VECTOR CALCULUS

There are a number of differences between the calculus of one and two variables. However, the calculus of functions of three or more variables differs only slightly from that of functions of two variables.

Scalar Triple Product

While the various vector operations can be combined in many ways, there are two combinations involving three operands that are of particular importance. We call attention first to the **scalar triple product**, of the form

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \qquad (3.12)$$

We can draw a number of conclusions from this highly symmetric determinantal form. To start, we see that the determinant contains no vector quantities, so it must evaluate to an ordinary number. Because the left-hand side of Eq. (3.12) is a rotational invariant, the number represented by the determinant must also be rotationally invariant, and can therefore be identified as a scalar. Since we can permute the rows of the determinant (with a sign change for an odd permutation, and with no sign change for an even permutation), we can permute the vectors **A**, **B**, and **C** to obtain

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = -\mathbf{A}\cdot\mathbf{C}\times\mathbf{B}, \quad \text{etc.} \qquad (3.13)$$

3.5 DIFFERENTIAL VECTOR OPERATORS

We already have a standard name for a simple algebraic quantity f(x,y,z) which is assigned a value at all points of a spatial region (it is called a **function**). Iin physics contexts it may also be referred to as a **scalar field**.

The term **field** refers to a quantity that has values at all points of a region;
if the quantity is a vector, its distribution is described as a **vector field**.

While physicists need to be able to characterize the rate at which the values of vectors (and also scalars) change with position, and this is most effectively done by introducing differential vector operator concepts. It turns out that there are a large number of relations between these differential operators, and it is our current objective to identify such relations and learn how to use them.

Gradient, ∇

Our first differential operator is that known as the **gradient**, which characterizes the change of a scalar quantity, here φ, with position. Working in $\mathbb{R}^3$, and labeling the coordinates x_1, x_2, x_3, we write $\varphi(\mathbf{r})$ as the value of φ at the point $\mathbf{r} = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3$, and consider the effect of small changes dx_1, dx_2, dx_3, respectively, in x_1, x_2, and x_3. This situation corresponds to that discussed in Section 1.9, where we introduced **partial derivatives** to describe how a function of several variables (there x, y, and z) changes its value when these variables are changed by respective amounts dx, dy, and dz. The equation governing this process is Eq. (1.141).

To first order in the differentials dx_i, φ in our present problem changes by an amount

$$d\varphi = \left(\frac{\partial\varphi}{\partial x_1}\right)dx_1 + \left(\frac{\partial\varphi}{\partial x_2}\right)dx_2 + \left(\frac{\partial\varphi}{\partial x_3}\right)dx_3, \qquad (3.38)$$

which is of the form corresponding to the dot product of

$$\nabla\varphi = \begin{pmatrix} \partial\varphi/\partial x_1 \\ \partial\varphi/\partial x_2 \\ \partial\varphi/\partial x_3 \end{pmatrix} \quad \text{and} \quad d\mathbf{r} = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}.$$

These quantities can also be written as

$$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x_1}\right)\hat{\mathbf{e}}_1 + \left(\frac{\partial\varphi}{\partial x_2}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial\varphi}{\partial x_3}\right)\hat{\mathbf{e}}_3, \qquad (3.39)$$

$$d\mathbf{r} = dx_1\hat{\mathbf{e}}_1 + dx_2\hat{\mathbf{e}}_2 + dx_3\hat{\mathbf{e}}_3, \qquad (3.40)$$

in terms of which we have

$$d\varphi = (\nabla\varphi)\cdot d\mathbf{r}. \qquad (3.41)$$

Having now established the legitimacy of the form $\nabla\varphi$, we proceed to give ∇ a life of its own. We therefore define (calling the coordinates x, y, z)

$$\nabla = \hat{\mathbf{e}}_x\frac{\partial}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial}{\partial y} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z}. \qquad (3.44)$$

We note that ∇ is a **vector differential operator**, capable of operating on a scalar (such as φ) to produce a vector as the result of the operation. Because a differential operator only operates on what is to its right, we have to be careful to maintain the correct order in expressions involving ∇, and we have to use parentheses when necessary to avoid ambiguity as to what is to be differentiated.

The gradient of a scalar is extremely important in physics and engineering, as it expresses the relation between a force field **F**(**r**) experienced by an object at **r** and the related potential *V*(**r**),

$\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$. (3.45)

The minus sign in Eq. (3.45) is important; it causes the force exerted by the field to be in a direction that lowers the potential. We consider later (in Section 3.9) the conditions that must be satisfied if a potential corresponding to a given force can exit.

Example: Potential at a point P(x, y, z) of a charge placed at origin V = kq/r

Electric field at same point P(x, y, z) of a charge placed at origin $\mathbf{E} = \hat{\mathbf{r}}\, kq/r^2 = -\nabla V$

'e 3.5.1 GRADIENT OF r^n

As a first step toward computation of ∇r^n, let's look at the even simpler ∇r. We begin by writing $r = (x^2 + y^2 + z^2)^{1/2}$, from which we get

$$\frac{\partial r}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}. \qquad (3.46)$$

From these formulas we construct

$$\nabla r = \frac{x}{r}\hat{\mathbf{e}}_x + \frac{y}{r}\hat{\mathbf{e}}_y + \frac{z}{r}\hat{\mathbf{e}}_z = \frac{1}{r}(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z) = \frac{\mathbf{r}}{r}. \qquad (3.47)$$

The result is a unit vector in the direction of **r**, denoted $\hat{\mathbf{r}}$. For future reference, we note that

$$\hat{\mathbf{r}} = \frac{x}{r}\hat{\mathbf{e}}_x + \frac{y}{r}\hat{\mathbf{e}}_y + \frac{z}{r}\hat{\mathbf{e}}_z \qquad (3.48)$$

and that Eq. (3.47) takes the form

$$\nabla r = \hat{\mathbf{r}}. \qquad (3.49)$$

The geometry of **r** and $\hat{\mathbf{r}}$ is illustrated in Fig. 3.8.

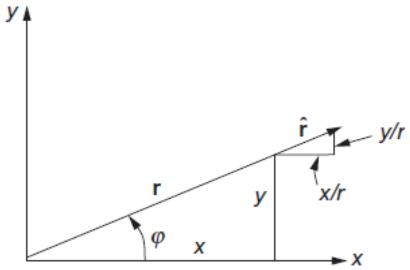

FIGURE 3.8 Unit vector $\hat{\mathbf{r}}$ (in xy-plane).

Continuing now to ∇r^n, we have

$$\frac{\partial r^n}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x},$$

with corresponding results for the y and z derivatives. We get

$$\nabla r^n = nr^{n-1}\nabla r = nr^{n-1}\hat{\mathbf{r}}. \tag{3.50}$$

■

3.5.2 Coulomb's Law

In electrostatics, it is well known that a point charge produces a potential proportional to $1/r$, where r is the distance from the charge. To check that this is consistent with the Coulomb force law, we compute

$$\mathbf{F} = -\nabla\left(\frac{1}{r}\right).$$

This is a case of Eq. (3.50) with $n = -1$, and we get the expected result

$$\mathbf{F} = \frac{1}{r^2}\hat{\mathbf{r}}.$$

■

3.5.3 General Radial Potential

Another situation of frequent occurrence is that the potential may be a function only of the radial distance from the origin, i.e., $\varphi = f(r)$. We then calculate

$$\frac{\partial \varphi}{\partial x} = \frac{df(r)}{dr}\frac{\partial r}{\partial x}, \quad \text{etc.},$$

which leads, invoking Eq. (3.49), to

$$\nabla\varphi = \frac{df(r)}{dr}\nabla\mathbf{r} = \frac{df(r)}{dr}\hat{\mathbf{r}}. \tag{3.51}$$

This result is in accord with intuition; the direction of maximum increase in φ must be radial, and numerically equal to $d\varphi/dr$. ■

Divergence, $\nabla\cdot$

The **divergence** of a vector $\mathbf{A}$ is defined as the operation

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{3.52}$$

The above formula is exactly what one might expect given both the vector and differential-operator character of ∇.

After looking at some examples of the calculation of the divergence, we will discuss its physical significance.

le 3.5.4 DIVERGENCE OF COORDINATE VECTOR

Calculate $\nabla \cdot \mathbf{r}$:

$$\nabla \cdot \mathbf{r} = \left(\hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\right) \cdot (\hat{\mathbf{e}}_x x + \hat{\mathbf{e}}_y y + \hat{\mathbf{e}}_z z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z},$$

which reduces to $\nabla \cdot \mathbf{r} = 3$. ■

le 3.5.5 DIVERGENCE OF CENTRAL FORCE FIELD

Consider next $\nabla \cdot f(r)\hat{\mathbf{r}}$. Using Eq. (3.48), we write

$$\nabla \cdot f(r)\hat{\mathbf{r}} = \left(\hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\right) \cdot \left(\frac{xf(r)}{r}\hat{\mathbf{e}}_x + \frac{yf(r)}{r}\hat{\mathbf{e}}_y + \frac{zf(r)}{r}\hat{\mathbf{e}}_z\right).$$

$$= \frac{\partial}{\partial x}\left(\frac{xf(r)}{r}\right) + \frac{\partial}{\partial y}\left(\frac{yf(r)}{r}\right) + \frac{\partial}{\partial z}\left(\frac{zf(r)}{r}\right).$$

Using

$$\frac{\partial}{\partial x}\left(\frac{xf(r)}{r}\right) = \frac{f(r)}{r} - \frac{xf(r)}{r^2}\frac{\partial r}{\partial x} + \frac{x}{r}\frac{df(r)}{dr}\frac{\partial r}{\partial x} = f(r)\left[\frac{1}{r} - \frac{x^2}{r^3}\right] + \frac{x^2}{r^2}\frac{df(r)}{dr}$$

and corresponding formulas for the y and z derivatives, we obtain after simplification

$$\nabla \cdot f(r)\hat{\mathbf{r}} = 2\frac{f(r)}{r} + \frac{df(r)}{dr}. \tag{3.53}$$

In the special case $f(r) = r^n$, Eq. (3.53) reduces to

$$\nabla \cdot r^n \hat{\mathbf{r}} = (n+2)r^{n-1}. \tag{3.54}$$

For $n = 1$, this reduces to the result of Example 3.5.4. For $n = -2$, corresponding to the Coulomb field, the divergence vanishes, except at $r = 0$, where the differentiations we performed are not defined. ■

If a vector field represents the flow of some quantity that is distributed in space, its divergence provides information as to the accumulation or depletion of that quantity at the point at which the divergence is evaluated. To gain a clearer picture of the concept, let us suppose that a vector field $\mathbf{v}(\mathbf{r})$ represents the velocity of a fluid[5] at the spatial points $\mathbf{r}$, and that $\rho(\mathbf{r})$ represents the fluid density at $\mathbf{r}$ at a given time t. Then the direction and magnitude of the flow rate at any point will be given by the product $\rho(\mathbf{r})\mathbf{v}(\mathbf{r})$.

Our objective is to calculate the net rate of change of the fluid density in a volume element at the point $\mathbf{r}$. To do so, we set up a parallelepiped of dimensions dx, dy, dz centered at $\mathbf{r}$ and with sides parallel to the xy, xz, and yz planes. See Fig. 3.9. To first order (infinitesimal $d\mathbf{r}$ and dt), the density of fluid exiting the parallelepiped per unit time

Curl, $\nabla\times$

Another possible operation with the vector operator ∇ is to take its cross product with a vector. Using the established formula for the cross product, and being careful to write the derivatives to the left of the vector on which they are to act, we obtain

$$\nabla\times\mathbf{V}=\hat{\mathbf{e}}_x\left(\frac{\partial}{\partial y}V_z-\frac{\partial}{\partial z}V_y\right)+\hat{\mathbf{e}}_y\left(\frac{\partial}{\partial z}V_x-\frac{\partial}{\partial x}V_z\right)+\hat{\mathbf{e}}_z\left(\frac{\partial}{\partial x}V_y-\frac{\partial}{\partial y}V_x\right)$$

$$=\begin{vmatrix}\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ V_x & V_y & V_z\end{vmatrix}. \tag{3.58}$$

This vector operation is called the **curl** of $\mathbf{V}$. Note that when the determinant in Eq. (3.58) is evaluated, it must be expanded in a way that causes the derivatives in the second row to be applied to the functions in the third row (and not to anything in the top row); we will encounter this situation repeatedly, and will identify the evaluation as being **from the top down**

le 3.5.6 CURL OF A CENTRAL FORCE FIELD

Calculate $\nabla\times[f(r)\hat{\mathbf{r}}]$. Writing

$$\hat{\mathbf{r}}=\frac{x}{r}\hat{\mathbf{e}}_x+\frac{y}{r}\hat{\mathbf{e}}_y+\frac{z}{r}\hat{\mathbf{e}}_z,$$

and remembering that $\partial r/\partial y=y/r$ and $\partial r/\partial z=z/r$, the x-component of the result is found to be

$$\begin{aligned}\left[\nabla\times[f(r)\hat{\mathbf{r}}]\right]_x&=\frac{\partial}{\partial y}\frac{zf(r)}{r}-\frac{\partial}{\partial z}\frac{yf(r)}{r}\\&=z\left(\frac{d}{dr}\frac{f(r)}{r}\right)\frac{\partial r}{\partial y}-y\left(\frac{d}{dr}\frac{f(r)}{r}\right)\frac{\partial r}{\partial z}\\&=z\left(\frac{d}{dr}\frac{f(r)}{r}\right)\frac{y}{r}-y\left(\frac{d}{dr}\frac{f(r)}{r}\right)\frac{z}{r}=0.\end{aligned}$$

By symmetry, the other components are also zero, yielding the final result

$$\nabla\times[f(r)\hat{\mathbf{r}}]=0. \tag{3.59}$$

We close the discussion by noting that a vector whose curl is zero everywhere is termed **irrotational**. This property is in a sense the opposite of solenoidal, and deserves a parallel degree of emphasis:

$$\nabla\times\mathbf{B}=0\ \text{ everywhere}\quad\longrightarrow\quad\mathbf{B}\text{ is irrotational}. \tag{3.61}$$

25. If $f=f(x, y, z)$, show that (a) div $\mathbf{r}=3$ and (b) curl $\mathbf{r}=0$.

26. If f(x, y, z) has partial derivatives of order at least two, show that (a) $\nabla\times\nabla f=0$; (b) $\nabla\cdot(\nabla\times f)=0$; (c) $\nabla\cdot\nabla f=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)f$.

8. When $\mathbf{B} = xy^2\mathbf{i} + 2x^2yz\,\mathbf{j} - 3yz^2\mathbf{k}$, find (a) div $\mathbf{B}$ and (b) curl $\mathbf{B}$.

(a)
$$\text{div}\,\mathbf{B} = \nabla\cdot\mathbf{B} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(xy^2\mathbf{i} + 2x^2yz\,\mathbf{j} - 3yz^2\mathbf{k})$$
$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$$
$$= y^2 + 2x^2z - 6yz$$

(b)
$$\text{curl}\,\mathbf{B} = \nabla\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$
$$= \left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2)\right]\mathbf{j}$$
$$+ \left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]\mathbf{k}$$
$$= -(3z^2 + 2x^2y)\mathbf{i} + (4xyz - 2xy)\mathbf{k}$$

Now we must realize the existence of only two possibilities for something originating from one place.

a. It drifts away (diverges) to another place (no curling or irrotational) or

b. it comes back (curls) to from where it originated (no divergence or solenoidal).

Irrotational Vector: a vector field whose curl is zero everywhere is termed **irrotational**.

This property is in a sense the opposite of solenoidal.

$\vec{\nabla}\times\vec{V} = 0$ everywhere , $\vec{V} = \vec{\nabla}\varphi$ is **irrotational**. (3.57)

Solenoidal Vector: a vector field whose divergence is zero everywhere is termed solenoidal.

$\vec{\nabla}\cdot\vec{A} = 0$ everywhere , $\vec{V} = \vec{\nabla}\times\vec{A}$ is solenoidal. (3.57)

A CENTRAL FORCE FIELD : scalar field : $\varphi = f(r)$ or vector field : $\vec{V} = f(r)\,\hat{r}$

1. Gradient of a CENTRAL FORCE FIELD : scalar field : $\varphi = f(r)$

$$\vec{\nabla}\, f(r) = \hat{\imath}\frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial x} + \hat{\jmath}\frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial z} = \frac{\partial f(r)}{\partial r}\left(\hat{\imath}\frac{\partial r}{\partial x} + \hat{\jmath}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right) = \frac{\partial f(r)}{\partial r}\vec{\nabla}\,r$$

2. Divergence OF A CENTRAL FORCE FIELD : vector field : $\vec{\mathbf{V}} = f(r)\,\hat{\boldsymbol{r}}$

$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{r}} = 3$

3. CURL OF A CENTRAL FORCE FIELD : vector field : $\vec{\mathbf{V}} = f(r)\,\hat{\boldsymbol{r}}$

$\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{V}} = \vec{\boldsymbol{\nabla}}\times f(r)\,\hat{\boldsymbol{r}} = 0$ (3.59)

Exercises 3.5 Differential Vector Operators

.1 If $S(x,y,z) = (x^2+y^2+z^2)^{-3/2}$, find

(a) ∇S at the point $(1,2,3)$, Ans: (a) $-3(14)^{-5/2}(\hat{\mathbf{x}}+2\hat{\mathbf{y}}+3\hat{\mathbf{z}})$.

(b) the magnitude of the gradient of S, $|\nabla S|$ at $(1,2,3)$, and Ans: (b) $3/196$.

(c) the direction cosines of ∇S at $(1,2,3)$. Ans: (c) $-1/(14)^{1/2}, -2/(14)^{1/2}, -3/(14)^{1/2}$.

.2 (a) Find a unit vector perpendicular to the surface $x^2+y^2+z^2=3$ at the point $(1,1,1)$. Ans (a) $(\hat{\mathbf{e}}_x+\hat{\mathbf{e}}_y+\hat{\mathbf{e}}_z)/\sqrt{3}$,

3 Given a vector $\mathbf{r}_{12} = \hat{\mathbf{e}}_x(x_1-x_2)+\hat{\mathbf{e}}_y(y_1-y_2)+\hat{\mathbf{e}}_z(z_1-z_2)$, show that $\nabla_1 r_{12}$ (gradient with respect to x_1, y_1, and z_1 of the magnitude r_{12}) is a unit vector in the direction of $\mathbf{r}_{12}$.

Ans:

3. From $r_{12} = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$ we obtain $\nabla_1 r_{12} = \dfrac{\mathbf{r}_1-\mathbf{r}_2}{r_{12}} = \hat{\mathbf{r}}_{12}$ by differentiating componentwise.

.5 Show that $\nabla(uv) = v\nabla u + u\nabla v$, where u and v are differentiable scalar functions of x, y, and z.

Ans:

.5. $\nabla(uv) = v\nabla u + u\nabla v$ follows from the product rule of differentiation.

.6 For a particle moving in a circular orbit $\mathbf{r} = \hat{\mathbf{e}}_x r\cos\omega t + \hat{\mathbf{e}}_y r\sin\omega t$:

(a) Evaluate $\mathbf{r}\times\dot{\mathbf{r}}$, with $\dot{\mathbf{r}} = d\mathbf{r}/dt = \mathbf{v}$.

(b) Show that $\ddot{\mathbf{r}}+\omega^2\mathbf{r}=0$ with $\ddot{\mathbf{r}} = d\mathbf{v}/dt$. *Hint.* The radius r and the angular velocity ω are constant.

Ans:

6. (a) From $\dot{\mathbf{r}} = \omega r(-\hat{\mathbf{x}}\sin\omega t + \hat{\mathbf{y}}\cos\omega t)$, we get $\mathbf{r}\times\dot{\mathbf{r}} = \hat{\mathbf{z}}\omega r^2(\cos^2\omega t + \sin^2\omega t) = \hat{\mathbf{z}}\omega r^2$.

(b) Differentiating $\dot{\mathbf{r}}$ above we get $\ddot{\mathbf{r}} = -\omega^2 r(\hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t) = -\omega^2\mathbf{r}$.

.8 Show, by differentiating components, that just like the derivative of the product of two algebraic functions.

(a) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \dfrac{d\mathbf{A}}{dt}\cdot\mathbf{B}+\mathbf{A}\cdot\dfrac{d\mathbf{B}}{dt}$, (b) $\dfrac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \dfrac{d\mathbf{A}}{dt}\times\mathbf{B}+\mathbf{A}\times\dfrac{d\mathbf{B}}{dt}$,

Ans:

The product rule directly implies (a) and (b).

.9 Prove $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$. *Hint*. Treat as a scalar triple product.

Ans:

The product rule of differentiation in conjunction with $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ etc. gives the required result

10 Classically, orbital angular momentum is given by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p}$ is the linear momentum. To go from classical mechanics to quantum mechanics, $\mathbf{p}$ is replaced (in units with $\hbar = 1$) by the operator $-i\boldsymbol{\nabla}$. Show that the quantum mechanical angular momentum operator has Cartesian components

$$L_x = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right),\; L_y = -i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right),\; L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Ans:

If $\mathbf{L} = -i\mathbf{r} \times \boldsymbol{\nabla}$, then the determinant form of the cross product gives above results.

11 Using the angular momentum operators previously given, show that they satisfy commutation relations of the form $[L_x, L_y] \equiv L_x L_y - L_y L_x = iL_z$ and hence $\mathbf{L} \times \mathbf{L} = i\mathbf{L}$.

These commutation relations will be taken later as the defining relations of an angular momentum operator.

Ans:

Carry out the indicated operations, remembering that derivatives operate on everything to their right in the current expression as well as on the function to which the operator is applied. Therefore,

$$L_xL_y = -\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right]\left[z\frac{\partial}{\partial x} - x,\frac{\partial}{\partial z}\right] = -\left[y\frac{\partial}{\partial x} + yz\frac{\partial^2}{\partial z\partial x} - z^2\frac{\partial^2}{\partial y\partial x} - xy\frac{\partial^2}{\partial z^2} + zx\frac{\partial^2}{\partial y\partial z}\right].$$

$$L_yL_x = -\left[z\frac{\partial}{\partial x} - x,\frac{\partial}{\partial z}\right]\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right] = -\left[zy\frac{\partial^2}{\partial x\partial z} - xy\frac{\partial^2}{\partial z^2} - z^2\frac{\partial^2}{\partial x\partial y} + xz\frac{\partial^2}{\partial z\partial y} + x\frac{\partial}{\partial y}\right].$$

Combining the above, we get

$$L_xL_y - L_yL_x = x\frac{\partial}{\partial y} - u\frac{\partial}{\partial x} = iL_z.$$

3.6 DIFFERENTIAL VECTOR OPERATORS: FURTHER PROPERTIES

We initially had (i) a scalar field φ and
(ii) two vectors: vector field $\vec{A}$, (differential operator) vector $\vec{\nabla}$

→ These yielded $\vec{\nabla}\varphi$ and the two possible operations between vectors: $\vec{\nabla}\cdot\vec{A}$ and $\vec{\nabla}\times\vec{A}$
We now have the 4 vectors: $\vec{\nabla}$, $\vec{\nabla}\varphi$ and $\vec{\nabla}\times\vec{A}$ vector field $\vec{A}$ and
2 scalars: $\vec{\nabla}\cdot\vec{A}$ and scalar field φ

→ Further combinations: Successive Applications of $\vec{\nabla}$

(i) $\vec{\nabla}$ with vector $\vec{\nabla}\varphi$ → dot product -- $\vec{\nabla}\cdot\vec{\nabla}\varphi$ and cross product -- $\vec{\nabla}\times\vec{\nabla}\varphi$
(ii) $\vec{\nabla}$ with vector $\vec{\nabla}\times\vec{A}$ → dot product-- $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})$ and cross product -- $\vec{\nabla}\times\vec{\nabla}\times\vec{A}$
(iii) $\vec{\nabla}$ with scalar $\vec{\nabla}\cdot\vec{A}$ → $\vec{\nabla}(\vec{\nabla}\cdot\vec{A})$

Special cases: $\vec{\nabla}\times\vec{E}$ Vector $\vec{E}$ is irrotational
$\vec{\nabla}\cdot\vec{B}$ Vector $\vec{B}$ is solenoidal

➢ Prove that
(a) $\vec{\nabla}\times\vec{\nabla}\varphi = 0$ Gradient cannot curl (it is irrotational)
(b) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ curl cannot diverge (it is solenoidal)

Ans: can be proven by direct application of determinant form.

Laplacian

The first of these expressions, $\nabla \cdot \nabla\varphi$, the divergence of the gradient, is named the Laplacian of φ. We have

$$\nabla \cdot \nabla\varphi = \left(\hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\right) \cdot \left(\hat{\mathbf{e}}_x \frac{\partial\varphi}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial\varphi}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial\varphi}{\partial z}\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}. \qquad (3.62)$$

When φ is the electrostatic potential, we have

$$\nabla \cdot \nabla\varphi = 0 \qquad (3.63)$$

at points where the charge density vanishes, which is Laplace's equation of electrostatics. Often the combination $\nabla \cdot \nabla$ is written ∇^2, or Δ in the older European literature.

Example 3.6.1 LAPLACIAN OF A CENTRAL FIELD POTENTIAL

Calculate $\nabla^2\varphi(r)$. Using Eq. (3.51) to evaluate $\nabla\varphi$ and then Eq. (3.53) for the divergence, we have

$$\nabla^2\varphi(r) = \nabla \cdot \nabla\varphi(r) = \nabla \cdot \frac{d\varphi(r)}{dr}\hat{\mathbf{e}}_r = \frac{2}{r}\frac{d\varphi(r)}{dr} + \frac{d^2\varphi(r)}{dr^2}.$$

We get a term in addition to $d^2\varphi/dr^2$ because $\hat{\mathbf{e}}_r$ has a direction that depends on $\mathbf{r}$.

In the special case $\varphi(r) = r^n$, this reduces to

$$\nabla^2 r^n = n(n+1)r^{n-2}.$$

This vanishes for $n = 0$ ($\varphi =$constant) and for $n = -1$ (Coulomb potential). For $n = -1$, our derivation fails for $\mathbf{r} = 0$, where the derivatives are undefined. ■

$$\nabla \times \nabla\varphi = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\varphi/\partial x & \partial\varphi/\partial y & \partial\varphi/\partial z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix} \varphi = 0.$$

Expression (d) is a scalar triple product that may be written

$$\nabla \cdot (\nabla \times \mathbf{V}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} = 0.$$

This determinant also has two identical rows and yields zero if $\mathbf{V}$ has sufficient continuity.

Maxwell's Equations The unification of electric and magnetic phenomena that is encapsulated in Maxwell's equations provides an excellent example of the use of differential vector operators. In SI units, these equations take the form

$$\nabla \cdot \mathbf{B} = 0, \qquad (3.66)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad (3.67)$$

$$\nabla \times \mathbf{B} = \varepsilon_0\mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}, \qquad (3.68)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \qquad (3.69)$$

Here $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic induction field, ρ is the charge density, $\mathbf{J}$ is the current density, ε_0 is the electric permittivity, and μ_0 is the magnetic permeability, so $\varepsilon_0\mu_0 = 1/c^2$, where c is the velocity of light.

Example 3.6.2 ELECTROMAGNETIC WAVE EQUATION (Optional)

Even in vacuum, Maxwell's equations can describe electromagnetic waves. To derive an electromagnetic wave equation, we start by taking the time derivative of Eq. (3.68) for the case J = 0, and the curl of Eq. (3.69). We then have

$$\frac{\partial}{\partial t}\nabla \times \mathbf{B} = \epsilon_0\mu_0\frac{\partial^2 \mathbf{E}}{\partial t^2},$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}\nabla \times \mathbf{B} = -\epsilon_0\mu_0\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla \cdot \nabla \mathbf{V}. \quad (3.70)$$

The term $\nabla \cdot \nabla \mathbf{V}$, which is called the **vector Laplacian** and sometimes written $\nabla^2\mathbf{V}$ in vacuum, $\nabla \cdot \mathbf{E} = 0$. The result is the vector electromagnetic wave equation for **E**,

$$\nabla^2 \mathbf{E} = \epsilon_0\mu_0\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}. \quad (3.71)$$

Equation (3.71) separates into three scalar wave equations, each involving the (scalar) Laplacian. There is a separate equation for each Cartesian component of $\vec{\boldsymbol{E}}$.

DIFERENTIAL VECTOR OPERATORS: FURTHER PROPERTIES : Problems

.1 Show that $\mathbf{u} \times \mathbf{v}$ is solenoidal if $\mathbf{u}$ and $\mathbf{v}$ are each irrotational.

Ans:

1. By definition, $\mathbf{u} \times \mathbf{v}$ is solenoidal if $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = 0$. But we have the identity

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = v \cdot (\nabla \times u) - u \cdot (\nabla \times v).$$

If a vector $\mathbf{w}$ is irrotational, $\nabla \times \mathbf{w} = 0$, so if $\mathbf{u}$ and $\mathbf{v}$ are both irrotational, the right-hand side of the above equation is zero, proving that $\mathbf{u} \times \mathbf{v}$ is solenoidal.

.2 If $\mathbf{A}$ is irrotational, show that $\mathbf{A} \times \mathbf{r}$ is solenoidal.

Ans:

.2. If $\nabla \times \mathbf{A} = 0$, then $\nabla \cdot (\mathbf{A} \times \mathbf{r}) = \mathbf{r} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot (\nabla \times \mathbf{r}) = 0 - 0 = 0$.

.3 A rigid body is rotating with constant angular velocity $\boldsymbol{\omega}$. Show that the linear velocity $\mathbf{v}$ is solenoidal.

Ans:

3. From $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ we get $\nabla \cdot (\boldsymbol{\omega} \times \mathbf{r}) = -\boldsymbol{\omega} \cdot (\nabla \times \mathbf{r}) = 0$.

.4 If a vector function $\mathbf{V}(x, y, z)$ is not irrotational, show that if there exists a scalar function $g(x, y, z)$ such that $g\mathbf{V}$ is irrotational, then

$$\mathbf{V} \cdot \nabla \times \mathbf{V} = 0.$$

Ans:

4. Forming the scalar product of $\mathbf{f}$ with the identity

$$\nabla \times (g\mathbf{f}) = g\nabla \times \mathbf{f} + (\nabla g) \times \mathbf{f} \equiv 0$$

we obtain the result, because the second term of the identity is perpendicular to $\mathbf{f}$.

5 Verify the vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{A}) + \mathbf{A}(\nabla \cdot \mathbf{B}).$$

Ans:

5. Applying the BAC-CAB rule naively we obtain $(\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B}$, where ∇ still acts on $\mathbf{A}$ and $\mathbf{B}$. Thus, the product rule of differentiation generates two terms out of each which are ordered so that ∇ acts only on what comes after the operator. That is, $(\nabla \cdot \mathbf{B})\mathbf{A} \to \mathbf{A}(\nabla \cdot \mathbf{B}) + (\mathbf{B} \cdot \nabla)\mathbf{A}$, and similarly for the second term. Hence the four terms.

.6 As an alternative to the vector identity of Example 3.6.4 show that

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \times \nabla) \times \mathbf{B} + (\mathbf{B} \times \nabla) \times \mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) + \mathbf{B}(\nabla \cdot \mathbf{A}).$$

Ans:

.6. Write the x components of all the terms on the right-hand side of this equation. We get

$$[(\mathbf{A} \times \nabla) \times \mathbf{B}]_x = A_z \frac{\partial B_z}{\partial x} - A_x \frac{\partial B_z}{\partial z} - A_x \frac{\partial B_y}{\partial y} + A_y \frac{\partial B_y}{\partial x},$$

$$[(\mathbf{B} \times \nabla) \times \mathbf{A}]_x = B_z \frac{\partial A_z}{\partial x} - B_x \frac{\partial A_z}{\partial z} - B_x \frac{\partial A_y}{\partial y} + B_y \frac{\partial A_y}{\partial x},$$

$$[\mathbf{A}(\nabla \cdot \mathbf{B})]_x = A_x \frac{\partial B_x}{\partial x} + A_x \frac{\partial B_y}{\partial y} + A_x \frac{\partial B_z}{\partial z},$$

$$[\mathbf{B}(\nabla \cdot \mathbf{A})]_x = B_x \frac{\partial A_x}{\partial x} + B_x \frac{\partial A_y}{\partial y} + B_x \frac{\partial A_z}{\partial z}.$$

All terms cancel except those corresponding to the x component of the left-hand side of the equation.

.8 If $\mathbf{A}$ and $\mathbf{B}$ are constant vectors, show that

$$\nabla(\mathbf{A} \cdot \mathbf{B} \times \mathbf{r}) = \mathbf{A} \times \mathbf{B}.$$

Ans:

.8. $$\nabla(\mathbf{A} \cdot \mathbf{B} \times \mathbf{r}) = \nabla(\mathbf{r} \cdot \mathbf{A} \times \mathbf{B}) = \hat{\mathbf{e}}_x(\mathbf{A} \times \mathbf{B})_x + \hat{\mathbf{e}}_y(\mathbf{A} \times \mathbf{B})_y + \hat{\mathbf{e}}_z(\mathbf{A} \times \mathbf{B})_z$$

$$= \mathbf{A} \times \mathbf{B}.$$

.9 Verify Eq. (3.70),

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla \cdot \nabla \mathbf{V},$$

by direct expansion in Cartesian coordinates.

Ans:

9. It suffices to check one Cartesian component; we take x. The x component of the left-hand side of Eq. (3.70) is

$$\frac{\partial}{\partial y}(\nabla \times \mathbf{V})_z - \frac{\partial}{\partial z}(\nabla \times \mathbf{V})_y = \frac{\partial^2 V_y}{\partial y \partial x} - \frac{\partial^2 V_x}{\partial y^2} - \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_z}{\partial z \partial x}.$$

The x component of the right-hand side is

$$\frac{\partial}{\partial x}\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right] - \left[\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right].$$

After canceling the two right-hand-side occurrences of $\partial^2 V_x/\partial x^2$ these two expressions contain identical terms.

10 Prove that $\nabla \times (\varphi \nabla \varphi) = 0$.

Ans:

10. $\nabla \times (\varphi \nabla \varphi) = \nabla \varphi \times \nabla \varphi + \varphi \nabla \times (\nabla \varphi) = 0 + 0 = 0.$

11 You are given that the curl of **F** equals the curl of **G**. Show that **F** and **G** may differ by (a) a constant and (b) a gradient of a scalar function.

Ans:

11. (a) If **F** or **G** contain an additive constant, it will vanish on application of any component of ∇.

(b) If either vector contains a term ∇f, it will not affect the curl because $\nabla \times (\nabla f) = 0$.

.12 The Navier-Stokes equation of hydrodynamics contains a nonlinear term of the form $(\mathbf{v} \cdot \nabla)\mathbf{v}$. Show that the curl of this term may be written as $-\nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})]$.

Ans:

.**12.** Use the identity $\mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)\mathbf{v}$. Taking the curl and noting that the first term on the right-hand side then vanishes, we obtain the desired relation.

13 Prove that $(\nabla u) \times (\nabla v)$ is solenoidal, where u and v are differentiable scalar functions.

Ans:

.**13.** Using Exercise 3.5.9,

$$\nabla \cdot (\nabla u \times \nabla v) = (\nabla v) \cdot (\nabla \times \nabla u) - (\nabla u) \cdot (\nabla \times \nabla v) = 0 - 0 = 0.$$

.14 The function φ is a scalar satisfying Laplace's equation, $\nabla^2\varphi = 0$. Show that $\nabla\varphi$ is **both** solenoidal and irrotational.

Ans:

.**14.** $\nabla^2\varphi = \nabla \cdot \nabla\varphi = 0$, and $\nabla \times \nabla\varphi = 0$.

.15 Show that any solution of the equation

$$\nabla \times (\nabla \times \mathbf{A}) - k^2\mathbf{A} = 0$$

automatically satisfies the vector Helmholtz equation

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = 0$$

and the solenoidal condition

$$\nabla \cdot \mathbf{A} = 0.$$

Hint. Let $\nabla \cdot$ operate on the first equation.

Ans:

.**15.** From Eq. (3.70), $\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2\mathbf{A}$ if $\nabla \cdot \mathbf{A} = 0$.

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