Methods for determining the optical parameters of elliptic retarders*

F. El-Hosseiny

Pulp and Paper Research Institute of Canada, Pointe Claire, P. Q., Canada H9R 3J9
(Received 1 March 1975)

Two new methods are described for measurement of retardation introduced between the two components of polarized light that passes through an elliptically birefringent retarder. In one method, a quarter-wave-plate compensator is used. In the other, no compensators are needed and the calculations are made from the flux of light that passes through the specimen between a polarizer and an analyzer.

Index Headings: Polarization; Birefringence.

Light incident on certain crystalline birefringent materials is split into two plane-polarized components that travel at different speeds, so that one component is retarded with respect to the other. The polarization directions are orthogonal; they are related to the crystal symmetry and to the direction of propagation of the light. Such a birefringent material is termed a linear retarder. It allows plane-polarized light to pass without a change of its state of polarization, provided that the direction of polarization is parallel to one of the principal axes of the material. The retarder is characterized by two parameters, the phase retardation between the two components and the direction of the principal axes.

There is, however, another group of birefringent materials that are not often considered, but which occur surprisingly frequently in practice. These are termed elliptic retarders. An elliptic retarder can transmit two orthogonal elliptically polarized vibrations without change of polarization, but at different speeds. An incident beam that has any other state of polarization is split into components that have these characteristic polarizations; they recombine, on emergence, to give light with a different polarization from the incident light. Three parameters are needed to characterize such a material. One is the phase retardation between the two components. The other two are the direction of the major axis and the ellipticity of one of the two characteristic orthogonal elliptic vibrations. Certain crystals are elliptic retarders, and also two superimposed linear retarders behave in this way; this is a common occurrence in nature. Thus, a cotton or wood fiber, with superimposed oriented layers, or a uniaxially oriented fiber that has been twisted, are examples of elliptic retarders. Sheets used in photoelastic studies also behave as elliptic retarders if the condition of plane stress is not met.

In this paper, two methods are introduced for the characterization of retarders. In one method, a quarter-wave-plate compensator is utilized, whereas in the other no compensator is needed.

THEORY AND EXPERIMENTAL VERIFICATION

Two main mathematical techniques are used to determine the change of the state of polarization of light as it passes through a birefringent material. In one, linear algebra is used, including matrix analysis[1,2] or complex variables and bilinear transformations.[3] In the other, the states of polarization of polarized light are represented by points on the surface of a sphere of unit radius known as the Poincaré sphere[4] and calculations are made by use of spherical trigonometry. The second technique is utilized here.

On the Poincaré sphere, every state of polarization is uniquely represented by a point.[5] The ellipticity and azimuth are determined from the latitude and longitude of the point, respectively. Left-polarized beams are represented by points on the upper hemisphere and right-polarized beams by points on the lower hemisphere. Points on the equator represent plane-polarized light. Horizontal and vertical plane-polarized beams are represented by two points diametrically opposite to each other. Light with plane of polarization inclined at an angle χ to the horizontal is represented by a point that makes an angle 2χ with the point that represents the horizontal.

The merit of this technique lies in the many theorems that have been established for its use. Two are particularly relevant to this discussion.[6]

In Fig. 1, if R represents the state of polarization of the fast component of an elliptic retarder, if Δ is the retardation introduced by the retarder, and if P_1 represents the state of polarization of the light that is incident on the retarder, then the state of polarization P_2 of the emergent light is obtained by rotating P_1 counterclockwise through Δ, about a diameter through R.

If P represents the state of polarization of light incident on an analyzer whose transmission axis is at A, the flux of light transmitted, expressed as a fraction of the incident flux, is $\cos^2(PA/2)$.

Methods of measurement

The unknowns of an elliptic retarder are the state of polarization of light that passes unchanged through the retarder, and the retardation introduced by the retarder. The former is specified by the direction of its major axis and its ellipticity ϵ. It can be demonstrated, using the two theorems, that the minimum flux transmitted through a retarder between crossed polarizers occurs when the major axis of the fast (or slow) component is parallel to the transmission axis of the polarizer. The direction of the axis can, therefore, be determined by rotating the retarder between crossed polarizers until a minimum transmitted flux is obtained. It remains to determine the ellipticity ϵ and retardation Δ.

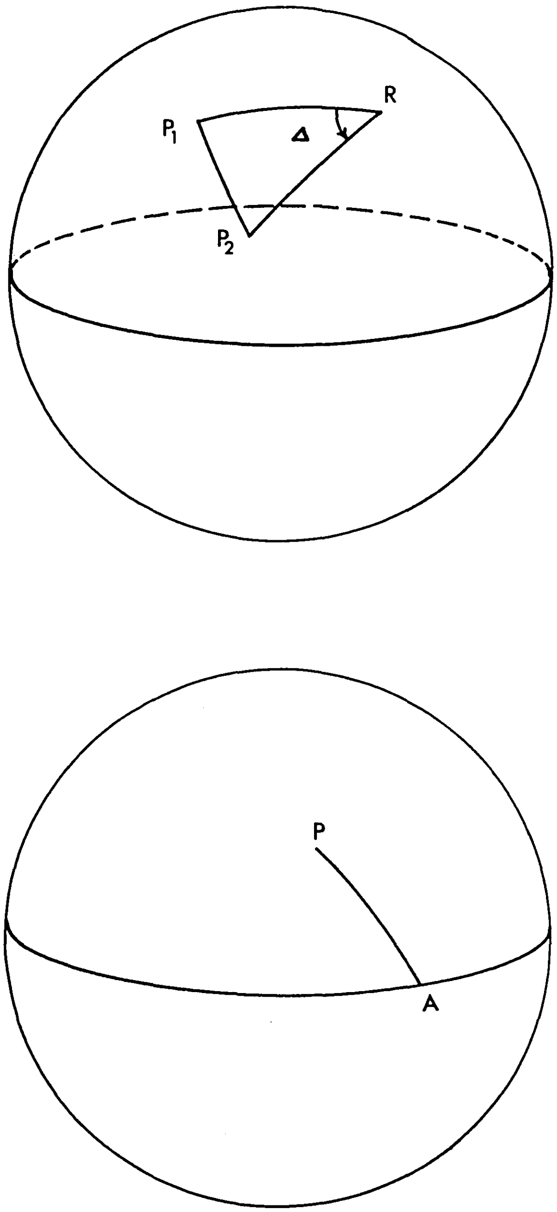

FIG. 1. Effect of a retarder whose fast axis is represented by R, on polarized light represented by P_1, is a rotation through an angle Δ. The flux transmitted by an analyzer at A is proportional to $\cos^2 PA/2$.

1. The generalized Sénarmont method

In the method of Sénarmont, the fast axis of the retarder is positioned at an angle 45° to the transmission axis of the polarizer, and a quarter-wave plate is placed after it, with its fast axis parallel to transmission axis

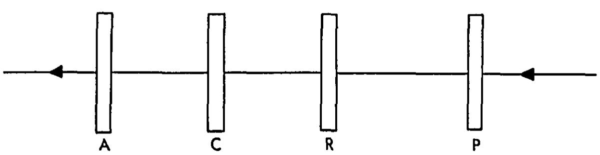

FIG. 2. Arrangement of the optical components in the generalized Sénarmont method. P, R, C and A are polarizer, retarder, compensator, and analyzer, respectively.

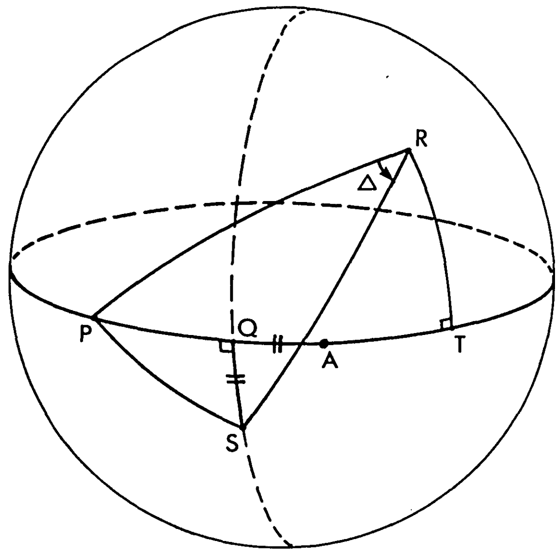

FIG. 3. Generalized Sénarmont method.

of the polarizer. The analyzer is rotated until complete extinction is obtained. The angle between the polarizer and analyzer is then half the retardation of the retarder.

The Sénarmont method does not give extinction for any position of the analyzer if an elliptic retarder is used. However, if both the quarter-wave plate and the analyzer are rotated independently, an extinction position can be found. The positions of these components give information on ϵ and Δ but do not uniquely define either. If the procedure is repeated, for various rotational positions of the specimen, ϵ and Δ can be determined. We have termed this the generalized Sénarmont method. A schematic diagram showing the arrangement of the optical components is shown in Fig. 2.

This method is similar to the one introduced by Azzam and Bashara.[7] In their method the quarter-wave plate is placed before the retarder; extinction is achieved by the rotation of the polarizer, compensator and the analyzer. One advantage of our method is the simplicity of calculations. To develop the theory for our method, let the angle between the polarizer and the major axis of the fast component of the retarder be χ.

The effect of the optical elements on the incident light P is shown in Fig. 3. The retarder R rotates the plane of polarization P through an angle Δ, to S. At extinction, the quarter-wave plate is positioned at Q on the same longitude as S and rotates S to A on the equator. Extinction is achieved when the extinction postion of the analyzer is at A

Spherical trigonometry gives, from the triangles RPS and TRP,

$$\cos PS = \cos^2\epsilon(1-\cos\Delta)\cos^2 2\chi + \cos\Delta \qquad (1)$$

and from the triangle PQS,

$$\cos PS = \cos 2\alpha \cos 2\beta \, . \qquad (2)$$

Thus,

$$\cos 2\alpha \cos 2\beta = \cos^2\epsilon(1-\cos\Delta)\cos^2 2\chi + \cos\Delta \, . \qquad (3)$$

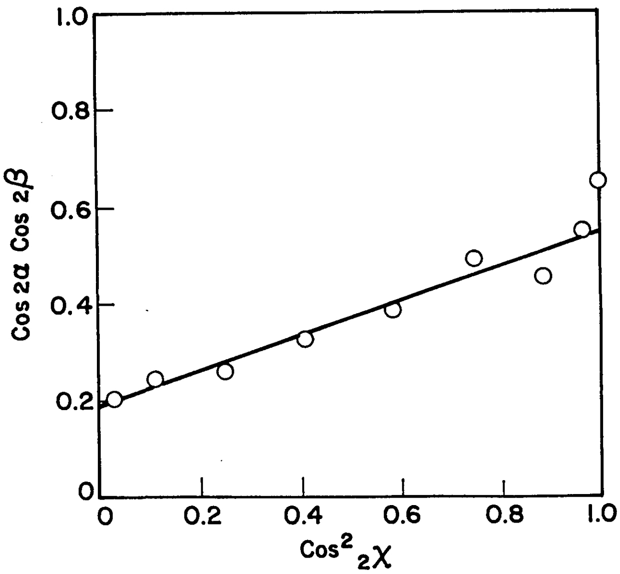

FIG. 4. $\cos 2\alpha \cos 2\beta$ vs $\cos^2 2\chi$ for a cotton fiber. $\Delta = 79.3°$, $\epsilon = 47.3°$.

Experimentally, the specimen is rotated through an angle χ from its minimum-flux position between crossed polarizers. The quarter-wave plate is introduced and the extinction position is found by alternately rotating the quarter-wave plate and the analyzer. Then α is the angle between the plane of polarization transmitted by the polarizer and the fast axis of the quarter-wave plate, β is the angle between the fast axis of the quarter-wave plate and the plane of polarization extinguished by the analyzer; β and α are determined for a range of values of χ. A plot of $\cos 2\alpha \cos 2\beta$ against $\cos^2 2\chi$ gives directly $\cos^2\epsilon$ and $\cos\Delta$. Figure 4 shows the results for a cotton fiber, which is an elliptic retarder. The data fit a straight line, from which Δ and ϵ can be determined.

Figure 5 shows the method used for a linear retarder,

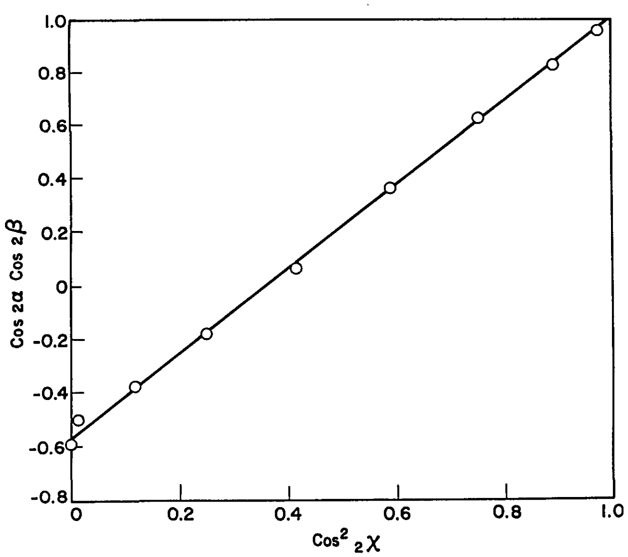

FIG. 5. $\cos 2\alpha \cos 2\beta$ vs $\cos^2 2\chi$ for a sheet of mica. $\Delta = 235.5°$, $\epsilon = 0°$.

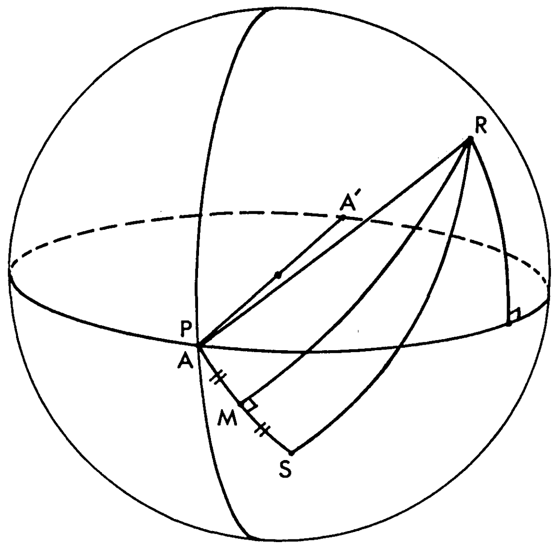

FIG. 6. Flux method.

a sheet of mica. For a linear retarder, the line should pass through (1, 1); this is confirmed experimentally.

2. The flux method

In this method,[8] the retarder is placed above the polarizer and the analyzer is placed first in the crossed position and then in the parallel position. The flux of the light transmitted is measured in the two cases, for various positions of rotation of the retarder. From these fluxes, the ellipticity ϵ and retardation Δ can be determined. In Fig. 6, let P represent the state of polarization of the incident light and S the state of polarization of the light that emerges from the retarder R. Spherical trigonometry gives

$$\sin^2 PM = \sin^2 \tfrac{1}{2}\Delta - \sin^2 \tfrac{1}{2}\Delta \cos^2\epsilon \cos^2 2\chi \ . \qquad (4)$$

$\sin^2 PM$ can be determined from the transmitted fluxes for the two positions of the analyzer. The planes of polarization of the light transmitted by the analyzer in the

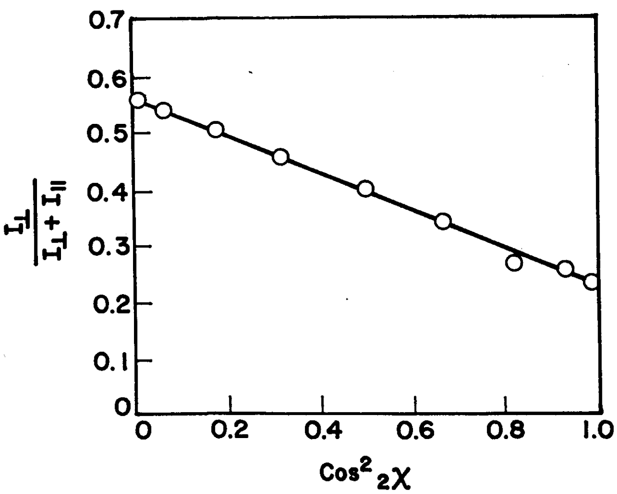

FIG. 7. $I_\perp/(I_\perp + I_\parallel)$ vs $\cos^2 2\chi$ for a cotton fiber. $\Delta = 98.2°$, $\epsilon = 37.5°$.

parallel and crossed positions are represented on the sphere by A and A', respectively. By the second theorem of the sphere, $I_\perp$ and $I_\parallel$ are

$$I_\perp = I_0 \cos^2 \frac{AS}{2} = I_0 \sin^2 PM \,, \tag{5}$$

$$I_\parallel = I_0 \cos^2 \frac{A'S}{2} = I_0 \cos^2 PM \,. \tag{6}$$

Thus

$$\frac{I_\perp}{I_\parallel + I_\perp} = \sin^2 \frac{\Delta}{2} - \sin^2 \frac{\Delta}{2} \cos^2\epsilon \cos^2 2\chi \,. \tag{7}$$

A plot of $I_\perp/(I_\parallel + I_\perp)$ vs $\cos^2 2\chi$ is a straight line whose intercept and slope determine $\sin^2 \frac{1}{2}\Delta$ and $\cos^2\epsilon$.

Figure 7 shows the result for a cotton fiber as an elliptic retarder. Figure 8 shows the result for a quarter-wave plate as an unknown linear retarder; as theoretically predicted, the line passes through (0, 0.5) and (1, 0).

Ambiguity of ϵ and Δ

For simplicity, the equations have been based on the assumption that the direction of the fast axis of the retarder is known. The preliminary determination of the extinction position between crossed polarizers gives only the two major axes; it is not simple, with an elliptic retarder, to determine which of these is the fast axis. Equations (3) and (4) are, however, correct whether the fast or slow axis is represented by the point R. If it is the fast axis, ϵ lies between $-90°$ and $+90°$; if it is the slow axis, ϵ lies between 90° and 270°. The angles ϵ and Δ are determined from the trigonometric functions, $\cos^2\epsilon$ and $\cos\Delta$. Therefore, it is not possible to say which quadrant contains ϵ or which half contains Δ. Neither is it possible to determine how many orders of retardation are contained in Δ.

These ambiguities can be resolved. The quadrant of Δ can be determined, in both the generalized Sénarmont and flux methods, by measuring $\cos\Delta$ at two slightly different wavelengths. The magnitude of $\cos\Delta$ increases with increasing wavelength when Δ is in the first and third quadrants and decreases when Δ is in the second and fourth quadrants. The number of orders n contained in Δ can be determined from the magnitude of the change of Δ with wavelength according to

$$n = \frac{(\Delta_1 - \Delta_2)\lambda_2}{(\lambda_2 - \lambda_1)} - \Delta_1 \,, \tag{8}$$

where λ_1 and λ_2 are the different wavelengths that give the retardations Δ_1 and Δ_2.

ϵ can be determined uniquely only by using the generalized Sénarmont method. The signs of the angles α and β are observed for the case in which $\chi = 45°$. The convention is adopted that α and β are less than 90°. They are measured with respect to the fast axis of the quarter-wave plate and are considered positive for counterclockwise rotations. The quadrant of ϵ in terms of Δ and the signs of α and β are given in Table I.

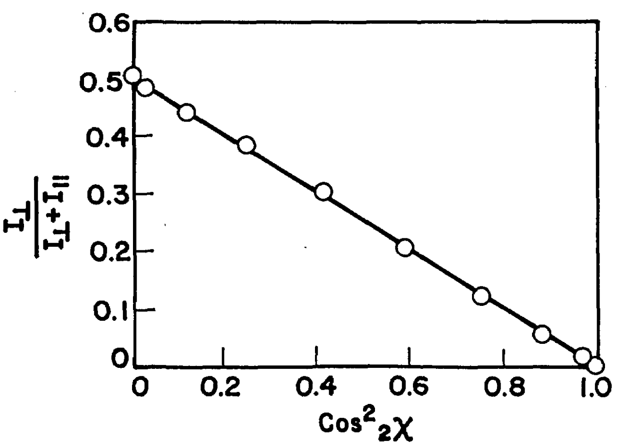

FIG. 8. $I_\perp/(I_\perp + I_\parallel)$ vs $\cos^2 2\chi$ for a quarter-wave plate. $\Delta = 90°$, $\epsilon = 0°$.

TABLE I. Quadrant of ϵ in terms of Δ and the signs of α and β.

Half containing Δ	Sign of β	Sign of α	Quadrant containing ϵ
First	Negative	Positive	First
Second	Positive	Negative	First
First	Negative	Negative	Second
Second	Positive	Positive	Second
First	Positive	Negative	Third
Second	Negative	Positive	Third
First	Positive	Positive	Fourth
Second	Negative	Negative	Fourth

CONCLUSION

The two methods presented can be used to measure the retardation and ellipticity of an elliptic retarder. The methods are, in principle, precise because calculations are made from the slope and intercept of a linear plot.

ACKNOWLEDGMENT

I wish to express my appreciation to H. J. Woods and Dr. D. H. Page for valuable discussions and also to Dr. D. H. Page for assistance in preparing the manuscript.

*This work was initiated at the University of Leeds, as part of a Ph.D. thesis, and completed at PPRIC.

[1] R. C. Jones, J. Opt. Soc. Am. 31, 488 (1941).

[2] H. Muller, J. Opt. Soc. Am. 38, 661 (1948).

[3] R. M. A. Azzam and N. M. Bashara, J. Opt. Soc. Am. 62, 222 (1972).

[4] H. Poincaré, *Théorie Mathématique de la Lumière* (Gauthiers-Villars, Paris, 1892), Vol. II.

[5] W. A. Shurcliff, *Polarized Light* (Harvard U. P., Cambridge, Mass., 1962).

[6] G. N. Ramachandran and S. Ramaseshan, in *Encyclopedia of Physics, Vol. XXV/1*, edited by S. Flügge (Springer, Berlin, 1961).

[7] R. M. A. Azzam and N. M. Bashara, J. Opt. Soc. Am. 62, 336 (1972).

[8] F. El-Hosseiny, Ph.D. thesis (Leeds University, England, 1969).