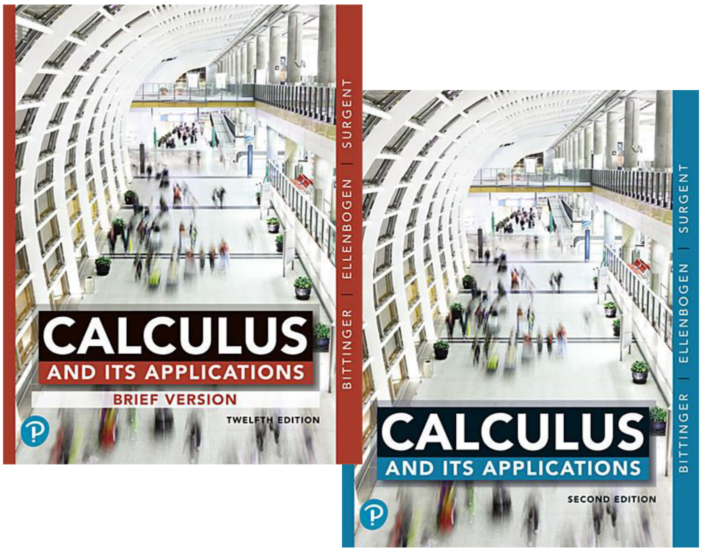
Chapter 1

Differentiation



1.1 Limits: A Numerical and Graphical Approach

Objective

- Find limits of functions, if they exist, using numerical or graphical methods.

1.1 Limits: A Numerical and Graphical Approach

Example 1: For each sequence, determine its limit, and rewrite the sequence in the form $x \to a^-$ or $x \to a^+$.

a) 2.24, 2.249, 2.2499, 2.24999, …
b) 5.51, 5.501, 5.5001, 5.50001,…
c) $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots$

1.1 Limits: A Numerical and Graphical Approach

Example 1 (concluded):

a) These numbers are approaching the limit 2.25. Since each number in the sequence is less than 2.25, we write $x \to 2.25^-$, read "x approaches 2.25 from the left."
b) These numbers are approaching the limit 5.5. Since each number in the sequence is greater than 5.5, we write $x \to 5.5^+$, read "x approaches 5.5 from the right."
c) These numbers are approaching the limit 1. Since each number in the sequence is less than 1, we write $x \to 1^-$, read "x approaches 1 from the left."

1.1 Limits: A Numerical and Graphical Approach

DEFINITION:

As x approaches a, the **limit** of $f(x)$ is L, written

$$\lim_{x \to a} f(x) = L,$$

if all values of $f(x)$ are close to L for values of x that are sufficiently close, but not equal to, a.

1.1 Limits: A Numerical and Graphical Approach

THEOREM 1

As x approaches a, the **limit** of $f(x)$ is L, if the limit from the left exists and the limit from the right exists and both limits are L. That is, if

1) $$\lim_{x \to a^-} f(x) = L,$$

and

2) $$\lim_{x \to a^+} f(x) = L,$$

then

$$\lim_{x \to a} f(x) = L.$$

1.1 Limits: A Numerical and Graphical Approach

Quick Check 1

Let $f(x) = \dfrac{x^2 - 9}{x - 3}$.

a) What is $f(3)$?

b) What is the limit of $f(x)$ as x approaches 3?

1.1 Limits: A Numerical and Graphical Approach

Quick Check 1 Solution a)

1.) Since $f(x) = \dfrac{x^2 - 9}{x - 3}$, we will substitute 3 in for x, giving us the new equation $f(3) = \dfrac{3^2 - 9}{3 - 3}$.

2.) Solving for $f(3)$, we get $f(3) = \dfrac{3^2 - 9}{3 - 3} = \dfrac{9 - 9}{3 - 3} = \dfrac{0}{0}$.

Thus $f(3)$ does not exist.

1.1 Limits: A Numerical and Graphical Approach

Quick Check 1 Solution b)

First let x approach 3 from the left:

$x \to 3^-$	2	2.5	2.9	2.99	2.999
$f(x)$	5	5.5	5.9	5.99	5.999

Thus it appears that $\lim_{x \to 3^-} f(x)$ is 6.

Next let x approach 3 from the right:

$x \to 3^+$	4	3.5	3.1	3.01	3.001
$f(x)$	7	6.5	6.1	6.01	6.001

Thus it appears that $\lim_{x \to 3^+} f(x)$ is 6.

Since both the left-hand and right-hand limits agree, $\lim_{x \to 3} f(x) = 6$.

1.1 Limits: A Numerical and Graphical Approach

Example 2: Consider the function H given by

$$H(x) = \begin{cases} 2x+2 & \text{for} \quad x < 1 \\ 2x-4 & \text{for} \quad x \geq 1 \end{cases}.$$

Graph the function and find each of the following limits, if they exist. When necessary, state that the limit does not exist.

a) $\lim_{x \to 1} H(x)$ b) $\lim_{x \to -3} H(x)$

1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically**

First, let x approach 1 from the left:

$x \to 1^-$	0	0.5	0.8	0.9	0.99	0.999
$H(x)$	2	3	3.6	3.8	3.98	3.998

Thus, it appears that $\lim_{x \to 1^-} H(x) = 4.$

1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically (continued)**

Then, let x approach 1 from the right:

$x \to 1^+$	2	1.8	1.1	1.01	1.001	1.0001
$H(x)$	0	–0.4	–1.8	–1.98	–1.998	–1.9998

Thus, it appears that $\lim_{x \to 1^+} H(x) = -2.$

1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically (concluded)**

Since 1) $\lim_{x \to 1^-} H(x) = 4$

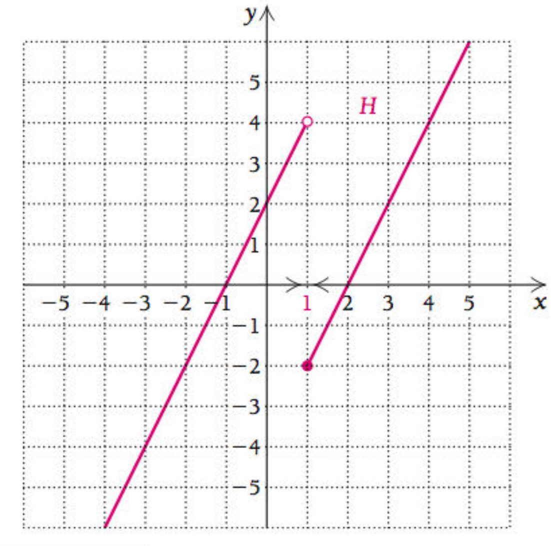
and

2) $\lim_{x \to 1^+} H(x) = -2,$

then, $\lim_{x \to 1} H(x)$ does not exist.

1.1 Limits: A Numerical and Graphical Approach

a) **Limit Graphically**



Observe on the graph that

1) $\lim_{x \to 1^-} H(x) = 4$

and

2) $\lim_{x \to 1^+} H(x) = -2.$

Therefore,

$\lim_{x \to 1} H(x)$ does not exist.

1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically**

First, let x approach -3 from the left:

$x \to -3^-$	-4	-3.5	-3.1	-3.01	-3.001
$H(x)$	-6	-5	-4.2	-4.02	-4.002

Thus, it appears that $\lim\limits_{x \to -3^-} H(x) = -4.$

1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically (continued)**

Then, let x approach -3 from the right:

$x \to -3^+$	-2	-2.5	-2.9	-2.99	-2.999
$H(x)$	-2	-3	-3.8	-3.98	-3.998

Thus, it appears that $\lim\limits_{x \to -3^+} H(x) = -4.$

1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically (concluded)**

Since 1) $\lim_{x \to -3^-} H(x) = -4$

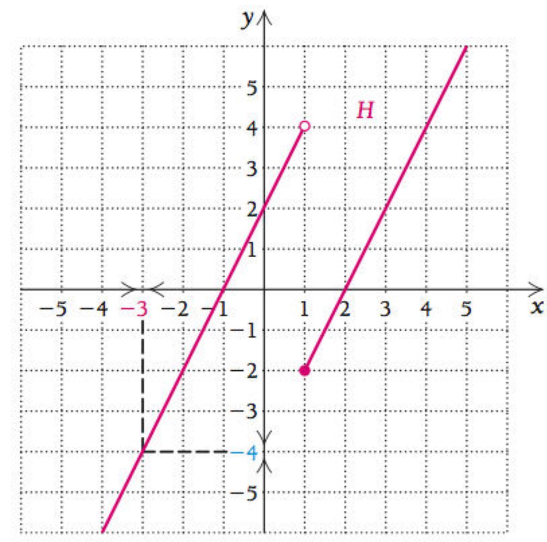
and

2) $\lim_{x \to -3^+} H(x) = -4,$

then, $\lim_{x \to -3} H(x) = -4.$

1.1 Limits: A Numerical and Graphical Approach

b) **Limit Graphically**



Observe on the graph that

1) $\lim_{x \to -3^-} H(x) = -4$

and

2) $\lim_{x \to -3^+} H(x) = -4.$

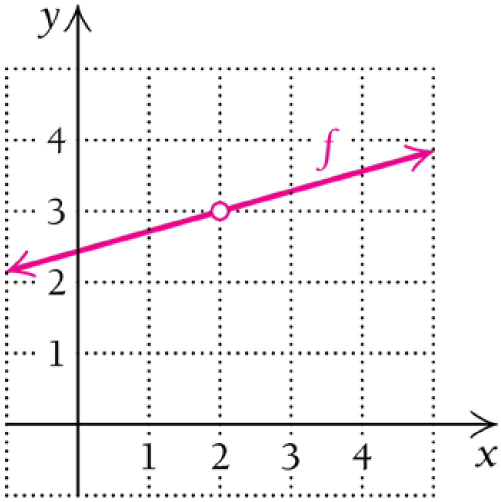
Therefore,

$$\lim_{x \to -3} H(x) = -4.$$

1.1 Limits: A Numerical and Graphical Approach

Quick Check 2

Calculate the following limits based on the graph of f.



a.) $\lim\limits_{x\to 2^-} f(x)$

b.) $\lim\limits_{x\to 2^+} f(x)$

c.) $\lim\limits_{x\to 2} f(x)$

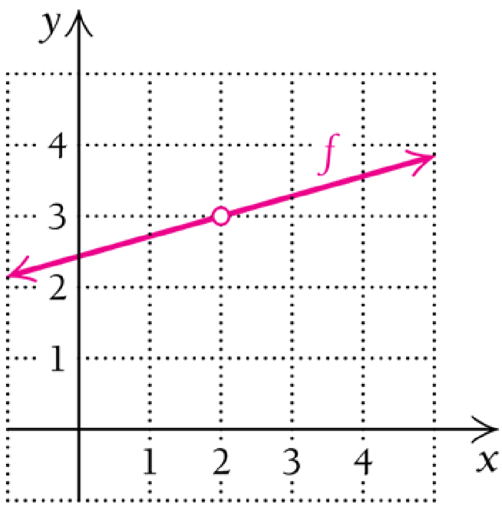
1.1 Limits: A Numerical and Graphical Approach

Quick Check 2 Solution

a.) $\lim\limits_{x\to 2^-} f(x)$: By looking at the graph, as x approaches 2 from the left, we can see that the $\lim\limits_{x\to 2^-} f(x)=3$.

b.) $\lim\limits_{x\to 2^+} f(x)$: By looking at the graph, as x approaches 2 from the right, we can see that the $\lim\limits_{x\to 2^+} f(x)=3$.

c.) Based on the solutions to parts a.) and b.), we know that the $\lim\limits_{x\to 2} f(x)=3$.



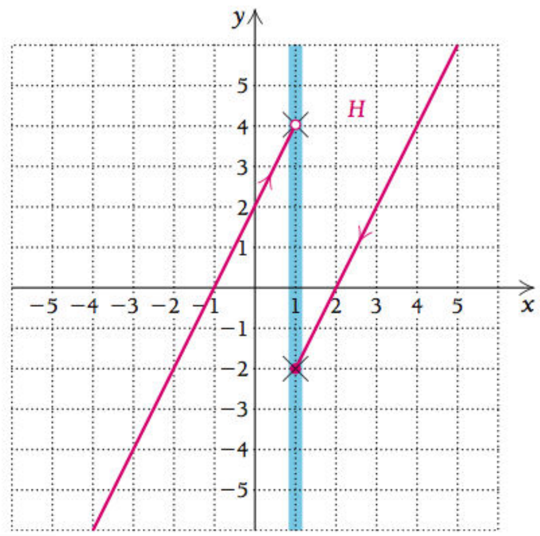
1.1 Limits: A Numerical and Graphical Approach

The "Wall" Method:

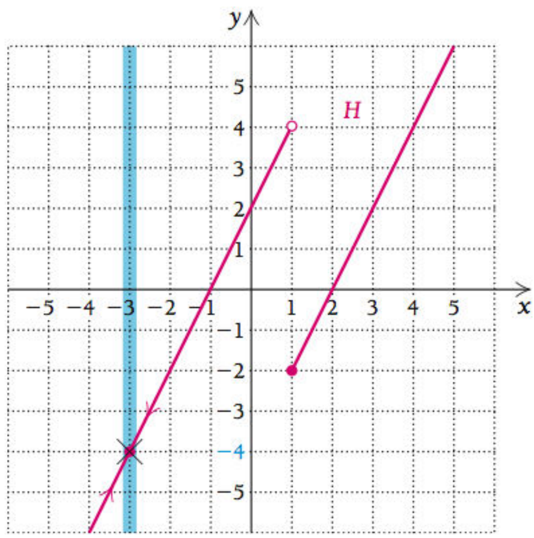
As an alternative approach to Example 1, we can draw a "wall" at $x = 1$, as shown in blue on the following graphs. We then follow the curve from left to right with pencil until we hit the wall and mark the location with an ×, assuming it can be determined. Then we follow the curve from right to left until we hit the wall and mark that location with an ×. If the locations are the same, we have a limit. Otherwise, the limit does not exist.

1.1 Limits: A Numerical and Graphical Approach

Thus, for Example 2:



$\lim_{x\to 1} H(x)$ does not exist



$$\lim_{x\to -3} H(x) = -4$$

1.1 Limits: A Numerical and Graphical Approach

Example 3: Consider the function f given by

$$f(x) = \frac{1}{x-2} + 3.$$

Graph the function, and find each of the following limits, if they exist. If necessary, state that the limit does not exist.

a) $\lim\limits_{x \to 3} f(x)$ b) $\lim\limits_{x \to 2} f(x)$

1.1 Limits: A Numerical and Graphical Approach

a) **Limit Numerically**

Let x approach 3 from the left and right:

$x \to 3^-$	2.1	2.5	2.9	2.99
$f(x)$	13	5	$4.1\overline{1}$	$4.\overline{01}$

$\Rightarrow \lim\limits_{x \to 3^-} f(x) = 4$

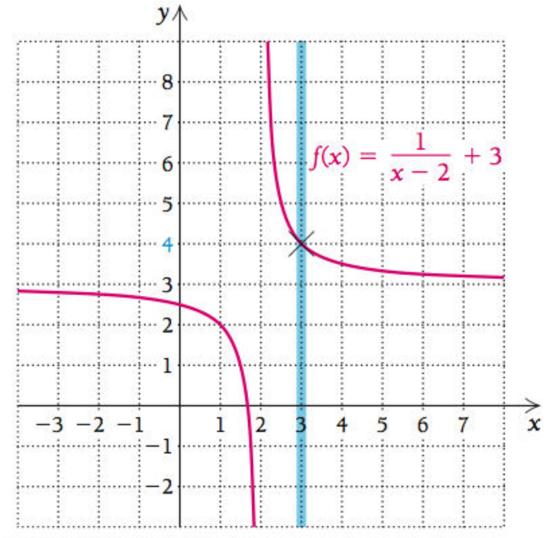
$x \to 3^+$	3.5	3.2	3.1	3.01
$f(x)$	$3.6\overline{6}$	$3.8\overline{3}$	$3.90\overline{90}$	$3.\overline{9900}$

$\Rightarrow \lim\limits_{x \to 3^+} f(x) = 4$

Thus, $\lim\limits_{x \to 3} f(x) = 4.$

1.1 Limits: A Numerical and Graphical Approach

a) **Limit Graphically**



Observe on the graph that:

1) $\lim_{x \to 3^-} f(x) = 4$

and

2) $\lim_{x \to 3^+} f(x) = 4$

Therefore,

$$\lim_{x \to 3} f(x) = 4.$$

1.1 Limits: A Numerical and Graphical Approach

b) **Limit Numerically**

Let x approach 2 from the left and right:

$x \to 2^-$	1.5	1.9	1.99	1.999
$f(x)$	1	–7	–97	–997

$\Rightarrow \lim_{x \to 2^-} f(x) = -\infty$

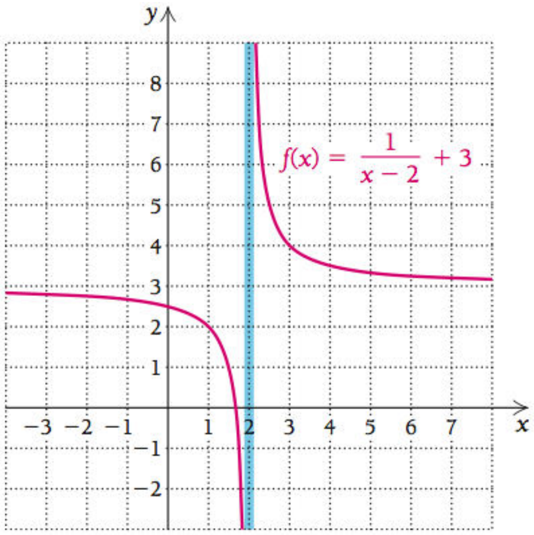
$x \to 2^+$	2.5	2.1	2.01	2.001
$f(x)$	5	13	103	1003

$\Rightarrow \lim_{x \to 2^+} f(x) = \infty$

Thus, $\lim_{x \to 2} f(x)$ does not exist.

1.1 Limits: A Numerical and Graphical Approach

b) **Limit Graphically**



Observe on the graph that

1) $\lim\limits_{x \to 2^-} f(x) = -\infty$

and

2) $\lim\limits_{x \to 2^+} f(x) = \infty.$

Therefore,

$\lim\limits_{x \to 2} f(x)$ does not exist.

1.1 Limits: A Numerical and Graphical Approach

Example 4: Consider again the function f given by

$$f(x) = \frac{1}{x-2} + 3.$$

Find $\lim\limits_{x \to \infty} f(x)$.

1.1 Limits: A Numerical and Graphical Approach

Limit Numerically

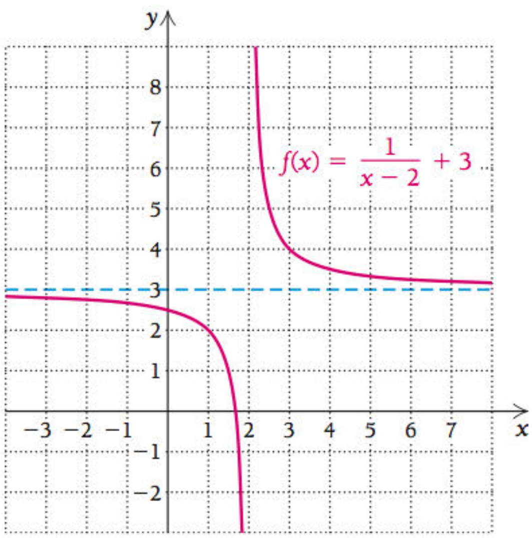
Note that you can only approach ∞ from the left:

$x \to \infty$	5	10	100	1000
$f(x)$	$3.\overline{3}$	3.125	3.0102	3.001

Thus, $\lim\limits_{x\to\infty} f(x) = 3.$

1.1 Limits: A Numerical and Graphical Approach

Limit Graphically



Observe on the graph that, again, you can only approach ∞ from the left.

Therefore,

$$\lim_{x\to\infty} f(x) = 3.$$

1.1 Limits: A Numerical and Graphical Approach

Quick Check 3

Let $h(x) = \frac{1}{1-x} + 6.$ Find the following limits:

a.) $\lim_{x \to 1} h(x)$

b.) $\lim_{x \to 2} h(x)$

c.) $\lim_{x \to \infty} h(x)$

1.1 Limits: A Numerical and Graphical Approach

Quick Check 3 Solution

a.) $\lim_{x \to 1} h(x)$: Find the left-hand and right-hand limits as x approaches 1:

Left-hand Limit

$x \to 1^-$	$h(x)$
0	7
0.5	8
0.9	16
0.99	106
0.999	1006

Right-hand Limit

$x \to 1^+$	$h(x)$
2	5
1.5	4
1.1	–4
1.01	–94
1.001	–994

Since the Left-Hand Limit goes to ∞ and the Right-Side Limit goes to $-\infty$,

the $\lim_{x \to 1} h(x) =$ does not exist.

1.1 Limits: A Numerical and Graphical Approach

Quick Check Solution Continued

b.) $\lim_{x\to 2} h(x)$: Find both the left-hand and right-hand limits as x approaches 2.

Left-Hand Limit

$x \to 2^-$	$h(x)$
1.1	-4
1.5	4
1.9	$4.\overline{8}$
1.99	$4.\overline{98}$
1.999	$4.\overline{998}$

Right-Hand Limit

$x \to 2^+$	$h(x)$
3	5.5
2.5	$5.\overline{3}$
2.1	$5.\overline{09}$
2.01	$5.\overline{0099}$
2.001	$5.\overline{000999}$

Since both the Left-Side Limit and Right-Side Limit agree, the $\lim_{x\to 2} h(x) = 5.$

Quick Check Solution Concluded

c.) $\lim_{x\to\infty} h(x)$: Find the limit as x approaches ∞:

$x \to \infty$	$h(x)$
5	5.75
10	$5.\overline{8}$
100	$5.\overline{98}$
1000	$5.\overline{998}$

Since both the Left-Side Limit and Right-Side Limit agree, the $\lim_{x\to\infty} h(x) = 6.$

1.1 Limits: A Numerical and Graphical Approach

Section Summary

•The *limit* of a function f , as x approaches a, is written $\lim_{x \to a} f(x) = L$.

This means that as the values of x approach a the corresponding values of $f(x)$ approach L. The value of L must be a unique, finite number.

• A *left-hand limit* is written $\lim_{x \to a^-} f(x)$.

The values of x are approaching a from the left, that is, $x < a$.

•A *right-hand limit* is written $\lim_{x \to a^+} f(x)$.

The values of x are approaching a from the right, that is, $x > a$.

• If the left-hand and right-hand limits (as x approaches a) are *not* equal, the limit does *not* exist. On the other hand, if the left-hand and right-hand limits are equal, the limit does exist.

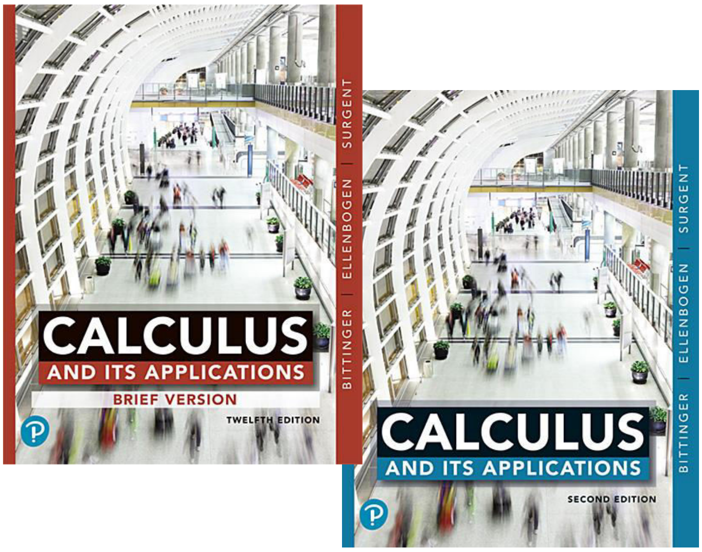
• A limit $\lim_{x \to a} f(x)$ may exist even though the function value $f(a)$ does not. (See Example 1.)

• A limit $\lim_{x \to a} f(x)$ may exist and be different from the function value $f(a)$. (See Example 3b.)

• Graphs and tables are useful tools in determining limits.

Chapter 1

Differentiation



1.2 Algebraic Limits and Continuity

OBJECTIVE

- Develop and use the Limit Principles to calculate limits.
- Determine whether a function is continuous at a point.

1.2 Algebraic Limits and Continuity

LIMIT PROPERTIES:

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, and c is any constant, then we have the following:

L.1

The limit of a constant is the constant: $\lim_{x \to a} c = c$

1.2 Algebraic Limits and Continuity

LIMIT PROPERTIES (continued):

L.2 The limit of a power function is the limit of the base, raised to that power.

That is, for any positive integer n,

$$\lim_{x \to a} \left[f(x) \right]^n = \left[\lim_{x \to a} f(x) \right]^n = L^n,$$

and

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L},$$

assuming that $L \geq 0$ when n is even.

1.2 Algebraic Limits and Continuity

LIMIT PROPERTIES (continued):

L.3 The limit of a sum or difference is the sum or difference of the limits.

$$\lim_{x \to a} \left[f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M.$$

L.4 The limit of a product is the product of the limits.

$$\lim_{x \to a} \left[f(x) \cdot g(x) \right] = \left[\lim_{x \to a} f(x) \right] \cdot \left[\lim_{x \to a} g(x) \right] = L \cdot M.$$

1.2 Algebraic Limits and Continuity

LIMIT PROPERTIES (concluded):

L.5 The limit of a quotient is the quotient of the limits.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}, \quad M \neq 0.$$

L.6 The limit of a constant times a function is the constant times the limit.

$$\lim_{x \to a} \left[cf(x) \right] = c \cdot \lim_{x \to a} f(x) = cL.$$

1.2 Algebraic Limits and Continuity

Example 1: Use the limit properties to find

$$\lim_{x \to 4} \left(x^2 - 3x + 7 \right)$$

We know that $\lim_{x \to 4} x = 4.$

By Limit Property L4,

$$\lim_{x \to 4} x^2 = \lim_{x \to 4} x \cdot \lim_{x \to 4} x = 4 \cdot 4 = 16.$$

1.2 Algebraic Limits and Continuity

Example 1 (concluded):

By Limit Property L6,

$$\lim_{x\to 4}(-3x) = -3\cdot\lim_{x\to 4} x = -3\cdot 4 = -12.$$

By Limit Property L1,

$$\lim_{x\to 4} 7 = 7.$$

Thus, using Limit Property L3, we have

$$\lim_{x\to 4}\left(x^2 - 3x + 7\right) = 16 - 12 + 7 = 11.$$

1.2 Algebraic Limits and Continuity

THEOREM 2: LIMITS OF RATIONAL FUNCTIONS

For any rational function F, with a in the domain of F,

$$\lim_{x\to a} F(x) = F(a).$$

1.2 Algebraic Limits and Continuity

Example 2: Find $\lim_{x \to 0} \sqrt{\left(x^2 - 3x + 2\right)}$

The Theorem on Limits of Rational Functions and Limit Property L2 tell us that we can substitute to find the limit:

$$\lim_{x \to 0} \sqrt{\left(x^2 - 3x + 2\right)} = \sqrt{0^2 - 3 \cdot 0 + 2} = \sqrt{2}$$

1.2 Algebraic Limits and Continuity

Quick Check 1

Find the following limits and note the Limit Property you use at each step:

a.) $\lim_{x \to 1} 2x^3 + 3x^2 - 6$

b.) $\lim_{x \to 4} \dfrac{2x^2 + 5x - 1}{3x - 2}$

c.) $\lim_{x \to 2} \sqrt{1 + 3x^2}$

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1.2 Algebraic Limits and Continuity

Quick Check 1 Solution a.) $\lim_{x\to 1} 2x^3 + 3x^2 - 6$

We know that the $\lim_{x\to 1} x = 1$.

1.) $\lim_{x\to 1} x^3 = \lim_{x\to 1} x \cdot \lim_{x\to 1} x \cdot \lim_{x\to 1} x = 1\cdot 1\cdot 1 = 1$	Limit Property L4
2.) $\lim_{x\to 1} 2x^3 = 2(\lim_{x\to 1} x \cdot \lim_{x\to 1} x \cdot \lim_{x\to 1} x) = 2$	Limit Property L6
3.) $\lim_{x\to 1} x^2 = \lim_{x\to 1} x \cdot \lim_{x\to 1} x = 1\cdot 1 = 1$	Limit Property L4
4.) $\lim_{x\to 1} 3x^2 = 3(\lim_{x\to 1} x \cdot \lim_{x\to 1} x) = 3$	Limit Property L6
5.) $\lim_{x\to 1} 6 = 6$	Limit Property L1
6.) Combining steps 2.), 4.), and 5.) we get	
$\lim_{x\to 1} 2x^3 + 3x^2 - 6 = 2 + 3 - 6 = -1$	Limit Property L3

1.2 Algebraic Limits and Continuity

Quick Check 1 solution b.) $\lim_{x\to 4} \dfrac{2x^2 + 5x - 1}{3x - 2}$

We know that $\lim_{x\to 4} x = 4$.

1.) $\lim_{x\to 4} 2x^2 = 2(\lim_{x\to 4} x \cdot \lim_{x\to 4} x) = 2(4\cdot 4) = 2\cdot 16 = 32$	Limit Properties L4 and L6
2.) $\lim_{x\to 4} 5x = 5\cdot \lim_{x\to 4} x = 5\cdot 4 = 20$	Limit Property L6
3.) $\lim_{x\to 4} 1 = 1$	Limit Property L1
4.) Combine above steps: $\lim_{x\to 4} 2x^2 + 5x - 1 = 32 + 20 - 1 = 51$	Limit Property L3
5.) $\lim_{x\to 4} 3x = 3\cdot \lim_{x\to 4} x = 3\cdot 4 = 12$	Limit Property L6
6.) $\lim_{x\to 4} 2 = 2$	Limit Property L1
7.) Combine above steps: $\lim_{x\to 4} 3x - 2 = 12 - 2 = 10$	Limit Property L3
8.) Combine steps 4.) and 7.)	
$\lim_{x\to 4} \dfrac{2x^2 + 5x - 1}{3x - 2} = \dfrac{51}{10} = 5.1$	Limit Property L5

1.2 Algebraic Limits and Continuity

Quick Check 1 solution c.) $\lim_{x\to 2}\sqrt{1+3x^2}$

We know that $\lim_{x\to 2} x = 2.$

1.) $\lim_{x\to 2} 1 = 1$ Limit Property L1

2.) $\lim_{x\to 2} 3x^2 = 3(\lim_{x\to 2} x \cdot \lim_{x\to 2} x) = 3(2\cdot 2) = 3\cdot 4 = 12$ Limit Properties L4 and L6

3.) Combine above steps: $\lim_{x\to 2} 1+3x^2 = 1+12 = 13$ Limit Property L3

4.) Using step 3.)

$$\lim_{x\to 2}\sqrt{1+3x^2} = \sqrt{1+12} = \sqrt{13}$$ Limit Property L2

1.2 Algebraic Limits and Continuity

Example 3: Find $\lim_{x\to -3}\dfrac{x^2-9}{x+3}$

Note that the Theorem on Limits of Rational Functions does not immediately apply because –3 is not in the domain of $\dfrac{x^2-9}{x+3}$.

However, if we simplify $\dfrac{x^2-9}{x+3}$ first, the result can be evaluated at $x = -3$.

1.2 Algebraic Limits and Continuity

Example 3 (concluded):

$$\begin{aligned}\lim_{x\to -3}\frac{x^2-9}{x+3} &= \lim_{x\to -3}\frac{\cancel{(x+3)}(x-3)}{\cancel{x+3}}\\ &= \lim_{x\to -3} x-3\\ &= -3-3\\ &= -6\end{aligned}$$

This means that the limit exists as x approaches -3, but the actual point (from previous slide) does not.

1.2 Algebraic Limits and Continuity

Example 4: Find $\lim_{x\to\infty}\left(\dfrac{x^2+4x-5}{2x^2+x+1}\right)$.

First, divide the numerator and the denominator by the highest power of the denominator, x^2.

$$\lim_{x\to\infty}\left(\frac{x^2+4x-5}{2x^2+x+1}\right) = \lim_{x\to\infty}\left(\frac{1+\frac{4}{x}-\frac{5}{x^2}}{2+\frac{1}{x}+\frac{1}{x^2}}\right)$$

Using our Limit Properties, we get:

$$= \frac{\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{4}{x}-\lim_{x\to\infty}\frac{5}{x^2}}{\lim_{x\to\infty}2+\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{1}{x^2}} = \frac{1+0-0}{2+0+0} = \frac{1}{2}$$

1.2 Algebraic Limits and Continuity

Quick Check 2:

Find $\lim_{x\to\infty}\frac{2x^3+5x^2+4x-1}{3x^3+6x^2-7}$.

First, divide the numerator and the denominator by the highest power of the denominator, x^3.

$$\lim_{x\to\infty}\frac{2x^3+5x^2+4x-1}{3x^3+6x^2-7}=\lim_{x\to\infty}\left(\frac{2+\frac{5}{x}+\frac{4}{x^2}-\frac{1}{x^3}}{3+\frac{6}{x}-\frac{7}{x^3}}\right)$$

Using our Limit Properties, we get:

$$=\frac{\lim_{x\to\infty}2+\lim_{x\to\infty}\frac{5}{x}+\lim_{x\to\infty}\frac{4}{x^2}-\lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty}3+\lim_{x\to\infty}\frac{6}{x}-\lim_{x\to\infty}\frac{7}{x^3}}=\frac{2+0+0-0}{3+0-0}=\frac{2}{3}$$

1.2 Algebraic Limits and Continuity

DEFINITION:

A function f is **continuous** at $x = a$ if:

1) $f(a)$ exists, (The output at a exists.)

2) $\lim_{x\to a} f(x)$ exists, (The limit as $x\to a$ exists.)

3) $\lim_{x\to a} f(x)=f(a)$. (The limit is the same as the output.)

A function is **continuous over an interval** $c < x < d$ if it is continuous at each point in that interval.

1.2 Algebraic Limits and Continuity

Example 5: Is the function f given by

$$f(x) = x^2 - 5$$

continuous at $x = 3$? Why or why not?

1) $f(3) = 3^2 - 5 = 9 - 5 = 4$

2) By the Theorem on Limits of Rational Functions,

$$\lim_{x \to 3} x^2 - 5 = 3^2 - 5 = 9 - 5 = 4$$

3) Since $\lim_{x \to 3} f(x) = f(3)$ f is continuous at $x = 3$.

1.2 Algebraic Limits and Continuity

Example 6: Is the function g given by

$$g(x) = \begin{cases} \frac{1}{2}x + 3, & \text{for} \quad x < -2 \\ x - 1, & \text{for} \quad x \geq -2 \end{cases}$$

continuous at $x = -2$? Why or why not?

1) $g(-2) = -2 - 1 = -3$

2) To find the limit, we look at left and right-side limits.

$$\lim_{x \to -2^-} g(x) = \frac{1}{2} \cdot -2 + 3 = -1 + 3 = 2$$

1.2 Algebraic Limits and Continuity

Example 6 (concluded):

3) $\lim_{x \to -2^+} g(x) = -2 - 1 = -3$

Since $\lim_{x \to -2^-} g(x) \neq \lim_{x \to -2^+} g(x)$ we see that the

$\lim_{x \to -2} g(x)$ does not exist.

Therefore, g is not continuous at $x = -2$.

1.2 Algebraic Limits and Continuity

Quick Check 3

Let $g(x) = \begin{cases} 3x - 5, & \text{for } x < 2 \\ 2x + 1, & \text{for } x \geq 2 \end{cases}$

Is g continuous at $x = 2$? Why or why not?

1.) $g(2) = 2(2) + 1 = 4 + 1 = 5$

2.) To find the limit, we look at both the left-hand and right-hand limits:

Left-hand: $\lim_{x \to 2^-} g(x) = 3(2) - 5 = 6 - 5 = 1$

Right-hand: $\lim_{x \to 2^+} g(x) = 2(2) + 1 = 4 + 1 = 5$

Since $\lim_{x \to 2^-} g(x) \neq \lim_{x \to 2^+} g(x)$ we see that $\lim_{x \to 2} g(x)$ does not exist.

Therefore g is not continuous at $x = 2$.

1.2 Algebraic Limits and Continuity

Quick Check 4a

Let $h(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & \text{for } x\neq 3\\ 7, & \text{for } x=3\end{cases}$ Is h continuous at $x=3$? Why or why not?

In order for $h(x)$ to continuous, $\lim\limits_{x\to3}h(x)=h(3)$. So lets start by finding $\lim\limits_{x\to3}h(x)$.

$$\lim_{x\to3}h(x)=\lim_{x\to3}\frac{x^2-9}{x-3}=\lim_{x\to3}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to3}x+3=6$$

So the $\lim\limits_{x\to3}h(x)=6$. However, $h(3)=7$, and thus $\lim\limits_{x\to3}h(x)\neq h(3)$. Therefore $h(x)$ is not continuous at $x=3$.

1.2 Algebraic Limits and Continuity

Quick Check 4b

Let $p(x)=\begin{cases}\dfrac{x^2-25}{x-5}, & \text{for } x\neq 5\\ c, & \text{for } x=5\end{cases}$ Determine c such that p is continuous at $x = 5$.

In order for p to be continuous at $x=5$, $\lim\limits_{x\to5}p(x)=p(5)=c$. So if we find $\lim\limits_{x\to5}p(x)$, we can determine what c is. Let's find $\lim\limits_{x\to5}p(x)$:

$$\lim_{x\to5}p(x)=\lim_{x\to5}\frac{x^2-25}{x-5}=\lim_{x\to5}\frac{(x-5)(x+5)}{x-5}=\lim_{x\to5}x+5=10$$

So $\lim\limits_{x\to5}p(x)=10$. Therefore, in order for p to be continuous at $x=5$, $c=10$.

1.2 Algebraic Limits and Continuity

Section Summary

- For a rational function for which a is in the domain, the limit as x approaches a can be found by direct evaluation of the function at a.
- If direct evaluation leads to the *indeterminate form* $0/0$, the limit may still exist: algebraic simplification and/or a table and graph are used to find the limit.
- Informally, a function is *continuous* if its graph can be sketched without lifting the pencil off the paper.

1.2 Algebraic Limits and Continuity

Section Summary Continued

- Formally, a function is continuous at $x = a$ if:
  1. The function value $f(a)$ exists
  2. The limit as x approaches a exists
  3. The function value and the limit are equal
  4. This can be summarized as $\lim_{x \to a} f(x) = f(a)$.
- If any part of the continuity definition fails, then the function is *discontinuous* at $x = a$.