FOURIER ANALYSIS

Fourier analysis : is a field dealing with representation of periodic or nonperiodic functions.

Fourier series : are used extensively to represent periodic functions especially wave forms for signal processing and those arising in other periodic phenomena involving waves, rotating machines (harmonic motion), or other repetitive driving forces.

Fourier series : are a basic tool for solving ordinary differential equations (ODEs) and partial differential equations (PDEs) with periodic boundary conditions.

Fourier integrals : are used to represent nonperiodic phenomena.

GENERAL PROPERTIES

A Fourier series is defined as an expansion of a function or representation of a function in a series of sines and cosines, such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \qquad (19.1)$$

The coefficients a_0, a_n, and b_n are related to $f(x)$ by definite integrals:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \cos ns \, ds, \quad n = 0, 1, 2, \ldots, \qquad (19.2)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \sin ns \, ds, \quad n = 1, 2, \ldots, \qquad (19.3)$$

which are subject to the requirement that the integrals exist.

Show that, by expressing $\cos nx$ and $\sin nx$ in exponential form, Eq. (19.1) can be recast as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad (19.4)$$

in which

$$c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n), \quad n > 0, \qquad (19.5)$$

and

$$c_0 = \frac{1}{2} a_0. \qquad (19.6)$$

The form of the series is inherently periodic; the expansions in Eqs. (19.1) and (19.4) are periodic with period 2π, with $\sin nx$, $\cos nx$, and $\exp(inx)$, each completing n cycles of oscillation in that interval. Thus, while the coefficients in a Fourier expansion are determined from an interval of length 2π, the expansion itself (if the function involved is actually periodic) applies for an indefinite range of x.

Example 19.1.1 SAWTOOTH WAVE

Consider the expansion of a sawtooth wave form (shown in Fig. 19..1) and described by the following function

$$f(x)=\begin{cases} x, & 0 \le x < \pi, \\ x-2\pi, & \pi < x \le 2\pi. \end{cases} \qquad (19.8)$$

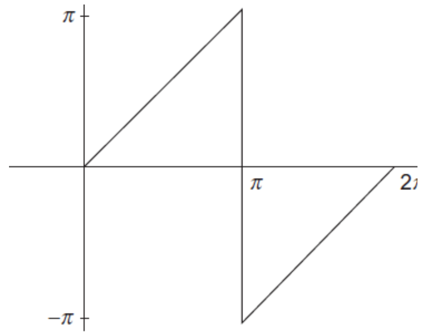

FIGURE 19.1 Sawtooth wave form.

Using Eqs. (19.2) and (19.3), we find the expansion to be

$$f(x)=2\left[\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3} - \cdots + (-1)^{n+1}\frac{\sin nx}{n}+\cdots\right]. \qquad (19.9)$$

- The periodicity also means that the interval used for determining the coefficients need not be 0 to 2π but may be any other interval of that length.
- Often one encounters situations in which the formulas in Eqs. (19.2) and (19.3) are changed so that their integrations run between $-\pi$ and π.

- *Assgnment :* ***Redo Example 19.1.1*** for SAWTOOTH WAVE in interval $[-\pi , \pi]$ by considering the expansion of a sawtooth wave form described by the function: $f(x) = x$, for $-\pi < x < \pi$.

- Another way of writing Fourier series
- In actual situations, the natural interval for a Fourier expansion can also be the wavelength of our waveform, so it may make sense to redefine our Fourier series so that Eq. (19.1) becomes

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad (19.10)$$

with

$$a_n=\frac{1}{L}\int_{-L}^{L} f(s)\cos\frac{n\pi s}{L}\,ds, \quad n=0,1,2,\ldots, \qquad (19.11)$$

$$b_n=\frac{1}{L}\int_{-L}^{L} f(s)\sin\frac{n\pi s}{L}\,ds, \quad n=1,2,\ldots. \qquad (19.12)$$

SYMMETRY

Suppose we have a function *f* (x) that is either an even or an odd function of *x*. If it is even, then its Fourier expansion cannot contain any odd terms. Our expansion, developed for the interval [- π. π] then must take the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad f(x) \text{ even.} \tag{19.13}$$

On the other hand, if *f* (x) is odd, we must have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad f(x) \text{ odd.} \tag{19.14}$$

In both cases, when determining the coefficients $\mathbf{a_n}$ and $\mathbf{b_n}$ we only need consider the interval [0 , π], referring to Eqs. (19.2) and (19.3), as the adjoining interval of length π will make a contribution identical to that considered.

- The series in Eq.(19.13) is called **Fourier cosine** series.
- The series in Eq.(19.14) is called **Fourier sine** series.

-------------------- Proof of Eq.(19.13)--------------------------

$$\mathbf{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\; dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\; dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\; dx$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(-x)\cos(-nx)\; d(-x) + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\; dx$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} f(-x)\cos(-nx)\; d(-x) + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\; dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} f(-x)\cos nx\; dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\; dx$$

$\mathbf{a_n}$ is either $= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\; dx$ if $f(-x) = f(x)$ (even)

or $= 0$ if $f(-x) = -f(x)$ (odd)

---------------------- Proof of Eq.(19.14) --------------------------

$$\mathbf{b_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\; dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\; dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\; dx$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(-x)\sin(-nx)\; d(-x) + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\; dx$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} f(-x)\sin(-nx)\; d(-x) + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\; dx$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} f(-x)\sin nx\; dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\; dx$$

$\mathbf{b_n}$ is either $= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\; dx$ if $f(-x) = -f(x)$ (odd)

or $= 0$ if $f(-x) = f(x)$ (even)

------------------------ end Proof ---------------------------

Example 19.1.2 EXPANSIONS OF $f(x) = x$ in Fourier cosine and Fourier sine series :

We consider two possible ways to expand $f(x) = x$ based on its values on the range $[0, \pi]$:

1. *Assignment:* Show that its Fourier cosine series will have coefficients determined from

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \begin{cases} \pi, & n = 0, \\ -\dfrac{4}{n^2\pi}, & n = 1,\ 3,\ 5, \ldots, \\ 0, & n = 2,\ 4,\ 6, \ldots, \end{cases}$$

corresponding to the expansion

$$f(x) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{4}{\pi}\frac{\cos(2n+1)x}{(2n+1)^2}.$$

2. *Assignment:* Show that its Fourier sine series will have the form given in Eq. (19.9).

Note: All of these expansions represent $f(x) = x$ well in the range of definition, $[0, \pi]$, but their behaviour becomes strikingly different outside that range.

Operations on Fourier Series

Fourier series may be integrated or differentiated term by term.

➢ INTEGRATION : Term-by-term integration of the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad (19.15)$$

yields

$$\int_{x_0}^{x} f(x)\,dx = \left.\frac{a_0 x}{2}\right|_{x_0}^{x} + \sum_{n=1}^{\infty} \left.\frac{a_n}{n}\sin nx\right|_{x_0}^{x} - \sum_{n=1}^{\infty} \left.\frac{b_n}{n}\cos nx\right|_{x_0}^{x}. \qquad (19.16)$$

Clearly, the effect of integration is to place an additional power of *n* in the denominator of each coefficient. This results in more rapid convergence than before. Consequently, a convergent Fourier series may always be integrated term by term, the resulting series converging uniformly to the integral of the original function. Indeed, term-by-term integration may be valid even if the original series, Eq. (19.15), is not itself convergent. The function $f(x) = x$ need only be integrable. Strictly speaking, Eq. (19.16) may not be a Fourier series; that is, if $a_0 \neq 0$, there will be a term $\frac{1}{2}a_0 x$. However,

$$\int_{x_0}^{x} f(x)dx - \frac{1}{2}a_0 x \qquad (19.17)$$

will still be a Fourier series.

➢ DIFFERENTIATION : The situation regarding differentiation is quite different from that of integration. Here the word is caution.

Consider the series for

$$f(x) = x, \quad -\pi < x < \pi. \tag{19.18}$$

We readily found (in Example 19.1.1) that the Fourier series is

$$x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}, \quad -\pi < x < \pi. \tag{19.19}$$

Differentiating term by term, we obtain $1 = 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos nx,$ which is not convergent.

Warning: Check your derivative of FS for convergence.

➢

A triangular wave (Fig. 19.4) is represented by

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ -x, & -\pi < x < 0. \end{cases}$$

and shown in Fig. 19.4

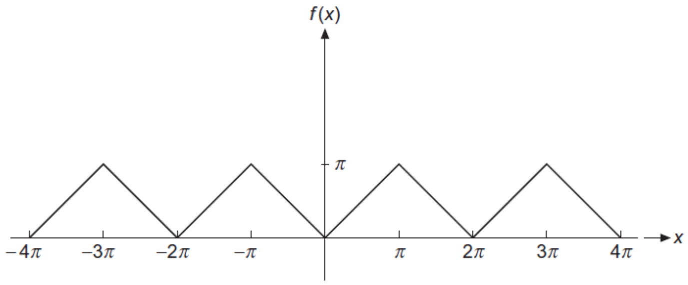

FIGURE 19.4 Triangular wave.

a. Assignment: Show that its Fourier expansion is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,\text{odd}}^{\infty}\frac{\cos nx}{n^2}, \tag{19.21}$$

which converges more rapidly than the expansion of Eq. (19.19); in fact, it exhibits uniform convergence. Differentiating term by term we get

$$f'(x) = \frac{4}{\pi}\sum_{n=1,\text{odd}}^{\infty}\frac{\sin nx}{n}, \tag{19.22}$$

b. Assignment: Show that Eq.(19.22) is the Fourier expansion of a square wave,

$$f'(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0. \end{cases} \tag{19.23}$$

➢

One application of Fourier series, the analysis of a "square" wave (Fig. 19.7) in terms of its Fourier components, occurs in electronic circuits designed to handle sharply rising pulses. Suppose that our wave is defined by

$$\begin{aligned} f(x) &= 0, \quad -\pi < x < 0, \\ f(x) &= h, \quad 0 < x < \pi. \end{aligned} \tag{19.25}$$

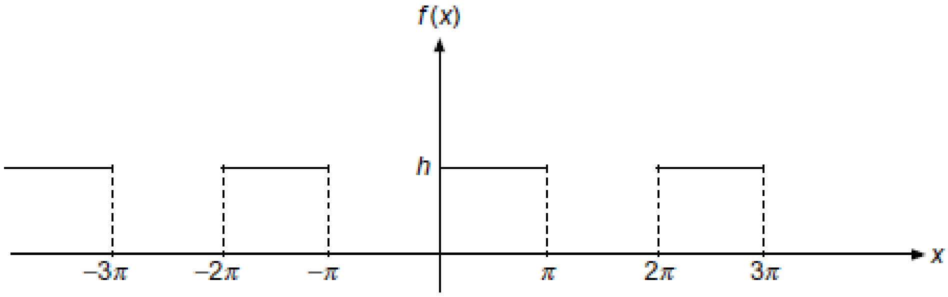

Fig.19.7: Square Wave

From Eqs. (19.2) and (19.3), we find

$$a_0 = \frac{1}{\pi}\int_0^{\pi} h\,dt = h,$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} h\cos nt\,dt = 0, \quad n = 1, 2, 3, \ldots,$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} h\sin nt\,dt = \frac{h}{n\pi}(1 - \cos n\pi)$$

$$= \begin{cases} \dfrac{2h}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

The resulting series is

$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right). \tag{19.26}$$

Except for the first term, which represents an average of $f(x)$ over the interval $[-\pi, \pi]$, all the cosine terms have vanished. Since $f(x) - h/2$ is odd, we have a Fourier sine series. Although only the odd terms in the sine series occur, they fall only as n^{-1}. This **conditional convergence** is like that of the alternating harmonic series. Physically this means that our square wave contains a lot of **high-frequency components**. If the electronic apparatus will not pass these components, our square-wave input will emerge more or less rounded off, perhaps as an amorphous blob. ■

PROBLEMS

1
A different sawtooth wave is described by

$$f(x) = \begin{cases} -\frac{1}{2}(\pi + x), & -\pi \leq x < 0 \\ +\frac{1}{2}(\pi - x), & 0 < x \leq \pi. \end{cases}$$

Show that $f(x) = \sum_{n=1}^{\infty} (\sin nx / n)$.

2
A triangular wave (Fig. 19.4) is represented by

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ -x, & -\pi < x < 0. \end{cases} \quad \textit{ANS.} \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{\cos nx}{n^2}.$$

Represent $f(x)$ by a Fourier series.

3
(a) Find the Fourier series representation of

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ x, & 0 \leq x < \pi. \end{cases}$$

(b) From the Fourier expansion show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

4
Integrate the Fourier expansion of the unit step function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi. \end{cases}$$

5
In the analysis of a complex waveform (ocean tides, earthquakes, musical tones, etc.), it might be more convenient to have the Fourier series written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx - \theta_n).$$

Show that this is equivalent to Eq. (19.1) with

$$\begin{aligned} a_n &= \alpha_n \cos\theta_n, & \alpha_n^2 &= a_n^2 + b_n^2, \\ b_n &= \alpha_n \sin\theta_n, & \tan\theta_n &= b_n / a_n. \end{aligned}$$

Note. The coefficients α_n^2 as a function of n define what is called the **power spectrum**. The importance of α_n^2 lies in their invariance under a shift in the phase θ_n.

.6 Develop the Fourier series representation of

$$f(t) = \begin{cases} 0, & -\pi \le \omega t \le 0, \\ \sin \omega t, & 0 \le \omega t \le \pi. \end{cases}$$

This is the output of a simple half-wave rectifier. It is also an approximation of the solar thermal effect that produces "tides" in the atmosphere.

ANS. $f(t) = \frac{1}{\pi} + \frac{1}{2}\sin \omega t - \frac{2}{\pi} \sum_{n=2,4,6,\dots}^{\infty} \frac{\cos n\omega t}{n^2 - 1}$.

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FOURIER TRANSFORM

Fourier transform of a function *f(t)* is defined as

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt. \qquad (20.10)$$

If we rewrite the exponential in Eq. (20.10) in terms of the sine and cosine, and then restrict consideration to functions that are assumed to be either even or odd functions of *x*, we obtain *variants of the original form* that are also useful integral transforms:

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t \, dt, \qquad (20.11)$$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t \, dt. \qquad (20.12)$$

These formulas define the **Fourier cosine** and **Fourier sine** transforms. Their kernels, which are real, are natural for use in studies of wave motion and for extracting information from waves, particularly when phase information is involved. The output of a stellar interferometer, for instance, involves a Fourier transform of the brightness across a stellar disk. The electron distribution in an atom may be obtained from a Fourier transform of the amplitude of scattered x-rays.

Example : SOME FOURIER TRANSFORMS

1. $f(t) = e^{-\alpha|t|}$, with $\alpha > 0$. To deal with the absolute value, we break the transform integral into two regions:

$$g(\omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{0} e^{\alpha t + i\omega t} dt + \sqrt{\frac{1}{2\pi}} \int_0^{\infty} e^{-\alpha t + i\omega t} dt$$

$$= \sqrt{\frac{1}{2\pi}} \left[\frac{1}{\alpha + i\omega} + \frac{1}{\alpha - i\omega} \right] = \sqrt{\frac{1}{2\pi}} \frac{2\alpha}{\alpha^2 + \omega^2}. \qquad (20.13)$$

We note two features of this result: (1) It is real; from the form of the transform, we can see that if $f(t)$ is even, its transform will be real. (2) The more localized is $f(t)$, the less localized will be $g(\omega)$. The transform will have an appreciable value until $\omega \gg \alpha$; larger α corresponds to greater localization of $f(t)$.

Example 2 FOURIER TRANSFORM OF GAUSSIAN

The Fourier transform of a Gaussian function e^{-at^2}, with $a > 0$,

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-at^2} e^{i\omega t}\, dt,$$

can be evaluated analytically by completing the square in the exponent,

$$-at^2 + i\omega t = -a\left(t - \frac{i\omega}{2a}\right)^2 - \frac{\omega^2}{4a},$$

which we can check by evaluating the square. Substituting this identity and changing the integration variable from t to $s = t - i\omega/2a$, we obtain (in the limit of large T)

$$g(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/4a} \int_{-T-i\omega/2a}^{T-i\omega/2a} e^{-as^2}\, ds. \tag{20.18}$$

The s integration, shown in Fig. 20.3, is on a path parallel to, but below the real axis by an amount $i\omega/2a$. But because connections from that path to the real axis at $\pm T$ make negligible contributions to a contour integral and since the contours in Fig. 20.3 enclose no

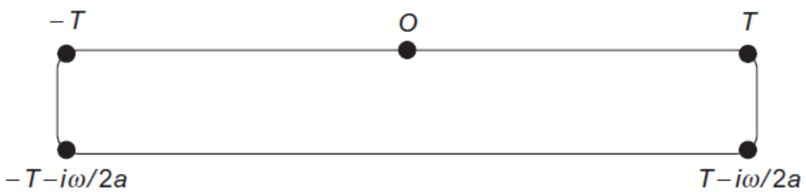

FIGURE 20.3 Contour for transform of Gaussian in Example 20.2.2.

singularities, the integral in Eq. (20.18) is equivalent to one along the real axis. Changing the integration limits to $\pm\infty$ and rescaling to the new variable $\xi = s/\sqrt{a}$, we reach

$$\int_{-\infty}^{\infty} e^{-as^2} dt = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\frac{\pi}{a}},$$

where we have used Eq. (1.148) to evaluate the error-function integral. Substituting these results we find

$$g(\omega) = \frac{1}{\sqrt{2a}} \exp\left(-\frac{\omega^2}{4a}\right), \tag{20.19}$$

again a Gaussian, but in ω-space. An increase in a makes the original Gaussian e^{-at^2} narrower, while making wider its Fourier transform, the behavior of which is dominated by the exponential $e^{-\omega^2/4a}$. ■

2. $f(t) = \delta(t)$. We easily find

$$g(\omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{i\omega t} \, dt = \sqrt{\frac{1}{2\pi}}. \tag{20.14}$$

This is the ultimately localized $f(t)$, and we see that $g(\omega)$ is completely delocalized; it has the same value for all ω.

INVERSE FOURIER TRANSFORM

Eq. (20.23) given below is a formula for the **inverse Fourier transform**.

Note the difference is in the sign of the complex exponential.

$$f(t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega, \tag{20.23}$$

The analysis of the preceding subsection can also be applied to the Fourier cosine and sine transforms. For convenience, we summarize the formulas for all three varieties of the Fourier transform and their respective inverses.

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt, \tag{20.25}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega, \tag{20.26}$$

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos \omega t \, dt, \tag{20.27}$$

$$f_c(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(\omega) \cos \omega t \, d\omega, \tag{20.28}$$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \omega t \, dt, \tag{20.29}$$

$$f_s(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(\omega) \sin \omega t \, d\omega. \tag{20.30}$$

Note that the Fourier sine and cosine transforms only use data for $0 \le t < \infty$.

FOURIER INTEGRAL REPRESENTATION

We found above that $f(t) = e^{-\alpha|t|}$ has Fourier transform $g(\omega) = \sqrt{1/2\pi}\, 2\alpha/(\alpha^2 + \omega^2)$.
If we substitute these data into Eq. (20.23), we obtain

$$e^{-\alpha|t|} = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\alpha e^{-i\omega t}}{\alpha^2 + \omega^2} d\omega = \frac{\alpha}{\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\alpha^2 + \omega^2} d\omega.$$

Above equation provides an integral representation for $f(t) = e^{-\alpha|t|}$ that contains no absolute value signs and may constitute a useful starting point for various analytical manipulations.

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