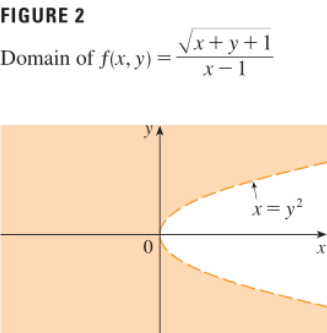# 1 Functions of Several Variables

Functions of Two Variables

> **Definition**
>
> A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D (domain subset of $\mathbb{R}^2$) of $f(x, y)$ (range subset of $\mathbb{R}$).
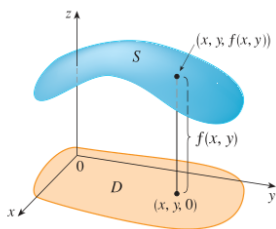
FIGURE 2

Domain of $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$

`EXAMPLE.`

Evaluate $f(3, 2)$, find and sketch the domain of $f(x, y) = \ln (y^2 - x)$.

$$f(3, 2) = 3 \ln (2^2 - 3) = 3 \ln 1 = 0$$

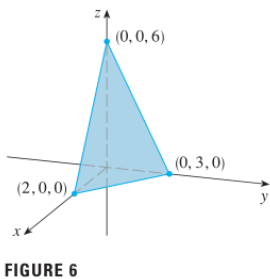The domain of f is $D = \{(x, y) \mid x < y^2\}$.

Graph

The graph of $f(x, y)$ is the set of all points (x, y, z) in $\mathbb{R}^3$.

FIGURE 5

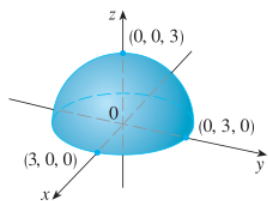`EXAMPLE.` Sketch the graph of $f(x, y) = 6 - 3x - 2y$.

FIGURE 6

`A linear function...`

The equation of the graph is $3x + 2 + z = 6$, which represents a plane. To graph it, find the *intercepts* by setting 2 of the 3 variables to 0.

`EXAMPLE.` Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

FIGURE 7

Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Square both sides of this equation to obtain $x^2 + y^2 + z^2 = 9$. Since $z \geq 0$, this is the upper part of a sphere whose center the origin and radius 3.

Computer programs can graph functions $f(x,y)$. Traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k and parts of the graph are eliminated using hidden line removal.
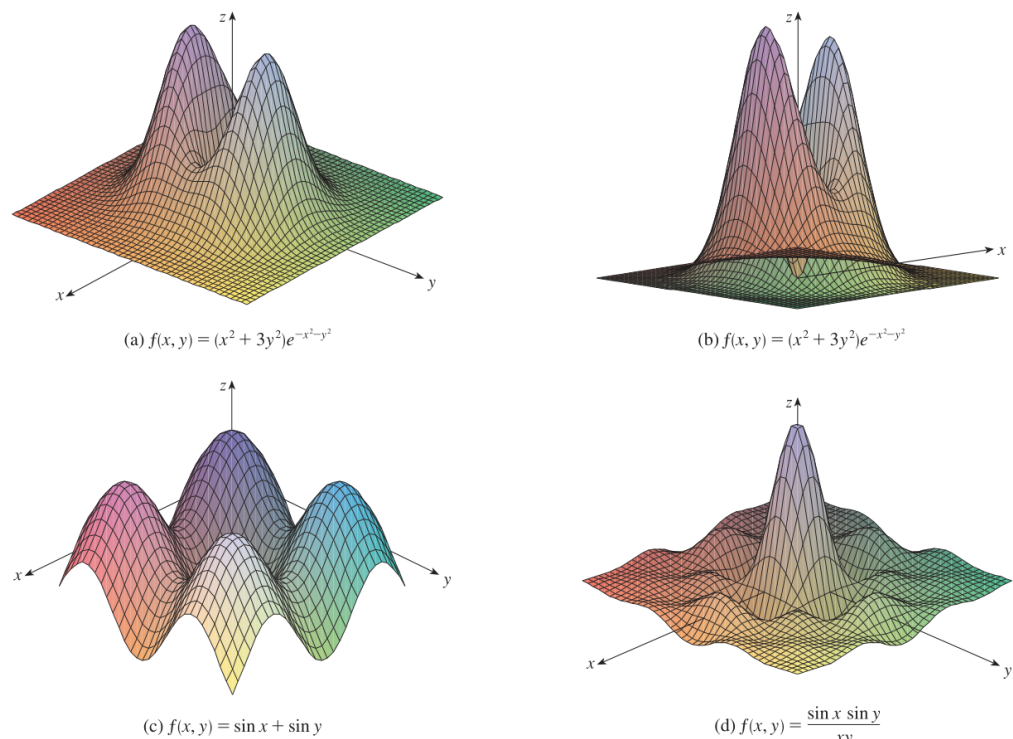

(a) $f(x,y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(b) $f(x,y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(c) $f(x,y) = \sin x + \sin y$

(d) $f(x,y) = \dfrac{\sin x \sin y}{xy}$

FIGURE 10

Level Curves

Beside arrow diagrams and graphs, we visualize a function using *level curves*, or *contour lines*, formed by a contour map on which points of constant elevation are joined.
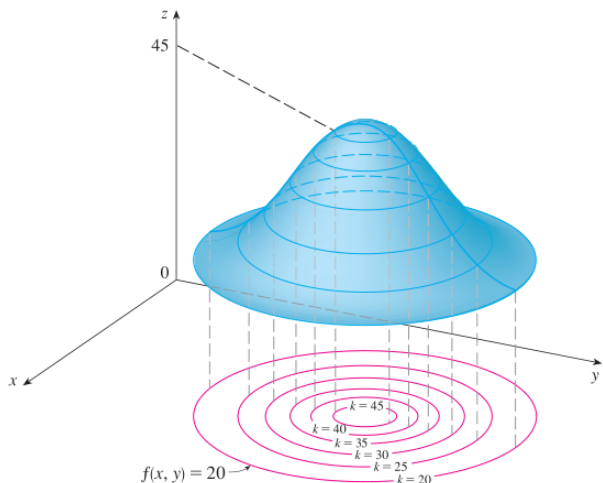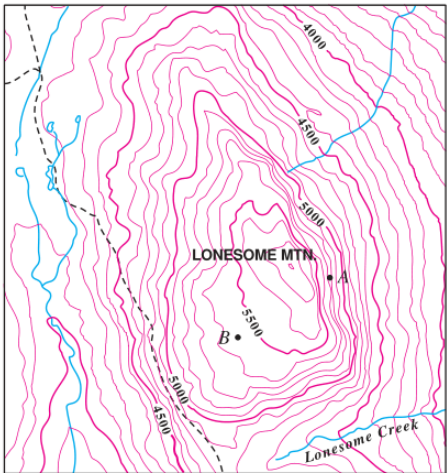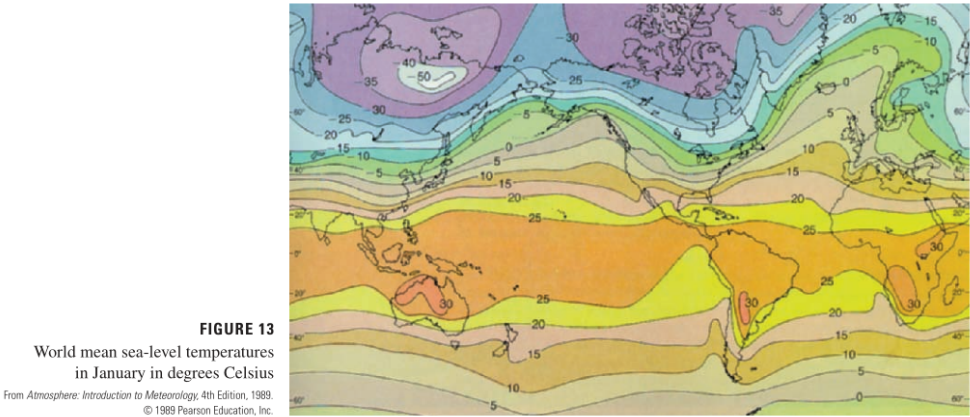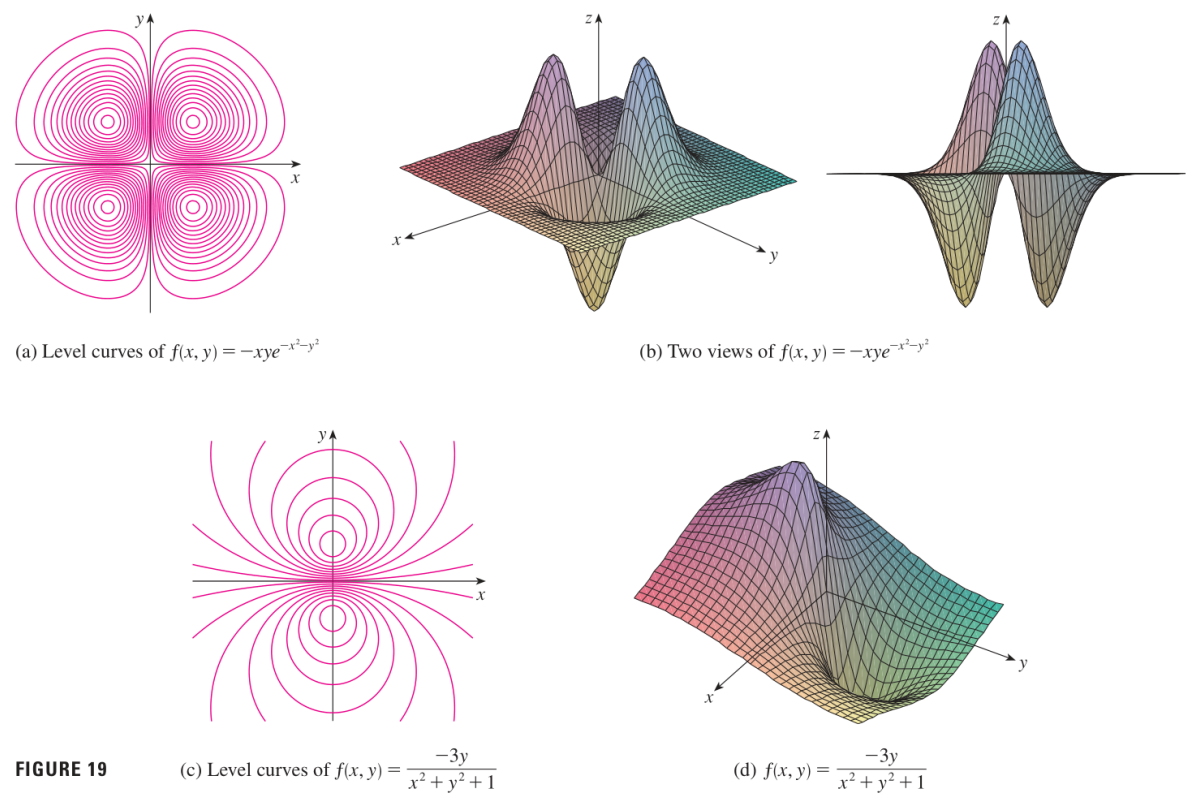

FIGURE 11

FIGURE 12

Another example of the temperature functions, the level curves are **isothermals**.
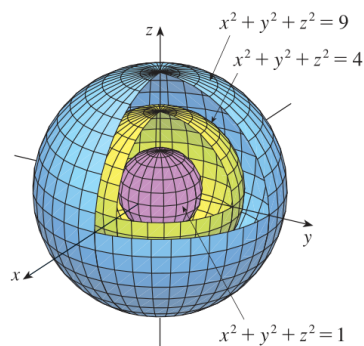

FIGURE 13
World mean sea-level temperatures
in January in degrees Celsius

2

For some purpuses, a contour map is more useful than a graph.

(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$

(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$

FIGURE 19

(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2+y^2+1}$

(d) $f(x, y) = \dfrac{-3y}{x^2+y^2+1}$

Functions of Three or More Variables

It's hard to visualize $f(x, y, z)$ by its graph (four-dimensional space). We examine its **level surfaces**, which are the surfaces of $f(x, y, z) = k$.

EXAMPLE. Find the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.

$x^2 + y^2 + z^2 = 9$
$x^2 + y^2 + z^2 = 4$
$x^2 + y^2 + z^2 = 1$

The level surfaces are $x^2 + y^2 + z^2 = k \geq 0$, which forms a family of concentric spheres with radius $\sqrt{k}$.

FIGURE 20

> **Definition**
>
> A **function of n variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$. The set of all n-tuples is $\mathbb{R}^n$. We can look at it as a function of
>
> - n real variables $x_1, x_2, \ldots, x_n$.
>
> - A single point $(x_1, x_2, \ldots, x_n)$.
>
> - A single vector $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$

2 Limits and Continuity

Let's compare 2 functions as (x, y) approach the origin.

$$f(x,y) = \frac{\sin x^2 + y^2}{x^2 + y^2} \quad \text{and} \quad g(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

TABLE 1 Values of $f(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

TABLE 2 Values of $g(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

It appears that $f(x, y)$ are approaching 1 whereas $g(x, y)$ aren't approaching any number.

> **Definition : Limit**
>
> The domain D includes points arbitrarily close to (a, b). The **limit of** $f(x, y)$ **as** (x, y) **approaches** (a, b) is L.
>
> $$\lim_{(x,y) \to (a,b)} f(x,y) = L$$
>
> if for $\forall \varepsilon > 0$, there is a corresponding $\delta > 0$ such that
>
> if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $f(x,y) - L| < \varepsilon$

Note. $|f(x, y) - L|$ is the distance between $f(x, y)$ and L.

$\sqrt{(x-a)^2 + (y-b)^2}$ is the distance between the point (x, y) and (a, b) .

If $(L - \varepsilon, L + \varepsilon)$ is given, we can find a disk D_δ with a center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ (except possibly (a, b)) into $(L - \varepsilon, L + \varepsilon)$.
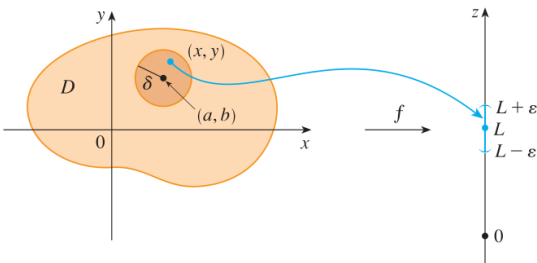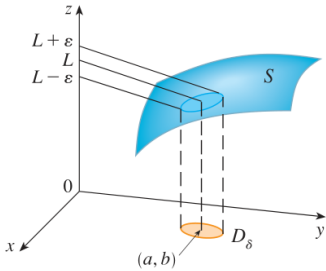

FIGURE 1

FIGURE 2

1 variable. Recall that for $f(x)$, there are only 2 directions of approach, from the left *or* from the right. And if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $lim_{x \to a} f(x)$ does not exist.

2 variables. We can't just let (x, y) approach (a, b) from an infinite number of directions. But if the limit exists, the $f(x, y)$ must approach the **same limit** no matter how.

> If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y) \to (a,b)} f(x, y)$ does not exist.

EXAMPLE. Show that this does not exist
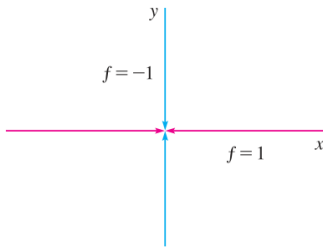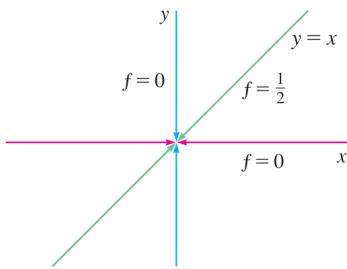
$$\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

4

Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$.

☐ First, let approach $(0, 0)$ along the x-axis.

Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for $\forall x \neq 0$.

$$f(x, y) \to 1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

☐ Now, approach along the y-axis by putting $x = 0$.

Then $f(0, y) = -y^2/y^2 = -1$ for $\forall y \neq 0$.

$$f(x, y) \to -1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

Since f has 2 different limits along 2 different lines, the given limit does not exist.

FIGURE 4

EXAMPLE. Does this limit exist?

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

FIGURE 5

☐ If $y = 0$, then $f(x, 0) = 0/x2$

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

☐ If $x = 0$, then $f(0, y) = -/y^2 = 0$, so

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$