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Master 2 of Quantitative Finance, **Numerical Finance**

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Dynamic Hedging Strategy in the Black and Scholes framework

Due for November 12^{th} *2020.*

Let us consider a (complete) financial market with two assets: a risky one $(S_t)_{t\geq 0}$ and a non risky one $(S_t^0)_{t\geq 0}$. It is assumed that the interest rate is constant and denoted by r. Thus, the dynamics of the non risky asset (or bond) writes:

$$dS_t^0 = rS_t^0 dt. \tag{B}$$

It is also assumed that the risky asset follows a Black and Scholes dynamics, i.e.

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{B\&S}$$

where μ is the *market trend* of the asset, σ the *volatility* and W stands for a standard Brownian motion, associated with a filtration $(\mathcal{F}_t)_{t\geq 0}$, under the *market probability* $\mathbb{P}$.

The goal of this exercise is twofold. First, to give the fair price of an option on S. Second, from the fair price, to implement an investment strategy that will guarantee the option is indeed replicated at maturity.

Let $T > 0$ be a fixed final time horizon and let Φ be a non-negative pay-off function which has polynomial growth, i.e. there exists $p \geq 1$ and $C \geq 1$ such that for all $x \in \mathbb{R}_+,\ 0 \leq \Phi(x) \leq C(1+|x|^p)$.

1. Justify that for all given $T > 0$, the quantity $\Phi(S_T) \in L^1(\mathbb{P})$, i.e. that $\mathbb{E}^{\mathbb{P}}[\Phi(S_T)] < +\infty$, where $\mathbb{E}^{\mathbb{P}}$ denotes the expectation under $\mathbb{P}$.
2. Establish that the fair price at time $t \in [0,T]$ of the option on S with pay-off Φ and maturity T writes:
$$V_t := \exp(-r(T-t))\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)|\mathcal{F}_t] = \exp(-r(T-t))\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)|S_t], \tag{P}$$
where $\mathbb{Q}$ stands for the *risk-neutral* probability, under which the dynamics of S rewrites for $s \geq t$ as
$$dS_s = S_s(rds + \sigma dW_s^{\mathbb{Q}}), \tag{B\&S$_{\mathbb{Q}}$}$$
where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$ Brownian motion.
3. Justify from (P) that $V_t = v(t, S_t)$ for some function v to be specified. Assuming now that v is *smooth*, namely that $v \in C^{1,2}([0,T) \times \mathbb{R}_+^*, \mathbb{R}) \cap C^0([0,T) \times \mathbb{R}_+^*, \mathbb{R})$, with $\mathbb{R}_+^* := (0, +\infty)$, prove that
$$\begin{cases} \partial_t v(t,x) + rx\partial_x v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 v(t,x) - rv(t,x) = 0, (t,x) \in [0,T) \times \mathbb{R}_+^*, \\ v(T,x) = \Phi(x),\ x \in \mathbb{R}_+^*. \end{cases} \tag{PDE}$$

The Partial Differential Equation (PDE) is called the *valuation PDE* and provides an alternative to the estimation of the expectation in (P) for the numerical approximation of the fair price.

Bonus: prove that if Φ is also continuous, then v indeed satisfies the indicated smoothness conditions.

4. The point is now to implement a dynamic strategy to replicate from the initial wealth V_0 the final pay-off $\Phi(S_T)$ at time T. The idea is to invest in continuous time partly in the risky asset, partly in the non risky one and to construct a *self financing* portfolio $(P_t)_{t\in[0,T]}$ achieving this goal. Thus from $P_0 = V_0$, initial amount of money received by the manager corresponding to the *fair price* of the option, the point is to find $(\delta_t)_{t\geq 0}, (\delta_t^0)_{t\geq 0}$, corresponding respectively to the quantities of risky and non-risky asset at time $t \in [0,T]$, such that $P_T = V_T = \Phi(S_T)$. The value at time t of the portfolio is

$$P_t = \delta_t S_t + \delta_t^0 S_t^0, \tag{P}$$

and the *self financing* constraint writes:

$$dP_t = \delta_t dS_t + \delta_t^0 dS_t^0. \tag{SF}$$

We carefully specify that the dynamics in (SF) does not come from an Itô type expansion but simply expresses that, once the strategy is chosen at some time t, then the portfolio feels the infinitesimal variations of the asset S and the bond S^0.

A natural idea to replicate the option is to construct a portfolio P such that its value at time t corresponds to the option price at the same time. In other words, it is tempting to have $P_t = V_t$. The remaining point consists in exhibiting an investment strategy such that (SF) is satisfied and $P_t = V_t$. To this end, we will use the valuation PDE (PDE). Fix $0 \leq u < t < T$. Expand then v with Itô's formula, recall that $v \in C^{1,2}([u,t]\times\mathbb{R}_+^*, \mathbb{R})$:

$$\begin{aligned}
v(t,S_t) &= v(u,S_u) + \int_u^t \partial_s v(s,S_s)ds + \int_u^t \partial_x v(s,S_s)dS_s + \frac{1}{2}\int_u^t \partial_x^2 v(s,S_s)d\langle S\rangle_s \\
&= v(u,S_u) + \int_u^t \partial_s v(s,S_s)ds + \int_u^t \partial_x v(s,S_s)dS_s + \frac{1}{2}\int_u^t \sigma^2 S_s^2 \partial_x^2 v(s,S_s)ds \\
&= v(u,S_u) + \int_u^t \partial_x v(s,S_s)dS_s + \int_u^t \Big(\partial_s v(s,S_s) + rS_s\partial_x v(s,S_s) + \sigma^2 S_s^2\partial_x^2 v(s,S_s)\Big)ds \\
&\quad - \int_u^t rS_s\partial_x v(s,S_s)ds \\
&= v(u,S_u) + \int_u^t \partial_x v(s,S_s)dS_s + \int_u^t \Big(rv(s,S_s) - rS_s\partial_x v(s,S_s)\Big)ds \\
&= v(u,S_u) + \int_u^t \partial_x v(s,S_s)dS_s + \int_u^t \Big(v(s,S_s) - S_s\partial_x v(s,S_s)\Big)rds,
\end{aligned}$$

where we have used (PDE) for the last but one equality. This equivalently reads in differential form:

$$dv(t,S_t) = \partial_x v(t,S_t)dS_t + \Big(v(t,S_t) - S_t\partial_x v(t,S_t)\Big)rdt. \tag{1}$$

This equality precisely allows to identify P_t and V_t taking $\delta_t = \partial_x v(s,S_s)$ observing as well that for this choice, from (P), $\delta_t^0 S_t^0 = V_t - \delta_t S_t$ so that from (1) we indeed get:

$$dV_t = dv(t,S_t) = \partial_x v(t,S_t)dS_t + \delta_t^0 S_t^0 rdt = \delta_t dS_t + \delta_t^0 dS_t^0 = dP_t. \tag{I}$$

This thus means that, if one could update in practice the investment strategy in continuous time, then it would be precisely possible to replicate the pay-off following the dynamics (I).

In practice, we are going to discretize the dynamics on a time grid.

5. Let $N \in \mathbb{N}$ be fixed and consider the grid $\Lambda_N := \{(t_i)_{i\in[\![0,N]\!]},\ t_i := ih,\ h := \frac{T}{N}\}$. Starting from $V_0^h := v(0,S_0) = \exp(-rT)\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)|S_0]$ the discretization of (I) reads for all $i \in [\![0,N-1]\!]$:

$$\begin{aligned}
V_{t_{i+1}}^h - V_{t_i}^h &= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + \delta_{t_i}^0(S_{t_{i+1}}^0 - S_{t_i}^0) = \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + \delta_{t_i}^0 S_{t_i}^0(\exp(rh)-1) \\
&= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + (V_{t_i}^h - \delta_{t_i}S_{t_i})(\exp(rh)-1).
\end{aligned} \tag{I$_h$}$$

Implement now for $\Phi(x) = (x-K)^+$, $S_0 = 100$, $\sigma = 0.2, \mu = 0.03, r = 0.015$, $K = 97$, $T = 1$, $N \in \{10, 100, 500, 1000, 5000, 10000\}$ the strategy (I$_h$). Pay attention that this requires:

i. To simulate $(S_{t_i})_{i\in[\![1,N]\!]}$ under the market probability $\mathbb{P}$. Namely, given a collection of standard i.i.d. Gaussian variables $(N_i)_{i\in\mathbb{N}}$ s.t. $N_1 \overset{\text{(law)}}{=} \mathcal{N}(0,1)$, this can be done for all $i \in [\![0, N-1]\!]$ from the recursion:

$$S_{t_{i+1}} = S_{t_i} \exp(\sigma\sqrt{h}N_{i+1} + (\mu - \frac{\sigma^2}{2})h).$$

This represents the pathwise evolution of the asset under $\mathbb{P}$.

ii. To update at any discretization time $\delta_{t_i} := \partial_x v(t, S_{t_i})$ which in turn implies to differentiate for the current example the Black and Scholes formula (price under the risk neutral probability measure $\mathbb{Q}$).

You should eventually observe

$$V_T^h - V_T = V_T^h - \Phi(S_T) \xrightarrow[h\to 0]{a.s.} 0. \tag{C$_H$}$$

Namely, the discretized strategy converges *almost surely* to the final pay-off at maturity when the rebalancing step tends to 0. To illustrate (C_H) you could for instance run, for fixed h, a number M of independent simulations $\big((V_T^h - V_T)^i\big)_{i\in[\![1,M]\!]}$ and observe that the associated standard deviation goes to zero with M.

6. Bonus: Prove that (C_H) holds. You might use a regularization of the pay-off or an estimate of the probability that S_T belongs to a neighborhood where the pay-off is not *smooth*.