QUESTION 1

Determine whether the matrix is symmetric.

$$A = \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 3 & 4 \\ 0 & 4 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} -5 & 3 & 4 \\ 3 & 7 & -2 \\ 4 & -2 & 3 \end{bmatrix}$$

(a) Yes. (b) No. (c) No. (d) Yes.

QUESTION 2

Find the eigenvalues with their eigenvectors and the dimensions of the corresponding eigenspaces.

(a) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 \\ -3 & \lambda - 1 \end{vmatrix} = (\lambda - 4)(\lambda + 2)$

- For $\lambda_1 = 4$: $4I - A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

Let $s = x_2 \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\implies \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (\dim = 1)$

- For $\lambda_2 = -2$: $-2I - A = \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Let $s = x_2 \implies \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\implies \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (\dim = 1)$

(b) $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda & -2 \\ -2 & \lambda \end{vmatrix}$

Let $\lambda = \dfrac{-2(\lambda' - 1)}{-3}$, the characteristic polynomial has been solved for λ' in (a).

- For $\lambda_1 = \dfrac{2(\lambda_1' - 1)}{3} = 2, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (\dim = 1)$

- For $\lambda_2 = \dfrac{2(\lambda_2' - 1)}{3} = -2, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (\dim = 1)$

(c) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & -1 \\ -1 & \lambda - 2 & -1 \\ -1 & -1 & \lambda - 2 \end{vmatrix}$

Let $\lambda' = \lambda - 2$. The determinant become the one in (f), we already solved for λ'.

- For $\lambda_1 = \lambda_1' + 2 = 1, \quad \begin{cases} \mathbf{u}_1 = (-1, 1, 0) \\ \mathbf{u}_2 = (-1, 0, 1) \end{cases} \quad (\dim = 2)$
- For $\lambda_2 = \lambda_2' + 2 = 4, \quad \mathbf{u}_3 = (1, 1, 1) \quad (\dim = 1)$

(d) $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda & -2 & -2 \\ -2 & \lambda & -2 \\ -2 & -2 & \lambda \end{vmatrix} = (\lambda + 2)^2(\lambda - 4)$

- For $\lambda_1 = -2$: $2I - A = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_2$ and $t = x_3$: $\mathbf{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\implies \begin{cases} \mathbf{u}_1 = (-1, 1, 0) \\ \mathbf{u}_2 = (-1, 0, 1) \end{cases} \quad (\dim = 2)$

- For $\lambda_2 = 4 : 4I - A = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_3 : \mathbf{x} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\implies \mathbf{u}_3 = (1, 1, 1) \quad (\dim = 1)$

(e) $A = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 2 & 0 \\ 4 & 0 & -2 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -4 & -4 \\ -4 & \lambda - 2 & 0 \\ -4 & 0 & \lambda + 2 \end{vmatrix} = \lambda(\lambda - 6)(\lambda + 6)$$

• For $\lambda_1 = 0 : 0I - A = \begin{bmatrix} 0 & -4 & -4 \\ -4 & -2 & 0 \\ -4 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_3 : \mathbf{x} = s \begin{bmatrix} 1/2 \\ -1 \\ 1 \end{bmatrix} \implies \mathbf{w}_1 = (\frac{1}{2}, -1, 1) \quad (\dim = 1)$

• For $\lambda_2 = 6 : 6I - A = \begin{bmatrix} 6 & -4 & -4 \\ -4 & 4 & 0 \\ -4 & 0 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_3 : \mathbf{x} = s \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \implies \mathbf{w}_2 = (2, 2, 1) \quad (\dim = 1)$

• For $\lambda_3 = -6 : -6I - A = \begin{bmatrix} -6 & -4 & -4 \\ -4 & -8 & 0 \\ -4 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_3 : \mathbf{x} = s \begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix} \implies \mathbf{w}_3 = (-1, \frac{1}{2}, 1) \quad (\dim = 1)$

(f) $A_{(f)} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \frac{1}{2} A_{(d)} \implies A_{(d)}\mathbf{x} = 2\lambda x$

This implies that the eigenvalues of $A_{(f)}$ are equal to half of the ones from (d) with the corresponding eigenvectors.

$\bullet \lambda_1 = \frac{1}{2}(-2) = -1, \quad \begin{cases} \mathbf{u}_1 = (-1, 1, 0) \\ \mathbf{u}_2 = (-1, 0, 1) \end{cases} \quad (\dim = 2)$

$\bullet \lambda_2 = \frac{1}{2}(4) = 2, \quad \mathbf{u}_3 = (1, 1, 1) \quad (\dim = 1)$

(g) $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 1 & 1 \\ 1 & \lambda - 2 & 1 \\ 1 & 1 & \lambda - 2 \end{vmatrix}$$

Let $\lambda' = 2 - \lambda$, then we already solved for λ' in (f).

• For $\lambda_1 = 2 - \lambda'_1 = 3, \quad \begin{cases} \mathbf{u}_1 = (-1, 1, 0) \\ \mathbf{u}_2 = (-1, 0, 1) \end{cases} \quad (\dim = 2)$

• For $\lambda_2 = 2 - \lambda'_2 = 0, \quad \mathbf{u}_3 = (1, 1, 1) \quad (\dim = 1)$

(h) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

• For $\lambda_1 = 3$: the dimension of the eigenspace of λ_1 is 1.

$3I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_1 : \mathbf{x} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies \mathbf{u}_1 = (1, 0, 0)$

• For $\lambda_2 = 1$: since λ_2 has a multiplicity of 2, the corresponding eigenspace has dimesion 2.

$I - A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_2$ and $t = x_3 : \mathbf{x} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies \begin{cases} \mathbf{u}_2 = (0, 1, 0) \\ \mathbf{u}_3 = (0, 0, 1) \end{cases}$

QUESTION 3

Determine whether the matrix is **orthogonal**.

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(a) Yes.
The column vectors of A form an orthonormal basis.

$$\begin{bmatrix} 2/3 & -2/3 \\ 2/3 & 1/3 \end{bmatrix}$$

(b) No.

$$\|a_1\| = \frac{2\sqrt{2}}{3} \neq 1$$

$$A = \begin{bmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix}$$

(c) No.

$$\|a_1\| = 5 \neq 0$$

$$A = \begin{bmatrix} -4/5 & 0 & 3/5 \\ 0 & 1 & 0 \\ 3/5 & 0 & 4/5 \end{bmatrix}$$

(d) Yes.
The column vectors of A form an orthonormal basis.

QUESTION 4

Find an orthogonal matrix P that diagonalizes A and P^TAP.

(a) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda - 2)\lambda$

• For $\lambda_1 = 2$: $2I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

Let $s = x_2 \implies \mathbf{x} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies \mathbf{u}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

• For $\lambda_2 = 0$: $0I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Let $s = x_2 \implies \mathbf{x} = \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix} \implies \mathbf{u}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P^TAP = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$|\lambda I - A| = \begin{vmatrix} \lambda - 4 & 2 \\ 2 & \lambda - 4 \end{vmatrix} = (\lambda - 6)(\lambda - 2)$

• For $\lambda_1 = 6$: $6I - A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Let $s = x_2 : \mathbf{x} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix} \implies \mathbf{u}_1 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

• For $\lambda_2 = 2$: $2I - A = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

Let $s = x_2 : \mathbf{x} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies \mathbf{u}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P^TAP = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -3 & 0 \\ -3 & \lambda & -4 \\ 0 & -4 & \lambda \end{vmatrix} = \lambda(\lambda-5)(\lambda+5)$$

• For $\lambda_1 = 0$: $0I - A = \begin{bmatrix} 0 & -3 & 0 \\ -3 & 0 & -4 \\ 0 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_3 : \mathbf{x} = \begin{bmatrix} -\frac{4}{3}s \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix} \implies \mathbf{u}_1 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$

• For $\lambda_2 = 5$: $5I - A = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & -4 \\ 0 & -4 & 5 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & -3 \\ 0 & 4 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_3 : \mathbf{x} = \begin{bmatrix} \frac{3}{4}s \\ \frac{5}{4}s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{3}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix} \implies \mathbf{u}_2 = \left(\frac{3}{5\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{4}{5\sqrt{2}}\right)$

• For $\lambda_3 = -5$: $5I - A = \begin{bmatrix} -5 & -3 & 0 \\ -3 & -5 & -4 \\ 0 & -4 & -5 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_3 : \mathbf{x} = \begin{bmatrix} \frac{3}{4}s \\ -\frac{5}{4}s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{3}{4} \\ -\frac{5}{4} \\ 1 \end{bmatrix} \implies \mathbf{u}_3 = \left(\frac{3}{5\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{4}{5\sqrt{2}}\right)$

$$P = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{3}{5} & \frac{4}{5\sqrt{2}} & \frac{4}{5\sqrt{2}} \end{bmatrix}$$

$$P^T AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

(d) $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 1 & -2 \\ 1 & \lambda-1 & -2 \\ -2 & -2 & \lambda-2 \end{vmatrix} = (\lambda-4)(\lambda-2)(\lambda+2)$$

• For $\lambda_1 = 4$: $4I - A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 3 & -2 \\ -2 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_3 : \mathbf{x} = s \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \implies \mathbf{u}_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$

• For $\lambda_2 = 2$: $2I - A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_2 : \mathbf{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \implies \mathbf{u}_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

• For $\lambda_3 = -2$: $-2I - A = \begin{bmatrix} -3 & 1 & -2 \\ 1 & -3 & -2 \\ -2 & -2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Let $s = x_3 : \mathbf{x} = s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \implies \mathbf{u}_3 = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$P = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$P^T AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(e) $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \implies |\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 & 0 & 0 \\ -1 & \lambda-1 & 0 & 0 \\ 0 & 0 & \lambda-1 & -1 \\ 0 & 0 & -1 & \lambda-1 \end{vmatrix} = \left[(\lambda-1)^2 - 1\right]^2 = (\lambda-2)^2\lambda^2$

• For $\lambda_1 = 2$: $2I - A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

Let $s = x_2$ and $t = x_4$: $\mathbf{x} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \implies \begin{cases} \mathbf{u}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right) \\ \mathbf{u}_2 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \end{cases}$

• For $\lambda_2 = 0$: $0I - A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

Let $s = x_2$ and $t = x_4$: $\mathbf{x} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{cases} \mathbf{u}_3 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right) \\ \mathbf{u}_4 = \left(0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \end{cases}$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P^T AP = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

QUESTION 5

$$A = \begin{bmatrix} 1 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad a \in \mathbb{R}$$

(a) Find the eigenvalues of A, then determine conditions on a such that A has 3 distinct eigenvalues.

SOLUTION. The characteristic polynomial of A is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 1 & -a & 0 \\ -a & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 3 \end{vmatrix} = (\lambda + 3) \begin{vmatrix} \lambda - 1 & -a \\ -a & \lambda - 1 \end{vmatrix} \\ &= (\lambda + 3)[\lambda - (a+1)][\lambda - (-a+1)] \end{aligned}$$

$$A \text{ has 3 distinct eigenvalues} \Leftrightarrow \begin{cases} a + 1 \neq -3 \\ -a + 1 \neq -3 \\ a + 1 \neq -a + 1 \end{cases} \implies \begin{cases} a \neq \pm 4 \\ a \neq 0 \end{cases}$$

(b) Let $a = 2$. Find an orthogonal matrix P such that $P^T AP$ is diagonal and find that diagonal matrix.

SOLUTION. Substuting $a = 2$ into the characteristic polynomial, obtain $\begin{cases} \lambda_1 = -3 \\ \lambda_2 = 3 \\ \lambda_3 = -1 \end{cases}$

- For $\lambda_1 = -3$:

$$-3I - A = \begin{bmatrix} -4 & -2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \text{ Let } s = x_3, \text{ then } \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies \mathbf{u}_1 = (0, 0, 1)$$

- For $\lambda_2 = 3$:

$$3I - A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \text{ Let } s = x_2, \text{ then } \mathbf{x} = \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \implies \mathbf{u}_2 = \frac{1}{\sqrt{2}}(1, 1, 0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

- For $\lambda_3 = -1$:

$$\lambda I - A = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \text{ Let } s = x_2, \text{ then } \mathbf{x} = \begin{bmatrix} -s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \implies \mathbf{u}_3 = \frac{1}{\sqrt{2}}(-1, 1, 0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$P = [\mathbf{u}_1 \vdots \mathbf{u}_2 \vdots \mathbf{u}_3] = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^T AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$