors in basis is important, and this will be emphasized by referring to the basis B as an *ordered* basis.

Coordinate Representation Relative to a Basis.
Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an *ordered* basis for V, and let $\mathbf{x} \in V$ such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2 v_2 + \cdots + c_n v_n$$

The scalar $c_1, c_2, \ldots, c_n$ are the **coordinates of x relative to the basis B**. The **coordinate matrix** (or **coordinate vector**) **of x relative to *B*** is the column matrix in $\mathbb{R}^n$ whose components are the coordinates of **x**.

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Coordinate Representation in $\mathbb{R}^n$

Writing a vector in R^n as $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ means that the x_i's are the coordinates of **x** *relative to the standard basis* S in R^n. So you have

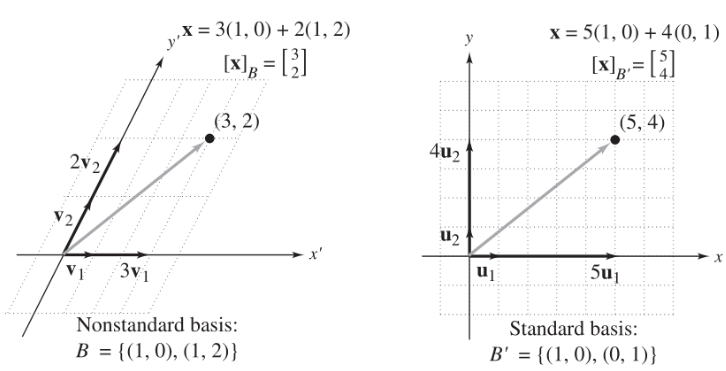
$$[\mathbf{x}]_S = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example. The coordinate matrix of **x** in R^2 relative to the (nonstandard) ordered basis $B = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1,0),(1,2)\}$ is

$$[\mathbf{x}_B] = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find the coordinate of **x** relative to the standard basis $B' = \{\mathbf{u}_1, \mathbf{u}_2\} = \{(1,0),(0,1)\}$.

ANSWER. $[\mathbf{x}_B] = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad [\mathbf{x}]_{B'} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$



Example. Find the coordinate matrix of $\mathbf{x} = (1, 2, -1)$ in R^3 relative to the (nonstandard) basis

$$B' = \{(u)_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1,0,1),(0,-1,2),(2,3,-5)\}$$

SOLUTION. Begin by writing **x** as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$:

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$
$$(1,2,-1) = c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5)$$

Equating corresponding components produces the following system of linear equations

$$\begin{cases} c_1 \qquad\quad + 2c_3 = 1 \\ \quad - \; c_2 + 3c_3 = 2 \\ c_1 + 2c_2 - 5c_3 = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 3 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

The solution of this system is $c_1 = 5, c_2 = -8, c_3 = -2$. So, the coordinate matrix of $\mathbf{x}$ relative to B' is

$$[\mathbf{x}_{B'}] = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

7.1 Change of Basis in R^n

In the previous example,

$$\underset{P}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}} \underset{[\mathbf{x}]_{B'}}{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}} = \underset{[\mathbf{x}]_B}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}.$$

The matrix P is the **transition matrix from *B'* to *B***.

$$P[\mathbf{x}_{B'}] = []Bx_B$$

To perform a change of basis from B to B', use the matrix P^{-1}

$$[\mathbf{x}_{B'}] = P^{-1}[\mathbf{x}_B]$$

This means that the change of basis problem in Eg.3 can be represented by the matrix equation

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underset{P^{-1}}{\begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}} \underset{[\mathbf{x}]_B}{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} = \underset{[\mathbf{x}]_{B'}}{\begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}}.$$

THEOREM 4.20 **The Inverse of a Transition Matrix.**
P : *transition matrix* from $B' \to B$ in R^n, then P is **invertible** and P^{-1} is the transition matrix from $B \to B'$.

My own thoughts. To perform a change of basis from B to S, use B itself

$$B[\mathbf{x}_B] = \mathbf{x}_S$$

Hence, for $[\mathbf{x}]_B$ and $[\mathbf{x}]_{B'}$ that is the same point

$$B[\mathbf{x}_B] = B'[\mathbf{x}_{B'}] = [\mathbf{x}_S]$$

To perform a change of basis from B' to B, simply

$$[\mathbf{x}_B] = B^{-1}B'[\mathbf{x}_{B'}]$$

In other words, $P = B^{-1}B'$ (from $[\mathbf{x}_{B'}] \to [\mathbf{x}_B]$). And $P^{-1} = B'^{-1}B$, we got a theorem.

THEOREM 4.21 **Transition Matrix from *B* to *B'*.**
The matrix P^{-1} from B to B^{-1} can be found by using Gauss-Jordan elimination on

$$[B' \vdots B] \to [I_n \vdots P^{-1}]$$

7.2 Coordinate Representation in General n-Dimensional Spaces

Example. $p = 3x^3 - 2x^2 + 4$ has the coordinate matrix $[p]_S = \begin{bmatrix} 4 \\ 0 \\ -2 \\ 3 \end{bmatrix}$.