13. GAMMA FUNCTION

The gamma function is probably the special function that occurs most frequently in the discussion of problems in physics. For integer values, as the factorial function, it appears in every Taylor expansion. As we shall later see, it also occurs frequently with half-integer arguments, and is needed for general nonintegral values in the expansion of many functions, e.g., Bessel functions of noninteger order. The gamma function (by itself) does not usually describe a physical quantity of interest, but rather tends to appear as a factor in expansions of physically relevant quantities.

13.1 DEFINITIONS, PROPERTIES

At least *three different* convenient definitions of the gamma function are in common use.

Our first task is to state these definitions, to develop some simple, direct consequences and to show the equivalence of the *three* forms.

1. Infinite Limit (*Euler*): The first definition, named after Euler, is

$$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z, \quad z \neq 0, -1, -2, -3, \ldots . \qquad (13.1)$$

Here ***z*** may be either *real* or *complex*. Replacing ***z*** with ***z +1***, we have

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{(z+1)(z+2)(z+3)\cdots(z+n+1)} n^{z+1}$$

$$= \lim_{n\to\infty} \frac{nz}{z+n+1} \cdot \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z = z\Gamma(z). \qquad (13.2)$$

This is the basic functional relation for the gamma function. It should be noted that it is a **difference** equation.

- *Assignment*: Prove the above relation Eq.(13.2).

➢ Also, from the definition,

$$\Gamma(1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{1\cdot 2\cdot 3\cdots n(n+1)} n = 1. \qquad (13.3)$$

Now, repeated application of Eq. (13.2) gives

$$\Gamma(2) = 1, \quad \Gamma(3) = 2\,\Gamma(2) = 2, \quad \Gamma(4) = 3\,\Gamma(3) = 2\cdot 3, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots \quad \Gamma(n) = 1\cdot 2\cdot 3\cdots(n-1) = (n-1)!. \qquad (13.4)$$

- *Assignment*: Prove the above relation Eq.(13.3).

2. Definite Integral (Euler): A second definition, also frequently called the Euler integral is

$$\Gamma(z) \equiv \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re e(z) > 0. \qquad (13.5)$$

3. Infinite Product (Weierstrass): The third definition (Weierstrass' form) is the infinite product

$$\frac{1}{\Gamma(z)} \equiv z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}, \qquad (13.15)$$

where γ is the Euler-Mascheroni constant $\gamma = 0.5772156619\cdots$,

- **Functional Relation**

In Eq. (13.2) we already obtained the most important functional relation for the gamma function,

$$\Gamma(z+1) = z\,\Gamma(z). \qquad (13.22)$$

The gamma function satisfies several other functional relations, of which one of the most interesting is the **reflection formula**,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}. \qquad (13.23)$$

A special case of Eq. (13.23) results if we set $z = 1/2$. Then (taking the positive square root), we get

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \qquad (13.26)$$

Another functional relation is **Legendre's duplication formula**,

$$\Gamma(1+z)\,\Gamma\left(z+\frac{1}{2}\right) = 2^{-2z}\sqrt{\pi}\;\Gamma(2z+1), \qquad (13.27)$$

Assuming z to be a nonnegative integer n, we start the proof by writing $\Gamma(n+1) = n!$, $\Gamma(2n+1) = (2n)!$, and

$$\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)\cdot\left[\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}\right] = \sqrt{\pi}\,\frac{1\cdot 3\cdots(2n-1)}{2^n} = \sqrt{\pi}\,\frac{(2n-1)!!}{2^n}, \qquad (13.28)$$

where we have used Eq. (13.26) and the double factorial notation first introduced in Eqs. (1.75) and (1.76). The double factorial notation is used frequently enough in physics applications that a familiarity with it is essential, and will from here on be used without comment. Making the further observation that $n! = 2^{-n}(2n)!!$, Eq. (13.27) follows directly.

Incidentally, we call attention to the fact that gamma functions with half-integer arguments appear frequently in physics problems, and Eq. (13.28) shows how to write them in closed form.

- Note: The "double factorial" notation indicates products of even or odd positive integers as follows:

$$\begin{aligned} 1\cdot 3\cdot 5\cdots(2n-1) &= (2n-1)!! \\ 2\cdot 4\cdot 6\cdots(2n) &= (2n)!!. \end{aligned} \qquad (1.75)$$

These are related to the regular factorials by

$$(2n)!! = 2^n\, n! \quad \text{and} \quad (2n-1)!! = \frac{(2n)!}{2^n\, n!}. \qquad (1.76)$$

Note that these relations include the special cases $0!! = (-1)!! = 1$.