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Теоретические задачи.

(1)

Доказать, что

$$E_{x,y} E_{X^\ell} \left(y - a_{X^\ell}(x)\right)^2 = noise + bias + variance,$$

где

$E_{x,y}\left(y - E(y|x)\right)^2$ – noise

$E_{x,y}\left(E(y|x) - E_{X^\ell} a_{X^\ell}(x)\right)^2$ – bias

$E_{x,y} E_{X^\ell}\left(a_{X^\ell}(x) - E_{X^\ell} a_{X^\ell}(x)\right)^2$ – variance

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$$E_{x,y} E_{X^\ell}\left(y - a_{X^\ell}(x)\right)^2 =$$

$$= E_{x,y} E_{X^\ell}\left(y - E(y|x) + E(y|x) - a_{X^\ell}(x)\right)^2 =$$

$$= \underbrace{E_{x,y} E_{X^\ell}\left[y - E(y|x)\right]^2}_{\substack{noise, \\ \text{т.к. не зависит от } X^\ell}} + E_{x,y} E_{X^\ell}\Big[a_{X^\ell} -$$

$$- E_{X^\ell} a_{X^\ell} + E_{X^\ell} a_{X^\ell} - E(y|x)\Big]^2 +$$

$$+ 2 E_{x,y} E_{X^\ell}\left[\left(y - E(y|x)\right)\left(E(y|x) - a_{X^\ell}(x)\right)\right] =$$

$$= noise + E_{x,y} E_{X^\ell}\Big[\overbrace{E_{X^\ell} a_{X^\ell} - E(y|x)}^{bias}\Big]^2 +$$

$$+ E_{x,y} E_{X^e} \overbrace{[a_{X^e} - E_{X^e} a_{X^e}]^2}^{variance} +$$

$$+ 2E_{x,y} E_{X^e} [(a_{X^e} - E_{X^e} a_{X^e})(E_{X^e} a_{X^e} - E(y|x)] +$$

$$+ \underbrace{2E_{x,y} (y - E(y|x)) E_{X^e} (E(y|x) - a_{X^e}(x))}_{0} =$$

$$= noise + bias + variance +$$

$$+ 2E_{x,y} [E_{X^e} a_{X} - E(y|x)] \underbrace{E_{X^e} [E_{X^e} a_{X^e} + a_{X^e}]}_{0} =$$

$$= noise + bias + variance$$ ☐

(2) $$a(x) = \frac{1}{M} \sum_{m=1}^{M} a_m(x)$$

• $$bias(a(x)) = E_{x,y} (E(y|x) - E_{X^e} (\frac{1}{M} \sum_{m=1}^{M} a_m(x))^2 =$$

по линейности матожи.:
и равномерно распределены $a_i(x)$

$$= E_{x,y} (E(y|x) - \frac{1}{M} \cdot M E_{X^e} (a_i(x)))^2 =$$

$$= bias(a_i(x)) \quad \forall i \in \{1, \dots M\}$$

• $$variance(a(x)) = E_{x,y} E_{X^e} (\frac{1}{M} \sum_{i=1}^{M} a_i(x) -$$

$$- E_{X^e} \frac{1}{M} \sum_{i=1}^{M} a_i(x))^2 =$$

$$= \frac{1}{M^2} E_{x,y} E_{x\ell} \left(\sum_{i=1}^{M} (a_i(x) - E_{x\ell}(a_i(x)) \right)^2 =$$

$\circledast$ [$a_i(x)$ - независимы $\Rightarrow$ некоррелируемы $\Rightarrow$

$E_{x\ell}(a_i(x) - E_{x\ell}(a_i(x))(a_j(x) - E_{x\ell}(a_j(x)) = 0$]

$\forall i,j : j \neq i$

$$\stackrel{\circledast}{=} \frac{1}{M^2} E_{x,y} E_{x\ell} M (a_i(x) - E_{x\ell}(a_i(x))) =$$

$$= \frac{1}{M} \text{variance}(a_i(x)) \quad \forall i \in \{1, \dots M\}$$

③ $X_1 \dots X_M \sim$ о.р. сл. в с дисперсией σ

$\forall i,j \; i \neq j \quad \text{corr}(X_i, X_j) = \rho. \quad EX_1 = a$

$$D\overline{X} = D\left(\frac{1}{M}\sum_{i=1}^{M} X_i\right) = E\left(\frac{1}{M}\sum_{i=1}^{M} X_i - a\right)^2 =$$

$$= \frac{1}{M^2} E\left(\sum_{i=1}^{M}(X_i - a)\right)^2 = \frac{1}{M^2}\left(\sum_{i=1}^{M} E(X_i - a)^2 + \right.$$

$$\left. + \sum_{\substack{i,j=1 \\ i \neq j}}^{M} (x_i - a)(x_j - a)\right) = \frac{1}{n}\sigma^2 + \frac{1}{n^2} 2C_M^2 \rho\sigma^2 =$$

$$= \frac{\sigma^2}{M} + \frac{M(M-1)}{M^2}\rho\sigma^2 =$$

$$= \rho\sigma^2 + (1-\rho)\frac{\sigma^2}{M}$$