An introduction to Group theory

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Winter semester 2017/2018

These notes contain a part of the lessons of the course "Mathematical Methods of Physics 3" held by Professor Barbara Kraus in the winter semester 2017/2018 at the university of Innsbruck. The numbering of theorems and sections is different from class, due to a little reorganization of contents and sections, however I tried to keep the notation as close as possible to the one used in class. Unavoidably there are errors and I am not responsible for those, hence I recommend a critical and careful reading. Any comments or suggestions are welcome and appreciated.

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Chapter 1

Group theory and representation of groups

1.1 Fundamentals of groups

Group theory is a branch of mathematics that studies *Groups*, groups are useful, for instance, to describe symmetries in a system. In physics, symmetries have a central role, a problem can be simplified in presence of symmetries and some physical quantities are directly correlated to symmetries. Therefore, the study of groups is indeed important in theoretical physics. Let us start from the beginning by defining what is a group

Definition 1.1.1. Let G be a set and u a map $u : G \times G \to G$, then the pair (G, u) is called a **group** if the following conditions hold

   i. (Associative) $g(fh) = (gf)h$;

  ii. (Neutral element) $\exists e \in G : eg = ge = g \quad \forall g \in G$;

 iii. (Inverse element) $\forall g \in G \ \exists g^{-1} \in G : gg^{-1} = g^{-1}g = e$;

notice that the map u is usually represented like the standard multiplication $gh := u(g, h)$ and very often we call G a group instead of the pair (G, u).

Definition 1.1.2. A group G is called **Abelian** if $gh = hg \quad \forall g, h \in G$.

A group can have several properties, one of the most important is the order of a group, which can be defined as follow

Definition 1.1.3. The **order** of a group G is the number of elements of G (cardinality) and it is denoted as $|G|$.

There are two cases, either the order of a group is finite (or countably infinite), and in this case the group is called discrete, or the order in not finite (uncountably infinite) and we say that the group is continuous. In order to clarify what is a group let us give some interesting examples of a group:

- $G = \mathbb{Z}_2 = \{-1, 1\}$ and u the standard multiplication.
  We need to check that this is a group, first of all G must be closed with respect to u, this is easy to show, in fact there are only two elements and with the standard multiplication we know that multiplying -1 by 1 or vice versa will always end up with -1. The standard multiplication is also associative. The neutral element of this group is 1 and the inverse of $g \in G$ in this group is g itself. Therefore, this is a finite group.

- $G = \{z \in \mathbb{C} : |z| = 1\}$, u standard multiplication.
  This is also a group, in fact we can represent a complex number with a complex exponential $z = e^{i\theta}$, multiplying two complex number will only change the phase of the number, not the modulus, thus it is closed. It is also associative, the neutral element is 1 and the inverse element is the complex conjugate $zz^* = |z|^2 = 1$.

- $G = \{e, a, a^2, a^3\}$ with $a^4 = e$ and also $a^0 = e$, with the map defined as $u(a^k, a^l) = a^{k+l}$.
  It is closed, associative, the neutral element is e and the inverses are: $aa^3 = e$, $a^2a^2 = e$, $a^3a = e$. Therefore, this is a group of order 4.

- $G = GL(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) : \det A \neq 0\}$ and u is matrix multiplication. Note that $GL(n, \mathbb{C})$ is usually called General Linear group and is the set of $n \times n$ invertible matrices.
  u is closed, associative, the neutral element is $\mathbb{1}$ and the inverse is the inverse matrix $A^{-1} \in G$. Hence, this is a group.

- $G = \{U \in M_n : U^{-1}U = \mathbb{1}\}$, i.e. the set of unitary matrices with the matrix multiplication.
  The product of two unitary matrices is still unitary, so the map is closed. It is also associative, the neutral element is $\mathbb{1}$ and the inverse is $U^\dagger$. Therefore, it is a group.

A particular class of Group is the so called symmetric groups. Given n elements $\overline{G} = \{1, 2, \ldots, n\}$, the symmetric group is defined as the set of permutations $\Pi : \overline{G} \to \overline{G}$, which we can write as $S_n = \{\Pi : \overline{G} \to \overline{G} :$ where Π is a bijective permutation$\}$. If u is the concatenation of two permutations, then (S_n, u) is a symmetric group. In fact u is associative, the neutral element is the identity map, and the inverse element is Π^{-1}, which exists because permutations are bijective. Note also that the order of a symmetric group is $n!$, indeed there are $n!$ possible permutations in a set with n elements. For instance, let us take S_3, we will denote the permutation as $\Pi_{1,2} \equiv (1, 2)$, i.e the permutation that swaps 1 with 2: $1 \to 2, 2 \to 1, 3 \to 3$, and $(1, 2, 3)$ as the permutation that changes every number like this: $1 \to 2, 2 \to 3, 3 \to 1$. Therefore the set S_3 is the following

$$S_3 = \{\mathbb{1}, (1, 2), (2, 3), (1, 2, 3), (1, 3, 2), (1, 3)\}$$

there are of course $3! = 6$ elements which correspond to the symmetry of a triangle, as we will see later in more detail.
We can also define a dihedral group $D_n, n \geq 3$ which denotes the orthogonal symmetries of a regular polygon, which has n vertices and is centered in the origin. The vertices

are denoted by $V_0, V_1, \ldots, V_{n-1} \in \mathbb{R}^2$ in counter clockwise direction. The symmetries of a polygon are: $C_n = R_{\frac{2\pi}{n}}$, that is a counter clockwise rotation of angle $2\pi/n$; $\sigma^{(i)}$, that is a reflection with respect to the line through V_i and the center. Therefore, we can write D_n as

$$D_n = \{\mathbb{1}, C_n, C_n^2, \ldots, C_n^{n-1}, \sigma^{(i)}, C_n\sigma^{(i)}, \ldots C_n^{n-1}\sigma^{(i)}\}$$

as can be easily seen, the order of this group is $|D_n| = 2n$. In the case $n = 3$ we would have $D_3 = \{\mathbb{1}, C_3, C_3^2, \sigma^{(1)}, C_3\sigma^{(1)}, C_3\sigma^{(1)}\}$, notice that for instance $\sigma^{(3)} = C_3\sigma^{(1)}$. Moreover, in this case we have that $S_3 = D_3$, in fact we can denotes every vertex of a triangle with a number and exchanging numbers is equivalent of rotating of reflecting the triangle.

In the case of finite group, u doesn't have to follow any algebraic rules. Hence, it is sometimes convenient to write $u(g_i, g_j)$ for $g_i, g_j \in G$ into a table. Due to the existence of the neutral element, there exist one row (column) which coincides with the first row (column). Furthermore, due to the existence of the inverse element, each row and each column contains all n elements. In fact, suppose that there exist i such that one element occurs

G	g_1	g_2		g_n
g_1	g_1g_1	g_1g_2	$\cdots$	g_1g_n
g_2	g_2g_1	g_2g_2		g_2g_n
$\vdots$				
g_n	g_ng_1	g_ng_2		g_ng_n

at least twice in the i-th row, i.e. $\exists j \neq k : g_ig_j = g_ig_k$. Then it also true that $g_i^{-1}g_ig_j = g_i^{-1}g_ig_k \implies g_j = g_k$ which is a contradiction. Therefore, all the n elements of the i-th row must be different, that is each row contains all n element of G. The same argument can be done with a column. From this we can say that the multiplication of g_i corresponds to a permutation Π_{g_i} of the n elements of G.

1.2 Subgroups

A subgroup of a group is basically a subset of a group which is itself a group, or more formally

Definition 1.2.1. Let G be a group, then $H \subset G$ is a **subgroup** of G if

i. $e_g \in H$

ii. $\forall g, h \in H, gh \in H$

iii. $\forall g \in H, g^{-1} \in H$

To distinguish between subgroup and subset, there is a different notation, so if H is a subgroup of G we write $H \leq G$, while if H is only a subset the notation is the standard $H \subset G$. A couple of examples of subgroups, which are relevant for their applications in physics, are

- $SL(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) : |A| = 1\} \leq GL(n, \mathbb{C})$, SL is called the special linear group and it is a subgroup of the general linear group of matrices which have determinant equals to one.

- $SO(n) = \{A \in SL(n, \mathbb{R}) : A^\dagger A = \mathbb{1}\} \leq SL(n, \mathbb{R}) \leq GL(n, \mathbb{C})$

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- $U(n) = \{A \in GL(n, \mathbb{C}) : A^\dagger A = \mathbb{1}\} \leq GL(n, \mathbb{C})$

A property of subgroups is that if we take $H_i \leq G$ subgroups, with an index $i \in I$ the the intersection of the these subgroup is still a subgroup, that is

$$H = \bigcap_{i \in I} H_i \leq G.$$

The proof is straightforward, we only need to check if the intersection has the properties of a group. Associative property is obvious. Since H_i are groups, the neutral element is in each of them, so the neutral element is also in the intersection. The same argument goes for the inverse, $g \in H_i \implies g^{-1} \in H_i$, which means that $\forall g \in H \implies g^{-1} \in H$. And lastly with the same logic as before if $g, h \in H_i \implies gh \in H_i$ then $gh \in H$ if $g, h \in H$.

As already said, subsets and subgroups are different, but it is also possible to generate a subgroup from a subset. Let G be a group and $E \subset G$ a subset, then the subgroup generated by E is

$$\langle E \rangle = \bigcap_{\substack{E \subseteq H \\ H \leq G}} H,$$

which is the smallest subgroup that contains E. It can be easily computed by taking the elements of E, all their inverse and all their products. For instance, let us consider the set $E = \{C_3\}$, the inverse of this element is C_3^2 and the neutral element is $\mathbb{1}$. Therefore the subgroup generated by E is

$$\langle E \rangle = \{C_3, C_3^2, \mathbb{1}\}.$$

It is also easy to show that the following symmetric group can be written as $S_3 = \langle \{C_3, \sigma^{(1)}\} \rangle$, or that $\mathbb{Z}$ can be generated by $\mathbb{Z} = \langle \{1\} \rangle = \langle \{-1\} \rangle$, where the map u is the addition. Last example shows also that a generate set of a subgroup is not necessarily unique. Moreover it is called cyclic, the general definition is the following

Definition 1.2.2. A group is called **cyclic** if it is generated by a single element.

Cyclic groups G can be divided in two classes:

- If $|G|$ is finite, then $\exists n : g^n = e$, let k be the minimal element of $\mathbb{Z}$ such as $g^k = e$, then $G = \{g^0, g^1, \dots, g^{k-1}\}$ and therefore $|G| = k$. The proof is done by contradiction: suppose $\exists l, j : g^l = g^j \implies g^{l-j} = e$, but also $g^{j-l} = e$, but either $j > l$ or $l > j$. This is a contradiction and therefore it must be $|G| = k$.

- $|G|$ is infinite In this case $G = \{g^k : k \in \mathbb{Z}\}$ and $\nexists n : g^n = e$.

1.2.1 Cosets and conjugacy classes

Definition 1.2.3. Let $H \leq G$ and $a \in G$, then the **right (left) coset** of G is defined as

$$Ha = \{ha, h \in H\} \quad (aH = \{ah, h \in H\})$$

Notice that the cosets are not necessarily subgroups, but two cosets of a subgroup H, either coincide or are disjoint. Mathematically: let $a, b \in G$ with $a \neq b$, if $\exists g \in bH, aH \implies aH = bH$. This can be proved by contradiction, suppose that $\exists g \in bH, aH$, this implies that $\exists h, h' : g = bh = ah' \implies b = ah'h^{-1}$ and, because H is a subgroup, $h'h^{-1} \in H$, so $b \in aH$. Now we can take $\widetilde{h} \in H$ and let us consider $b\widetilde{h} = ah'h^{-1}\widetilde{h} \in aH$, since this holds for every $\widetilde{h}$, then $bH \subseteq aH$. The same argument can be done in order to show that also $aH \subseteq bH$. Therefore the only possibility is that $aH = bH$.

Furthermore, it is also clear that $\forall g \in G$, there exists $a \in G$ such that $g \in aH$. In fact, H is a subgroup, so you can take $e \in H$, which implies $a = g$.

Considering these last two observations, we have that, for any subgroup $H \leq G$, G can be written as

$$G = \overset{\cdot}{\bigcup_{i \in I}} a_i H, \tag{1.1}$$

where the dot represents disjoint union. This means that we can indeed decompose our group.

Theorem 1.2.4. *(Lagrange's theorem) Let H be a subgroup of G, then $|H|$ divides G, i.e.*

$$\frac{|G|}{|H|} \in \mathbb{N}$$

The proof is straightforward, since G can be decomposed as (1.1), then we have that $|G| = |I| \cdot |H|$, and of course $|I| \in \mathbb{N}$.

Definition 1.2.5. Let $H \leq G$ and $g \in G$, then $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a subgroup of G and it is called the **subgroup conjugate** to H.

The proof that it is a group is easy and therefore not done.

Definition 1.2.6. Let $N \leq G$ be a subgroup, then it is called **normal group** (or normalizer) if

$$gNg^{-1} = N \quad \forall g \in G.$$

(The notation could also be $GNG^{-1} = N$)

Any group has two trivial normalizers, namely $N = \{e\}$ and $N = G$. Moreover, if G is abelian, then any subgroup of G is a normalizer.

Definition 1.2.7. Let G be a group, then the following group is called **central group** of G:

$$Z = \{z \in G : zg = gz \, \forall g \in G\}$$

Z is always not empty, in fact it contains at least the neutral element of G. Furthermore, if G is abelian, then $Z = G$, and for "strong" non abelian group $Z = \{e\}$.

Definition 1.2.8. Conjugate elements $a, b \in G$ are called **conjugate** to each other if $\exists g \in G : a = gbg^{-1}$.

Before continuing, let us recall what an equivalence relation is. It is denoted by $\sim$ and it satisfies the following properties

1. (reflexive) $a \sim a$

2. (symmetric) $a \sim b \iff b \sim a$

3. (transitive) $a \sim b$ and $b \sim c \implies a \sim c$

Let us go back to conjugation, it is easy to prove that conjugation defines an equivalent relation. In fact:

- $a \sim a$, we can choose $g = e$, and we have $a = a$

- $a \sim b \implies \exists g \in G : a = gbg^{-1} \implies b = g^{-1}ag \implies b \sim a$

- $a = g_1 b g_1^{-1}, b = g_2 c g_2^{-1} \implies a = g_1 g_2 c g_1^{-1} g_2^{-1} \implies a = g_3 c g_3$

Definition 1.2.9. The **conjugacy class** of an element $a \in G$ is defined as

$$K_a = \{gag^{-1} : g \in G\}$$

If we choose $g = e$, we immediately find that $a \in K_a$.

Theorem 1.2.10. *Any group can be decomposed in disjoint conjugacy classes K_i. Let $n(K_i)$ denote the cardinality of K_i, then it holds that $|G| = \sum_i n(K_i)$*

The proof is left to the reader as exercise.

Theorem 1.2.11. *The number of elements of a conjugacy class $n(K_i)$ divides $|G|$*

The proof is left to the reader as exercise.

Corollary 1.2.12. *If $|G|$ is prime, then $n(K_i) = 1 \forall K_i \implies G$ is abelian.*

1.3 Homomorphism and isomorphism

Before jumping into representation theory is necessary to define and understand what homomorphisms and isomorphisms are in group theory.

Definition 1.3.1. Let G, H be groups, then a function $f : G \to H$ is called **group homomorphism** (GH) if

$$f(g_1 g_2) = f(g_1)f(g_2) \in H \quad \forall g_1, g_2 \in G$$

Definition 1.3.2. Let G, H be groups, then a function $f : G \to H$ is called **group isomorphism** (GI) if it is bijective and an homomorphism. And we write $G \simeq H$

Furthermore, the following hold

1. $f(e_G) = e_H$, in fact $f(g) = f(e_G g) = f(e_G)f(g) \forall g$, hence $f(e_G)$ must be e_H.

2. $f(g^{-1}) = f(g)^{-1}$, in fact $f(gg^{-1}) = f(e_G) = e_H \implies f(g)f(g^{-1}) = e_H \implies f(g^{-1}) = f(g)^{-1}$.

3. if f is a GI, then f^{-1} is a GI.

Let us do some examples with of homomorphisms and isomorphisms.

- The determinant of a matrix $\det : GL(n, \mathbb{C}) \to \mathbb{C}$ is a GH, in fact from the well known property of determinant $|AB| = |A||B|$.

- Let G be a group and $g \in G$, the function $f : \mathbb{Z} \to G, k \mapsto g^k$ is a GH. In fact, $f(k_1 k_2) = g^{k_1 + k_2} = g^{k_1} g^{k_2} = f(k_1) f(k_2)$. (Recall that for the group $\mathbb{Z}$ we can choose as map the addition)

- the exponential $\mathbb{R} \to \mathbb{R}^+, x \mapsto e^x$ is a GI.

Definition 1.3.3. Let G, H be groups and $f : G \to H$ a GH, then we define the **kernel** of f as

$$\mathrm{Ker}(f) = \{g \in G : f(g) = e_H\}$$

Moreover, we define the **range** of f as

$$R(f) = \{f(a) : a \in G\}$$

From this definitions it is easy to derive that the kernel is a subgroup of G, while the range is a subgroup of H. Indeed, if $g_1, g_2 \in \mathrm{Ker}(f) \implies f(g_1) = f(g_2) = e_H$, but due to the fact that f is a GH, it is also true that $f(g_1 g_2) = f(g_1) f(g_2) = e_H$. Therefore, $g_1 g_2 \in \mathrm{Ker}(f)$. Furthermore, $e_G \in \mathrm{Ker}(f)$, in fact $f(e_G))e_H$. And finally if $g \in \mathrm{Ker}(f) \implies f(g^{-1}) = f(g)^{-1} = e_H^{-1} = e_H$, therefore $g^{-1} \in \mathrm{Ker}(f)$. This proves the properties of a group and thus $\mathrm{Ker}(f) \leq G$. Similarly it can be demonstrated also that $R(f) \leq H$. Moreover, if f is also injective, then $\mathrm{Ker}(f) = \{e_G\}$ and if f is surjective, then $R(f) = H$.

In the first section of this chapter we gave some examples of groups, in particular we studied the symmetric group. Its central importance in group theory is due to the following theorem that connects finite groups to the symmetric group through an isomorphic map. Furthermore, groups can be represented by matrices via an homomorphic map, as we will see later.

Theorem 1.3.4. *(Theorem of Cayley for finite groups) Every finite group of order n is isomorphic to a subgroup of the symmetric group S_n.*

The proof is based on the representation of a group with a group table. Each column and each row contains all elements of G. Hence, multiplication of a row or a columns with an element of G corresponds to a permutation of the element of G. $g \mapsto \Pi_g, G \to H \leq S_n$ is a GI.

1.4 Representation of groups

The elements of a group can be very abstract objects with which it could be difficult to work. However, it is possible to map groups into matrices that can be manipulated more easily. This mapping is the aim of representation theory, it can be proved that a group has infinitely many representations. Let us start as usual, by defining a representation.

Definition 1.4.1. A representation D of a group G is a GH $D : G \to GL_k(V)$, where $GL_k(V)$ is the group of general linear functions over a vector space V.

It can be showed that $GL_k(V)$ is isomorphic to $GL_n(\mathbb{C})$, which are invertible matrices. Therefore, D maps the group G into invertible $n \times n$ matrices, where $n = \dim(V)$. Some remarks, V is called carrier (representation) space and $\dim(V)$ is called the dimension of the representation. As notation we use D_G or $D(G)$ for the representation and in the rest of this work we will consider only finite dimensional representation. Every group G has a *trivial representation*, that is $D : G \to \mathbb{C} \simeq GL_1, g \mapsto 1$. It easy to check that this is indeed a representation, however all the information about the group is lost. This trivial representation is one dimensional, but it is also possible to define a trivial representation in higher dimension.

Definition 1.4.2. A representation D is called **faithful** if D is injective.

As already said, a group can have different representations, but two representations $D_1 : G \to GL(V), D_2 : G \to GL(W)$ are called equivalent if there exists a mapping $f : W \to V$ which is an isomorphism

$$D_1(g) = f \circ D_2(g) \circ f^{-1} \quad \forall g \in G.$$

Before looking at some examples, notice that, since D is a GH, it holds that $D(g_1 g_2) = D(g_1)D(g_2)$ for any representation D. Therefore, $D(g)$ can be computed from $D(g_i)$, where g_i are the generating elements, $G = \langle \{g_i\} \rangle$. For instance, let us consider $S_3 \simeq D_3 = \langle \{\sigma^1, C_3\} \rangle$, we can compute $D_3(C_3^2) = D_3(C_3)D_3(C_3)$, where $D_3(C_3)$ is known. The same for $D_3(C_3^{-1}) = D_3(C_3)^{-1}$.
Let us go through some instances of representations of the group S_3.

- A two dimensional representation of S_3 could be the following $D_2 : S_3 \to GL_2$:

$$C_3 \mapsto \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & \sin\left(\frac{2\pi}{3}\right) \\ -\sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

  This representation is also faithful.

- A one dimensional representation is the map

$$C_3 \mapsto 1 \qquad \sigma^1 \mapsto 1,$$

  which is a trivial representation.

- Another non trivial one dimensional representation is

$$C_3 \mapsto 1 \qquad \sigma^1 \mapsto -1.$$

  The number are not chosen randomly, but they are the determinant of previous matrices. This representation is not faithful and can be generalized. In fact, it can be demonstrated that if $D : F \to GL_n$ is a representation of G, then $D : G \to \mathbb{C}, g \mapsto \det(D(g))$ is a one dimensional representation of G.

- A three dimensional representation can be done by adding one row and one column to the two dimensional representation, as follows

$$C_3 \mapsto \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & \sin\left(\frac{2\pi}{3}\right) & 0 \\ -\sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This can be also done for higher dimensional representations, but they are not very interesting. However, there are representations called irreducible with which every representation can be constructed, as we will see later.

- There is a more interesting three dimensional representation of S_3. First let us recall that the action of a group with representation $D_3 : S_3 \to GL(\mathbb{C}^3)$ can be written using the standard basis $\{\mathbf{e_i}\}$ of $\mathbb{C}^3$, in fact $D_3(g)\mathbf{e_i} = \mathbf{e_g}$

$$D_3(g_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad D_3(g_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad D_3(g_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$D_3(g_4) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad D_3(g_5) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{1.2}$$

and the identity matrix which is the neutral element. In order to show that this is indeed a representation, it has to fulfil $D_3(g_ig_j) = D_3(g_i)D_3(g_j)$, i.e. the representation is a GO. This can be done easily with the Dirac notation $D(g_ig_j) = \sum_i |e_{g_i}\rangle \langle e_{g_i}|$. This last example is called regular representation, which we are going to focus on in the next section.

1.4.1 Regular representations

A regular representation is defined for any finite group, and if $|G| = n$, then the representation $R(p) \in M_{n,n}(\mathbb{R})$. Moreover, the regular representation contains in a sense all the representation.

Definition 1.4.3. Let G be a group, then a **regular representation** $R : G \to GL_n(\mathbb{R})$ is defined as follows

$$R(g_i)_{kj} = \begin{cases} 1 & \text{if } g_kg_j^{-1} = g_i \\ 0 & \text{otherwise} \end{cases}$$

This definition can be better understood by considering the group table. Since the order of the elements is not relevant, we can sort the table such that, on the diagonal, there is only the neutral element. $R(g_i)$ is then defined as the $n \times n$ matrix that we obtain from the table by replacing g_i with one, and $g_j \neq g_i$ with zeros on each row. It is possible to prove that R is a representation, this is left as an exercise for the reader.

G	g_1	g_2		g_n
g_1^{-1}	e	$g_1^{-1}g_2$	$\cdots$	$g_1^{-1}g_n$
g_2^{-1}	$g_2^{-1}g_1$	e		$g_2^{-1}g_n$
$\vdots$			$\ddots$	
g_n^{-1}	$g_n^{-1}g_1$	$g_n^{-1}g_2$		e

Definition 1.4.4. Let $D : G \to GL(V)$ be a representation, then D is called **unitary** if

$$D(g)^\dagger = D(g)^{-1} = D(g^{-1}) \quad \forall g \in G$$

Let us consider two representations D_1, D_2 of G, then $D_1 \oplus D_2 : G \to GL(V_1 \oplus V_2), g \mapsto D_1(g) \oplus D_2(g)$ is a representation. Therefore, it is possible to build representations. This also means that there exist infinitely many representations, as we are going to see, for finite groups only very few representations are necessary to construct all of them. Notice that if $D_1 \oplus D_2$ is a representation, also $S(D_1 \oplus D_2)S^{-1}$ is a representation for any $S \in GL(\dim(V_1) + \dim(V_2))$.

1.4.2 Irreducible representations

We have already remarked that a representation can be built with others. However, there are some representations that constitute the basic blocks for building other representations, and cannot be obtained from other representations. These representations are called irreducible representations. Before jumping into their definition, we need to discuss a particular class of spaces.

Definition 1.4.5. Let (D, V) be a representation, then a subspace $W \subseteq V$ is called **invariant subspace** if

   i. $D(g)W \subseteq W, \forall g \in G$, i.e. $D(g)w \in W$ for $w \in W$.

   ii. $W \neq \{0\}$ and $W \neq V$.

Definition 1.4.6. A representation (D, V) is called **irreducible** if there exist no non-trivial invariant subspaces, i.e. there exists no invariant subspace W with $W \neq \{0\}$ and $W \neq V$. Otherwise (D, V) is called **reducible**.

Invariant subspaces are important because we can actually work inside them, in fact, let W be an invariant subspace of a representation (D, V), then D restricted to W is a representation

$$D\big|_W : G \to GL(W) \qquad g \mapsto D\big|_W(g).$$

Definition 1.4.7. A representation (D, V) is called **completely reducible** if there exist invariant subspaces $V_i \subseteq V$ such that $V_1 \oplus V_2 \oplus \cdots \oplus V_n = V$. If $(D\big|_{V_i}, V_i)$ are irreducible, then $V_1 \oplus V_2 \oplus \cdots \oplus V_n = V$ is called **decomposition** into irreducible representations. Furthermore, we can write our representation as

$$D = \bigoplus_i D\big|_{V_i}$$

Notice that not any representation is completely reducible, for instance let us consider the following representation

$$D(\mathbb{R}, +) \to GL_2 \quad x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

If we apply our matrix on the vector $(1, 0)^T$ we obtain $(1, 0)^T$. Therefore, this vector is invariant, but there exist no v such that $D(x)v$ is proportional to v, that is the matrix is not diagonalizable, thus we cannot write $\mathbb{R}$ as a direct sum of two invariant subspaces. Hence D is not completely reducible.
Nevertheless, for this example we considered a non finite group, but we will see that for a finite group, any representation is completely reducible.
A relevant case for physics is when V is a Hilbert space, and therefore there exists a scalar product on such space, therefore the orthogonal complement $W^\perp$ is well defined and can be used.

Lemma 1.4.8. *A representation (D, V) is completely reducible if and only if for any invariant subspace W, there exists an invariant subspace W' such that $W \oplus W' = V$*

Proof:

$\implies$) (D, V) is completely reducible $\implies V = V_1 \oplus V_2 \oplus \dots V_n$, where V_i are invariant subspaces and does not contain any non-trivial invariant subspace. If W is an invariant subspace, then $W \cap V_i$ is either $\{0\}$ or V_i this implies that $W = \bigoplus_{i \in I} V_i$ for some indices I, thus $W^\perp = \bigoplus_{i \notin I} V_i \implies W^\perp$ is invariant.

$\impliedby$) The proof is by induction on the dimension of V. If $\dim(V) = 1$, the statement is obvious $V = V_1$. Now let us assume that it is true for $\dim(V) = d$, and we prove that it is true for $\dim(V) = d + 1$. V is either irreducible or there exists an invariant subspace $W \subset V$ with $1 \leq \dim(W) \leq d$. By assumption $\exists W' : V = W \oplus W'$ and W' is irreducible and also $\dim(W') \leq d$ which implies that W, W' are completely irreducible, i.e. V is completely reducible. $\qquad\square$

As an example let us study the symmetric group S_3, recall the three dimensional representation $D_3(g)$ which can be written as $D_3(g)e_i = e_{g_i}$. Our goal is to find an invariant subspace $W \subset \mathbb{R}^3$ such that $D_g W = W \ \forall g \in G$. For instance

$$W_1 = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

is an invariant subspace with dimension one, in fact applying a permutation to this space does not change it. Another is

$$W_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \sum_i x_i = 0 \right\}.$$

This is a two dimensional subspace. The point now is: can we reduce further this space? If the answer is no we decomposed D_3, otherwise we can further decompose it. The answer is no, there is not, and we can show it. Consider a hypothetical invariant subspace $\overline{W}$ of W that is non trivial. The dimension of $\overline{W}$ must be one and $D_g \overline{W} \propto \overline{W}$, for instance we we take the permutation $2 \to 1$

$$D_g \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \propto \begin{pmatrix} w_2 \\ w_1 \\ w_3 \end{pmatrix} \quad \forall g \in G.$$

This must hold for every permutation. Therefore, $\omega_1 \propto \omega_2 \propto \omega_3$ and $\sum_i \omega_i = 0$. The only solution for ω_i is $\omega_i = 0$ that is a trivial space. In conclusion, W_1 e W_2 are the only invariant subspaces which implies that D_3 is completely reducible with $V = W_1 \oplus W_2$ and more importantly $D = D\big|_{W_1} \oplus D\big|_{W_2}$. We will return on this example in the next section.

1.5 Finite groups

Theorem 1.5.1. *Any irreducible representation of a finite group has finite dimension.*

Proof:

The proof can be done by contradiction. Suppose that V is infinite dimensional and irreducible, then we can define the space $W_a = \{D_g \ket{a} : g \in G, a \in V\}$. W_a is finite because G is finite and it is invariant which is a contradiction of the assumption. $\quad\square$

Theorem 1.5.2. *Any representation of finite group is equivalent to a unitary representation, i.e.*

$$\forall (D,V), \forall g \in G \ \exists S : SDS^{-1} = \widetilde{D}_g \ \text{ and } \ \widetilde{D}_g^{-1} = \widetilde{D}_g^\dagger$$

Proof:
Let D be a representation of G, we define the positive[1] operator $T = \sum\limits_{g \in G} D_g^\dagger D_g$. Then we show the following

$$D_h^\dagger T D_h = D_h^\dagger \sum_{g \in G} D_g^\dagger D_g D_h = \sum_{g \in G} D_h^\dagger D_g^\dagger D_g D_h = \sum_{g \in G} D_g^\dagger D_g \equiv T, \qquad (1.3)$$

where in the last step we used the fact that $D_g D_h = D_{gh}$, and we have already showed that gh is just a permutation of the elements, therefore the sum over every term D_{gh} is equivalent to the sum D_g by the commutativity of the sum. Obviously, equation (1.3) holds for every $h \in G$. Now we define $S = T^{\frac{1}{2}}$, which is well defined because T is positive[2], and we rewrite equation (1.3) as $D_h^\dagger S^2 D_h = S^2$, which can be rewritten as $D_h^\dagger S = S^2 D_h^{-1} S^{-1}$. Multiplying this by S^{-1} leads to $S^{-1} D_h^\dagger S = S D_h^{-1} S^{-1}$ and finally taking the inverse $(S^{-1} D_h^\dagger S)^{-1} = S^{-1} D_h S$, in the right hand side we can write instead of S, $S^\dagger$, since they are the same and we obtain the final result $(S^{-1} D_h^\dagger S)^{-1} = (S^{-1} D_h^\dagger S)^\dagger$ $\hfill (?:/)\square$

Theorem 1.5.3. *Any unitary representation of a finite group is completely reducible*

Proof:
Let W be invariant, and let $a \in W^\perp, b \in W, g \in G$, since $D_g b \in W$ then $0 = \langle a | D_g | b \rangle = \langle D_g^\dagger a | b \rangle = \langle D_{g^{-1}} a | b \rangle = 0 \implies D_g W^\perp \subseteq W^\perp$. According to lemma 1.4.8 a representation (D,V) is completely reducible if and only if for every invariant subspace W also $W^\perp$ is invariant. $\hfill \square$

In summary, we showed the following propositions

1. Any representation of a finite group is finite

2. Any representation of a finite group is equivalent to a unitary group

3. Any unitary representation is completely reducible

as a corollary of these we have the most important result

> Any representation of a finite group
> is completely reducible

That is, if G is finite, let us consider $D : G \to GL(V)$, then

$$\exists S \in GL(V) : D_g = S \begin{pmatrix} D_1(g) & & \\ & D_2(g) & \\ & & \ddots \end{pmatrix} S^{-1},$$

---

[1] in fact $\langle \psi | D_g^\dagger D_g | \psi \rangle = \|\psi\|^2$, and sum of positive terms is always positive.
[2] $T = T^\dagger \implies \exists u, d : T = u d u^\dagger$, where d is diagonal. That is $T^{1/2} = U \sqrt{D} U^\dagger$

where D_i are irreducible representations.

Let us continue with the usual example of the representation of S_3. We have already showed how it can be decomposed into two invariant spaces W_1 and W_2, now let us try to find the representations restricted to those spaces. This can be done projecting the operator into the subspace

$$D_3(g)\big|_{W_1} = P_{W_1} D_3(g) P_{W_1} = |w_1\rangle \langle w_1| D_3(G) |w_1\rangle \langle w_1| = |w_1\rangle \langle w_1| \quad \forall g \in G.$$

In fact a permutation over a state with the same components does not change anything. Hence $D_3(g)\big|_{W_1}$ is the trivial representation $D_3(g)\big|_{W_1} : G \to 1$. More interesting is of course the space W_2, consider a basis of this space, for instance

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Then the projector is $P_{W_2} = |v_1\rangle \langle v_1| + |v_2\rangle \langle v_2|$, hence

$$D_3(g)\big|_{W_2} = P_{W_2} D_3(g) P_{W_2} = \sum_{ij} |v_i\rangle \langle v_i| D_3(g) |v_j\rangle \langle v_j|,$$

which can be calculated by hand for every permutation. A faster way is to consider The decomposition of D as

$$D_s(g) = S \begin{pmatrix} D_3(g)\big|_{W_1} & 0 \\ 0 & D_3(g)\big|_{W_2} \end{pmatrix} S^{-1} = S \begin{pmatrix} 1 & 0 \\ 0 & D_3(g)\big|_{W_2} \end{pmatrix} S^{-1}.$$

$D_3(g)\big|_{W_2}$ must be a two dimensional representation of S_3 which we have already calculated in a previous example.

Let us finish this section by showing why it is possible to decompose a representation as a direct sum of restricted representations. For ease we will show the case of S_3, where the vector space is decomposed as $W = W_1 \oplus W_2 = W_1 \oplus W_1^\perp$. Consider the projector $P_1 = \sum_i^{\dim(W_1)} |v_i\rangle \langle v_i|$, where $\{v_i\}$ is a basis of W_1 and respectively $P_2 = \sum_i^{\dim(W_2)} |v_i\rangle \langle v_i|$, where $\{v_i\}$ is a basis of W_2. From the completeness of the basis we can write $\mathbb{1} = P_1 + P_2$, thus we can also write $D(g) = \mathbb{1} D(g) \mathbb{1}$, substituting the identity operator with the sum of projectors yields

$$D(g) = (P_1 + P_2) D(g)(P_1 + P_2) = (P_1 D(g) + P_2 D(g))(P_1 + P_2)$$
$$P_1 D(g) P_1 + P_1 D(g) P_2 + P_2 D(g) P_1 + P_2 D(g) P_2. \quad (1.4)$$

In this expression all the mixed terms are zero, since W_1 and W_2 are invariant subspaces, hence $D(g)W_1 \subseteq W_1$ and the same for W_2, i.e. $D(g)P_1$ will be in W_1 for all g in G. Therefore, if you project W_1 into W_2 you will obtain 0, since the spaces are orthogonal. In conclusion we get

$$D(g) = P_1 D(g) P_1 + P_2 D(g) P_2,$$

which is exactly the same as $D(g) = D(g)\big|_{W_1} \oplus D(g)\big|_{W_2}$.

1.6   Schur's lemma

Lemma 1.6.1. *(Schur's lemma) Let (D_1, V_1) and (D_2, V_2) be two irreducible representations of a finite group G, and let A be a linear operator between V_1 and V_2, $A \in \mathfrak{L}(V_1, V_2)$ such that*

$$AD_1(g) = D_2(g)A \qquad \forall g \in G,$$

then A is either an isomorphism, or $A = 0$.

Proof:
In order to prove this lemma, we are going to show that both $\mathrm{Ker}(A)$ and $R(A)$ are invariant subspaces. In fact, if $\mathrm{Ker}(A)$ is an invariant subspace under D_1, then, since $D_1(g)$ is irreducible,

$$\mathrm{Ker}(A) = \begin{cases} \{0\} & \text{or} \\ D_1 \end{cases}.$$

The same holds for the range, if $R(A)$ is an invariant subspace under D_2, then

$$R(A) = \begin{cases} \{0\} & \text{or} \\ D_2 \end{cases}.$$

Let us take a look at the case $\mathrm{Ker}(A) = D_1$ and $R(A) = 0$, this means that every element of D_1 is mapped to 0, therefore $A = 0$. In the other case $\mathrm{Ker}(A) = \{0\}$ means that A is injective and $R(A) = D_2$ means that A is surjective, hence A is bijective, so it is a isomorphism.

Therefore, in order to prove the lemma we must prove the $\mathrm{Ker}(A)$ and $R(A)$ are invariant subspaces.

- $\mathrm{Ker}(A)$ is an invariant subspace of V_1:

$$D_1(g)\mathrm{Ker}(A) = \mathrm{Ker}(A) \qquad \forall g \in G,$$

  let $|a\rangle \in \mathrm{Ker}(A)$, then $AD_1(g)|a\rangle = D_2(g)A|a\rangle = 0$, which means that $D_1(g)|a\rangle \in \mathrm{Ker}(A)$ for all g and for all a. Hence $D_1(g)\mathrm{Ker}(A) \subseteq \mathrm{Ker}(A)$

- $R(A)$ is an invariant subspace of V_2:
  let $|b\rangle \in R(A)$, then it exists $|x\rangle$ such that $|b\rangle = A|x\rangle$, therefore we can write

$$AD_1(g)|x\rangle = D_2(g)A|x\rangle \qquad \forall g \in G, \forall b \in R,$$

  but we can notice that $AD_1(g)|x\rangle \in R(A)$, and also $A|x\rangle \in R(A)$, thus $D_2(g)R(A) \subseteq R(A)$.

$\square$

This lemma has a special case, i.e. when $V_1 = V_2$, which is particularly interesting for physics. In fact it can be used to diagonalize operators on Hilbert space very easily, as we will see in an example later.

Lemma 1.6.2. *Let $D : G \to GL(V)$ be an irreducible representation of G, and $A \in \mathfrak{L}(V)$, such that*

$$AD(g) = D(g)A \qquad \forall g \in G,$$

then $A = \lambda \mathbb{1}$, with $\lambda \in \mathbb{C}$.

The proof of this lemma is so straightforward that any reader can do it by exercise. It is also possible to find λ very easily, consider the trace of the operator

$$\mathrm{Tr}(A) = \lambda \mathrm{Tr}(\mathbb{1}) \implies \lambda = \frac{\mathrm{Tr}(A)}{\dim(V)},$$

where we used the fact that $\mathrm{Tr}(\mathbb{1}) = \dim(V)$.

Theorem 1.6.3. *Any irreducible finite dimension representation of an abelian group is one dimensional.*

The proof is left as an exercise.

1.7 Applications

In quantum mechanics, it is common to diagonalize a certain operator in order to obtain the possible outcomes of a measurement. Moreover, it is also useful to diagonalize an operator, especially the Hamiltonian, since the evolution of a state is dictated by the evolution operator $U(t) = e^{-iHt}$. Group theory, in particular representation, and all the theory we developed so far can help to perform this diagonalization. Let us make the hypothesis that there is a symmetry on the system which commutes with the Hamiltonian, i.e. $HD(g) = D(g)H$, then if $D(g)$ is irreducible, we can express our Hamiltonian as $H = \lambda \mathbb{1}$ by the lemmas of the previous section. This is not very interesting, but if $D(g)$ is reducible, we can write it as $S \oplus_i D_i(g)S^{-1}$, where $D_i(g)$ are now irreducible and can be written as $D_i(g) = \lambda_i \mathbb{1}$. This will give us the eigenvalues of the Hamiltonian. For instance we can consider the following matrix

$$A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} \qquad a, b \in \mathbb{C}.$$

It is easy to check that $AD_3(g) = D_3(g)A$. We have already seen the decomposition of D_3 $D_3 = S(D_1 \oplus D_2)S^{-1}$, we can use this relation in the previous and we found

$$AS(D_1 \oplus D_2)S^{-1} = S(D_1 \oplus D_2)S^{-1}A,$$

we can multiply this expression by S on the right and S^{-1} on the left, so we obtain

$$S^{-1}AS(D_1 \oplus D_2) = (D_1 \oplus D_2)S^{-1}AS. \qquad (1.5)$$

$S^{-1}AS$ is not known, but we can express it as a matrix with unknown coefficients

$$B = S^{-1}AS = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \qquad B_{11} \in \mathbb{C}, B_{22} \in M_{2,2}, B_{1,2} \in M_{1,2}, B_{2,1} \in M_{2,1}.$$

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Notice that B commutes with $D_3(g)$ as shown in equation (1.5). Now, we can calculate the matrix multiplication between B and D_3 of equation (1.5):

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

we end up with

$$\begin{pmatrix} B_{11}D_1 & B_{12}D_2 \\ B_{21}D_1 & B_{22}D_2 \end{pmatrix} = \begin{pmatrix} D_1 B_{11} & D_1 B_{12} \\ D_2 B_{21} & D_2 B_{22}. \end{pmatrix}$$

The equality for every term of the matrix is

$$\begin{cases} B_{11}D_1 = D_1 B_{11} \implies B_{11} = \lambda_1 \mathbb{1} \\ B_{12}D_2 = D_2 B_{12} \implies B_{12} = 0 \\ B_{21}D_1 = D_1 B_{21} \implies B_{21} = 0 \\ B_{22}D_2 = D_2 B_{22} \implies B_{22} = \lambda_2 \mathbb{1} \end{cases}$$

where we used lemma 1.6.2. In conclusion we found that

$$B = S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \mathbb{1} \end{pmatrix},$$

which is the matrix A diagonalized.

1.8 Orthogonality relations

Let (D^α, W^α) and (D^β, W^β) be two irreducible representations of G and $X \in \mathfrak{L}(W^\alpha, W^\beta)$, we define the following operator

$$A = \sum_{g \in G} D_g^\alpha X D_{g^{-1}}^\beta,$$

which has the following propriety

$$D_h^\alpha A = A D_h^\beta \quad \forall h \in G.$$

This can be proved easily, in fact

$$A D_h^\beta = \sum_{g \in G} D_g^\alpha X D_{g^{-1}}^\beta D_h^\beta = \sum_{g \in G} D_g^\alpha X D_{g^{-1}h}^\beta,$$

but since we are in a group $g^{-1}h \in G$, and since we are summing over all the elements we have that it is the same to write $g^{-1}h$ or g in the sum. The same argument can be made for $D_h^\alpha A$ and the equality is verified. Now if $D^\alpha = D^\beta$, we can use lemma 1.6.2 and therefore $A = \lambda \mathbb{1}$, or in the case that D^α and D^β are inequivalent, by Schur's lemma $A = 0$. We can write both of these cases as $A = \lambda \mathbb{1} \delta_{\alpha\beta}$ using the definition we gave of A we can write

$$\sum_{g \in G} D_g^\alpha X D_{g^{-1}}^\beta = \lambda \mathbb{1} \delta_{\alpha\beta}, \tag{1.6}$$

taking the trace on both sides of this equation leads to

$$\lambda \delta_{\alpha,\beta} \mathrm{Tr}(\mathbb{1}) = \sum_{g \in G} \mathrm{Tr}(D_g^\alpha X D_{g^{-1}}^\beta) \tag{1.7}$$

for $\alpha = \beta$ we have

$$\lambda \mathrm{Tr}(\mathbb{1}) = \sum_{g \in G} \mathrm{Tr}(D_g X D_{g^{-1}})$$

we can use the property of the trace that we can move the factors inside the trace, and since $D_g D_{g^{-1}} = \mathbb{1}$, we obtain

$$\lambda \mathrm{Tr}(\mathbb{1}) = \sum_{g \in G} \mathrm{Tr}(X) \implies \lambda = \frac{\mathrm{Tr}(X)}{\dim(W)} |G|. \tag{1.8}$$

This equation holds for any X, so we can choose a basis in W^α, consider X in that basis, and then choose $X_{ij} = \delta_{ij} \delta_{jm}$. If we insert this expression in equation (1.6) we obtain

$$\lambda \mathbb{1} \delta_{\alpha,\beta} = \sum_{g \in G} \left(D_g^\alpha\right)_{il} \left(D_{g^{-1}}^\beta\right)_{mj} = \frac{|G|}{\dim(W^\alpha)} \delta_{\alpha\beta} \delta_{ij} \delta_{ml}.$$

In the case where D^β is a unitary representation the previous equation can be written as

$$\sum_{g \in G} \left(D_g^\alpha\right)_{il} \left(D_g^\beta\right)_{mj}^\dagger = \frac{|G|}{\dim(W^\alpha)} \delta_{\alpha\beta} \delta_{ij} \delta_{ml}.$$

These expressions depend on the basis, unless we consider the equation with the trace, this leads to the definition of character of representations, which aims to characterize how many irreducible representations exist and how they can be characterized.

Definition 1.8.1. Let $D : G \to \mathrm{GL}(V)$ be a representation of a group G, the character of this representation is defined as

$$\chi : G \to \mathbb{C} \quad g \to \mathrm{Tr}(D_g) = \chi(g) = \sum_i D_{ii}(g).$$

If D is irreducible, then χ is called simple character.

The character has the following proprieties

- $\chi(e) = \mathrm{Dim}(V)$.

- $\forall g, h \in G \; \chi(g, hg^{-1}) = \chi(h)$, which is elements within the same conjugacy class have the same character.

- $\forall g \in G, \chi(g^{-1}) = \chi(g)^*$, Proof: $\chi(g^{-1}) = \mathrm{Tr}(D(g^{-1})) = \mathrm{Tr}(SD(g^{-1})S^{-1}) = \mathrm{Tr}(\widetilde{D}(g^{-1}))$, where $\widetilde{D}$ is a unitary representation from theorem 1.5.2. $\mathrm{Tr}(\widetilde{D}(g^{-1})) = \mathrm{Tr}(\widetilde{D}(g)^\dagger) = \chi(g)^*$

- $\chi_{D_1 \oplus D_2}(g) = \chi_{D_1}(g) + \chi_{D_2}(g)$

Another property of orthogonality can be found by considering the previous orthogonality relations and with $i = l$, $j = m$, and summing over i and j, we can obtain the following

$$\sum_{g \in G} \chi^\alpha(g)\chi^\beta(g^{-1}) = \sum_{g \in G} \chi^\alpha(g)\chi^\beta(g)^* = \delta_{\alpha\beta}\frac{|G|}{\mathrm{Dim}(W^\alpha)}\sum_{ij}\delta_{ij} = \delta_{\alpha\beta}|G|, \qquad (1.9)$$

where we used the fact that $\sum_{ij}\delta{ij} = \mathrm{Dim}(W^\alpha)$. We can notice that the left hand side of the equations is equivalent to a scalar product, hence the characters of inequivalent irreducible representations are orthogonal. Introducing the following notation

$$\vec{\chi} = \begin{pmatrix} \chi_D(g_1) \\ \vdots \\ \chi_D(g_n) \end{pmatrix},$$

we can rewrite equation (1.9) as

$$\langle \vec{\chi}^\alpha, \vec{\chi}^\beta \rangle = \delta_{\alpha\beta}|G|.$$

Since $\vec{\chi}$ is a G-dimensional vector, we have that there can be at most $|G|$ inequivalent irreducible representations. The vector $\vec{\chi}$ can be reduced with the conjugacy classes, in fact as already stated, elements of a conjugacy class have the same characteristic. Therefore we define the following

$$\overline{\chi}_D = \begin{pmatrix} \sqrt{C_1}\chi_D(g_1) \\ \vdots \\ \sqrt{C_r}\chi_D(g_r) \end{pmatrix},$$

where r is the number of different conjugacy classes, and C_i is the number of elements of a conjugacy class. The orthogonality relation becomes

$$\langle \vec{\chi}^\alpha, \vec{\chi}^\beta \rangle = \langle \overline{\chi}^\alpha, \overline{\chi}^\beta \rangle = \delta_{\alpha\beta}|G|.$$

Furthermore, $\overline{\chi}_D$ is an r-dimensional vector, thus there exist at most r different conjugacy classes.

Theorem 1.8.2. *The number of irreducible representations of a finite group coincides with the number of different conjugacy classes.*

1.9 Analysis of representations

The idea of character introduced can be very useful in the study of representations, we already saw that any representation D of a finite group is completely reducible, i.e. $D = \bigoplus_i D_i = m_1 D_1 + m_2 D_m \cdots = \sum_\alpha m_\alpha D^\alpha$, where D_i are irreducible representations. Moreover we saw that $\chi^\alpha \perp \chi^\beta$ for D^α and D^β. From the properties of the character we have that $\chi_D = \sum_\alpha \chi^\alpha$ and the equivalent vector expression $\vec{\chi}_D = \sum_\alpha \vec{\chi}^\alpha$. If we take the scalar product with $\vec{\chi}^\beta$ and with the orthogonality relation, we obtain

$$m_\beta = \frac{\langle \vec{\chi}^\beta | \vec{\chi}_D \rangle}{|G|}.$$

This result can be used to prove a useful criterion

Criterion for characters of irreducible representation

Let D be a representation of G, then D is irreducible $\iff \langle \vec{\chi}_D | \vec{\chi}_D \rangle = \sum_{g \in G} |\chi_D(g)|^2 = |G|$.

The proof is the following:

$\implies$) If D is irreducible, then $D = m_1 D_1$ with $m_1 = 1$, then $1 = \frac{\langle \vec{\chi}_D | \vec{\chi}_D \rangle}{|G|} \implies \langle \vec{\chi}_D | \vec{\chi}_D \rangle = |G|$

$\impliedby$) $\chi_D = \sum_\alpha m_\alpha \chi^\alpha \implies \langle \chi_D | \chi_D \rangle = \sum m_\alpha m_\beta \langle \chi_\alpha | \chi_\beta \rangle$, now we use the orthogonality relation $\langle \chi_\alpha | \chi_\beta \rangle = \delta_{\alpha\beta} |G|$ and we get $\sum m_\alpha m_\beta \langle \chi_\alpha | \chi_\beta \rangle = |G| \sum m_\alpha^2 = |G| \implies \sum m_\alpha^2 = 1$. Since $m\alpha \in \mathbb{N}, \exists! \alpha : m_\alpha = 1$, while $m_\beta = 0$ for $\beta \neq \alpha$ which implies that D is irreducible. $\qquad \square$

For small groups this criterion can be very useful, for instance, consider S_3 and its three dimensional representation D_3 defined with equation (1.2), we can easily check that $\chi_{D_3}(e) = 3$, while $\chi_{D_3}(g_1) = \chi_{D_3}(g_2) = \chi_{D_3}(g_3) = 1$, and $\chi_{D_3}(g_4) = \chi_{D_3}(g_5) = 0$. With these values we can check whether D_3 is irreducible or not. If it is irreducible, then $\langle \chi_{D_3} | \chi_{D_3} \rangle = |G| = 6$, but summing the squares of all the characters calculated we get $\langle \vec{\chi}_{D_3} | \vec{\chi}_{D_3} \rangle = 12$, thus D_3 is not irreducible. With the criterion we can also draw some interesting conclusions, i.e. let us consider D_3 as sum of irreducible representations $D_3 = \bigoplus_\alpha m_\alpha D_\alpha$, then

$$\langle \vec{\chi}_{D_3} | \vec{\chi}_{D_3} \rangle = \sum_\alpha m_\alpha^2 \langle \vec{\chi_{D_i}} | \vec{\chi}_{D_i} \rangle = |G| \sum_\alpha m_\alpha^2,$$

that with the values $|G| = 6$ and $\langle \vec{\chi}_{D_3} | \vec{\chi}_{D_3} \rangle = 12$ can be written as $12 = 6 \sum_\alpha m_\alpha^2$, and since m_α are integers, the only solution is $m_1 = m_2 = 1, m_i = 0$ for $i \neq 1, 2$. In conclusion the representation D_3 can be written as sum of two irreducible representations $D_3 = D_1 \oplus D_2$.

We can also study the character of a regular representation R, that recalling section 1.4.1 is defined as

$$R(g_i)_{kj} = \begin{cases} 1 & \text{if } g_k g_j^{-1} = g_i \\ 0 & \text{otherwise} \end{cases} .$$

We want to prove that $R = \bigoplus_i m_i D_i$, where $m_i = \text{Dim}(D_i)$, and D_i is an irreducible representation. The character of a regular representation can be easily calculated $\chi_R(g_i) = \text{Tr}(R(g_i)) = \delta_{i1} |G|$, where $g_1 = e$. We can express the regular representation as a sum: $\vec{\chi}_R = \sum_i m_i \vec{\chi}_i$, where $\vec{\chi}_i$ is a simple character. We can also calculate the following scalar product

$$\langle \vec{\chi}_R | \vec{\chi}_j \rangle = \sum_i [\chi_R(g_i)]^* \chi_j(g_i) = |G| \chi_j(e) = |G| \dim(D_j) \equiv |G| n_j,$$

where we used the fact that $\chi_R(g_i) \delta_{i1} |G|$. Moreover if we use $\vec{\chi}_R = \sum_i m_i \vec{\chi}_i$ the same scalar product is

$$\langle \vec{\chi}_R | \vec{\chi}_j \rangle = m_j \langle \vec{\chi}_j | \vec{\chi}_j \rangle = |G| m_j.$$

From the last two equations we obtain the desired result $m_j = n_j \equiv \dim(D_j)$. Therefore, the regular representation contains all irreducible representations of G with multiplicity $n_i = \dim(D_i)$. From the character of a regular representation we can also find

the following useful relation

$$\chi_R(g_i) = \sum_{j=1} n_j \chi_j(g_i) = \delta_{i1}|G| \implies |G| = \sum_{j=1} n_j^2$$

. Using this fact we can state that for instance

- if $|G| = 2$, then the group has exactly two one-dimensional representations.

- if $|G| = 3$, then the group has exactly three one-dimensional representations.

- if $|G| = 6$, then the group has either two one-dimensional representations and one two-dimensional, or six one-dimensional representations.

Theorem 1.9.1. *If a group G has only one-dimensional representations, then G is abelian.*

The theorem is not the case of S_3, since it has a two-dimensional representation.

1.10 Summary of finite groups and their representations

The important results of the previous sections can be summarized as follows.
Consider a finite group $G = \{g_1, \ldots, g_n\}$, since it is a group $g_i g_j = g_k$. A representation is a function $D : G \to GL(n)$ with the property $D(hg) = D(h)D(g)$. Moreover, we have the following result

- Any representation is equivalent to a unitary group $D(g)^\dagger = D(g^{-1}) = D(g)^{-1}$.

- Any unitary representation is completely reducible.

- Any representation is completely reducible, i.e. $D = \bigoplus D_i$, with D_i irreducible representations.

- There exist r irreducible representations, where r is the number of conjugacy classes.

- D_i is irreducible $\iff \langle \chi_i | \chi_i \rangle = |G|$

- $\vec{\chi}_D = \sum_i m_i \vec{\chi}_i$, and $\vec{\chi}_R = \sum_i n_i \vec{\chi}_i$, $D_R = \bigoplus_i n_i D^i$.

1.11 Lie Group

A Lie group has a central role in physics, so in this chapter we will give a short introduction. We only treated finite groups, but many of the obtained result cold be generalized to continuous groups G, that is group whose order is uncountably infinite. The elements of G can be parametrized as $g(\mathbf{a}) = g(a_1, a_2, \ldots, a_n)$, with $\mathbf{a} \in \mathbb{R}^n$. For the multiplication of two elements, we have that $g(\mathbf{a})g(\mathbf{b}) = g(\mathbf{c})$, where $\mathbf{c} = f(\mathbf{a}, \mathbf{b})$.

Definition 1.11.1. Let $G = \{(a_1, a_2, \ldots, a_n)\}$, be a continuous group, and f be an analytic function, i.e. there exists a converging Taylor series for f, then G is called an n-parameter **Lie Group**.

Definition 1.11.2. A Lie group is called compact if its parameter space is compact in $\mathbb{R}^n$.

Examples of Lie groups are

- $U(1) = \{e^{i\alpha}\}$, with $\alpha \in [0, 2\pi]$, which is a compact Lie group.

- GL_n is also a Lie group.

- $SU(2) = \{e^{i\alpha\sigma_x} e^{i\beta\sigma_y} e^{i\alpha\sigma_x}\}$ is a three-parameter compact Lie group.

- $SO(2) = \{A \in SL(2, \mathbb{R}) : AA^\dagger = A^\dagger A = \mathbb{1}\}$, this is a one-parameter compact Lie group, which we can parametrize as $R(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$

The idea of Lie was to consider infinitesimal transformations, any element of the group can be decomposed into those infinitesimal transformations.
We can also find the generator of the last example, we can expand in a Taylor series $R(\phi)$ around $\phi = 0$

$$R(\phi) = R(0) + \left.\frac{dR}{d\phi}\right|_{\phi=0} \phi + \frac{1}{2}\left.\frac{d^2R}{d\phi^2}\right|_{\phi=0} \phi^2 + \ldots,$$

the first derivative is easy to calculate

$$\left.\frac{dR}{d\phi}\right|_{\phi=0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv X,$$

and similarly the other derivatives, we find that

$$R(\phi) = \mathbb{1} + X\phi + \frac{1}{2}X^2\phi^2 + \cdots = e^{\phi X} = e^{i\phi\sigma_y}.$$

X is called the generator of the group.