Population Growth on the Solow Model[1]

Case A : without population growth.

Although the initial level of capital, K_t, is given, future levels of capital can be influenced through investment. In particular, investment in period t yields new capital in period $t+1$. Furthermore, some existing capital depreciates ($0 < \delta < 1$ is the depreciation rate) during production. Formally, capital accumulates according to:

$$K_{t+1} = I_t + K_t - \delta K_t \tag{1}$$

As in lecture slides/notes capital per worker is $k_t = \frac{K_t}{L_t}$ and the output per worker is $y_t = \frac{Y_t}{L_t} = c_t + sy_t$ and it is divided between consumption per worker, c_t and investment per worker, sy_t where s is the saving rate and $0 < s < 1$. The Solow model assumes that investment is a constant fraction of output. In particular, $I_t = sY_t$. Again, we assume that there is no net export and government expenditure. Every period households save a constant fraction, s of their income y_t and a certain fraction, δ, of the capital stock wears out. Then Eq-(1) becomes

$$K_{t+1} = \overbrace{sY_t}^{Investment} + \overbrace{K_t - \delta K_t}^{Non-depreciated\ capital} \tag{2}$$

This equation is know as the capital accumulation equation, or sometimes as a "*law of motion*" for capital. This equation says that your capital stock in $t+1$ equals your investment in period t plus the non-depreciated stock of capital with which you started.

Divide both sides of the Eq-(2) by L_t, number of workers.

$$\begin{aligned}\frac{K_{t+1}}{L_t} &= \frac{sY_t}{L_t} + \frac{K_t}{L_t} - \frac{\delta K_t}{L_t} \\ &= sy_t + k_t - \delta k_t\end{aligned}$$

Multiply and divide the left hand side by L_{t+1}

$$\frac{K_{t+1}}{L_t}\frac{L_{t+1}}{L_{t+1}} = sy_t + k_t - \delta k_t \tag{3}$$

where $\frac{L_{t+1}}{L_t} = 1$, and $\frac{K_{t+1}}{L_{t+1}} = k_{t+1}$

$$k_{t+1} = sy_t + (1-\delta)k_t \tag{4}$$

Equation-4 is the central equation of the Solow model. It describes how capital per worker evolves over time, given an initial value of the capital stock, an exogenous value of parameter s and δ.

To transform the central equation of the Solow model, into first differences. In particular, define

$$\Delta k_{t+1} = k_{t+1} - k_t$$

[1]Explanations: -Mostly- Intermediate Macroeconomics by Julio Garin, Robert Lester, Eric Sims, Graphs: `https://www.elabs.academy`

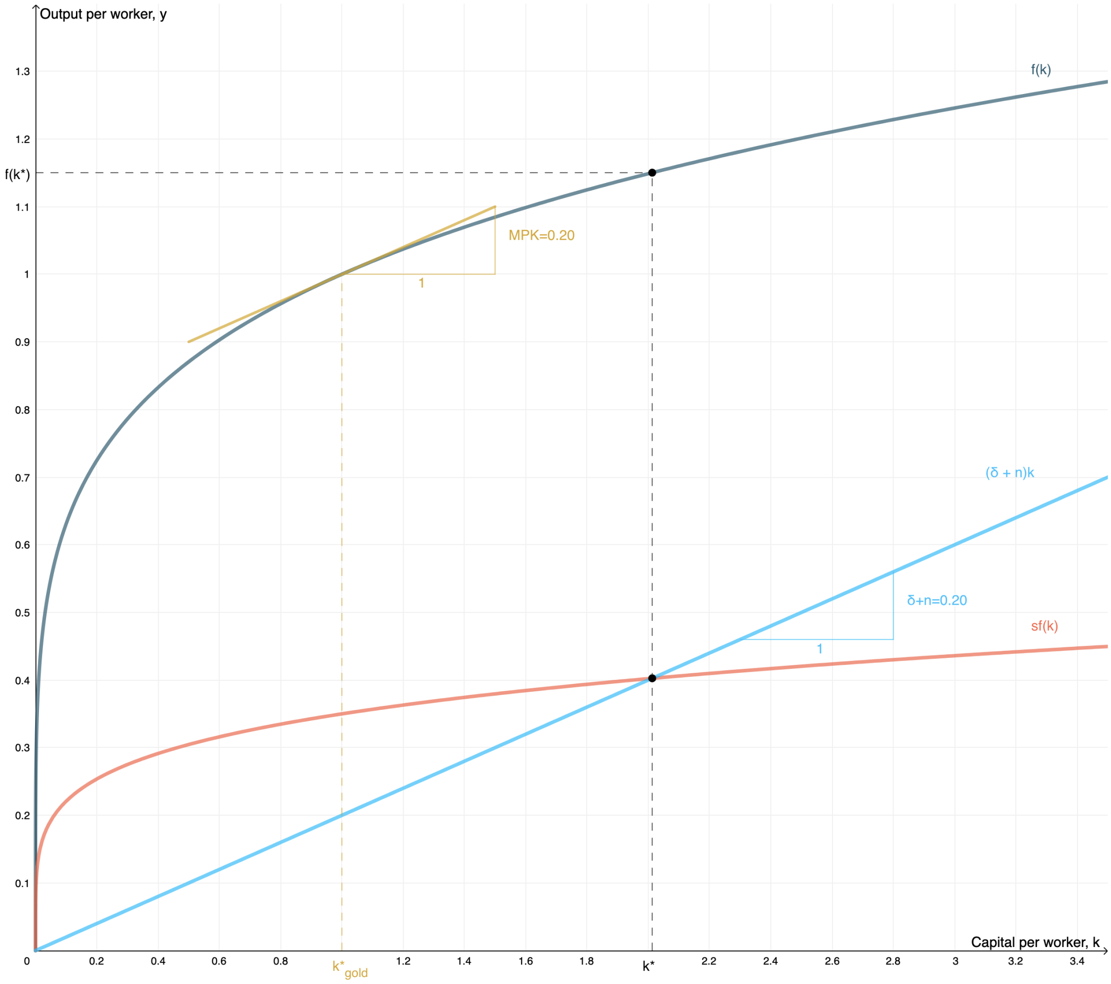
and subtracting k_t from both sides of equation-4

$$\Delta k_{t+1} = sy_t - \delta k_t \tag{5}$$

This equation says that the change in the capital stock is equal to the difference between investment and depreciation.

Let us see all these in a graph. Assume that $s = 0.3$, $\delta = 0.2$ and $k = 2$. Thus at the current depreciation rate, capital per worker and saving rate, 0.4 units of capital depreciate every period and again the effective depreciation rate $(n + \delta)$ is 0.4.

Figure 1: Without Population Growth, $n = 0$



Case B : with population growth.

We will allow L_t to grow over time to account for population growth. In particular, let's assume:

$$L_{t+1} = L_t(1 + n), \quad n \geq 0$$

In other words, we allow L_t to grow over time, where $n \geq 0$ is the growth rate between two periods. If we iterate back to period 0, and normalize the initial level $L_t = 1$, then we get:

$$L_t = (1+n)^t \tag{6}$$

In particular, equation-6 shows that if $n = 0$, then $L_t = 1$ at all times. In other words constant growth rate. The blue line on the Figure-1 represents this situation. However, we are now interested in the effect of population growth. Let assume time begins in period $t = 0$ with a representative household which supplies $L_0 = 1$ unit of labour. Over time, the size of this household grows at rate n, but each member of the household continues to supply 1 unit of labour inelastically each period.

Therefore, we need to consider equation-3 again since $\frac{L_{t+1}}{L_t} \neq 1$

$$\frac{K_{t+1}}{L_t}\frac{L_{t+1}}{L_{t+1}} = sy_t + k_t - \delta k_t \tag{7}$$

where $\frac{L_{t+1}}{L_t} = 1 + n$, and $\frac{K_{t+1}}{L_{t+1}} = k_{t+1}$

$$k_{t+1}(1+n) = sy_t + (1-\delta)k_t \tag{8}$$

After some algebra equation-5 becomes

$$\Delta k_{t+1}(1+n) = sy_t - (n+\delta)k_t \tag{9}$$

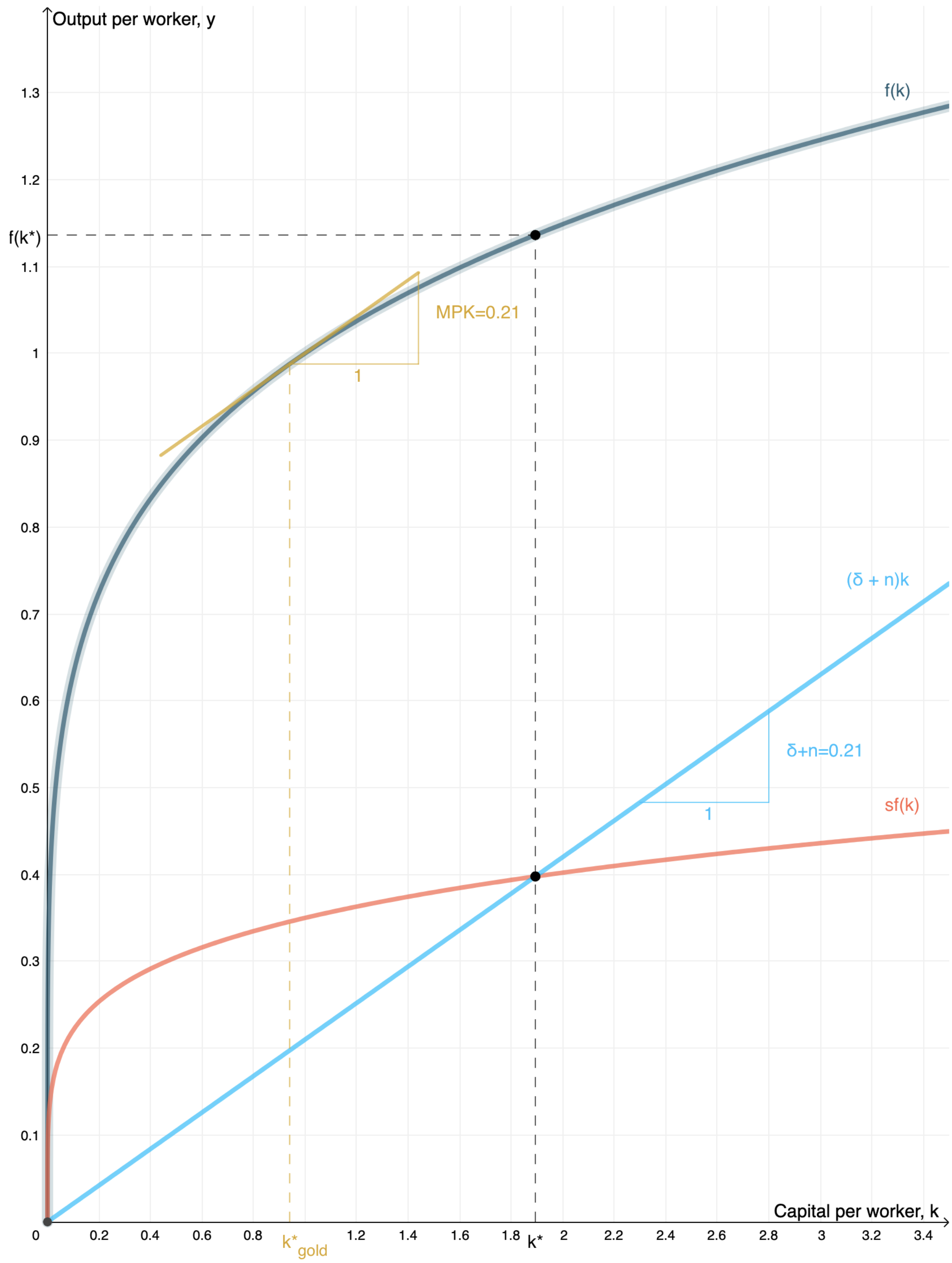
Again, thhe equation-9 is known as the fundamental dynamic equations of the Solow model and states that capital-labour ratio increases per capita saving,sy_t and decreases with the "effective" depreciation rate, $n+\delta$. We can understand that the "effective" depreciation of capital per worker is larger when the number of workers increses over time. Thus, a given amount of investment at time t generates less capital per worker at $t+1$ if the number of workers increases between t and $t+1$.

The Figure-2 shows this situation. The blue line which represent population growth and depreciation shifts upward. The new steady state has a lower level of per worker than the initial steady state. Therefore, the model which includes population growth predicts that economies with higher levels of populations growth will have lower levels of capital per worker and therefore lower income.

In the presence of population growth, capital per worker and output per worker remain constant in the steady state situation. Since the number of workers is increasing at the rate n, total capital (K_t) and total output ($Y_t = y_t L_t$) must also grow at the same rate if a steady state situation is to be maintained. Thus, population growth can explain sustained growth in GDP. However, since GDP per worker ($y_t = f(k^*)$) remains constant in the steady state, population growth cannot explain improvement in the standard of living.

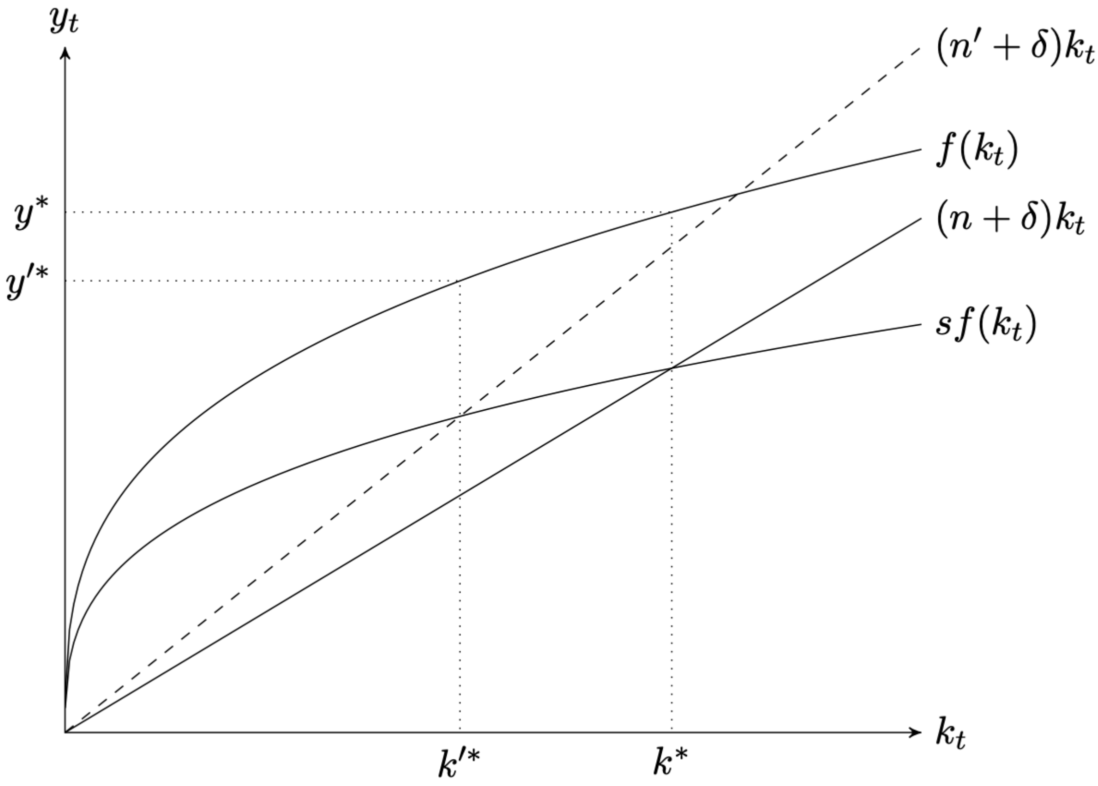
Again, in the Solow model, an increase in the population growth rate raises the growth rate of aggregate output but has no permanent effect on the growth rate of output per worker. An increase in the population growth rate lowers the steady-state level of output per worker.

Figure 2: With Population Growth, $n = 0.01$



Some Questions

- What happens in the steady state when the population growth rate, n changes?



- Population growth should allow economy to grow faster because labour input is growing faster, but given the saving rate it will be harder to accumulate capital per worker because the higher birth rate means more new workers must be equipped.
- **Graphically:** Let assume that the economy initially are in the steady state (k^*, y^*). Then the population growth rate changes from n to $n^{'}$. This changes can be seen the slope of the line denoting break-even investments—it becomes steeper. The new intersection between the break-even investment curve and the savings curve (the steady state) is now at the point $(k^{'*}, y^{'*})$, and we see that $k^{'*} < k^*$ and $y^{'*} < y^*$. The long term effect of a change in population growth from n to $n^{'}$, where $n^{'} > n$, is hence that GDP per capita falls from y^* to $y^{'*}$. We have thus found that a change in the population growth rate has a *level* effect, the steady state level of y changes as n changes. The *growth* of y in *steady state* does not change, however. Population growth hence has no growth effect (in steady state). In the *transition*-short tem from the old to the new steady state, there will be negative growth in y.

  Now why is this the case, why does the steady state level of y become lower as the population growth increases? To maintain the same level of capital *per capita* next period, the current generation has to invest more than before (the 'cake' has to be larger if everybody is to have the same amount of cake as before). This will only be possible if the savings rate s changed, which it by assumption does not do here. This means that the amount of capital per capita has to go down over time, as the the amount of investment is less that what the society need to invest in order to keep the capital stock constant. But the key here is that as the capital stock becomes lower and lower, the marginal return of the capital increases (due to the assumption about decreasing marginal return to capital), and this is what ensures that this process of capital de-accumulation will converge and hit the new steady state.

- What is the growth rate of steady state k^*
  - Zero. Because it is a steady state, it will not move from there.
- What is the growth rate of y^*
  - Zero. Because $k = k^*$ in the long run, output per worker is constant at $y^* = f(k^*)$
- So, can we say that there is no growth?
  - Of course there is growth in this economy! In the long run, when $k = k^*$ all real aggregate quantities grow at the rate n.
  - The aggregate quantity of capital is $K_t = k_t \times L_t$. Since k^* is constant and L grows at a rate n, K should grow at a rate n too.
  - Similarly, aggregate real output is $Y_t = y_t \times L_t = f(k^*)L$, where the the growth rate of y^* zero and the growth rate of L is n hence Y also grows at a rate n.
- What does the model explain?

  Solow's model, even in a rudimentary version without technical change (g), explains
  - positive correlation of investment rates and per capita income
  - negative correlation of population growth and per capita income
- What will happen if a war reduces the labour force through casualties,
  - The L falls but Capital-labour ratio $k = K/L$ rises. The production function tells us that total output ($Y = y \times L$) falls because there are fewer workers. Output per worker increases, however, since each worker has more capital.
  - The reduction in the labour force means that the capital stock per worker is higher after the war. Therefore, if the economy were in a steady state prior to the war, then after the war the economy has a capital stock that is higher than the steady-state level. So in the short run there will be an increase in capital per worker from k_1 to k_2 where $k_2 > k_1$. As the economy returns to the steady state in the long run, the capital stock per worker falls from k_2 back to k_2, so output per worker also falls.
  - Thus, an increase(decrease) in the labour force with a given saving and technological progress rate, cause an decrease(increase) in capital per worker so output per worker also increases(falls).
- In this way, the Solow growth model is an exogenous growth model. Only technological progress can lead to improvements in output per worker. Growth in this model is "exogenous": output growth $= n + g$, both of which are taken as given from outside the model.