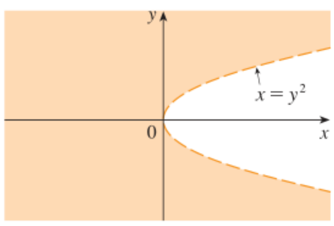
1 Functions of Several Variables

Functions of Two Variables

Definition

A **function** f **of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D (domain subset of $\mathbb{R}^2$) of $f(x, y)$ (range subset of $\mathbb{R}$).

FIGURE 2
Domain of $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$



EXAMPLE.
Evaluate $f(3, 2)$, find and sketch the domain of $f(x, y) = \ln(y^2 - x)$.

$$f(3, 2) = 3\ln(2^2 - 3) = 3\ln 1 = 0$$

The domain of f is $D = \{(x, y) \mid x < y^2\}$.

Graph

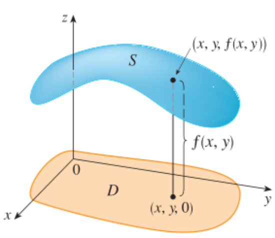


FIGURE 5

The graph of $f(x, y)$ is the set of all points (x, y, z) in $\mathbb{R}^3$.

EXAMPLE. Sketch the graph of $f(x, y) = 6 - 3x - 2y$.

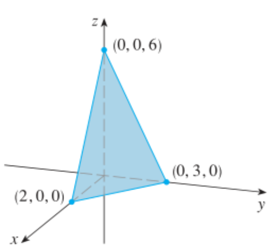


FIGURE 6

A linear function...
The equation of the graph is $3x + 2 + z = 6$, which represents a plane. To graph it, find the *intercepts* by setting 2 of the 3 variables to 0.

EXAMPLE. Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

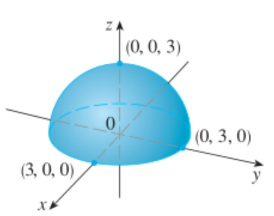
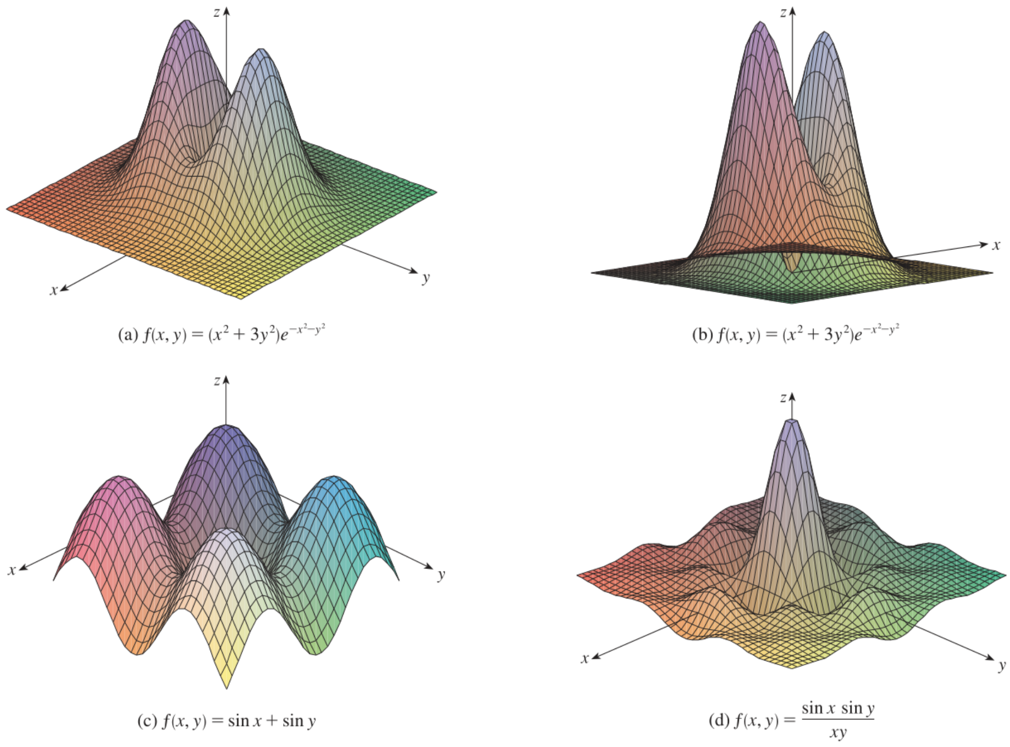


FIGURE 7
Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Square both sides of this equation to obtain $x^2 + y^2 + z^2 = 9$. Since $z \geq 0$, this is the upper part of a sphere whose center the origin and radius 3.

Computer programs can graph functions $f(x, y)$. Traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k and parts of the graph are eliminated using hidden line removal.



(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(c) $f(x, y) = \sin x + \sin y$

(d) $f(x, y) = \dfrac{\sin x \sin y}{xy}$

FIGURE 10

Level Curves

Beside arrow diagrams and graphs, we visualize a function using *level curves*, or *contour lines*, formed by a contour map on which points of constant elevation are joined.

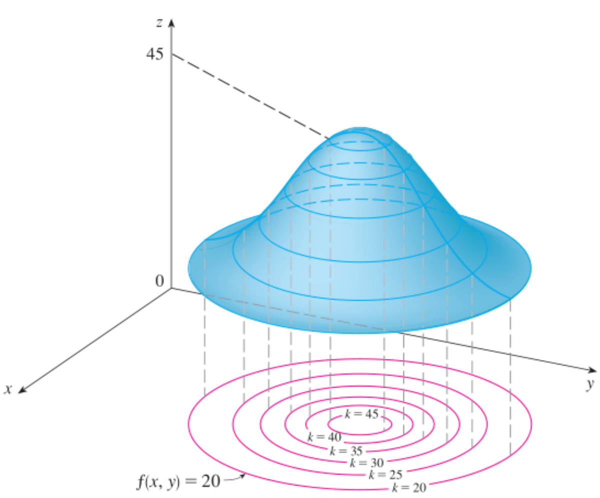


FIGURE 11

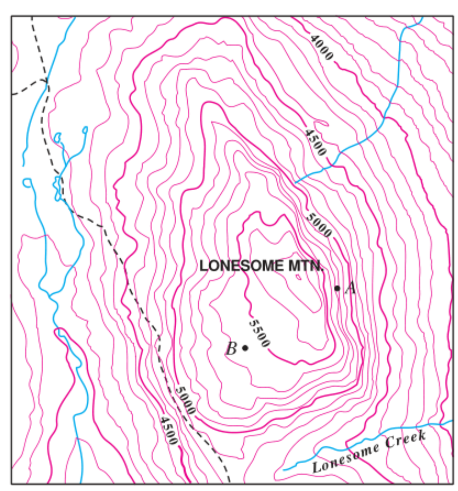


FIGURE 12

Another example of the temperature functions, the level curves are **isothermals**.

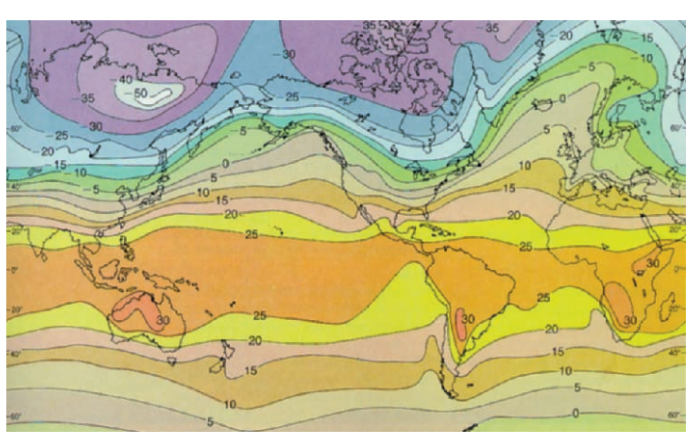
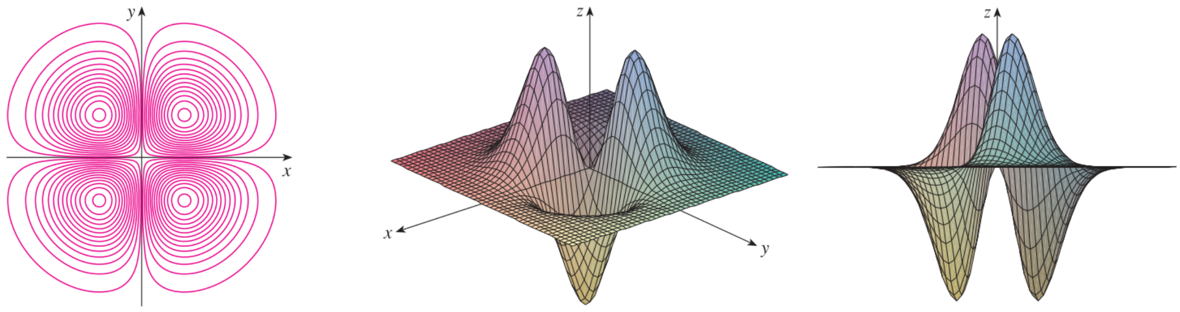


FIGURE 13
World mean sea-level temperatures in January in degrees Celsius
From *Atmosphere: Introduction to Meteorology*, 4th Edition, 1989. © 1989 Pearson Education, Inc.

For some purpuses, a contour map is more useful than a graph.



(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$

(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$

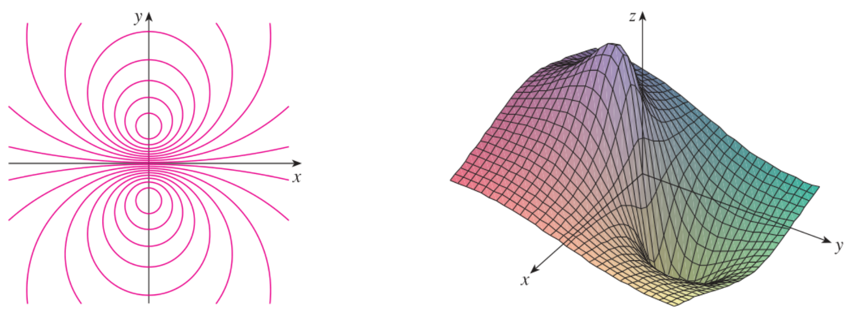


FIGURE 19

(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2+y^2+1}$

(d) $f(x, y) = \dfrac{-3y}{x^2+y^2+1}$

Functions of Three or More Variables

It's hard to visualize $f(x, y, z)$ by its graph (four-dimensional space). We examine its **level surfaces**, which are the surfaces of $f(x, y, z) = k$.

EXAMPLE. Find the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$.

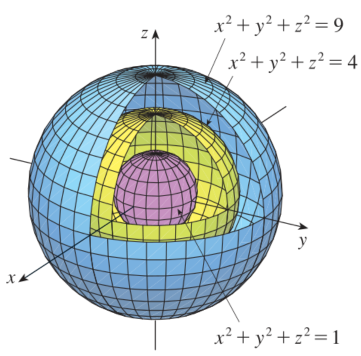


FIGURE 20

The level surfaces are $x^2 + y^2 + z^2 = k \geq 0$, which forms a family of concentric spheres with radius $\sqrt{k}$.

Definition

A **function of n variables** is a rule that assigns a number $z = f(x_1, x_2, \cdots, x_n)$ to an n-tuple $(x_1, x_2, \cdots, x_n)$. The set of all n-tuples is $\mathbb{R}^n$. We can look at it as a function of

- n real variables $x_1, x_2, \cdots, x_n$.
- A single point $(x_1, x_2, \cdots, x_n)$.
- A single vector $\mathbf{x} = \langle x_1, x_2, \cdots, x_n \rangle$

2 Limits and Continuity

Let's compare 2 functions as (x, y) approach the origin.

$$f(x,y) = \frac{\sin x^2 + y^2}{x^2 + y^2} \quad \text{and} \quad g(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

TABLE 1 Values of $f(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

TABLE 2 Values of $g(x, y)$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

It appears that $f(x, y)$ are approaching 1 whereas $g(x, y)$ aren't approaching any number.

Definition : Limit

The domain D includes points arbitrarily close to (a, b). The **limit of** $f(x, y)$ **as** (x, y) **approaches** (a, b) is L.

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for $\forall \varepsilon > 0$, there is a corresponding $\delta > 0$ such that

$$\text{if } (x,y) \in D \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \text{ then } f(x,y) - L| < \varepsilon$$

`Note.` $|f(x, y) - L|$ is the distance between $f(x, y)$ and L.

$\sqrt{(x-a)^2 + (y-b)^2}$ is the distance between the point (x, y) and (a, b) .

If $(L - \varepsilon, L + \varepsilon)$ is given, we can find a disk D_δ with a center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ (except possibly (a, b)) into $(L - \varepsilon, L + \varepsilon)$.

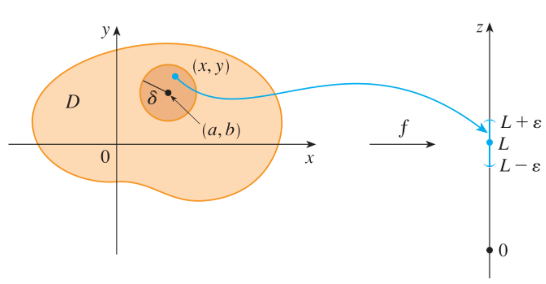


FIGURE 1

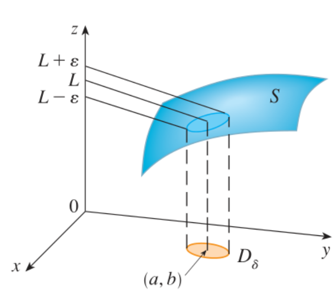


FIGURE 2

1 variable. Recall that for $f(x)$, there are only 2 directions of approach, from the left *or* from the right. And if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $lim_{x\to a} f(x)$ does not exist.

2 variables. We can't just let (x, y) approach (a, b) from an infinite number of directions. But if the limit exists, the $f(x, y)$ must approach the **same limit** no matter how.

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

`EXAMPLE.` Show that this does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

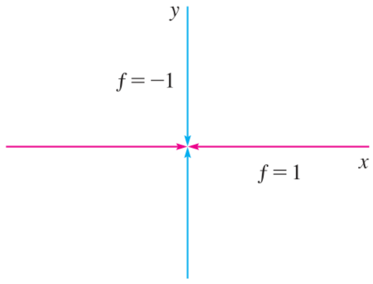


FIGURE 4

Let $f(x,y) = (x^2 - y^2)/(x^2 + y^2)$.

☐ First, let approach $(0,0)$ along the x-axis.

Then $\quad y = 0$ gives $f(x,0) = x^2/x^2 = 1$ for $\forall x \neq 0$.

$$f(x,y) \to 1 \quad \text{as} \quad (x,y) \to (0,0) \text{ along the } x\text{-axis}$$

☐ Now, approach along the y-axis by putting $x = 0$.

Then $\quad f(0,y) = -y^2/y^2 = -1$ for $\forall y \neq 0$.

$$f(x,y) \to -1 \quad \text{as} \quad (x,y) \to (0,0) \text{ along the } y\text{-axis}$$

Since f has 2 different limits along 2 different lines, the given limit does not exist.

EXAMPLE. Does this limit exist?

$$\lim_{(x,y)\to(0,0)} f(x,y) = xy/(x^2 + y^2)$$

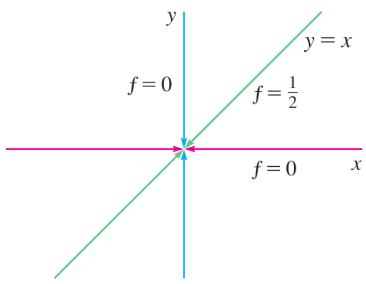


FIGURE 5

☐ If $y = 0$, then $f(x,0) = 0/x^2$

$$f(x,y) \to 0 \quad \text{as} \quad (x,y) \to (0,0) \text{ along the } x\text{-axis}$$

☐ If $x = 0$, then $f(0,y) = -/y^2 = 0$, so

$$f(x,y) \to 0 \quad \text{as} \quad (x,y) \to (0,0) \text{ along the } y\text{-axis}$$

Although we have obtained identical limits along the axes, that does not show the answer is 0.

☐ Let's approach $(0,0)$ along another line, say $y = x$. For all $x \neq 0$,

$$f(x,x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore $\quad f(x,y) \to \frac{1}{2} \quad$ as $\quad (x,y) \to (0,0)$ along $y = x$. The given limit **does not exist**.

The ridge that occurs above the line $y = x$ correspond to the fact that $f(x,y) = \frac{1}{2}$ for all points (x,y) on that line **except the origin**.

TEC In Visual 14.2 a rotating line on the surface in Figure 6 shows different limits at the origin from different directions.

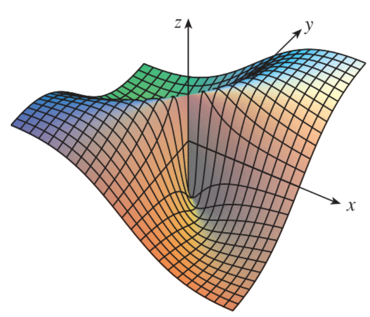


FIGURE 6

$f(x,y) = \dfrac{xy}{x^2 + y^2}$

EXAMPLE. Find the limit.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$$

Let's approach along any nonvertical line through the origin. Then $y = mx$, where m is the slope, then

$$f(x,y) = f(x,mx) = \frac{x(mx)^2}{x^2 + (mx)^4} = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}$$

So $\quad f(x,y) \to 0 \quad$ as $\quad (x,y) \to (0,0)$ along $y = mx$.

Thus f has the same limiting value along *every nonvertical line* through the origin. But that does not show that the answer is 0, if we now let $(x,y) \to (0,0)$ along the parabola $x = y^2$, we have

$$f(x,y) = f(y^2,y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

Hence, the limit **does not exist**.

What limits that do exist then?

The limit of a **sum** the sum of the limits, so does a **product**. These equations are true.

$$\lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c$$

The Squeeze Theorem also holds.

EXAMPLE. Find the limit.

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$$

☐ As the previous example, we see that the limit along any line through the origin is 0. Plus, the limits along the parabolas $y = x^2$ and $x = y^2$ are also 0, so we suspect the limit exist and equal to 0.
☐ Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if} \quad 0 < \sqrt{x^2+y^2} < \delta \quad \text{then} \quad \left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon$$

That is,

$$\text{if} \quad 0 < \sqrt{x^2+y^2} < \delta \quad \text{then} \quad \frac{3x^2|y|}{x^2+y^2} < \varepsilon$$

Since $x^2 \leq x^2 + y^2$, so $x^2/(x^2+y^2) \leq 1$ and therefore

$$\frac{3x^2|y|}{x^2+y^2} \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2} < 3\delta$$

Thus we choose $\delta = \varepsilon/3$ and let $0 < \sqrt{x^2+y^2} < \delta$, then

$$\frac{3x^2|y|}{x^2+y^2} \leq 3\sqrt{x^2+y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

Hence, by the **Definition: Limit**,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

Continuity

Limits of *continuous* functions is easy evaluated by direct substitution.

Definition : Continuity

A function f is **continuous at** (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say f is **continuous on** D if f is continuous at *every* point (a,b) in D.

A **polynomial function of 2 variables** is a sum of terms cx^my^n, and a **rational function** is a ratio of polynomials. For instance,

$$f(x,y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial, whereas

$$g(x,y) = \frac{2xy+1}{x^2+y^2}$$

is a rational function. A rational function is *continuous* on its domain because it's a quotient of continuous functions (polynomial).
EXAMPLE.

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$

EXAMPLE. Let

$$g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

Here g is defined at $(0,0)$ but still discontinuous because $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist.

Composition. If $f(x, y)$ is continuous, and so is $g(x)$, then the composite function $h = g \circ f$, defined by $h(x, y) = g(f(x, y))$ is continuous.

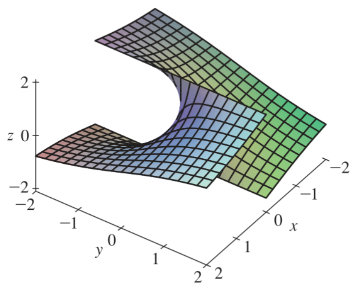


FIGURE 9
The function $h(x, y) = \arctan(y/x)$ is discontinuous where $x = 0$.

EXAMPLE. Where is the function $h(x, y) = \arctan(y/x)$ continuous?
The function $f(x, y) = y/x$ is a rational function, and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan t$ is continuous everywhere. So the composite function $g(f(x, y))$ is continuous except where $x = 0$.

Functions of Three or More Variables

It's just the same. For every number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y, z) \text{ is in the domain of } f \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$$
$$\text{then} \quad |f(x, y, z) - L| < \varepsilon$$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c)$$

EXAMPLE. The function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function, so it's continuous at every point of $\mathbb{R}^3$ except where $x^2 + y^2 + z^2 = 1$. In other words, it's discontinuous on the sphere with center the origin and radius 1.

Definition : Limit in $\mathbb{R}^n$

If $\lim_{x\to a} f(\mathbf{x}) = L$, then it means that for every number $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$\text{if} \quad x \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$