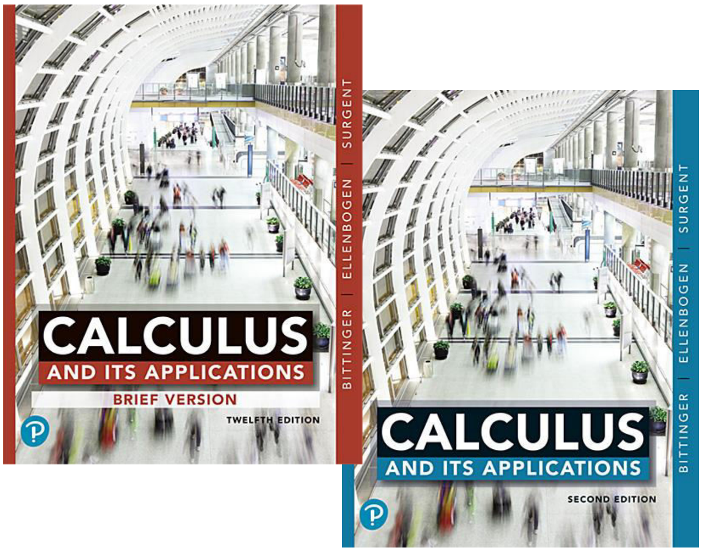
Chapter 4

Integration



Pearson

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

OBJECTIVE

- Use the properties of definite integrals to find the area between curves.
- Solve applied problems involving definite integrals.
- Determine the average value of a function.
- Find the moving average of a function

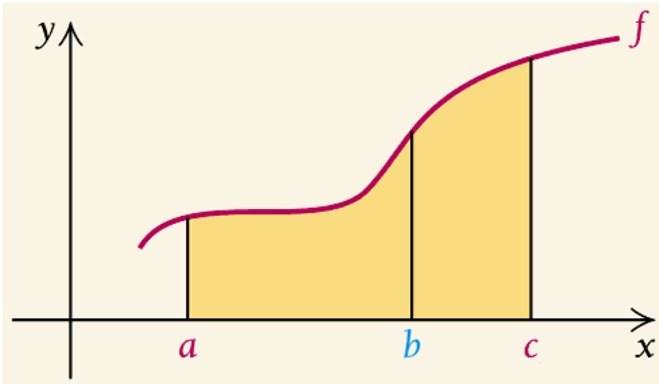
Pearson

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

THEOREM 5

For $a < b < c$, $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$

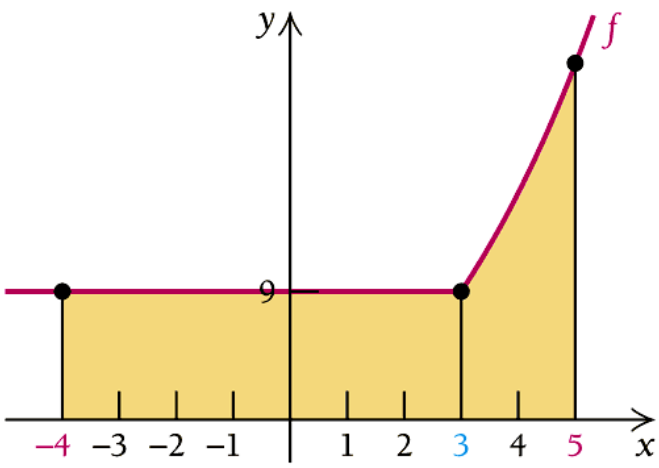
For any number b between a and c, the integral from a to c is the integral from a to b plus the integral from b to c.



4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 1: Find the area under the graph of $y = f(x)$ from –4 to 5, where

$$f(x) = \begin{cases} 9, & \text{for } x < 3, \\ x^2, & \text{for } x \geq 3. \end{cases}$$



4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 1 (concluded):

$$\begin{aligned}\int_{-4}^{5} f(x)\,dx &= \int_{-4}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx \\ &= \int_{-4}^{3} 9\ dx + \int_{3}^{5} x^2 dx \\ &= 9[x]_{-4}^{3} + \left[\frac{x^3}{3}\right]_{3}^{5} \\ &= 9\left(3-(-4)\right) + \left(\frac{5^3}{3} - \frac{3^3}{3}\right) \\ &= 95\frac{2}{3}\end{aligned}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Quick Check 1

Find the area under the graph of $y = g(x)$ from -3 to 6, where

$$g(x) = \begin{cases} x^2, & \text{for } x \le 2 \\ 8 - x, & \text{for } x > 2 \end{cases}$$

$$\int_{-3}^{6} g(x)\ dx = \int_{-3}^{2} g(x)\ dx + \int_{2}^{6} g(x)\ dx = \int_{-3}^{2} x^2 dx + \int_{2}^{6} (8-x)\,dx$$

$$= \left[\frac{1}{3}x^3\right]_{-3}^{2} + \left[8x - \frac{1}{2}x^2\right]_{2}^{6}$$

$$= \left(\frac{1}{3}(2)^3\right) - \left(\frac{1}{3}(-3)^3\right) + \left(\left(8(6) - \frac{1}{2}(6)^2\right) - \left(8(2) - \frac{1}{2}(2)^2\right)\right)$$

$$= \left(\frac{8}{3} + \frac{27}{3}\right) + (30 - 14) \ = 11\frac{2}{3} + 16 = 27\frac{2}{3}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Quick Check 2

Evaluate $\int_0^7 |2x-1|\,dx.$

The function $f(x)=|2x-1|$ is defined piecewise as follows

$$f(x)=|2x-1|=\begin{cases} 2x-1, & \text{for } x \ge \dfrac{1}{2} \\ 1-2x, & \text{for } x < \dfrac{1}{2} \end{cases}.$$

From here we can use our piecewise integration technique to solve.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Quick Check 2 Concluded

$$\int_0^7 |2x-1|\,dx = \int_0^{1/2}(1-2x)\,dx + \int_{1/2}^{7}(2x-1)\,dx$$

$$= \left[x-x^2\right]_0^{1/2} + \left[x^2+x\right]_{1/2}^{7}$$

$$= \left(\left(\frac{1}{2}-\left(\frac{1}{2}\right)^2\right)-\left(0-0^2\right)\right)+\left(\left(7^2-7\right)-\left(\left(\frac{1}{2}\right)^2-\frac{1}{2}\right)\right)$$

$$= \left(\frac{1}{4}-0\right)+\left(42+\frac{1}{4}\right)$$

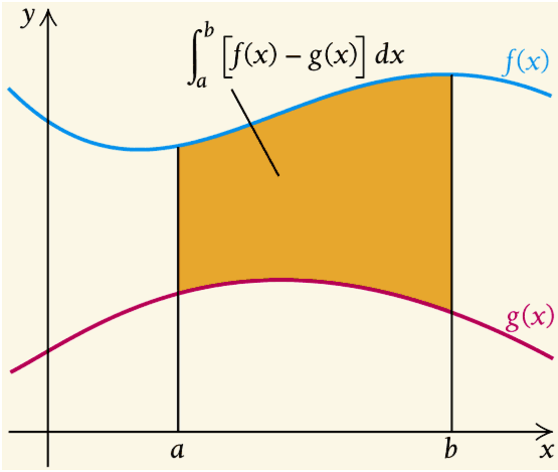
$$= 42\frac{1}{2}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

THEOREM 6

Let f and g be continuous functions and suppose that $f(x) \geq g(x)$ over the interval $[a, b]$. Then the area of the region between the two curves, from $x = a$ to $x = b$, is

$$\int_a^b [f(x) - g(x)]\, dx.$$



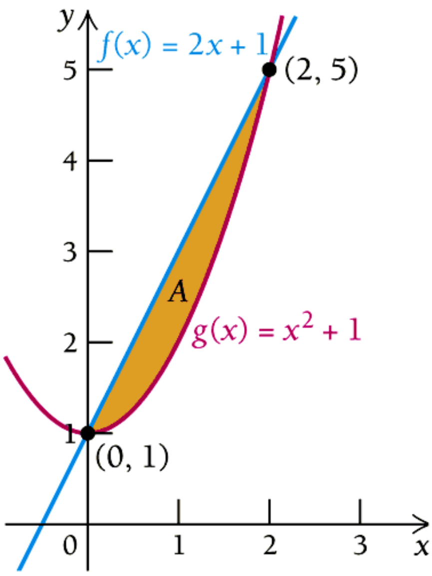
4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 2: Find the area of the region that is bounded by the graphs of

$f(x) = 2x + 1$ and

$g(x) = x^2 + 1.$

First, look at the graph of these two functions.



4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 2 (continued):

Second, find the points of intersection by setting $f(x) = g(x)$ and solving.

$$\begin{aligned} f(x) &= g(x) \\ 2x+1 &= x^2+1 \\ 0 &= x^2-2x \\ 0 &= x(x-2) \end{aligned}$$

$$x = 0 \quad \text{or} \quad x = 2$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 2 (concluded):

Lastly, compute the integral. Note that on [0, 2], $f(x)$ is the upper graph.

$$\begin{aligned} \int_0^2 \left[(2x+1)-(x^2+1)\right] dx &= \int_0^2 (2x-x^2)\ dx \\ &= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= \left(2^2 - \frac{2^3}{3}\right) - \left(0^2 + \frac{0^3}{3}\right) \\ &= 4 - \frac{8}{3} - 0 + 0 = \frac{4}{3} \end{aligned}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 3: A clever college student develops an engine that is believed to meet all state standards for emission control. The new engine's rate of emission is given by

$$E(t) = 2t^2,$$

where $E(t)$ is the emissions, in billions of pollution particulates per year, at time t, in years. The emission rate of a conventional engine is given by

$$C(t) = 9 + t^2.$$

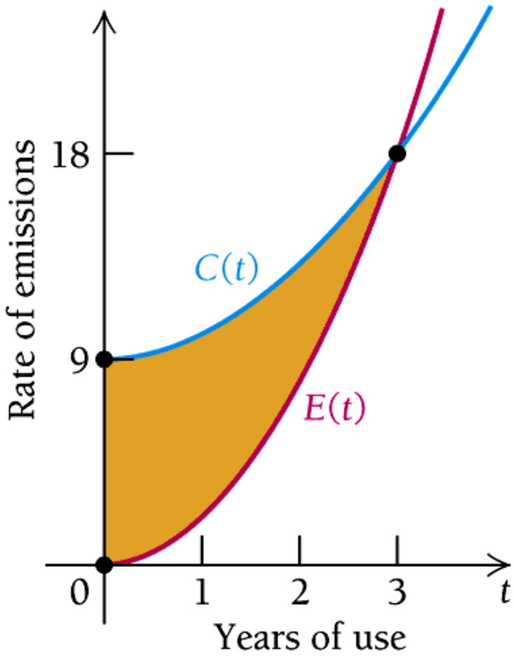
4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 3 (continued):

The graphs of both curves are shown at the right.

a) At what point in time will the emission rates be the same?

b) What is the reduction in emissions resulting from using the student's engine?



4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 3 (continued):

a) The emission rates will be the same when $E(t) = C(t)$.

$$\begin{aligned} 2t^2 &= 9+t^2 \\ t^2-9 &= 0 \\ (t-3)(t+3) &= 0 \\ t=3 \quad \text{or} \quad t &= -3 \end{aligned}$$

Since negative time has no meaning in this problem, the emission rates will be the same when $t = 3$ yr.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 3 (concluded):

b) The reduction in emissions is represented by the area between the graphs of $E(t)$ and $C(t)$ from $t = 0$ to $t = 3$.

$$\int_0^3 \left[(9+t^2-2t^2\right]\, dt = \int_0^3 (9-t^2)\, dt = \left[9t-\frac{t^3}{3}\right]_0^3$$

$$= \left(9\cdot 3-\frac{9^3}{3}\right)-\left(9\cdot 0-\frac{0^3}{3}\right)$$

$$= 27-9 = 18$$

$= 18$ billion pollution particulates per year.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

DEFINITION:

Let f be a continuous function over a closed interval $[a, b]$. Its **average value, y_{av}**, over $[a, b]$ is given by

$$y_{\text{av}} = \frac{1}{b-a}\int_a^b f(x)\ dx.$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

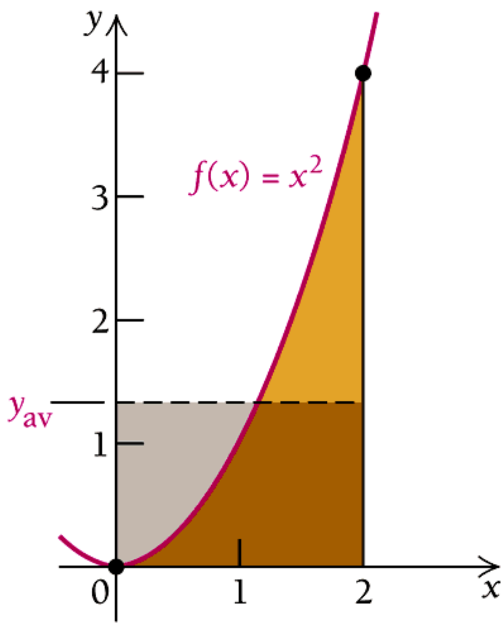
Example 4: Find the average value of $f(x) = x^2$ over the interval $[0, 2]$.

$$\frac{1}{2-0}\int_0^2 x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{2}\left(\frac{2^3}{3} - \frac{0^3}{3}\right)$$

$$= \frac{1}{2}\cdot\frac{8}{3}$$

$$= \frac{4}{3} = 1\frac{1}{3}$$



4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 5: Rico's speed, in miles per hour, t minutes after entering the freeway is given by

$$v(t) = -\frac{1}{200}t^3 + \frac{3}{20}t^2 - \frac{3}{8}t + 60, \quad t \le 30.$$

From 5 min after entering the freeway to 25 min, what was Rico's average speed? How far did he travel over that time interval?

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 5 (continued):

The average speed is

$$\frac{1}{25-5}\int_5^{25}\left(-\frac{1}{200}t^3 + \frac{3}{20}t^2 - \frac{3}{8}t + 60\right)dt$$

$$= \frac{1}{20}\left[-\frac{1}{800}t^4 + \frac{1}{20}t^3 - \frac{3}{16}t^2 + 60t\right]_5^{25}$$

$$= \frac{1}{20}\left(\frac{53{,}625}{32} - \frac{9625}{32}\right)$$

$$= 68\frac{3}{4}\text{ mph.}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 5 (concluded):

To find how far Rico traveled over the time interval [5, 25], we first note that t is given in minutes, not hours. Since 25 min – 5 min = 20 min is 1/3 hr, the distance traveled over [5, 25] is

$$\frac{1}{3}\cdot 68\frac{3}{4} = 22\frac{11}{12}\text{ mi.}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Quick Check 3

The temperature, in degrees Fahrenheit, in Minneapolis on a winter's day is modeled by the function

$$f(x) = -0.012x^3 + 0.38x^2 - 1.99x - 10.1,$$

where x is the number of hours from midnight $(0 \le x \le 24)$. Find the average temperature in Minneapolis during this 24-hour period.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Quick Check 3 Concluded

The average temperature is

$$= \frac{1}{24-0}\int_0^{24}\left(-0.012x^3 + 0.38x^2 - 1.99x - 10.1\right)dx$$

$$= \frac{1}{24}\left[-\frac{0.012}{4}x^4 + \frac{0.38}{3}x^3 - \frac{1.99}{2}x^2 - 10.1x\right]_0^{24}$$

$$= \frac{1}{24}\left(-\frac{0.012}{4}(24)^4 + \frac{0.38}{3}(24)^3 - \frac{1.99}{2}(24)^2 - 10.1(24)\right)$$

$$= \frac{1}{24}\left(-995.328 + 1751.04 - 573.12 - 242.4\right)$$

$$\approx -2.5$$

So the average temperature is approximately -2.5° F.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Definition

Let f be a continuous function over $[a, b]$. The **moving average function** of f is given by

$$f_{av}(x) = \frac{1}{L}\int_x^{x+L} f(t)\,dt,$$

where L is a constant representing a fixed interval of time.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6: Business: Moving Average. The weekly revenue, $R(x)$, in thousands of dollars, of Trux Rentals x weeks since the start of the year is given by

$$R(x) = 0.05x^4 - 1.2x^3 + 9.6x^2 - 28.6x + 38.3,$$

Where $0 \le x \le 10$. Find the moving average over 3-week intervals.

Solution: We have $L = 3$. To find the moving average for the first 3 weeks, we set $x = 0$:

$$R_{av}(x) = \frac{1}{L}\int_x^{x+L} R(t)\,dt = R_{av}(0) = \frac{1}{3}\int_0^{0+3} R(t)\,dt = \frac{1}{3}\int_0^3 R(t)\,dt$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6 continued:

$$R_{av}(0) = \frac{1}{3}\int_0^3 (0.05t^4 - 1.2t^3 + 9.6t^2 - 28.6t + 38.3)dt$$

$$= \frac{1}{3}\Big[0.01t^5 - 0.3t^4 + 3.2t^3 - 14.3t^2 + 38.3t\Big]_0^3$$

$$\approx \frac{1}{3}(50.73) = 16.91.$$

Thus, over the first 3 weeks of the year, Trux Rentals had an average revenue of $16,910 per week.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6 continued:

The moving average over the second 3 weeks—that is, between $x = 1$ and $x = 4$ is:

$$R_{av}(1) = R_{av}(1) = \frac{1}{3}\int_1^{1+3} R(t)\,dt = \frac{1}{3}\int_1^4 R(t)\,dt$$

$$= \frac{1}{3}\int_1^4 (0.05t^4 - 1.2t^3 + 9.6t^2 - 28.6t + 38.3)dt$$

$$= \frac{1}{3}\left[0.01t^5 - 0.3t^4 + 3.2t^3 - 14.3t^2 + 38.3t\right]_1^4$$

$$\approx \frac{1}{3}(37.73) = 11.91.$$

Thus, over these weeks, Trux Rentals had an average revenue of \$16,910 per week.

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6 continued:

The moving average over successive 3 week intervals can be found in a similar fashion.

Weeks 2 – 5: $R_{av}(2) = \frac{1}{3}\int_2^5 R(t)\,dt \approx 12.41$ or \$12,410/week

Weeks 3 – 6: $R_{av}(3) = \frac{1}{3}\int_3^6 R(t)\,dt \approx 14.81$ or \$14,810/week

Weeks 4 – 7: $R_{av}(4) = \frac{1}{3}\int_4^7 R(t)\,dt \approx 16.71$ or \$16,710/week

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6 continued:

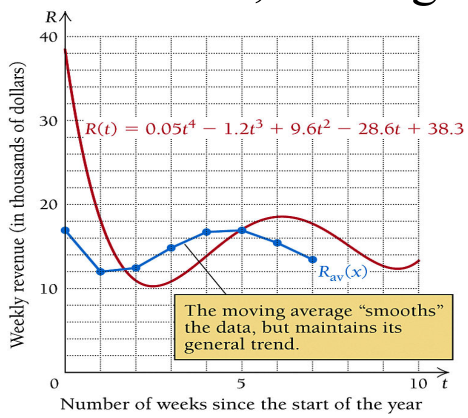
$$\text{Weeks } 5-8\text{: } R_{av}(5) = \frac{1}{3}\int_5^8 R(t)\,dt \approx 16.91 \quad \text{or } \$16{,}910/\text{week}$$

$$\text{Weeks } 6-9\text{: } R_{av}(6) = \frac{1}{3}\int_6^9 R(t)\,dt \approx 15.41 \quad \text{or } \$15{,}410/\text{week}$$

$$\text{Weeks } 7-10\text{: } R_{av}(7) = \frac{1}{3}\int_7^{10} R(t)\,dt \approx 13.41 \quad \text{or } \$13{,}410/\text{week}$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Example 6 concluded: The graph below displays *R*, along with a graph of the moving average, R_{av}, over these seven 3-week periods of time. The graph of the moving average retains the general shape of the graph of *R* but reduces the fluctuations, making the general trend easier to see.



4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Section Summary

- The *additive property of definite integrals* states that a definite integral can be expressed as the sum of two (or more) other definite integrals. If *f* is continuous on $[a,c]$ and we chose b such that $a < b < c$, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

- The area of a region bounded by the graphs of two functions, $f(x)$ and $g(x)$, where $f(x) \geq g(x)$ over an interval $[a,b]$, is

$$A = \int_a^b \left[f(x) - g(x)\right]dx.$$

4.4 Properties of Definite Integrals: Additive Property, Average Value and Moving Averages

Section Summary Concluded

- The *average value* of a continuous function *f* over an interval $[a,b]$ is

$$y_{AV} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

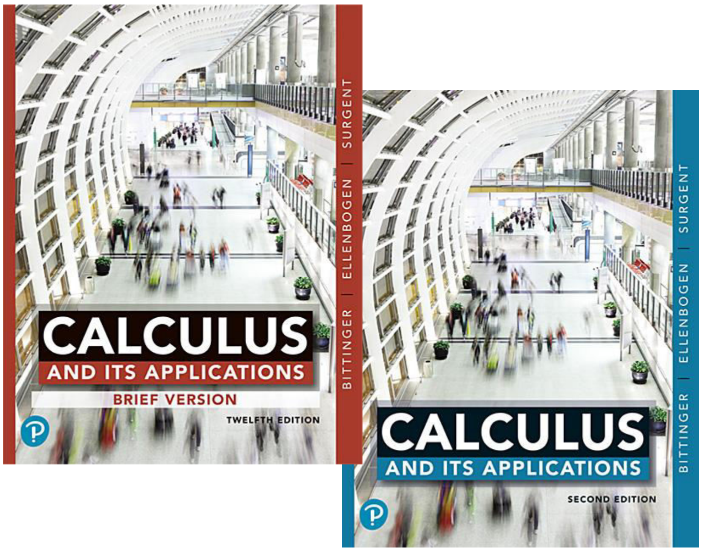
- Let *f* be a continuous function over [*a*, *b*]. The **moving average function** of *f* is given by

$$f_{av}(x) = \frac{1}{L}\int_x^{x+L} f(t)\,dt,$$

where L is a constant representing a fixed interval of time.

Chapter 5

Applications of Integration



5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

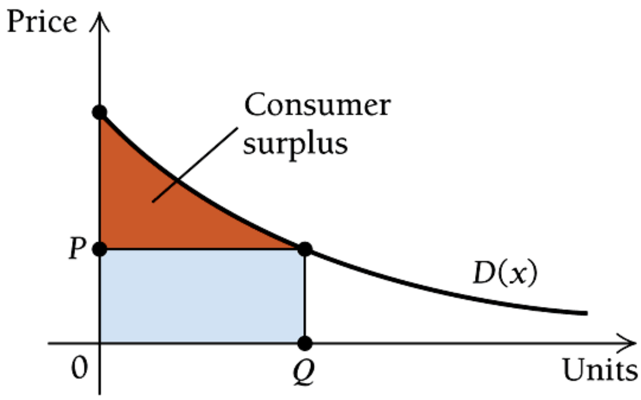
OBJECTIVE

- Given demand and supply functions, find the consumer surplus and the producer surplus at the equilibrium point.
- Find the deadweight loss for consumers and producers when a price ceiling or price floor is imposed.

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

DEFINITION:

Suppose that $p = D(x)$ describes the demand function for a product. Then, the **consumer surplus for** Q **units** of the product, at price P per unit, is $\int_0^Q D(x)dx - QP$.



5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 1: Find the consumer surplus for the demand function given by $D(x) = (x-5)^2$ when $x = 3$.

When $x = 3$, we have $D(3) = (3-5)^2 = 4$. Then,

$$\text{Consumer Surplus} = \int_0^{x_E} D(x)dx - Q \cdot P$$

$$= \int_0^3 (x-5)^2 dx - 3 \cdot 4$$

$$= \int_0^3 (x^2 - 10x + 25)dx - 12$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 1 (concluded):

$$= \int_0^3 (x^2 - 10x + 25)dx - 12$$

$$= \left[\frac{x^3}{3} - 5x^2 + 25x \right]_0^3 - 12$$

$$= \left(\frac{3^3}{3} - 5 \cdot 3^2 + 25 \cdot 3 \right) - \left(\frac{0^3}{3} - 5 \cdot 0^2 + 25 \cdot 0 \right) - 12$$

$$= (9 - 45 + 75) - 0 - 12$$

$$= \$27.00$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Quick Check 1

Find the consumer surplus for the demand function given by $D(x) = x^2 - 6x + 16$ when $x = 1$.

Finding $D(x)$ when $x = 1$, we get:

$$D(1) = 1^2 - 6(1) + 16 = 1 - 6 + 16 = 11.$$

Then, consumer surplus is:

$$CP = \int_0^Q D(x)dx - QP.$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Quick Check 1 Concluded

$$CP = \int_0^1 \left(x^2 - 6x + 16\right) dx - 1\cdot 11$$

$$CP = \frac{1}{3}x^3 - 3x^2 + 16x\Big|_0^1 - 11$$

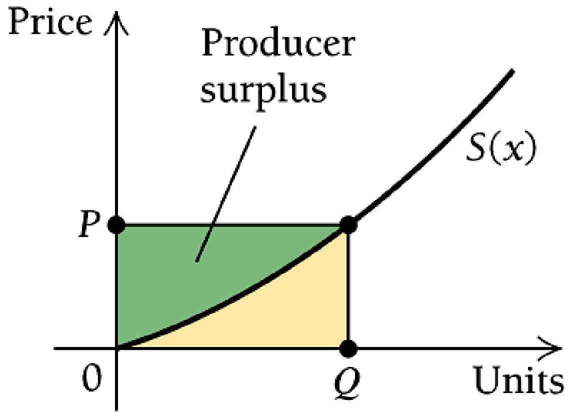
$$CP = \left(\frac{1}{3}(1)^3 - 3(1)^2 + 16(1)\right) - \left(\frac{1}{3}(0)^2 - 3(0)^2 + 16(0)\right) - 11$$

$$CP = \left(\frac{1}{3} - 3 + 16\right) - 11 = 2\frac{1}{3} \text{ or } \$2.33$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

DEFINITION:

Suppose that $p = S(x)$ is the supply function for a product. Then, the **producer surplus** Q units of the product, at price P per unit, is $QP - \int_0^Q S(x)\ dx.$



5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 2: Find the producer surplus for $S(x)=x^2+x+3$ when $x=3$.

When $x=3$, $S(3)=3^2+3+3=15$. Then,

$$\text{Producer Surplus} = Q\cdot P-\int_0^{x_E} S(x)dx$$

$$= 3\cdot 15-\int_0^3 (x^2+x+3)dx$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 2 (continued):

$$=3\cdot 15-\int_0^3 (x^2+x+3)dx$$

$$=45-\left[\frac{x^3}{3}+\frac{x^2}{2}+3x\right]_0^3$$

$$=45-\left(\frac{3^3}{3}+\frac{3^2}{2}+3\cdot 3\right)-\left(\frac{0^3}{3}+\frac{0^2}{2}+3\cdot 0\right)$$

$$=45-\left(9+\frac{9}{2}+9\right)-0$$

$$=\$22.50$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Quick Check 2

Find the producer surplus for $S(x)=\frac{1}{3}x^2+\frac{4}{3}x+4$ when $x=1$.

When $x=1$, $S(1)=\frac{1}{3}(1)^2+\frac{4}{3}(1)+4=5\frac{2}{3}$. Then,

$$\text{Producer Surplus} = QP-\int_0^Q S(x)\,dx.$$

$$= 1\cdot 5\frac{2}{3}-\int_0^1\left(\frac{1}{3}x^2+\frac{4}{3}x+4\right)dx$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Quick Check 2 Concluded

$$= 1\cdot 5\frac{2}{3}-\int_0^1\left(\frac{1}{3}x^2+\frac{4}{3}x+4\right)dx$$

$$=5\frac{2}{3}-\left(\frac{1}{9}x^3+\frac{2}{3}x^2+4x\Big|_0^1\right)$$

$$=5\frac{2}{3}-\left(\frac{1}{9}(1)^3+\frac{2}{3}(1)^2+4(1)\right)-\left(\frac{1}{9}(0)^3+\frac{2}{3}(0)^2+4(0)\right)$$

$$=5\frac{2}{3}-\left(\frac{1}{9}+\frac{2}{3}+4\right)=5\frac{2}{3}-4\frac{7}{9}=\frac{8}{9}$$

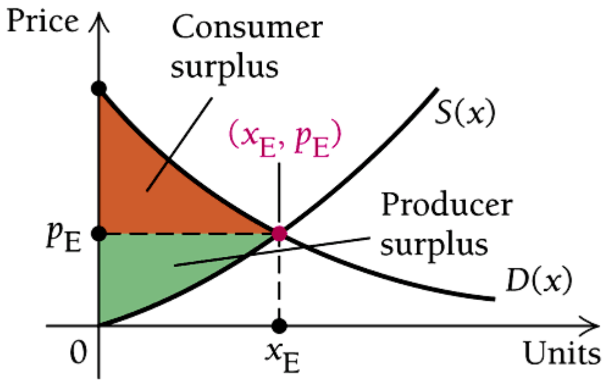
$$=\$0.89$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

DEFINITION:

The equilibrium point, (x_E, p_E), is the point at which the supply and demand curves intersect.

It is that point at which sellers and buyers come together and purchases and sales actually occur.



5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 3: Given

$D(x) = (x-5)^2$ and $S(x) = x^2 + x + 3$,

find each of the following:

a.) The equilibrium point.

b.) The consumer surplus at the equilibrium point.

c.) The producer surplus at the equilibrium point.

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 3 (continued):

a) To find the equilibrium point, set $D(x) = S(x)$ and solve.

$$\begin{aligned}
(x-5)^2 &= x^2 + x + 3 \\
x^2 - 10x + 25 &= x^2 + x + 3 \\
-10x + 25 &= x + 3 \\
22 &= 11x \\
2 &= x
\end{aligned}$$

Thus, $x_E = 2$. To find p_E, substitute x_E into either $D(x)$ or $S(x)$ and solve.

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 3 (continued):

If we choose $D(x)$, we have

$$\begin{aligned}
p_E = D(x_E) &= D(2) \\
&= (2-5)^2 \\
&= (-3)^2 \\
&= \$9
\end{aligned}$$

Thus, the equilibrium point is (2, \$9).

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 3 (continued):

b) The consumer surplus at the equilibrium point is

$$\int_0^2 (x-5)^2\,dx - 2\cdot 9 = \left[\frac{(x-5)^3}{3}\right]_0^2 - 18$$

$$= \frac{(2-5)^3}{3} - \frac{(0-5)^3}{3} - 18$$

$$= -\frac{27}{3} + \frac{125}{3} - 18 = \frac{44}{3}$$

$$\approx \$14.67$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 3 (concluded):

c) The producer surplus at the equilibrium point is

$$2\cdot 9 - \int_0^2 (x^2 + x + 3)\,dx = 18 - \left[\frac{x^3}{3} + \frac{x^2}{2} + 3x\right]_0^2$$

$$= 18 - \left(\frac{(2)^3}{3} + \frac{(2)^2}{2} + 3\cdot 2\right) - \left(\frac{(0)^3}{3} + \frac{(0)^2}{2} + 3\cdot 0\right)$$

$$= 18 - \left(\frac{8}{3} + \frac{4}{3} + 6\right) - 0 = \frac{22}{3}$$

$$\approx \$7.33$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Quick Check 3

Given $D(x) = x^2 - 6x + 16$ and $S(x) = \frac{1}{3}x^2 + \frac{4}{3}x + 4$, find each of the following. Assume $x \leq 5$.

a.) The equilibrium point

b.) The consumer surplus at the equilibrium point

c.) The producer surplus at the equilibrium point

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Quick Check 3 Continued

a.) To find the equilibrium point, set $D(x) = S(x)$ and solve.

$$x^2 - 6x + 16 = \frac{1}{3}x^2 + \frac{4}{3}x + 4$$

$$\frac{2}{3}x^2 - 7\frac{1}{3}x + 12 = 0$$

$$2x^2 - 22x + 36 = 0$$

$$(2x - 4)(x - 9) = 0$$

$$x = 2 \text{ or } x = 9$$

Since we assume that $x \leq 5$, we know that $x_E = 2$. To find p_E, substitute x_E into either $D(x)$ or $S(x)$ and solve.

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Quick Check 3 Continued

If we choose $D(x)$, we have

$$p_E = D(x) = D(2)$$
$$= (2)^2 - 6(2) + 16$$
$$= 4 - 12 + 16$$
$$= 8$$

So the equilibrium point is (2, \$8).

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Quick Check 3 Continued

b.) The consumer surplus at the equilibrium point is $\int_0^{x_E} D(x)\,dx - x_E \cdot p_E$.

$$\int_0^2 \left(x^2 - 6x + 16\right) dx - 2 \cdot 8$$
$$= \frac{1}{3}x^3 - 3x^2 + 16x \Big|_0^2 - 16$$
$$= \left(\frac{1}{3}(2)^3 - 3(2)^2 + 16(2)\right) - \left(\frac{1}{3}(0)^3 - 3(0)^2 + 16(0)\right) - 16$$
$$= \left(\frac{8}{3} - 12 + 32\right) - 16 = 22\frac{2}{3} - 16 = \$6.67$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Quick Check 3 Concluded

c.) The producer surplus at the equilibrium point is $x_E p_E - \int_0^{x_E} S(x)\,dx.$

$$2\cdot 8-\int_0^2 \frac{1}{3}x^2+\frac{4}{3}x+4dx$$

$$=16-\left(\frac{1}{9}x^3+\frac{2}{3}x^2+4x\Big|_0^2\right)$$

$$=16-\left(\frac{1}{9}(2)^3+\frac{2}{3}(2)^2+4(2)\right)-\left(\frac{1}{9}(0)^3+\frac{2}{3}(0)^2+4(0)\right)$$

$$=16-\left(\frac{8}{9}+\frac{8}{3}+8\right)=16-11\frac{5}{9} \quad =\$4.44$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

DEFINITION:

A **price ceiling** is a price p_C such that $p_C < p_E$, and p_C is the maximum price for which the product may be sold.

A **price floor** is a price p_F such that, $p_F > p_E$ and p_F is the minimum price for which the product may be sold.

The loss in surplus is called the **deadweight loss**.

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 4: **Business: Price Ceiling.** In a small isolated mountain town, demand for propane is given by $D(x) = 6.50 - 0.25x$ and supply is given by $S(x) = 2.10 + 0.15x$, where x is in gallons per month per customer and $D(x)$ and $S(x)$ are dollars per gallon.

a) Find the equilibrium point, (x_E, p_E).

b) Find the consumer surplus and the producer surplus at the equilibrium point.

c) Assume a price ceiling of \$3 per gallon of propane is imposed. Find the point (x_C, p_C).

d) Find the new producer surplus and the new consumer surplus at (x_C, p_C).

e) Find the deadweight loss.

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 4 continued:

Solution:

a) To find the equilibrium point (x_E, p_E), we set $D(x) = S(x)$:

$$6.50 - 0.25x = 2.10 + 0.15x$$
$$4.40 = 0.40x$$
$$11 = x$$

Thus,

$D(11) = 6.50 - 0.25(11) = S(11) = 2.10 + 0.15(11) = 3.75$

So, the equilibrium point is $(11, 3.75)$.

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 4 continued:

b) The consumer surplus is:

$$\int_0^{11}(6.50-0.25x)dx-11(3.75)$$

$$=6.50x-0.125x^2\Big|_0^{11}-41.25$$

$$=6.50(11)-0.125(11)^2-0-41.25$$

$$=56.375-41.25=15.125$$

$$\approx \$15.13$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 4 continued:

b) The producer surplus is:

$$11(3.75)-\int_0^{11}(2.10+0.15x)dx$$

$$=41.25-[2.10x+0.075x^2]_0^{11}$$

$$=41.25-[2.10(11)-0.075(11)^2-0]$$

$$=41.25-32.175=9.075$$

$$\approx \$9.08$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 4 continued:

c) To find where the price ceiling $y = p_C = 3$ intersects the supply curve, we set $S(x) = 3$ and solve:

$$2.10 + 0.15x = 3$$

$$0.15x = 0.90$$

$$x = 6 = x_C$$

Thus, $(x_C, p_C) = (6,3)$.

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 4 continued:

d) With the price ceiling in effect, the new producer surplus is found over the interval [0, 6]:

$$6(3) - \int_0^6 (2.10 + 0.15x)\, dx$$

$$= 18 - [2.10x + 0.075x^2]_0^6$$

$$= 18 - [2.10(6) + 0.075(6)^2 - 0]$$

$$= 18 - 15.30$$

$$= \$2.70$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 4 continued:

d) Continued

The new consumer surplus is:

$$\int_0^6 (6.50 - 0.25x)dx - 6(3)$$

$$= 6.50x + 0.125x^2 \Big|_0^6 - 18$$

$$= 6.50(6) + 0.125(6)^2 - 0 - 18$$

$$= \$16.50$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Example 4 concluded:

e) The deadweight loss is the area bounded above by $D(x)$ and below by $S(x)$ over the interval [6, 11]:

$$\int_6^{11} [(6.50 - 0.25x) - (2.10 + 0.15x)]dx$$

$$= \int_6^{11} (4.40 - 0.40x)dx = 4.40x - 0.20x^2 \Big|_6^{11}$$

$$= 4.40(11) - 0.20(11)^2 - (4.40(6) - 0.20(6)^2)$$

$$= \$5$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Section Summary

- *A demand curve* is the graph of a function $p = D(x)$, which represents the unit price p a consumer is willing to pay for x items. It is usually a decreasing function.
- *A supply curve* is the graph of a function $p = S(x)$, which represents the unit price p a producer is willing to accept for x items. It is usually an increasing function.

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Section Summary Continued

- *Consumer surplus* at point (Q, P) is defined as

$$\int_0^Q D(x)\,dx - QP.$$

- *Producer surplus* at point (Q, P) is defined as

$$QP - \int_0^Q S(x)\,dx.$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Section Summary Continued

• The *equilibrium point* (x_E, p_E) is the point at which the supply and demand curves intersect. The consumer surplus at the equilibrium point is

$$\int_0^{x_E} D(x)\,dx - x_E p_E.$$

The producer surplus at the equilibrium point is

$$x_E p_E - \int_0^{x_E} S(x)\,dx.$$

5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Section Summary Continued

- A *price ceiling*, p_C, is the maximum price for which a product can be sold. It is always less than the equilibrium price, p_E. That is, $p_C < p_E$.
  - The price ceiling p_C intersects the supply curve at (x_C, p_C).
  - The new producer surplus is
    $$(x_C)(p_C) - \int_0^{x_C} S(x)\,dx.$$
  - The new consumer surplus is
    $$\int_0^{x_C} D(x)\,dx - (x_C)(p_C).$$
  - The *deadweight loss* is
    $$\int_{x_C}^{x_E} (D(x) - S(x))\,dx.$$

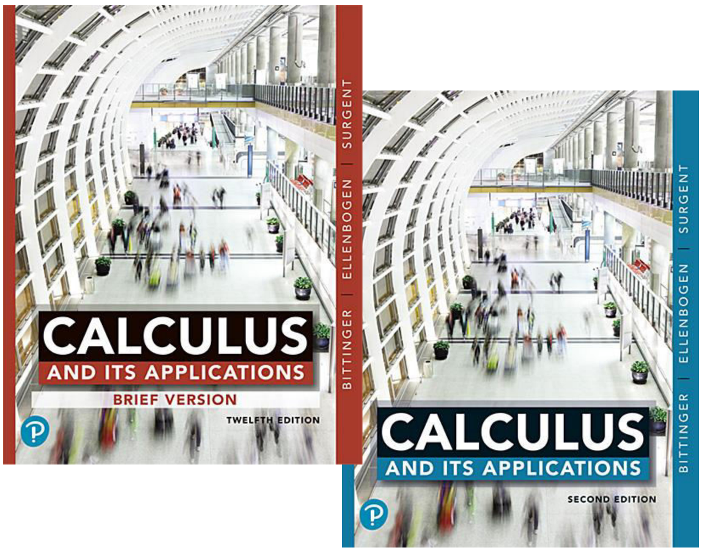
5.1 Consumer and Producer Surplus; Price Floors, Price Ceilings and Deadweight Loss

Section Summary Concluded

- A *price floor*, p_F, is the minimum price for which a product can be sold. It is always greater than the equilibrium price, p_E. That is, $p_F > p_E$.
  - The price floor p_F intersects the demand curve at (x_F, p_F).
  - The new producer surplus is
  $$(x_F)(p_F) - \int_0^{x_F} S(x)dx.$$
  - The new consumer surplus is
  $$\int_0^{x_F} D(x)dx - (x_F)(p_F).$$
  - The *deadweight loss* is
  $$\int_{x_F}^{x_E} (D(x) - S(x))dx.$$

Chapter 6

Functions of Several Variables



6.1 Functions of Several Variables

OBJECTIVE

- Find a function value for a function of several variables.

6.1 Functions of Several Variables

DEFINITION:

A **function of two variables** assigns to each input pair, (x, y), exactly one output number, $f(x, y)$.

6.1 Functions of Several Variables

Example 1: For the profit function, $P(x, y) = 4x + 6y$, find $P(25, 10)$.

$$\begin{aligned} P(25,\ 10) &= 4 \cdot 25 + 6 \cdot 10 \\ &= 100 + 60 \\ &= \$160 \end{aligned}$$

6.1 Functions of Several Variables

Quick Check 1

A company's cost function is given by $C(x, y) = 6.5x + 7.25y$.

Find $C(10,15)$.

$$C(10,15) = 6.5(10) + 7.25(15)$$
$$C(10,15) = 65 + 108.75$$
$$C(10,15) = \$173.75$$

6.1 Functions of Several Variables

Example 2: The total cost to a company, in thousands of dollars, is given by

$$C(x, y, z, w) = 4x^2 + 5y + z - \ln(w+1),$$

where x dollars is spent for labor, y dollars for raw materials, z dollars for advertising, and w dollars for machinery. Find $C(3, 2, 0, 10)$.

$$\begin{aligned} C(3, 2, 0, 10) &= 4 \cdot 3^2 + 5 \cdot 2 + 0 - \ln(10+1) \\ &\approx 36 + 10 - 2.397895 \\ &\approx \$43.6 \text{ thousand, or } \$43{,}600 \end{aligned}$$

6.1 Functions of Several Variables

Example 3: A business purchases a piece of storage equipment that costs C_1 dollars and has capacity V_1. Later it wishes to replace the original with a new piece of equipment that costs C_2 dollars and has capacity V_2. Industrial economists have found that in such cases, the cost of the new piece of equipment can be estimated by the function of three variables

$$C_2 = \left(\frac{V_2}{V_1}\right)^{0.6} C_1.$$

6.1 Functions of Several Variables

Example 3 (concluded):

For $45,000, a beverage company buys a manufacturing tank that has a capacity of 10,000 gallons. Later it decides to buy a tank with double the capacity of the original. Estimate the cost of the new tank.

$$\begin{aligned} C_2 &= \left(\frac{20,000}{10,000}\right)^{0.6}(45,000) \\ &= 2^{0.6}(45,000) \\ &\approx \$68,207.25 \end{aligned}$$

6.1 Functions of Several Variables

Quick Check 2

a.) Repeat Example 3 assuming that the company buys a tank with a capacity of 2.75 times that of the original.

b.) What is the percentage increase in cost for this tank compared to the cost of the original tank?

a.) Using the cost formula from Example 3, we get the following.

$$C_2 = \left(\frac{V_2}{V_1}\right)^{0.6} C_1$$

$$C_2 = \left(\frac{27,500}{10,000}\right)^{0.6} (\$45,000) \approx \$82,568.07$$

6.1 Functions of Several Variables

Quick Check 2 Concluded

b.) What is the percentage increase in cost for this tank compared to the cost of the original tank?

The formula for finding a percentage increase is $p = \frac{C_2}{C_1} - 1,$ where p is the percentage increase, C_2 is the new cost, and C_1 is the old cost. So,

$$p = \frac{82{,}568.07}{45{,}000} - 1 \approx 0.835 = 83.5\%.$$

6.1 Functions of Several Variables

Example 4: As the populations of two cities grow, the number of telephone calls between the cities increases, much like the gravitational pull will increase between two growing objects in space. The average number of telephone calls per day between two cities is given by

$$N(d, P_1, P_2) = \frac{2.8P_1P_2}{d^{2.4}},$$

where d is the distance, in miles, between the cities and P_1 and P_2 are their populations.

6.1 Functions of Several Variables

Example 4 (concluded):
The cities of Dallas and Ft. Worth are 30 mi apart and have populations of 1,213,825 and 624,067, respectively. Find the average number of calls per day between the two cities.

$$N(30,\ 1{,}213{,}825,\ 624{,}067) = \frac{2.8(1{,}213{,}825)(624{,}067)}{30^{2.4}}$$

$$\approx 604{,}580{,}752$$

6.1 Functions of Several Variables

Quick Check 3

Find the average number of calls per day between Phoenix, Arizona (population 1,552,300) and Tucson, Arizona (population 541,800), given that the distance between the two cities is 120 mi.

Using the formula from Example 4, $N(d, P_1, P_2) = \dfrac{2.8P_1P_2}{d^{2.4}}$, we find that

$$N(120{,}1552300{,}541800) = \frac{2.8(1552300)(541800)}{(120)^{2.4}}$$

$$\approx 24{,}095{,}597 \text{ calls per day.}$$

6.1 Functions of Several Variables

Example 5: For $f(x, y) = -3$, find $f(5, 7)$ and $f(-2, 0)$.

Since f is a constant function, it has the value of –3 for any values of x and y. Thus,

$$f(5,\ 7) = -3 \quad \text{and} \quad f(-2,\ 0) = -3$$

6.1 Functions of Several Variables

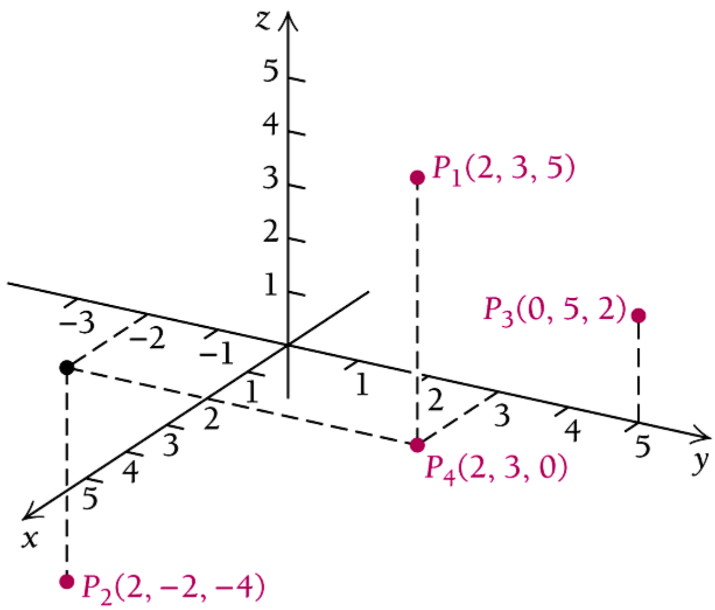
Example 6: Plot these points:

$P_1(2,\ 3,\ 5)$,

$P_2(2,\ -2,\ -4)$,

$P_3(0,\ 5,\ 2)$, and

$P_4(2,\ 3,\ 0)$.



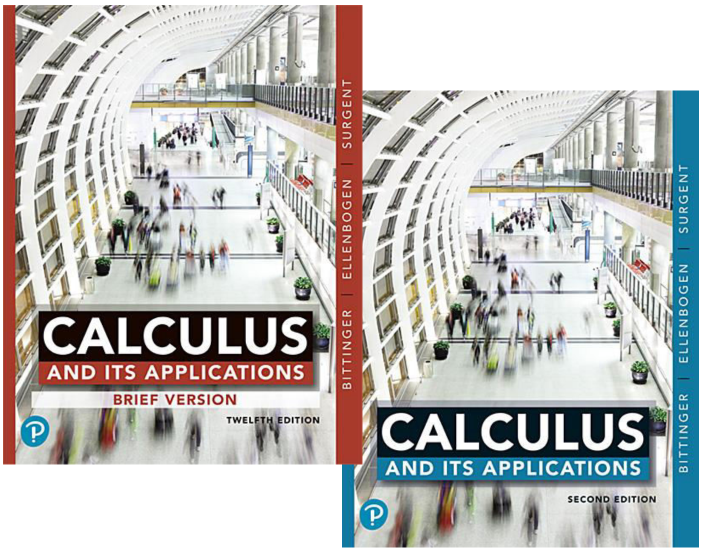
6.1 Functions of Several Variables

Section Summary

- A *function of two variables* assigns to each input pair, (x, y), exactly one output number, $f(x, y)$.
- A function of two variables generates points (x, y, z), where $z = f(x, y)$.
- The graph of a function of two variables is a *surface* and requires a three-dimensional coordinate system.
- The *domain* of a function of two variables is the set of points in the xy-plane for which the function is defined.

Chapter 6

Functions of Several Variables



6.2 Partial Derivatives

OBJECTIVE

- Find the partial derivatives of a given function.
- Evaluate partial derivatives.
- Use differentials of a multivariable function to estimate the change in the function.
- Find the four second-order partial derivatives of a function in two variables.

6.2 Partial Derivatives

DEFINITION:

For $z = f(x, y)$, the partial derivatives with respect to x and y are

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

$$\text{and} \quad \frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

6.2 Partial Derivatives

Example 1: For $w = x^2 - xy + y^2 + 2yz + 2z^2 + z,$

find $\dfrac{\partial w}{\partial x}, \quad \dfrac{\partial w}{\partial y}, \quad$ and $\quad \dfrac{\partial w}{\partial z}.$

In order to find $\partial w/\partial x$, we regard *x* as the variable and *y* and *z* as constants.

$$\frac{\partial w}{\partial x} = 2x - y$$

6.2 Partial Derivatives

Example 1 (concluded):
Similarly, from $w = x^2 - xy + y^2 + 2yz + 2z^2 + z,$
we get

$$\frac{\partial w}{\partial y} = -x + 2y + 2z$$

and $$\frac{\partial w}{\partial z} = 2y + 4z + 1$$

6.2 Partial Derivatives

Quick Check 1

For $u = x^2y^3z^4$, find $\frac{du}{dx}$, $\frac{du}{dy}$, and $\frac{du}{dz}$.

$$\frac{du}{dx} = 2xy^3z^4$$

$$\frac{du}{dy} = 3x^2y^2z^4$$

$$\frac{du}{dz} = 4x^2y^3z^3$$

6.2 Partial Derivatives

Example 2:

For $f(x, y) = 3x^2y + xy^2$, find f_x and f_y.

$f_x = 6xy + y^2$ Treating y^2 and y as constants

$f_y = 3x^2 + 2xy$ Treating x^2 and x as constants

6.2 Partial Derivatives

Quick Check 2

For $f(x,y)=7x^3y^2-\dfrac{x}{y}$, find f_x and f_y.

$$f_x=21x^2y^2-\frac{1}{y}$$ Treating y^2 and y as constants.

$$f_y=14x^3y+\frac{x}{y^2}$$ Treating x^3 and x as constants.

6.2 Partial Derivatives

Example 3:

For $f(x,y)=e^{xy}+y\ln x$, find f_x and f_y.

$$\begin{aligned} f_x &= y\cdot e^{xy}+y\cdot\frac{1}{x} \\ &= ye^{xy}+\frac{y}{x} \end{aligned}$$

$$\begin{aligned} f_y &= x\cdot e^{xy}+1\cdot\ln x \\ &= xe^{xy}+\ln x \end{aligned}$$

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6.2 Partial Derivatives

Example 4: A cellular phone company has the following production function for a certain product:

$$p(x, y) = 50x^{2/3}y^{1/3},$$

where p is the number of units produced with x units of labor and y units of capital.

a) Find the number of units produced with 125 units of labor and 64 units of capital.

b) Find the marginal productivities.

c) Evaluate the marginal productivities at $x = 125$ and $y = 64$.

6.2 Partial Derivatives

Example 4 (continued):

$$\text{a) } p(125, 64) = 50(125)^{2/3}(64)^{1/3} = 50(25)(4) = 5000 \text{ units}$$

b) Marginal Productivity of Labor

$$= \frac{\partial p}{\partial x} = p_x$$

$$= 50 \cdot \frac{2}{3} x^{-1/3} y^{1/3} = \frac{100y^{1/3}}{3x^{1/3}}$$

6.2 Partial Derivatives

Example 4 (continued):

Marginal Productivity of Capital

$$= \frac{\partial p}{\partial y} = p_y$$

$$= 50 \cdot \frac{1}{3} x^{2/3} y^{-2/3} = \frac{50x^{2/3}}{3y^{2/3}}$$

6.2 Partial Derivatives

Example 4 (continued):

c) Marginal Productivity of Labor

$$= p_x(125, 64)$$

$$= \frac{100(64)^{1/3}}{3(125)^{1/3}}$$

$$= \frac{100 \cdot 4}{3 \cdot 5}$$

$$= 26\frac{2}{3}$$

6.2 Partial Derivatives

Example 4 (concluded):

Marginal Productivity of Capital

$$= p_y(125, 64)$$

$$= \frac{50(125)^{2/3}}{3(64)^{2/3}}$$

$$= \frac{50 \cdot 25}{3 \cdot 16}$$

$$= 26\frac{1}{24}$$

6.2 Partial Derivatives

Example 5: Tolerances. A circular swimming pool has a radius of 12 ft and a depth of 4 ft. Both measurements have a tolerance of ±3 in. $\left(\pm 1/4\ ft\right)$.

a) Find the volume of this pool assuming a radius of 12 ft and a depth of 4 ft.

b) Use differentials to estimate the change in volume, assuming that both measurements include the extra 3 in.

c) Compare the result in part (b) to the actual change in volume, assuming that both measurements include the extra 3 in.

6.2 Partial Derivatives

Example 5 Continued:

Solution: The formula for the volume of a right circular cylinder of radius *r* and depth *h* is

$$V(r,h) = \pi r^2 h.$$

a) We have

$$V(12,4) = \pi(12)^2(4)$$

$$= 576\pi \quad \text{or about } 1809.6 \text{ ft}^3.$$

6.2 Partial Derivatives

Example 5 Continued:

b) To approximate the change in *V*, we first determine

$$\frac{\partial V}{\partial r} \text{ and } \frac{\partial V}{\partial h}.$$

$$\frac{\partial V}{\partial r} = 2\pi rh$$

$$\frac{\partial V}{\partial h} = \pi r^2$$

6.2 Partial Derivatives

Example 5 Continued:

We then have

$$\Delta V \approx V_r(r,h)dr + V_h(r,h)dh$$

$$= 2\pi rh \cdot dr + \pi r^2 \cdot dh \qquad \text{Substituting}$$

Thus, for $r = 12$, $h = 4$, and $\Delta r = \Delta h = \frac{1}{4}$, we have

$$\Delta V \approx 2\pi(12)(4) \cdot \frac{1}{4} + \pi(12)^2 \cdot \frac{1}{4} \qquad \text{Substituting}$$

$$= 60\pi, \text{ or about } 188.5 \text{ ft}^3.$$

6.2 Partial Derivatives

Example 5 Concluded:

c) The actual change in V is found by evaluating

$$V\left(r+\Delta r, h+\Delta h\right) - V(r,h).$$

Thus, we have

$$\Delta V = V\left(r+\Delta r, h+\Delta h\right) - V(r,h)$$

$$= V\left(12+\frac{1}{4}, 4+\frac{1}{4}\right) - V(12,4) \qquad \text{Substituting}$$

$$= \pi(12.25)^2(4.25) - \pi(12)^2(4)$$

$$\approx 194 \text{ ft}^3.$$

6.2 Partial Derivatives

DEFINITION:

Second-Order Partial Derivatives

1. $$\frac{\partial^2 z}{\partial x \partial x} = \frac{\partial^2 f}{\partial x \partial x} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

Take the partial with respect to x, and then with respect to x again.

2. $$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

Take the partial with respect to x, and then with respect to y.

6.2 Partial Derivatives

DEFINITION (concluded):

3. $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

Take the partial with respect to y, and then with respect to x.

4. $$\frac{\partial^2 z}{\partial y \partial y} = \frac{\partial^2 f}{\partial y \partial y} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Take the partial with respect to y, and then with respect to y again.

6.2 Partial Derivatives

Example 6: For $z = f(x,y) = x^2y^3 + x^4y + xe^y$, find the four second-order partial derivatives.

$$\text{a) } \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}(2xy^3 + 4x^3y + e^y) = 2y^3 + 12x^2y$$

$$\text{b) } \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = \frac{\partial}{\partial y}(2xy^3 + 4x^3y + e^y) = 6xy^2 + 4x^3 + e^y$$

6.2 Partial Derivatives

Example 6 (concluded):

$$z = f(x,y) = x^2y^3 + x^4y + xe^y$$

$$\text{c) } \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 + x^4 + xe^y) = 6xy^2 + 4x^3 + e^y$$

$$\text{d) } \frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 + x^4 + xe^y) = 6x^2y + xe^y$$

6.2 Partial Derivatives

Quick Check 3

For $z = g(x, y) = 6x^2 + 3xy^4 - y^2$, find the four second-order partial derivatives.

$$\frac{\partial^2 z}{\partial x^2} = f_{xx} = \frac{\partial z}{\partial x} 12x + 3y^4 = 12$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{xy} = \frac{\partial z}{\partial y} 12x + 3y^4 = 12y^3$$

$$\frac{\partial^2 z}{\partial x \partial y} = f_{yx} = \frac{\partial z}{\partial x} 12xy^3 - 2y = 12y^3$$

$$\frac{\partial^2 z}{\partial y^2} = f_{yy} = \frac{\partial z}{\partial y} 12xy^3 - 2y = 36xy^2 - 2$$

6.2 Partial Derivatives

Section Summary

- For $z = f(x, y)$, the *partial derivatives with respect to x and y* are, respectively:

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \text{ and}$$

$$\frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

- Simpler notations for partial derivatives are $f_x(x, y)$ or f_x,

$f_y(x, y)$ or $f_y \frac{\partial y}{\partial x}$, z_x for $\frac{\partial z}{\partial x}$ and z_y for $\frac{\partial z}{\partial y}$.

6.2 Partial Derivatives

Section Summary Concluded

- For a surface $z = f(x, y)$ and a point (x_0, y_0, z_0) on this surface, the partial derivative of f with respect to x gives the slope of the tangent line at (x_0, y_0, z_0) in the positive x-direction. Similarly, the partial derivative of f with respect to y gives the slope of the tangent line at (x_0, y_0, z_0) in the positive y-direction.
- For $z = f(x, y)$, the second-order partial derivatives are

$$f_x = \frac{\partial^2 f}{\partial x^2}, f_{xy} = \frac{\partial^2 f}{\partial y \partial x}, f_{yx} = \frac{\partial^2 f}{\partial x \partial y}, \text{ and } f_{yy} = \frac{\partial^2 f}{\partial y^2}.$$

Often (but not always), $f_{xy} = f_{yx}$.