# Master of Quantitative Finance,
# Université d'Evry-Paris Saclay,
# Numerical finance,
# Homework 2

December 17, 2020

**Homework due January 14$^{\text{th}}$ , 2021**.

**Exercise 1. Weak and strong error**

1. Solve explicitly the SDE

$$X_t = x + \int_0^t bX_s ds + \int_0^t \sigma X_s dW_s,\ (b,\sigma) \in \mathbb{R}^2. \tag{1}$$

2. Simulate a path of $(X_t)_{t\in[0,1]}$ using the explicit formula and an Euler scheme on a time grid with step $2^{-4}, 2^{-8}, 2^{-10}$. Plot on three separate figures the paths of the solution of the scheme for a given time step. et ses approximations

3. Let us now consider a scalar SDE with dynamics,

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \tag{2}$$

with $b, \sigma$ Lipschitz continuous, for which (1) is a particular case. Let us formally write a first order expansion of the diffusion coefficient in the stochastic integral. We get:

$$\begin{aligned}\int_{t_k}^{t_{k+1}} \sigma(X_s)dW_s &\simeq \int_{t_k}^{t_{k+1}} \sigma(X_{t_k} + b(X_{t_k})(s-t_k) + \sigma(X_{t_k})(W_s - W_{t_k}))dW_s \\ &\simeq \sigma(X_{t_k})(W_{t_{k+1}} - W_{t_k}) + \left(\sigma'\sigma\right)(X_{t_k}) \int_{t_k}^{t_{k+1}} (W_s - W_{t_k})dW_s, \end{aligned} \tag{3}$$

where for a given time step $h$ we denoted $t_k = kh,\ k \in \mathbb{N}^*$.

a) How to simulate exactly $\int_{t_k}^{t_{k+1}}(W_s - W_{t_k})dW_s$ from the realisation of $W_{t_{k+1}} - W_{t_k}$? One can think about the Itô formula.

b) Write the numerical scheme associated with the dynamics (2) when the stochastic integral is approximated by the right hand side of (3). This scheme is called the "Milstein scheme" and will be denoted by $(X_{t_j}^{N,M})_{j\in[\![1,N]\!]}$.

c) Prove that for $T > 0$ such that $h = T/N$, and $p \geq 1$ are fixed

$$\exists C := C(p,T,x),\ \mathbb{E}[\sup_{s\in[0,T]} |X_s^{N,M} - X_s|^p] \leq Ch^p,$$

when the Milstein scheme is extended in a natural way to continuous time. The proof can be inspired from the one for the Euler scheme.

d) Go back to question 1. using the Milstein scheme. Emphasize the difference in terms of strong error, i.e. the one which takes into account the supremum of the difference along the whole path.

4. Compute directly $\mathbb{E}[X_1]$. Define $E_{n,N} := \frac{1}{n}\sum_{j=1}^{n} \tilde{X}_1^{j,N}$ where $\tilde{X}_1^{j,N}$ stands for the realisation a time 1 of an Euler or Milstein scheme with step $1/N,\ N \in \mathbb{N}^*$ where the realizations $(\tilde{X}_1^{j,N})_{j\in[\![1,n]\!]}$ are independent. For independent realizations of $E_{n,N}$ compute

$$\frac{1}{m}\sum_{i=1}^{m} |E^i_{n,N} - \mathbb{E}[X_1]|$$

for the Euler and Milstein scheme with $n \in \{(10^i)_{i\in[\![2,6]\!]}\},\ m = 100$. Plot the figures representing the logarithme of the error in terms of the logarithm of the step. What do you observe? Comment.

**Exercise 2. Sensitivity of Option prices by the Monte-Carlo method**

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space endowed with a scalar standard Brownian motion $W$. We assume that a financial asset $(S_t)_{t\geq 0}$ follows the Black and Scholes dynamics, i.e. under the risk neutral proability $S_t = S_0 \exp((r - \sigma^2/2)t + \sigma W_t)$, where $S_0$ is the initial price of the asset, $r$ stands for the interest rate supposed to be constant, $\sigma$ is the volatility.

The goal of this exercise is to approximate the $\Delta$ and $\Gamma$ of a vanilla option with pay-off $f$, assumed to have polynomial growth, and maturity $T > 0$ within this framework using a Monte-Carlo method. Let us first recall that

$$\Delta = \partial_{S_0}\mathbb{E}[f(S_T)],\ \Gamma = \partial^2_{S_0}\mathbb{E}[f(S_T)].$$

To this end, we will present and introduce three techniqueson va présenter et mettre en oeuvre trois techniques: finite differences, flow method, approach derived from Malliavin calculus.
In particular, the empirical variances associated to each of the approach will be discussed.

# 1 Finite Differences Estimation

For this approach we consider a parameter $\varepsilon > 0$ for which we define

$$\begin{aligned}
\Delta_\varepsilon &= (2\varepsilon)^{-1}\Big(\mathbb{E}_{S_0+\varepsilon}[f(S_T)] - \mathbb{E}_{S_0-\varepsilon}[f(S_T)]\Big),\\
\Gamma_\varepsilon &= \varepsilon^{-2}\Big(\mathbb{E}_{S_0+\varepsilon}[f(S_T)] - 2\mathbb{E}_{S_0}[f(S_T)] + \mathbb{E}_{S_0-\varepsilon}[f(S_T)]\Big).
\end{aligned}$$

1. Prove that the regularity of the function $v(t,s) = \mathbb{E}_s[f(S_{T-t})]$ for $t > 0$ yields $\Delta_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \Delta,\ \Gamma_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \Gamma$.

2. Specify a convergence rate in function of $\varepsilon$ using a Taylor series expansion.

3. Implement a function `Simulate_Hedge_FD` to estimate $\Delta_\varepsilon, \Gamma_\varepsilon$ with a Monte Carlo method:
   (a) Using the same Gaussian realizations for each of the terms under the expectation.
   (b) Using independent Gaussian realizations.

4. Comment the empirical variances obtained and validate your results for a call with parameters $S_0 = K = 100, r = .02, \sigma = .35, T = 1$.

5. Specify how to equilibrate $M$, number of paths used to estimate $\Delta_\varepsilon, \Gamma_\varepsilon$ and $\varepsilon$ to obtain a global error of fixed order$\eta > 0$ in the two considered cases. Justify the various arguments that could involve the smoothness of the above function $v$.

## 2 Estimation by flow technique

1. Prove that $f \in C(\mathbb{R}, \mathbb{R})$ is Lipschitz continuous, and thus differentiable almost everywhere, one can write $\Delta = \mathbb{E}[f'(S_T)\partial_{S_0} S_T] = \mathbb{E}[f'(S_T)\frac{S_T}{S_0}]$.

   **Remark:** Observe that the $\Delta$ can be computed this way but not directly the $\Gamma$. Recall indeed that the $\Gamma$ of a call can be viewed as the $\Delta$ of a digital option whose pay-off writes $\mathbb{I}_{S_T > K}$.

2. Implémenter une fonction `Simulate_Hedge_Flow` to obtain an estimation of the $\Delta$ by a flow method.

3. Compare the results obtained to those of the previous section for the same set of parameters.

## 3 Estimation by a Malliavin calculus type approach

1. Prove that in the Black et Scholes setting, for $f \in L^2(\mu)$, where $\mu$ stands for the law of $S_T$, one has:
$$\Delta = \mathbb{E}[f(S_T)\frac{W_T}{S_0\sigma T}],\ \Gamma = \mathbb{E}[\frac{f(S_T)}{S_0^2\sigma T}\left(\frac{W_T^2}{\sigma T} - W_T - \frac{1}{\sigma}\right)].$$

2. Write a function `Simulate_Hedge_Malliavin` in order to implement the weighted estimator of the $\Delta$. Compare the results with the previous ones, especially in terms of variance.

3. If now $f$ satisfies the assumptions of Section 2 prove that it can be derived that
$$\Gamma = \mathbb{E}[f'(S_T)\frac{S_T W_T}{S_0^2\sigma T} - f(S_T)\frac{W_T}{S_0^2\sigma T}].$$

   Deduce an estimator for the $\Gamma$ of a call.

4. Compare the results with those associated with finite difference estimator of the l $\Gamma$.

## 4 Variance Reduction

1. Propose a variance reduction method in terms of: a control variate, importance sampling.

2. For $R > 0$ introduce the function $\psi_R$ which precisely regularizes $x \mapsto \psi(x) = (x - K)^+$ close to $K$. Namely:
$$\psi_R(x) = \begin{cases} x - K,\ x > K + R, \\ (x - (K - R))^2/4R,\ x \in (K - R, K + R], \\ 0,\ x\ \le K - R. \end{cases}$$

   Study the empirical variance associated with the estimator deriving from the identity
$$\Delta = \mathbb{E}[\psi_R'(S_T)\frac{S_T}{S_0}] + \mathbb{E}[(\psi - \psi_R)(S_T)\frac{W_T}{S_0\sigma T}].$$

3. Which value of $R$ gives the optimal variance?

4. Answer the previous question for the $\Gamma$.