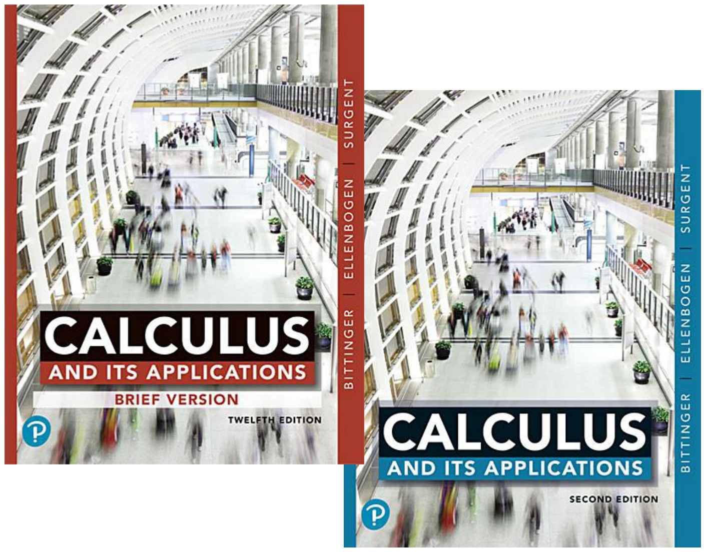
Chapter 4

Integration



4.1 Antidifferentiation

OBJECTIVE

- Find an antiderivative of a function.
- Evaluate indefinite integrals using basic rules of antidifferentiation.
- Use initial conditions to determine an antiderivative.

4.1 Antidifferentiation

THEOREM 1

The **antiderivative** of $f(x)$ is the set of functions $F(x)+C$ such that

$$\frac{d}{dx}\left[F(x)+C\right]=f(x).$$

The constant C is called the **constant of integration.**

4.1 Antidifferentiation

Integrals and Integration

Antidifferentiating is often called integration.

To indicate the antiderivative of x^2 is $x^3/3 + C$, we write $\int x^2 dx = \frac{x^3}{3} + C$, where the notation $\int f(x)\,dx$ is used to represent the antiderivative of $f(x)$.

More generally, $\int f(x)\,dx = F(x) + C$, where $F(x) + C$ is the general form of the antiderivative of $f(x)$.

4.1 Antidifferentiation

Example 1: Determine these indefinite integrals. That is, find the antiderivative of each integrand:

a.) $\int 8dx = 8x + C$ $\quad$ $Check: \frac{d}{dx}(8x+C) = 8$

b.) $\int 3x^2 dx = x^3 + C$ $\quad$ Check: $\frac{d}{dx}\left(x^3 + C\right) = 3x^2$

c.) $\int e^x dx = e^x + C$ $\quad$ Check: $\frac{d}{dx}\left(e^x + C\right) = e^x$

d.) $\int \frac{1}{x} dx = \ln x + C$ $\quad$ Check: $\frac{d}{dx}\left(\ln x + C\right) = \frac{1}{x}$

4.1 Antidifferentiation

THEOREM 2: Basic Integration Formulas

1. $\int k \; dx = kx + C \quad$ (k is a constant)

2. $\int x^r dx = \frac{x^{r+1}}{r+1} + C,\quad$ provided $r \neq -1$

(To integrate a power of x other than -1, increase the power by 1 and divide by the increased power.)

4.1 Antidifferentiation

THEOREM 2: Basic Integration Formulas (continued)

3. $\int x^{-1}dx = \int \frac{1}{x}dx = \int \frac{dx}{x} = \ln x + C, \quad x > 0$

$\int x^{-1}dx = \ln|x| + C, \quad x < 0$

(We will generally assume that $x > 0$.)

4. $\int be^{ax}dx = \frac{b}{a}e^{ax} + C$

4.1 Antidifferentiation

Example 2: Use the Power Rule of Antidifferentiation to determine these indefinite integrals:

a) $\int x^7 dx$; b) $\int x^{99} dx$; c) $\int \sqrt{x}dx$; d) $\int \frac{1}{x^3}dx$

a.) $\int x^7 dx = \frac{x^{7+1}}{7+1} + C = \frac{1}{8}x^8 + C$

Check: $\frac{d}{dx}\left(\frac{1}{8}x^8 + C\right) = \frac{1}{8}\left(8x^7\right) = x^7$

b.) $\int x^{99} dx = \frac{x^{99+1}}{99+1} + C = \frac{1}{100}x^{100} + C$

Check: $\frac{d}{dx}\left(\frac{1}{100}x^{100} + C\right) = \frac{1}{100}\left(100x^{99}\right) = x^{99}$

4.1 Antidifferentiation

Example 2 (Continued)

c.) We note that $\sqrt{x} = x^{1/2}$. Therefore

$$\int \sqrt{x}\, dx = \int x^{1/2} = \frac{x^{1/2+1}}{\frac{1}{2}+1} + C = \frac{2}{3}x^{3/2} + C$$

Check: $\frac{d}{dx}\left(\frac{2}{3}x^{3/2} + C\right) = \frac{2}{3}\left(\frac{3}{2}x^{1/2}\right) = x^{1/2} = \sqrt{x}$

4.1 Antidifferentiation

Example 2 (Concluded)

d.) We note that $\frac{1}{x^3} = x^{-3}$. Therefore

$$\int \frac{1}{x^3}\, dx = \int x^{-3} dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2}x^{-2} + C = -\frac{1}{2x^2} + C$$

Check: $\frac{d}{dx}\left(-\frac{1}{2}x^{-2} + C\right) = -\frac{1}{2}\left(-2x^{-3}\right) = x^{-3} = \frac{1}{x^3}$

4.1 Antidifferentiation

Quick Check 1

Determine these indefinite integrals:

a.) $\int x^{10}dx = \dfrac{x^{10+1}}{10+1} + C = \dfrac{1}{11}x^{11} + C$

b.) $\int x^{200}dx = \dfrac{x^{200+1}}{200+1} + C = \dfrac{1}{201}x^{201} + C$

c.) $\int \sqrt[6]{x} \cdot dx = \int x^{1/6}dx = \dfrac{x^{1/6+1}}{\frac{1}{6}+1} + C = \dfrac{6}{7}x^{7/6} + C = \dfrac{6}{7}\sqrt[6]{x^7} + C$

d.) $\int \dfrac{1}{x^4}dx = \int x^{-4}dx = \dfrac{x^{-4+1}}{-4+1} + C = \dfrac{1}{-3}x^{-3} + C = -\dfrac{1}{3x^3} + C$

4.1 Antidifferentiation

Example 3: Determine the indefinite integral $\int e^{4x}dx$.

Since we know that $\dfrac{d}{dx}e^x = e^x$, it is reasonable to make this initial guess:

$$\int e^{4x}dx = e^{4x} + C.$$

But this is (slightly) wrong, since

$$\frac{d}{dx}\left(e^{4x} + C\right) = 4e^{4x}$$

4.1 Antidifferentiation

Example 3 (Concluded): We modify our guess by inserting $\frac{1}{4}$ to obtain the correct antiderivative:

$$\int e^{4x}dx = \frac{1}{4}e^{4x} + C$$

This checks:

$$\frac{d}{dx}\left(\frac{1}{4}e^{4x} + C\right) = \frac{1}{4}\left(4e^{4x}\right) = e^{4x}$$

4.1 Antidifferentiation

Quick Check 2

Find each antiderivative:

a.) $\int e^{-3x}dx = \frac{1}{-3}e^{-3x} + C$

b.) $\int e^{(1/2)x}dx = 2e^{(1/2)x} + C$

4.1 Antidifferentiation

THEOREM 3

Properties of Antidifferentiation

P1. $\int c f(x)dx = c\int f(x)dx$

(The integral of a constant times a function is the constant times the integral of the function.)

P2. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

(The integral of a sum or difference is the sum or difference of the integrals.)

4.1 Antidifferentiation

Example 4: Determine these indefinite integrals. Assume $x > 0$.

a) $\int \left(3x^5 + 7x^2 + 8\right)dx$; b) $\int \frac{4 + 3x + 2x^4}{x}dx$

a.) We antidifferentiate each term separately:

$$\int \left(3x^5 + 7x^2 + 8\right)dx = \int 3x^5 dx + \int 7x^2 dx + \int 8dx$$

$$= 3\left(\frac{1}{6}x^6\right) + 7\left(\frac{1}{3}x^3\right) + 8x$$

$$= \frac{1}{2}x^6 + \frac{7}{3}x^3 + 8x + C$$

4.1 Antidifferentiation

Example 4 (Concluded):

b) We algebraically simplify the integrand by noting that *x* is a common denominator and then reducing each ratio as much as possible:

$$\frac{4+3x+2x^4}{x} = \frac{4}{x} + \frac{3x}{x} + \frac{2x^4}{x} = \frac{4}{x} + 3 + 2x^3$$

Therefore,

$$\int \frac{4+3x+2x^4}{x}\,dx = \int\left(\frac{4}{x} + 3 + 2x^3\right)dx$$

$$= 4\ln x + 3x + \frac{1}{2}x^4 + C$$

4.1 Antidifferentiation

Quick Check 3

Determine these indefinite integrals:

a.) $\int\left(2x^4 + 3x^3 - 7x^2 + x - 5\right)dx = \frac{2}{5}x^5 + \frac{3}{4}x^4 - \frac{7}{3}x^3 + \frac{1}{2}x^2 - 5x + C$

b.) $\int (x-5)^2\,dx = \int x^2 - 10x + 25dx = \frac{1}{3}x^3 - 5x^2 + 25x + C$

c.) $\int \frac{x^2 - 7x + 2}{x^2}\,dx = \int 1 - \frac{7}{x} + \frac{2}{x^2}\,dx = x - 7\ln x - \frac{2}{x} + C$

4.1 Antidifferentiation

Example 5: Find the function f such that

$$f'(x) = x^2 \text{ and } f(-1) = 2.$$

First find $f(x)$ by integrating.

$$f(x) = \int x^2 dx$$

$$f(x) = \frac{x^3}{3} + C$$

4.1 Antidifferentiation

Example 5 (concluded):

Then, the initial condition allows us to find C.

$$f(-1) = \frac{(-1)^3}{3} + C = 2$$

$$-\frac{1}{3} + C = 2$$

$$C = \frac{7}{3}$$

Thus, $f(x) = \dfrac{x^3}{3} + \dfrac{7}{3}$.

4.1 Antidifferentiation

Section Summary

- The *antiderivative* of a function $f(x)$ is a set of functions
$$\frac{d}{dx}\left[F(x)+C\right]=f(x),$$
where the constant C is called the *constant of integration*.
- An antiderivative is denoted by an *indefinite integral* using the integral sign, $\int$. If $F(x)$ is an antiderivative of $f(x)$ we write
$$\int f(x)\,dx=F(x)+C.$$
We check the correctness of an antiderivative we have found by differentiating it.

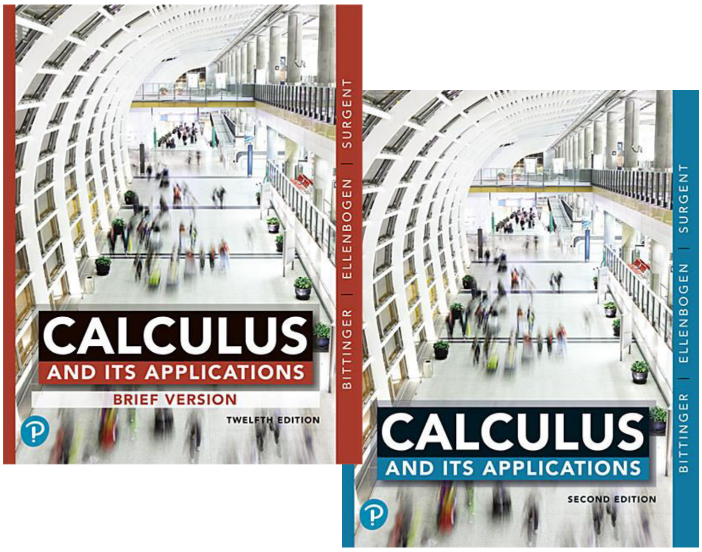
4.1 Antidifferentiation

Section Summary Continued

- The *Constant Rule of Antidifferentiation* is $\int k\cdot dx=kx+C.$
- The *Power Rule of Antidifferentiation* is
$$\int x^n\,dx=\frac{1}{n+1}x^{n+1}+C,\text{ for } n\neq -1.$$
- The *Natural Logarithm Rule of Antidifferentiation* is
$$\int \frac{1}{x}\,dx=\ln x+C,\ \text{ for } x>0.$$
- The *Exponential Rule* (base *e*) *of Antidifferentiation* is
$$\int e^{ax}=\frac{1}{a}e^{ax}+C,\text{ for } a\neq 0.$$
- An *initial condition* is an ordered pair that is a solution of a particular antiderivative of an integrand.

Chapter 4

Integration



4.2 Antiderivatives as Areas

OBJECTIVE

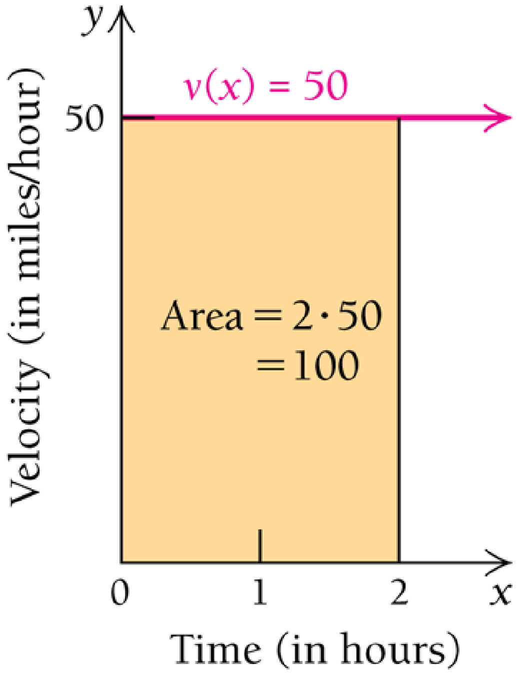
- Find the area under a graph and use it to solve real-world problems
- Use rectangles to approximate the area under a graph.

4.2 Antiderivatives as Areas

Example 1: A vehicle travels at 50 mi/hr for 2 hr. How far has the vehicle traveled?

The answer is 100 mi. We treat the vehicle's velocity as a function, $v(x) = 50.$ We graph this function, sketch a vertical line at $x = 2,$ and obtain a rectangle. This rectangle measures 2 units horizontally and 50 units vertically. Its area is the distance the vehicle has traveled:

$$2 \text{ hr} \cdot \frac{50 \text{ mi}}{1 \text{ hr}} = 100 \text{ mi}.$$



4.2 Antiderivatives as Areas

Example 2: The velocity of a moving object is given by the function $v(x) = 3x,$ where x is in hours and v is in miles per hour. Use geometry to find the area under the graph, which is the distance the object has traveled:

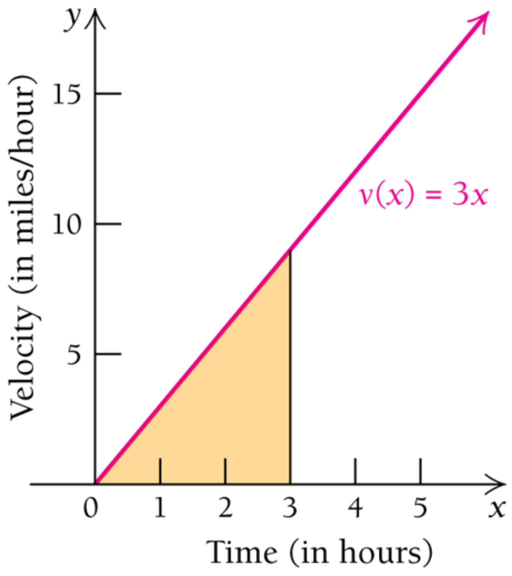
a.) during the first 3 hr $(0 \le x \le 3);$

b.) between the third hour and the fifth hour $(3 \le x \le 5).$

4.2 Antiderivatives as Areas

Example 2 (continued):

a.) The graph of the velocity function is shown at the right. We see the region corresponding to the time interval $0 \le x \le 3$ is a triangle with base 3 and height 8 (since $v(3) = 9$). Therefore, the area of this region is



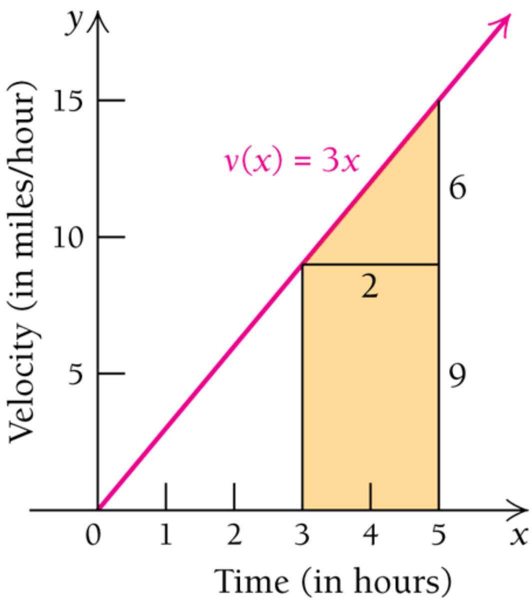
$$A = \frac{1}{2}(3)(9) = \frac{27}{2} = 13.5.$$

The object traveled 13.5 mi during the first 3 hr.

4.2 Antiderivatives as Areas

Example 2 (Continued):

b.) The region corresponding to the time interval $3 \le x \le 5$ is a trapezoid. It can be decomposed into a rectangle and a triangle as indicated in the figure to the right. The rectangle has a base 2 and height 9, and thus an area $A = (2)(9) = 18.$



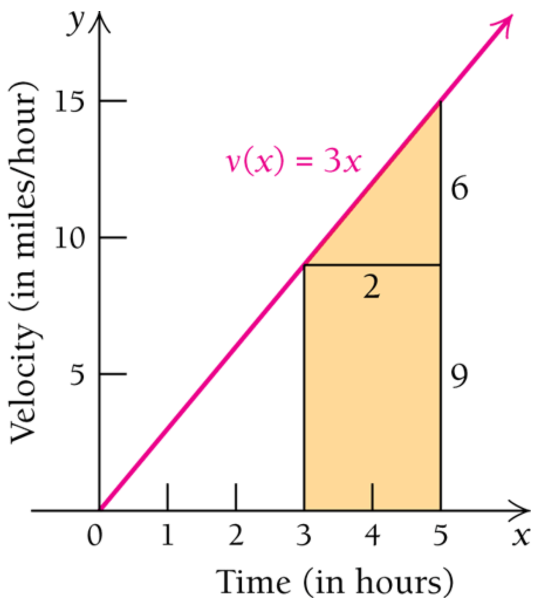
4.2 Antiderivatives as Areas

Example 2 (Concluded):

b.) The triangle has base 2 and height 6, for an area

$$A = \frac{1}{2}(2)(6) = 6.$$

Summing the two areas, we get 24. Therefore, the object traveled 24 mi between the third hour and the fifth hour.



4.2 Antiderivatives as Areas

Quick Check 1

An object moves with a velocity of $v(t) = \frac{1}{2}t$, where t is in minutes and v is in feet per minute.

a.) How far does the object travel during the first 30 min?

b.) How far does the object travel between the first hour and the second hour?

a.) We know that this is a linear function, so the region corresponding to the time interval $(0 \le t \le 30)$ is a triangle with base 30 and height 15 (since $v(30) = 15$). Therefore the area of this region is $A = \frac{1}{2}(30)(15) = 225$. The object traveled 225 feet in the first 30 minutes.

4.2 Antiderivatives as Areas

Quick Check 1 Concluded

b.) How far does the object travel between the first hour and the second hour?

The region corresponding to the time interval $60 \le t \le 120$ is a trapezoid. It can be decomposed into a rectangle and a triangle. The rectangle has base 60 and height 30 (since $v(60) = 30$). The triangle is has base 60 and height 30 (since $v(120) - v(60) = 60 - 30 = 30$). Therefore the area of this region is $A = (60)(30) + \frac{1}{2}(60)(30) = 2700$. Thus the object traveled 2,700 feet between the first hour and second hour.

4.2 Antiderivatives as Areas

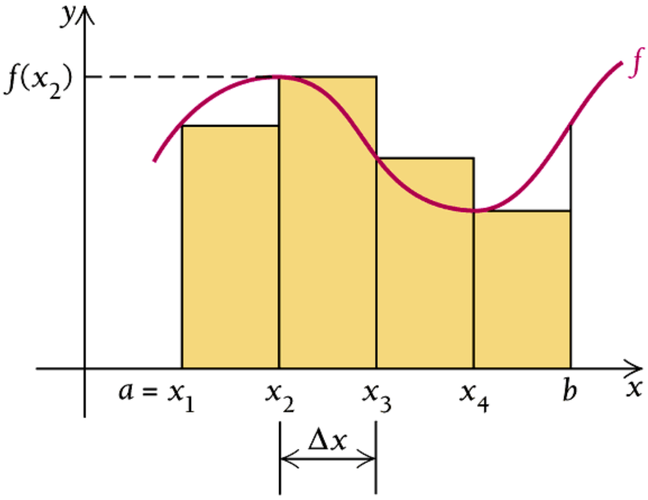
Riemann Sums:

The last two examples, the area function is an antiderivative of the function that generated the graph. Is this always true? Is the formula for the area under the graph of any function the antiderivative of that function? How do we handled curved graphs for which area formulas may not be known? We investigate the questions using geometry, in a procedure called **Riemann summation**.

4.2 Antiderivatives as Areas

Riemann Sums (continued):

In the following figure, $[a, b]$ is divided into four subintervals, each having width $\Delta x = (b - a)/4$.



The heights of the rectangles are $f(x_1)$, $f(x_2)$, $f(x_3)$ and $f(x_4)$.

4.2 Antiderivatives as Areas

Riemann Sums (concluded):

The area of the region under the curve is approximately the sum of the areas of the four rectangles:

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x.$$

We can denote this sum with summation, or sigma, notation, which uses the Greek capital letter sigma, or Σ:

$$\sum_{i=1}^{4} f(x_i)\Delta x, \text{ or } \textstyle\sum_{i=1}^{4} f(x_i)\Delta x.$$

This is read "the sum of the product $f(x_i)\Delta x$ from $i = 1$ to $i = 4$." To recover the original expression, we substitute the numbers 1 through 4 successively for i in $f(x_i)\Delta x$ and write plus signs between the results.

4.2 Antiderivatives as Areas

Example 3: Write summation notation for

$$2 + 4 + 6 + 8 + 10.$$

Note that we are adding consecutive values of 2.

$$2+4+6+8+10=\sum_{i=1}^{5} 2i$$

4.2 Antiderivatives as Areas

Quick Check 2

Write the summation notation for each expression

a.) $5+10+15+20+25$

Note that we are adding consecutive multiples of 5. Thus,

$$5+10+15+20+25 = \sum_{i=1}^{5} 5i$$

b.) $33+44+55+66$

Note that we are adding consecutive multiples of 11. Thus,

$$33+44+55+66 = \sum_{i=3}^{6} 11i$$

4.2 Antiderivatives as Areas

Example 4: Write summation notation for:

$$g(x_1)\Delta x + g(x_2)\Delta x + \cdots + g(x_{19})\Delta x$$

$$g(x_1)\Delta x + g(x_2)\Delta x + \cdots + g(x_{19})\Delta x = \sum_{i=1}^{19} g(x_i)\Delta x$$

4.2 Antiderivatives as Areas

Example 5: Express $\sum_{i=1}^{4} 3^i$ without using summation notation.

$$\sum_{i=1}^{4} 3^i = 3^1 + 3^2 + 3^3 + 3^4 = 120$$

4.2 Antiderivatives as Areas

Quick Check 3

Express $\sum_{i=1}^{6}\left(i^2+i\right)$ without using summation notation.

$$\sum_{i=1}^{6}\left(i^2+i\right)=\left(1^2+1\right)+\left(2^2+2\right)+\left(3^2+3\right)+\left(4^2+4\right)+\left(5^2+5\right)+\left(6^2+6\right)$$

$$=2+6+12+20+30+42$$

$$=112$$

4.2 Antiderivatives as Areas

Example 6: Express $\sum_{i=1}^{30} h(x_i)\Delta x$ without using summation notation.

$$\sum_{i=1}^{30} h(x_i)\Delta x = h(x_1)\Delta x + h(x_2)\Delta x + \cdots + h(x_{30})\Delta x$$

4.2 Antiderivatives as Areas

Example 7: Consider the graph of

$$f(x) = 600x - x^2$$

over the interval [0, 600].

a) Approximate the area by dividing the interval into 6 subintervals.
b) Approximate the area by dividing the interval into 12 subintervals.

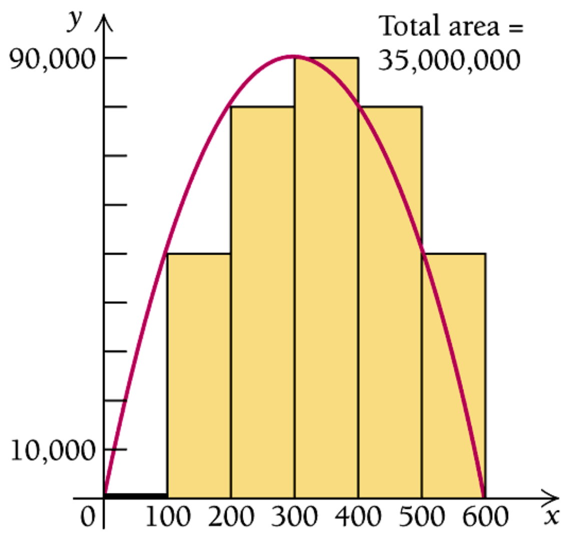
4.2 Antiderivatives as Areas

Example 7 (continued):

a) We divide [0, 600] into 6 intervals of size

$$\Delta x = \frac{600 - 0}{6} = 100,$$

with x_i ranging from $x_1 = 0$ to $x_6 = 500$.



4.2 Antiderivatives as Areas

Example 7 (continued):

Thus, the area under the curve is approximately

$$\sum_{i=1}^{6} f(x_i)\Delta x = f(0)\cdot 100 + f(100)\cdot 100 + f(200)\cdot 100$$
$$+ f(300)\cdot 100 + f(400)\cdot 100 + f(500)\cdot 100$$
$$= 0\cdot 100 + 50{,}000\cdot 100 + 80{,}000\cdot 100$$
$$+ 90{,}000\cdot 100 + 80{,}000\cdot 100 + 50{,}000\cdot 100$$
$$= 35{,}000{,}000$$

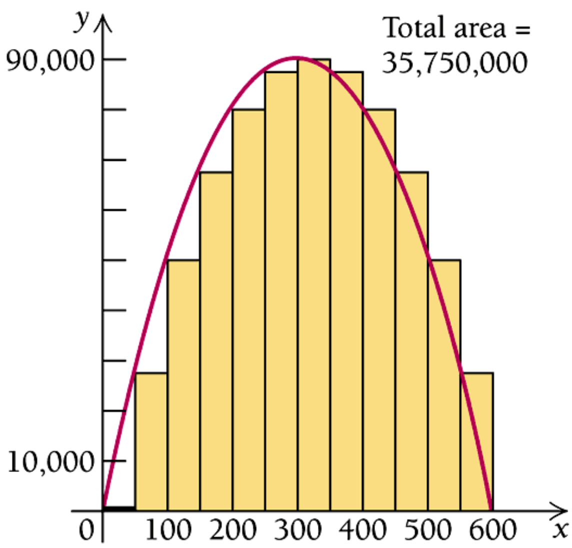
4.2 Antiderivatives as Areas

Example 7 (continued):

b) We divide [0, 600] into 12 intervals of size

$$\Delta x = \frac{600-0}{12} = 50,$$

with x_i ranging from $x_1 = 0$ to $x_{12} = 550$.



4.2 Antiderivatives as Areas

Example 7 (concluded):

Thus, the area under the curve is approximately

$$\sum_{i=1}^{12} f(x_i)\Delta x = f(0)\cdot 50 + f(50)\cdot 50 + f(100)\cdot 50 + f(150)\cdot 50$$
$$+ f(200)\cdot 50 + f(250)\cdot 50 + f(300)\cdot 50 + f(350)\cdot 50$$
$$+ f(400)\cdot 50 + f(450)\cdot 50 + f(500)\cdot 50 + f(550)\cdot 50$$
$$= 0\cdot 50 + 27{,}500\cdot 50 + 50{,}000\cdot 50 + 67{,}500\cdot 50$$
$$+80{,}000\cdot 50 + 87{,}500\cdot 50 + 90{,}000\cdot 50 + 87{,}500\cdot 50$$
$$+80{,}000\cdot 50 + 67{,}500\cdot 50 + 50{,}000\cdot 50 + 27{,}500\cdot 50$$
$$= 35{,}750{,}000$$

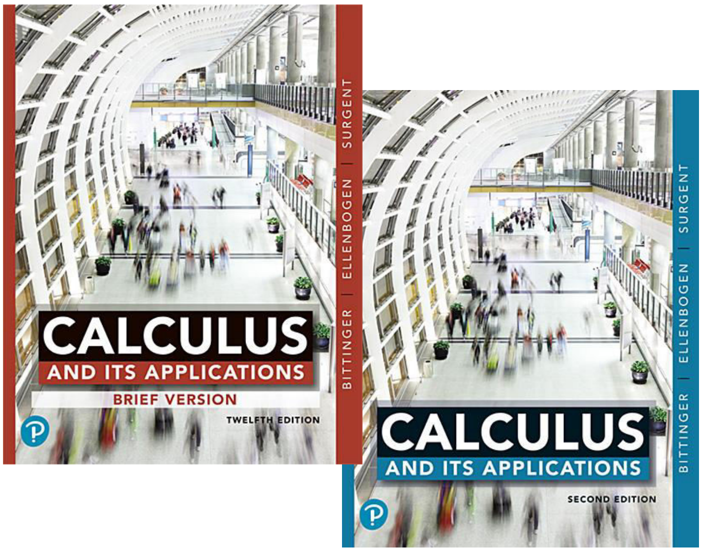
4.2 Antiderivatives as Areas

Section Summary

- The area under a curve can often be interpreted in a meaningful way.
- The units of the area are found by multiplying the units of the input variable by the units of the output variable. It is crucial that the units are consistent.
- Geometry can be used to find areas of regions formed by graphs of linear functions.
- A *Riemann sum* uses rectangles to approximate the area under a curve. The more rectangles, the better approximation.
- The *definite integral* $\int_a^b f(x)\,dx$, is a representation of the exact area under the graph of a continuous function $y = f(x)$, where $f(x) \geq 0$, over an interval $[a,b]$.

Chapter 4

Integration



4.3 Area and Definite Integrals

OBJECTIVE

- Find the area under the graph of a nonnegative function over a given closed interval.
- Evaluate a definite integral.
- Solve applied problems involving definite integrals.

4.3 Area and Definite Integrals

To find the area under the graph of a nonnegative, continuous function f over the interval $[a, b]$:

1. Find any antiderivative $F(x)$ of $f(x)$. (The simplest is the one for which the constant of integration is 0.)
2. Evaluate $F(x)$ using b and a, and compute $F(b) - F(a)$. The result is the area under the graph over the interval $[a, b]$.

4.3 Area and Definite Integrals

Example 1: Find the area under the graph of $y = x^2 + 1$ over the interval $[-1, 2]$.

1. Find any antiderivative $F(x)$ of $f(x)$. We choose the simplest one.

$$F(x) = \frac{x^3}{3} + x$$

4.3 Area and Definite Integrals

Example 1 (concluded):

2. Substitute 2 and –1, and find the difference $F(2) - F(-1)$.

$$\begin{aligned} F(2)-F(-1) &= \left[\frac{2^3}{3}+2\right]-\left[\frac{(-1)^3}{3}+(-1)\right] \\ &= \frac{8}{3}+2-\left[-\frac{1}{3}-1\right] \\ &= \frac{8}{3}+2+\frac{1}{3}+1 \\ &= 6 \end{aligned}$$

4.3 Area and Definite Integrals

Quick Check 1

Refer to the function in Example 1.

a.) Calculate the area over the interval $[0,5]$.

b.) Calculate the area over the interval $[-2,2]$.

c.) Can you suggest a shortcut for part (b)?

a.) $f(x) = x^2 + 1$, so $F(x) = \dfrac{x^3}{3} + x$.

Substitute 0 and 5, and find the difference $F(5) - F(0)$.

$$F(5)-F(0) = \left(\frac{5^3}{3}+5\right)-\left(\frac{0^3}{3}+0\right) = \frac{125}{3}+5 \quad = 46\frac{2}{3}$$

4.3 Area and Definite Integrals

Quick Check 1 Concluded

b.) Calculate the area over the interval $[-2, 2]$.

$$F(2) - F(-2) = \left(\frac{2^3}{3} + 2\right) - \left(\frac{(-2)^3}{3} + (-2)\right) \quad = 4\frac{2}{3} - \left(-4\frac{2}{3}\right)$$

$$= 9\frac{1}{3}$$

c.) Can you suggest a shortcut for part (b)?

Note that $F(-2) = -F(2)$. Then we have

$$F(2) - F(-2) = F(2) - (-F(2)) = 2F(2).$$

Thus we would integrate from 0 to 2, then double the results.
This is because the graph of *f* is symmetric with the y-axis.

4.3 Area and Definite Integrals

DEFINITION:

Let f be any continuous function over the interval $[a, b]$ and F be any antiderivative of f. Then, the **definite integral** of f from a to b is

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

4.3 Area and Definite Integrals

Example 2: Evaluate $\int_a^b x^2 dx.$

Using the antiderivative $F(x) = x^3/3$, we have

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}.$$

It is convenient to use an intermediate notation:

$$\int_a^b f(x)\ dx = \left[F(x)\right]_a^b = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

4.3 Area and Definite Integrals

Example 3: Evaluate each of the following:

a) $\int_{-1}^{4} (x^2 - x)\ dx;$

b) $\int_0^3 e^x\ dx;$

c) $\int_1^e \left(1 + 2x - \frac{1}{x}\right) dx$ (assume $x > 0$).

4.3 Area and Definite Integrals

Example 3 (continued):

$$\text{a) } \int_{-1}^{4} (x^2 - x)\, dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{4}$$

$$= \left(\frac{4^3}{3} - \frac{4^2}{2}\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2}\right)$$

$$= \frac{64}{3} - \frac{16}{2} - \left(-\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{64}{3} - 8 + \frac{1}{3} + \frac{1}{2} = 14\frac{1}{6}$$

4.3 Area and Definite Integrals

Example 3 (continued):

$$\text{b) } \int_{0}^{3} e^x\, dx = \left[e^x\right]_0^3 = e^3 - e^0$$

$$= e^3 - 1$$

$$\approx 19.086$$

4.3 Area and Definite Integrals

Example 3 (concluded):

c) $$\int_1^e \left(1 + 2x - \frac{1}{x}\right) dx = \left[x + x^2 - \ln x\right]_1^e$$

$$= (e + e^2 - \ln e) - (1 + 1^2 - \ln 1)$$

$$= (e + e^2 - 1) - (1 + 1 - 0)$$

$$= e + e^2 - 1 - 1 - 1$$

$$= e + e^2 - 3$$

$$\approx 7.107$$

4.3 Area and Definite Integrals

Quick Check 2

Evaluate each of the following:

a.) $\int_2^4 \left(2x^3 - 3x\right) dx$

b.) $\int_0^{\ln 4} 2e^x \, dx$

c.) $\int_1^5 \frac{x-1}{x} dx$

4.3 Area and Definite Integrals

Quick Check 2 Continued

a.) $\int_2^4 \left(2x^3 - 3x\right) dx = \left[\frac{1}{2}x^4 - \frac{3}{2}x^2\right]_2^4$

$$= \left(\frac{1}{2}(4)^4 - \frac{3}{2}(4)^2\right) - \left(\frac{1}{2}(2)^4 - \frac{3}{2}(2)^2\right)$$

$$= (128 - 24) - (8 - 6)$$

$$= 102$$

4.3 Area and Definite Integrals

Quick Check 2 Continued

b.) $\int_0^{\ln 4} 2e^x dx = \left[2e^x\right]_0^{\ln 4} = \left(2e^{\ln 4}\right) - \left(2e^0\right) = 2(4) - 2(1) = 6$

c.) $\int_1^5 \frac{x-1}{x} dx = \int_1^5 \left(1 - \frac{1}{x}\right) dx = \left[x - \ln x\right]_1^5 = (5 - \ln 5) - (1 - \ln 1)$

$$= (5 - \ln 5) - (1 - 0) = 4 - \ln 5 \approx 2.39$$

4.3 Area and Definite Integrals

THE FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

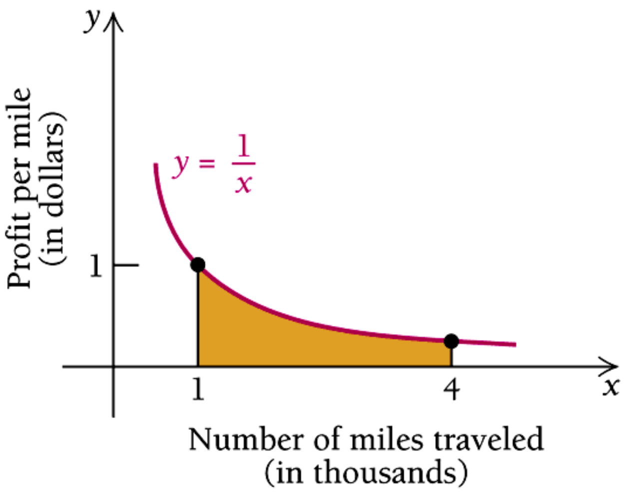
If a continuous function f has an antiderivative F over $[a, b]$, then

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\ dx = F(b) - F(a).$$

4.3 Area and Definite Integrals

Example 4: Suppose that y is the profit per mile traveled and x is number of miles traveled, in thousands. Find the area under $y = 1/x$ over the interval [1, 4] and interpret the significance of this area.

$$\int_1^4 \frac{dx}{x} = \left[\ln x\right]_1^4$$
$$= \ln 4 - \ln 1$$
$$= \ln 4 - 0$$
$$\approx 1.3863$$



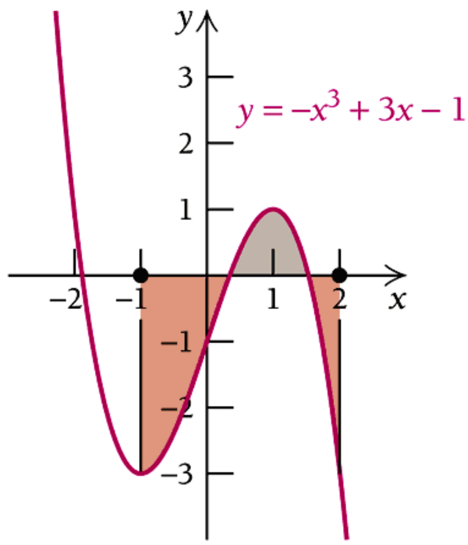
4.3 Area and Definite Integrals

Example 4 (concluded):
The area represents a total profit of \$1386.30 when the miles traveled increase from 1000 to 4000 miles.

4.3 Area and Definite Integrals

Example 5: Consider $\int_{-1}^{2}(-x^3+3x-1)dx$. Predict the sign of the integral by examining the graph, and then evaluate the integral.

From the graph, it appears that there is considerably more area below the *x*-axis than above. Thus, we expect that the sign of the integral will be negative.



4.3 Area and Definite Integrals

Example 5 (concluded):
Evaluating the integral, we have

$$\int_{-1}^{2}\left(-x^3+3x-1\right)dx=\left[-\frac{x^4}{4}+\frac{3}{2}x^2-x\right]_{-1}^{2}$$

$$=\left(-\frac{2^4}{4}+\frac{3}{2}\cdot 2^2-2\right)-\left(-\frac{(-1)^4}{4}+\frac{3}{2}\cdot(-1)^2-(-1)\right)$$

$$=(-4+6-2)-\left(-\frac{1}{4}+\frac{3}{2}+1\right)$$

$$=0-2\frac{1}{4}=-2\frac{1}{4}$$

4.3 Area and Definite Integrals

Example 6: Northeast Airlines determines that the marginal profit resulting from the sale of x seats on a jet traveling from Atlanta to Kansas City, in hundreds of dollars, is given by

$$P'(x)=\sqrt{x}-6.$$

Find the total profit when 60 seats are sold.

4.3 Area and Definite Integrals

Example 6 (continued):

We integrate to find $P(60)$.

$$\begin{aligned} P(60) &= \int_0^{60} P'(x)\,dx \\ &= \int_0^{60} \left(\sqrt{x} - 6\right) dx \\ &= \left[\frac{2}{3}x^{3/2} - 6x\right]_0^{60} \\ &= \left(\frac{2}{3}\cdot 60^{3/2} - 6\cdot 60\right) - \left(\frac{2}{3}\cdot 0^{3/2} - 6\cdot 0\right) \\ &\approx -50.1613 \end{aligned}$$

4.3 Area and Definite Integrals

Example 6 (concluded):

When 60 seats are sold, Northeast's profit is –\$5016.13. That is, the airline will lose \$5016.13 on the flight.

4.3 Area and Definite Integrals

Quick Check 3

Referring to Example 6, find the total profit of Northeast Airlines when 140 seats are sold.

From Example 6, we have $P'(x) = \sqrt{x} - 6.$

We integrate to find $P(140).$

$$P(140) = \int_0^{140} \left(\sqrt{x} - 6\right) dx = \left[\frac{2}{3}x^{3/2} - 6x\right]_0^{140}$$

$$= \left(\frac{2}{3}(140)^{3/2} - 6(140)\right) - \left(\frac{2}{3}(0)^{3/2} - 6(0)\right) \approx 264.3349$$

When 140 seats are sold, Northeast Airlines makes

$$\$100 \cdot 264.3349 = \$26.433.49.$$

4.3 Area and Definite Integrals

Example 7: A particle starts out from some origin. Its velocity, in miles per hour, is given by

$$v(t) = \sqrt{t} + t,$$

where t is the number of hours since the particle left the origin. How far does the particle travel during the second, third, and fourth hours (from $t = 1$ to $t = 4$)?

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4.3 Area and Definite Integrals

Example 7 (continued):

Recall that velocity, or speed, is the rate of change of distance with respect to time. In other words, velocity is the derivative of the distance function, and the distance function is an antiderivative of the velocity function. To find the total distance traveled from $t = 1$ to $t = 4$, we evaluate the integral

$$\int_1^4 \left(\sqrt{t} + t\right)\, dt.$$

4.3 Area and Definite Integrals

Example 7 (concluded):

$$\begin{aligned}
\int_1^4 \left(\sqrt{t} + t\right) dt &= \int_1^4 \left(t^{1/2} + t\right) dt \\
&= \left[\frac{2}{3}t^{3/2} + \frac{1}{2}t^2\right]_1^4 \\
&= \frac{2}{3}\cdot 4^{3/2} + \frac{1}{2}\cdot 4^2 - \left(\frac{2}{3}\cdot 1^{3/2} + \frac{1}{2}\cdot 1^2\right) \\
&= \frac{16}{3} + \frac{16}{2} - \frac{2}{3} - \frac{1}{2} = \frac{14}{3} + \frac{15}{2} = \frac{73}{6} \\
&= 12\frac{1}{6} \text{ mi.}
\end{aligned}$$

4.3 Area and Definite Integrals

Section Summary

- The exact area between the x-axis and the graph of the nonnegative continuous function $y = f(x)$ over the interval $[a,b]$ is found by evaluating the *definite integral*

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is an antiderivative of f.

- If a function has areas both below and above the x-axis, the definite integral gives the net total area, or the difference between the sum of the areas above the x-axis and the sum of the areas below the x-axis.

4.3 Area and Definite Integrals

Section Summary Concluded

- If there is more area above the x-axis than below, the definite integral will be positive.
- If there is more area below the x-axis than above, the definite integral will be negative.
- If the areas above and below the x-axis are the same, the definite integral will be 0.