Part I

Eigenvalues and Eigenvectors

Definition : Eigenvalue and Eigenvector

Let A be an $n \times n$ matrix. The scalar λ is called an **eigenvalue** of A if there is a *nonzero* vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

The vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

Example 1. Verifying Eigenvalues and Eigenvectors

For the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix},$$

$\mathbf{x}_1 = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$, and $\mathbf{x}_2 = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

Example 2. Verifying Eigenvalues and Eigenvectors

For the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

find the eigenvalues corresponding to these eigenvectors

$$\mathbf{x}_1 = (-3, -1, 1) \quad \text{and} \quad \mathbf{x}_2 = (1, 0, 0)$$

SOLUTION. Multiplying these vectors by A produces

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

So, $\mathbf{x}_1 = (-3, -1, 1)$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 0$.

$$A\mathbf{x}_2 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So, $\mathbf{x}_2 = (1, 0, 0)$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 1$.

1 Eigenspaces

In fact, if A is an $n \times n$ matrix with an eigenvalue λ and a corresponding eigenvector $\mathbf{x}$, then every nonzero scalar multiple of $\mathbf{x}$ is also an eigenvector of A.

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x})$$

It is also true that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors corresponding to the same eigenvalue λ, then their sum is also an eigenvector corresponding to λ.

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \lambda\mathbf{x}_1 + \lambda\mathbf{x}_2 = \lambda(\mathbf{x}_1 + \mathbf{x}_2)$$

THEOREM 7.1 **Eigenvectors of λ Form a Subspace**
If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ, together with the zero vector

$$\{\mathbf{0}\} \cup \{\mathbf{x} : \mathbf{x} \text{ is an eigenvector of } \lambda\},$$

is a subspace of $\mathbb{R}^n$. This subspace is called the **eigenspace** of λ.

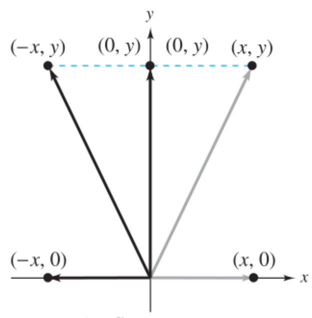
Example 3. **An Example of Eigenspaces in the Plane**

Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

SOLUTION. Geometrically, multiplying a vector $\mathbf{v} = (x, y)$ in $\mathbb{R}^2$ by A corresponds to a reflection in the y-axis.

$$A\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$



A reflects vectors in the y-axis.

Notice that only vectors reflected onto scalar multiples of themselves are those lying on either the x-axis or the y-axis.

For a vector on the x-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1 \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Eigenvalue is $\lambda_1 = -1$.

For a vector on the y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1 \begin{bmatrix} 0 \\ y \end{bmatrix}$$

Eigenvalue is $\lambda_2 = 1$.

This implies that, the eigenspace corresponding to $\lambda_1 = -1$ is the x-axis and the eigenspace corresponding to $\lambda_2 = 1$ is the y-axis.

2 Finding Eigenvalues and Eigenvectors

THEOREM 7.2 **Eigenvalues and Eigenvectors of a Matrix**
Let A be an $n \times n$ matrix.

1. An eigenvalue of A is a scalar λ such that

$$\det(\lambda I - A) = 0$$

2. The eigenvectors of A corresponding to λ are the nonzero solution of

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

The equation $\det(\lambda I - A) = 0$ is called the **characteristic equation** of A. Moreover, when expanded to polynomial

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

is called the **characteristic polynomial** of A. This tells us that the eigenvalues of an $n \times n$ matrix A correspond to the roots of the characteristic polynomial of A. Because the characteristic polynomial of A is of degree n, A can have at most n distinct eigenvalues.

Example 4. **Finding Eigenvalues and Eigenvectors**

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

SOLUTION. The characteristic polynomial of A is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} \\ &= (\lambda - 2)(\lambda + 5) + 12 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda + 1)(\lambda + 2) \end{aligned}$$

So, the characteristic equation is $(\lambda + 1)(\lambda + 2) = 0$, which gives $\lambda_1 = -1$ and $\lambda_2 = -2$ as the eigenvalues of A. Next, solve the homogeneous linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

• For $\lambda_1 = -1$, the coefficient matrix after row reducing is $\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$, showing that $x_1 - 4x_2 = 0$. Letting $x_2 = t$, we can conclude that every eigenvector of λ_1 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, t \neq 0$$

• For $\lambda_2 = -2$, the coefficient matrix after row reducing is $\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$, showing that $x_1 - 3x_2 = 0$. Letting $x_2 = t$, we can conclude that every eigenvector of λ_1 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, t \neq 0$$

***SUMMARY* : Finding Eigenvalues and Eigenvectors**

Let A be an $n \times n$ matrix.
1. Form the characteristic equation $|\lambda I - A| = 0$, which is a polynomial equation of degree n in the variable λ.
2. Find the real roots of the characteristic equation. These are the eigenvalues of A.
3. For each eigenvalue λ_i, find the corresponding eigenvectors by solving the homogeneous system

$$(\lambda_i I - A)\mathbf{x} = \mathbf{0}$$

Example 5. Finding Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

What is the dimension of the eigenspace of each eigenvalues?

SOLUTION. The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3$$

So, the characteristic equation is $(\lambda - 2)^3 = 0$. Thus, the only eigenvalue is $\lambda = 2$.

$$\lambda I - A = 2I - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies that $x_2 = 0$. Using the parameters $s = x_1$ and $t = x_3$, we can find that the eigenvectors of $\lambda = 2$ are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s \text{ and } t \text{ not both zero.}$$

Because $\lambda = 2$ has 2 linearly independent eigenvectors, the dimension of its eigenspace is 2.

If an eigenvalue λ_1 occurs as a *multiple root* (k times) of the characteristic polynomial, then λ_1 has **multiplicity** k. This implies that $(\lambda - \lambda_1)^k$ is a factor of the characteristic polynomial and $(\lambda - \lambda_1)^{k+1}$ is not.

For instance, in Example 5 the eigenvalue $\lambda = 2$ has a multiplicity of 3. Also note that, the **dimension** of the eigenspace of $\lambda = 2$ is 2. In general, the *multiplicity* of λ_i is greater than or equal to the *dimension* of its eigenspace. (Tell me if you want a proof)

Example 6. Finding Eigenvalues and Eigenvectors

Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

and find a basis for each of the corresponding eigenspaces.

SOLUTION. The characteristic polynomial of A is

$$\begin{aligned} |\lambda I - A| &= \begin{bmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda - 1 & -5 & 10 \\ -1 & 0 & \lambda - 2 & 0 \\ -1 & 0 & 0 & \lambda - 3 \end{bmatrix} \\ &= (\lambda - 1)^2(\lambda - 2)(\lambda - 3) \end{aligned}$$

So, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. (Note that λ_1 has a multiplicity of 2.)

- For $\lambda_1 = 1$:

$$(1)I - A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $s = x_2$ and $t = x_4$ produces

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0s - 2t \\ s + 0t \\ 0s + 2t \\ 0s + t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

A basis for the eigenspace corresponding to $\lambda_1 = 1$ is

$$B_1 = \{(0, 1, 0, 0), (-2, 0, 2, 1)\}$$

- For $\lambda_2 = 2$, $B_2 = \{(0, 5, 1, 0)\}$.
- For $\lambda_3 = 3$, $B_3 = \{(0, -5, 0, 1)\}$.

Conclude

Computing those is kinda bruh if $n \geq 4$. The procedure followed in Example 6 is generally **inefficient** when used on a computer, because finding roots is both time comsuming and subject to roundoff errors.

Consequently, numerical methods of approximating the eigenvalues of large matrices are required. [Advanced Linear Algebra] [Numerical Analysis]

THEOREM 7.3 **Eigenvalues of Triangular Matrices**
If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

Its proof follows from the fact that the determinant of a triangular matrix is the product of its diagonal elements.

3 Eigenvalues and Eigenvectors of Linear Transformations

The number λ is called an **eigenvalue** of $T : V \to V$ if there exist a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an eigenvector of T corresponding to λ, and the set of all eigenvectors of λ (with $\mathbf{0}$) is the **eigenspace** of λ.

QUESTION

For a given T, can we find a basis B' whose corresponding matrix is **diagonal**?

Part II

Diagonalization

Definition : Diagonalizable Matrix

An $n \times n$ matrix A is **diagonalizable** if A is similar to a diagonal matrix. That is, there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

THEOREM 7.4 **Similar Matrices Have the Same Eigenvalues**
If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

PROOF Since A and B are similar, there exists P such that $B = P^{-1}AP$. By the properties of determinant, it follows that

$$\begin{aligned} |\lambda I - B| = |\lambda I - P^{-1}AP| &= |P^{-1}\lambda IP - P^{-1}AP| \\ &= |P^{-1}(\lambda I - A)P \\ &= |P^{-1}||\lambda I - A||P| \\ &= |\lambda I - A| \end{aligned}$$

THEOREM 7.5 **Condition for Diagonalization**
An $n \times n$ matrix A is diagonalizable if an only if it has n linearly independent eigenvectors.

PROOF The key result of this proof is the fact that for diagonalizable matrices, *the columns of P consist of the n linearly independent eigenvectors.*

Steps for Diagonalizing an $n \times n$ Square Matrix
Let A be an $n \times n$ matrix.
1. Find n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ for A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If n linearly independent eigenvectors do not exist, then A is not *diagonalizable.*
2. Let $P = [\mathbf{p}_1 \vdots \mathbf{p}_2 \vdots \cdots \vdots \mathbf{p}_n]$.
3. The diagonal matrix $D = P^{-1}AP$ will have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on its main diagonal.

For a square matrix A of order n to be diagonalizable, the sum of the dimensions of the eigenspaces must be equal to n. One way is stated in the next theorem.

THEOREM 7.6 **Sufficient Condition for Diagonalization**
If $A_{n\times n}$ has n *distinct* eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

4 Diagonalization and Linear Transformations

For $T : V \to V$, does there exist a basis B such that the matrix for T relative to B is diagonal? The answer is "yes", provided that the standard matrix for T is diagonalizable.

Example 8. Finding a Diagonal Matrix for a Linear Transformations

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation represented by

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_1 + 3x_2 + x_3, -3x_1 + x_2 - x_3)$$

If possible, find a basis B for $\mathbb{R}^3$ such that the matrix for T relative to B is diagonal.

SOLUTION. The standard matrix for T is represented by

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 3)$$

So, the eigenvalues are $\lambda_1 = 2, \lambda_2 = -2$, and $\lambda_3 = 3$.

$$\begin{array}{cccc} & & & \text{Eigenvector} \\ 2I - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\ -2I - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \\ 3I - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \end{array}$$

Form the matrix P whose columns are the eigenvectors just obtained.

$$P = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & 1 & \frac{1}{5} \end{bmatrix}$$

So, $B = \{(-1, 0, 1), (1, -1, 4), (-1, 1, 1)\}$. The matrix T relative to this basis is

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

5 Symmetric Matrices and Orthogonal Diagonalization

Most matrices requires much of diagonaliation process before we can finally determine whether it is possible. One exception is a triangular matrix with distinct entries on the main diagonal. In this section, we will study another type of matrix that is guaranteed to be **diagonalizable** - a **symmetric** matrix.

Definition : Symmetric Matrix

A square matrix A is **symmetric** if it is equal to its transpose:

$$A = A^T$$

THEOREM 7.7 **Eigenvalues of Symmetric Matrices**

If A is an $n \times n$ symmetric matrix, then the following properties are true.

1. A is diagonalizable.
2. All eigenvalues of A are real.
3. If λ is an eigenvalue of A with multiplicity k,then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k.

REMARK. The Theorem 7.7 is called the **Real Spectral Theorem**, and the set of eigenvalues of A is called the **spectrum** of A.

Example 2. **The Eigenvalues and Eigenvectors of a 2 × 2 Symmetric Matrix**

Prove that $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ is diagonalizable.

SOLUTION. The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = \lambda^2 - (a + b)\lambda + ab - c^2$$

As a quadratic in λ, this polynomial has a discriminant of

$$\begin{aligned}(a+b)^2 - 4(ab - c^2) &= a^2 + 2ab + b^2 - 4ab + 4c^2 \\ &= (a-b)^2 + 4c^2 \geq 0\end{aligned}$$

• If $\Delta = 0 \implies a = b, c = 0$, which implies that A is already diagonal. • If $\Delta > 0$, A has 2 distinct real eigenvalues, then A is diagonalizable also.

Example 5. Dimension of the Eigenspaces of a Symmetric Matrix

Find the eigenvalues of the symmetric matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

and determine the dimensions of the corresponding eigenspaces.

SOLUTION. The characteristic polynomial of A is represented by

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 2 & 0 & 0 \\ 2 & \lambda-1 & 0 & 0 \\ 0 & 0 & \lambda-1 & 2 \\ 0 & 0 & 2 & \lambda-1 \end{vmatrix} = (\lambda+1)^2(\lambda-3)^2$$

So, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$, which all have a multiplicity of 2, so the corresponding eigenspaces also have dimensions 2.

6 Orhogonal Matrices

To diagonalize a matrix A, we need to find P (*invertible*) such that $P^{-1}AP$ is diagonal. For **symmetric** matrices, P can be chosen to have a special properties.

Definition : Orthogonal Matrix

A square matrix P is called **orthogonal** if it is invertible and if

$$P^{-1} = P^T$$

THEOREM 7.8 **Property of Orthogonal Matrices**
An $n \times n$ matrix P is orthogonal if and only if its column vectors form an **orthonormal** set.

PROOF Suppose the column vectors of P form an orthonormal set. (p.469)

Example 5. An Orthogonal Matrix

Show that

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

is orthogonal by showing that $PP^T = 1$. Then show that the column vectors of P form an orthonormal set.

SOLUTION. $PP^T = 1$, we can conclude that P is orthogonal.

Moreover, letting

$$\mathbf{p}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{\sqrt{5}} \\ -\frac{2}{3\sqrt{5}} \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{5}} \\ -\frac{4}{3\sqrt{5}} \end{bmatrix}, \quad \text{and } \mathbf{p}_3 = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

produces

$$\mathbf{p}_1 \cdot \mathbf{p}_2 = \mathbf{p}_2 \cdot \mathbf{p}_3 = \mathbf{p}_3 \cdot \mathbf{p}_1 = 0$$

and

$$\|\mathbf{p}_1\| = \|\mathbf{p}_2\| = \|\mathbf{p}_3\| = 1$$

THEOREM 7.9 **Property of Symmetric Matrices**
Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are 2 distinct eigenvalues of A, then their corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are **orthogonal**.

PROOF (p471)

7 Orthogonal Diagonalization

A matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal.

THEOREM 7.10 **Fundamental Theorem of Symmetric Matrices**
Let A be an $n \times n$ matrix. Then A is **orthogonally diagonalizable** and has real eigenvalues if and only if A is **symmetric**.

PROOF (p472)

Example 7. Determining Whether a Matrix is Orthogonally Diagonalizable

Just choose the **symmetric** ones.

Orthogonal Diagonalization of a Symmetric Matrix
Let A be an $n \times n$ symmetric matrix.
1. Find all eigenvalues of A and determine the multiplicity of each.
2. **Normalize** the eigenvectors, make every set **orthonormal**. (Applying the Gram-Schmidt o.p)
4. Use n eigenvectors to form the columns of P. The matrix $D = P^{-1}AP = P^TAP$ is diagonal.

Example 8. Orthogonal Diagonalization

Find an orthogonal matrix P that orthogonally diagonalizes

$$A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$$

SOLUTION. The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda + 2 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 3)(\lambda - 2)$$

So, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$. For each of those, find an eigenvector by converting the matrix $|\lambda I - A$ to reduced row-echelon form.

$$\begin{aligned} -3I - A &= \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ 2I - A &= \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

Normalize these eigenvectors to produce an *orthonormal* basis.

$$\begin{aligned} \mathbf{p}_1 &= \frac{(-2,1)}{\|(-2,1)\|} = \frac{1}{\sqrt{5}}(-2,1) = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \\ \mathbf{p}_2 &= \frac{(1,2)}{\|(1,2)\|} = \frac{1}{\sqrt{5}}(1,2) = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) \end{aligned}$$

Using $\mathbf{p}_1$ and $\mathbf{p}_2$ as column vectors, construct the matrix P.

$$P = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Example 9. **Orthogonal Diagonalization**

Find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$$

SOLUTION. The characteristic polynomial of A, $|\lambda I - A| = (\lambda - 3)^2(\lambda + 6)$ yields the eigenvalues $\lambda_1 = -6$ (multiplicity of 1) and $\lambda_2 = 3$ (multiplicity of 2).

- For $\lambda_1 = -6$, an eigenvector is $\mathbf{v}_1 = (1, -2, 2)$, which normalizes to

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

- For $\lambda_2 = 3$, 2 eigenvectors are $\mathbf{v}_2 = (2, 1, 0)$ and $\mathbf{v}_3 = (-2, 1, 0)$. Note that $\mathbf{v}_1$ is *orthogonal* to them, but these 2 are **not** orthogonal. Applying the Gram-Schmidt process:

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{5}}(2, 1, 0) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$
$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{3\sqrt{5}}{5}\left(-\frac{2}{5}, \frac{4}{5}, 1\right) = \left(-\frac{2}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}\right)$$

Form the matrix P:

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$$