17. INTRODUCTION TO GROUP THEORY

Central to all the symmetry notions is the fact that complete sets of symmetry operations form what in mathematics are known as groups. The elements of a group may be finite in number, in which case the group is then termed **finite** or **discrete**, as for example the symmetry operations on an equilateral triangle or square. But alternatively, the symmetry operations may be infinite in number and described by continuously variable parameter(s); such groups are termed **continuous**. An example of a continuous group is the set of possible rotational displacements of a circular object about its axis (in which case the parameter is the rotation angle).

DEFINITION OF A GROUP

A group *G* is defined as a set of objects or operations (e.g., rotations, permutations or other transformations), called the elements of *G*, that may be combined, by a procedure to be called **multiplication** and denoted by *, to form a well-defined **product**, subject to the following four conditions:

1. If *a* and *b* are any two elements of *G*, then the product $a * b$ is also an element of *G*; more formally, $a * b$ associates an element of *G* with the ordered pair (a, b) of elements of *G*. In other words, *G* is **closed** under multiplication of its own elements.
2. This multiplication is associative: (a* b)*c = a*(b*c)
3. There is a unique identity element I in *G*, such that $I * a = a * I = a$ for every element *a* in *G*.
4. Each element *a* of *G* has an inverse, denoted a^{-1}, such that $a * a^{-1} = a^{-1} * a = I.$

The above simple rules have a number of direct consequences, including the following:

- It can be shown that the inverse of any element *a* is unique:

  If a^{-1} and $\hat{a}^{-1}$ are both inverses of *a*, then $\hat{a}^{-1} = \hat{a}^{-1} * (a * a^{-1}) = (\hat{a}^{-1} * a) * a^{-1} = a^{-1}.$
- The products $g * a$, where *a* is fixed and *g* ranges over all elements of the group, consist (in some order) of all the elements of the group. If *g* and *g*' produce the same element, then g *a = g'*a. Multiplying on the right by a^{-1}, we get $(g * a) * a^{-1} = (g' * a) * a^{-1}$, which reduces to g = g'.

  ➢ Here are some useful conventions and further definitions:

- The * for multiplication is tedious to write; when no ambiguity will result it is customary to drop it, and instead of $a * b$ we write *ab*.
- When *a* and *b* are operations, and *ab* is to be applied to an object appearing to their right, *b* is deemed to act first, with *a* then applied to the result of operation with *b*.
- If a *discrete* group possesses *n* elements (including *I*), its **order** is *n*.
- A *continuous* group of order *n* has elements that are defined by *n* parameters.

Abelian group:

- If $ab = ba$ for all a, b of G, the multiplication is **commutative**, and the group is called **abelian**.

Cyclic group:

- If a group possesses an element a such that the sequence $I,\ a,\ a^2(=aa),\ a^3,\ \cdots$ includes all elements of the group, it is termed **cyclic**.
- If a group is cyclic, it must also be abelian. However, not all abelian groups are cyclic.

Subgroup *H*

- If a subset H of G is closed under the multiplication defined for G, it is also a group and called a **subgroup** of G.
- The identity I of G always forms a subgroup of G.

Isomorphic and Homomorphic groups

- Two groups $\{I, a, b, \cdots\}$ and $\{I', a', b', \cdots\}$ are **isomorphic** if their elements can be put into one-to-one correspondence such that for all a and b, $ab = c \Longleftrightarrow a'b' = c'$.
- If the correspondence is many-to-one, the groups are **homomorphic**.

Examples of Groups:

1. SYMMETRY OF AN EQUILATERAL TRIANGLE - $\boldsymbol{D}_3$ (Dihedral) Group

The symmetry operations of an equilateral triangle form a finite group with six elements.

The triangle can be placed either side up, and with any vertex in the top position. Figure 17.1 is a schematic diagram indicating these symmetry operations, and Fig. 17.2 shows their result, with the vertices of the triangle numbered to show the effect of each operation. The six operations that convert the initial orientation into symmetry equivalents are:

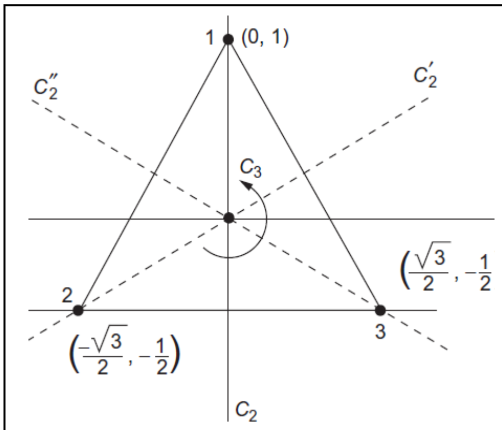


Fig. 17.1

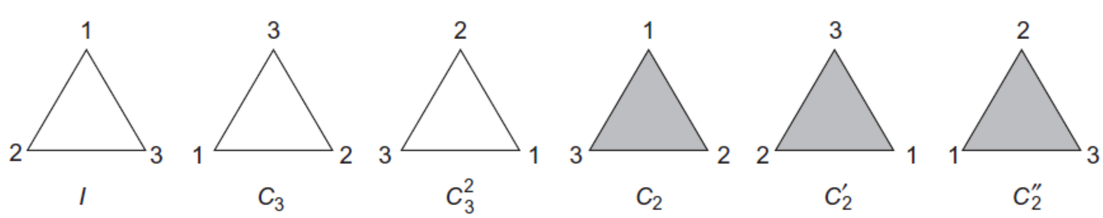


Fig. 17.2 : Result of applying the symmetry operations identified in Fig. 17.1 to an equilateral triangle. One side of the triangle is shaded to make it obvious when that side is up.

I - the identity operation that makes no orientation change,

C_3 - an operation which rotates the triangle counter-clockwise about the center by 120^o,

C_3^2 - corresponds to two successive C_3 operations,

C_2, - 1/2 revolution (by 180^o) about an axis passing through 1 in the plane of the triangle,

$C_2^{`}$, - 1/2 revolution (by 180^o) about an axis passing through 2 in the plane of the triangle, and

$C_2^{``}$ - 1/2 revolution (by 180^o) about an axis passing through 3 in the plane of the triangle.

The multiplication table for the group is shown in Table 17.1, where the product ab (which describes the result of first applying operation b, and then operation a) is the group element listed in row a and column b of the table.

From the multiplication table or by examination of the symmetry operations themselves, we can see that the inverse of I is I, the inverse of C_3 is C_3^2 (so the inverse of C_3^2 is C_3), and each C_2 is its own inverse. This group is not abelian; $C_3C_2 \neq C_2C_3$ ($C_3C_2 = C_2^{``}$, while $C_2C_3 = C_2^{`}$).

Table 17.1 Multiplication Table for Group D_3

	I	C_3	C_3^2	C_2	C_2'	C_2''
I	I	C_3	C_3^2	C_2	C_2'	C_2''
C_3	C_3	C_3^2	I	C_2''	C_2	C_2'
C_3^2	C_3^2	I	C_3	C_2'	C_2''	C_2
C_2	C_2	C_2'	C_2''	I	C_3	C_3^2
C_2'	C_2'	C_2''	C_2	C_3^2	I	C_3
C_2''	C_2''	C_2	C_2'	C_3	C_3^2	I

Operations are pictured in Fig. 17.2. The table entry for row a and columr is the product element ab. For example, $C_2C_3 = C_2'$.

2. **AN ABSTRACT GROUP** : The **Vierergruppe** (German: four-membered group) is a C_4 group

Groups do not need to represent geometric operations. Consider a set of four quantities (elements) I , A, B, C, with our knowledge about them only that when any two are multiplied, the result is an element of the set. The multiplication table of this four-element set is shown in Table 17.2. These elements form a group, because each has an inverse (itself), there is an identity element (I), and the set is closed under multiplication.

Multiplication Table for the Vierergruppe

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

Table 17.2

Assignment: Determine whether the Vierergruppe C_4 group is cyclic and whether it is abelian.

Assignment: (a) Show that the permutations of *3* distinct objects (abc) satisfy the group postulates.

(b) Construct the multiplication table for the permutations of these three objects (abc), giving each permutation a name of some sort. (Suggestion: Use *I* for the permutation that leaves the order unchanged.)

(c) Show that this permutation group (named S_3) is isomorphic with D_3 and identify corresponding operations. Is your identification unique?

❖ GROUP : C_4

The symmetry operations of a square that cannot be turned over form a four-membered group sometimes called C_4 whose elements can be named I , C_4 (90^o rotation), C_2 (180^o rotation), C_4^2 (270^o rotation).

Assignment: Write the multiplication table of this group (C_4).

❖ GROUP : I_4

The four complex numbers *{1, i , -1, -i }* also form a group when the group operation is ordinary multiplication.

❖ Isomorphism

The groups C_4 and I_4 are isomorphic in the sense that their elements can be put into (one to one) correspondence in two different ways:

$$I \leftrightarrow 1, \quad C_4 \leftrightarrow i, \quad C_2 \leftrightarrow -1, \quad C_4' \leftrightarrow -i$$

$$I \leftrightarrow 1, \quad C_4 \leftrightarrow -i, \quad C_2 \leftrightarrow -1, \quad C_4' \leftrightarrow i.$$

Assignment: Write the multiplication table of both groups (I_4 and C_4) and illustrate the isomerism by comparing them.

❖ Homorphism

If the correspondence between the elements of two groups is *not* one to one then they are homomorphic.

❖ Subgroup

If A subset H of a group *G satisfies group postulates, it is called* a subgroup *H* with elements h_i .

Note: Identity is a (trivial) subgroup of any group. So every group has at least one subgroup.

Assignment: Form various subgroups of the groups: $\mathbf{D_3}$, $\mathbf{C_4}$, $\mathbf{I_4}$, $\mathbf{V_4}$ (described below) etc.

❖ Conjugate Subgroup

Let x be a *fixed* element of the original group *G* and **not** a member of *H*.

The transform $x\, h_i\, x^{-1}$, (i = 1, 2, …….) generates a **conjugate subgroup:** $x\, H\, x^{-1}$ with respect to the element x .

Assignment: *(a)* Show that this conjugate subgroup $x\, H\, x^{-1}$ satisfies each of the four group postulates and therefore is a group.

(b) How many conjugate subgroups can one form for a given subgroup H ?

Problems:

1. The **Vierergruppe** $\mathbf{V_4}$ (German: four-membered group) is a group different from the the groups C_4 and I_4 . The Vierergruppe has the multiplication table shown here. Determine whether this group is cyclic and whether it is abelian.

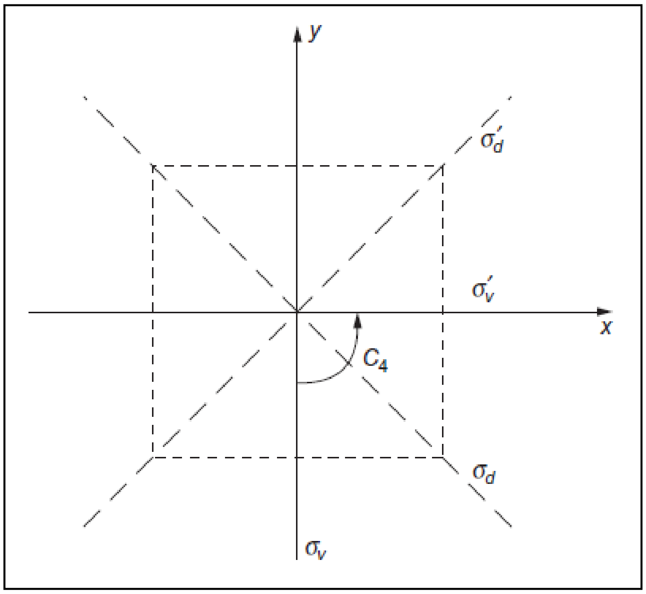
	I	*A*	*B*	*C*
I	*I*	*A*	*B*	*C*
A	*A*	*I*	*C*	*B*
B	*B*	*C*	*I*	*A*
C	*C*	*B*	*A*	*I*

2. **Rearrangement theorem:** Given a group of distinct elements, G = {*I, a, b, c,…….., n*}, show (*using a known group*) that the set of products {*aI*, *aa*, *ab*, *ac*, ……., *an*} reproduces all the group elements in a new order.

3. (a) A particular group is abelian. A second group is created by replacing g_i by g_i^{-1} for each element in the original group. Show that the two groups are isomorphic. *Note*. This means showing that if $ab = c$, then $a^{-1}b^{-1} = c^{-1}$.

(b) Continuing part (a), show that the second group is also abelian.

4. Write the multiplication table for set of symmetries of a square **C_{4v}** which in addition to rotations [C_4 (90^o rotation), C_2 (180^o rotation), C_4^2 (270^o rotation)] can also be turned (flipped) along x axis (σ_x), y axis (σ_y) and the two diagonals (σ_d) and (σ_e). Does this set form a Group ?



Note: For clarity we have changed σ_d' shown in fig to σ_e and also σ_y' shown in fig to σ_x .

CLASSES

It has been found useful to divide the elements of a finite group G into sets called **classes**.

Starting from a group element a_1, one can apply similarity transformations of the form ga_1g^{-1}, where g can be any member of G. If we let a_1 be transformed in this way, using all the elements g of G, the result will be a set of elements that we can denote a_1, ………………, a_k , where k may or may not be larger than 1. Certainly this set will include $a1$ itself, as that result is obtained when $g = I$ and also when $g = a_1$ or $g = a_1^{-1}$.

The set of elements obtained in this way is called a **class** of G, and can be identified by *specifying one of its members*. If we choose $a_1 = I$, we find that I is in a class all by itself; often classes will have larger numbers of members.

Note: A class will have the same members no matter which of its elements is assigned the role of a_1. This is clear, since if $a_1 = g\, a_1\, g^{-1}$ then also $a_1 = g^{-1}\, a_1\, g$, showing that we can get a_1 from any other element of the class, and therefrom all the elements reachable from a_1.

Example: CLASSES OF THE TRIANGULAR GROUP D_3 (Set of symmetries of an equilateral triangle)

This group has 3 classes

As observed already in general, one class of D_3 will consist solely of I.

The class including C_3 contains also $C_3^2{}_2$ (the result of $C_2C_3C_2^{-1}$).

Finally, C_2, C_2' and C_2'' constitute a third class.

Assignment: Find all the classes of each of the groups: $\mathbf{D_3}$ **,** $\mathbf{C_4}$ **,** $\mathbf{I_4}$ **,** $\mathbf{V_4}$, $\mathbf{C_{4v}}$ **.**
