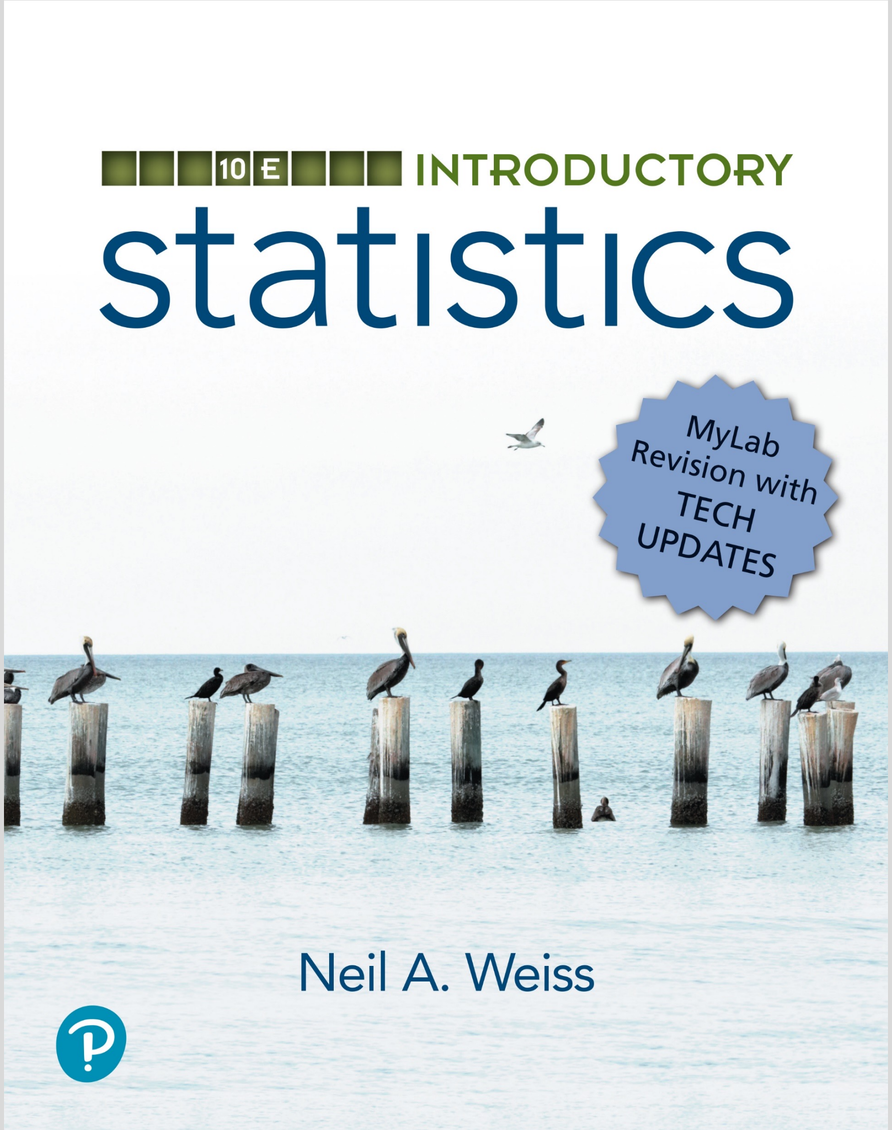
Chapter 10

Inferences for Two Population Means

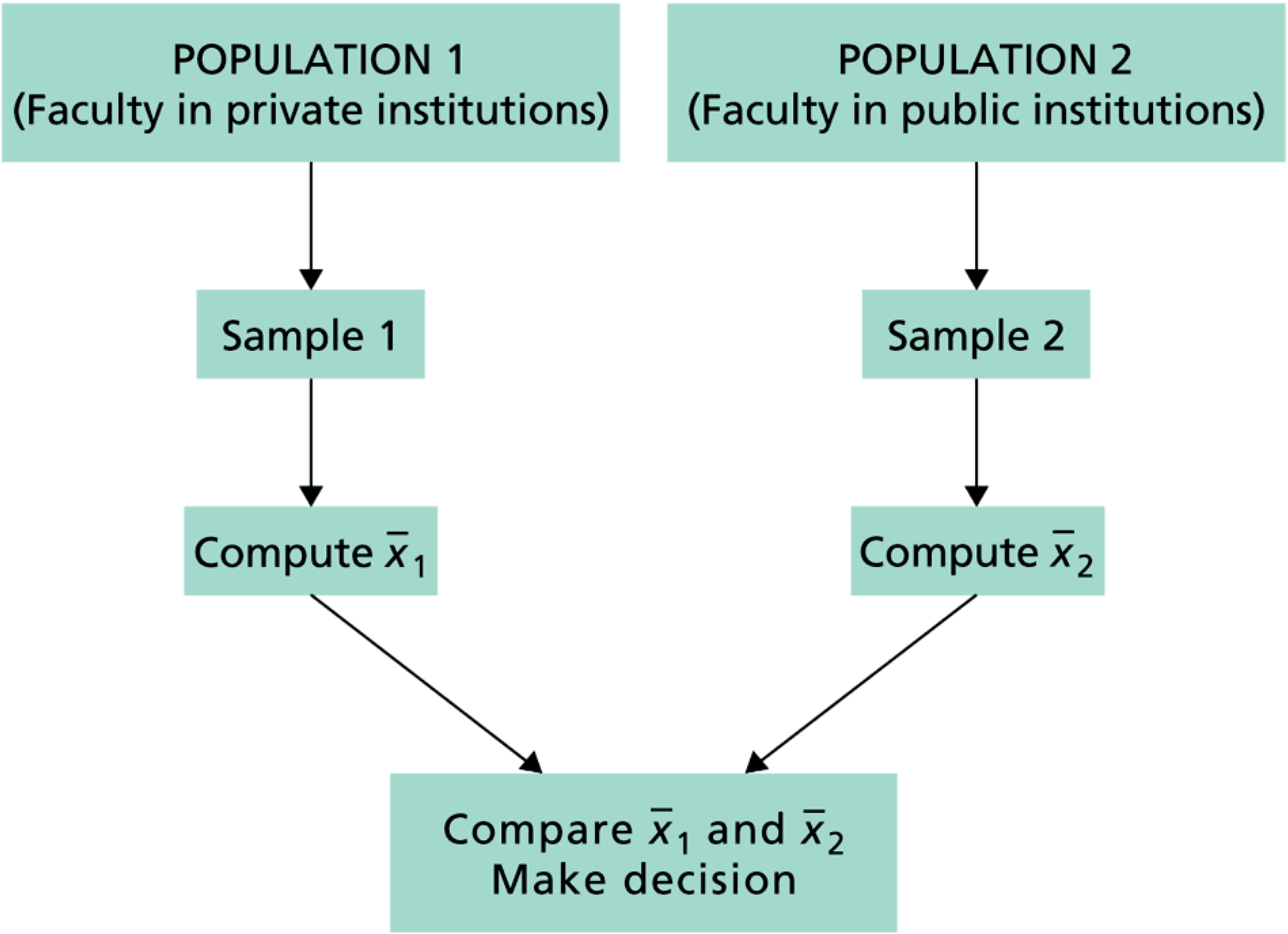


Section 10.1

The Sampling Distribution of the Difference between Two Sample Means for Independent Samples

Figure 10.1

Process for comparing two population means, using independent samples



Key Fact 10.1

The Sampling Distribution of the Difference between Two Sample Means for Independent Samples

Suppose that x is a normally distributed variable on each of two populations. Then, for independent samples of sizes n_1 and n_2 from the two populations,

- $\mu_{\bar{x}_1 - \bar{x}_2} = \mu_1 - \mu_2$,
- $\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{(\sigma_1^2/n_1) + (\sigma_2^2/n_2)}$, and
- $\bar{x}_1 - \bar{x}_2$ is normally distributed.

Section 10.2

Inferences for Two Population Means, Using Independent Samples: Standard Deviations Assumed Equal

Key Fact 10.2

Distribution of the Pooled *t*-Statistic

Suppose that x is a normally distributed variable on each of two populations and that the population standard deviations are equal. Then, for independent samples of sizes n_1 and n_2 from the two populations, the variable

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{s_p\sqrt{(1/n_1) + (1/n_2)}}$$

has the t-distribution with df $= n_1 + n_2 - 2$.

Procedure 10.1

Pooled *t*-Test

Purpose To perform a hypothesis test to compare two population means, μ_1 and μ_2

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations or large samples
4. Equal population standard deviations

Step 1 **The null hypothesis is H_0: $\mu_1 = \mu_2$, and the alternative hypothesis is**

H_a: $\mu_1 \neq \mu_2$	**or**	H_a: $\mu_1 < \mu_2$	**or**	H_a: $\mu_1 > \mu_2$
(Two tailed)		**(Left tailed)**		**(Right tailed)**

Step 2 **Decide on the significance level, α.**

Step 3 **Compute the value of the test statistic**

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{(1/n_1) + (1/n_2)}},$$

where

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}.$$

Denote the value of the test statistic t_0.

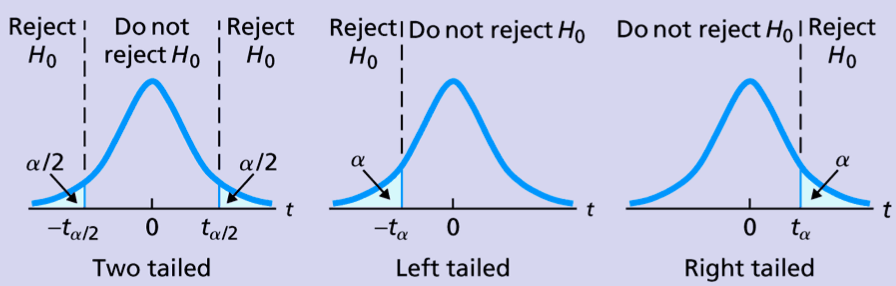
Procedure 10.1 (cont.)

CRITICAL-VALUE APPROACH	OR	*P*-VALUE APPROACH

Step 4 The critical value(s) are

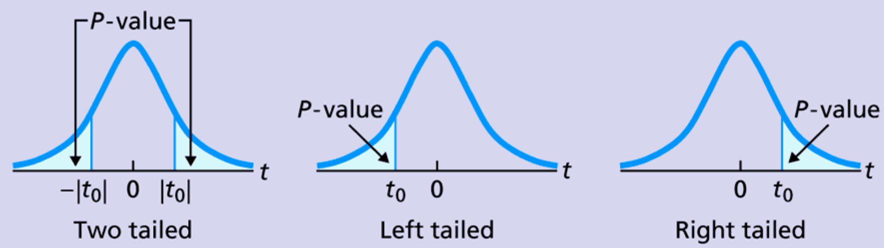
$\pm t_{\alpha/2}$ (Two tailed) or $-t_{\alpha}$ (Left tailed) or t_{α} (Right tailed)

with df $= n_1 + n_2 - 2$. Use Table IV to find the critical value(s).



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

Step 4 The t-statistic has df $= n_1 + n_2 - 2$. Use Table IV to estimate the P-value, or obtain it exactly by using technology.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Note: The hypothesis test is exact for normal populations and is approximately correct for large samples from nonnormal populations.

Procedure 10.2

Pooled t-Interval Procedure

Purpose To find a confidence interval for the difference between two population means, μ_1 and μ_2

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations or large samples
4. Equal population standard deviations

Step 1 **For a confidence level of $1 - \alpha$, use Table IV to find $t_{\alpha/2}$ with df $= n_1 + n_2 - 2$.**

Step 2 **The endpoints of the confidence interval for $\mu_1 - \mu_2$ are**

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \cdot s_p\sqrt{(1/n_1) + (1/n_2)},$$

where s_p is the pooled sample standard deviation.

Step 3 **Interpret the confidence interval.**

Note: The confidence interval is exact for normal populations and is approximately correct for large samples from nonnormal populations.

Section 10.3

Inferences for Two Population Means, Using Independent Samples: Standard Deviations Not Assumed Equal

Key Fact 10.3

Distribution of the Nonpooled t-Statistic

Suppose that x is a normally distributed variable on each of two populations. Then, for independent samples of sizes n_1 and n_2 from the two populations, the variable

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}}$$

has approximately a t-distribution. The degrees of freedom used is obtained from the sample data. It is denoted Δ and given by

$$\Delta = \frac{\left[(s_1^2/n_1) + (s_2^2/n_2)\right]^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}},$$

rounded down to the nearest integer.

Procedure 10.3

Nonpooled *t*-Test

Purpose To perform a hypothesis test to compare two population means, μ_1 and μ_2

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations or large samples

Step 1 **The null hypothesis is H_0: $\mu_1 = \mu_2$, and the alternative hypothesis is**

H_a: $\mu_1 \neq \mu_2$ (Two tailed)	or	H_a: $\mu_1 < \mu_2$ (Left tailed)	or	H_a: $\mu_1 > \mu_2$ (Right tailed)

Step 2 **Decide on the significance level, α.**

Step 3 **Compute the value of the test statistic**

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}}.$$

Denote the value of the test statistic t_0.

Procedure 10.3 (cont.)

CRITICAL-VALUE APPROACH	OR	*P*-VALUE APPROACH

CRITICAL-VALUE APPROACH

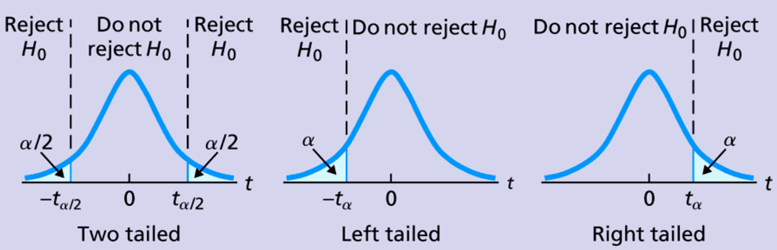
Step 4 The critical value(s) are

$$\pm t_{\alpha/2} \text{ (Two tailed)} \quad \text{or} \quad -t_{\alpha} \text{ (Left tailed)} \quad \text{or} \quad t_{\alpha} \text{ (Right tailed)}$$

with df $= \Delta$, where

$$\Delta = \frac{\left[(s_1^2/n_1) + (s_2^2/n_2)\right]^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}}$$

rounded down to the nearest integer. Use Table IV to find the critical value(s).



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

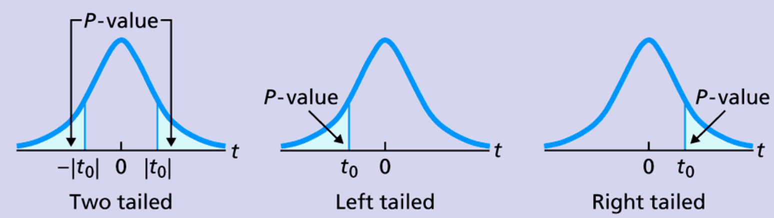
OR

***P*-VALUE APPROACH**

Step 4 The t-statistic has df $= \Delta$, where

$$\Delta = \frac{\left[(s_1^2/n_1) + (s_2^2/n_2)\right]^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}}$$

rounded down to the nearest integer. Use Table IV to estimate the P-value, or obtain it exactly by using technology.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Procedure 10.4

Nonpooled *t*-Interval Procedure

Purpose To find a confidence interval for the difference between two population means, μ_1 and μ_2

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations or large samples

Step 1 **For a confidence level of $1 - \alpha$, use Table IV to find $t_{\alpha/2}$ with df $= \Delta$, where**

$$\Delta = \frac{\left[(s_1^2/n_1) + (s_2^2/n_2)\right]^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}}$$

rounded down to the nearest integer.

Step 2 **The endpoints of the confidence interval for $\mu_1 - \mu_2$ are**

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2} \cdot \sqrt{(s_1^2/n_1) + (s_2^2/n_2)}.$$

Step 3 **Interpret the confidence interval.**

Key Fact 10.4

Choosing between a Pooled and a Nonpooled t-Procedure

Suppose you want to use independent simple random samples to compare the means of two populations. To decide between a pooled t-procedure and a nonpooled t-procedure, follow these guidelines: If you are reasonably sure that the populations have nearly equal standard deviations, use a pooled t-procedure; otherwise, use a nonpooled t-procedure.

Section 10.4

The Mann-Whitney Test

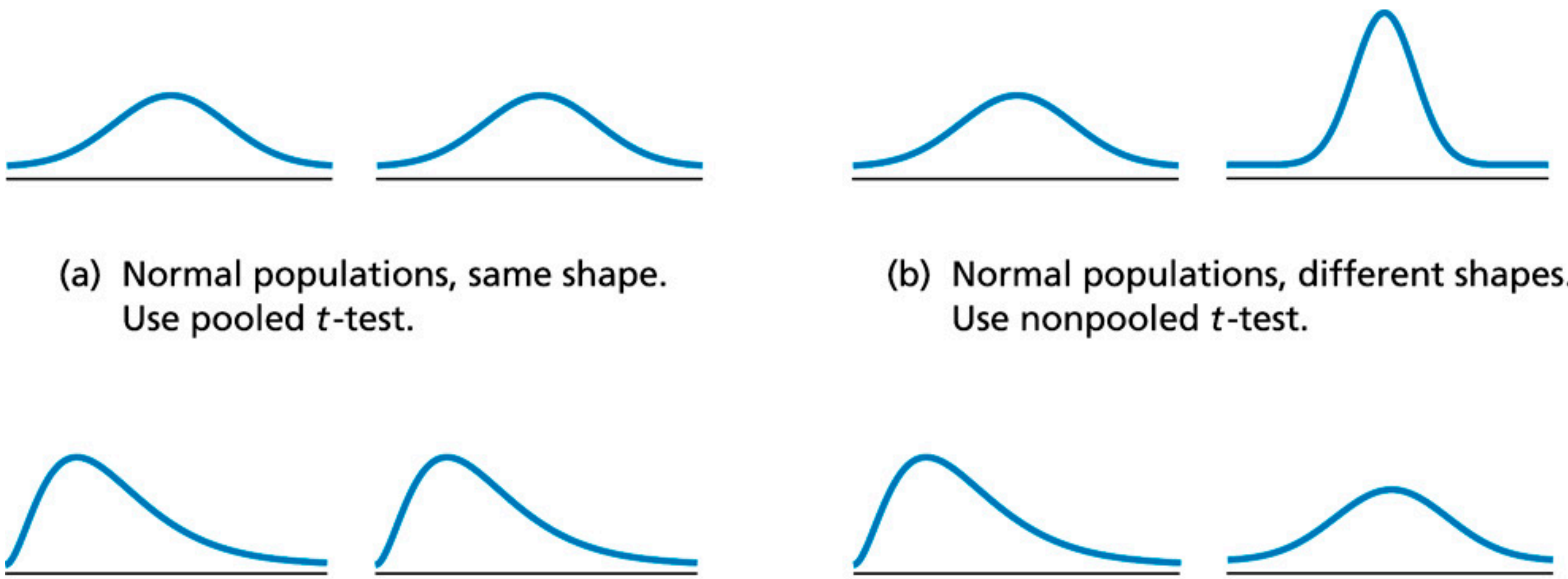
Definition 10.1

Distributions of the Same Shape

We say that two or more distributions have the same shape if they are identical except possibly for the locations of their centers.

Figure 10.9

Appropriate procedure for comparing two population means based on independent simple random samples



(a) Normal populations, same shape. Use pooled t-test.

(b) Normal populations, different shapes. Use nonpooled t-test.

(c) Nonnormal populations, same shape. Use Mann-Whitney test.

(d) Not both normal populations, different shapes. Use nonpooled t-test for large samples; otherwise, consult a statistician.

Table 10.9 & 10.10

Times, in minutes, required to learn how to use the system

Without experience	With experience
139	142
118	109
164	130
151	107
182	155
140	88
134	95
	104

Results of ranking the combined data from Table 10.9

Without experience	Overall rank	With experience	Overall rank
139	9	142	11
118	6	109	5
164	14	130	7
151	12	107	4
182	15	155	13
140	10	88	1
134	8	95	2
		104	3

Procedure 10.5

Mann–Whitney Test

Purpose To perform a hypothesis test to compare two population means, μ_1 and μ_2

Assumptions

1. Simple random samples
2. Independent samples
3. Same-shape populations

Step 1 **The null hypothesis is H_0: $\mu_1 = \mu_2$, and the alternative hypothesis is**

$$H_a\text{: } \mu_1 \neq \mu_2 \text{ (Two tailed)} \quad \text{or} \quad H_a\text{: } \mu_1 < \mu_2 \text{ (Left tailed)} \quad \text{or} \quad H_a\text{: } \mu_1 > \mu_2 \text{ (Right tailed)}$$

Step 2 **Decide on the significance level, α.**

Step 3 **Compute the value of the test statistic**

$$M = \text{sum of the ranks for sample data from Population 1}$$

and denote that value M_0. To do so, construct a work table of the following form.

Sample from Population 1	Overall rank	Sample from Population 2	Overall rank
.	.	.	.
.	.	.	.
.	.	.	.

Procedure 10.5 (cont.)

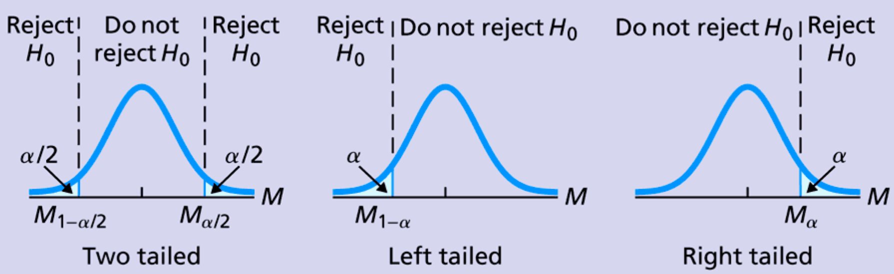
CRITICAL-VALUE APPROACH	OR	*P*-VALUE APPROACH

CRITICAL-VALUE APPROACH

Step 4 The critical value(s) are

$M_{1-\alpha/2}$ and $M_{\alpha/2}$ (Two tailed) or $M_{1-\alpha}$ (Left tailed) or M_{α} (Right tailed)

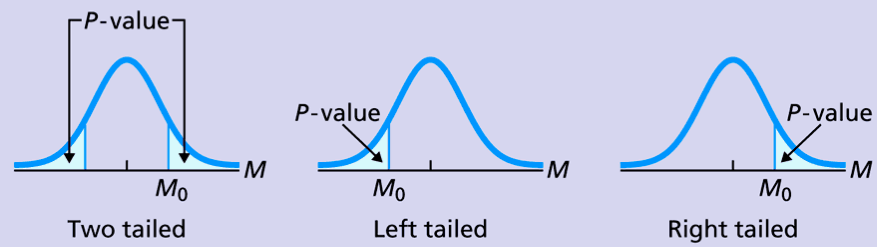
Use Table VI to find the critical value(s). For a left-tailed or two-tailed test, you will also need the relation $M_{1-A} = n_1(n_1 + n_2 + 1) - M_A$.



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

P-VALUE APPROACH

Step 4 Obtain the P-value by using technology.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Key Fact 10.5

The Mann-Whitney Test Versus the Pooled *t*-Test

Suppose that the distributions of a variable of two populations have the same shape and that you want to compare, using independent simple random samples, the two population means. When deciding between the pooled *t*-test and the Mann-Whitney test, follow these guidelines: If you are reasonably sure that the two distributions are normal, use the pooled *t*-test; otherwise, use the Mann-Whitney test.

Section 10.5

Inferences for Two Population Means, Using Paired Samples

Table 10.13

Ages, in years, of a random sample of 10 married couples

Couple	Husband	Wife	Difference, *d*
1	59	53	6
2	21	22	−1
3	33	36	−3
4	78	74	4
5	70	64	6
6	33	35	−2
7	68	67	1
8	32	28	4
9	54	41	13
10	52	44	8
			36

Key Fact 10.6

Distribution of the Paired t-Statistic

Suppose that x is a variable on each of two populations whose members can be paired. Further suppose that the paired-difference variable d is normally distributed. Then, for paired samples of size n, the variable

$$t = \frac{\bar{d} - (\mu_1 - \mu_2)}{s_d / \sqrt{n}}$$

has the t-distribution with df $= n - 1$.

Procedure 10.6

Paired *t*-Test

Purpose To perform a hypothesis test to compare two population means, μ_1 and μ_2

Assumptions

1. Simple random paired sample
2. Normal differences or large sample

Step 1 **The null hypothesis is H_0: $\mu_1 = \mu_2$, and the alternative hypothesis is**

H_a: $\mu_1 \neq \mu_2$	or	H_a: $\mu_1 < \mu_2$	or	H_a: $\mu_1 > \mu_2$
(Two tailed)		**(Left tailed)**		**(Right tailed)**

Step 2 **Decide on the significance level, α.**

Step 3 **Compute the value of the test statistic**

$$t = \frac{\bar{d}}{s_d/\sqrt{n}}$$

and denote that value t_0.

Procedure 10.6 (cont.)

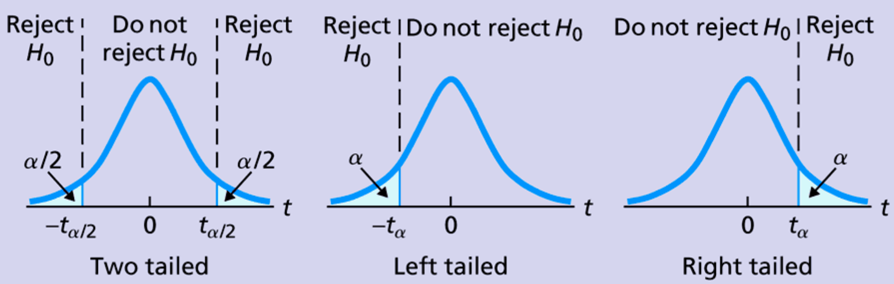
CRITICAL-VALUE APPROACH	OR	*P*-VALUE APPROACH

CRITICAL-VALUE APPROACH

Step 4 The critical value(s) are

$\pm t_{\alpha/2}$ (Two tailed) or $-t_{\alpha}$ (Left tailed) or t_{α} (Right tailed)

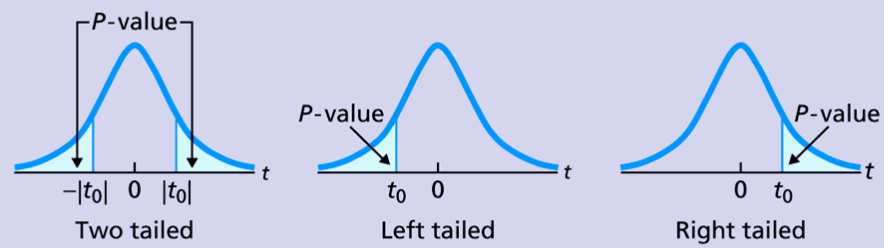
with df $= n - 1$. Use Table IV to find the critical value(s).



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

P-VALUE APPROACH

Step 4 The t-statistic has df $= n - 1$. Use Table IV to estimate the P-value, or obtain it exactly by using technology.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Note: The hypothesis test is exact for normal differences and is approximately correct for large samples and nonnormal differences.

Procedure 10.7

Paired *t*-Interval Procedure

Purpose To find a confidence interval for the difference between two population means, μ_1 and μ_2

Assumptions

1. Simple random paired sample
2. Normal differences or large sample

Step 1 **For a confidence level of $1 - \alpha$, use Table IV to find $t_{\alpha/2}$ with df $= n - 1$.**

Step 2 **The endpoints of the confidence interval for $\mu_1 - \mu_2$ are**

$$\bar{d} \pm t_{\alpha/2} \cdot \frac{s_d}{\sqrt{n}}.$$

Step 3 **Interpret the confidence interval.**

Note: The confidence interval is exact for normal differences and is approximately correct for large samples and nonnormal differences.

Section 10.6

The Paired Wilcoxon Signed-Rank Test

Procedure 10.8

Paired Wilcoxon Signed-Rank Test

Purpose To perform a hypothesis test to compare two population means, μ_1 and μ_2

Assumptions

1. Simple random paired sample
2. Symmetric differences

Step 1 **The null hypothesis is H_0: $\mu_1 = \mu_2$, and the alternative hypothesis is**

$$H_a\colon \mu_1 \neq \mu_2 \text{ (Two tailed)} \quad \text{or} \quad H_a\colon \mu_1 < \mu_2 \text{ (Left tailed)} \quad \text{or} \quad H_a\colon \mu_1 > \mu_2 \text{ (Right tailed)}$$

Step 2 **Decide on the significance level, α.**

Step 3 **Compute the value of the test statistic**

$$W = \text{sum of the positive ranks}$$

and denote that value W_0. To do so, first calculate the paired differences of the sample pairs, next discard all paired differences that equal 0 and reduce the sample size accordingly, and then construct a work table of the following form.

Paired difference d	$\lvert d \rvert$	Rank of $\lvert d \rvert$	Signed rank R
.	.	.	.
.	.	.	.
.	.	.	.

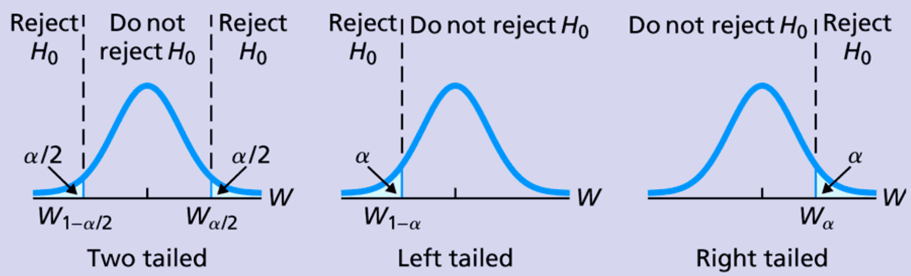
Procedure 10.8 (cont.)

CRITICAL-VALUE APPROACH	OR	*P*-VALUE APPROACH

Step 4 The critical value(s) are

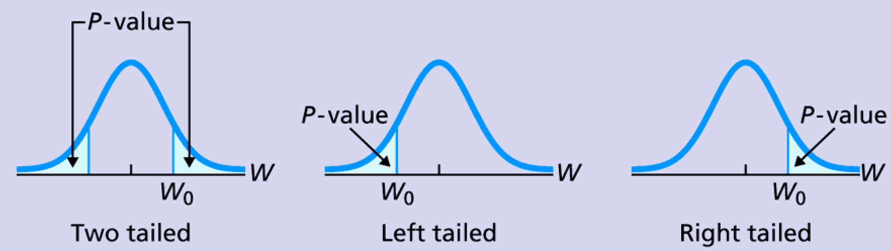
$W_{1-\alpha/2}$ and $W_{\alpha/2}$ (Two tailed) or $W_{1-\alpha}$ (Left tailed) or W_{α} (Right tailed)

Use Table V to find the critical value(s). For a left-tailed or two-tailed test, you will also need the relation $W_{1-A} = n(n+1)/2 - W_A$.



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

Step 4 Obtain the *P*-value by using technology.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Key Fact 10.7

Paired Wilcoxon Signed-Rank Test Versus the Paired *t*-Test

Suppose that you want to perform a hypothesis test using a paired sample to compare the means of two populations. When deciding between the paired *t*-test and the paired Wilcoxon signed-rank test, follow these guidelines:

- If you are reasonably sure that the paired-difference variable is normally distributed, use the paired *t*-test.
- If you are not reasonably sure that the paired-difference variable is normally distributed but are reasonably sure that it has a symmetric distribution, use the paired Wilcoxon signed-rank test.

Section 10.7

Which Procedure Should Be Used?

Table 10.15

Summary of hypothesis-testing procedures for comparing two population means. The null hypothesis for all tests is H_0

Type	Assumptions	Test statistic	Procedure to use
Pooled t-test	1. Simple random samples 2. Independent samples 3. Normal populations or large samples 4. Equal population standard deviations	$t = \dfrac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{(1/n_1) + (1/n_2)}}$ † $(\text{df} = n_1 + n_2 - 2)$	10.1 (page 441)
Nonpooled t-test	1. Simple random samples 2. Independent samples 3. Normal populations or large samples	$t = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{(s_1^2/n_1) + (s_2^2/n_2)}}$ ‡	10.3 (page 453)
Mann–Whitney test	1. Simple random samples 2. Independent samples 3. Same-shape populations	M = sum of the ranks for sample data from Population 1	10.5 (page 468)
Paired t-test	1. Simple random paired sample 2. Normal differences or large sample	$t = \dfrac{\bar{d}}{s_d/\sqrt{n}}$ $(\text{df} = n - 1)$	10.6 (page 481)
Paired W-test	1. Simple random paired sample 2. Symmetric differences	W = sum of positive ranks	10.8 (page 492)

† $s_p = \sqrt{\dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$ ‡ $\text{df} = \dfrac{[(s_1^2/n_1) + (s_2^2/n_2)]^2}{\dfrac{(s_1^2/n_1)^2}{n_1 - 1} + \dfrac{(s_2^2/n_2)^2}{n_2 - 1}}$

Figure 10.19

Flowchart for choosing the correct hypothesis-testing procedure for comparing two population means

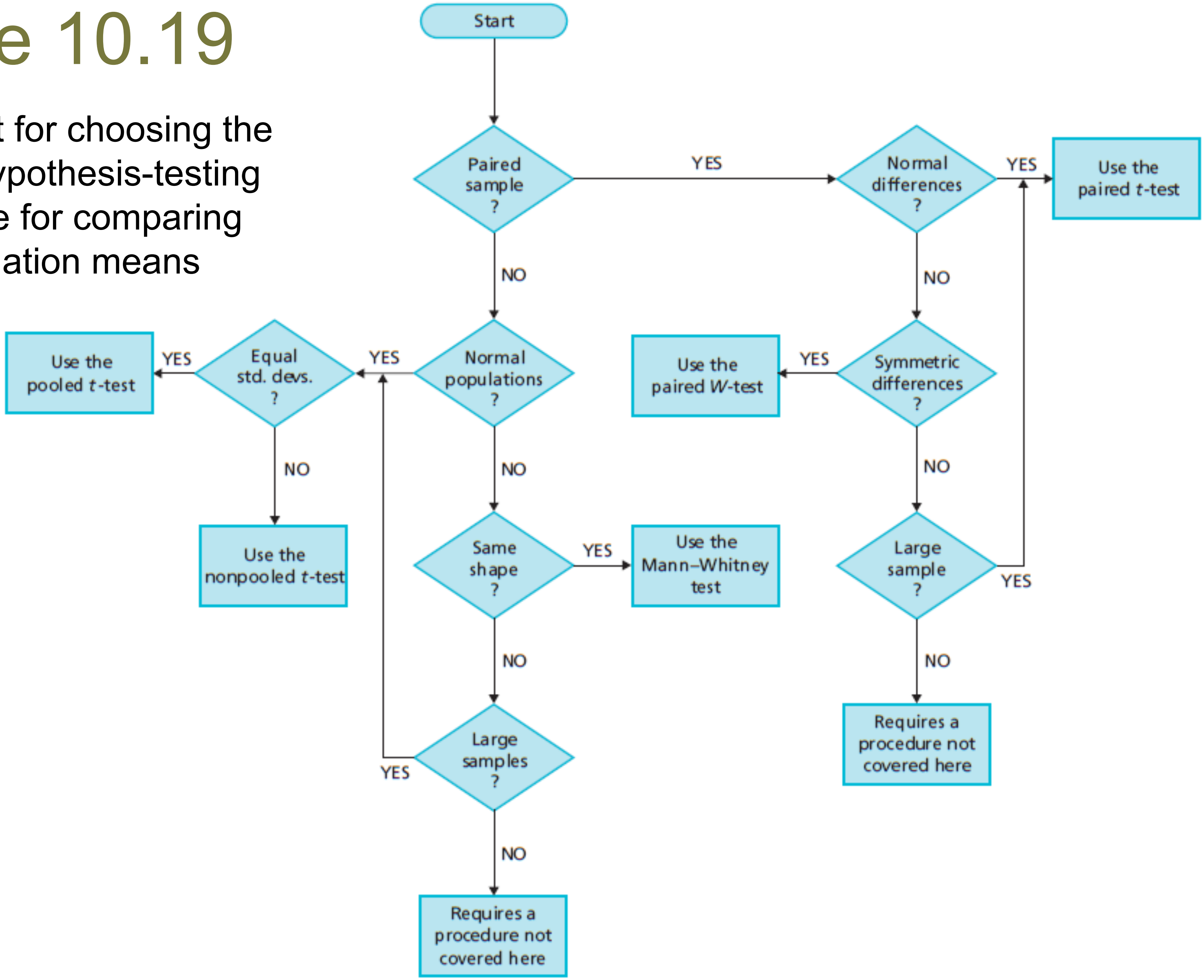


Figure 10.20

Normal probability plots of the sample data for (a) elite runners and (b) others

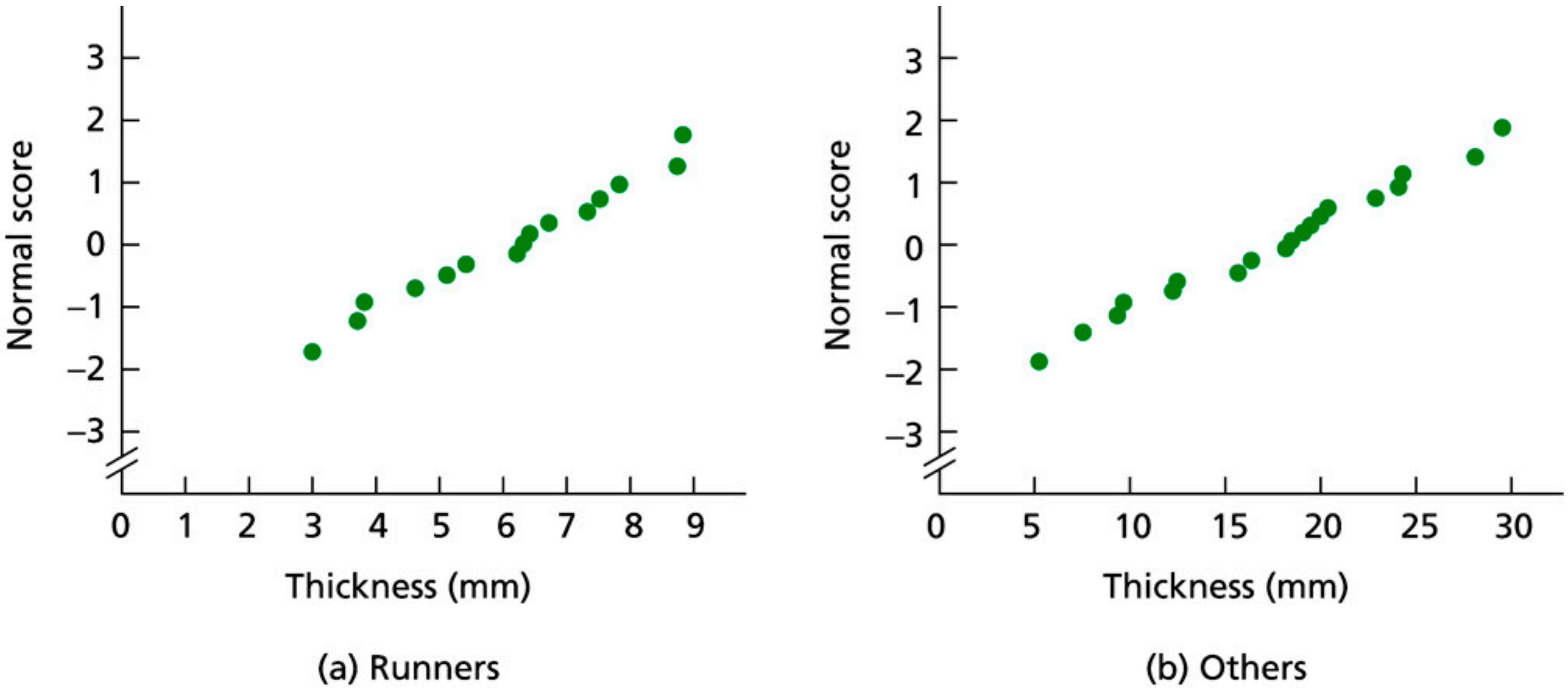


Figure 10.21

Boxplots of the sample data for elite runners and others