EC106: Introduction to Economics
– MACROECONOMICS –

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Lecture - 3 -

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This Week: Outline

1. Reminder

2. Steady State

3. Experiments

3.1 Increasing Saving s

3.2 Increasing Productivity A

4. Golden Rule

5. Cross-Country Income Differences

5.1 Convergence

What we have done?

Central Equation

$$K_{t+1} = sAF(K_t, L_t) + (1-\delta)K_t \tag{1}$$

- Multiply and divide the left-hand side by L_{t+1} and re-arranging terms so as to write it in terms of capital per worker

$$\frac{K_{t+1}}{L_t}\frac{L_{t+1}}{L_{t+1}} = sAf(k_t) + (1-\delta)k_t$$

- Since we are assuming that labour is constant across time, this means that $L_{t+1}/L_t = 1$ So we can write:

$$k_{t+1} = sAf(k_t) + (1-\delta)k_t \tag{2}$$

- **Equation-3 is the central equation of the Solow model.**
- It describes how capital per worker evolves over time, given an initial value of the capital stock, an exogenous value of A, and parameter values s and δ

Central Equation: Capital Per Worker Level

$$k_{t+1} = sAf(k_t) + (1-\delta)k_t \tag{3}$$

- We can define y_t, c_t, and i_t as output, consumption, and investment per worker. An again k_t capital per worker. Thus, in terms of k_t, we can re-write equations in page-**??**

$$y_t = Af(k_t) \tag{4}$$

$$c_t = (1-s)Af(k_t) \tag{5}$$

$$i_t = sAf(k_t) \tag{6}$$

$$w_t = Af'(k_t) \tag{7}$$

$$R_t = Af(k_t) - k_tAf'(k_t) \tag{8}$$

Central Equation: Capital Per Worker Level Cobb-Douglas

- Suppose that we have the Cobb-Douglas production function. The central equation of the Solow model (Equation-3) can be written:

$$k_{t+1} = sAk_t^{\alpha} + (1-\delta)k_t$$

- The other variables are determined as a function of k_t. These can be written:

$$y_t = Ak_t^{\alpha} \tag{9}$$

$$c_t = (1-s)Ak_t^{\alpha} \tag{10}$$

$$i_t = sAk_t^{\alpha} \tag{11}$$

$$w_t = (1-\alpha)Ak_t^{\alpha} \tag{12}$$

$$R_t = \alpha Ak_t^{\alpha-1} \tag{13}$$

Connections

Aggregate	Per Capita	Per Capita- Cobbb-Douglas
$K_{t+1} = sAF(K_t, L_t) + (1-\delta)K_t$	$k_{t+1} = sAf(k_t) + (1-\delta)k_t$	$k_{t+1} = sAk_t^{\alpha} + (1-\delta)k_t$
$Y_t = AF(K_t, L_t)$	$y_t = Af(k_t)$	$y_t = Ak_t^{\alpha}$
$Y_t = C_t + I_t$	$c_t = (1-s)Af(k_t)$	$c_t = (1-s)Ak_t^{\alpha}$
$K_{t+1} = I_t + (1-\delta)K_t$	$k_{t+1} = sy_t + (1-\delta)k_t$	$k_{t+1} = sk_t^{\alpha} + (1-\delta)k_t$
$I_t = sY_t$	$i_t = sAf(k_t)$	$i_t = sAk_t^{\alpha}$
$w_t = AF_L(K_t, L_t)$	$w_t = Af'(k_t)$	$w_t = (1-\alpha)Ak_t^{\alpha}$
$R_t = AF_K(K_t, L_t)$	$R_t = Af(k_t) - k_tAf'(k_t)$	$R_t = \alpha Ak_t^{\alpha-1}$

NOTE. Population growth rate is $n = 0$.

Assumption 1: Production Function

- Continuity
- Differentiability
- Positive and Diminishing Marginal Products
- Constant Returns to Scale

The production function $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ is twice differentiable in K and L, and satisfies

$$F_K(K, L) \equiv \frac{\partial F(K, L)}{\partial K} > 0, \quad F_L(K, L) \equiv \frac{\partial F(K, L)}{\partial L} > 0,$$
$$F_{KK}(K, L) \equiv \frac{\partial^2 F(K, L)}{\partial K^2} < 0, \quad F_{LL}(K, L) \equiv \frac{\partial^2 F(K, L)}{\partial L^2} < 0.$$

Moreover, F exhibits constant returns to scale in K and L.

A is constant

Dynamics of Solow: Graphical Analysis

- We will use both graphs and math to analyze the Solow model. Let's graph k_{t+1} as a function of k_t
- Remember that k_t is predetermined in period t and therefore exogenous. If $k_t = 0$, then $k_{t+1} = 0$ **Why?**
- Remember that we assumed capital is necessary for production. Thus the graph starts at the origin,
- The important question is how will k_{t+1} vary as k_t changes.

Dynamics of Solow: Graphical Analysis

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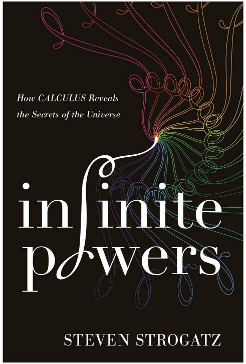
- Remember that we assumed capital is necessary for production. Thus the graph starts at the origin,
- The important question is how will k_{t+1} vary as k_t changes.
- Since Newton/Leibniz we know that to see this, we can take the derivative of k_{t+1} with respect to k_t:

$$\frac{dk_{t+1}}{dk_t} = sAf'(k_t) + (1 - \delta) \qquad (14)$$

- Equation-14 is for the slope of the graph of k_{t+1} against k_t.



STEVEN STROGATZ : Professor of Applied Mathematics at Cornell University

Dynamics of Solow: Graphical Analysis

$$\frac{dk_{t+1}}{dk_t} = sAf'(k_t) + (1-\delta)$$

- **The slope is positive** and k_{t+1} is increasing in k_t:

  Why?

Dynamics of Solow: Graphical Analysis

$$\frac{dk_{t+1}}{dk_t} = sAf'(k_t) + (1-\delta)$$

- **The slope is positive** and k_{t+1} is increasing in k_t:

  Why?

  $f'(k_t)$ is positive and $\delta < 1$

- $sAf'(k_t)$ **gets smaller as** k_t **gets bigger**

  Why?

Dynamics of Solow: Graphical Analysis

$$\frac{dk_{t+1}}{dk_t} = sAf'(k_t) + (1-\delta)$$

- **The slope is positive** and k_{t+1} is increasing in k_t:

  Why?

  $f'(k_t)$ is positive and $\delta < 1$
- $sAf'(k_t)$ **gets smaller as** k_t **gets bigger**

  Why?

  since $f''(k_t) < 0$,
- This means that k_{t+1} is an increasing (first derivation) function of k_t, but at a decreasing (second derivation) rate.

Inada Conditions

- In addition to these standard assumptions (in page-8) on the production function, the following boundary conditions, ***the Inada conditions,*** are often imposed in the analysis of economic growth and macroeconomic equilibria.
- The role of these conditions—especially in ensuring the existence of interior equilibria.
- They imply that the first units of capital and labour are highly productive and that when capital or labour is sufficiently abundant, their marginal products are close to zero.

Assumption 2 (Inada Conditions) F satisfies the Inada conditions

$$\lim_{K \to 0} F_K(K, L) = \infty \quad and \quad \lim_{K \to \infty} F_K(K, L) = 0 \quad for\ all\ L > 0\ and\ all\ A$$

,

$$\lim_{L \to 0} F_L(K, L) = \infty \quad and \quad \lim_{L \to \infty} F_L(K, L) = 0 \quad for\ all\ K > 0\ and\ all\ A$$

.

Moreover, $F(0, L) = 0$ for all L and A.

Inada Conditions

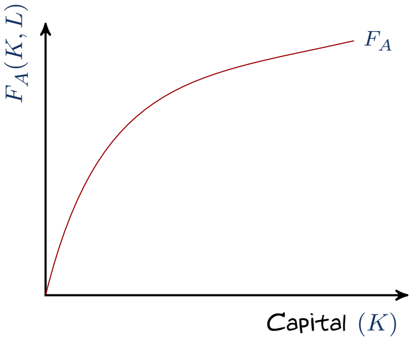
Simply:

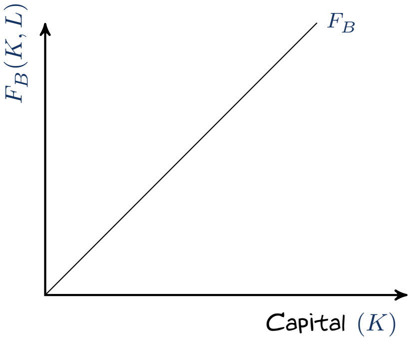
$$\lim_{k_t \to 0} f'(k_t) = \infty \tag{15}$$

$$\lim_{k_t \to \infty} f'(k_t) = 0 \tag{16}$$

- Equation-(15) says that the marginal product of capital is infinite when there is no capital,
- Equation-(16) says that the marginal product of capital goes to zero as the capital stock per worker gets infinitely large.
- These conditions together imply that $\frac{dk_{t+1}}{dk_t}$ starts out at the origin at positive infinity but eventually settles down to $1-\delta$, which is positive but less than one.

Inada Conditions





Production function **A** satisfies the Inada conditions in Assumption 2, while **B** does not.

Inada Conditions with Cobb-Douglas

- Suppose that the production function is Cobb-Douglas so that the central equation of the Solow model is given by Eq-(17) and the expression for the slope of the central equation is Eq-(18)

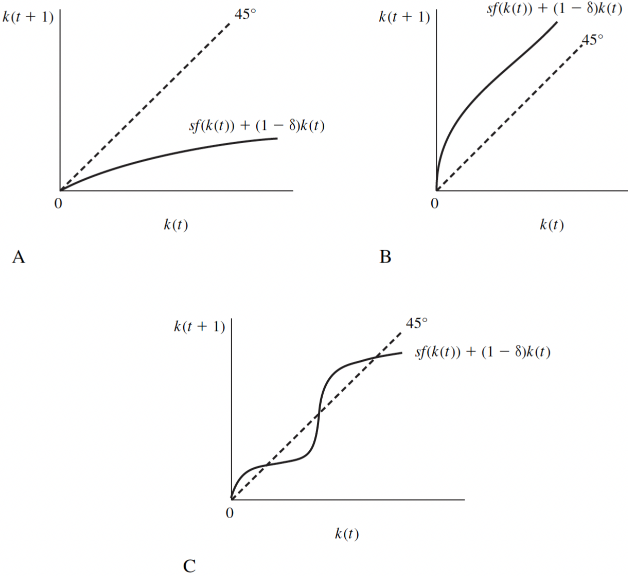
$$k_{t+1} = sAk_t^{\alpha} + (1-\delta)k_t \tag{17}$$

$$\frac{dk_{t+1}}{dk_t} = \alpha sAk_t^{\alpha-1} + (1-\delta) \tag{18}$$

This can equivalently be written as:

$$\frac{dk_{t+1}}{dk_t} = \alpha sA\left(\frac{1}{k_t}\right)^{1-\alpha} + (1-\delta). \tag{19}$$

- If $k_t \rightarrow 0$, then $\frac{1}{k_t} \rightarrow \infty$. Since $1-\alpha > 0$, and infinity raised to any positive number is infinity, the slope is infinity.
- Likewise, if $k_t \rightarrow \infty$, then $\frac{1}{k_t} \rightarrow 0$.
- 0 raised to any positive power is 0.
- Hence, the Inada conditions hold for the Cobb-Douglas production function.



EXAMPLES OF NONEXISTENCE AND NONUNIQUENESS OF INTERIOR STEADY STATES WHEN ASSUMPTIONS 1 AND 2 ARE NOT SATISFIED..

What is Steady State? Why it is very important?

Dynamics of Solow: Graphical Analysis (repeat)

$$\frac{dk_{t+1}}{dk_t} = sAf'(k_t) + (1 - \delta)$$

- **The slope is positive** and k_{t+1} is increasing in k_t:

  Why?

  $f'(k_t)$ is positive and $\delta < 1$

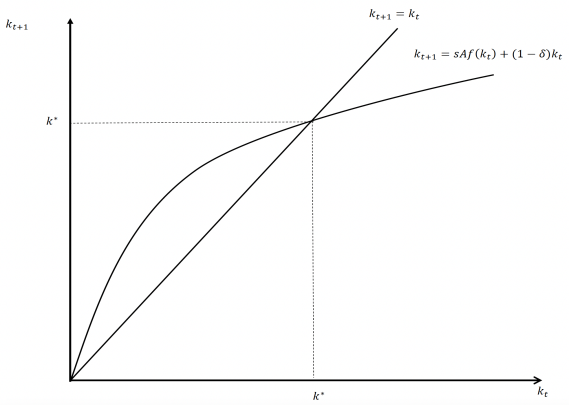
- $sAf'(k_t)$ **gets smaller as** k_t **gets bigger**

  Why?

  since $f''(k_t) < 0$,

- This means that k_{t+1} is an increasing (first derivation) function of k_t, but at a decreasing (second derivation) rate.

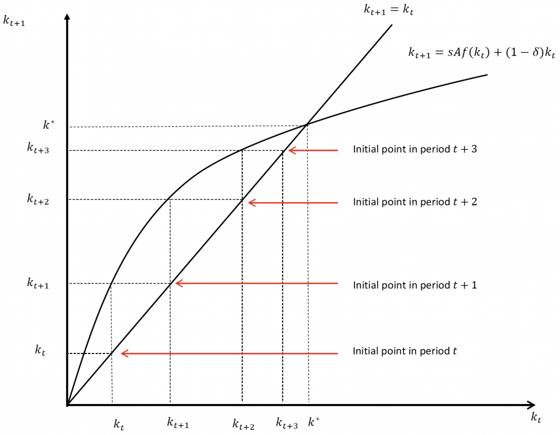
Central Equation and Steady State



Plot of Central Equation of Solow Model.

- As explained above the plot starts in the origin, starts out steep and flattens out as k_t gets bigger, eventually having a slope equal to $1 - \delta$
- The k_{t+1} plot starts out with a slope greater than 1, and hence initially lies above the 45-degree line. Eventually, the plot of k_{t+1} has a slope less than 1, and therefore lies below the 45-degree line.
- The plot of k_{t+1} crosses the 45-degree line exactly once away from the origin.
- We indicate this point with k^* this is the value of k_t for which k_{t+1} will be the same as k_t, i.e. $k_{t+1} = k_t = k^*$
- We will refer to this point, k, as the "**steady state**".

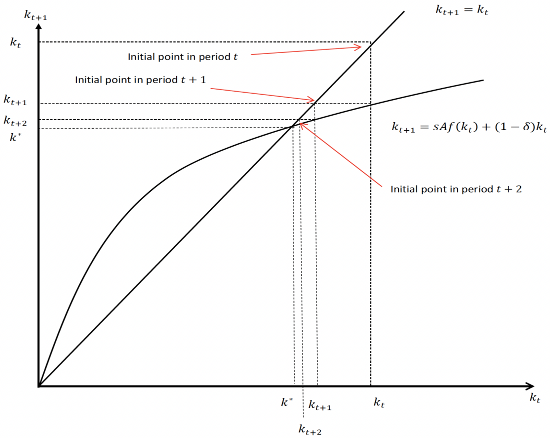
Why are we using the 45-degree line? Steady State $k_t < k^*$



Plot of Central Equation of Solow Model.

- The 45-degree line allows one to "reflect" the horizontal axis onto the vertical axis.
- Suppose that the economy begins with a period t capital stock below the steady state, i.e. $k_t < k^*$.
- One can read the current capital stock off of the vertical axis by reflecting it with the 45 degree line. This is labelled as "initial point in period t "
- The next period capital stock, k_{t+1}, is determined at the initial k_t from the curve. Since the curve lies above the 45 degree line in this region, we see that $k_{t+1} > k_t$.
- Then think about how the capital stock will evolve in future periods, we can functionally iterate the graph forward to another period.
- We can continue iterating with this procedure as we move "forward" in time. We observe that if k_t starts below k^*, then the capital stock will be expected to grow.

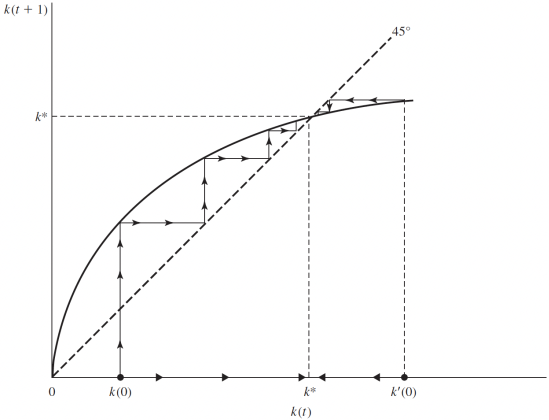
Convergence to Steady State from $k_t > k^*$



PLOT OF CENTRAL EQUATION OF SOLOW MODEL.

- Figure on the left side repeats the analysis but assumes the initial capital stock per worker lies about the steady state, $k_t > k^*$.
- The process plays out similar, but in reverse.
- Since in this region the line lies above the curve, the capital stock per worker will get smaller over time, eventually approaching the steady state point.

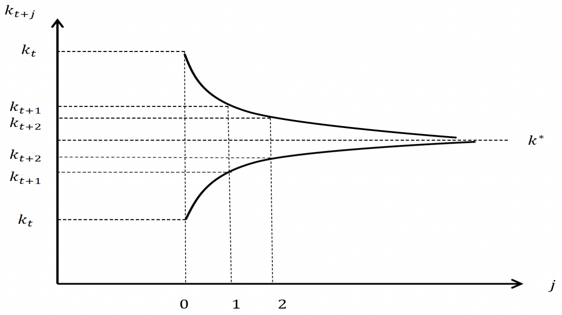
Transitional dynamics in the basic Solow model



Plot of Central Equation of Solow Model.

- For any non-zero starting value of k_t, the capital stock per worker ought to move toward k^* over time.
- Once the economy reaches $k_t = k^*$, it will stay there (since $k_{t+1} = k_t$ at that point), hence the term "**steady**".
- Furthermore, the capital stock will change quite a bit across time far from the steady state and will change very little when the initial capital stock is close to the steady state
- In summary, if the initial capital stock is $k(0) > 0$, which is below the steady-state level k^*, the economy grows toward k^* and experiences **capital deepening** -meaning that the capital-labour ratio increases. Together with capital deepening comes growth of per capita income.
- If instead, the economy were to start with $k_t > k^*$, it would reach the steady state by **decumulating capital** and contracting (i.e., by experiencing negative growth).

Central Equation and Steady State



PLOT OF CENTRAL EQUATION OF SOLOW MODEL.

- Figure plots hypothetical time paths of the capital stock, where in one case $k_t > k^*$ and in the other $k_t < k^*$.
- In the former case, k_t declines over time, approaching k^*.
- In the latter, k_t increases over time, also approaching k^*.
- The steady state is a natural point of interest.
- This is not because the economy is always at the steady state, but rather because, no matter where the economy starts it will naturally gravitate towards this point.

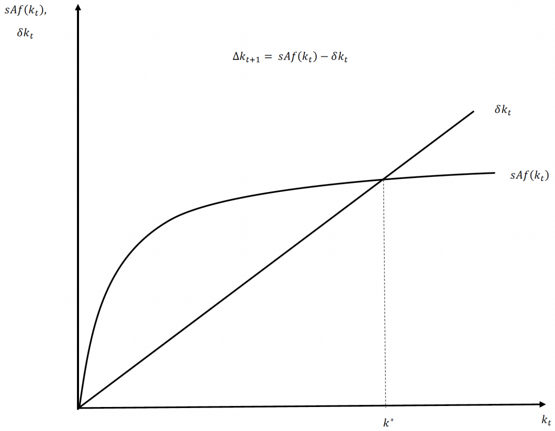
The Total Investment vs Total Depreciation

- An alternative way to graphically analyze the Solow model, define $\Delta k_{t+1} = k_{t+1} - k_t$. Subtracting k_t from both sides of $k_{t+1} = sAf(k_t) + (1-\delta)k_t$, one gets:

$$\Delta k_{t+1} = sAf\left(k_t\right) - \delta k_t. \tag{20}$$

- In Eq-(20), the first term on the right-hand side, $sAf\left(k_t\right)$, is **the total investment**.
- The second term, δk_t, is **total depreciation**.
- This equation says that *the change in the capital stock is equal to the difference between investment and depreciation.*
- Sometimes the term δk_t is called "break-even investment," because this is the amount of investment the economy must do so as to keep the capital stock from falling.

The Total Investment vs Total Depreciation



Plot of Central Equation of Solow Model.

- The first term, $sAf(k_t)$, starts at the origin, is increasing (since $f'(k_t) > 0$), but has diminishing slope (since $f''(k_t) < 0$).
- Eventually, as k_t gets big enough, the slope of this term goes to zero.
- The second term is just a line with slope δ, which is positive but less than one.
- The curve must cross the line at some value of k_t, call it k^*. This single crossing is guaranteed if the Inada conditions hold.
- For values of $k_t < k^*$, we have the curve lying above the line, which means that investment, $sAf(k_t)$, exceeds depreciation, δk_t, so that the capital stock will be expected to grow over time.
- Alternatively, if $k_t > k^*$, then depreciation exceeds investment, and the capital stock will decline over time.

Steady State with Cobb-Douglas Production

Suppose that the production function is Cobb-Douglas, so that the central equation of the model is given by Eq-(17). To algebraically solve for the steady state capital stock, take $k_{t+1} = sAk_t^{\alpha} + (1-\delta)k_t$ and set $k_{t+1} = k_t = k^*$:

$$k^* = sAk^{*\alpha} + (1-\delta)k^*$$

This is one equation in one unknown. k^* is:

$$k^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}} \tag{21}$$

We observe that k^* is increasing in s and A and decreasing in δ. All the other variables in the model can be written as functions of k_t and parameters. Hence, there will exist a steady state in these other variables as well. Plugging Eq-(21) in wherever k_t shows up, we get:

$$y^* = Ak^{*\alpha} \tag{22}$$

$$c^* = (1-s)Ak^{*\alpha} \tag{23}$$

$$i^* = sAk^{*\alpha} \tag{24}$$

$$w^* = (1-\alpha)Ak^{*\alpha} \tag{25}$$

$$R^* = \alpha Ak^{\star\alpha-1} \tag{26}$$

In summary: Steady State

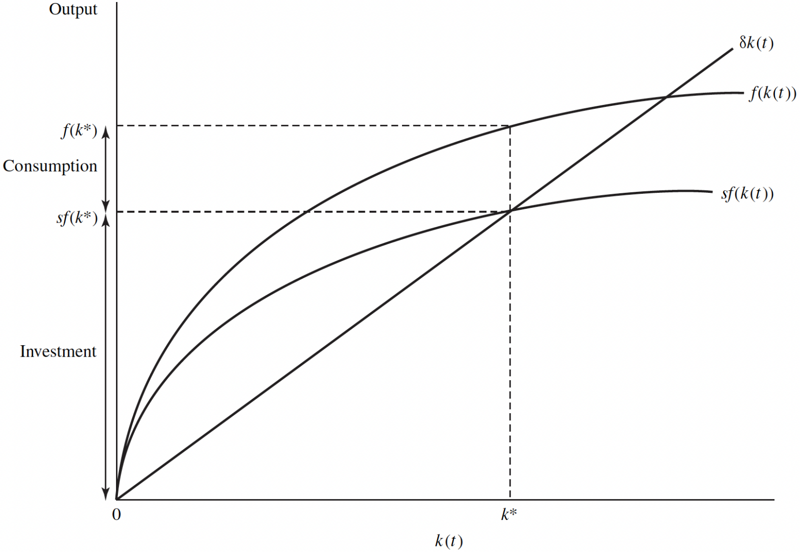
- Recall that when the economy starts with too little capital relative to its labour supply, the capital-labour ratio will increase.
- Thus the marginal product of capital will fall due to diminishing returns to capital and the wage rate will increase.
- Conversely, if it starts with too much capital, it will decumulate capital, and in the process, the wage rate will decline and the rate of return to capital will increase.
- The analysis has established that the Solow growth model has a number of nice properties: unique steady state, global (asymptotic) stability, and finally, simple and intuitive comparative statics.
- Yet so far it has no growth. **The steady state is the point at which there is no growth in the capital-labour ratio, no more capital deepening, and no growth in output per capita.**
- Consequently, the basic Solow model (without technological progress) can only generate economic growth along the transition path to the steady state (starting with $k_t < k^*$).
- However, this growth is not sustained: it slows down over time and eventually comes to an end.
- To sum up, the Solow model can incorporate economic growth by allowing exogenous technological change.

All Together

Aggregate	Per Capita	Per Capita- Cobbb-Douglas	Steady State
$K_{t+1} = sAF(K_t, L_t) + (1-\delta)K_t$	$k_{t+1} = sAf(k_t) + (1-\delta)k_t$	$k_{t+1} = sAk_t^\alpha + (1-\delta)k_t$	$k^* = sAk^{*\alpha} + (1-\delta)k^*$
$Y_t = AF(K_t, L_t)$	$y_t = Af(k_t)$	$y_t = Ak_t^\alpha$	$y^* = Ak^{*\alpha}$
$Y_t = C_t + I_t$	$c_t = (1-s)Af(k_t)$	$c_t = (1-s)Ak_t^\alpha$	$c^* = (1-s)Ak^{*\alpha}$
$K_{t+1} = I_t + (1-\delta)K_t$	$k_{t+1} = sy_t + (1-\delta)k_t$	$k_{t+1} = sk_t^\alpha + (1-\delta)k_t$	$k^* = sAk^{*\alpha} + (1-\delta)k^*$
$I_t = sY_t$	$i_t = sAf(k_t)$	$i_t = sAk_t^\alpha$	$i^* = sAk^{*\alpha}$
$w_t = AF_L(K_t, L_t)$	$w_t = Af'(k_t)$	$w_t = (1-\alpha)Ak_t^\alpha$	$w^* = (1-\alpha)Ak^{*\alpha}$
$R_t = AF_K(K_t, L_t)$	$R_t = Af(k_t) - k_t Af'(k_t)$	$R_t = \alpha Ak_t^{\alpha-1}$	$R^* = \alpha Ak^{\star\alpha-1}$

NOTE. Population growth rate is $n = 0$.

Investment and Consumption in the Steady-State

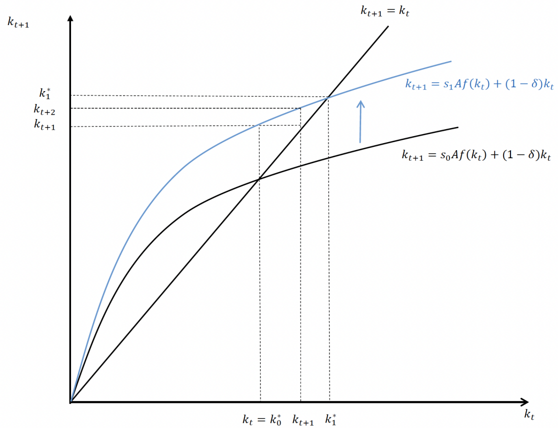


What would be happen if X changes?

Increasing Saving s

- We have set up the model and now we want to examine how the variables in the Solow model react dynamically to changes in parameters and exogenous variables.
- In the real world, increases in the saving rate could be driven by policy changes.
- Let's assume our hypothetical country increases the tax rates which encourages saving, also household has become pessimistic and simply just changed their preferences for example, households are keener on saving for the future.
- Let start with a change (increase) in s.
- This parameter is exogenous to the model.
- Suppose that the economy initial sits in a steady state, where the saving rate is s_0.
- Then, in period t, the saving rate increases to $s_1 > s_0$ and is forever expected to remain at s_1.

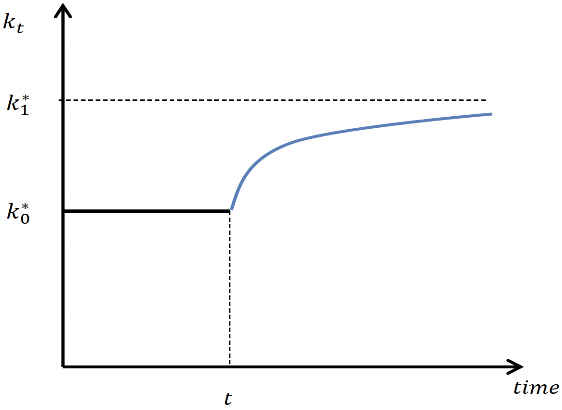
Increasing Saving s



Exogenous Increase in Saving s $s_1 > s_0$.

- In terms of the graph, an increase in s has the effect of shifting the curve plotting k_{t+1} against k_t up and also has the effect of making the curve steeper at every value of k_t.
- This effect can be seen in Figure left with the blue curve.
- The 45 degree line is unaffected. This means that the curve intersects the 45-degree line at a larger value, $k_1^* > k_0^*$.
- In other words, a higher value of the saving rate results in a larger steady-state capital stock.
- Overall, when s increases, the economy is suddenly below its steady state. **Hence, the capital stock will grow over time, eventually approaching the new, higher steady state.**

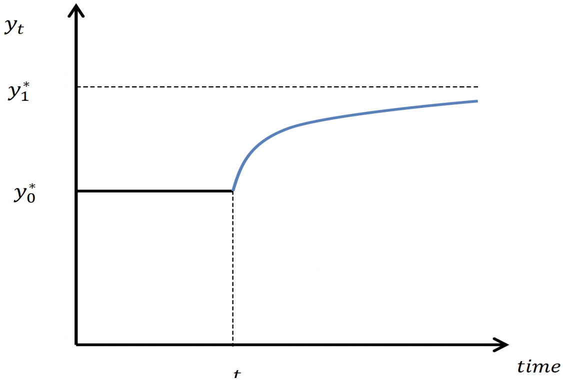
Increasing Saving s



Increasing Saving and impact on .

- Suppose that the economy initial sits in a steady state, where the saving rate is s_0.
- Then, in period t, the saving rate increases to $s_1 > s_0$ and is forever expected to remain at s_1.
- In period t (the period in which s increases), nothing happens to the capital stock per worker.
- It starts getting bigger in period $t + 1$ and continues to get bigger, though at a slower rate as time passes.
- Eventually, it will approach the new steady state associated with the higher saving rate, k_1^*.

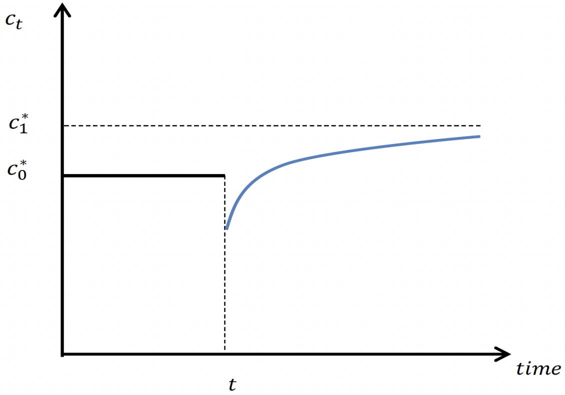
Increasing Saving s



INCREASING SAVING AND IMPACT ON .

- Once we have the dynamic path of k_t, we can back out the dynamic paths of all other variables in the model.
- Since $y_t = Af\left(k_t\right)$, output will follow a similar looking path to k_t– it will not change in period t, and then will grow for a while, approaching a new, higher steady-state value.
- Note that the response graphs in Figure are meant to be qualitative and are not drawn to scale, so do not interpret anything about the magnitudes of the responses of k_t and other variables.

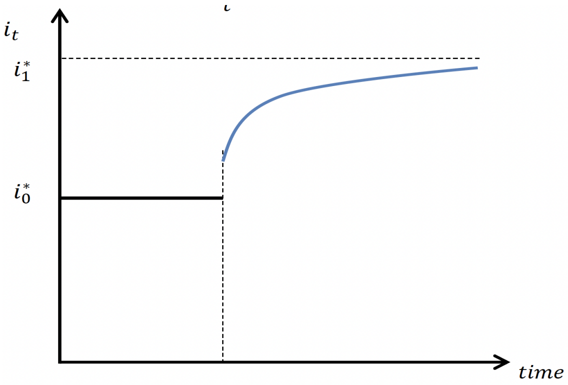
Increasing Saving s



INCREASING SAVING AND IMPACT ON .

- Since $c_t = (1-s)y_t$, consumption per worker must initially decline in the period in which the saving rate increases.
- Effectively, the "size of the pie," y_t, doesn't initially change, but a smaller part of the pie is being consumed.
- After the initial decrease, consumption will begin to increase, tracking the paths of k_t and y_t.
- Whether consumption ends up in a higher or lower steady state than where it began is unclear, though we have drawn the figure where consumption eventually ends up being higher.

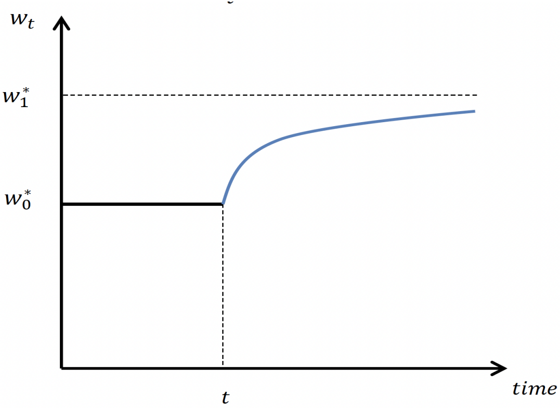
Increasing Saving s



INCREASING SAVING AND IMPACT ON .

- Investment is $i_t = sy_t$.
- Hence, investment per worker must jump up in the period in which the saving rate increases.
- It will thereafter continue to increase as capital accumulates and transitions to the new steady state.

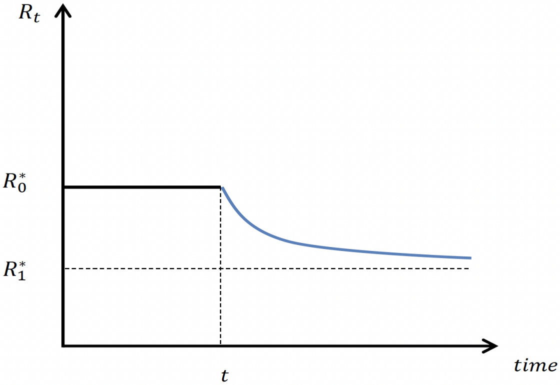
Increasing Saving s



Increasing Saving and impact on .

- w_t will not react in period t, but will follow a similar dynamic path as the other variables thereafter.
- This happens because of our underlying assumption that $F_{LK} > 0$ - so having more capital raises the marginal product of labour, and hence the wage.

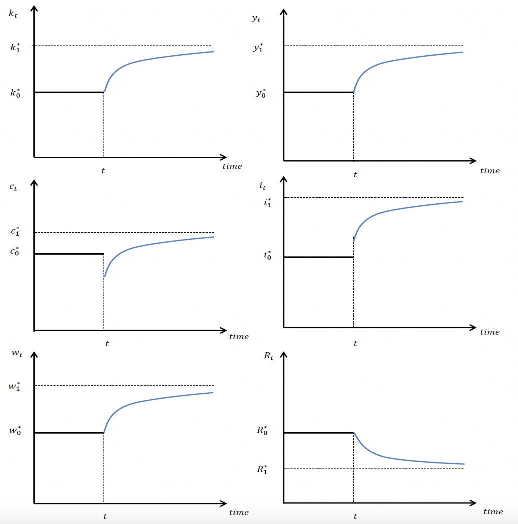
Increasing Saving s



INCREASING SAVING AND IMPACT ON .

- The rental rate on capital, R_t, will not react in the period s increases but then will decrease thereafter.
- This is driven by the assumption that $F_{KK} < 0$.
- As capital accumulates following the increase in the saving rate, the marginal product of capital falls.
- It will continue to fall and eventually ends up in a lower steady state.

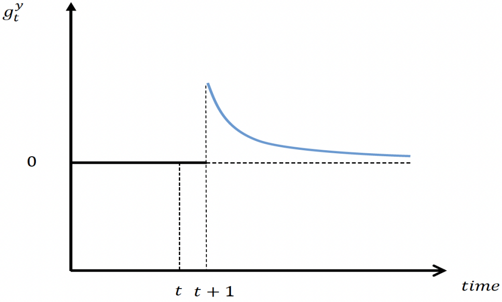
Increasing Saving s



INCREASING SAVING AND IMPACT ON .

- Make a summary by your self.

Increasing Saving s



INCREASING SAVING AND IMPACT ON .

- What happens to the growth rate of output after an increase in s?
- Using the approximation that the growth rate is approximately the log first difference of a variable, define $g_t^y = \ln y_t - \ln y_{t-1}$ as the growth rate of output.
- Since output per worker converges to a steady state, in steady state output growth is 0.
- In the period of the increase in s, nothing happens to output, so nothing happens to output growth.
- Since output begins to increase starting in period $t+1$, output growth will jump up to some positive value in period $t+1$.
- It will then immediately begin to decrease (though remain positive), as we transition to the new steady state, in which output growth is again zero.

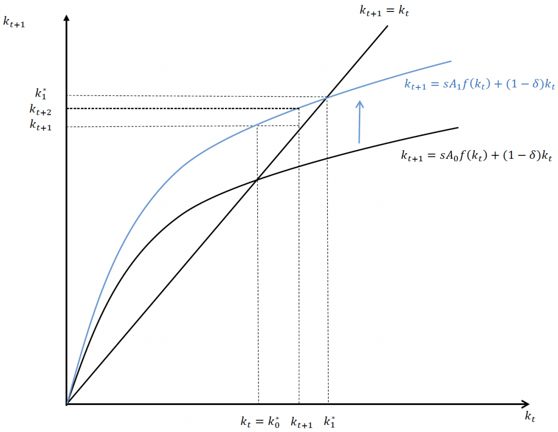
Increasing Saving s

- The analysis portrayed above has an important and powerful implication - **output will not forever grow faster if an economy increases the saving rate.**
- There will be an initial burst of higher-than-normal growth immediately after the increase in s, but this will dissipate and the economy will **eventually return to a steady state with no growth.**

Increasing Productivity A

- A is to capture improvements in the technological knowhow of the economy. There is little doubt that today human societies know how to produce many more goods and can do so more efficiently than in the past.
- Next, consider the experiment of an exogenous increase in A. Let assume there is new AI technology like ChatGPT and increase productivity in production.
- In particular, suppose that, prior to period t, the economy sits in a steady state associated with A_0.
- Then, suppose that A increases to A_1. This change is permanent, so all future values of A will equal A_1.
- How will this impact the economy?
- Can the Solow model generate sustained growth without technological progress? The answer is YES
- ****But how?****

Increasing Saving A



Increasing Productivity and impact on .

- In terms of the main graph plotting k_{t+1} against k_t, this has very similar effects to an increase in s.
- For every value of k_t, k_{t+1} will be higher when A is higher.
- In other words, the curve shifts up (and has a steeper slope at every value of k_t). This is shown in Figure 5.9 below.
- We can use the figure to think about the dynamic effects on k_t.
- Since the curve is shifted up relative to where it was with $A_{0,t}$, we know that the curve will intersect the 45 degree line at a higher value, meaning that the steady state capital stock will be higher, $k_1^* > k_0^*$.
- In period t, nothing happens to k_t. But since the curve is now shifted up, we will have $k_{t+1} > k_t$. Capital will continue to grow as it transitions toward the new, higher steady state.

Increasing Saving A

- How does one know that there is no long run effect of A on R_t ?
- Suppose that the production function is Cobb-Douglas. Then the expression for steady state R^* is:

$$R^* = \alpha A {k^*}^{\alpha-1}.$$

- Plug in the steady state expression for k^* :

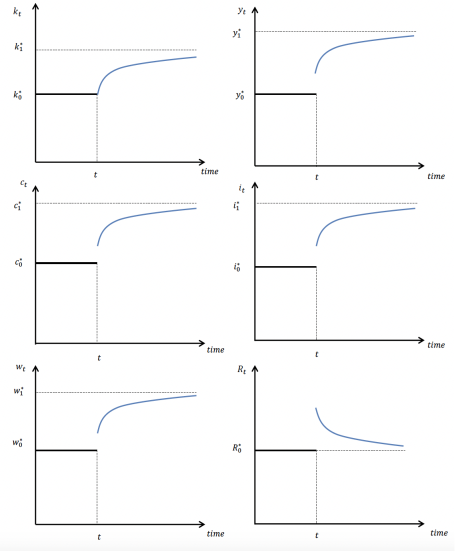
$$R^* = \alpha A \left(\frac{sA}{\delta}\right)^{\frac{\alpha-1}{1-\alpha}}.$$

- The exponent here is -1, which means we can flip numerator and denominator. In other words, the A cancel out, leaving:

$$R^* = \frac{\alpha\delta}{s}.$$

- As mentioned above, we can think about there being two effects of an increase in A on the variables of the model.
- There is the direct effect, which is what happens holding k_t fixed. Then there is an indirect effect that comes about because higher A triggers more capital accumulation.
- This indirect effect is qualitatively the same as what happens when s changes. What differs across the two cases is that the increase in A causes an immediate effect on the variables in the model.

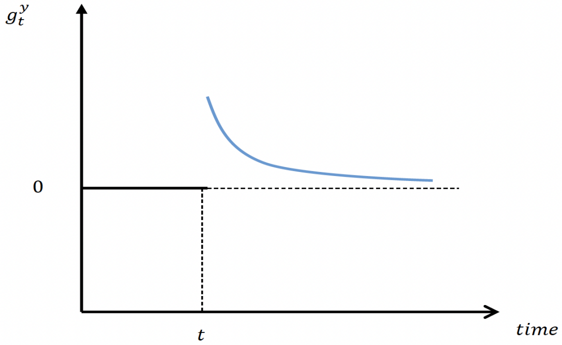
Increasing Saving A



INCREASING PRODUCTIVITY AND IMPACT ON .

- k_t does not jump in period t, but grows steadily, eventually approaching a new higher steady state.
- Since $y_t = Af(k_t)$, y_t jumps up initially in period t. This is the "direct effect" of the increase in A (unlike the case of an increase in s).
- But y_t continues to grow thereafter, due to the accumulation of more capital. It eventually levels off to a new higher steady state.
- c_t and i_t follow similar paths as y_t, since they are just fixed fractions of output.
- The wage jumps up initially, and then continues to grow as capital accumulates.
- R_t, initially jumps up. This is because higher A makes the marginal product of capital higher. But as capital accumulates, the marginal product of capital starts to decline.
- R_t eventually settles back to where it began - there is no effect of A on the steady state value of R_t

Increasing Saving A



INCREASING PRODUCTIVITY AND IMPACT ON .

- As in an increase in s, we can think about what happens to the growth rate of output following a permanent increase in A.
- Qualitatively, it looks similar to Figure in page-39, but the subtle difference is that output growth jumps up immediately in period t,
- Whereas in the case of an increase in s there is no increase in output growth until period $t+1$.
- In either case, the extra growth eventually dissipates, with output growth ending back up at zero.

Summary

- In summary, based on the Solow Model, the economy naturally converges to a steady state in which there is no growth.
- In other words, the basic model presented in this section gives us the insight of where that steady-state growth must come **from**
- Sustained growth cannot come from capital accumulation per se. As shown above, an increase in s triggers temporarily high growth because of capital accumulation, but this dissipates and eventually growth settles back down to zero.
- Even if an economy repeatedly increased its saving rate, it would eventually run out of room to do so, so even repeated increases in s cannot plausibly generate sustained growth over long periods of time.
- What about changes in A ? It is true that a one time change in A only generates a temporary burst of output growth which is magnified due to capital accumulation as the economy transitions to a new steady state.
- But unlike changes in s, there is no logical limit on productivity repeatedly increasing over time.
- This means that continual **productivity improvements could plausibly generate sustained growth in output per capita over time.**

Growth Rate in Steady State

What is the growth rate of k_t,

$$\gamma_{k_t} = \frac{k_{t+1} - k_t}{k_t} = ?$$

$$k_{t+1} = \frac{sAk_t^{\alpha} + (1-\delta)k_t}{1 + \boxed{n}}$$

we assume n=0

$$\gamma_{k_t} = \frac{\frac{sAk_t^{\alpha} + (1-\delta)k_t}{1+n} - k_t}{k_t}$$

$$\gamma_{k_t} = \frac{sAk_t^{\alpha-1} - (\delta + n)\, k_t}{1+n}$$

Growth Rate in Steady State

To find the steady state you can set $\gamma k_t = 0$ and solve for k

$$\gamma_{k_t} = \frac{sAk_t^{\alpha-1} - (\delta + n)\, k_t}{1+n}$$

$$sAk^{*\alpha-1} = (\delta + n)\, k^*$$

$$k^* = \left(\frac{sA}{\delta + n}\right)^{1/(1-\alpha)}$$

A very special case: The Golden Rule

The Golden Rule

- What is the effect of an increase in the saving rate on the steady state level of consumption per worker?

$$c^* = (1 - s)y_t^*$$

The Golden Rule

- What is the effect of an increase in the saving rate on the steady state level of consumption per worker?

$$c^* = (1 - s)y_t^*$$

- There is an ambiguous effect of an increase in the saving rate on the steady state level of consumption per worker.
- Increasing the saving rate always results in an increase in k^*, and hence an increase in y^*.
- In other words, a higher saving rate always results in a bigger "size of the pie."
- But increasing the saving rate means that households are consuming a smaller fraction of the pie.
- **Which of these effects dominates is unclear.**

The Golden Rule

- We can see these different effects at work in the expression for the steady state consumption per worker:

$$c^* = (1-s)Af\left(k^*\right)$$

- A higher s increases $f\left(k^*\right)$ (since a higher s increases k^*), but reduces $1-s$.
- We can see that if $s=0$, then $c^*=0$. This is because if $s=0$, then $k^*=0$, so there is nothing at all available to consume.
- Conversely, if $s=1$, then we can also see that $c^*=0$. While $f\left(k^*\right)$ may be big if $s=1$, there is nothing left for households to consume.
- We can therefore intuit that c^* must be increasing in s when s is near 0 , and decreasing in s when s is near 1 .

Golden Rule Saving Rate s

- *We can therefore intuit that c^* must be increasing in s when s is near 0, and decreasing in s when s is near 1 .*
- **But where is the near 0 or near 1?**
- For example, at the moment the US saving rate is about 25%. Suppose you were to calibrate the "*golden rule*" saving rate for the United States.
- What value would you use?
- In other words, what should the US increase or decrease the saving rate in order to consume more?

Golden Rule Saving Rate s

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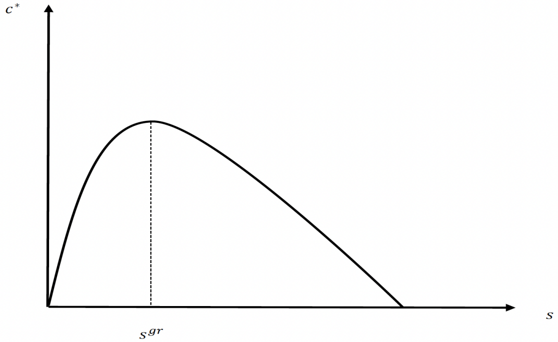


GOLDEN RULE SAVING AND CONSUMPTION .

Golden Rule Saving Rate s

We can derive an expression that must hold at the Golden Rule using the total derivative. The steady-state capital stock is implicitly defined by:

$$sAf\left(k^*\right) = \delta k^*. \tag{27}$$

Totally differentiate this expression about the steady state, allowing s to vary:

$$sAf'\left(k^*\right)dk^* + Af\left(k^*\right)ds = \delta dk^*. \tag{28}$$

Solve for dk^* :

$$\left[sAf'\left(k^*\right) - \delta\right]dk^* = -Af\left(k^*\right)ds \tag{29}$$

Steady-state consumption is implicitly defined by:

$$c^* = Af\left(k^*\right) - sAf\left(k^*\right). \tag{30}$$

Totally differentiate this expression:

$$dc^* = Af'\left(k^*\right)dk^* - sAf'\left(k^*\right)dk^* - Af\left(k^*\right)ds. \tag{31}$$

Golden Rule Saving Rate s

Re-arranging terms:

$$dc^* = \left[Af'\left(k^*\right) - sAf'\left(k^*\right)\right] dk^* - Af\left(k^*\right) ds \tag{32}$$

From Eq-(29), we know that $-Af\left(k^*\right) ds = \left[sAf'\left(k^*\right) - \delta\right] dk^*$ Plug this into Eq-(32) and simplify:

$$dc^* = \left[Af'\left(k^*\right) - \delta\right] dk^* \tag{33}$$

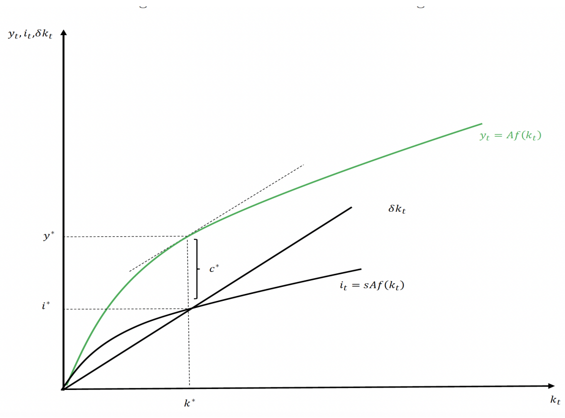
Divide both sides of Eq-(33) by ds :

$$\frac{dc^*}{ds} = \left[Af'\left(k^*\right) - \delta\right] \frac{dk^*}{ds} \tag{34}$$

For s to maximize c^*, it must be the case that $\frac{dc^*}{ds} = 0$. Since $\frac{dk^*}{ds} > 0$, this can only be the case if:

$$Af'\left(k^*\right) = \delta \tag{35}$$

Golden Rule Saving Rate s



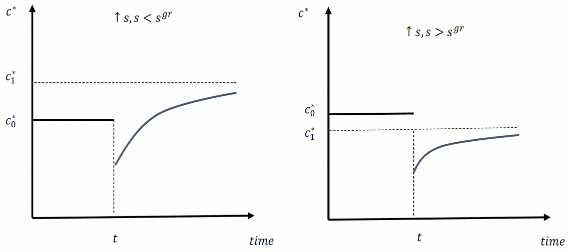
GOLDEN RULE SAVING AND CONSUMPTION .

- Figure graphically gives a sense of why (35) must hold.
- For a given k_t, the vertical distance between y_t and i_t is c_t, consumption.
- At the steady state, we must have $sAf\left(k^*\right) = \delta k^*$; in other words, the steady state is where the plot of i_t crosses the plot of δk_t.
- Steady state consumption is given by the vertical distance between the plot of y_t and the plot of i_t at this k^*.
- **The Golden rule saving rate is the s that maximizes this vertical distance.**
- Graphically, this must be where the plot of $y_t = Af\left(k_t\right)$ is tangent to the plot of δk_t (where $sAf\left(k_t\right)$ crosses in the steady state). To be tangent, the slopes must equal at that point, so we must have $Af'\left(k_t\right) = \delta$ at the Golden rule.
- In other words, at the Golden Rule, the marginal product of capital equals the depreciation rate on capital.

Why $Af'(k^*) = \delta$ but not $Af'(k^*) > \delta$. or $Af'(k^*) < \delta$?

- Suppose that, for a given s, that $Af'(k^*) > \delta$.
- This means that raising the steady state capital stock (by increasing s) raises output by more than it raises steady state investment (the change in output is the marginal product of capital, $Af'(k^*)$, and the change in steady state investment is δ).
- This means that consumption increases, so this s cannot be the s which maximizes steady state consumption.
- In contrast, if s is such that $Af'(k^*) < \delta$, then the increase in output from increasing the steady state capital stock is smaller than the increase in steady state investment, so consumption declines.
- Hence that s cannot be the s which maximizes steady state consumption. Only if $Af'(k^*) = \delta$ is s consistent with steady state consumption being as big as possible.

Dynamic effects of an increase in s,



Golden Rule Saving and Consumption .

- An increase in s always results in an immediate reduction in c_t in the short run - a larger fraction of an unchanged level of income is being saved.
- After the initial short-run decline, c_t starts to increase as the capital stock increases and hence income increases.
- Whether the economy ends up with more or less consumption, in the long run, depends on where s was initially relative to the Golden Rule.
- If initially $s < s^{gr}$, then a small increase in s results in a long run increase in consumption in the new steady state. If $s > s^{gr}$, then an increase in s results in a long run decrease in consumption in the new steady state.
- Prior to t, the economy sits in a steady state. Then, in period t, there is an increase in s.
- Qualitatively, the time path of c_t looks the same whether we are initially above or below the Golden Rule. What differs is whether c_t ends up higher or lower than where it began.

Golden Rule Saving Rate s Cobb-Douglas

Find the Golden Rule saving rate (s such that the steady state is k^{GR})

$\longrightarrow$ max $\bar{c} = (1-s)y^*$ with respect to s $\left[c^* = (1-s)\, A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}\right]$

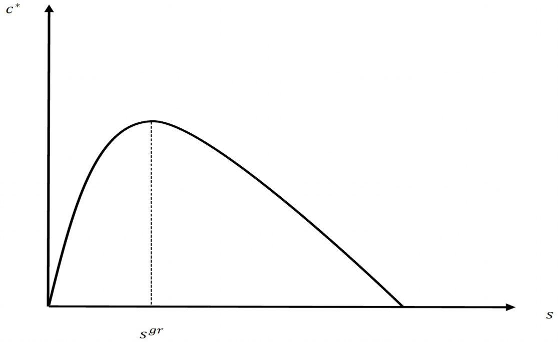
$$(1-s)\frac{\alpha}{1-\alpha}\frac{A^{\frac{1}{1-\alpha}}}{(\delta+n)^{\frac{\alpha}{1-\alpha}}}s^{\frac{2\alpha-1}{1-\alpha}} = A^{\frac{1}{1-\alpha}}\left(\frac{s}{(\delta+n)}\right)^{\frac{\alpha}{1-\alpha}}$$

$$\frac{\alpha}{1-\alpha} = \frac{s}{1-s}$$

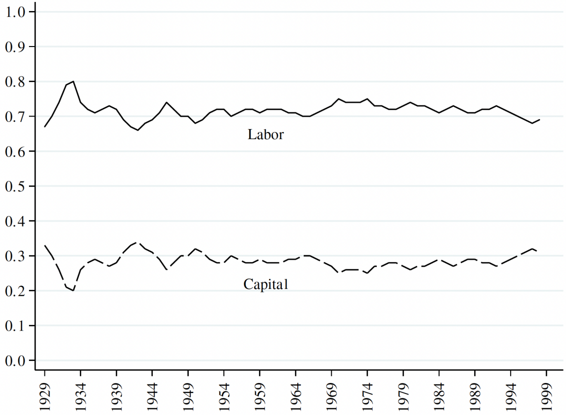
which implies:

$$\rightarrow s^{GR} = \alpha$$

Dynamic effects of an increase in s,



Golden Rule Saving and Consumption . $s^{GR} = \alpha$



Capital and labor share in the U.S. GDP: Daron Acemoglu.

If so why some countries are still too rich?

Cross-Country Income Differences

- There were and are enormous differences in GDP per capita across countries and over the time. However, we still do not have enough evidence to understand what can account for these large differences.
- **Solow:** poor countries are poor not because they lack capital, but because they are relatively unproductive.

Set up Model

- There are two countries and both have a Cobb-Douglas production function and the parameters α, δ are the same across countries.
- Three things are different - productivity levels (i.e. $A_1 \neq A_2$, saving rates (i.e. $s_1 \neq s_2$), and initial endowments of capital per worker (i.e. $k_{1,t} \neq k_{2,t}$).

Cross-Country Income Differences

Set up Model

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Cross-Country Income Differences

- At any point in time, output per capita in the two countries is:

$$y_{1,t} = A_1 k_{1,t}^{\alpha}$$

$$y_{2,t} = A_2 k_{2,t}^{\alpha}$$

- The ratio of output per capita in the two countries is:

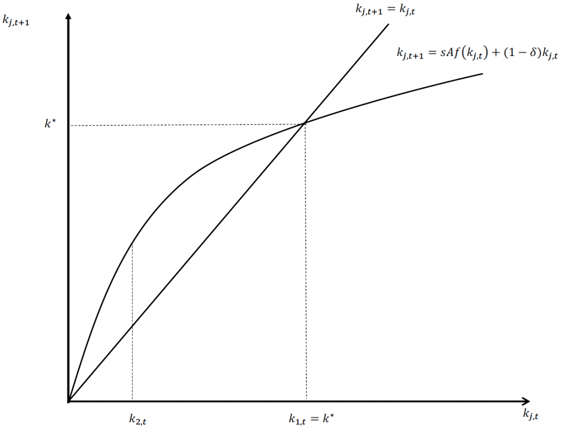
$$\frac{y_{1,t}}{y_{2,t}} = \frac{A_1}{A_2} \left(\frac{k_{1,t}}{k_{2,t}} \right)^{\alpha} \tag{36}$$

- From Eq-(36), we can see that there are really only two reasons why the countries could have different levels of output per capita - either the **productivity** levels are different or the **capital per worker** are different.
- **Which one is it, and what are the policy implications?**

Cross-Country Income Differences

- Assume that countries are fundamentally the same; the same productivity levels and the same saving rates.
- This means that the only thing that could potentially differ between the two countries is the initial endowment of capital stocks per worker.
- The steady-state capital stock per worker, $k^* = \left(\frac{sA}{\delta+n}\right)^{1/(1-\alpha)}$ and hence the steady-state level of output per worker $y^* = Ak^{*\alpha}$, does not depend on the initial endowment of capital.
- Starting from any (non-zero) initial endowment of capital, the economy converges to a steady state where the steady state is determined by productivity, the saving rate, the curvature of the production function, and the depreciation rate.
- If we are supposing that all of the features are the same for the two countries under consideration, it means that these countries will have the same steady state capital stocks per worker and hence identical steady state levels of output per capita.
- One hypothesis for why some countries are richer than others is that those countries are initially endowed with more capital than others.

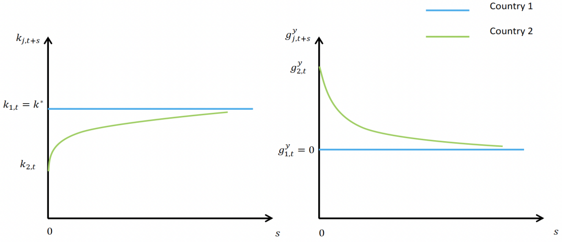
Cross-Country Income Differences



Country 1 Initially Endowed With More Capital than Country 2.

- Let country 1 be relatively rich and country 2 relatively poor, but the countries have identical productivity levels and saving rates.
- The main equation of the Solow model is the same for both countries, so $k_{j,t+1} = sAf\left(k_{j,t}\right) + (1-\delta)k_{j,t}$.
- Country 1 starts with a capital stock equal to the steady state capital stock, so $k_{1,t} = k^*$. Country 2 starts out with $k_{2,t} < k^*$. In this scenario, country 2 is poor because it is initially endowed with little capital.
- Because country 2 is initially endowed with less capital than country 1, it will initially produce less output than country 1.
- Country 2'capital will grow over time, whereas the capital stock for country 1 will be constant.
- This means that, if country 2 is poor relative to country 1 only because it is initially endowed with less capital than country 1, it will grow faster and will eventually catching up to country 1

Cross-Country Income Differences



COUNTRY 1 INITIALLY ENDOWED WITH MORE CAPITAL THAN COUNTRY 2.

- Figure shows dynamic paths of the capital stock in each country from the assumed initial starting positions - i.e. it plots $k_{j,t+s}$ for $j = 1, 2$ and $s \geq 0$.
- Since it starts in steady state, country 1's capital per worker simply remains constant across time.
- Country 2 starts with a capital below steady state, but its capital should grow over time, eventually catching up to country 1.
- In the right panel, we plot the growth rate of output per worker in each country across time, $g^y_{j,t+s}$. Because it starts in steady state, country 1's growth rate will simply remain constant at zero
- In contrast, country 2 will start out with a high growth rate - this is because it is accumulating capital over time, which causes its output to grow faster.
- Eventually, country 2's growth rate should settle down to 0, in line with country 1's growth rate.
- **the Solow model predicts convergence if two countries have the same saving rates and the same levels of productivity.**

Cross-Country Income Differences

- Assume that two countries have the same α and same δ, but potentially differ in terms of saving rates and productivity levels. Under these assumptions, the steady state output per worker in country $j = 1, 2$ is given by:

$$y_j^* = A_j^{\frac{1}{1-\alpha}} \left(\frac{s_j}{\delta}\right)^{\frac{\alpha}{1-\alpha}} \text{ for } j = 1, 2$$

- The ratio of steady state output per capita across the two economies is then:

$$\frac{y_1^*}{y_2^*} = \left(\frac{A_1}{A_2}\right)^{\frac{1}{1-\alpha}} \left(\frac{s_1}{s_2}\right)^{\frac{\alpha}{1-\alpha}}$$

- Very persistent differences in output per capita across the two countries can be driven either by differing productivity levels (i.e. $A_1 \neq A_2$) or different saving rates (i.e. $s_1 \neq s_2$).
- **Can differences in s alone account for large and persistent differences in output per capita?**
- To see this concretely, suppose that the two countries in question have the same level of productivity, i.e. $A_1 = A_2$. From (7.5), their relative outputs are then:

$$\frac{y_1^\star}{y_2^\star} = \left(\frac{s_1}{s_2}\right)^{\frac{\alpha}{1-\alpha}}. \tag{37}$$

Cross-Country Income Differences: The Role of Saving

- Our objective is to see how different saving rates across countries would have to be to account for a given difference in per capita output.
- Let's consider a comparison between a "middle-income" country like Mexico and the U.S. In 2011 the US GDP per capita is $\$42,426$ and Mexico $\$12,648$
- Then $\frac{y_1^*}{y_2^*} = 4$. Let's then solve Eq-(37) for s_2 in terms of s_1, given this income difference. We obtain:

$$s_2 = 4^{\frac{\alpha-1}{\alpha}} s_1. \tag{38}$$

- A plausible value of α is $1/3$. With this value of α, $4^{\frac{\alpha-1}{\alpha}} = 0.0625$.
- What Equation Eq-(38) then tells us is that to account for Mexican GDP that is one-fourth of the U.S.'s, the Mexican saving rate would have to be 0.0625 times the U.S. saving rate - i.e. the Mexican saving rate would have to be about 6 per cent of the US saving rate.
- If the U.S. saving rate is $s_1 = 0.2$, this would then mean that the Mexican saving rate would have to be $s_2 = 0.0125$.
- This means that Mexico would essentially have to be saving nothing if the only thing that differed between Mexico and the U.S. was the saving rate.
- **This is not plausible.**

Cross-Country Income Differences: The Role of Productivity

- This result simply says that productivity must be the primary driver of long-run growth, not saving rates.
- But - are there large differences in productivity across countries? We can come up with empirical measures of A across countries by assuming a functio form for the production function. In particular, suppose that the production function is Cobb-Douglas:

$$Y_t = AK_t^{\alpha} N_t^{1-\alpha}. \tag{39}$$

- Take natural logs of eq-(39) and re-arrange terms to yield:

$$\ln A = \ln Y_t - \alpha \ln K_t - (1-\alpha) \ln L_t.$$

- If we can observe empirical measures of Y_t, K_t, and L_t across countries, and if we are willing to take a stand on a value of α, we can recover an empirical estimate of $\ln A$.
- In essence, $\ln A$ is a residual - it is the part of output which cannot be explained by observable capital and labour inputs.
- Consequently, this measure of $\ln A$ is sometimes called the "Solow residual." It is also called "total factor productivity" (or TFP for short).

Cross-Country Income Differences: The Role of Productivity

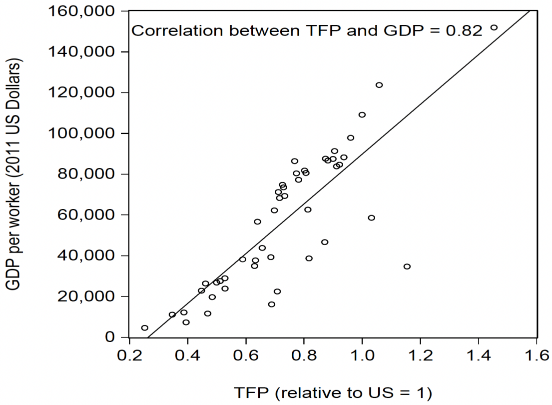
$$\frac{y_{t+1}}{y_t} = \frac{As^{\alpha} y_t^{\alpha-1}}{(1+n)^{\alpha}}$$

taking logs: $$\ln \frac{y_{t+1}^i}{y_t^i} = \ln(1+\gamma_{y_t}^i)$$
$$= \ln A + \alpha \ln s^i - (1-\alpha)\ln y_t^i - \alpha \ln\left(1+n^i\right)$$

where $\ln(1+\gamma_{y_t}^i) \approx \gamma_{y_t}^i$ and $\ln(1+n^i) \approx n^i$

- A is the constant term - β_0
- s is positively correlated with γ_{y_t}
- n is negatively correlated with γ_{y_t}
- y_t is negatively correlated with $\gamma_{y_t} \implies \beta$ convergence

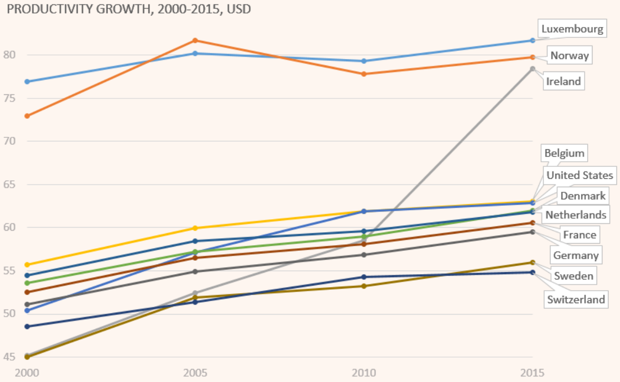
Cross-Country Income Differences: The Role of Productivity



TFP AND GDP PER WORKER IN 2011.

- Figure presents a scatter plot of GDP per worker (measured in 2011 U.S. dollars) against TFP for the year 2011.
- There is an extremely tight relationship between TFP and GDP per capita. In particular, the correlation between the two series is 0.82. By and large, rich countries (countries with high GDP per worker) have high TFP (i.e. are very productive) and poor countries have low TFP (i.e. are not very productive).
- In summary, the Solow model suggests that the best explanation for large differences in standards of living is that there are large differences in productivity across countries.
- If some countries were poor simply because they were initially endowed without much capital, the Solow model would predict that these countries would converge to the GDP per capita of richer countries.

TOP-10 MOST PRODUCTIVE COUNTRIES

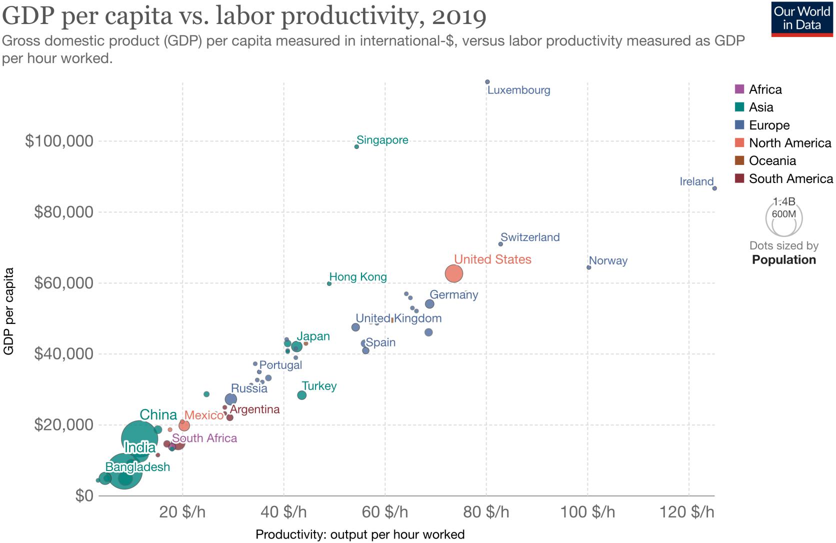


Top-10 Most Productive Countries (GDP per hour worked, USD, 2015)		
1	Luxembourg	81.7
2	Norway	79.8
3	Ireland	78.4
4	Belgium	63.0
5	United States	62.9
6	Denmark	61.9
7	Netherlands	61.8
8	France	60.6
9	Germany	59.5
10	Sweden	56.0

Source: OECD (2000-2015), Level of GDP per capita and productivity (extracted on 02 March 2017).

GDP per capita vs. labor productivity, 2019

Gross domestic product (GDP) per capita measured in international-$, versus labor productivity measured as GDP per hour worked.



Source: Data compiled from multiple sources by World Bank, Feenstra et al. (2015), Penn World Table 10.0

Solow Model and The Data

Figure I: GDP per capita (relative to the US) over time

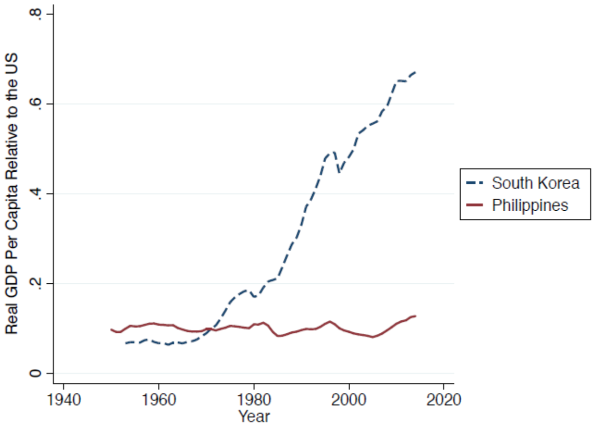
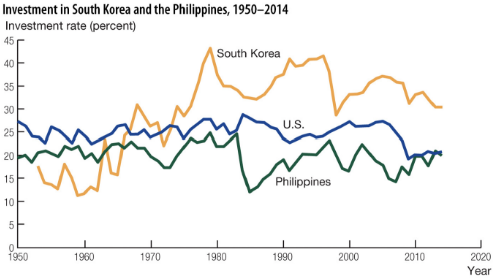


Figure II: Saving rate over time



Solow Model and The Data

- Is the growth experienced in the **Philippines** consistent with the predictions of the Solow model? Discuss why or why not.

Figure I: GDP per capita (relative to the US) over time

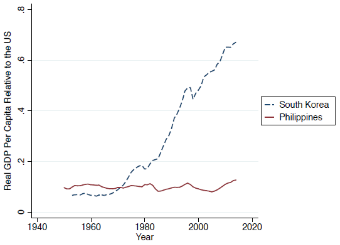
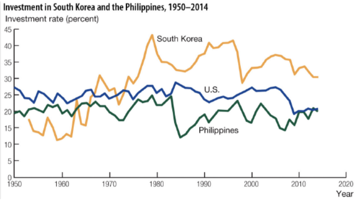


Figure II: Saving rate over time

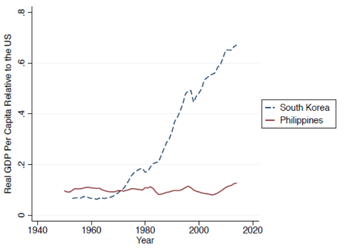


Solow Model and The Data

- Is the growth experienced in the **Philippines** consistent with the predictions of the Solow model? Discuss why or why not.

- **Yes,** if the Philippines and the US have been on their respective steady states, which involves different levels of GDP per capita (because of differences in the investment rate and, more importantly, TFP) but similar growth rates.

Figure I: GDP per capita (relative to the US) over time



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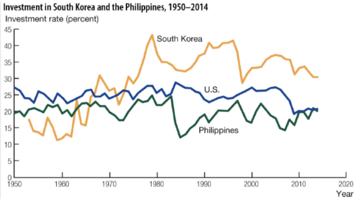


Solow Model and The Data

- Is the growth experience in **South Korea** consistent with the predictions of the Solow model? Discuss why or why not.

Figure I: GDP per capita (relative to the US) over time

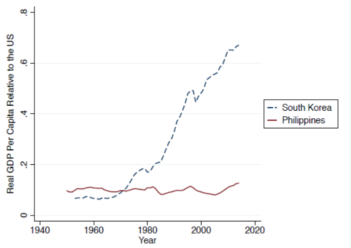
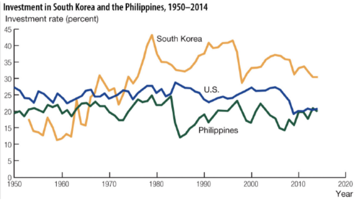


Figure II: Saving rate over time



Solow Model and The Data

- Is the growth experience in **South Korea** consistent with the predictions of the Solow model? Discuss why or why not.

  * Yes. The graphs are consistent with the view that South Korea was on a steady state before the 1970s, when the saving rate more than doubles. After that, South Korea has been converging to a higher steady state and, because of this, growing faster compared to the US.

Figure I: GDP per capita (relative to the US) over time

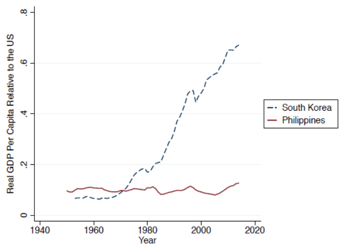
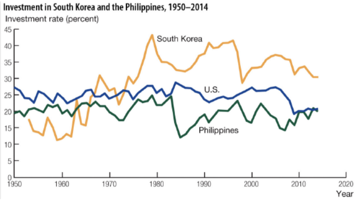


Figure II: Saving rate over time



Solow Model and The Data

- Today, the saving rate in South Korea is about 10 percentage points higher compared to the US. At the same time, income per capita in South Korea is about 70% of the US one. Can the Solow model reconcile these two facts?

Figure I: GDP per capita (relative to the US) over time

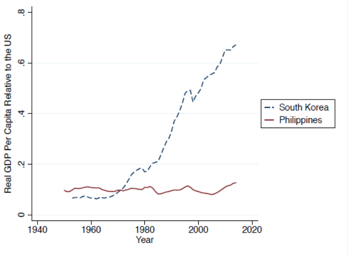
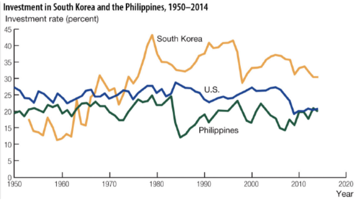


Figure II: Saving rate over time



Solow Model and The Data

- Today, the saving rate in South Korea is about 10 percentage points higher compared to the US. At the same time, income per capita in South Korea is about 70% of the US one. Can the Solow model reconcile these two facts?
  - A combination of two possibilities:
    - South Korea is still converging to its steady state, and eventually will have higher GDP per capita than the US.
    - South Korea is on a lower steady state because of a lower TFP, which more than compensate the higher saving rate.

Figure I: GDP per capita (relative to the US) over time

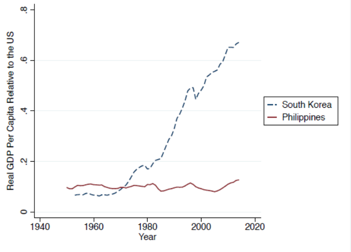


Figure II: Saving rate over time