1 Double Integrals over Rectangles

The Riemann sum

$$\int_a^b f(x)\,dx = \lim_{n\to\inf} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

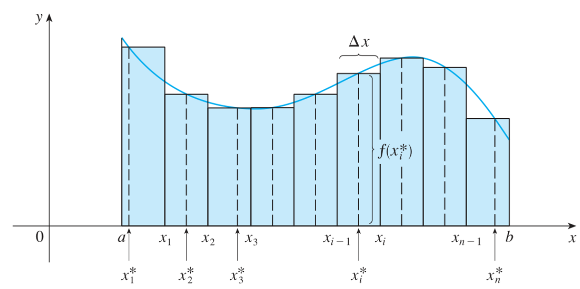


FIGURE 1

Volumes and Double Integrals

Form the subrectangles

$$F_{ij} = [x_{i-1}, x_y] \times [y_{i-1}, y_i] = \{(x,y) \mid x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$$

each with area $\Delta A = \Delta x \Delta y$.

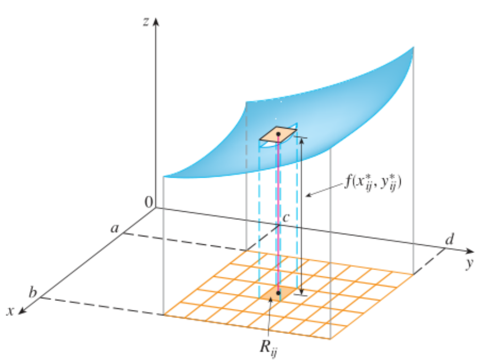


FIGURE 4

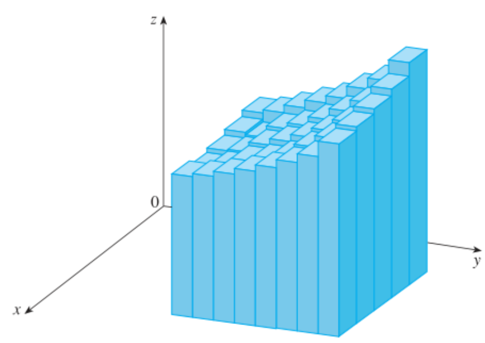


FIGURE 5

Definition : Double Integral

The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)\,dA = V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

EXAMPLE 1. Estimate the volume

$$R = [0,2]\times[0,2], \quad z = 16 - x^2 - 2y^2$$

Divide R into 4 squares and choose the sample point to be the upper right corner of each square R_{ij}.

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\Delta A \\ &= f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A \\ &= 13(1) + 7(1) + 10(1) + 4(1) = 34 \end{aligned}$$

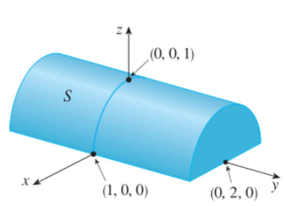


FIGURE 9

EXAMPLE. If $R = \{(x,y) | -1 \le x \le 1, -2 \le y \le 2\}$, evaluate

$$\iint_R \sqrt{1-x^2}\,dA$$

Since $\sqrt{1-x^2} \ge 0$, we can interpreting it as a volume. $x^2 + z^2 = 1$ and $z \ge 0$.

$$\iint_R \sqrt{1-x^2}\,dA = \frac{1}{2}\pi(1)^2 \times 4 = 2\pi$$

The Midpoint Rule

Take $(x_i*, y_i*) = (\overline{x_i}, \overline{y_i})$ (the middle point between x_i, x_{i-1}).

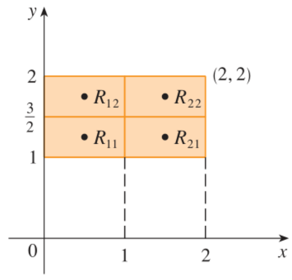


FIGURE 10

EXAMPLE. $m = n = 2,\ R = \{(x,y)|0 \le x \le 2, 1 \le y \le 2\}$

$$\iint_R (x - 3y^2)\, dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x_i}, \overline{y_i})\Delta A$$

$$= f(\overline{x_1}, \overline{y_1})\Delta A + \cdots + f(\overline{x_2}, \overline{y_2})\Delta A$$

$$= f(\frac{1}{2}, \frac{5}{4})\Delta A + f(\frac{1}{2}, \frac{7}{4})\Delta A + f(\frac{3}{2}, \frac{5}{4})\Delta A + f(\frac{3}{2}, \frac{7}{4})\Delta A$$

$$= -11.875$$

Note. Double integral as a bolume is valid only when f is a *positive* function. So in the previous example, the integral is not a volume.

Average Value

The average value of $f(x)$ on (a,b) is $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\, dx$.

Definition : Average Value

The **average value** of $f(x,y)$ on a rectangle R is

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\, dA$$

If $f(x,y) \ge 0$, the equation $A(R) \times f_{\text{ave}} = \iint_R f(x,y)\, dA$ says that it has the same V as a box with base R and height f_{ave}.

Properties of Double Integrals

The *linearity* of the integral $(+, c\times)$.

If $f(x,y) \ge g(x,y)$ for all $(x,y) \in R$, then

$$\iint_R f(x,y)\, dA \ge \iint_R g(x,y)\, dA$$

Iterated Integrals

$\int_c^d f(x,y)\, dy$ means that x is fixed and $f(x,y)$ is integrated with respecto y from $c \to d$. (*partial integration with respect to* y).

Definition : Iterated Integral

$$\int_c^d \int_a^b f(x,y)\, dx\, dy = \int_c^d \left[\int_a^b f(x,y)\, dx\right] dy$$

work from the inside out.

EXAMPLE.

(a)

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 \left[\int_1^2 x^2 y\, dy\right] dx$$

$$= \int_0^3 \frac{3}{2}x^2\, dx = \frac{x^3}{2} = \frac{27}{2}$$

(b)

$$\int_1^2 \int_0^3 x^2 y\, dx\, dy = \int_1^2 \left[\int_0^3 x^2 y\, dx\right] dy$$

$$= \int_1^2 9y\, dy = 9\frac{y^2}{2}\Big]_1^2 = \frac{27}{2}$$

Definition : Fubini's Theorem

If f is continuous on $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

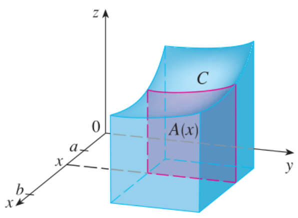


FIGURE 1

TEC Visual 15.2 illustrates Fubini's Theorem by showing an animation of Figures 1 and 2.

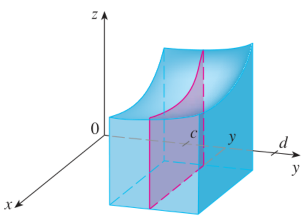


FIGURE 2

$$V = \int_a^b A(x)\,dx$$

where $A(x)$ is the area of the surface that is perpendicular to the $x-$axis.

$$A(x) = \int_c^d f(x,y)\,dy$$

Definition : Special case

In case $f(x,y) = g(x)\,h(y)$,

$$\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

EXAMPLE. $R = [0,\pi/2] \times [0,\pi/2]$, then

$$\iint_R \sin x \cos y\,dA = \int_0^{\pi/2} \sin x \int_0^{\pi/2} \cos y\,dy$$

$$= [-\cos x]_0^{\pi/2}\,[\sin y]_0^{\pi/2} = 1 \cdot 1 = 1$$

Double Integrals over General Regions

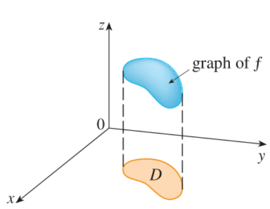


FIGURE 3

The **double integral of f over D** is

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

where $F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$

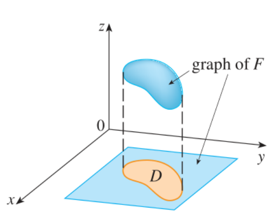


FIGURE 4

Type I: D **lies between 2 continuous function of** x

$$D = \{(x,y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

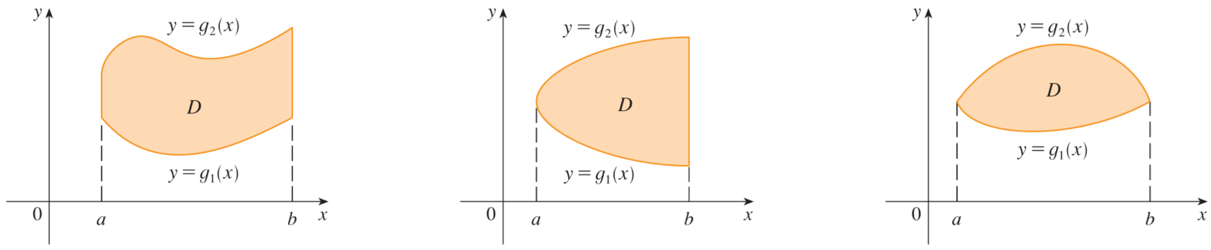


FIGURE 5 Some type I regions

Definition : Type I

If f is continuous on a type I region D such that

$$D = \{(x,y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

which leads to the definition for **Type II**,

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

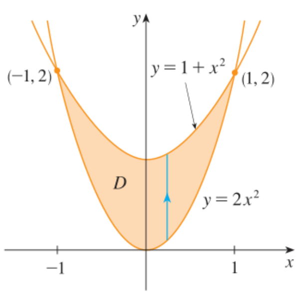


FIGURE 8

EXAMPLE. $y = 2x^2, y = 1 + x^2$, evaluate $\iint_D (x+2y)\,dA$.

$$\begin{aligned}
\int_D (x+2y)\,dA &= \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx \\
&= \int_{-1}^{1} \left[xy + y^2\right]_{y-2x^2}^{y=1+x^2}\,dx \\
&= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx \\
&= \frac{32}{15}
\end{aligned}$$

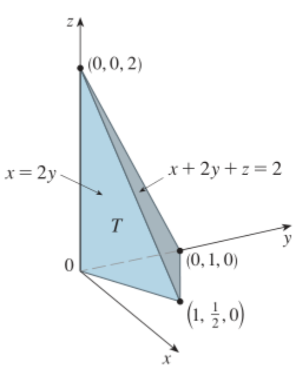


FIGURE 13

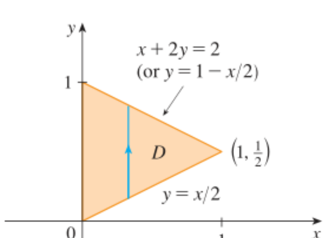


FIGURE 14

EXAMPLE. Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2, x = 2y.x = 0, z = 0$.

$$D = \left\{(x,y) \mid 0 \le x \le 1, x/2 \le y \le 1 - x/2\right\}$$

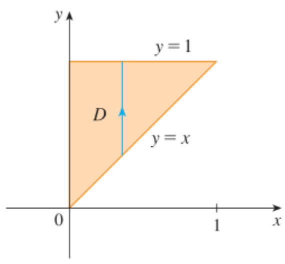


FIGURE 15
D as a type I region

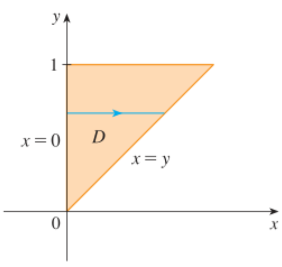


FIGURE 16
D as a type II region

EXAMPLE.

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA$$

$$D = \{(x,y) \mid 0 \le x \le 1, x \le y \le 1\}$$

can be transformed to

$$D = \{(x,y) \mid 0 \le y \le 1, 0 \le x \le y\}$$

$$\begin{aligned}\int_0^1 \int_0^y \sin(y^2)\, dx\, dy &= \int_0^1 \left[x \sin(y^2)\right]_{x=0}^{x=y} dy \\ &= \int_0^1 y \sin(y^2)\, dy \\ &= -\frac{1}{2}\cos(y^2)\Big]_0^1 = \frac{1}{2}(1 - \cos 1)\end{aligned}$$

Properties 1: Double Integrals

Beside sum and constant multiplier.

- If $f(x,y) \ge g(x,y)$ for all $(x,y) \in D$.

$$\iint_D f(x,y)\, dA \ge \iint_D g(x,y)\, dA$$

- If $D = D_1 \cup D_2$, and they don't overlap except perhaps on their bound daries

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

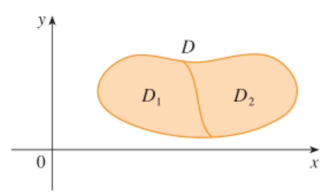


FIGURE 17

- Since $\iint_D 1\, dA = A(D)$, so if $m \le f(x,y) \le M$ for all $(x,y) \in D$.

$$mA(D) \le \iint_D f(x,y)\, DA \le MA(D)$$

EXAMPLE. Estimate $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and $r = 2$. Since $-1 \le \sin x \le 1$ and $-1 \le \cos y \le 1$, we have $-1 \le \sin x \cos y \le 1$. Therefore

$$e^{-1} \le e^{\sin x \cos y} \le e^1$$

$$\frac{4\pi}{e} \le \iint_D e^{\sin x \cos y}\, dA \le 4\pi e$$

2 Double Integrals in Polar Coordinate

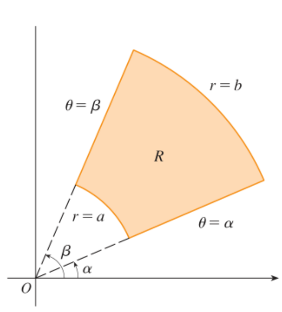


FIGURE 3 Polar rectangle

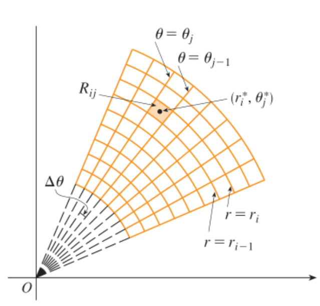


FIGURE 4 Dividing *R* into polar subrectangles

Divide into m subinterval $[r_{i-1}, r_i]$ of $\Delta r = (b-a)/m$ and n subinterval of $(\beta - \alpha)/n$.

- Then the center of the polar subrectangles has polar coordinate

$$r_i* = \frac{1}{2}(r_{i-1} + r_i), \quad \theta_j* = \frac{1}{2}(\theta_{j-1} + \theta_j)$$

- And the area

$$\begin{aligned}\Delta A_i &= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta \\ &= r_i^* \Delta r \Delta\theta\end{aligned}$$

Definition : Change to Polar Coordinates in a Double Integral

If f is continuous on a polar rectangle R $(0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi)$, then

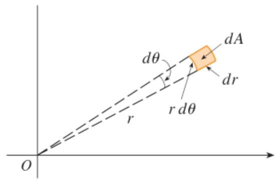


FIGURE 5

$$\iint_R f(x,y)\;dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

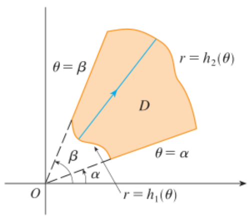


FIGURE 7
$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

EXAMPLE. Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

$$D = \{(r,\theta) \mid -\pi/4 \le \theta \le \pi/4, 0 \le r \le \cos 2\theta\}$$

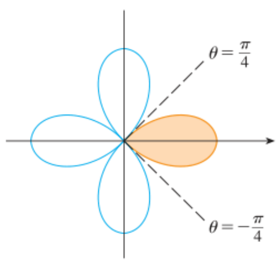
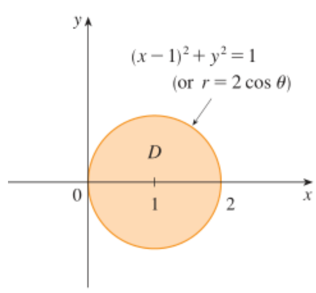


FIGURE 8

So the area is

$$A(D) = \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta$$
$$= \int_{-\pi/4}^{\pi/4} \left[\frac{1}{2}r^2\right]_0^{\cos 2\theta} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos 2\theta^2\,d\theta$$
$$= \frac{1}{4}\int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta)\,d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$$

EXAMPLE. $z = x^2 + y^2$, $x^2 + y^2 = 2x$.



IGURE 9

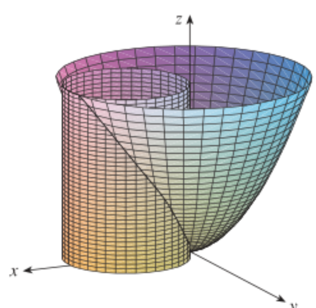


FIGURE 10

$$D = \{(r,\theta) \mid -\pi/2 \le \theta \le \pi/2, 0 \le r \le 2\cos\theta\}$$

$$V \iint_D (x^2 + y^2)\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{2 cos theta} r^2 r\,dr\,d\theta$$
$$= 4\int_{-\pi/2}^{\pi/2} \cos\theta^4\,d\theta = 8\int_0^{\pi/2} \cos\theta^4\,d\theta = 8\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$
$$= 2\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

Why split into 2 parts?

3 Surface Area

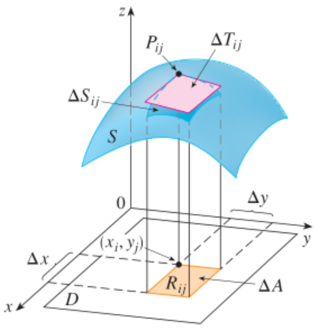


FIGURE 1

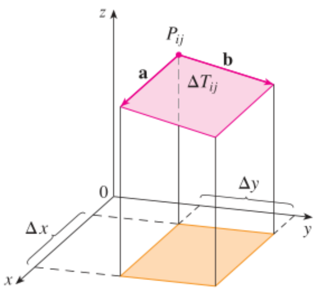


FIGURE 2

Divide into $m \times n$ square. Then $A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$. Since $\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}\ |$. Recall that $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij}.

$$\mathbf{a}\ = \Delta x\mathbf{i}\ + f_x(x_i, y_j)\Delta x\mathbf{k}$$
$$\mathbf{b}\ = \Delta y\ \mathbf{j}\ + f_y(x_i, y_j)\Delta y\mathbf{k}$$

$$\begin{aligned}
\mathbf{a}\ \times\ \mathbf{b}\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_i, y_j)\Delta x \\ 0 & \Delta y & f_y(x_i, y_j)\Delta y \end{vmatrix} \\
&= [-f_x(x_i, y_j)\ \mathbf{i}\ - f_y(x_i, y_j)\mathbf{j} + \mathbf{k}\ ]\Delta A \\
\Delta T_{ij} = |\ \mathbf{a}\ \times \mathbf{b}\ | &= \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\,\Delta A
\end{aligned}$$

Hence we have

Definition : The Area of the Surface

If f_x, f_y are continuous.

$$\begin{aligned}
A(S) &= \iint\limits_D \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\; dA \\
&= \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dA
\end{aligned}$$

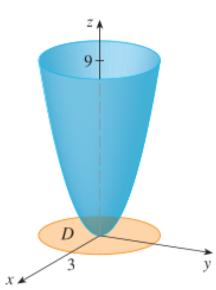


FIGURE 5

EXAMPLE. Area of $z = x^2 + y^2$ that lies under $z = 9$.

$$\begin{aligned}
A &= \iint\limits_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA \\
&= \iint\limits_D \sqrt{1 + 4(x^2 + y^2)}\, dA
\end{aligned}$$

Converting to polar coordinates, we obtain

$$\begin{aligned}
A &= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\
&= \int_0^{2\pi} d\theta \int_0^3 \frac{1}{8}\sqrt{1 + 4r^2}(8r)\, dr \\
&= 2\pi\left(\tfrac{1}{8}\right)\tfrac{2}{3}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{\pi}{6}(37\sqrt{37} - 1)
\end{aligned}$$

4 Triple Integrals

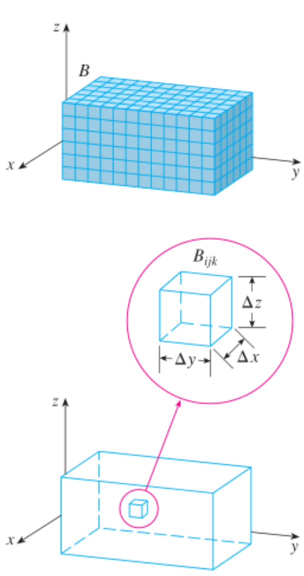


FIGURE 1

Divide into subboxes.

Definition : Triple Integrals

Let $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$, then

$$\iiint\limits_B \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}*, y_{ijk}*, z_{ijk}*)\Delta V$$

Fubini's Theorem. $\iiint\limits_B f(x, y, z)\, dV = \int_a^b \int_r^s \int_c^d f(x, y, z)\, dy\, dz\, dx$

Just the same, wrap E inside a box, and we got $\iiint\limits_B F(x,y,z)\,dV$.

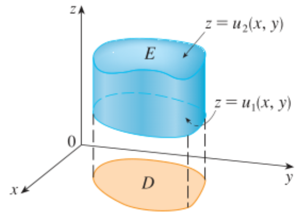


FIGURE 2
A type 1 solid region

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$$

$$\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

Definition : 3 Types of Triple Integrals

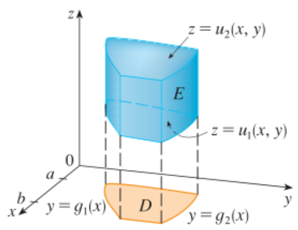


FIGURE 3
A type 1 solid region where the projection D is a type I plane region

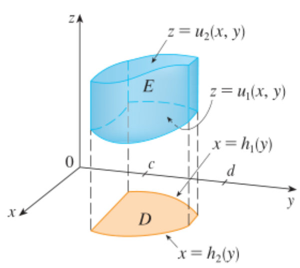


FIGURE 4
A type 1 solid region with a type II projection

- **Type I.** D is the projection on the xy-plane.

$$E = \{(x,y,z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\}$$

$$\iint\limits_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

- **Type II.** D is the projection on the yz-plane.
- **Type III.** D is the projection on the xz-plane.

EXAMPLE. $\iiint\limits_E \sqrt{x^2+z^2}\,dV$, where E bounded by $y = x^2+z^2$ and $y = 4$.

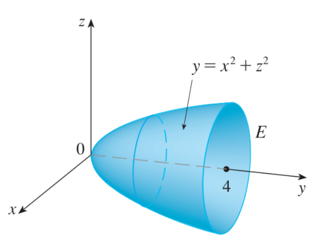


FIGURE 9
Region of integration

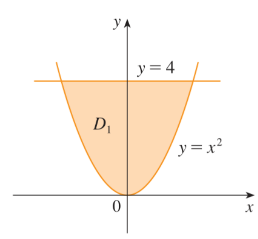


FIGURE 10
Projection onto xy-plane

$$\iint\limits_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right] dA = \iint\limits_{D_3} (4 - x^2 - z^2)\sqrt{x^2+z^2}\,dA$$

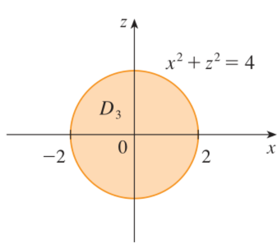


FIGURE 11
Projection onto xz-plane

Convert to polar coordinate in the xz-plane: $x = r\cos\theta, z = r\sin\theta$, which gives

$$\iiint\limits_E \sqrt{x^2+z^2}\,dV = \iint\limits_{D_3} (4 - x^2 - z^2)\sqrt{x^2+z^2}\,dA$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2) r\, r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4)\,dr$$

$$= 2\pi \left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 = \frac{128\pi}{15}$$

5 Applications of Triple Integrals

First, begin with the special case where $f(x,y,z) = 1$ for all points in E. That would be the volume of the shape.

6 Triple Integrals in Cylindrical Coordinates

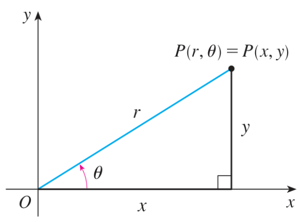


FIGURE 1

Recall the connection between polar and Cartesian coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

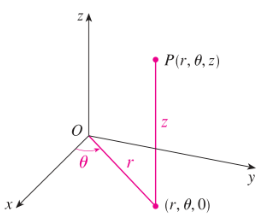


FIGURE 2
The cylindrical coordinates of a point

Cylindrical Coordinates

Represented by (r, θ, z),

- r, θ: polar coordinates of the **projection** of P onto the xy-plane.
- z: the directed distance from the xy-plane to P.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$
$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

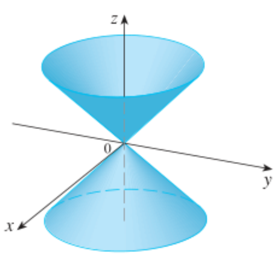
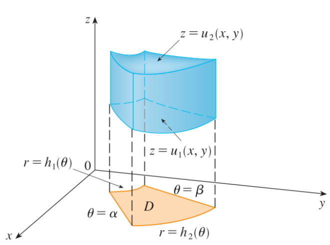


FIGURE 5
$z = r$, a cone

This is the surface of $z = r$.

Evaluating Triple Integrals with Cylindrical Coordinates



Suppose $E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$, and

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

Definition : Triple Integrals with Cylindrical Coordinates

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta}^{u_2(r\cos\theta, r\sin\theta))} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

7 Triple Integrals in Spherical Coordinates

Definition : Spherical Coordinates

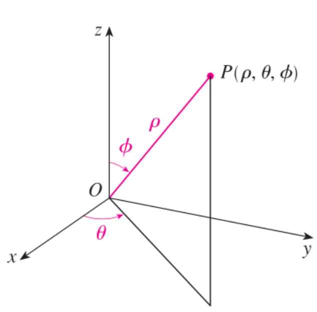


FIGURE 1
The spherical coordinates of a point

The **spherical coordinates** (ρ, θ, ϕ) of a point P:

- $\rho = |OP| \ge 0$: the distance from O to P.
- θ: the same angle as in cylindrical coordinates.
- $0 \le \phi \le \pi$: the angle between the positive z and OP.

Useful when there is symmetry about a point.

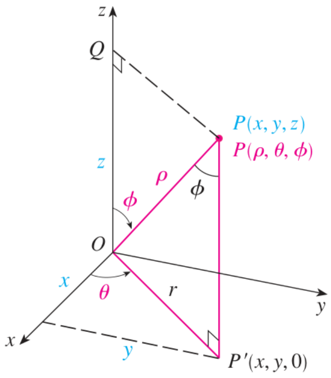


FIGURE 5

We have $z = \rho\cos\phi$ and $r = \rho\sin\phi$.

$$\boxed{x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi}$$

The distance formula

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

Definition : Evaluating Triple Integrals with Spherical Coordinates

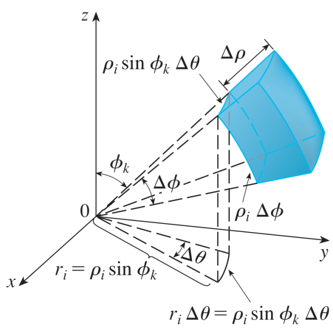


FIGURE 7

$$\iiint_E f(x,y,z)\,dV$$
$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

where E is a **spherical wedge**

$$E = \{(\rho,\theta,\phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$$

where $a \ge 0,\ \beta - \alpha \le 2\pi,\ d - c \le \pi$.
The formula can be extended as $g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi)$.

8 Change of Variables in Multiple Integrals

Consider a **transformation** T from the uv-plane to the xy-plane: $x = g(u,v)$, $y = h(u,v)$.
Assume T is a C^1 **transformation** (g, h have continuous 1st-order partial derivatives).

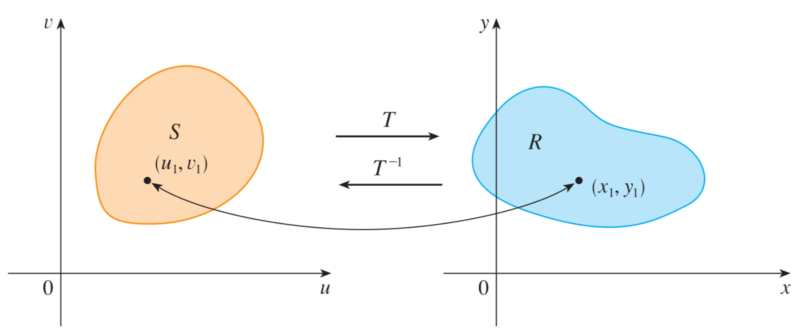


FIGURE 1

If T is **one-to-one**, the the **inverse transformation** T^{-1} exist: $u = G(x,y)$, $v = H(x,y)$.

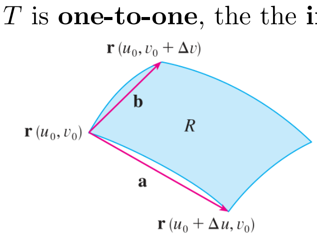


FIGURE 4

We approximate the image R by a parallelogram determined by

$$\begin{cases} \mathbf{a} &= \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u\mathbf{r}_u \\ \mathbf{b} &= \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v\mathbf{r}_v \\ S_R &= |(\Delta u\mathbf{r}_u) \times (\Delta v\mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\Delta u\,\Delta v \end{cases}$$

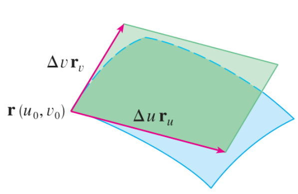


FIGURE 5

Computing the cross product

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}\mathbf{k}$$

Definition : The Jacobian

The **Jacobian** of the transformation T: $x = g(u,v)$, $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Hence we got $\Delta A \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \Delta u \Delta v \right|$ where the Jacobian is evaluated at (u_0, v_0).

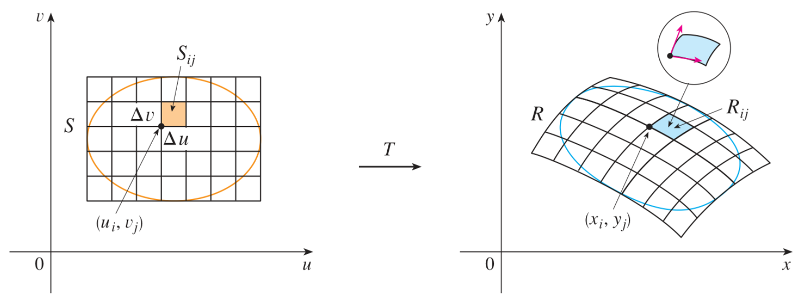


FIGURE 6

$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i,v_i), h(u_i,v_i)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$$

Definition : Change of Variables in a Double Integral

Suppose T is a C^1 transformation whose Jacobian is nonzero, map from uv to xy. R, S are type I, II, f is continuous.

$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$$

Triple Integral

$$\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases}$$

The **Jacobian** of T is the determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Definition : Triple Integration in Spherical Coordinates

$$\iiint_R f(x,y,z)\, dV = \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$