The Solow 1956 Model

(Desecrate time version)

Constant Technology

Discrete time: $t = 0, 1, 2, \ldots, \infty$

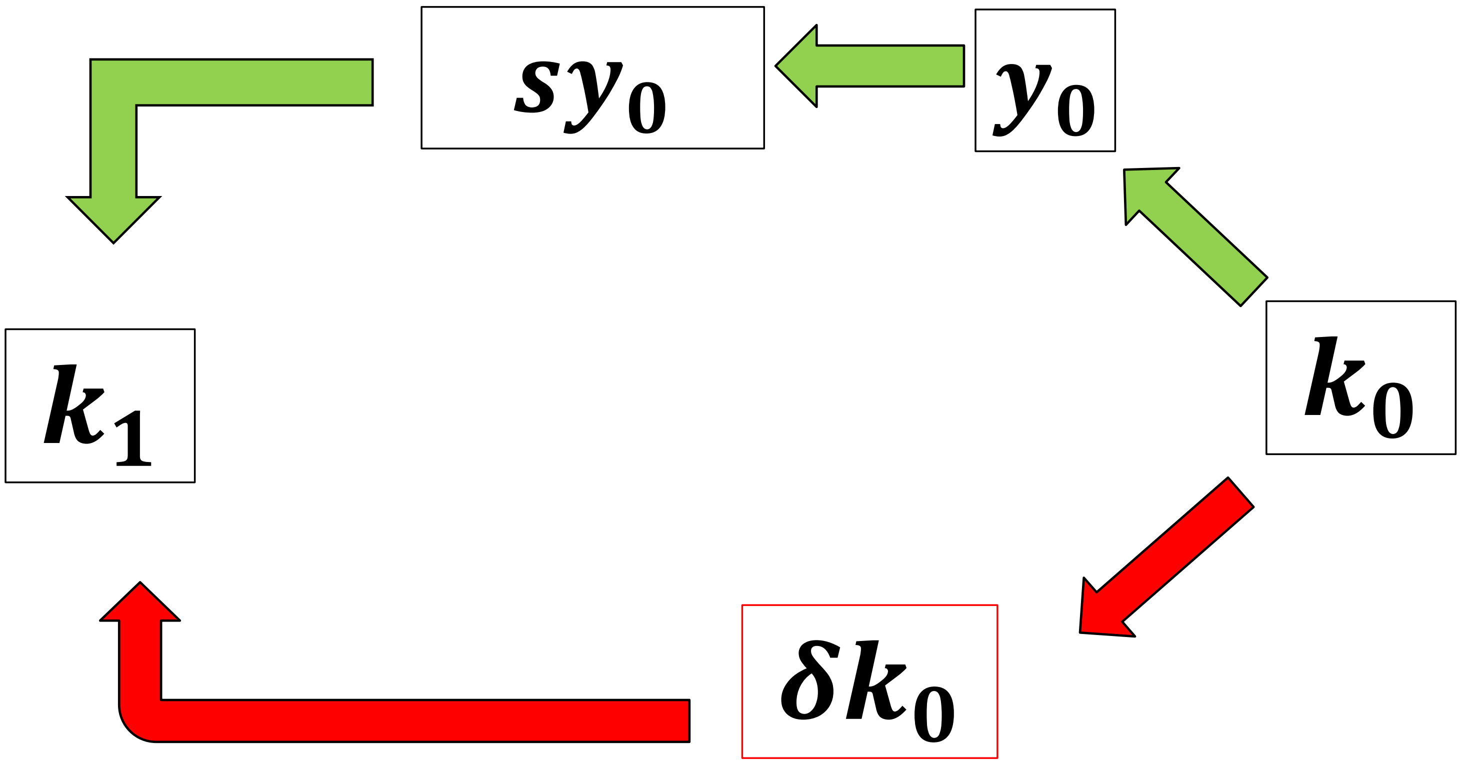
Two factors of production:

L_t - Labor

K_t - Capital

Produce one final good that can be used for consumption or as capital in the production process.

The dynamics of the Solow Model



Factor supply

Labor supply at $t+1$:

$$L_{t+1} = (1+n)L_t$$

where:

L_0 is given

$n > -1$

capital supply at $t+1$:

$$K_{t+1} = S_t + (1-\delta)K_t$$

where:

K_0 is given

S_t - aggregate saving

$\delta \in [0,1]$

A1:

$$n + \delta > 0$$

Production

output produced at time t :

$$Y_t = F(K_t, L_t)$$

A2:

$F_K(K_t, L_t), F_L(K_t, L_t) > 0,$
$F_{KK}(K_t, L_t), F_{LL}(K_t, L_t) < 0,$ for all
$K_t, L_t > 0$

$\lim_{K \to 0} F_K(K_t, L_t) = \infty$

$\lim_{K \to \infty} F_K(K_t, L_t) = 0$

$F(0, L_t) = 0$

$$\lambda F(K_t, L_t) = F(\lambda K_t, \lambda L_t)$$

$\rightarrow$

$$Y_t = F(K_t, L_t) = L_t F(K_t/L_t, 1) \equiv L_t f(k_t)$$

where $k_t \equiv K_t/L_t$

It follows from A2:

$$f(0) = 0 \quad (L_t f(0) = F(0, L_t) = 0)$$

for all $k_t > 0$:

$$f'(k_t) = F_K(K_t, L_t) > 0$$

$$F_K(K_t, L_t) = dL_t f(K_t/L_t)/dK_t = f'(k_t)$$

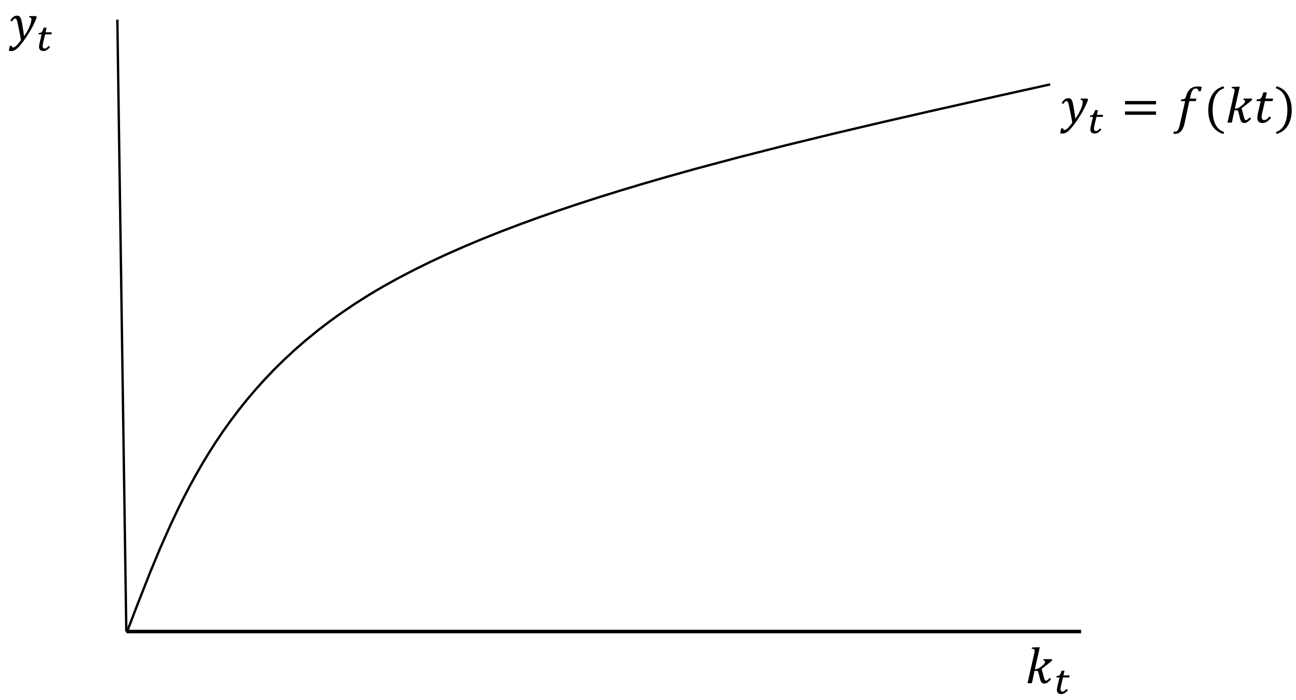
and

$$f''(k_t) = L_t F_{KK} < 0$$

$$F_{KK} = df'(K_t/L_t)/dK_t = f''(k_t)/L_t$$

$$\lim_{k_t \to 0} f'(k_t) = \infty \quad \lim_{k_t \to \infty} f'(k_t) = 0$$

Output per worker as a function of capital per worker



Moreover:

since:

$\lambda F(K_t, L_t) = F(\lambda K_t, \lambda L_t)$, differentiating with respect to λ :

$$F(K_t, L_t) = F_K K_t + F_L L_t$$

and dividing by L_t :

$$f(k_t) = f'(k_t)k_t + F_L$$

$\rightarrow$

$$f(k_t) - f'(k_t)k_t = F_L > 0$$

Remark:

In a competitive environment:

the rate of return per unit of capital (rental rate):

$$F_K = f'(k_t)$$

the wage rate per unit of labor:

$$F_L = f(k_t) - f'(k_t)k_t$$

Remark:

Since $F(K_t, L_t) = F_K K_t + F_L L_t$, it follows from differentiating with respect to L_t that

$$F_L = F_{KL} K_t + F_{LL} L_t + F_L$$

$\rightarrow$

$$F_{KL} K_t + F_{LL} L_t = 0$$

$\rightarrow$

$$F_{KL} > 0$$

Consumption, Saving and Investment

$$S_t = sY_t$$

where $s \in [0, 1]$

Capital Accumulation:

$$\begin{aligned} K_{t+1} &= S_t + (1-\delta)K_t \\ &= sL_t f(k_t) + (1-\delta)K_t \end{aligned}$$

$\rightarrow$

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{sL_t f(k_t) + (1-\delta)K_t}{L_t} \frac{L_t}{L_{t+1}}$$

$\rightarrow$

$$k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{1+n} \equiv \phi(k_t)$$

The Dynamical System

$\{k_t\}_0^\infty$ such that

$$k_{t+1} = \phi(k_t) \;\forall t$$

where k_0 is given

Let y_t be output per worker

$$y_t = Y_t/L_t = f(k_t)$$

$\rightarrow$

$\{k_t\}_0^\infty$ uniquely determines $\{y_t\}_0^\infty$

Properties of $\phi(k_t)$:

$$\phi(0) = 0$$

$$\phi'(k_t) = \frac{sf'(k_t) + (1-\delta)}{1+n} > 0 \ \forall k_t > 0$$

$$\phi''(k_t) = \frac{sf''(k_t)}{1+n} < 0 \ \forall k_t > 0$$

$$\lim_{k_t \to 0} \phi'(k_t) = \infty$$

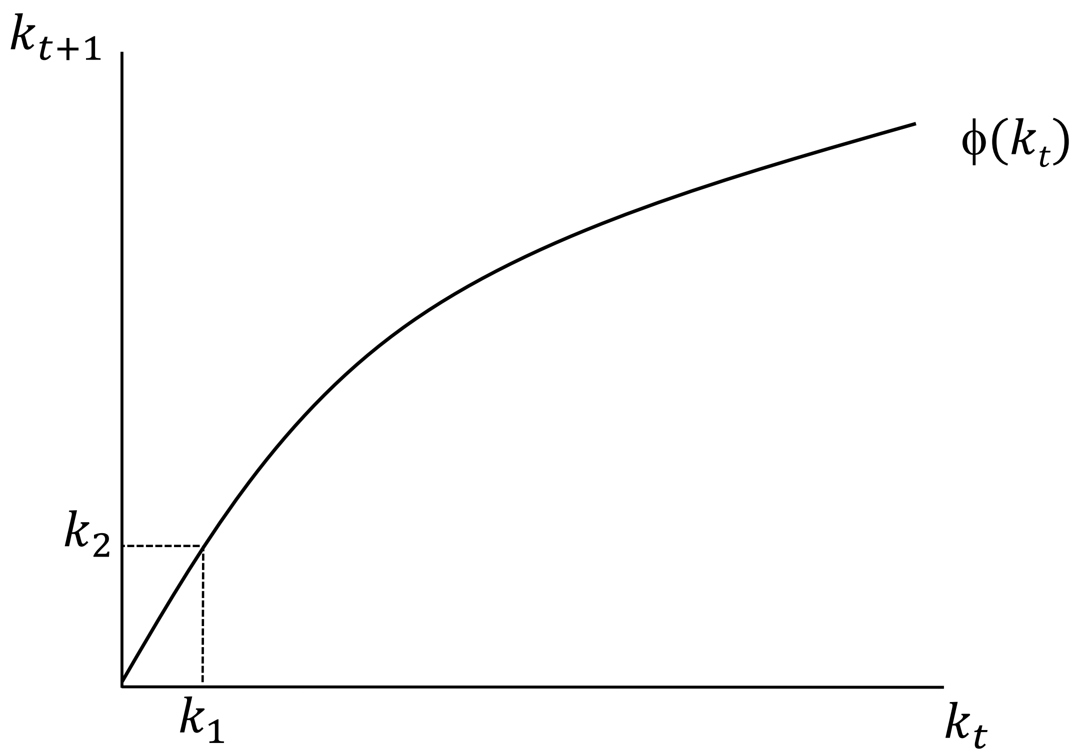
$$\lim_{k_t \to \infty} \phi'(k_t) = \frac{1-\delta}{1+n} \in [0,1)$$

Remark:

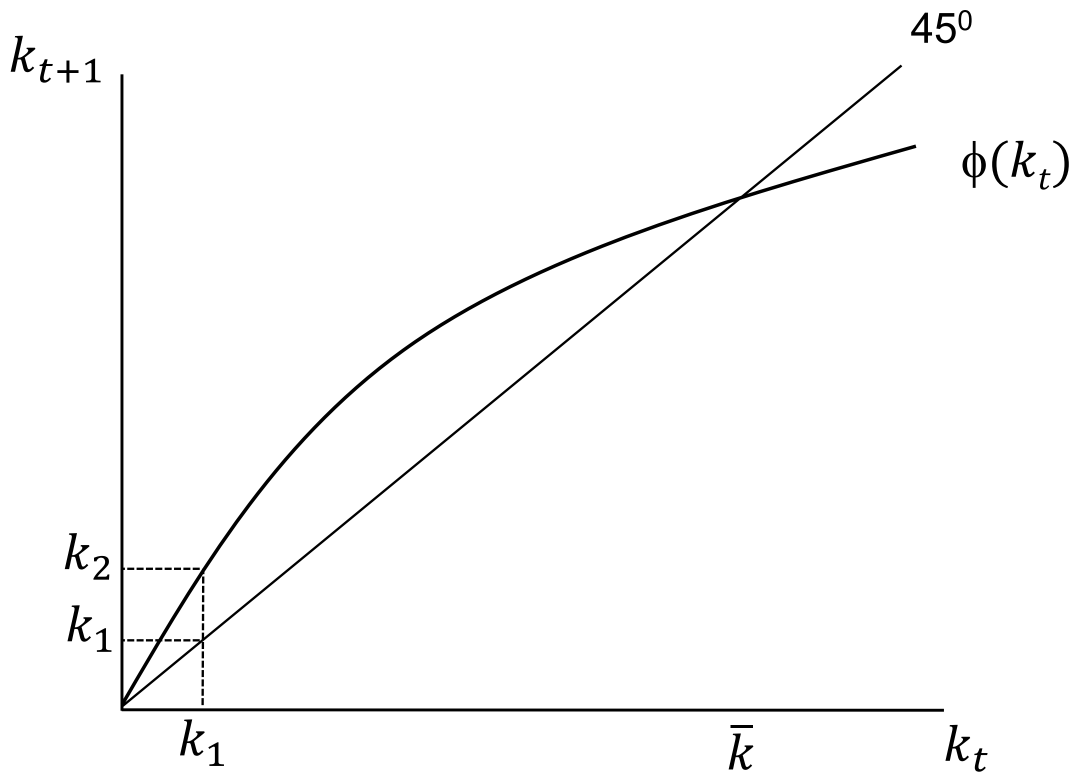
The strict concavity of $\phi(k_t)$ follows from:

1. the strict concavity of $f(k_t)$
2. saving is a constant fraction of output

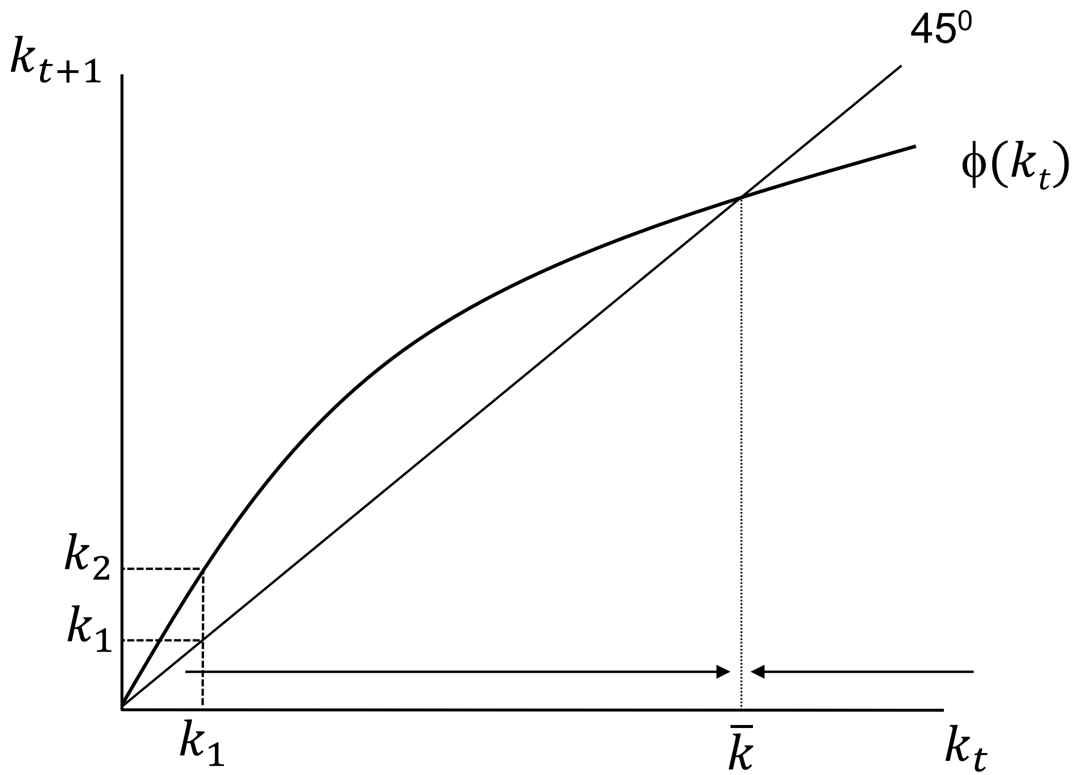
The dynamical system



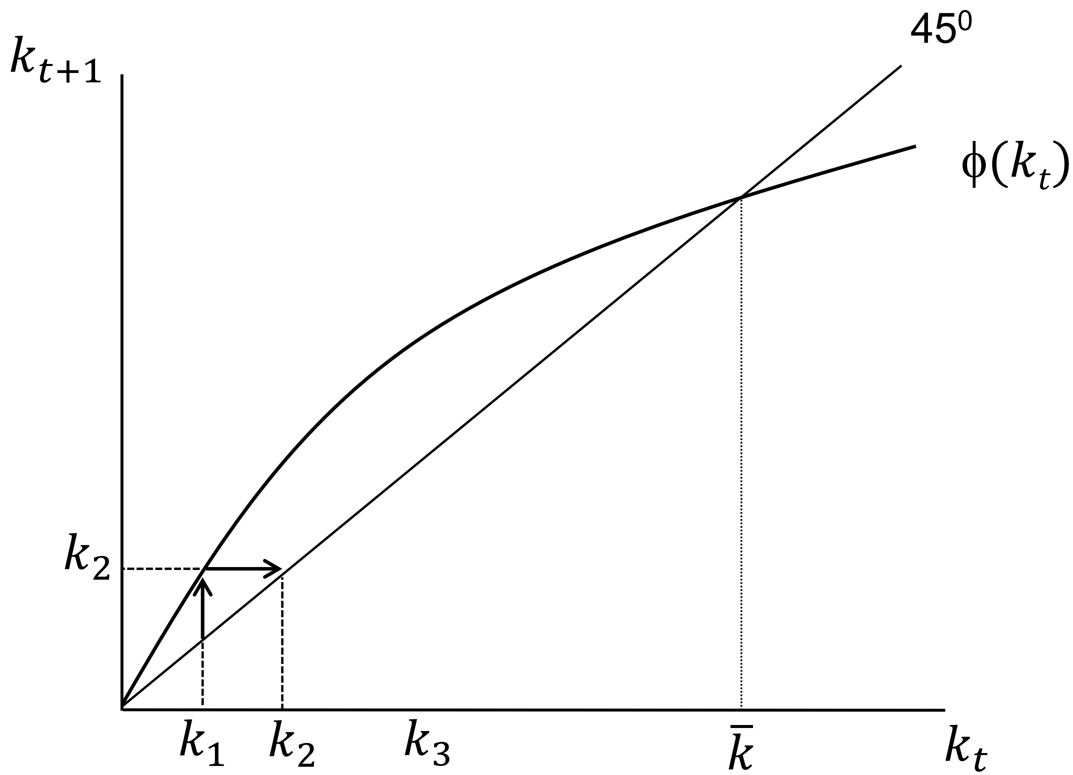
The dynamical system



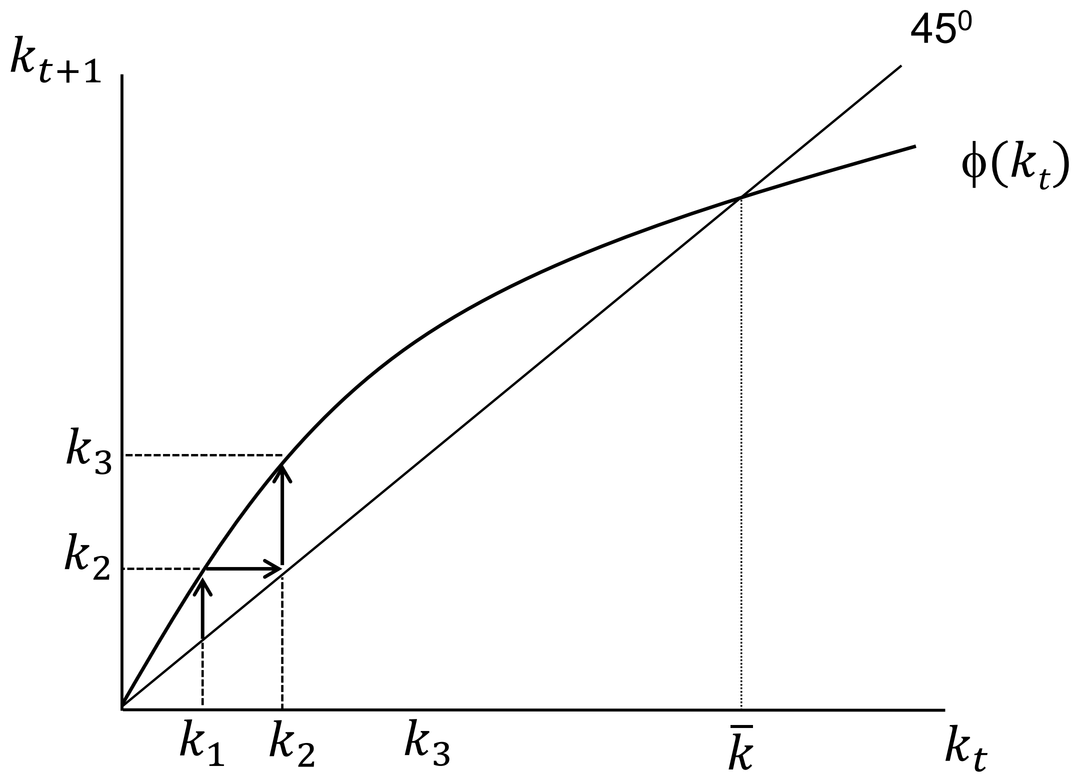
The dynamical system



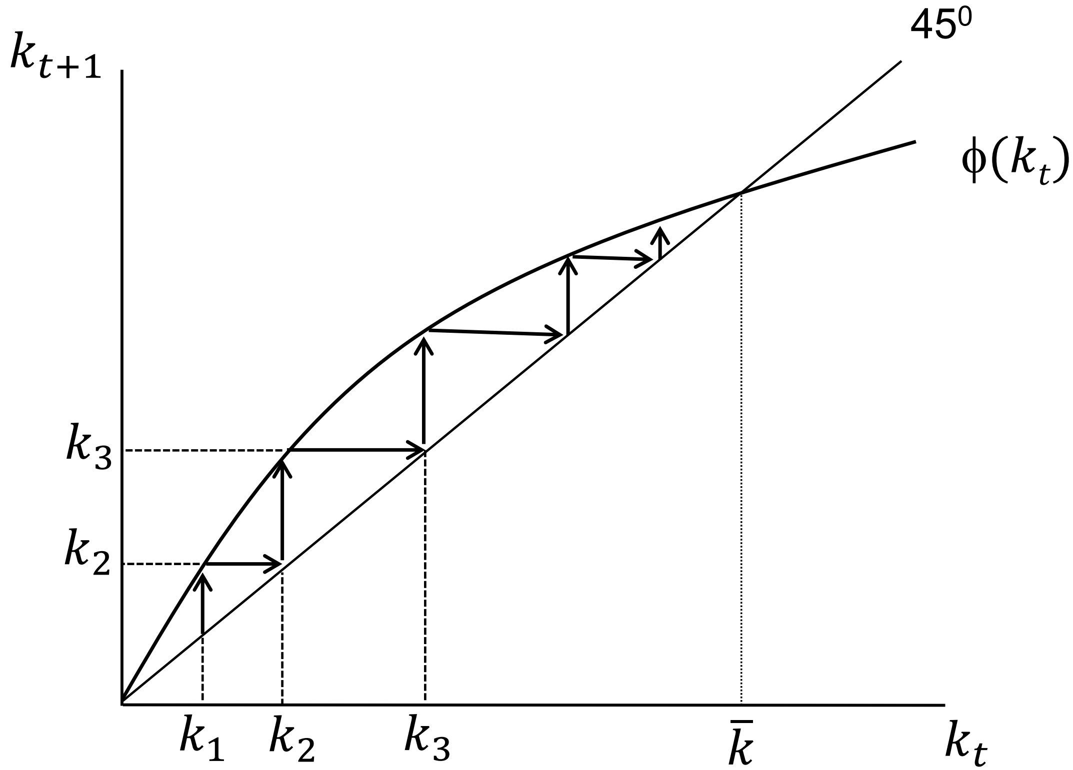
The dynamical system



The dynamical system



The dynamical system



Steady states

$\bar{k}$ such that:

$$\bar{k} = \phi(\bar{k}) = \frac{sf(\bar{k}) + (1-\delta)\bar{k}}{1+n}$$

$\rightarrow$

$$(n+\delta)\bar{k} = sf(\bar{k})$$

$\rightarrow$ there exist 2 steady states:

0 unstable

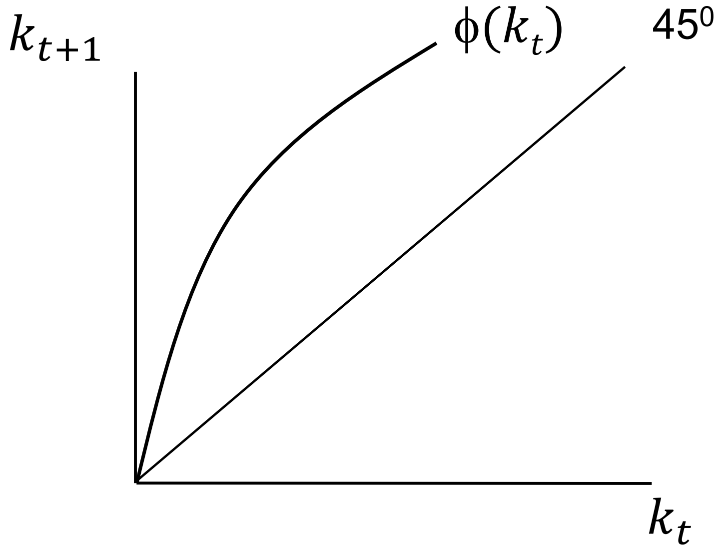
$\bar{k} > 0$ stable

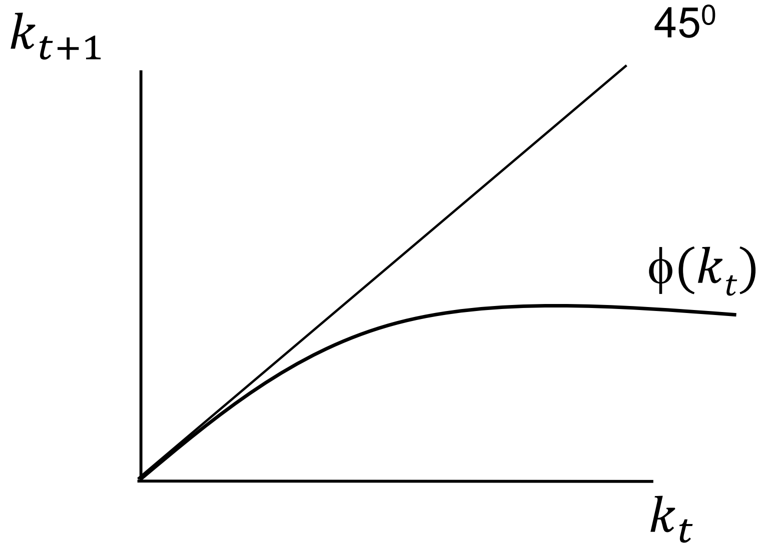
Properties of the equilibrium/steady state

- Existence
- Uniqueness
- Stability

Properties of the equilibrium/steady state

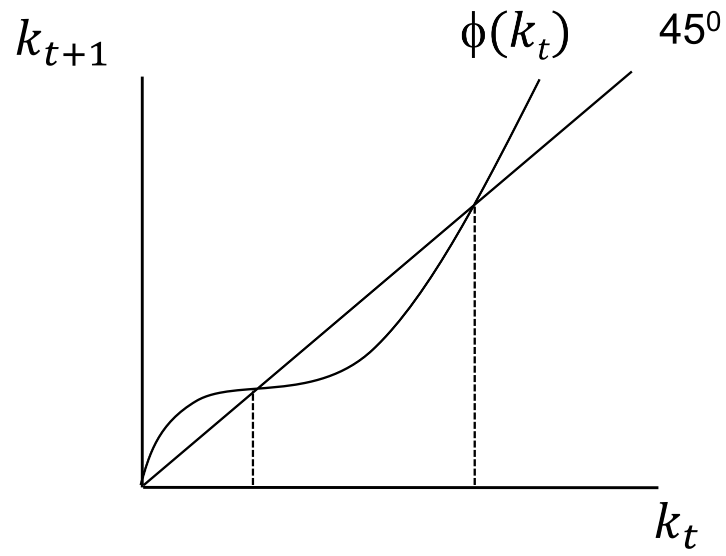
- Existence?





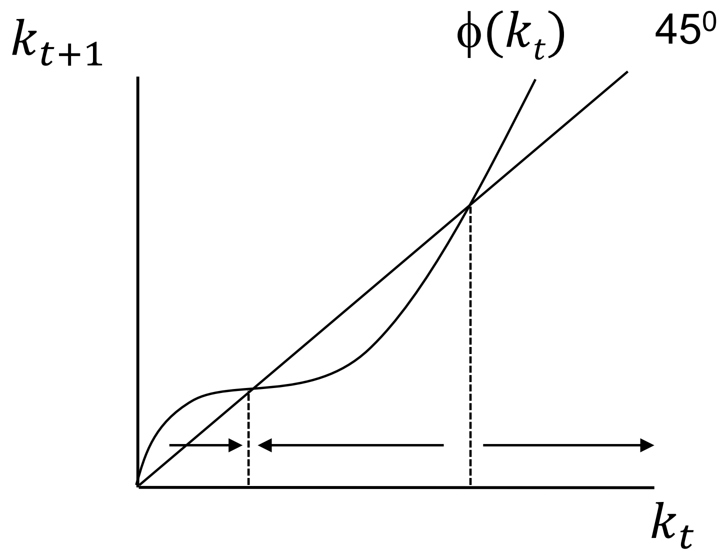
Properties of the equilibrium/steady state

- Uniqueness?



Properties of the equilibrium/steady state

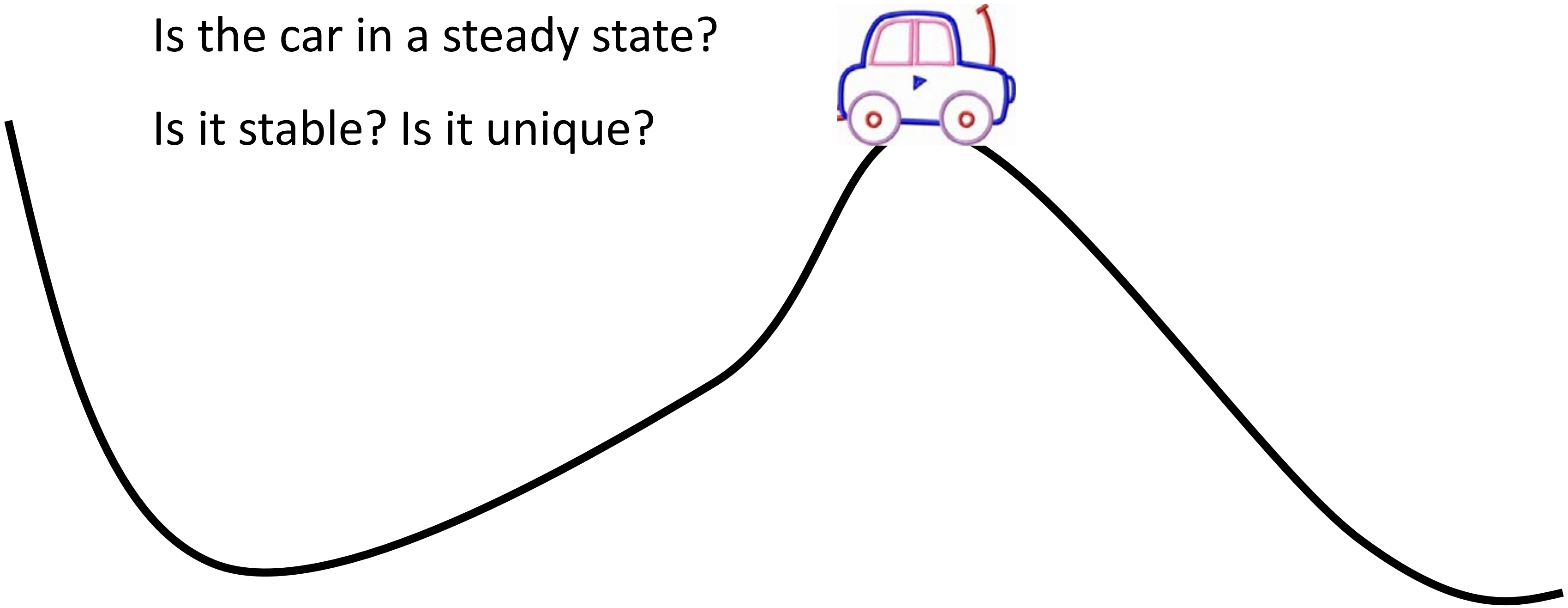
- Stability?



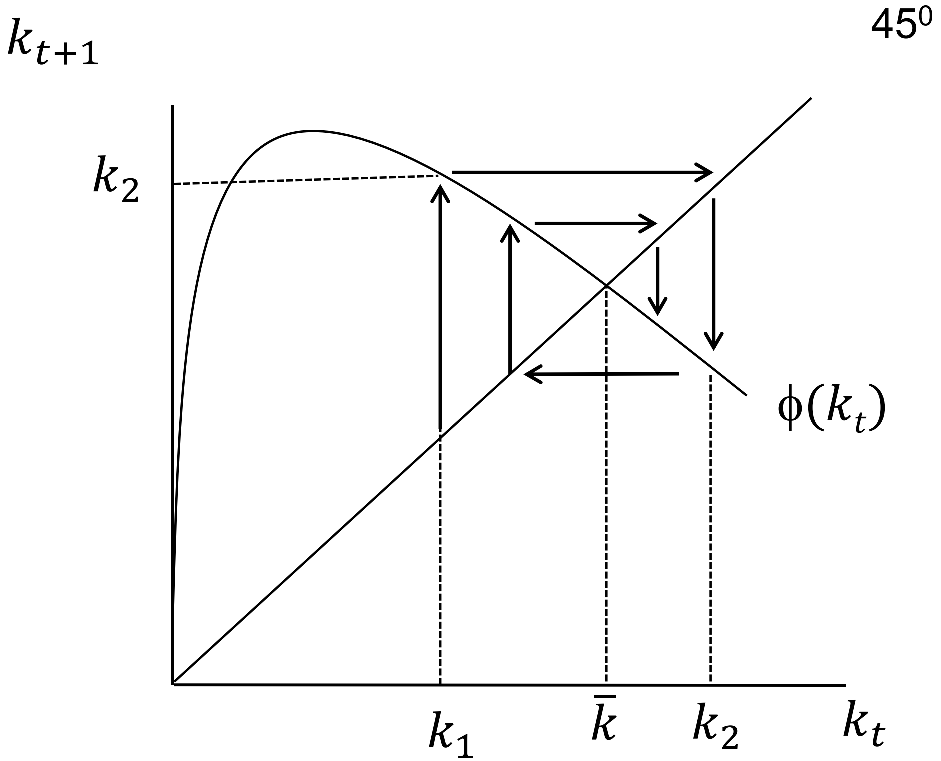
Properties of the equilibrium/steady state

Is the car in a steady state?

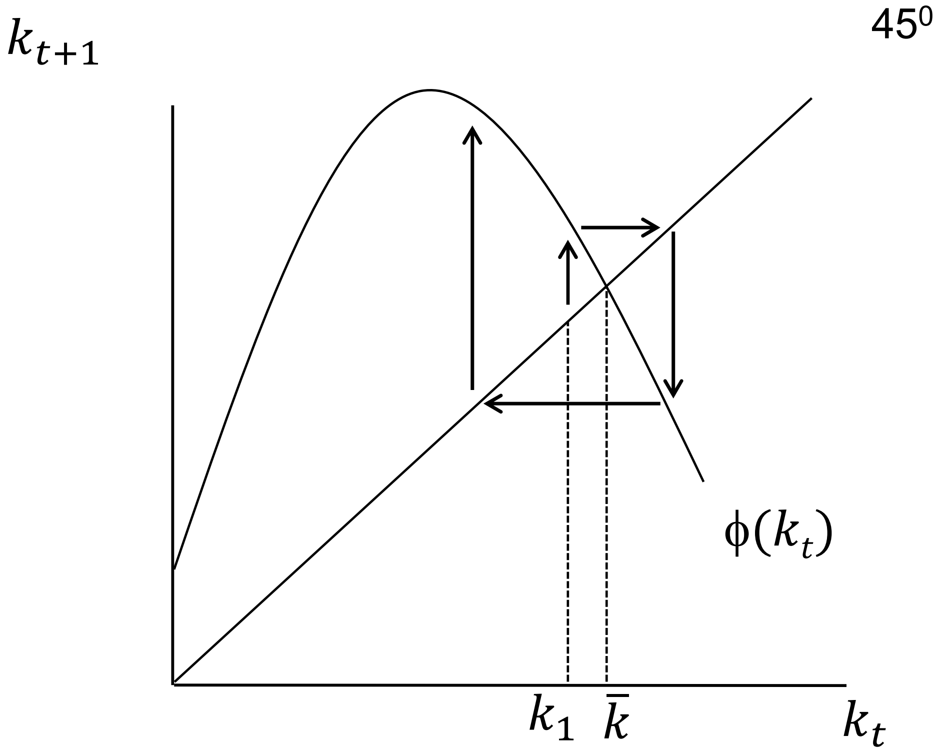
Is it stable? Is it unique?



Stability: convergence in oscillations



Stability: divergence in oscillations



Stability (Local)

$\phi'(\bar{k}) > 0$ → monotonic convergence or divergence

$\phi'(\bar{k}) < 0$ → oscillatory convergence or divergence

$|\phi'(\bar{k})| < 1$ → convergence

$|\phi'(\bar{k})| > 1$ → divergence

Comparative Statics

Proposition.

$$\frac{d\bar{k}}{dn} < 0$$

$$\frac{d\bar{k}}{ds} > 0$$

$$\frac{d\bar{k}}{dk_0} = 0$$

Proof.

Let

$$G(\bar{k}, n, s) \equiv (n+\delta)\bar{k} - sf(\bar{k}) = 0$$

$\rightarrow$

$$\frac{d\bar{k}}{dn} = -\frac{\frac{\partial G}{\partial n}}{\frac{\partial G}{\partial \bar{k}}} = -\frac{\bar{k}}{n+\delta - sf'(\bar{k})} < 0$$

$$\frac{d\bar{k}}{ds} = -\frac{\frac{\partial G}{\partial s}}{\frac{\partial G}{\partial \bar{k}}} = \frac{f(\bar{k})}{n+\delta - sf'(\bar{k})} > 0$$

initial condition do not matter since there exists a unique globally stable steady state equilibrium

Comparative Dynamics

Let

$$\gamma_{k_t} \equiv \frac{k_{t+1} - k_t}{k_t}$$

Proposition.

$$\frac{d\gamma_{k_t}}{dn} < 0$$

$$\frac{d\gamma_{k_t}}{ds} > 0$$

$$\frac{d\gamma_{k_t}}{dk_t} < 0$$

Proof.

$$\gamma_{k_t} = \left[\frac{sf(k_t) + (1-\delta)k_t}{1+n} - k_t \right] / k_t$$

$$= \frac{sf(k_t) - (n+\delta)k_t}{(1+n)k_t}$$

$$= \frac{sf(k_t)}{(1+n)k_t} - \frac{n+\delta}{1+n}$$

$\rightarrow$

$$\frac{d\gamma_{k_t}}{dn} = -\frac{sf(k_t)}{(1+n)^2 k_t} - \frac{1-\delta}{(1+n)^2} < 0$$

$$\frac{d\gamma_{k_t}}{ds} = \frac{f(k_t)}{(1+n)k_t} > 0$$

$$\frac{d\gamma_{k_t}}{dk_t} = -\frac{s}{(1+n)k_t^2}[f(k_t) - f'(k_t)k_t] < 0$$

Conclusion: no growth in the long-run without technological progress

Testable Implication:

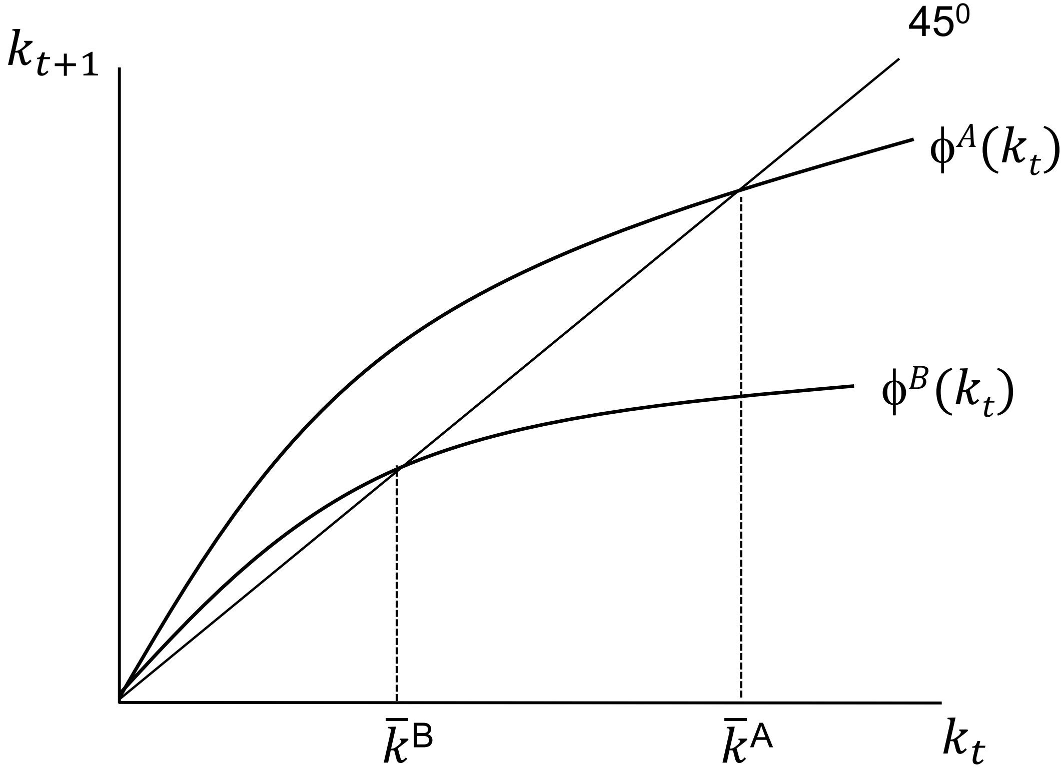
Conditional convergence, not global convergence

Testing for convergence:

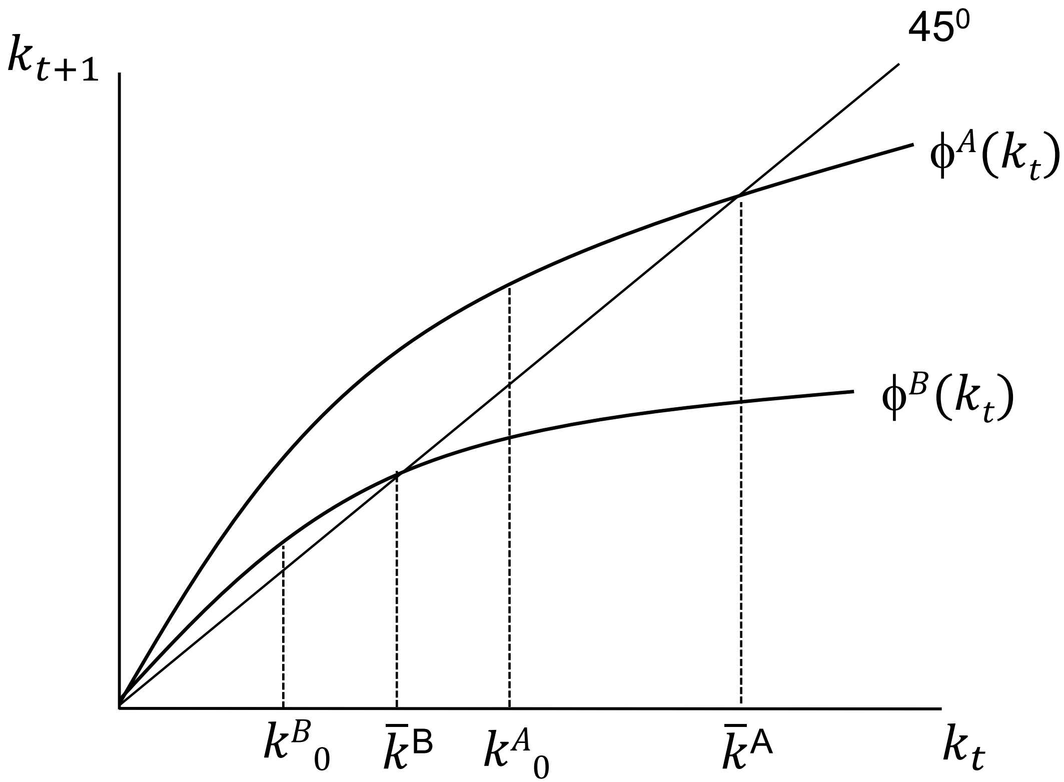
β convergence

σ convergence

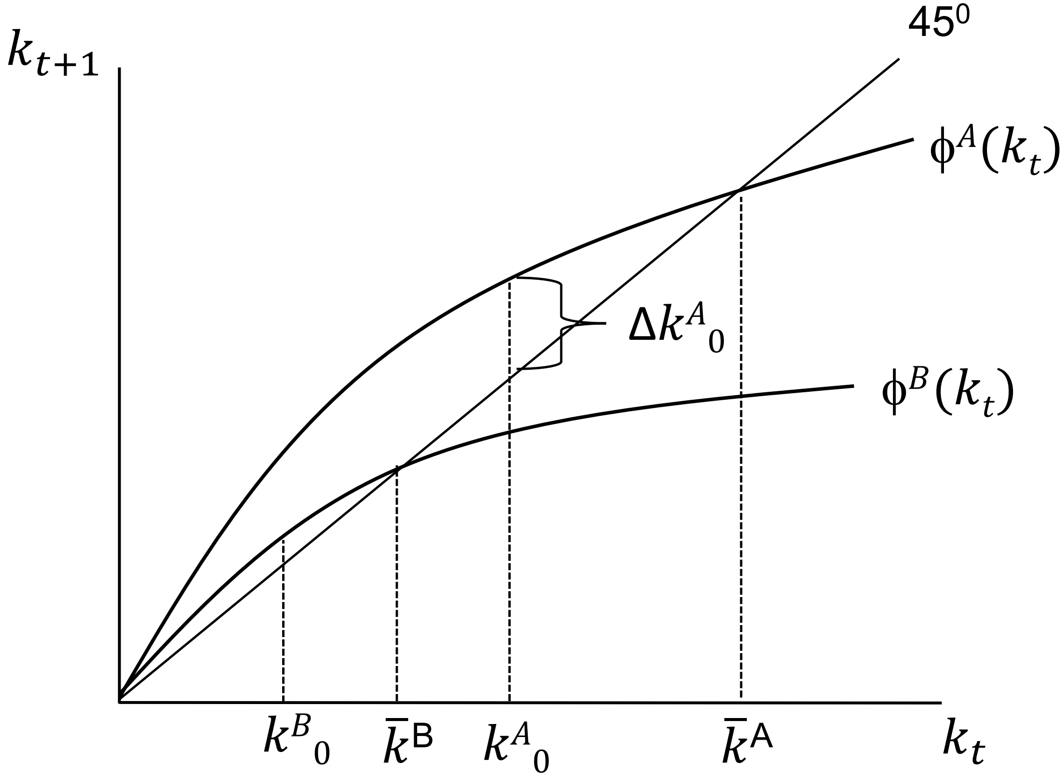
Convergence



Convergence



Convergence



β convergence

Global convergence:

$$\gamma^i_{1980-2020} = \alpha + \beta_0 y^i_{1980} + \varepsilon^i$$

Conditional convergence

$$\gamma^i_{1980-2020} = \alpha + \beta_0 y^i_{1980} + \beta_1 X^i_{1980} + \varepsilon^i$$

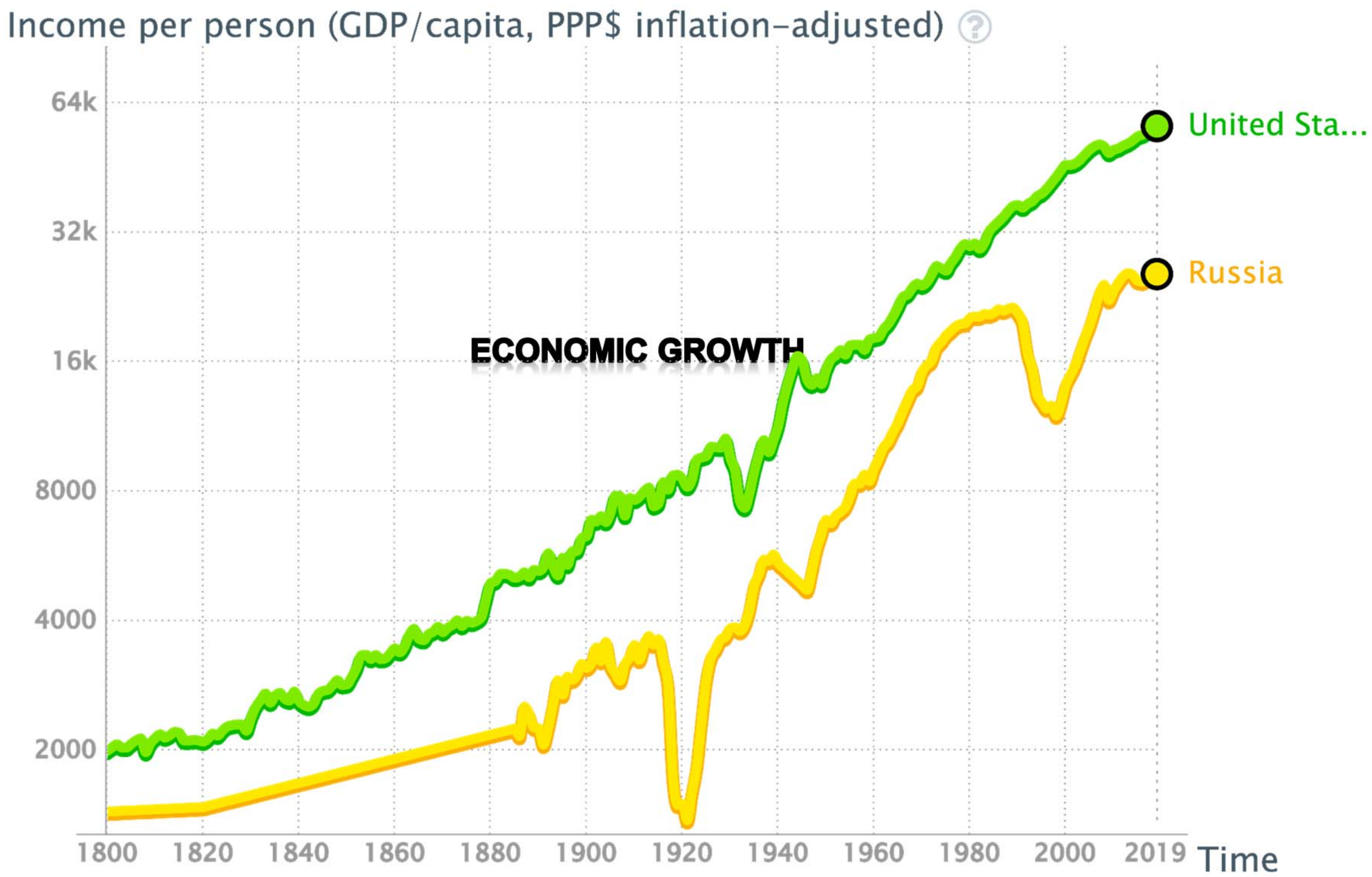
σ convergence

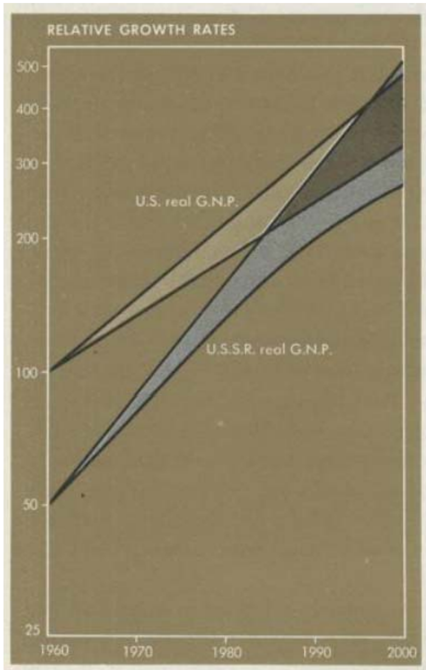
Sigma convergence occurs when the dispersion of income per capita between different countries tends to decrease over time

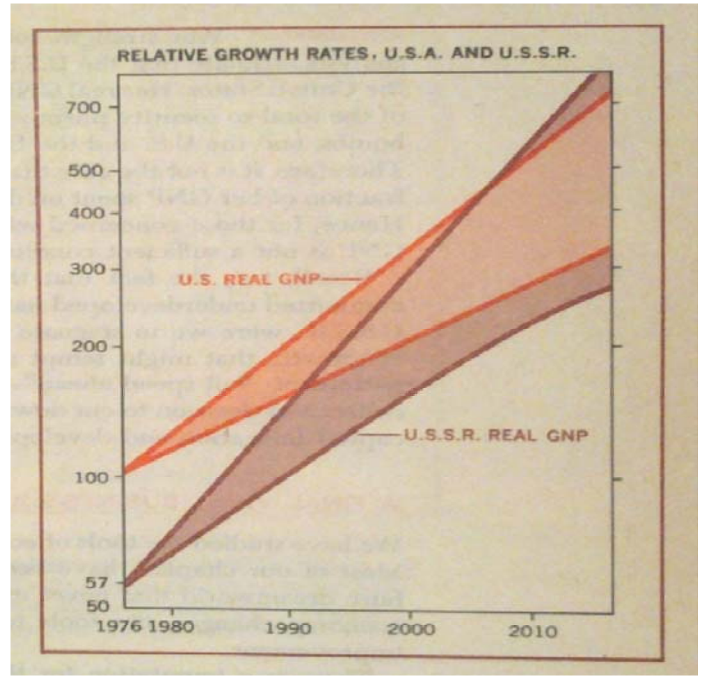

ЗАКОН
С ЧЕСТЬЮ
ВЫПОЛНИМ
И ПЕРЕВЫПОЛНИМ
НОВЫЙ СТАЛИНСКИЙ
ПЯТИЛЕТНИЙ ПЛАН!

VS.


Enjoy
Capitalism


Income per person (GDP/capita, PPP$ inflation-adjusted)
64k
32k
16k
8000
4000
2000
1800
1820
1840
1860
1880
1900
1920
1940
1960
1980
2000
2019
Time
United Sta...
Russia
ECONOMIC GROWTH





Paul Samuelson's predictions

Factors of Production vs. Productivity as Explanations for International Income Differences