REPRESENTATION OF GROUPS

All discrete groups and the continuous groups can also be represented by square matrices. That is, to each element of the group, we can associate a matrix.

- If U(a) is the matrix associated with a group element a and U(b) the matrix associated with element b, then the matrix product U(a)U(b) will be the matrix associated with product of the elements ab.
- In other words, the matrices have the same multiplication table as the group.
- We call these matrices U because they can be chosen to be unitary.
- It is not necessary that matrices U have a dimension *equal* to the *order* of the group.

Example: Here is a unitary representation of the group D_3 (symmetries of an equilateral triangle):

$$\mathsf{U}(I)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad \mathsf{U}(C_3)=\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\sqrt{3}\\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix},$$

$$\mathsf{U}(C_3^2)=\begin{pmatrix}-\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix},\qquad \mathsf{U}(C_2)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

$$\mathsf{U}(C_2')=\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{1}{2}\end{pmatrix},\qquad \mathsf{U}(C_2'')=\begin{pmatrix}-\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ -\frac{1}{2}\sqrt{3} & \frac{1}{2}\end{pmatrix}. \tag{17.2}$$

- Several features of this representation are apparent:
- The unit operation is represented by a unit matrix.
- The inverse of an operation is represented by the inverse of its matrix.

We can check that the U form a representation: From the multiplication table, we have

$$C_2\,C_3 = C_2'.$$

Let us now evaluate

$$\mathsf{U}(C_2)\mathsf{U}(C_3)=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\sqrt{3}\\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{1}{2}\end{pmatrix},$$

which is indeed $\mathrm{U}(\mathrm{C}_2')$.

Matrix multiplication is in general not commutative, and gives results that are consistent with the lack of commutativity of the group operations. For example consider

$$\mathrm{U}(\mathrm{C}_3)\mathrm{U}(\mathrm{C}_2)=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$$

which is $\mathrm{U}(\mathrm{C}_2'')$. Thus $\mathrm{U}(\mathrm{C}_3)\mathrm{U}(\mathrm{C}_2) \neq \mathrm{U}(\mathrm{C}_2)\mathrm{U}(\mathrm{C}_3)$

- The 2×2 representation shown above is **faithful**, meaning that each group element corresponds to a different matrix.
- In other words, our 2×2 representation of Eq,(17.2) is isomorphic with the original group D_3 .

Assignment: Verify that other products of group elements correspond to the products of the representation matrices shown in Eq.(17.2).

❖ Equivalent Representations

Representations that can be transformed into each other by application of a unitary transformation are termed **equivalent representations**.

Example: Consider what happens when we replace each U(g) by V U(g) V^{-1} where V is unitary.

Then the product U(g) . U(g'), which is some U(g''), becomes

$$V\,U(g)\,V^{-1} \,.\, V\,U(g')\,V^{-1} = V\,U(g)\,.\,U(g')\,V^{-1}$$

$$= V\,U(g'')\,V^{-1}$$

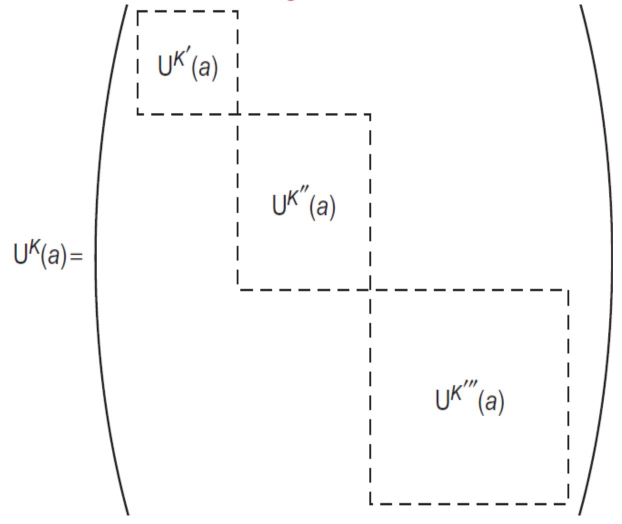
So the transformed matrices V U(g) V^{-1} still form a representation and it is **equivalent** representation (as U(g) is) of G.

REDUCIBLE AND IRREDUCIBLE REPRESENTATION (of G)

The possibility of unitary transformation also enables us to consider whether a representation of *G* is **reducible**.

An **irreducible** representation of *G* is defined as one that cannot be broken into a **direct sum** of representations of smaller dimension by application of the same unitary transformation to all members of the representation. What we mean by a direct sum of representations is that each matrix will be block diagonal (all with the same sequence of blocks). Since different blocks will not mix under matrix multiplication, corresponding blocks of the representation members will themselves define representations (see adjacent Fig. 17.4). If a representation named *K* is a direct sum of smaller representations K_1 and K_2, that fact can be indicated by the notation $K = K_1 \oplus K_2$.

Fig. 17.4



It is important to understand that **reducibility** implies the **existence** of a unitary transformation that brings ***all members of a representation to the same block-diagonal form;*** a reducible representation may not exhibit the block-diagonal form if it has not been subjected to a suitable unitary transformation.

Problems

1. Consider the following matrix representation of the symmetries of equilateral triangle: D_3

$$\mathsf{U}(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{U}(C_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathsf{U}(C_3^2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathsf{U}(C_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathsf{U}(C_2') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{U}(C_2'') = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{17.4}$$

Make unitary transformation $U' = V\,U\,V^{-1}$ of all the above matrices using the matrix

$$\mathsf{V} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -\sqrt{2/3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

. What is V^{-1}?

2. (a) Show that the following four matrices form a representation of the Vierergruppe, V_4

$$\mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{C} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

(b) Show that the above matrices 1, A, B, and C are reducible. Reduce them by transforming B and C to diagonal form (by the same unitary transformation using matrix, $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$).

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