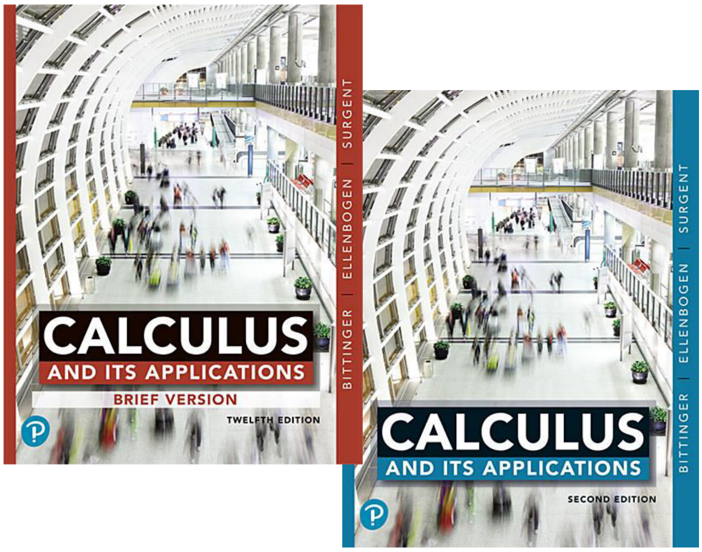
Chapter 6

Functions of Several Variables



6.3 Maximum-Minimum Problems

OBJECTIVE

- Find relative extrema of a function of two variables.
- Solve applied problems by finding the minimum or maximum value for a function of two variables.

6.3 Maximum-Minimum Problems

DEFINITION:

A function f of two variables:

1. has a **relative maximum** at (a, b) if
$$f(x, y) \le f(a, b)$$
for all points (x, y) in a region containing (a, b).

2. has a **relative minimum** at (a, b) if
$$f(x, y) \ge f(a, b)$$
for all points (x, y) in a region containing (a, b).

6.3 Maximum-Minimum Problems

THEOREM 1: The *D*-Test

If f is a differentiable function of x and y, to find the relative maximum and minimum values of f:

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}.
2. Solve the system of equations $f_x = 0$, $f_y = 0$. Let (a, b) represent a solution.
3. Evaluate D, where $D = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

6.3 Maximum-Minimum Problems

THEOREM 1 (concluded):

4. Then:

a) f has a maximum at (a, b) if $D > 0$ and $f_{xx}(a, b) < 0$.

b) f has a minimum at (a, b) if $D > 0$ and $f_{xx}(a, b) > 0$.

c) f has neither a maximum nor a minimum at (a, b) if $D < 0$. The function has a saddle point at (a, b).

d) This test is not applicable if $D = 0$.

6.3 Maximum-Minimum Problems

Example 1: Find the relative maximum or minimum values of $f(x,y) = x^2 + xy + y^2 - 3x$.

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}.

$$f_x = 2x + y - 3 \qquad f_y = x + 2y$$

$$f_{xx} = 2 \qquad f_{yy} = 2$$

$$f_{xy} = 1$$

6.3 Maximum-Minimum Problems

Example 1 (continued):

2. Solve the system of equations $f_x = 0$ and $f_y = 0$.

$$2x + y - 3 = 0 \qquad x + 2y = 0$$
$$x = -2y$$

Using substitution,

$$\begin{aligned} 2(-2y) + y - 3 &= 0 \\ -4y + y - 3 &= 0 \\ -3y &= 3 \\ y &= -1. \end{aligned}$$

6.3 Maximum-Minimum Problems

Example 1 (continued):

Then, substituting back,

$$\begin{aligned} x &= -2(-1) \\ x &= 2. \end{aligned}$$

Thus, (2, –1) is the only critical point.

3. Find D.

$$\begin{aligned} D &= f_{xx}(2,-1) \cdot f_{yy}(2,-1) - [f_{xy}(2,-1)]^2 \\ &= 2 \cdot 2 - [1]^2 \\ &= 3. \end{aligned}$$

6.3 Maximum-Minimum Problems

Example 1 (concluded):

4. Now $D = 3$ and $f_{xx}(2, -1) = 2$, so $D > 0$ and $f_{xx}(2, -1) > 0$, if follows from the D-Test that f has a relative minimum at $(2, -1)$. The minimum value is

$$\begin{aligned} f(2,-1) &= 2^2 + 2(-1) + (-1)^2 - 3\cdot 2 \\ &= 4 - 2 + 1 - 6 \\ &= -3 \end{aligned}$$

6.3 Maximum-Minimum Problems

Quick Check 1

Find the relative maximum and minimum values of

$$f(x,y) = x^2 + xy + 2y^2 - 7x.$$

1. Find $f_x, f_y, f_{xx}, f_{yy},$ and f_{xy} :

$f_x = 2x + y - 7$ $\quad f_y = x + 4y$ $\quad f_{xx} = 2$ $\quad f_{yy} = 4$ $\quad f_{xy} = 1$

2. Solve the system of equations $f_x = 0, f_y = 0$:

$$2x + y - 7 = 0,\ x + 4y = 0 \xrightarrow{\text{Use Substitution}} x = -4y$$

$$\begin{aligned} 2(-4y) + y - 7 &= 0 \\ -7y &= 7 \\ y &= -1 \end{aligned} \xrightarrow{\text{Substitute back}} \begin{aligned} x &= -4(-1) \\ x &= 4 \end{aligned}$$

Thus $(4, -1)$ is the only critical point.

6.3 Maximum-Minimum Problems

Quick Check 1 Concluded

3. Find D.

$$D = f_{xx}(4,-1)\cdot f_{yy}(4,-1) - \left[f_{xy}(4,-1)\right]^2$$
$$D = 2\cdot 4 - (1)^2$$
$$D = 7$$

4. Since $D = 7$ and $f_{xx} = 2,$ and since $D > 0$ and $f_{xx} > 0,$ it follows from the D-Test that f has a relative minimum at $(4,-1)$. The minimum value is

$$f(4,-1) = (4)^2 + 4(-1) + 2(-1)^2 - 7(4)$$
$$= 16 - 4 + 2 - 28$$
$$= -14.$$

6.3 Maximum-Minimum Problems

Example 2: Find the relative maximum and minimum values of $f(x,y) = xy - x^3 - y^2$.

1. Find $f_x,\ f_y,\ f_{xx}, f_{yy},$ and f_{xy}.

$$f_x = y - 3x^2 \qquad f_y = x - 2y$$
$$f_{xx} = -6x \qquad f_{yy} = -2$$
$$f_{xy} = 1$$

6.3 Maximum-Minimum Problems

Example 2 (continued):

2. Solve the system of equations $f_x = 0$ and $f_y = 0$.

$$y - 3x^2 = 0 \qquad x - 2y = 0$$

$$y = 3x^2$$

Using substitution,

$$\begin{aligned} x - 2(3x^2) &= 0 \\ x - 6x^2 &= 0 \\ x(1 - 6x) &= 0 \end{aligned}$$

$$x = 0 \quad \text{or} \quad x = \frac{1}{6}.$$

6.3 Maximum-Minimum Problems

Example 2 (continued):

Then, substitute back to find y for both values of x.

$$y = 3(0)^2 \quad \text{and} \quad y = 3\left(\frac{1}{6}\right)^2$$

$$y = 0 \quad \text{and} \quad y = \frac{1}{12}$$

Thus, (0,0) and (1/6, 1/12) are two critical points.

6.3 Maximum-Minimum Problems

Example 2 (continued):

3. Find D for both critical points. First, for (0,0)

$$\begin{aligned} D &= f_{xx}(0,0)\cdot f_{yy}(0,0)-[f_{xy}(0,0)]^2 \\ &= (-6\cdot 0)\cdot(-2)-[1]^2 \\ &= -1. \end{aligned}$$

Then, for (1/6, 1/12)

$$\begin{aligned} D &= f_{xx}\left(\frac{1}{6},\frac{1}{12}\right)\cdot f_{yy}\left(\frac{1}{6},\frac{1}{12}\right)-\left[f_{xy}\left(\frac{1}{6},\frac{1}{12}\right)\right]^2 \\ &= \left(-6\cdot\frac{1}{6}\right)\cdot(-2)-[1]^2 \\ &= 1. \end{aligned}$$

6.3 Maximum-Minimum Problems

Example 2 (concluded):

4. For (0,0), $D < 0$, so (0,0) is neither a maximum nor a minimum value but a saddle point.

For (1/6, 1/12), $D > 0$ and $f_{xx}(1/6, 1/12) < 0$.
Therefore, f has a relative maximum at (1/6, 1/12).

$$\begin{aligned} f\left(\frac{1}{6},\frac{1}{12}\right) &= \frac{1}{6}\cdot\frac{1}{12}-\left(\frac{1}{6}\right)^3-\left(\frac{1}{12}\right)^2 \\ &= \frac{1}{72}-\frac{1}{216}-\frac{1}{144} \\ &= \frac{1}{432} \end{aligned}$$

6.3 Maximum-Minimum Problems

Quick Check 2

Find the critical points of $g(x,y)=x^3+y^2-3x-4y+3$.
Then use the D-test to classify each point as a relative maximum, a relative minimum, or a saddle point.

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}.

$$f_x = 3x^2 - 3 \qquad f_y = 2y - 4$$

$$f_{xx} = 6x \qquad f_{yy} = 2$$

$$f_{xy} = 0$$

6.3 Maximum-Minimum Problems

Quick Check 2 Continued

2. Solve for the system of equations $f_x = 0, f_y = 0$:

$$3x^2 - 3 = 0 \qquad 2y - 4 = 0$$

$$3x^2 = 3 \qquad 2y = 4$$

$$x^2 = 1$$

$$x = \pm 1 \qquad y = 2$$

Thus, there are critical points at $(1,2)$ and $(-1,2)$.

6.3 Maximum-Minimum Problems

Quick Check 2 Continued

3. Find D for both critical points. First for $(1,2)$.

$$D = f_{xx}(1,2)\cdot f_{yy}(1,2) - \left[f_{xy}(1,2)\right]^2$$
$$D = 6\cdot 2 - (0)^2$$
$$D = 12$$

Then, for $(-1,2)$.

$$D = f_{xx}(-1,2)\cdot f_{yy}(-1,2) - \left[f_{xy}(-1,2)\right]^2$$
$$D = (-6)\cdot 2 - (0)^2$$
$$D = -12$$

6.3 Maximum-Minimum Problems

Quick Check 2 Concluded

4. For $(-1,2), D < 0,$ and $f_{xx}(-1,2) = -6 < 0$ so $(-1,2)$ is neither a minimum nor a maximum value, but is a saddle point.

For $(1,2), D > 0,$ and $f_{xx}(1,2) = 6 > 0.$

Therefore there is a relative minimum at $(1,2)$.The relative minimum is

$$\begin{aligned} f(1,2) &= 1^3 + 2^2 - 3(1) - 4(2) + 3 \\ &= 1 + 4 - 3 - 8 + 3 \\ &= -3. \end{aligned}$$

6.3 Maximum-Minimum Problems

Example 3: A firm produces two kinds of golf balls, one that sells for \$3 and one priced at \$2. The total revenue, in thousands of dollars, from the sale of *x* thousand balls at \$3 each and *y* thousand at \$2 each is given by

$$R(x, y) = 3x + 2y.$$

The company determines that the total cost, in thousands of dollars, of producing *x* thousand of the \$3 ball and *y* thousand of the \$2 ball is given by

$$C(x, y) = 2x^2 - 2xy + y^2 - 9x + 6y + 7.$$

6.3 Maximum-Minimum Problems

Example 3 (continued):
How many balls of each type must be produced and sold in order to maximize profit?

The total profit function $P(x,y)$ is given by

$$P(x, y) = R(x, y) - C(x, y)$$
$$P(x, y) = 3x + 2y - (2x^2 - 2xy + y^2 - 9x + 6y + 7)$$
$$P(x, y) = -2x^2 + 2xy - y^2 + 12x - 4y - 7.$$

6.3 Maximum-Minimum Problems

Example 3 (continued):

1. Find P_x, P_y, P_{xx}, P_{yy}, and P_{xy}.

$$P_x = -4x + 2y + 12 \qquad P_y = 2x - 2y - 4$$

$$P_{xx} = -4 \qquad P_{yy} = -2$$

$$P_{xy} = 2$$

2. Solve the system of equations $P_x = 0$ and $P_y = 0$.

$$\begin{aligned} -4x + 2y + 12 &= 0 \\ 2x - 2y - 4 &= 0 \end{aligned}$$

6.3 Maximum-Minimum Problems

Example 3 (continued):

Adding these two equations, we get $-2x + 8 = 0$.

Then,

$$\begin{aligned} -2x &= -8 \\ x &= 4. \end{aligned}$$

Now, substitute back into $P_x = 0$ or $P_y = 0$ to find y.

$$\begin{aligned} 2 \cdot 4 - 2y - 4 &= 0 \\ -2y + 4 &= 0 \\ -2y &= -4 \\ y &= 2 \end{aligned}$$

6.3 Maximum-Minimum Problems

Example 3 (continued):

Thus, (4, 2) is the only critical point.

3. Find D.

$$D = P_{xx}(4,2) \cdot P_{yy}(4,2) - [P_{xy}(4,2)]^2$$

$$D = (-4) \cdot (-2) - [2]^2$$

$$D = 4$$

6.3 Maximum-Minimum Problems

Example 3 (continued):

Thus, since $D > 0$ and $P_{xx}(4,2) < 0$, it follows that P has a relative maximum at (4,2). So in order to maximize profit, the company must produce and sell 4 thousand of the \$3 golf ball and 2 thousand of the \$2 golf ball. The maximum profit will be

$$P(4,2) = -2 \cdot 4^2 + 2 \cdot 4 \cdot 2 - 2^2 + 12 \cdot 4 - 4 \cdot 2 - 7$$

$$P(4,2) = 13 \text{ or } \$13 \text{ thousand.}$$

6.3 Maximum-Minimum Problems

Quick Check 3

Repeat Example 3 using the same cost function and assuming that the company's total revenue, in thousands of dollars, comes from the sale of x thousand balls at \$3.50 each and y thousand at \$3.75 each.

The total profit function $P(x,y)$ is given by

$$P(x,y) = R(x,y) - C(x,y)$$
$$P(x,y) = 3.50x + 2.75y - \left(2x^2 - 2xy + y^2 - 9x + 6y + 7\right)$$
$$P(x,y) = -2x^2 - y^2 + 2xy + 12.50x - 3.25y - 7.$$

6.3 Maximum-Minimum Problems

Quick Check 3 Continued

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}.

$$f_x = -4x + 2y + 12.50 \qquad f_y = -2y + 2x - 3.25$$
$$f_{xx} = -4 \qquad f_{yy} = -2$$
$$f_{xy} = 2$$

2. Solve the system of equations $f_x = 0, f_y = 0$.

$$-4x + 2y + 12.50 = 0 \qquad -2y + 2x - 3.25 = 0$$

Add the two equations together and you get $-2x + 8.75 = 0, x = 4.625.$

Substitute this into one of the equations and you get

$$-2y + 9.25 - 3.25 = 0,\ y = 3.$$

Thus the only critical value is at $(4.625, 3)$.

6.3 Maximum-Minimum Problems

Quick Check 3 Concluded

3. Find D.

$$D = f_{xx}(4.625,3)\cdot f_{yy}(4.625,3) - \left[f_{xy}(4.625,3)\right]^2$$
$$D = (-4)\cdot(-2) - (2)^2$$
$$D = 4$$

4. Thus, since $D > 0$ and $P_{xx}(4.625,3) < 0$, P has a relative maximum at $(4.625,3)$. The maximum is

$$P(4.625,3) = -2(4.625)^2 - 3^2 + 2(4.625)(3) + 12.50(4.625) - 3.25(3) - 7$$
$$= -42.78125 - 9 + 27.75 + 57.8125 - 9.75 - 7$$
$$= 17.03125.$$

Thus, the maximum profit is \$17.031 thousand when x = \$4.625 thousand and y = \$3 thousand.

6.3 Maximum-Minimum Problems

Example 4: Optimization with Constraints. Express Shipping requires that the sum of the length, width, and height of a large rectangular shipping box be 90 in. What is the maximum volume of such a box?

Solution: We let x, y, and z be, respectively, the length, width, and height of the box, in inches. Thus, the volume of the box is given by the function of three variables

$$V(x,y,z) = xyz.$$

6.3 Maximum-Minimum Problems

Example 4 continued:
The sum of the box's length, width, and height is 90 in. This is a constraint:

$$x + y + z = 90$$

Solving for z, we get:

$$z = 90 - x - y$$

Substituting, we now treat V as a function of two variables x and y:

$$V(x, y, 90 - x - y) = xy(90 - x - y)$$
$$= 90xy - x^2y - xy^2$$

6.3 Maximum-Minimum Problems

Example 4 continued:
To find possible critical points, we find the first- and second-order partial derivatives:

$$V_x = 90y - 2xy - y^2 \qquad V_y = 90x - x^2 - 2xy$$

$$V_{xx} = -2y \qquad V_{yy} = -2x$$

$$V_{xy} = 90 - 2x - 2y$$

6.3 Maximum-Minimum Problems

Example 4 continued:

We solve the system $V_x = 0$ and $V_y = 0$:

$$90y - 2xy - y^2 = 0$$
$$90x - x^2 - 2xy = 0$$

Since these are measurements, we can assume that *x, y*, and *z* are all positive. This allows us to simplify these equations as follows:

$$\frac{90y - 2xy - y^2}{y} = \frac{0}{y} \qquad \frac{90x - x^2 - 2xy}{x} = \frac{0}{x}$$
$$90 - 2x - y = 0 \qquad 90 - x - 2y = 0$$

6.3 Maximum-Minimum Problems

Example 4 continued:

$$90 - 2x - y = 0 \quad \to \quad y = 90 - 2x$$
$$90 - x - 2y = 0$$

Substituting for y in $90 - x - 2y = 0$, we get:

$$90 - x - 2(90 - 2x) = 0$$
$$90 - x - 180 + 4x = 0 \qquad \text{Distributing}$$
$$3x - 90 = 0 \qquad \text{Simplifying}$$
$$3x = 90$$
$$x = 30$$

6.3 Maximum-Minimum Problems

Example 4 continued:

Substituting $x = 30$ into $y = 90 - 2x$, we get:

$$y = 90 - 2(30)$$
$$y = 90 - 60$$
$$y = 30$$

Thus, (30, 30) is a critical point. To determine if this gives a maximum or minimum, we use the *D*-test:

$$D = V_{xx}(30,30) \cdot V_{yy}(30,30) - [V_{xy}(30,30)]^2$$

6.3 Maximum-Minimum Problems

Example 4 continued:

$$D = V_{xx}(30,30) \cdot V_{yy}(30,30) - [V_{xy}(30,30)]^2$$

$$= (-2(30)) \cdot (-2(30)) - [90 - 2(30) - 2(30)]^2 \quad \text{Substituting}$$

$$= (-60) \cdot (-60) - [90 - 60 - 60]^2$$

$$= 3600 - [-30]^2$$

$$= 3600 - 900$$

$$= 2700$$

6.3 Maximum-Minimum Problems

Example 4 concluded:

Since $D > 0$ and $V_{xx} < 0$, we conclude that $V(30,30)$ is a maximum.

$$V(30,30) = 30 \cdot 30[90 - 30 - 30] \qquad \text{Substituting}$$

$$= 30 \cdot 30[30]$$

$$= 27,000\,\text{in}^3.$$

Thus, the box will have a maximum volume of 27,000 cubic inches where each side measures 30 inches.

6.3 Maximum-Minimum Problems

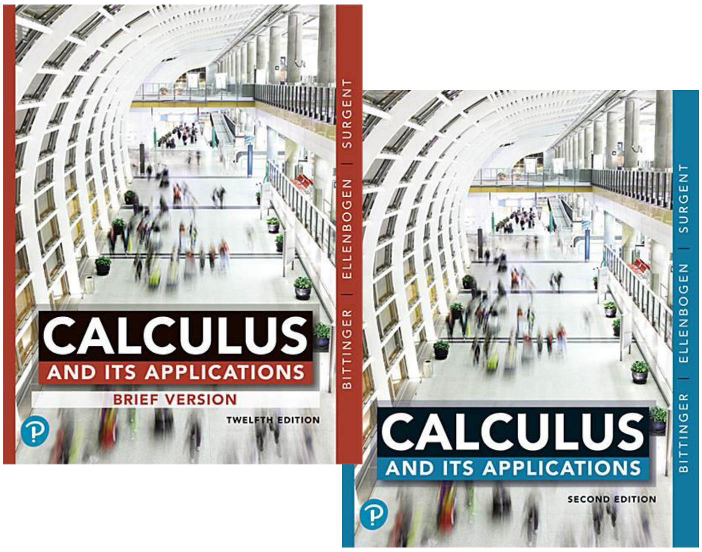
Section Summary

- A two-variable function f has a *relative maximum* at (a,b) if $f(x,y) \le f(a,b)$ for all points in a region containing (a,b) and has a *relative minimum* at (a,b) if $f(x,y) \ge f(a,b)$ for all points in a region containing (a,b).

- The *D-test* is used to classify a *critical point* as a relative minimum, a relative maximum, or a *saddle point*.

Chapter 6

Functions of Several Variables



6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

OBJECTIVE

- Find minimum and maximum values using Lagrange multipliers.
- Solve constrained optimization problems involving Lagrange multipliers.
- Use the Extreme Value Theorem for Two-Variable Functions to find absolute minimum and maximum values.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

The Method of Lagrange Multipliers

To find a maximum or minimum value of a function $f(x, y)$ subject to the constraint $g(x, y) = 0$:

1. Form a new function, called the Lagrange function:

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y).$$

The variable λ (lambda) is called a **Lagrange multiplier**.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

The Method of Lagrange Multipliers (continued)

2. Find the first partial derivatives F_x, F_y, and F_λ.

3. Solve the system

$$F_x = 0, \quad F_y = 0, \quad \text{and} \quad F_\lambda = 0,$$

Let (a, b, λ) represent a solution of this system. We then must determine whether (a, b) yields a maximum or minimum of the function f.

The method of Lagrange multipliers can be extended to functions of three (or more) variables.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 1: Find the maximum value of

$$A(x, y) = xy$$

subject to the constraint $x + y = 20$.

First note that $x + y = 20$ is equivalent to $x + y - 20 = 0$.

1. We form the new function, F, given by

$$F(x, y, \lambda) = xy - \lambda\cdot(x + y - 20).$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 1 (continued):

2. We find the first partial derivatives:

$$F_x = y - \lambda$$
$$F_y = x - \lambda$$
$$F_\lambda = -(x + y - 20)$$

3. We set each derivative equal to 0 and solve the resulting system:

$$\begin{aligned} y - \lambda &= 0 \\ x - \lambda &= 0 \\ -(x + y - 20) &= 0 \end{aligned}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 1 (concluded):

From the first two equations, we can see that $x = \lambda = y$. Substituting x for y in the last equation, we get

$$\begin{aligned} x + x - 20 &= 0 \\ 2x &= 20 \\ x &= 10 \end{aligned}$$

Thus, $y = x = 10$. The maximum value of A subject to the constraint occurs at (10, 10) and is

$$\begin{aligned} A(10,10) &= 10 \cdot 10 \\ &= 100 \end{aligned}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Quick Check 1

Find the maximum value of $A(x, y) = xy$ subject to the constraint $x + 2y = 30$.

First note $x + 2y = 30$ is equivalent to $x + 2y - 30 = 0$.

1. We form the new equation, F, given by

$$F(x, y, \lambda) = xy - \lambda(x + 2y - 30).$$

2. We find the first partial derivatives:

$$\begin{aligned} F_x &= y - \lambda \\ F_y &= x - 2\lambda \\ F_\lambda &= -(x + 2y - 30) \end{aligned}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Quick Check 1 Concluded

3. We set each derivative equal to 0 and solve the resulting system of equations.

$$x - 2\lambda = 0$$

$$y - \lambda = 0$$

$$-(x + 2y - 30) = 0$$

From the first two equations we can see that $x = 2\lambda$ and $y = \lambda$.

Thus $x = 2y$. Substituting this into the third equation we get

$$-(x + x - 30) = 0$$
$$x = 15.$$

Substituting x and solving for y we get that $y = 7.5$.

Thus the maximum is $A(15, 7.5) = 15(7.5) = 112.5$.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 2: Find the maximum value of

$$f(x, y) = 3xy$$

subject to the constraint $2x + y = 8$.

Note that we rewrite $2x + y = 8$ as $2x + y - 8 = 0$.

1. We form the new function, F, given by

$$F(x, y, \lambda) = 3xy - \lambda(2x + y - 8).$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 2 (continued):

2. We find the first partial derivatives:

$$F_x = 3y - 2\lambda$$
$$F_y = 3x - \lambda$$
$$F_\lambda = -(2x + y - 8)$$

3. We set each derivative equal to 0 and solve the resulting system:

$$\begin{aligned} 3y - 2\lambda &= 0 \\ 3x - \lambda &= 0 \\ -(2x + y - 8) &= 0 \end{aligned}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 2 (continued):

Solving the second equation for λ, we get

$$\lambda = 3x.$$

Substituting this into the first equation gives

$$\begin{aligned} 3y - 2\cdot 3x &= 0 \\ 3y &= 6x \\ y &= 2x. \end{aligned}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 2 (concluded):

Substituting $y = 2x$ into the third equation, we get

$$\begin{aligned} 2x + 2x - 8 &= 0 \\ 4x &= 8 \\ x &= 2. \end{aligned}$$

Then $y = 2 \cdot 2 = 4$, and the maximum value of f subject to the constraint occurs at (2, 4) and is

$$\begin{aligned} f(2,4) &= 3 \cdot 2 \cdot 4 \\ &= 24. \end{aligned}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Quick Check 2

Find the minimum value of $g(x,y) = x^2 + y^2$ subject to the constraint $3x - y = 1$.

First note that $3x - y = 1$ is equivalent to $3x - y - 1 = 0$.

1. Form a new function, G, given by

$$G(x,y,\lambda) = \left(x^2 + y^2\right) - \lambda\left(3x - y - 1\right).$$

2. Find the first partial derivatives

$$\begin{aligned} G_x &= 2x - 3\lambda \\ G_y &= 2y + \lambda \\ G_\lambda &= -\left(3x - y - 1\right) \end{aligned}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Quick Check 2 Continued

3. Set the partial derivatives equal to 0 and solve.

$$2x-3\lambda=0$$
$$2y+\lambda=0$$
$$-(3x-y-1)=0$$

Using the first two equations we get $\frac{2}{3}x=\lambda=-2y$. Thus $x=-3y$.

Substitute x into the third equation, and we get

$$-(3(-3y)-y-1)=0$$
$$-(-9y-y-1)=0$$
$$10y=-1$$
$$y=-\frac{1}{10}.$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Quick Check 2 Concluded

Substituting y back into the first equation, we get

$$x=-3\left(-\frac{1}{10}\right)$$
$$x=\frac{3}{10}.$$

Thus a minimum occurs when $x=\frac{3}{10}$ and $y=-\frac{1}{10}$. The minimum is

$$g\left(\frac{3}{10},-\frac{1}{10}\right)=\left(\frac{3}{10}\right)^2+\left(-\frac{1}{10}\right)^2=\frac{9}{100}+\frac{1}{100}=\frac{1}{10}.$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 3: The standard beverage can has a volume of 12 fl. oz, or 21.66 in^3. What dimensions yield the minimum surface area? Find the minimum surface area. (Assume the shape of the can is a right circular cylinder.)

We want to minimize the function s, given by

$$s(h, r) = 2\pi rh + 2\pi r^2$$

subject to the volume constraint $\pi r^2 h = 21.66$ or $\pi r^2 h - 21.66 = 0$.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 3 (continued):

1. We form the new function, S, given by

$$S(h, r, \lambda) = 2\pi rh + 2\pi r^2 - \lambda \cdot (\pi r^2 h - 21.66).$$

2. We find the first partial derivatives:

$$S_h = 2\pi r - \lambda \pi r^2$$
$$S_r = 2\pi h + 4\pi r - 2\lambda \pi rh$$
$$S_\lambda = -(\pi r^2 h - 21.66)$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 3 (continued):

3. We set each derivative equal to 0 and solve the resulting system:

$$\begin{aligned} 2\pi r - \lambda \pi r^2 &= 0 \\ 2\pi h + 4\pi r - 2\lambda \pi r h &= 0 \\ -(\pi r^2 h - 21.66) &= 0 \end{aligned}$$

Since π is a constant, solve the first equation for r.

$$\pi r(2 - \lambda r) = 0$$

$$\pi r = 0 \quad \text{or} \quad 2 - \lambda r = 0$$

$$r = 0 \quad \text{or} \quad r = \frac{2}{\lambda}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 3 (continued):

Since $r = 0$ cannot be a solution to the problem, we will continue by substituting $2/\lambda$ into the second equation.

$$\begin{aligned} 2\pi h + 4\pi \cdot \frac{2}{\lambda} - 2\lambda\pi \cdot \frac{2}{\lambda} \cdot h &= 0 \\ 2\pi h + \frac{8\pi}{\lambda} - 4\pi h &= 0 \\ \frac{8\pi}{\lambda} &= 2\pi h \\ \frac{4}{\lambda} &= h \end{aligned}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 3 (continued):

Since $r = 2/\lambda$ and $h = 4/\lambda$, it follows that $h = 2r$. Substituting $2r$ for h in the third equation yields

$$\pi r^2(2r) - 21.66 = 0$$

$$2\pi r^3 = 21.66$$

$$r^3 = \frac{10.83}{\pi}$$

$$r = \sqrt[3]{\frac{10.83}{\pi}} \approx 1.51 \text{ in}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 3 (concluded):

Thus, when $r = 1.51$ in., we have $h = 3.02$ in, and the surface area is then a minimum and is approximately

$$2\pi(1.51)(3.02) + 2\pi(1.51)^2 \approx 42.98 \text{ in}^2$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Quick Check 3

Repeat Example 3 for a right circular cylinder with a volume of 500 mL. (*Hint:* $1 \text{ mL} = 1 \text{ cm}^3$.)
We want to minimize the function s given by $s(h,r) = 2\pi rh + 2\pi r^2$ subject to the constraint $\pi r^2 h = 500$ or $\pi r^2 h - 500 = 0$.

1. Form the function S given by

$$S(h,r,\lambda) = 2\pi rh + 2\pi r^2 - \lambda(\pi r^2 h - 500).$$

2. Find the first partial derivatives.

$$S_h = 2\pi r - \lambda \pi r^2$$
$$S_r = 2\pi h + 4\pi r - 2\lambda \pi rh$$
$$S_\lambda = -(\pi r^2 h - 500)$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Quick Check 3 Continued

3. Set each derivative equal to 0 and solve the system of equations.

$$2\pi r - \lambda \pi r^2 = 0$$
$$2\pi h + 4\pi r - 2\lambda \pi rh = 0$$
$$-(\pi r^2 h - 500) = 0$$

Since π is a constant, we can solve equation 1 for r.

$$\pi r(2 - \lambda r) = 0$$
$$\pi r = 0 \text{ or } 2 - \lambda r = 0$$
$$r = 0 \text{ or } r = \frac{2}{\lambda}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Quick Check 3 Continued

$r = 0$ cannot be a solution, so substitute $\frac{2}{\lambda}$ for r in equation 2.

$$2\pi h + 4\pi\frac{2}{\lambda} - 2\lambda\pi\frac{2}{\lambda}h = 0$$

$$2\pi h + \frac{8\pi}{\lambda} - 4\pi h = 0$$

$$\frac{8\pi}{\lambda} - 2\pi h = 0$$

$$-2\pi h = -\frac{8\pi}{\lambda}$$

$$h = \frac{4}{\lambda}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Quick Check 3 Concluded

Since $h = \frac{4}{\lambda}$ and $r = \frac{2}{\lambda}$, it follows that $h = 2r$. Substitute $2r$ for h:

$$\pi r^2(2r) - 500 = 0$$

$$2\pi r^3 - 500 = 0$$

$$2\pi r^3 = 500$$

$$r^3 = \frac{250}{\pi}$$

$$r \approx 4.3 \text{ cm}$$

Thus when $r \approx 4.3$ cm, we have $h \approx 8.6$ cm. The surface area is then a minimum and is approximately

$$2\pi(4.3)(8.6) + 2\pi(4.3)^2,$$

or about 348.53 cm^2.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Extreme-Value Theorem for Two-Variable Functions

If $f(x, y)$ is continuous for all (x, y) within a region of feasibility that is closed and bounded, then f has both an absolute maximum value and an absolute minimum value over that region.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Recall that a continuous function f defined on a closed interval $[a, b]$ must have an absolute maximum value and an absolute minimum value over $[a, b]$. In such cases, it is possible that the absolute maximum or minimum value occurs at an endpoint of the interval. Similarly, absolute maximum or minimum values may occur on the boundaries of a region of feasibility.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4:Business: Maximizing Revenue. Kim creates embroidered knit caps; one style has a script x on the front, the other a script y on the front. She sells them to raise funds for the Math Team, and her weekly revenue is modeled by the two-variable function

$$R(x, y) = -x^2 - xy - y^2 + 20x + 22y - 25,$$

where x is the number of x-caps produced and sold and y is the number of y-caps produced and sold. Kim spends 2 hr working on each x-cap, and 4 hr on each y-cap, and she works no more than 40 hr per week. How many of each style should she produce and sell in order to maximize her weekly revenue? Assume $x \geq 0$ and $y \geq 0$.

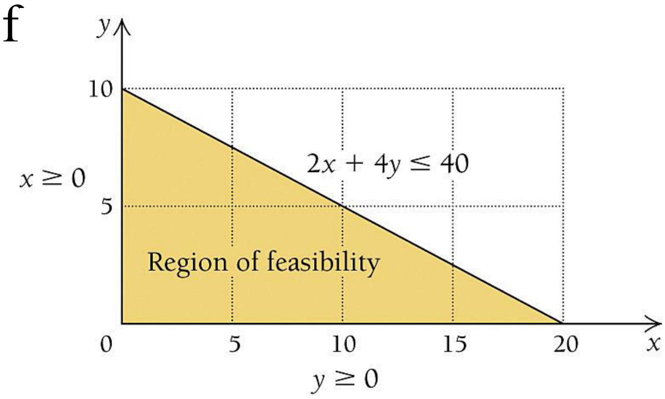
6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

Solution: The number of hours that Kim works is a constraint:

$$2x + 4y \leq 40.$$

We can sketch the region of feasibility, using this constraint along with the fact that $x \geq 0$ and $y \geq 0$.



6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

The region of feasibility is closed and bounded. Since R is continuous for all x and y in this region, by the Extreme-Value Theorem for Two-Variable Functions, R has an absolute maximum value.

We first check for possible critical points within the interior of the region using techniques from Section 6.3. Differentiating R with respect to x and to y, we have:

$$R_x = -2x - y + 20$$

$$R_y = -x - 2y + 22$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

Setting these expressions equal to 0, we solve the system for x and y:

$$-2x - y + 20 = 0 \rightarrow -2x - y = -20 \rightarrow 2x + y = 20$$

$$-x - 2y + 22 = 0 \rightarrow -x - 2y = -22 \rightarrow 2x + 4y = 44$$

Solving this system, we get: $3y = 24$

$$y = 8$$

Back-substituting, we get: $2x + 4(8) = 44 \rightarrow 2x + 32 = 44$

Thus, $2x = 12$

$$x = 6$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

The solution of this system is (6, 8). However, this point lies outside the region of feasibility, since 2(6)+4(8)=44, which is greater than 40. Thus, this point is ignored.

The boundaries must also be checked for possible critical points. Points at which two constraints intersect will also be considered as critical points.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

Along the y-axis, we have $x = 0$ for $0 \le y \le 10$.
To find possible critical points, we substitute $x = 0$ into R:

$$R(0, y) = -y^2 + 22y - 25.$$

The first derivative is $R_y = -2y + 22$. To find critical points, we set $R_y = 0$ and solve:

$$\begin{aligned} -2y + 22 &= 0 \\ -2y &= -22 \\ y &= 11 \end{aligned}$$

But $y = 11$ which is not in the region of feasibility. However, the two endpoints of the interval, (0,0) and (0,10) remain possibilities for critical points.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

Along the x-axis, we have y = 0 for $0 \le x \le 20$.

To find possible critical points, we substitute $y = 0$ into R:

$$R(x,0) = -x^2 + 20x - 25$$

The first derivative is $R_x = -2x + 20$. Setting this equal to zero to try to find critical points, we get:

$$-2x + 20 = 0$$
$$x = 10$$

Since $x = 10$ is in the region of feasibility, the point (10,0) is a critical point.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

The two endpoints of the interval, (0,0) and (20,0) are also considered possible critical points.

Along the line $2x + 4y = 40$, we have $0 \le x \le 20$ and $0 \le y \le 10$.

The endpoints, (0,10) and (20,0) are potential critical points, identified earlier.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

To check for critical points along the line $2x + 4y = 40$, we can use the method of Lagrange multipliers. The constraint is written as $2x + 4y - 40 = 0$, and we have:

$$F(x,y,\lambda) = R(x,y) - \lambda(2x+4y-40)$$
$$= -x^2 - xy - y^2 + 20x + 22y - 25 - 2\lambda x - 4\lambda y + 40\lambda$$

The first derivatives are:

$$F_x = -2x - y + 20 - 2\lambda$$
$$F_y = -x - 2y + 22 - 4\lambda$$
$$F_\lambda = -2x - 4y + 40$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

We set each partial derivative equal to 0:

$$-2x - y + 20 - 2\lambda = 0 \quad \rightarrow \quad \lambda = -x - \frac{1}{2}y + 10$$
$$-x - 2y + 22 - 4\lambda = 0 \quad \rightarrow \quad \lambda = -\frac{1}{4}x - \frac{1}{2}y + \frac{11}{2}$$
$$-2x - 4y + 40 = 0$$

Setting these last two equations that are solved for λ equal to each other and solving for x, we get:

$$-x - \frac{1}{2}y + 10 = -\frac{1}{4}x - \frac{1}{2}y + \frac{11}{2}$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

$$-x-\frac{1}{2}y+10=-\frac{1}{4}x-\frac{1}{2}y+\frac{11}{2}$$

$$-x+10=-\frac{1}{4}x+\frac{11}{2}$$

$$-4x+40=-x+22 \qquad \text{Multiplying by 4}$$

$$-3x=-18$$

$$x=6$$

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

Substituting x=6 into $-2x-4y+40=0$, we get:

$$-2(6)-4y+40=0$$

$$-12-4y+40=0$$

$$-4y+28=0$$

$$-4y=-28$$

$$y=7$$

Since (6, 7) is in the region of feasibility, it is a critical point.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 continued:

We now have five critical points, which we evaluate in the revenue function *R(x,y)*:

Critical point (x, y)	$R(x, y)$
(0, 0)	−25
(0, 10)	95
(10, 0)	75
(20, 0)	−25
(6, 7)	122 ← Maximum

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Example 4 concluded:

Thus, revenue is maximized when Kim produces 6 of the x-caps and 7 of the y-caps, for a weekly revenue of \$122. If there were no constraints, the maximum weekly revenue of \$123 would occur at $x = 6$ and $y = 8$. Kim might think it is not worth working an extra 4 hr for one more dollar in revenue.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Section Summary

- If input variables x and y for a function $f(x, y)$ are related by another equation, that equation is a *constraint*.
- *Constrained optimization* is a method of determining maximum and minimum points on a surface represented by $z = f(x, y)$, subject to given restrictions (constraints) on the input variables x and y.
- The *method of Lagrange multipliers* allows us to find a maximum or minimum value of a function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

6.5 Constrained Optimization: Lagrange Multipliers and the Extreme Value Theorem

Section Summary Concluded

- If the constraints are inequalities, the set of points that satisfy all the constraints simultaneously is called the *region of feasibility*.
- If the region of feasibility is closed and bounded and the surface $z = f(x, y)$ is continuous over the region, then the *Extreme-Value Theorem* guarantees that f will have both an absolute maximum and an absolute minimum value over that region.
- Critical points may be located at vertices, along a boundary, or in the interior of a region of feasibility.