TENSORS

All of the equations of physics can be written in terms of tensors. In fact, in order to do relativity – either special or general – they must be.

Tensors constitute a generalization of quantities previously introduced: scalars and vectors. We identified a **scalar** as an quantity that remained invariant under rotations of the coordinate system and which could be specified by the value of a single real number. **Vectors** were identified as quantities that had a number of real components equal to the dimension of the coordinate system, with the components transforming like the coordinates of a fixed point when a coordinate system is rotated.

Calling scalars **tensors of rank** 0 and vectors **tensors of rank** 1, we identify a tensor of rank *n* in a *d*-dimensional space as an object with the following properties:

- It has components labeled by *n* indices, with each index assigned values from 1 through *d*, and therefore having a total of d^n components.

  Example: A^{ij} has 3^2 components, A^{ijkl} has 3^4 components and so on.
- The components transform in a specified manner under coordinate transformations.

The *behaviour under coordinate transformation* is of central importance for tensor analysis and conforms to the physicist's notion that physical observables must not depend on the choice of coordinate frames.

Covariant and Contravariant Tensors

The rotational transformation of a vector $\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$ from the Cartesian system defined by $\hat{\mathbf{e}}_i\ (i = 1, 2, 3)$ into a rotated coordinate system defined by $\hat{\mathbf{e}}'_i$, with the same vector **A** then represented as $\mathbf{A}' = A'_1\hat{\mathbf{e}}'_1 + A'_2\hat{\mathbf{e}}'_2 + A'_3\hat{\mathbf{e}}'_3$. The components of **A** and $\mathbf{A}'$ are related by

$$A'_i = \sum_j (\hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j) A_j, \qquad (4.1)$$

where the coefficients $(\hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j)$ are the projections of $\hat{\mathbf{e}}'_i$ in the $\hat{\mathbf{e}}'_i$ $\hat{\mathbf{e}}_j$ directions. Because the $\hat{\mathbf{e}}'_i$ and the $\hat{\mathbf{e}}_j$ are linearly related, we can also write

$$A'_i = \sum_j \frac{\partial x'_i}{\partial x_j} A_j. \qquad (4.2)$$

The formula of Eq. (4.2) corresponds to the application of the chain rule to convert the set A_j into the set A'_i, and is valid for arbitrary magnitude because both vectors depend linearly on their components.

We have also previously noted that the gradient of a scalar φ has in the unrotated Cartesian coordinates the components $(\nabla\varphi)_j = (\partial\varphi/\partial x_j)\hat{\mathbf{e}}_j$, meaning that in a rotated system we would have

$$(\nabla\varphi)'_i \equiv \frac{\partial\varphi}{\partial x'_i} = \sum_j \frac{\partial x_j}{\partial x'_i}\frac{\partial\varphi}{\partial x_j}, \qquad (4.3)$$

Quantities transforming according to Eq. (4.2) are called **contravariant** vectors, while those transforming according to Eq. (4.3) are termed **covariant**. When non-Cartesian systems may be in play, it is therefore customary to distinguish these transformation properties by writing the index of a contravariant vector as a superscript and that of a covariant vector as a subscript. This means, among other things, that the components of the position vector **r**, which is contravariant, must now be written (x^1, x^2, x^3). Thus, summarizing,

$$(A')^i = \sum_j \frac{\partial (x')^i}{\partial x^j} A^j \quad \mathbf{A}\text{, a contravariant vector,} \qquad (4.4)$$

$$A'_i = \sum_j \frac{\partial x^j}{\partial (x')^i} A_j \quad \mathbf{A}\text{, a covariant vector.} \qquad (4.5)$$

It is useful to note that the occurrence of subscripts and superscripts is systematic; the **free** (i.e., unsummed) index i occurs as a superscript on both sides of Eq. (4.4), while it appears as a subscript on both sides of Eq. (4.5), if we interpret an upper index in the denominator as equivalent to a lower index. The summed index occurs once as upper and once as lower (again treating an upper index in the denominator as a lower index). A frequently used shorthand (the **Einstein convention**) is to omit the summation sign in formulas like Eqs. (4.4) and (4.5) and to understand that when the same symbol occurs both as an upper and a lower index in the same expression, it is to be summed.

Tensors of Rank 2

Now we proceed to define **contravariant, mixed, and covariant tensors of rank 2** by the following equations for their components under coordinate transformations:

$$\begin{aligned}
(A')^{ij} &= \sum_{kl} \frac{\partial (x')^i}{\partial x^k}\frac{\partial (x')^j}{\partial x^l} A^{kl}, \\
(B')^i_j &= \sum_{kl} \frac{\partial (x')^i}{\partial x^k}\frac{\partial x^l}{\partial (x')^j} B^k_l, \qquad (4.6) \\
(C')_{ij} &= \sum_{kl} \frac{\partial x^k}{\partial (x')^i}\frac{\partial x^l}{\partial (x')^j} C_{kl}.
\end{aligned}$$

Clearly, the rank goes as the number of *partial derivatives* (or *direction cosines*) in the definition: 0 for a scalar, 1 for a vector, 2 for a second-rank tensor, and so on. Each index (subscript or superscript) ranges over the number of dimensions of the space. The number of indices (equal to the rank of tensor) is not limited by the dimensionality of the space. We see that A_{kl} is contravariant with respect to both indices, C_{kl} is covariant with respect to both indices, and B_{kl} transforms contravariantly with respect to the index k but covariantly with respect to the index l. Once again, if we are using Cartesian coordinates, all three forms of the tensors of second rank, contravariant, mixed, and covariant are the same.

As with the components of a vector, the transformation laws for the components of a tensor, Eq.(4.6), cause its physically relevant properties to be independent of the choice of reference frame. This is what makes tensor analysis important in physics. The independence relative to reference frame (invariance) is ideal for expressing and investigating universal physical laws.

The second-rank tensor A (with components *Akl*) may be conveniently represented by writing out its components in a square array (3 _ 3 if we are in three-dimensional (3-D) space):

$$\mathsf{A} = \begin{pmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{pmatrix}. \qquad (4.7)$$

This does not mean that any square array of numbers or functions forms a tensor. The essential condition is that the components transform according to Eq. (4.6).

We can view each of Eq. (4.6) as a matrix equation. For A, it takes the form

$$(A')^{ij} = \sum_{kl} S_{ik} A^{kl} (S^T)_{lj}, \quad \text{or} \quad \mathsf{A}' = \mathsf{S}\mathsf{A}\mathsf{S}^{\mathsf{T}}, \qquad (4.8)$$

a construction that is known as a **similarity transformation** and is discussed in Section 5.6.

- *In summary,* ***tensors are systems of components*** *organized by one or more indices that* ***transform*** *according to* ***specific rules under a set of transformations****. The number of indices is called the rank of the tensor.*

Addition and Subtraction of Tensors

The addition and subtraction of tensors is defined in terms of the individual elements, just as for vectors. If

$$\mathsf{A} + \mathsf{B} = \mathsf{C}, \qquad (4.9)$$

then, taking as an example A, B, and C to be contravariant tensors of rank 2,

$$A^{ij} + B^{ij} = C^{ij}. \qquad (4.10)$$

In general, of course, A and B must be tensors of the same rank (of both contra- and co-variance) and in the same space.

Symmetry

The order in which the indices appear in our description of a tensor is important. In general, A^{mn} is independent of A^{nm}, but there are some cases of special interest. If, for all *m* and *n*,

$$A^{mn} = A^{nm}, \quad \mathsf{A} \text{ is } \textbf{symmetric}. \qquad (4.11)$$

If, on the other hand,

$$A^{mn} = -A^{nm}, \quad \mathsf{A} \text{ is } \textbf{antisymmetric}. \qquad (4.12)$$

Clearly, every (second-rank) tensor can be resolved into symmetric and antisymmetric parts by the identity

$$A^{mn} = \frac{1}{2}(A^{mn} + A^{nm}) + \frac{1}{2}(A^{mn} - A^{nm}), \qquad (4.13)$$

the first term on the right being a symmetric tensor, the second, an antisymmetric tensor.

CONTRACTION

When dealing with vectors, we formed a scalar product by summing products of corresponding components:

$$\mathbf{A} \cdot \mathbf{B} = \sum_i A_i B_i .$$

The generalization of this expression in tensor analysis is a process known as contraction. Two indices, one covariant and the other contravariant, are set equal to each other, and then (as implied by the summation convention) we sum over this repeated index. For example, let us contract the second-rank mixed tensor B_i^j by setting j to i , then summing over i . To see what happens, let's look at the transformation formula that converts B into B_i^j. Using the summation convention,

$$(B')_i^i = \frac{\partial (x')^i}{\partial x^k} \frac{\partial x^l}{\partial (x')^i} B_l^k = \frac{\partial x^l}{\partial x^k} B_l^k ,$$

where we recognized the i summation as an instance of the chain rule for differentiation. Then, because the x_i are independent, we may use Eq. (4.14) to reach

$$(B')_i^i = \delta_k^l B_l^k = B_k^k . \qquad (4.16)$$

Remembering that the repeated index (i or k) is summed, we see that the contracted B is invariant under transformation and is therefore a scalar.

- *In general, the operation of contraction reduces the rank of a tensor by 2.*

Direct Product

The components of two tensors (of any ranks and covariant/contravariant characters) can be multiplied, component by component, to make an object with all the indices of both factors. The new quantity, termed the **direct product** of the two tensors, can be shown to be a tensor whose rank is the sum of the ranks of the factors, and with covariant/contravariant character that is the sum of those of the factors. We illustrate:

$$C_{klm}^{ij} = A_k^i B_{lm}^j , \qquad F_{kl}^{ij} = A^j B_{lk}^i .$$

Note that the index order in the direct product can be defined as desired, but the covariance/contravariance of the factors must be maintained in the direct product.

Example: DIRECT PRODUCT OF TWO VECTORS

Let's form the direct product of a *covariant* vector a_i (rank-1 tensor) and a *contravariant* vector b^j (also a rank-1 tensor) to form a mixed tensor of rank 2, with components $C_i^j = a_i b^j$. To verify that C_i^j is a tensor, we consider what happens to it under transformation:

$$(C')^j_i = (a')_i (b')^j = \frac{\partial x^k}{\partial (x')_i} a_k \frac{\partial (x')^j}{\partial x^l} b_l = \frac{\partial x^k}{\partial (x')_i} \frac{\partial (x')^j}{\partial x^l} C^l_k, \qquad (4.17)$$

confirming that C_i^j is the mixed tensor indicated by its notation.

If we now form the contraction $\sum_i C_i^i$ (written as C_i^i and keeping in mind that i is summed), we obtain the scalar product $a_i b^i$. From Eq. (4.17) it is easy to see that $a_i b^i = (a')_i (b')^i$, indicating the invariance required of a scalar product.

Note: The direct product concept gives a meaning to quantities such as $\vec{\nabla}\ \vec{E}$, which was not defined within the framework of vector analysis. However, this and other tensorlike quantities involving differential operators must be used with caution, because their transformation rules are simple only in Cartesian coordinate systems. In non-Cartesian systems, operators act also on the partial derivatives in the transformation expressions and alter the tensor transformation rules.

We summarize the key idea of this subsection:

The direct product is a technique for creating new, higher-rank tensors.

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