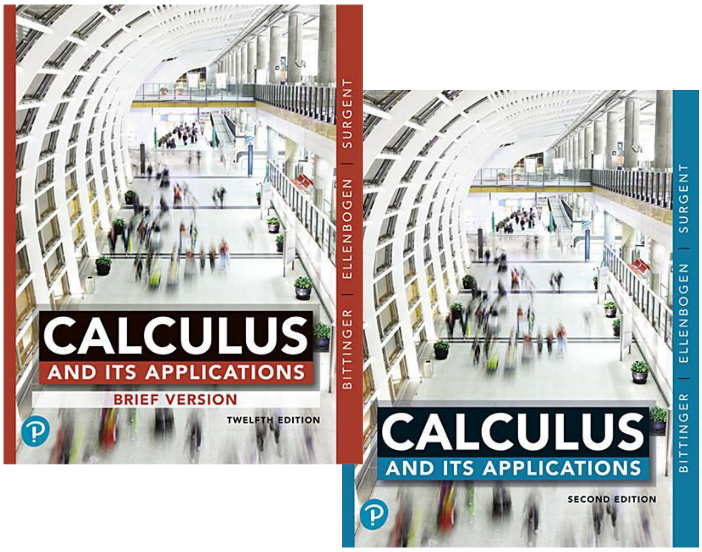
Chapter 1

Differentiation



1.3 Average Rates of Change

OBJECTIVE

- Compute an average rate of change.
- Find a simplified difference quotient.

1.3 Average Rates of Change

DEFINITION:

As x approaches a, the **limit** of $f(x)$ is L, written

$$\lim_{x \to a} f(x) = L,$$

if all values of $f(x)$ are close to L for values of x that are sufficiently close, but not equal to, a.

1.3 Average Rates of Change

DEFINITION:

The **average rate of change of y with respect to x**, as x changes from x_1 to x_2, is the ratio of the change in output to the change in input:

$$\frac{y_2 - y_1}{x_2 - x_1}, \qquad \text{where } x_2 \neq x_1.$$

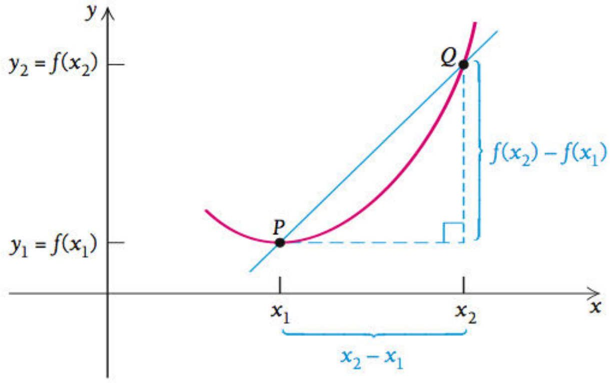
1.3 Average Rates of Change

DEFINITION (concluded):

If we look at a graph of the function, we see that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

which is both the average rate of change *and* the slope of the line from P(x_1, y_1) to Q(x_2, y_2).



The line through P and Q, $\overleftrightarrow{PQ}$, is called a **secant line**.

1.3 Average Rates of Change

Example 1: For $y = f(x) = x^2$ find the average rate of change as:

a) x changes from 1 to 3.

b) x changes from 1 to 2.

c) x changes from 2 to 3.

a) When $x_1 = 1$, $\quad y = f(x_1) = f(1) = 1^2 = 1.$

When $x_2 = 3$, $\quad y = f(x_2) = f(3) = 3^2 = 9.$

Thus, the average rate of change is

$$\frac{9-1}{3-1} = \frac{8}{2} = 4.$$

1.3 Average Rates of Change

Example 1 (concluded):

b) When $x_1 = 1$, $y = f(x_1) = f(1) = 1^2 = 1.$

When $x_2 = 2$, $y = f(x_1) = f(2) = 2^2 = 4.$

Thus, the average rate of change is

$$\frac{4-1}{2-1} = \frac{3}{1} = 3.$$

c) When $x_1 = 2$, $y = f(x_1) = f(2) = 2^2 = 4.$

When $x_2 = 3$, $y = f(x_1) = f(3) = 3^2 = 9.$

Thus, the average rate of change is

$$\frac{9-4}{3-2} = \frac{5}{1} = 5.$$

1.3 Average Rates of Change

Quick Check 1

State the average rate of change for each situation in a short sentence. Be sure to include units.

a.) It rained 4 inches over a period of 8 hours.

The average rate of change is $\frac{4\text{ in} - 0\text{ in}}{8\text{ hr} - 0\text{ hr}} = \frac{4\text{ in}}{8\text{ in}} = \frac{1\text{ in}}{2\text{ hr}}.$

The average rate of rain fall was 0.5 inches of rain every hour.

b.) Your car travels 250 miles on 20 gallons of gas.

The average rate of change is $\frac{250\text{ mi} - 0\text{ mi}}{20\text{ gal} - 0\text{ gal}} = \frac{250\text{ mi}}{20\text{ gal}} = \frac{25\text{ mi}}{2\text{ gal}}.$

The average miles traveled on a gallon of gas was 12.5 miles every gallon.

c.) At 2 p.m., the temperature was 82 degrees. At 5 p.m., the temperature was 76 degrees.

The average rate of change is $\frac{82 - 76\text{ degrees}}{5\text{ p.m.} - 2\text{ p.m.}} = \frac{-6\text{ degrees}}{3\text{ hours}} = -\frac{2\text{ degrees}}{1\text{ hour}}.$

The average change in temperature was -2 degrees every hour.

1.3 Average Rates of Change

Quick Check 2

For $f(x) = x^3$, find the average rate of change between:

a.) $x = 1$ and $x = 4$;

b.) $x = 1$ and $x = 2$;

c.) $x = 1$ and $x = 1.2$.

a.) When $x_1 = 1$, $y_1 = f(x_1) = f(1) = 1^3 = 1$.

When $x_2 = 4$, $y_2 = f(x_2) = f(4) = 4^3 = 64$.

Thus the rate of change is $\dfrac{64-1}{4-1} = \dfrac{63}{3} = 21$.

1.3 Average Rates of Change

Quick Check 2 Continued

b.) When $x_1 = 1$, $y_1 = f(x_1) = f(1) = 1^3 = 1$.

When $x_2 = 2$, $y_2 = f(x_2) = f(2) = 2^3 = 8$.

Thus the average rate of change is $\dfrac{8-1}{2-1} = 7$.

c.) When $x_1 = 1$, $y_1 = f(x_1) = f(1) = 1^3 = 1$.

When $x_2 = 1.2$, $y_2 = f(x_2) = f(1.2) = 1.2^3 = 1.728$.

Thus the average rate of change is

$$\frac{1.728-1}{1.2-1} = 3.64.$$

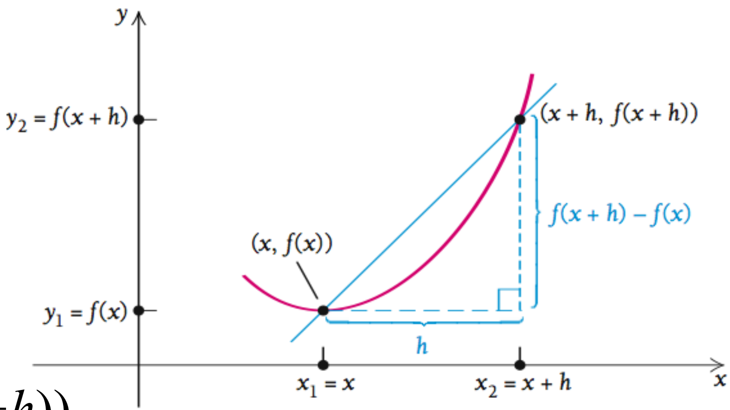
1.3 Average Rates of Change

DEFINITION:

The average rate of change of f with respect to x is also called the **difference quotient**. It is given by

$$\frac{f(x+h)-f(x)}{h}, \quad \text{where } h \neq 0.$$

The difference quotient is equal to the slope of the secant line from $(x, f(x))$ to $(x+h, f(x+h))$.



1.3 Average Rates of Change

Example 2: For $f(x) = x^2$ find the difference quotient when:

a) $x = 5$ and $h = 3$.

b) $x = 5$ and $h = 0.1$.

a) We substitute $x = 5$ and $h = 3$ into the formula:

$$\frac{f(x+h)-f(x)}{h} = \frac{f(5+3)-f(5)}{3} = \frac{f(8)-f(5)}{3}$$

$$= \frac{8^2-5^2}{3} = \frac{64-25}{3} = \frac{39}{3} = 13$$

1.3 Average Rates of Change

Example 2 (concluded):

b) We substitute $x = 5$ and $h = 0.1$ into the formula:

$$\frac{f(x+h)-f(x)}{h} = \frac{f(5+0.1)-f(5)}{0.1} = \frac{f(5.1)-f(5)}{0.1}$$

$$= \frac{5.1^2-5^2}{0.1} = \frac{26.01-25}{0.1} = \frac{1.01}{0.1} = 10.1.$$

1.3 Average Rates of Change

Example 3: For $f(x)=x^3$ find a simplified form of the difference quotient.

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^3-x^3}{h}$$

$$= \frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$$

$$= \frac{\cancel{h}\left(3x^2+3xh+h^2\right)}{\cancel{h}}$$

$$= 3x^2+3xh+h^2, \quad h \neq 0.$$

1.3 Average Rates of Change

Quick Check 3

Use the result of Example 3 to calculate the slope of the secant line (average rate of change) at $x = 2$, for $h = 0.1$, $h = 0.01$, and $h = 0.001$.

Use the formula found in Example 6 ($3x^2 + 3xh + h^2, h \neq 0$).

For $h = 0.1$: $3(2)^2 + 3(2)(0.1) + 0.1^2 = 12 + 0.6 + 0.01 = 12.61$

For $h = 0.01$: $3(2)^2 + 3(2)(0.01) + 0.01^2 = 12 + 0.06 + 0.0001 = 12.0601$

For $h = 0.001$: $3(2)^2 + 3(2)(0.001) + 0.001^2 = 12 + 0.006 + 0.000001$

$$= 12.006001$$

1.3 Average Rates of Change

Example 4: For $f(x) = \dfrac{3}{x}$ find a simplified form of the difference quotient.

$$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{3}{x+h} - \dfrac{3}{x}}{h} = \frac{\dfrac{3x - 3(x+h)}{x(x+h)}}{h}$$

$$= \frac{\dfrac{\cancel{3x} - \cancel{3x} - 3h}{x(x+h)}}{h} = \frac{\dfrac{-3\cancel{h}}{x(x+h)}}{\cancel{h}}$$

$$= \frac{-3}{x(x+h)}, \quad h \neq 0.$$

1.3 Average Rates of Change

Section Summary

- An *average rate of change* is the slope of a line between two points.

If the points are (x_1, y_1) and (x_2, y_2), then the average rate of change is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

- If the two points are solutions to a single function, an equivalent form of the slope formula is $\dfrac{f(x+h)-f(x)}{h}$, where h is the horizontal difference between the two x-values. This is called the *difference quotient*. The line connecting these two points is called a *secant line.*

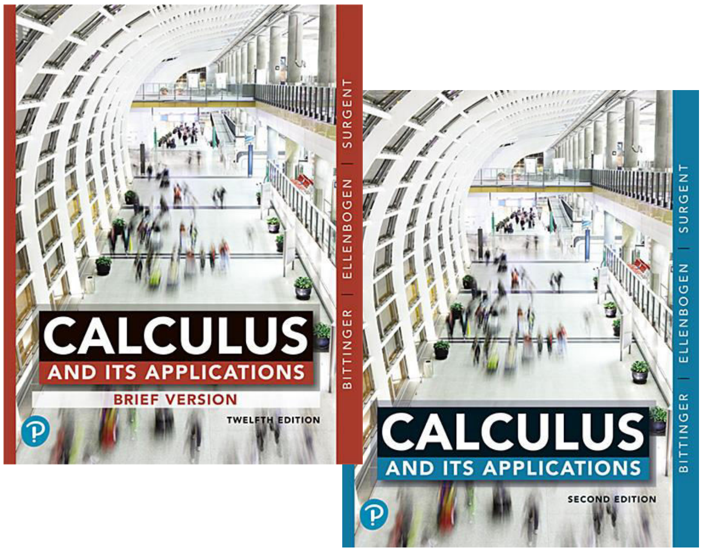
1.3 Average Rates of Change

Section Summary Continued

- The difference quotient is the same as the slope formula. Both give the slope of the line between two points.
- The difference quotient gives the *average rate of change* between two points on a graph, represented by the secant line.
- It is preferable to simplify a difference quotient algebraically before evaluating it for particular values of x and h.

Chapter 1

Differentiation



1.4 Differentiation Using Limits of Difference Quotients

OBJECTIVE

- Find derivatives and values of derivatives
- Find equations of tangent lines

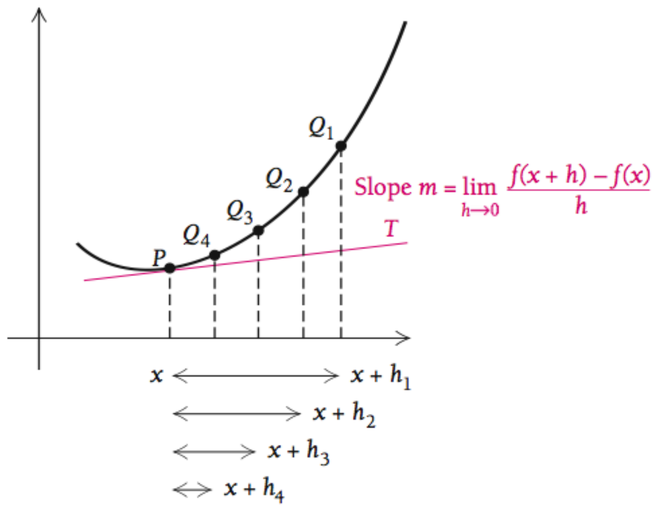
1.4 Differentiation Using Limits of Difference Quotients

DEFINITION:

The **slope of the tangent line** at $(x, f(x))$ is

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This limit is also the **instantaneous rate of change** of $f(x)$ at x.



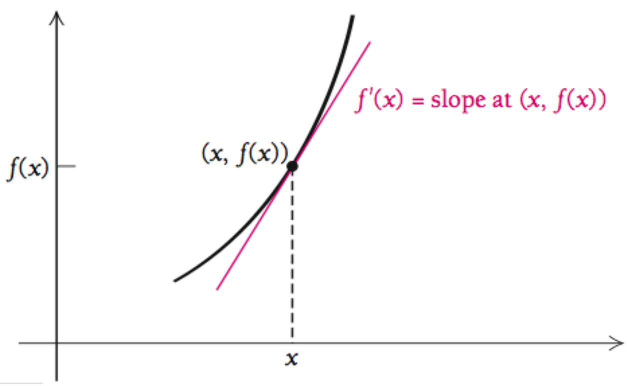
1.4 Differentiation Using Limits of Difference Quotients

DEFINITION:

For a function $y = f(x)$, its **derivative** at x is the function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.
If $f'(x)$ exists, then we say that f is **differentiable** at x.



1.4 Differentiation Using Limits of Difference Quotients

Example 1: For $f(x) = x^2$, find $f'(x)$. Then find $f'(-3)$ and $f'(4)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$f'(x) = \lim_{h \to 0} 2x + h$$

$$f'(x) = 2x$$

1.4 Differentiation Using Limits of Difference Quotients

Example 1 (concluded):

$$f'(x) = 2x$$

$$f'(-3) = 2(-3) = -6$$

$$f'(4) = 2(4) = 8$$

1.4 Differentiation Using Limits of Difference Quotients

Example 2: For $f(x)=x^3$, find $f'(x)$.
Then find $f'(-1)$ and $f'(1.5)$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\cancel{h}\left(3x^2+3xh+h^2\right)}{\cancel{h}}=\lim_{h\to 0}3x^2+3xh+h^2$$

$$f'(x)=3x^2$$

1.4 Differentiation Using Limits of Difference Quotients

Example 2 (concluded):

$$f'(x)=3x^2$$

$$f'(-1)=3(-1)^2=3(1)=3$$

$$f'(x)=3(1.5)^2=3(2.25)=6.75$$

1.4 Differentiation Using Limits of Difference Quotients

Quick Check 1

Use the results from Examples 1 and 2 to find the derivative $f(x)=x^3+x^2$ and then calculate $f'(-2)$ and $f'(4)$. Interpret these results.

From Example 1, we know that the derivative of x^2 is 2x , and from Example 2, we know that the derivative of x^3 is $3x^2$. Using the Limit Property L3, we then know that $f'(x)=3x^2+2x$.

1.4 Differentiation Using Limits of Difference Quotients

Quick Check 1 Concluded

Now, we plug in $x=-2$ into our new derivative formula:

$$f'(-2)=3(-2)^2+2(-2)\ =12-4\ =8$$

Next, we plug in $x=4$ into our new derivative formula:

$$f'(4)=3(4)^2+2(4)=48+8\ =56$$

These results mean that when $x=-2$, the slope of the tangent line is 8, and when $x=4$, the slope of the tangent line is 56.

1.4 Differentiation Using Limits of Difference Quotients

Example 3: For $f(x)=\dfrac{3}{x}$:

a) Find $f'(x)$.

b) Find $f'(2)$.

c) Find an equation of the tangent line to the curve at $x = 2$.

1.4 Differentiation Using Limits of Difference Quotients

Example 3 (continued):

a) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{3}{x+h}-\dfrac{3}{x}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{3x-3(x+h)}{x(x+h)}}{h}=\lim_{h\to 0}\frac{\dfrac{\cancel{3x}-\cancel{3x}-3h}{x(x+h)}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{-3\cancel{h}}{x(x+h)}}{\cancel{h}}=\lim_{h\to 0}\frac{-3}{x(x+h)}=-\frac{3}{x^2}.$$

1.4 Differentiation Using Limits of Difference Quotients

Example 3 (continued):

b) $$f'(x) = -\frac{3}{x^2}$$

$$f'(2) = -\frac{3}{2^2} = -\frac{3}{4}$$

1.4 Differentiation Using Limits of Difference Quotients

Example 3 (concluded):

c) $x = 2,\ m = f'(2) = -\frac{3}{4},\ \ y = f(2) = \frac{3}{2}$

$$\begin{aligned} y &= mx + b \\ \frac{3}{2} &= -\frac{3}{4}\cdot 2 + b \\ \frac{3}{2} &= -\frac{3}{2} + b \\ 3 &= b \end{aligned}$$

Thus, $y = -\frac{3}{4}x + 3$

is the equation of the tangent line.

1.4 Differentiation Using Limits of Difference Quotients

Quick Check 2

Repeat Example 3a for $f(x) = -\frac{2}{x}$. What are the similarities in your method?

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{-2}{x+h}-\frac{-2}{x}}{h} = \lim_{h\to 0}\frac{\frac{-2x+2(x+h)}{x(x+h)}}{h}$$

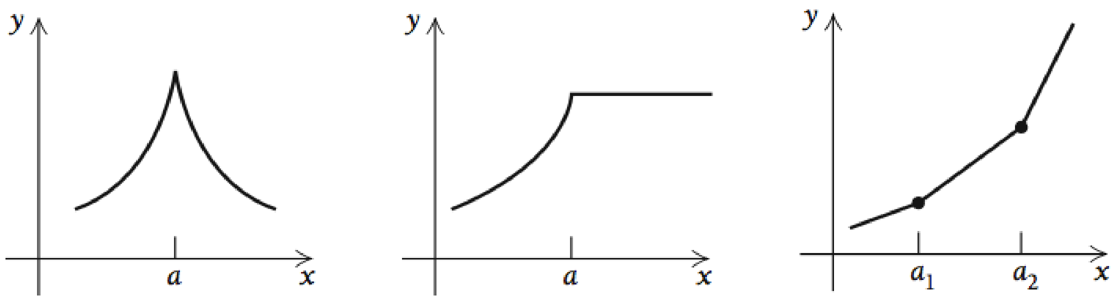
$$= \lim_{h\to 0}\frac{\frac{-2\cancel{x}+2\cancel{x}+2h}{x(x+h)}}{h} = \lim_{h\to 0}\frac{\frac{2\cancel{h}}{x(x+h)}}{\cancel{h}} = \lim_{h\to 0}\frac{2}{x(x+h)} = \frac{2}{x^2}$$

Both methods had the same basic principle. You start by using the derivative formula, then you break it down until you do not have an h anywhere in the equation.

1.4 Differentiation Using Limits of Difference Quotients

Where a Function is Not Differentiable:

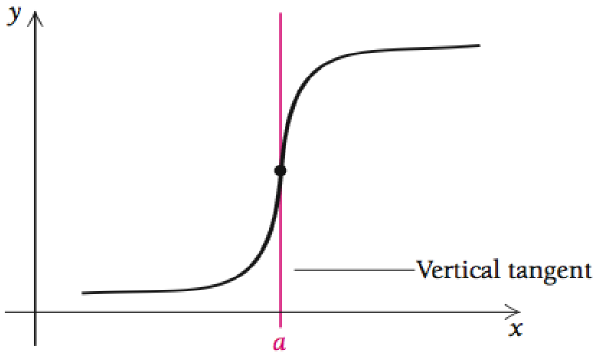
1) A function $f(x)$ is not differentiable at a point $x = a$, if there is a "corner" at a.



1.4 Differentiation Using Limits of Difference Quotients

Where a Function is Not Differentiable:

2) A function $f(x)$ is not differentiable at a point $x = a$, if there is a vertical tangent at a.

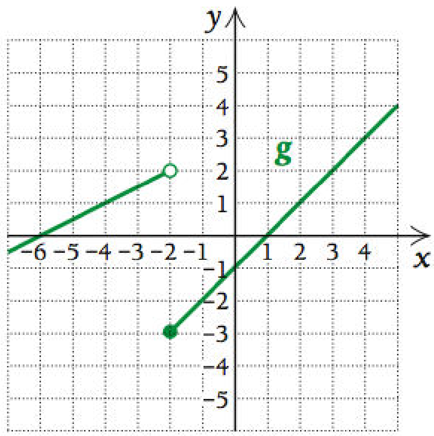


1.4 Differentiation Using Limits of Difference Quotients

Where a Function is Not Differentiable:

3) A function $f(x)$ is not differentiable at a point $x = a$, if it is not continuous at a.

Example: $g(x)$ is not continuous at $–2$, so $g(x)$ is not differentiable at $x = –2$.



1.4 Differentiation Using Limits of Difference Quotients

Quick Check 3

Where is $f(x)=|x-6|$ not differentiable? Why?

$f(x)=|x-6|$ is not differentiable at $x=6$. This is the vertex of the function, and is considered a corner of the function. Therefore $f(x)=|x-6|$ is not differentiable at $x=6$.

1.4 Differentiation Using Limits of Difference Quotients

Section Summary

- A *tangent line* is a line that touches a (smooth) curve at a single point, the *point of tangency*. See Fig. 3 (on p. 133) for examples of tangent lines (and lines that are not considered tangent lines).

- The *derivative* of a function $f(x)$ is defined by

$$f'(x)=\lim_{x\to 0}\frac{f(x+h)-f(x)}{h}.$$

1.4 Differentiation Using Limits of Difference Quotients

Section Summary Continued

- The *slope* of the tangent line to the graph of $y = f(x)$ at $x = a$ is the value of the derivative at $x = a$; that is, the slope of the tangent line at $x = a$ is $f'(a)$.
- Slopes of tangent lines are interpreted as *instantaneous rates of change.*
- The equation of a tangent line at $x = a$ is found by simplifying $y - f(a) = f'(a)(x - a)$
- If a function is differentiable at a point $x = a$, then it is *continuous* at $x = a$. That is, differentiability implies continuity.

1.4 Differentiation Using Limits of Difference Quotients

Section Summary Concluded

- However, continuity at a point $x = a$ does *not* imply differentiability at $x = a$. A good example is the absolute-value function, $f(x) = |x|$, or any function whose graph has a corner. Continuity alone is not sufficient to guarantee differentiability.
- A function is not differentiable at a point $x = a$ if:

  1) There is a discontinuity at $x = a$
  2) There is a corner at $x = a$, or
  3) There is a vertical tangent at $x = a$