Rocket Equation and Multi-Staging

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Rocket Equation & Multistaging

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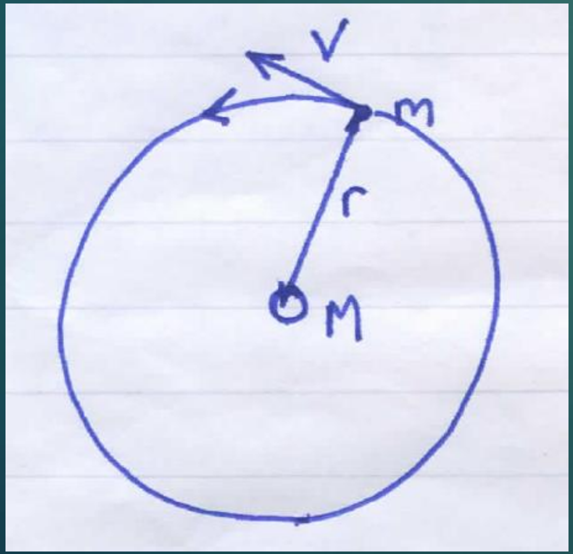
Orbital Mechanics & the Escape Velocity

- The motion of a space craft is that of a body with a certain momentum in a gravitational field
- The spacecraft moves under the combines effects of its momentum and the gravitational attraction towards the centre of the earth
- For a circular orbit $V = wr$ (1)



$F = mrw^2$ = centripetal acceleration (2)

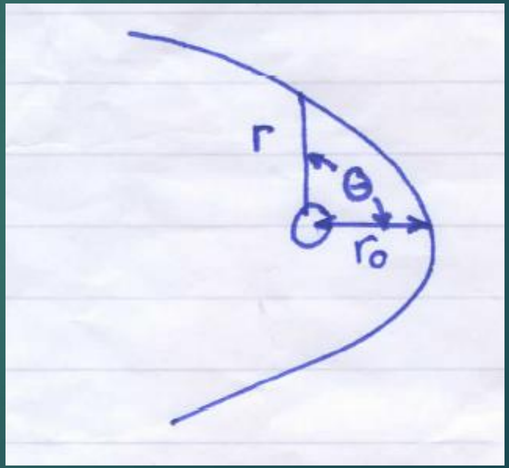
And this is balanced by the gravitational attraction

$F = mMG / r^2$ (3)

$$w = \left(\frac{GM}{r^3}\right)^{1/2} \; or \; V = \left(\frac{GM}{r}\right)^{1/2} \qquad (4)$$

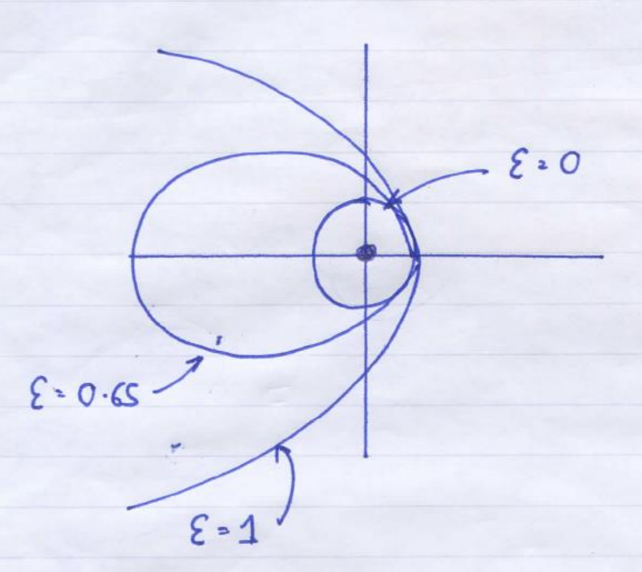
Orbital Velocity V

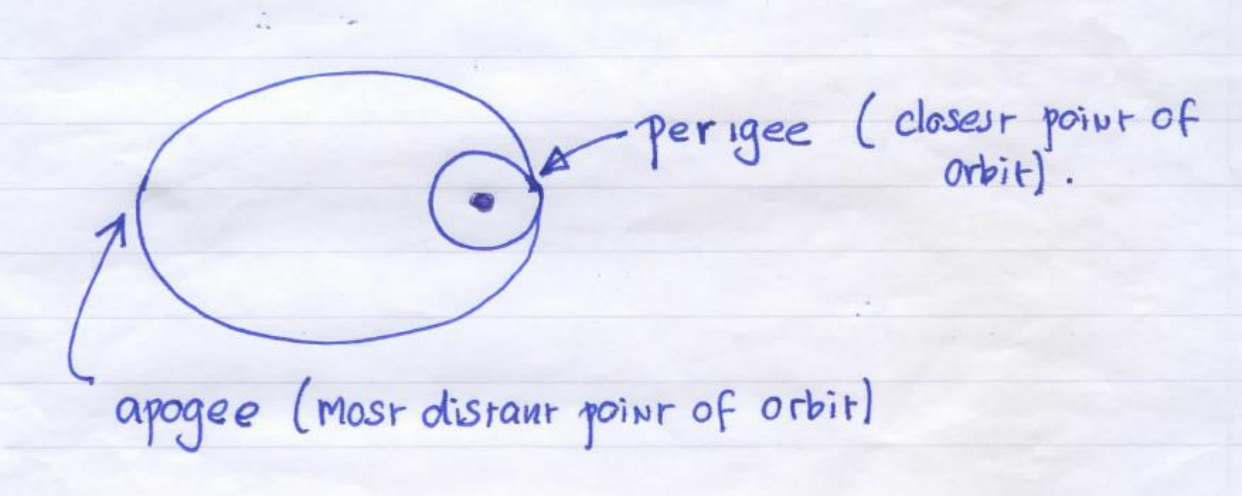
- $G = 6.67 \times 10^{-11}\ \mathrm{Nm}^2 kg^2$ Gravitational constant
- M = 5.97$\times 10^{24}$ kg Mass of the Earth
- r_0 = 6.371$\times 10^6\ \ m$
- Hence V = 7900 m/s
- For a non-circular (eccentric) orbit
- $\frac{1}{r} = \frac{GMm^2}{h^2}(1 + \varepsilon\ cos\theta)$ (5)
- $h = mrV$ angular momentum (6)
- $\varepsilon = \frac{h^2}{GMm^2 r_0} - 1$ eccentricity (7)
- $V = \left[\frac{GM}{r}(1 + \varepsilon\ cos\theta)\right]^{\frac{1}{2}}$ (8)



Orbit type

- For ε = 0 circular orbit
- For ε = 1 parabolic orbit
- For ε > 1 hyperbolic orbit (interplanetary fly-by)



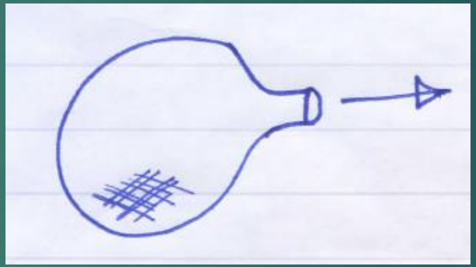


Escape Velocity

- For $\varepsilon = 1$ a parabolic orbit, the orbit ceases to be closed and the space vehicle will not return
- In this case at r_0 using equation (7)
- $1 = \frac{h^2}{GMm^2 r_0} - 1$ but $h = mrv$ (9)
- $2 = \frac{m^2 V_0^2 r_0^2}{GMm^2 r_0}$ therefore $V_0 = \left(\frac{2GM}{r_0}\right)^{\frac{1}{2}}$ (10)
- $G = 6.67 \times 10^{-11}\ \mathrm{N}m^2 kg^2$ Gravitational constant
- M = 5.975× 10^{24} kg Mass of the Earth
- r_0 = 6.371× 10^6 m Mean earth radius
- $V_0 = \left(\frac{2GM}{r_0}\right)^{\frac{1}{2}}$ Escape velocity (11)
- V_0= 11,185 m/s (~ 25,000 mph)
- 7,900 m/s to achieve a circular orbit (~18,000 mph)

Newton's 3rd Law & the Rocket Equation

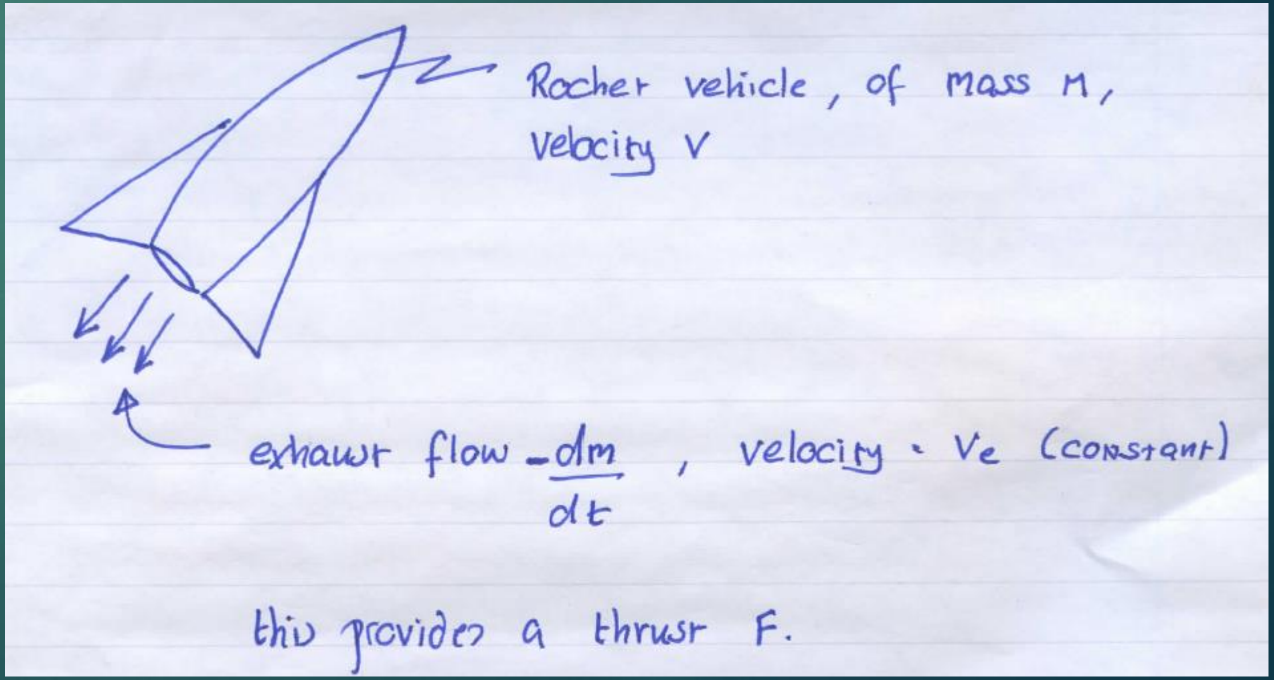
- N3: "to every action there is an equal and opposite reaction"
- A rocket is a device that propels itself by emitting a jet of matter.
- The momentum carried away results in a force acting to accelerate the rocket in a direction opposite to that of the jet
- Like a balloon expelling its gas and providing thrust



- A rocket is different to a gun because a bullet is given all its energy at the beginning of its flight. The energy of the bullet then decreases with time due to the losses against air friction.
- A cannon shell or a bullet is a projectile
- A rocket is a vehicle

Rocket Equation

- Thrust $F = -V_e \frac{dm}{dt}$ negative because the mass of the rocket decreases with time (14)
- The acceleration of the rocket under this force is given by Newton's 2nd Law
- $F = m\frac{dV}{dt}$ (15)
- Therefore $\frac{dV}{dt} = -\frac{1}{M} V_e \frac{dM}{dt}$ (16)
- $dV = -V_e \frac{dM}{M}$ (17)
- Integrate between limits of zero and V, for a change in mass M_0 to M gives the result



Rocket Equation

- $\int_0^V dV = -V_e \int_{M_0}^{M} \frac{dM}{M}$ (18)
- $V = -V_e\, log_e\left(\frac{M}{M_0}\right)$ (19)
- $V = \ V_e\, log_e\left(\frac{M_0}{M}\right)$ (20)
- This is Tsiolkovsk's Rocket Equation
- The rocket equation shows that the final speed depends upon only two numbers
  - The final mass ratio
  - The exhaust velocity

It does not depend on the thrust; size of engine; time of burn

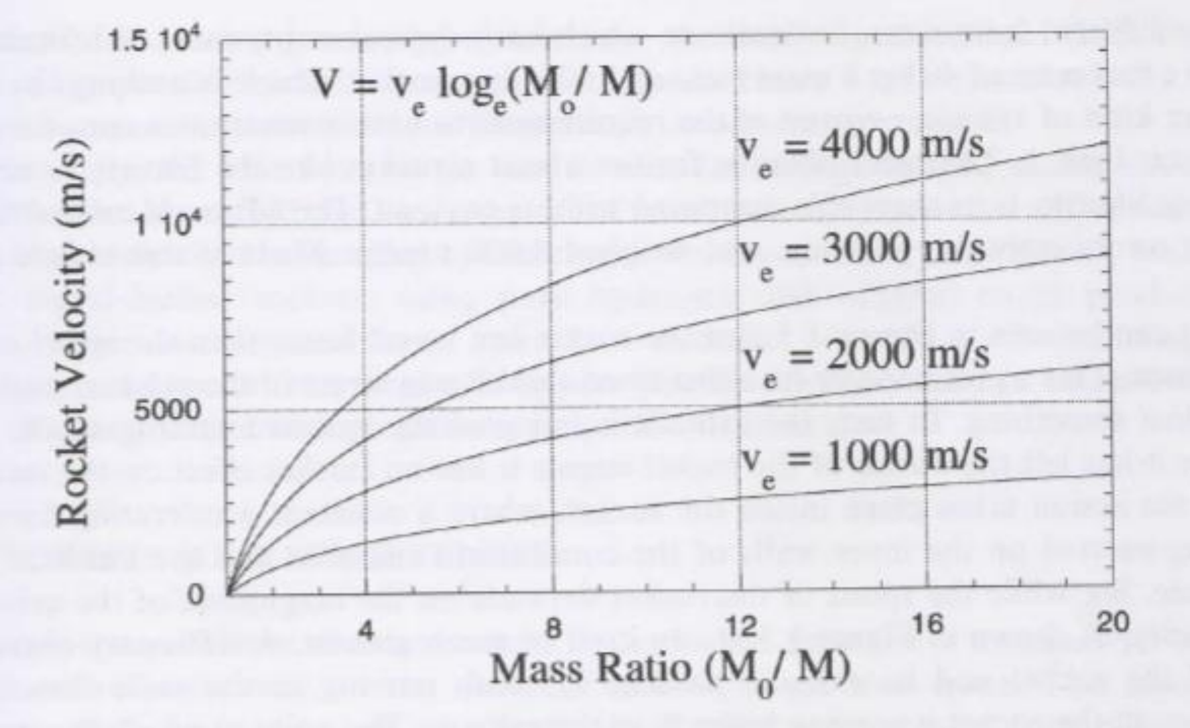


Figure 1.2. Tsiolkovsky's rocket equation.

Exit velocity depends on the fuel

- Gunpowder $V_e \approx 2000 \frac{m}{s}$
- Liquid fuel $V_e \approx 4500 \frac{m}{s}$
- Mass ratio = $\frac{Vehicle\ mass + propellant\ mass}{Vehicle\ mass} = \frac{M_0}{M}$ (21)
- $\frac{M_0}{M_V} = 20\ \ implies\ 95\%\ of\ the\ initial\ mass\ is\ fuel$ (22)

A rocket can travel faster than its exhaust speed

- A rocket can travel faster than its exhaust speed V_e
- This appears to be counter intuitive if we think of the exhaust as pushing against something. But this is not the case
- All the action and reaction takes place inside the rocket where an accelerating force is being developed against the walls of the combustion chamber and the inside of the nozzle
- A rocket will exceed its exhaust speed when
- $log_e\left(\frac{M_0}{M}\right) = 1$ (25)
- ie $\frac{M_0}{M} = e = 2.718$ (26)

Diminishing returns

- It is clear that increasing the mass ratio, that is: increasing the mass of fuel leads to diminishing returns
- For V_e = 1000 m/s, V $\Rightarrow 3000\ \mathrm{m/s}$
- A higher mass ratio will produce a higher velocity but only with a diminishing return
- To escape the earth's gravitational field a velocity of around 11km/s is required.
- This can only be achieved with a high exhaust velocity and a large mass ratio

Gravity loss

- Our main result neglects the so called “gravity loss” that is the work done against gravity. If this were included
- $V = \ V_e \, log_e\left(\frac{M_0}{M}\right) - gt$ (27)
- $V = \ V_e \, log_e\left(\frac{M_0}{M}\right) - g\,\frac{M_0}{\dot{m}}\left(1 - \frac{M}{M_0}\right)$ (28)
- $g\,\frac{M_0}{\dot{m}}\left(1 - \frac{M}{M_0}\right)$ can account for 1200 m/s

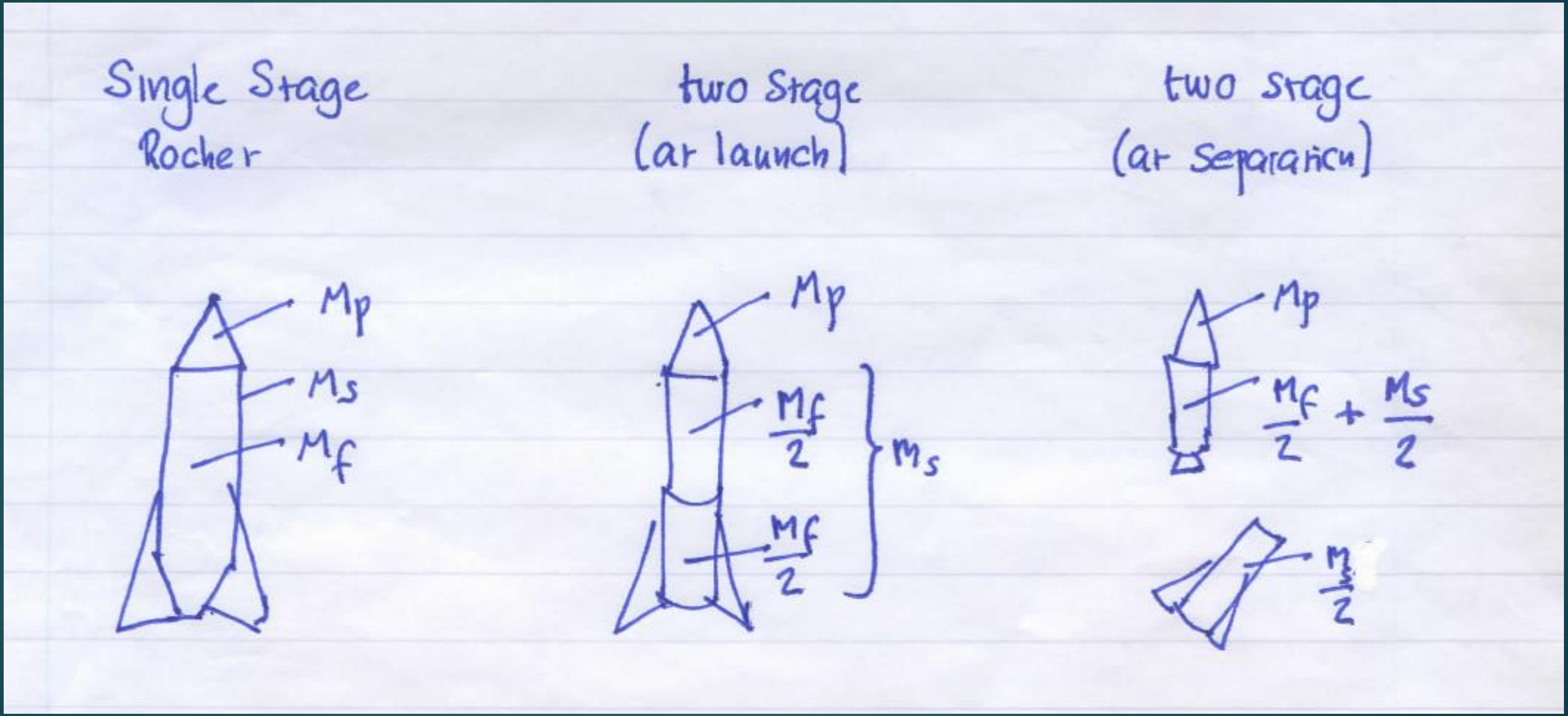
Multi stage Rockets

- As previously demonstrated a velocity of about 11 km/s is required to achieve escape from earth's gravity
- A velocity of 8 km/s is required to achieve a circular orbit
- For a single stage rocket, with modern fuel $V_e \sim 4\ \mathrm{km/s}$
- This implies a mass ratio of:
  - ~ 16 to achieve escape velocity 15/16th fuel $\sim 94\%$
  - ~ 7.4 to achieve orbit 6.4/ 7.4 fuel ~ 86%
- Although the latter is currently possible, the former, ie escape velocity can only be achieved by multi stage rockets

Multi-stage rockets

- For a single stage rocket
- $R_0 = \frac{M_0}{M} = \frac{M_S + M_F + M_P}{M_S + M_P}$
- M_F = fuel mass
- M_P = payload mass
- M_S = structural mass (depends on design – engines, pumps, fuel tanks, control systems)
- In general, we mayexpect the structural mass is kept to a minimum and is a constant proportion of the fuel mass for stages using the same fuel.

Multi-stage rockets



Two-stage rocket

- This rocket is divided into two stages
- The first rocket stage is ignited and burns until all its fuel is exhausted, this gives the whole stack a velocity defined by the rocket equation, with the mass ratio of:
- $R_1 = \frac{M_0}{M} = \frac{M_S + M_F + M_P}{M_S + \frac{M_F}{2} + M_P}$ $\quad\quad \frac{Before}{After}$
- The first stage burns out, is dropped off and the 2nd stage is ignited. It then gains additional velocity defined again by the rocket equation with mass ratio
- $R_2 = \frac{\frac{1}{2}M_S + \frac{1}{2}M_F + M_P}{\frac{1}{2}M_S + M_P}$ $\quad\quad \frac{Before}{After}$
- The second stage begins its burn with the payload, half the structural mass and half the fuel mass and ends with half the structural mass and the payload
- The final velocity is the sum of the two velocity increments

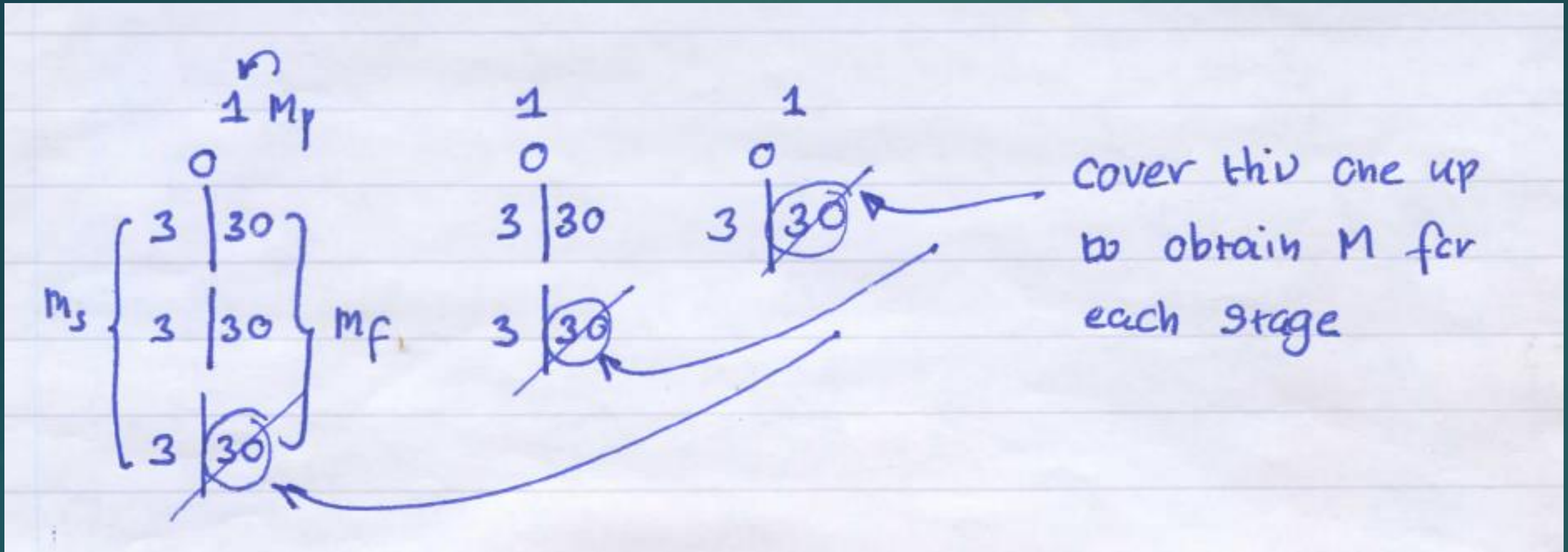
Single stage and two stage rocket compared

- So to compare the performance of single stage and two stage rockets we need to calculate:
- $V_0 = V_e \, log_e(R_0)$
- $V = V_e \, log_e(R_1) + V_e \, log_e(R_2)$
- For example:
- Total mass of 100 tonnes; Payload of 1tonne
- $V_e = 2.7 \times 10^3$ m/s
- $M_S = 10\% \; of \; fuel \; mass$
- Therefore M_F= 90 tonnes; M_S= 9 tonnes; M_P = 1 tonne

Single stage and two stage rocket compared

- $V_0 = 2700\, log_e \left(\frac{9+90+1}{9+1}\right)$ = 6217 m/s single stage final velocity
- Now divide the rocket into two smaller ones, each with half the fuel and the structural mass shared equally
- $V_1 = 2700\, log_e \left(\frac{9+90+1}{9+45+1}\right)$ = 1614 m/s
- $V_2 = 2700\, log_e \left(\frac{4.5+45+1}{4.5+1}\right)$ = 5986 m/s
- Total velocity increment = $V_1 + V_2$ = 7600 m/s

Three stage rocket



- $R_1 = \frac{90+9+1}{60+9+1} = 1.4286;$ $V_1 = 2700\ log_e R_1 = 963\ m/s$
- $R_2 = \frac{60+6+1}{30+6+1} = 1.8108;$ $V_2 = 2700\ log_e R_2 = 1603\ m/s$
- $R_3 = \frac{30+3+1}{3+1} = 8.5;$ $V_3 = 2700\ log_e R_3 = 5778\ m/s$
- Total velocity increment = $V_1 + V_2 + V_3$ = 963+1603+5778 = 8344 m/s

Multi stage rocket summary comparison

- Single stage Velocity increment = 6217 m/s
- Two stage Velocity increment = 7600 m/s
- Three stage Velocity increment = 8344 m/s

End