PARTIAL DERIVATIVES : CHAIN RULES

Partial derivatives also can be defined for functions of three or more variables.
Note that partial derivatives may fail to exist when the required limits do not exist.

- **13G** ***Chain rule for $f(g(x, y))$:*** $\dfrac{\partial f}{\partial x} = \dfrac{df}{dg}\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial f}{\partial y} = \dfrac{df}{dg}\dfrac{\partial g}{\partial y}$. (1)

- If z(x,y)

  In Problems 5–9, find the first partial derivatives.

5. $z = 2x^2 - 3xy + 4y^2$.

Treating y as a constant and differentiating with respect to x yields $\dfrac{\partial z}{\partial x} = 4x - 3y$.

Treating x as a constant and differentiating with respect to y yields $\dfrac{\partial z}{\partial y} = -3x + 8y$

6. $z = \dfrac{x^2}{y} + \dfrac{y^2}{x}$.

Treating y as a constant and differentiating with respect to x yields $\dfrac{\partial z}{\partial x} = \dfrac{2x}{y} - \dfrac{y^2}{x^2}$..

Treating x as a constant and differentiating with respect to y yields $\dfrac{\partial z}{\partial y} = -\dfrac{x^2}{y^2} + \dfrac{2y}{x}$..

7. $z = \sin(2x + 3y)$.

$$\frac{\partial z}{\partial x} = 2\cos(2x+3y) \quad \text{and} \quad \frac{\partial z}{\partial y} = 3\cos(2x+3y)$$

8. $z = \tan^{-1}(x^2 y) + \tan^{-1}(xy^2)$.

$$\frac{\partial z}{\partial x} = \frac{2xy}{1+x^4y^2} + \frac{y^2}{1+x^2y^4} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{x^2}{1+x^4y^2} + \frac{2xy}{1+x^2y^4}$$

9. $z = e^{x^2+xy}$

$$\frac{\partial z}{\partial x} = e^{x^2+xy}(2x+y) \quad \text{and} \quad \frac{\partial z}{\partial y} = xe^{x^2+xy}$$

*** In Problems 11–13, find the first partial derivatives of *z* with respect to the independent variables *x* and *y*.**

11. $x^2 + y^2 + z^2 = c$

Differentiate implicitly with respect to x, treating y as a constant, to obtain:

$$2x + 2z\frac{\partial z}{\partial x} = 0. \quad \text{Hence,} \quad \frac{\partial z}{\partial x} = -\frac{x}{z}$$

Differentiate implicitly with respect to y, treating x as a constant:

$$2y + 2z\frac{\partial z}{\partial y} = 0. \quad \text{Hence,} \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

Q13: xy + yz + zx = c

Differentiating with respect to x yields $y+y\dfrac{\partial z}{\partial x}+x\dfrac{\partial z}{\partial x}+z=0$, whence $\dfrac{\partial z}{\partial x}=-\dfrac{y+z}{x+y}$.

Differentiating with respect to y yields $x+y\dfrac{\partial z}{\partial y}+z+x\dfrac{\partial z}{\partial y}=0$, whence $\dfrac{\partial z}{\partial y}=-\dfrac{x+z}{x+y}$.

14. Considering x and y as independent variables, find $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}, \dfrac{\partial \theta}{\partial x}, \dfrac{\partial \theta}{\partial y}$ when $x=e^{2r}\cos\theta$, $y=e^{3r}\sin\theta$.

First differentiate the given relations with respect to x:

$$1=2e^{2r}\cos\theta\frac{\partial r}{\partial x}-e^{2r}\sin\theta\frac{\partial \theta}{\partial x} \qquad \text{and} \qquad 0=3e^{3r}\sin\theta\frac{\partial r}{\partial x}+e^{3r}\cos\theta\frac{\partial \theta}{\partial x}$$

Then solve simultaneously to obtain $\dfrac{\partial r}{\partial x}=\dfrac{\cos\theta}{e^{2r}(2+\sin^2\theta)}$ and $\dfrac{\partial \theta}{\partial x}=-\dfrac{3\sin\theta}{e^{2r}(2+\sin^2\theta)}$.

Now differentiate the given relations with respect to y:

$$0=2e^{2r}\cos\theta\frac{\partial r}{\partial y}-e^{2r}\sin\theta\frac{\partial \theta}{\partial y} \qquad \text{and} \qquad 1=3e^{3r}\sin\theta\frac{\partial r}{\partial y}+e^{3r}\cos\theta\frac{\partial \theta}{\partial y}$$

Then solve simultaneously to obtain $\dfrac{\partial r}{\partial y}=\dfrac{\sin\theta}{e^{3r}(2+\sin^2\theta)}$ and $\dfrac{\partial \theta}{\partial y}=\dfrac{2\cos\theta}{e^{3r}(2+\sin^2\theta)}$.

In Problems 16 and 17, find all second partial derivatives of *z*

16. $x^2 + y^2 + 3xy$

$$\frac{\partial z}{\partial x}=2x+3y, \qquad \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=2, \qquad \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=3$$

$$\frac{\partial z}{\partial y}=3x+2y, \qquad \frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=2, \qquad \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=3$$

Note that $\dfrac{\partial^2 z}{\partial y\partial x}=\dfrac{\partial^2 z}{\partial x\partial y}$.

17. x cosy – y cosx

$$\frac{\partial z}{\partial x}=\cos y+y\sin x, \qquad \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=y\cos x$$

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=-\sin y+\sin x$$

$$\frac{\partial z}{\partial y}=-x\sin y-\cos x, \qquad \frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=-x\cos y$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=-\sin y+\sin x$$

Note that $\dfrac{\partial^2 z}{\partial y\partial x}=\dfrac{\partial^2 z}{\partial x\partial y}$.

Total Differential. Differentiability. Chain Rules

Total Differential

Let $z=f(x,y)$. Let Δx and Δy be any numbers. Δx and Δy are called *increments of x and y*, respectively. For these increments of x and y, the corresponding *change in z*, denoted Δz, is defined by

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y) \qquad (49.1)$$

The *total differential dz* is defined by:

$$dz=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y=f_x(x,y)\,\Delta x+f_y(x,y)\,\Delta y \qquad (49.2)$$

Note that, if $z = f(x, y) = x$, then $\frac{\partial z}{\partial x} = 1$ and $\frac{\partial z}{\partial y} = 0$, and, therefore, $dz = \Delta x$ So, $dx = \Delta x$.

Similarly, $dy = \Delta y$. Hence, equation (49.2) becomes

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = f_x(x,y)\, dx + f_y(x,y)\, dy \qquad (49.3)$$

Notation: dz is also denoted df.

These definitions can be extended to functions of three or more variables. For example, if $u = f(x, y, z)$, then we get:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = f_x(x,y,z)\, dx + f_y(x,y,z)\, dy + f_z(x,y,z)\, dz$$

EXAMPLE 49.1: Let $z = x \cos y - 2x^2 + 3$. Then $\frac{\partial z}{\partial x} = \cos y - 4x$ and $\frac{\partial z}{\partial y} = -x \sin y$. Then the total differential for z is

$dz = (\cos y - 4x)\, dx - (x \sin y)\, dy$.

In the case of a function of one variable $y = f(x)$, we used the approximation principle $\Delta y \sim f'(x)\, \Delta x = dy$ to estimate values of f. However, in the case of a function $z = f(x, y)$ of two variables, the function f has to satisfy a special condition in order to make good approximations possible.

Chain Rules Chain Rule ($2 \to 1$)

Let $z = f(x, y)$, where f is differentiable, and let $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of one variable.

Then $z = f(g(t)$ is a differentiable function of one variable, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \qquad (49.6)$$

warning: Note the double meaning of z, x, and y in (49.6). In $\frac{dz}{dt}$, z means $f(g(t), h(t))$, whereas, in $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, z means $f(x, y)$.

In $\frac{\partial z}{\partial x}$, x is an independent variable, whereas, in $\frac{dz}{dt}$, x means $g(t)$. Likewise, y has two different meanings.

To prove (49.6), note first that, by (49.4), $\Delta z = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$

Then $\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}$. Letting $\Delta t \to 0$, we obtain

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + 0(\Delta x) + 0(\Delta y) = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

(Note that, since g and h are differentiable, they are continuous.

Hence, as $\Delta t \to 0$, $\Delta x \to 0$ and $\Delta y \to 0$ and, therefore, $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$.)

13H ***Chain rule for $f(x(t), y(t))$:*** $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \qquad (3)$$

For $f(x, y, t)$ the chain rule is $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}. \qquad (4)$$

EXAMPLE 49.4: Let $z = xy + \sin x$ and let $x = t^2$ and $y = \cos t$. Note that $\frac{\partial z}{\partial x} = y + \cos x$ and $\frac{\partial z}{\partial y} = x$.

Moreover, $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = -\sin t$. Now, as a function of $z = t^2 \cos t + \sin(t^2)$.

In Problems 1 and 2, find the total differential.

1. $z = x^3y + x^2y^2 + xy^3$

We have $\dfrac{\partial z}{\partial x} = 3x^2y + 2xy^2 + y^3$ and $\dfrac{\partial z}{\partial y} = x^3 + 2x^2y + 3xy^2$

Then $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = (3x^2y + 2xy^2 + y^3)\,dx + (x^3 + 2x^2y + 3xy^2)\,dy$

2. $z = x\sin y - y\sin x$

We have $\dfrac{\partial z}{\partial x} = \sin y - y\cos x$ and $\dfrac{\partial z}{\partial y} = x\cos y - \sin x$

Then $dz = \dfrac{\partial z}{\partial x}\,dx + \dfrac{\partial z}{\partial y}\,dy = (\sin y - y\cos x)\,dx + (x\cos y - \sin x)\,dy$

10. Find dz/dt, given $z = x^2 + 3xy + 5y^2$; $x = \sin t$, $y = \cos t$.

Since

$$\frac{\partial z}{\partial x} = 2x + 3y,\quad \frac{\partial z}{\partial y} = 3x + 10y,\quad \frac{dx}{dt} = \cos t,\quad \frac{dy}{dt} = -\sin t$$

we have $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (2x + 3y)\cos t - (3x + 10y)\sin t$

11. Find dz/dt, given $z = \text{In}\,(x^2 + y^2)$; $x = e^{-t}$, $y = e^t$.

Since

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2},\quad \frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2},\quad \frac{dx}{dt} = -e^{-t},\quad \frac{dy}{dt} = e^t$$

we have $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = \dfrac{2x}{x^2 + y^2}(-e^{-t}) + \dfrac{2y}{x^2 - y^2}e^t = 2\dfrac{ye^t - xe^{-t}}{x^2 + y^2}$

12. Find $\dfrac{\partial z}{\partial x}$, given $z = f(x, y) = x^2 + 2xy + 4y^2$, $y = e^{ax}$.

$$\frac{dz}{dx} = f_x + f_y\frac{dy}{dx} = (2x + 2y) + (2x + 8)ae^{ax} = 2(x + y) + 2a(x + 4y)e^{ax}$$

13. Find (a) $\dfrac{dz}{dx}$ and (b) $\dfrac{dz}{dy}$, given $z = f(x, y) = xy^2 + yx^2$, $y = \text{In}\,x$.

(a) Here x is the independent variable:

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = (y^2 + 2xy) + (2xy + x^2)\frac{1}{x} = y^2 + 2xy + 2y + x$$

(b) Here y is the independent variable:

$$\frac{dz}{dy} = \frac{\partial f}{\partial x}\frac{dx}{dy} + \frac{\partial f}{\partial y} = (y^2 + 2xy)x + (2xy + x^2) = xy^2 + 2x^2y + 2xy + x^2$$

16. Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$, given $z = x^2 + xy + y^2$, $x = 2r + s$, $y = r - 2s$.

Here

$$\frac{\partial z}{\partial x} = 2x + y, \quad \frac{\partial z}{\partial y} = x + 2y, \quad \frac{\partial x}{\partial r} = 2, \quad \frac{\partial x}{\partial s} = 1, \quad \frac{\partial y}{\partial r} = 1, \quad \frac{\partial y}{\partial s} = -2$$

Then $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (2x + y)(2) + (x + 2y)(1) = 5x + 4y$$

and $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x + y)(1) + (x + 2y)(-2) = -3y$$

17. Find $\frac{\partial u}{\partial \rho}$, $\frac{\partial u}{\partial \beta}$, and $\frac{\partial u}{\partial \theta}$, given $u = x^2 + 2y^2 + 2z^2$ $x = \rho \sin \beta \cos \theta$ $y = \rho \sin \beta \sin \theta$ $z = \rho \cos \beta$.

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \rho} = 2x \sin\beta\cos\theta + 4y\sin\beta\sin + 4z\cos\beta$$

$$\frac{\partial u}{\partial \beta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \beta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \beta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \beta} = 2x\rho\cos\beta\cos\theta + 4y\rho\cos\beta\sin\theta - 4z\rho\sin\beta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} = -2x\rho\sin\beta\sin\theta + 4y\rho\sin\beta\cos\theta$$

18. Find $\frac{du}{dx}$, given $u = f(x, y, z) = xy + yz + zx$; $y = \frac{1}{x}$, $z = x^2$.

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx} = (y + z) + (x + z)\left(-\frac{1}{x^2}\right) + (y + x)2x = y + z + 2x(x + y) - \frac{x + z}{x^2}$$

19. Use implicit differentiation (formula (49.8)) to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, given $F(x, y, z) = x^2 + 3xy - 2y^2 + 3xz + z^2 = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x + 3y + 3z}{3x + 2z} \quad \text{and} \quad \frac{\partial z}{\partial x} = -\frac{F_y}{F_z} = -\frac{3x - 4y}{3x + 2z}$$

20. Use implicit differentiation (formula (49.8)) to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, given $\sin xy + \sin yz + \sin zx = 1$.

Set $F(x, y, z) = \sin xy + \sin yz + \sin zx - 1$ then

$$\frac{\partial F}{\partial x} = y\cos xy + z\cos zx, \quad \frac{\partial F}{\partial y} = x\cos xy + z\cos yz, \quad \frac{\partial F}{\partial z} = y\cos yz + x\cos zx$$

and $$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = -\frac{y\cos xy + z\cos zx}{y\cos yz + x\cos zx}, \quad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} = -\frac{x\cos xy + z\cos yz}{y\cos yz + x\cos zx}$$

28. Find *du/dt*, given:

(a) $u = x^2y^3 \; x = 2t^3 \cdot \; y = 3t^2$

Ans. $6xy^2t(2yt + 3x)$

(b) $u = x\cos y + y\sin x \; x = \sin 2t \cdot \; y = \cos 2t$

Ans. $2(\cos y + y\cos x)\cos 2t - 2(-x\sin y + \sin x)\sin 2t$

(c) $u = xy + yz + zx; x = e^t \cdot \; y = e^{-t} \cdot \; z = e^t + e^{-t}$

Ans. $(x + 2y + z)e^t - (2x + y + z)e^{-t}$

24. Find the total differential of the following functions:

(a) $z = xy^3 + 2xy^3$ *Ans.* $dz = (3x^2 + 2y^2)\,dx + (x^2 + 6y^2)\,dy$

(b) $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ *Ans.* $d\theta = \frac{x\,dy - y\,dx}{x^2 + y^2}$

(c) $z = e^{x^2 - y^2}$ *Ans.* $dz = 2z(x\,dx - y\,dy)$

(d) $z = x(x^2 + y^2)^{-1/2}$ *Ans.* $dz = \frac{y(y\,dx - x\,dy)}{(x^2 + y^2)^{3/2}}$

24. (a) is not correct actual function is $z = x^3 y + 2xy^3$