WAVES AND OSCILLATIONS

PROBLEM SET 7 April 02, 2018

1. A scientist wants to suspend an atomic force microscope (AFM) of mass 5.4 kg (including the mass of the platform) using a rubber bungee cord of equilibrium length 1.2 m. If the scientist wants $\omega_0 = 10$ rad/s, what is the required diameter for the rubber cord? (Assume the mass of the cord is negligible compared to that of the AFM, Young's modulus for rubber is 0.002×10^9 N/m^2).

2. A massless spring with no mass attached to it hangs from the ceiling. Its length is 20 cm. A mass M is now hung on the lower end of the spring. Support the mass with your hand so that the spring remains relaxed, then suddenly remove your supporting hand. The mass and spring oscillate. The lowest position of the mass during the oscillation is 10 cm below the place it was resting when you supported it. (a) What is the frequency of oscillation? (b) What is the velocity when the mass is 5 cm below its original resting place?

3. The CO_2 molecule can be crudely and classically modeled as a system with a central mass $m_2 = 12$ AMU connected by equal springs of spring constant k to two masses $m_1 = m_3 = 16$ AMU, constrained to move only along the line joining their centers. Set up and solve the equations for the two normal modes in which the masses oscillate along that line.

4. Two simple harmonic motions of same angular frequency ω, $x_1 = a_1 sin\omega t$ and $x_2 = a_2 sin(\omega t + \phi)$ act on a particle along the x-axis simultaneously. Find amplitude, phase angle and hence the displacement of the resultant motion.

5. Consider a mass subjected to a restoring spring force $F_s = -kx$, a damping force $F_d = -b\dot{x}$, an oscillating drive force $F_D = F_0 sin\omega_D t$. Setup the differential equation of motion that describes the system and the general steady-state solution.

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7-1 given, $m = 5.4\ kg$

$l = 1.2\ m$ & required $\omega_0 = 10\ rad/s$.

required diameter of the bungee rubber cord?

$\omega_0 = \sqrt{k/m}$ (or) $k = \omega_0^2 m$.

$= (10)^2\ rad/s \times 5.4\ kg$.

$\underline{k = 540\ N/m}$.

$k = E\dfrac{A}{l} \longrightarrow A = \dfrac{kl}{E}$

young's modulus.

given, $E = 0.002 \times 10^9\ N/m^2$.

$\therefore A = \dfrac{540 \times 1.2}{0.002 \times 10^9} = \underline{3.2\ cm^2}$.

$A = \pi (d/2)^2 = \dfrac{\pi d^2}{4}$

$d = \sqrt{\dfrac{4A}{\pi}} = \underline{2.0\ cm}$ diameter of the cord.

7-2 Mass on a vertical spring.

(a) ~~a~~ lowest position of the mass is 10 cm below the place it was resting $\Rightarrow$ equilibrium point is 5 cm below the starting point

i.e. $\delta x = 5\,cm$.

$$\therefore \quad Mg = k\,\delta x \quad \& \qquad k = \frac{Mg}{\delta x}$$

(b) By energy conservation,

$$\tfrac{1}{2} k\,\delta x^2 = \tfrac{1}{2} M v^2$$

$$\text{(or)} \quad v = \sqrt{\frac{k \times \delta x^2}{M}} = \sqrt{g\,\delta x}$$

7-3 Normal mode of a CO_2 molecule

The eqns. of motion for the CO_2 molecule in a matrix form:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = k \begin{bmatrix} -1/m_1 & 1/m_1 & 0 \\ 1/m_2 & -2/m_2 & 1/m_2 \\ 0 & 1/m_1 & -1/m_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

inserting values of $m_1 = 16$ & $m_2 = 12$ amu,

eigen values & eigen vectors of this matrix:

$$\psi_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} ; \quad \omega_1^2 = k/16$$

$$\psi_2 = \begin{pmatrix} 3 \\ -8 \\ 3 \end{pmatrix} ; \quad \omega_2^2 = \frac{11k}{48}$$

7-4 Two SHMs of freq. 'ω'.

$x_1 = a_1 \sin \omega t$

$x_2 = a_2 \sin(\omega t + \phi)$.

Resultant displacement

$$X = x_1 + x_2 = \sin \omega t \, (a_1 + a_2 \cos\phi) + \cos \omega t \, (a_2 \sin\phi)$$

let, $R\cos\theta = a_1 + a_2 \cos\phi$

& $R\sin\theta = a_2 \sin\phi$

$\therefore R^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos\phi$

$$\tan\theta = \frac{a_2 \sin\phi}{a_1 + a_2\cos\phi}$$

& Resultant displacement

$$X = R\sin(\omega t + \theta).$$

also a SHM along x-axis with same ang. freq. 'ω'.

7-5

given Spring force $F_s = -kx$, $(-kx)$

damping force $F_d = -b\dot{x}$ α
$-\beta\dot{x}$

oscillating drive force $F_D = F_0 \sin\omega_D t$.
$F \sin pt$

differential eqn. of motion.

$$m\ddot{x} = -kx - b\dot{x} + F_0 \sin\omega_D t \quad -(1)$$

(or) $$\ddot{x} + 2\beta\dot{x} + \omega^2 x = f_0 \sin\omega_D t \quad -(2)$$

Where, $\beta = \frac{b}{2m}$, $\omega^2 = \frac{k}{m}$ & $f_0 = F_0/m$.

general soln. for 'x' is $x = x_1 + x_2$

x_1: general soln. of the homogeneous eqn.

$$\ddot{x}_1 + 2\beta\dot{x}_1 + \omega^2 x_1 = 0$$

& x_2 is any particular integral of eqn. (2)

soln. 'x_1': displacement of the damped H.O.

$$x_1 = e^{-\beta t}\left[A_1 \exp\left(\sqrt{\beta^2-\omega^2}\,t\right) + A_2 \exp\left(-\sqrt{\beta^2-\omega^2}\,t\right)\right]$$

'x_1' is the same as damped osc. $(\beta < \omega)$, dead beat motion $(\beta > \omega)$ or critically damped motion $(\beta = \omega)$.

To find x_2, let the soln. be

$$x_2 = A \sin(\omega_D t - \alpha).$$

$$\dot{x}_2 = A\omega_D \cos(\omega_D t - \alpha) \;\&$$

$$\ddot{x}_2 = -A\omega_D^2 \sin(\omega_D t - \alpha).$$

Substitute in eqn. (2).

$$A(\omega^2 - \omega_D^2)\sin(\omega_D t - \alpha) + 2A\beta\omega_D \cos(\omega_D t - \alpha)$$

$$= f_0 \sin\{(\omega_D t - \alpha) + \alpha\}$$

$$= f_0 \sin(\omega_D t - \alpha)\cos\alpha + f_0 \cos(\omega_D t - \alpha)\sin\alpha.$$

$\because$ above eqn. is true for all values of t, equate Coeffs. of $\sin(\omega_D t - \alpha)$ & $\cos(\omega_D t - \alpha)$ from both sides,

$$A(\omega^2 - \omega_D^2) = f_0 \cos\alpha$$

$$2A\beta\omega_D = f_0 \sin\alpha.$$

$$\therefore \quad A = \frac{f_0}{\sqrt{(\omega^2 - \omega_D^2)^2 + 4\beta^2\omega_D^2}}$$

$$\& \tan\alpha = \frac{2\beta\omega_D}{\omega^2 - \omega_D^2}.$$

& $\sin\alpha = \dfrac{2\beta\omega_D}{\sqrt{(\omega^2-\omega_D^2)^2+4\beta^2\omega_D^2}}$

$\cos\alpha = \dfrac{\omega^2-\omega_D^2}{\sqrt{(\omega^2-\omega_D^2)^2+4\beta^2\omega_D^2}}$

$\because$ α is never negative, range of α :

$$0 \le \alpha \le \pi$$

$\therefore$ complete solution is

$$x = x_1 + \frac{f_0}{\sqrt{(\omega^2-\omega_D^2)^2+4\beta^2\omega_D^2}} \sin(\omega_D t-\alpha)$$

When $\beta < \omega$, 1st part x_1 : rep. natural oscillations of damped H.O.

& becomes negligible as amplitude diminishes exponentially with time.

after long time,

$$x = \frac{f_0}{\sqrt{(\omega^2-\omega_D^2)^2+4\beta^2\omega_D^2}} \sin(\omega_D t-\alpha).$$

Steady state soln. ; sustained forced vibrations.