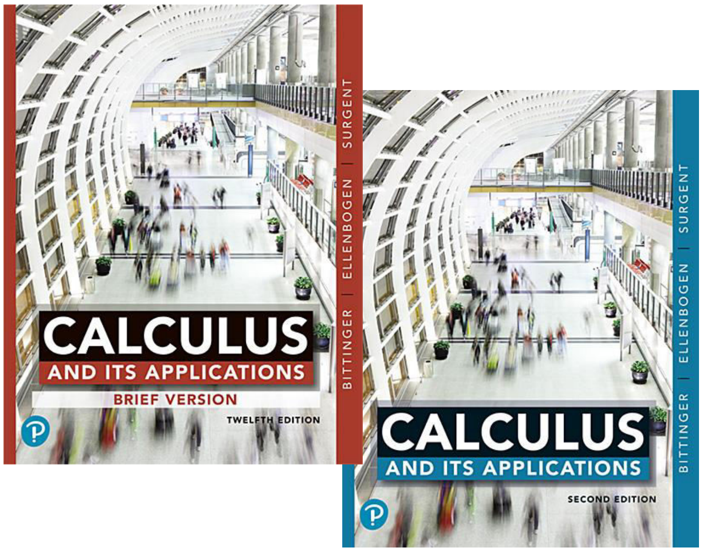
Chapter 3

Applications of Differentiation



3.3 Graph Sketching: Asymptotes and Rational Functions

OBJECTIVE

- Find limits involving infinity.
- Determine the asymptotes of a function's graph.
- Graph rational functions.

3.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION:

A **rational function** is a function f that can be described by

$$f(x)=\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials, with $Q(x)$ not the zero polynomial. The domain of f consists of all inputs x for which $Q(x)\neq 0$.

3.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION:

The line $x = a$ is a **vertical asymptote** if any of the following limit statements are true:

$$\lim_{x\to a^-} f(x)=\infty,\qquad \lim_{x\to a^-} f(x)=-\infty,$$

$$\lim_{x\to a^+} f(x)=\infty,\quad \text{or}\quad \lim_{x\to a^+} f(x)=-\infty.$$

3.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION (continued):

The graph of a rational function *never* crosses a vertical asymptote. If the expression that defines the rational function f is simplified, meaning that it has no common factor other that –1 or 1, then if a is an input that makes the denominator 0, the line $x = a$ is a vertical asymptote.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 1: Determine the vertical asymptotes of the function given by

$$f(x) = \frac{P(x)}{Q(x)}$$

$$f(x) = \frac{x(x-2)}{x(x-1)(x+1)}$$

$$f(x) = \frac{(x-2)}{(x-1)(x+1)}$$

Since $x = 1$ and $x = -1$ make the denominator 0, $x = 1$ and $x = -1$ are vertical asymptotes.

3.3 Graph Sketching: Asymptotes and Rational Functions

Quick Check 1

Determine the vertical asymptotes: $f(x) = \dfrac{1}{x(x^2 - 16)}$

$$f(x) = \frac{1}{x(x^2 - 16)}$$

$$f(x) = \frac{1}{x(x+4)(x-4)}$$

After factoring out the denominator, we see that $x = 0$, $x = 4$, and $x = -4$ make the denominator 0. Thus, there are vertical asymptotes at $x = 0$, $x = 4$, and $x = -4$.

3.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION:

The line $y = b$ is a **horizontal asymptote** if either or both of the following limit statements are true:

$$\lim_{x \to -\infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to \infty} f(x) = b.$$

3.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION (continued):

The graph of a rational function may or may not cross a horizontal asymptote. Horizontal asymptotes occur when the degree of the numerator is less than or equal to the degree of the denominator. (The degree of a polynomial in one variable is the highest power of that variable.)

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 2: Determine the horizontal asymptote of the function given by

$$f(x) = \frac{3x^2 + 2x - 4}{2x^2 - x + 1}.$$

First, divide the numerator and denominator by x^2.

$$f(x) = \frac{3 + \frac{2}{x} - \frac{4}{x^2}}{2 - \frac{1}{x} + \frac{1}{x^2}}$$

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 2 (continued):

Second, find the limit as $|x|$ gets larger and larger.

$$\lim_{x\to-\infty}\frac{3+\frac{2}{x}-\frac{4}{x^2}}{2-\frac{1}{x}+\frac{1}{x^2}}=\frac{3}{2} \quad \text{and} \quad \lim_{x\to\infty}\frac{3+\frac{2}{x}-\frac{4}{x^2}}{2-\frac{1}{x}+\frac{1}{x^2}}=\frac{3}{2}$$

Thus, the line $y=\frac{3}{2}$ is a horizontal asymptote.

3.3 Graph Sketching: Asymptotes and Rational Functions

Quick Check 2

Determine the horizontal asymptote of the function given by

$$f(x)=\frac{(2x-1)(x+1)}{(3x+2)(5x+6)}.$$

First we should multiply both the numerator and denominator out:

$$f(x)=\frac{(2x-1)(x+1)}{(3x+2)(5x+6)}=\frac{2x^2+x-1}{15x^2+28x+12}$$

3.3 Graph Sketching: Asymptotes and Rational Functions

Quick Check 2 Concluded

Since both the numerator and denominator have the same power of x, we can divide both by that power:

$$f(x) = \frac{2x^2 + x - 1}{15x^2 + 28x + 12} = \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{15 + \frac{28}{x} + \frac{12}{x^2}}$$

Now we can see that as $|x|$ gets very large, the numerator approaches 2 and the denominator approaches 15. Therefore the value of the function gets very close to $\frac{2}{15}$. Thus, $\lim_{x \to -\infty} f(x) = \frac{2}{15}$ and $\lim_{x \to \infty} f(x) = \frac{2}{15}$.

Therefore there is a horizontal asymptote at $y = \frac{2}{15}$.

3.3 Graph Sketching: Asymptotes and Rational Functions

DEFINITION:

A linear asymptote that is neither vertical nor horizontal is called a **slant**, or **oblique**, **asymptote**. For any rational function of the form $f(x) = p(x)/q(x)$, a slant asymptote occurs when the degree of $p(x)$ is exactly 1 more than the degree of $q(x)$. A graph can cross a slant asymptote.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 3: Find the slant asymptote:

$$f(x)=\frac{x^2-4}{x-1}$$

First, divide the numerator by the denominator.

$$\begin{array}{r|l} & x+1 \\ \hline x-1 & x^2 \quad -4 \\ & \underline{x^2-x} \\ & \quad x-4 \\ & \quad \underline{x-1} \\ & \quad\; -3 \end{array}$$

$$\Rightarrow \quad f(x)=\frac{x^2-4}{x-1}=(x+1)+\frac{-3}{x-1}$$

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 3 (concluded):

Second, now we can see that as $|x|$ gets very large, $-3/(x-1)$ approaches 0. Thus, for very large $|x|$, the expression $x+1$ is the dominant part of

$$(x+1)+\frac{-3}{x-1}$$

thus $y=x+1$ is the slant asymptote.

3.3 Graph Sketching: Asymptotes and Rational Functions

Quick Check 3

Find the slant asymptote: $g(x) = \dfrac{2x^2 + x - 1}{x - 3}$

Use polynomial division to solve for this:

$$
\begin{array}{r}
2x \quad +7 \\
x-3 \overline{\big) 2x^2 + x - 1} \\
-\underline{(2x^2 - 6x)}\downarrow \\
7x \quad -1 \\
-(\underline{7x - 21}) \\
20
\end{array}
$$

Since we have a remainder of 20, we can see that as $|x|$ gets very large, the remainder approaches 0. Thus the dominant part of $2x + 7 + \dfrac{20}{x-3}$ is $2x = 7$.

Therefore, there is slant asymptote at $y = 2x + 7$

3.3 Graph Sketching: Asymptotes and Rational Functions

Asymptotes of Exponential and Logarithmic Functions

An exponential function of the form $f(x) = ae^{bx}$, for $b > 0$, has an horizontal asymptote of $y = 0$ (the x-axis) as $x \to -\infty$.

An exponential function of the form $f(x) = ae^{bx}$, for $b < 0$, has an horizontal asymptote of $y = 0$ (the x-axis) as $x \to \infty$.

A logarithmic function of the form $f(x) = \ln(x - a)$, has a vertical asymptote at $x = a$ as $x \to a^+$.

A logarithmic function of the form $f(x) = \ln(a - x)$, has a vertical asymptote at $x = a$ as $x \to a^-$.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 4: Find the horizontal asymptotes of $f(x)=\dfrac{200}{1+25e^{-0.04x}}$.

Solution: As

$$\text{Thus,}\quad \lim_{x\to\infty}\frac{200}{1+25e^{-.04x}}=\frac{200}{1+0}=200.$$

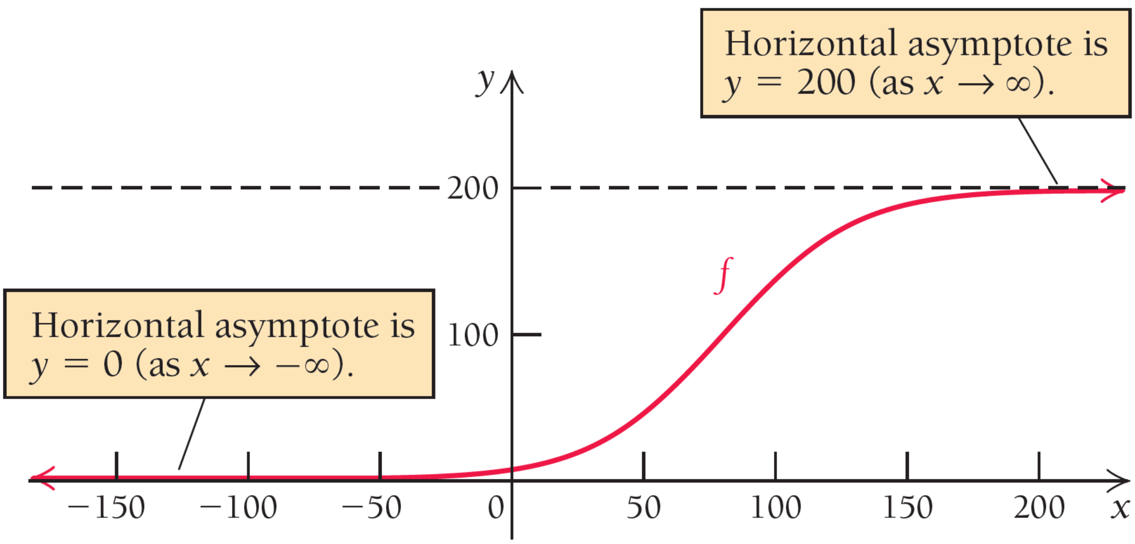
The line $y = 200$ is a horizontal asymptote of f.
Now, as $x \to -\infty$, $25e^{-0.04x} \to \infty$.

$$\text{Thus,}\quad \lim_{x\to-\infty}\frac{200}{1+25e^{-.04x}}=\frac{200}{1+\infty}=0.$$

The line $y = 0$ is also a horizontal asymptote of f.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 4 concluded: The graph of $f(x)=\dfrac{200}{1+25e^{-0.04x}}$ is shown below:



3.3 Graph Sketching: Asymptotes and Rational Functions

Strategy for Sketching Graphs:

a) *Intercepts*. Find the x-intercept(s) and the y-intercept of the graph.

b) *Asymptotes*. Find any vertical, horizontal, or slant asymptotes.

c) *Derivatives and Domain*. Find $f'(x)$ and $f''(x)$. Find the domain of f.

d) *Critical Values of f*. Find any inputs for which $f'(x)$ is not defined or for which $f'(x) = 0$.

3.3 Graph Sketching: Asymptotes and Rational Functions

Strategy for Sketching Graphs (continued):

e) *Increasing and/or decreasing; relative extrema*. Substitute each critical value, x_0, from step (d) into f''(x), and apply the Second Derivative Test. If no critical value exists, use f' and test values to find where f is increasing or decreasing.

f) *Inflection Points*. Determine candidates for inflection points by finding x-values for which $f''(x)$ does not exist or for which $f''(x) = 0$. Find the function values at these points.

3.3 Graph Sketching: Asymptotes and Rational Functions

Strategy for Sketching Graphs (concluded):

g) *Concavity*. Use the values c from step (f) as endpoints of intervals. Determine the concavity by checking to see where f' is increasing – that is, $f''(x) > 0$ – and where f' is decreasing – that is, $f''(x) < 0$. Do this by selecting test points and substituting into $f''(x)$. Use the results of step (d).

h) *Sketch the graph*. Use the information from steps (a) – (g) to sketch the graph, plotting extra points as needed.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 5: Sketch the graph of $f(x) = \dfrac{8}{x^2 - 4}$.

a) *Intercepts*. The x-intercepts occur at values for which the numerator equals 0. Since $8 \neq 0$, there are no x-intercepts. To find the y-intercept, we find $f(0)$.

$$f(0) = \frac{8}{0^2 - 4} = \frac{8}{-4} = -2$$

Thus, we have the point (0, –2).

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 5 (continued):

b) *Asymptotes*.

Vertical:
$$x^2 - 4 = 0$$
$$(x-2)(x+2) = 0$$

So, $x = 2$ and $x = -2$ are vertical asymptotes.

Horizontal: The degree of the numerator is less than the degree of the denominator. So, the x-axis, $y = 0$ is the horizontal asymptote.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 5 (continued):

Slant: There is no slant asymptote since the degree of the numerator is not 1 more than the degree of the denominator.

c) *Derivatives and Domain*. Using the Quotient Rule, we get

$$f'(x) = \frac{-16x}{(x^2-4)^2} \quad \text{and} \quad f''(x) = \frac{16(3x^2+4)}{(x^2-4)^3}.$$

The domain of f is all real numbers, $x \neq 2$ and $x \neq -2$.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 5 (continued):

d) *Critical Values of f.* $f'(x)$ equals 0 where the numerator equals 0 and does not exist where the denominator equals 0.

$$\begin{aligned} -16x &= 0 \\ x &= 0 \end{aligned}$$

$$\begin{aligned} (x^2-4)^2 &= 0 \\ x^2-4 &= 0 \\ (x-2)(x+2) &= 0 \end{aligned}$$

$$x = 2 \quad or \quad x = -2$$

However, since f does not exist at $x = 2$ or $x = –2$, $x = 0$ is the only critical value.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 5 (continued):

e) *Increasing and/or decreasing; relative extrema*.

$$f''(0) = \frac{16(3\cdot 0^2+4)}{(0^2-4)^3} = \frac{64}{-64} = -1 < 0$$

Thus, $x = 0$ is a relative maximum and f is increasing on (–2, 0) and decreasing on (0, 2).

Since f'' does not exist at $x = 2$ and $x = –2$, we use f' and test values to see if f is increasing or decreasing on (–∞, 2) and (2, ∞).

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 5 (continued):

$$f'(-3) = \frac{-16(-3)}{\left((-3)^2 - 4\right)^2} = \frac{48}{25} > 0$$

So, f is increasing on $(-\infty, 2)$.

$$f'(3) = \frac{-16(3)}{\left((3)^2 - 4\right)^2} = \frac{-48}{25} < 0$$

So, f is decreasing on $(2, \infty)$.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 5 (continued):

f) *Inflection points*. f'' does not exist $x = 2$ and $x = -2$. However, neither does f. Thus we consider where f'' equals 0.

$$16(3x^2 + 4) = 0$$

Note that $16(3x^2 + 4) > 0$ for all real numbers x, so there are no points of inflection.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 5 (continued):

g) *Concavity*. Since there are no points of inflection, the only places where f could change concavity would be on either side of the vertical asymptotes.

Note that we already know from step (e) that f is concave down at $x = 0$. So we need only test a point in $(-\infty, 2)$ and a point in $(2, \infty)$.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 5 (continued):

$$f''(-3) = \frac{16(3\cdot(-3)^2+4)}{((-3)^2-4)^3} = \frac{496}{125} > 0$$

Thus, f is concave up on $(-\infty, 2)$.

$$f''(3) = \frac{16(3\cdot(3)^2+4)}{((3)^2-4)^3} = \frac{496}{125} > 0$$

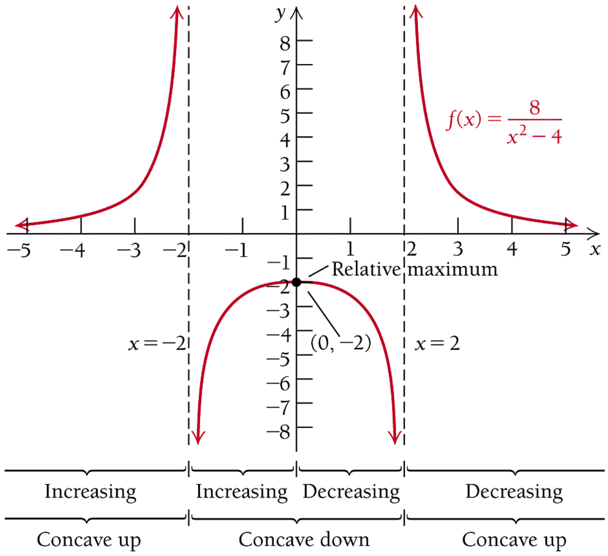
Thus, f is concave up on $(2, \infty)$.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 5 (concluded):

h) *Sketch the graph*. Using the information in steps (a) – (g), the graph follows.

x	$f(x)$ approximately
−5	0.38
−4	0.67
−3	1.6
−1	−2.67
0	−2
1	−2.67
3	1.6
4	0.67
5	0.38



3.3 Graph Sketching: Asymptotes and Rational Functions

Example 6: Sketch the graph of $f(x) = e^{-x^2}$.

a) *Intercepts*. Since $g(0) = e^{-0^2} = 1$, the y-intercept occurs at (0, 1). There are no x-intercepts since $e^{-x^2} > 0$ for all x.

b) *Asymptotes.*

$$\lim_{x\to\infty} e^{-x^2} = \lim_{x\to\infty}\left(\frac{1}{e^{x^2}}\right) = 0$$

$$\lim_{x\to-\infty} e^{-x^2} = \lim_{x\to-\infty}\left(\frac{1}{e^{x^2}}\right) = 0$$

Thus, the line $y = 0$ is a horizontal asymptote.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 6 (continued):

c) *Derivatives and Domain*.

The domain of f is all real numbers.

The derivatives are:

$$f'(x) = -2xe^{-x^2},$$

$$f''(x) = -2x(-2xe^{-x^2}) + (-2)e^{-x^2}$$

$$= 4x^2e^{-x^2} - 2e^{-x^2}$$

The domains of f' and f'' are also all real numbers.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 6 (continued):

d) *Critical Values of f*. Here the only critical values will occur where $f'(x)$ equals 0.

$$-2xe^{-x^2} = 0$$

$$x = 0$$

Thus, there is a critical value at (0, 1) or the y-intercept.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 6 (continued):

e) *Increasing and/or decreasing; relative extrema*.

The critical value at $x = 0$ creates two intervals: $(-\infty, 0)$ and $(0, \infty)$.

Testing a point in each interval, we get:

$$f'(-1) = -2(-1)e^{-(-1)^2} = \frac{2}{e} > 0,$$

$$f'(1) = -2(1)e^{-(1)^2} = -\frac{2}{e} < 0,$$

Thus, a relative maximum occurs at $x = 0$. That is the point $(0, 1)$ is a relative maximum. f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 6 (continued):

f) *Inflection points*. Since f'' exists for all x, the only inflection points will occur where f'' equals 0.

$$4x^2e^{-x^2} - 2e^{-x^2} = 0$$ Factoring, we get:

$$2e^{-x^2}(2x^2 - 1) = 0$$

Thus, $2x^2 - 1 = 0$

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{\sqrt{2}}{2}$$

Evaluating $g(x)$ at these x-values gives us the following second coordinates:

$$\left(-\frac{\sqrt{2}}{2}, e^{-1/2}\right) \text{ and } \left(\frac{\sqrt{2}}{2}, e^{-1/2}\right).$$

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 6 (continued):

g) *Concavity*.

Testing a point in each interval to determine the sign of f'':

$\left(-\infty, -\frac{\sqrt{2}}{2}\right)$ $\quad f''(-1) = 4(-1)^2 e^{-(-1)^2} - 2e^{-(-1)^2} = \frac{2}{e} > 0.$

$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $\quad f''(0) = 4(0)^2 e^{-(0)^2} - 2e^{-(0)^2} = -2 < 0.$

$\left(\frac{\sqrt{2}}{2}, \infty\right)$ $\quad f''(1) = 4(1)^2 e^{-(1)^2} - 2e^{-(1)^2} = \frac{2}{e} > 0.$

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 6 (continued):

Thus, f is concave up on $\left(-\infty, -\frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, \infty\right)$.

Thus, f is concave down on $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Notice, there is a change in concavity at both $x = \pm\frac{\sqrt{2}}{2}$.

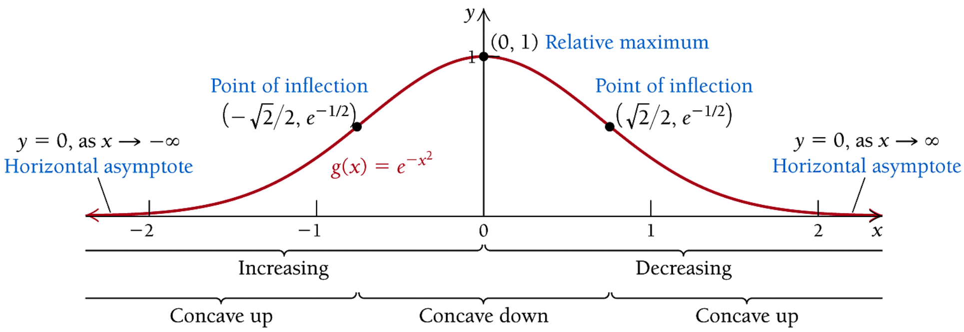
Thus, the inflection points are at $\left(-\frac{\sqrt{2}}{2}, e^{-1/2}\right)$ and $\left(\frac{\sqrt{2}}{2}, e^{-1/2}\right)$.

3.3 Graph Sketching: Asymptotes and Rational Functions

Example 6 (concluded):

h) *Sketch the graph*. Using the information in steps (a) – (g), the graph follows.

x	$g(x)$
-2	0.0183
-1	0.3679
0	1
1	0.3679
2	0.0183



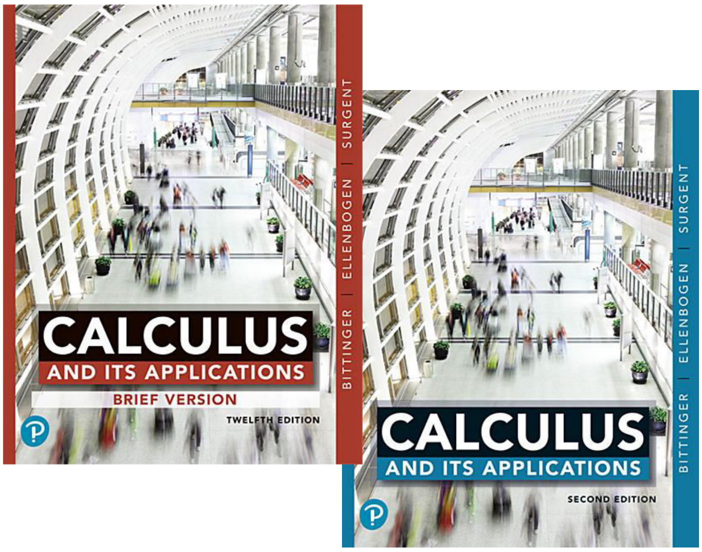
3.3 Graph Sketching: Asymptotes and Rational Functions

Section Summary

- A line $x = a$ is a *vertical asymptote* if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$
- A line $y = b$ is a *horizontal asymptote* if $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$
- A graph may cross a horizontal asymptote but never a vertical asymptote.
- A *slant asymptote* occurs when the degree of the numerator is 1 greater than the degree of the denominator. Long division of polynomials can be used to determine the equation of the slant asymptote.
- Vertical, horizontal, and slant asymptotes can be used as guides for accurate curve sketching. Asymptotes are not a part of a graph but are visual guides only.

Chapter 3

Applications of Differentiation



3.4 Optimization: Finding Absolute Maximum and Minimum Values

OBJECTIVE

- Find absolute extrema using Maximum-Minimum Principle 1.
- Find absolute extrema using Maximum-Minimum Principle 2.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

DEFINITION:

Suppose that f is a function with domain I.

$f(c)$ is an **absolute minimum** if $f(c) \leq f(x)$ for all x in I.

$f(c)$ is an **absolute maximum** if $f(c) \geq f(x)$ for all x in I.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

THEOREM 7: The Extreme Value Theorem

A continuous function f defined over a closed interval $[a, b]$ must have an absolute maximum value and an absolute minimum value over $[a, b]$.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

THEOREM 8: Maximum-Minimum Principle 1

Suppose that f is a continuous function defined over a closed interval $[a, b]$. To find the absolute maximum and minimum values over $[a, b]$:

a) First find $f'(x)$.

b) Then determine all critical values in $[a, b]$. That is, find all c in $[a, b]$ for which

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

3.4 Optimization: Finding Absolute Maximum and Minimum Values

THEOREM 8: Maximum-Minimum Principle 1 (continued)

c) List the values from step (b) and the endpoints of the interval:

$$a, c_1, c_2, \ldots, c_n, b.$$

d) Evaluate $f(x)$ for each value in step (c):

$$f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b).$$

The largest of these is the **absolute maximum of f over [a, b]**. The smallest of these is the **absolute minimum of f over [a, b]**.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 1: Find the absolute maximum and minimum values of $f(x)=x^3-3x+2$ over the interval $[-2, \frac{3}{2}]$.

a) $f'(x)=3x^2-3$

b) Note that $f'(x)$ exists for all real numbers.

$$3x^2-3=0$$
$$3x^2=3$$
$$x^2=1$$
$$x=\pm 1$$

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 1 (continued):

c) $-2, -1, 1, \frac{3}{2}$

d)
$$\begin{aligned}
f(-2) &= (-2)^3-3(-2)+2 &= 0\\
f(-1) &= (-1)^3-3(-1)+2 &= 4\\
f(1) &= (1)^3-3(1)+2 &= 0\\
f\left(\frac{3}{2}\right) &= \left(\frac{3}{2}\right)^3-3\left(\frac{3}{2}\right)+2 &= \frac{7}{8}
\end{aligned}$$

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 1 (concluded):

Thus, the absolute maximum value of $f(x)$ on $[-2, \frac{3}{2}]$ is 4, which occurs at $x = 1$. The absolute minimum value of $f(x)$ on $[-2, \frac{3}{2}]$ is 0, which occurs at $x = 2$ and $x = 1$.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 1

Find the absolute maximum and minimum values of the function given in Example 1 over the interval $[0,3]$.

Recall that the function in Example 1 was $f(x) = x^3 - 3x + 2$.

So, $f'(x) = 3x^2 - 3$. Set this equal to zero and you get:

$$\begin{aligned} 3x^2 - 3 &= 0 \\ 3x^2 &= 3 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 1 Concluded

Next we check the equation at $x=-1,0,1,3$ to find the minimum and maximum:

$$f(-1)=(-1)^3-3(-1)+2 \ =-1+3+2 \ =4$$

$$f(0)=(0)^3-3(0)+2 \ =0+0+2 \ =2$$

$$f(1)=(1)^3-3(1)+2 \ =1-3+2 \ =0$$

$$f(3)=(3)^3-3(3)+2 \ =27-9+2 \ =20$$

The absolute maximum is 20 at $x=3$.

The absolute minimum is 0 at $x=1$.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

THEOREM 9: Maximum-Minimum Principle 2

Suppose that f is a function such that $f'(x)$ exists for every x in an interval I, and that there is *exactly one* (critical) value c in I, for which $f'(c) = 0$. Then

$f(c)$ is the absolute maximum value over I if $f''(c) < 0$
or
$f(c)$ is the absolute minimum value over I if $f''(c) > 0$.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 2: Find the absolute maximum and minimum values of $f\left(x\right)=2e^{3x}$ over $\left[-2,\frac{1}{2}\right]$.

Solution: $f'(x)=6e^{3x}$ is defined for all real numbers.

Note that $6e^{3x}>0$ for all x.

Thus, there are no critical values and the absolute extrema must occur at the endpoints of the interval:

$$f(-2)=2e^{3(-2)}=2e^{-6}\approx 0.005$$

$$f(0.5)=2e^{3(0.5)}=2e^{1.5}\approx 8.963$$

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 2 concluded:

The absolute maximum of $f\left(x\right)=2e^{3x}$ over $\left[-2,\frac{1}{2}\right]$ is:

$2e^{1.5}\approx 8.963$ and occurs at $x=\frac{1}{2}$.

The absolute minimum of $f\left(x\right)=2e^{3x}$ over $\left[-2,\frac{1}{2}\right]$ is:
$2e^{-6}\approx 0.005$ and occurs at $x=-2$.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 3: Find the absolute maximum and minimum values of $f(x) = 4x - x^2$.

When no interval is specified, we consider the entire domain of the function. In this case, the domain is the set of all real numbers.

a) $f'(x) = 4 - 2x$

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 3 (continued):

b) $4 - 2x = 0$

$$-2x = -4$$

$$x = 2$$

c) Since there is only one critical value, we can apply Maximum-Minimum Principle 2 using the second derivative.

$$f''(x) = -2$$

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 3 (concluded):

Since $f''(x) < 0$ and $f(2) = 4(2) - (2)^2 = 4$, $f(x)$ has an absolute maximum value of 4 at $x = 2$.

$f(x)$ has no absolute minimum value.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 2

Find the absolute maximum and minimum values of $f(x) = x^2 - 10x$ over each interval:

a.) $[0,6]$

b.) $[4,10]$

Looking at the function, we can see that the absolute minimum over $(-\infty,\infty)$ is -25 located at $x = 5$. Since $x = 5$ is in both intervals, the absolute minimum for both intervals will be -25 located at $x = 5$.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 2 Concluded

Now we need to check both endpoints for both intervals to find the absolute maximum:

a.) $f(0) = 0^2 - 10(0) = 0 - 0 = 0$

$f(6) = 6^2 - 10(6) = 36 - 60 = -24$

So the absolute maximum over $[0,6]$ is 0 located at $x = 0$.

b.) $f(4) = 4^2 - 10(4) = 16 - 40 = -24$

$f(10) = 10^2 - 10(10) = 100 - 100 = 0$

So the absolute maximum over $[4,10]$ is 0 located at $x = 10$.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

A Strategy for Finding Absolute Maximum and Minimum Values:

To find absolute maximum and minimum values of a continuous function over an interval:

a) Find $f'(x)$.

b) Find the critical values.

c) If the interval is closed and there is more than one critical value, use Maximum-Minimum Principle 1.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

A Strategy for Finding Absolute Maximum and Minimum Values (continued):

d) If the interval is closed and there is exactly one critical value, use either Maximum-Minimum Principle 1 or Maximum-Minimum Principle 2. If it is easy to find $f''(x)$, use Maximum-Minimum Principle 2.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

A Strategy for Finding Absolute Maximum and Minimum Values (concluded):

e) If the interval is not closed, such as $(-\infty, \infty)$, $(0, \infty)$, or (a, b), and the function has only one critical value, use Maximum-Minimum Principle 2. In such a case, if the function has a maximum, it will have no minimum; and if it has a minimum, it was have no maximum.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 4: Find the absolute maximum and minimum values of $f(x) = 5x + \dfrac{35}{x}$ over the interval $(0, \infty)$.

a) $f'(x) = 5 - \dfrac{35}{x^2}$

b) Since $f'(x)$ exists for all values of x in $(0, \infty)$, the only critical values are those for which $f'(x) = 0$.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 4 (continued):

$$5 - \frac{35}{x^2} = 0$$
$$5 = \frac{35}{x^2}$$
$$5x^2 = 35$$
$$x^2 = 7$$
$$x = \sqrt{7}$$

c) The interval $(0, \infty)$ is not closed, and the only critical value is $\sqrt{7}$. Thus, we can use Maximum-Minimum Principle 2 using the second derivative.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 4 (continued):

$$f''(x) = \frac{70}{x^3}$$

$$f''\left(\sqrt{7}\right) = \frac{70}{\left(\sqrt{7}\right)^3} > 0$$

Thus $f(x)$ has an absolute minimum at $x = \sqrt{7}$.

$$f\left(\sqrt{7}\right) = 5 \cdot \sqrt{7} + \frac{35}{\sqrt{7}}$$

$$f\left(\sqrt{7}\right) = 5\sqrt{7} + \frac{35\sqrt{7}}{7}$$

$$f\left(\sqrt{7}\right) = 5\sqrt{7} + 5\sqrt{7} = 10\sqrt{7}$$

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Example 4 (concluded):

Thus, the absolute minimum of $f(x)$ is $10\sqrt{7}$, which occurs at $x = \sqrt{7}$.

$f(x)$ has no absolute maximum value.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 3

Find the absolute maximum and minimum values of $g(x) = \dfrac{2x^2 + 18}{x}$ over the interval $(0, \infty)$.

First we want to find $g'(x)$:

$$g'(x) = \frac{x(4x) - (2x^2 + 18)}{x^2} = \frac{4x^2 - 2x^2 - 18}{x^2} = \frac{2x^2 - 18}{x^2}$$

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 3 Continued

Next we find out where $g'(x) = 0$.

$$\frac{2x^2 - 18}{x^2} = \frac{2x^2}{x^2} - \frac{18}{x^2} = 2 - \frac{18}{x^2} = 0$$

$$2 = \frac{18}{x^2}$$

$$2x^2 = 18$$

$$x^2 = 9$$

$$x = \pm 3$$

Since $x = -3$ is not in the interval, the only critical value is at $x = 3$.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Quick Check 3 Concluded

We can now apply the Maximum-Minimum Principle 2:

$$f''(x) = \frac{36}{x^3}$$

$$f''(3) = \frac{36}{3^3} > 0$$

So an absolute minimum occurs at $x = 3$, and is 12.

Since this is on an open interval, there is no absolute maximum.

3.4 Optimization: Finding Absolute Maximum and Minimum Values

Section Summary

- An *absolute minimum* of a function f is a value $f(c)$ such that $f(c) \le f(x)$ for all x in the domain of f.
- An *absolute maximum* of a function f is a value $f(c)$ such that $f(c) \ge f(x)$ for all x in the domain of f.
- If the domain of f is a closed interval and f is continuous over that domain, then the *Extreme-Value Theorem* guarantees the existence of both an absolute minimum and an absolute maximum.

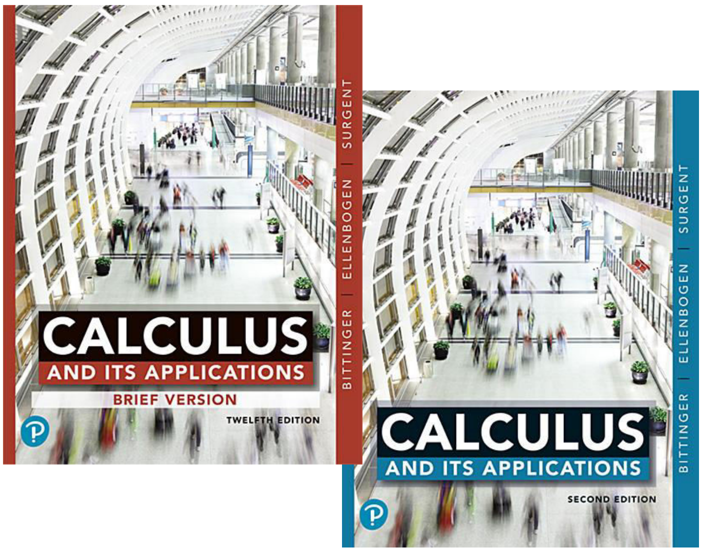
3.4 Optimization: Finding Absolute Maximum and Minimum Values

Section Summary Concluded

- Endpoints of a closed interval may be absolute extrema, but not relative extrema.
- If there is exactly one critical value c such that $f'(c)=0$ in the domain of f, then *Maximum-Minimum Principle 2* may be used. Otherwise, *Maximum-Minimum Principle 1* has to be used.

Chapter 3

Applications of Differentiation



3.5 Optimization: Business, Economic and General Applications

OBJECTIVE

- Solve optimization (maximum–minimum) problems using calculus.

3.5 Optimization: Business, Economic and General Applications

A Strategy for Solving Maximum-Minimum Problems:

1. Read the problem carefully. If relevant, make a drawing.
2. Make a list of appropriate variables and constants, noting what varies, what stays fixed, and what units are used. Label the measurements on your drawing, if one exists.

3.5 Optimization: Business, Economic and General Applications

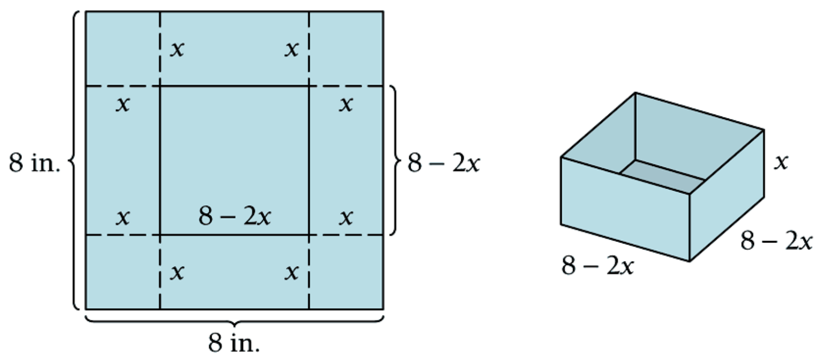
A Strategy for Solving Maximum-Minimum Problems (concluded):

3. Translate the problem to an equation involving a quantity Q to be maximized or minimized. Try to represent Q in terms of the variables of step (2).
4. Try to express Q as a function of one variable. Use the procedures developed in sections 2.1 – 2.3 to determine the maximum or minimum values and the points at which they occur.

3.5 Optimization: Business, Economic and General Applications

Example 1: From a thin piece of cardboard 8 in. by 8 in., square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? What is the maximum volume?

First, make a drawing in which x is the length of each square to be cut.



3.5 Optimization: Business, Economic and General Applications

Example 1 (continued):

Next write an equation for the volume of the box.

$$\begin{aligned} V &= l \cdot w \cdot h \\ V(x) &= (8-2x)\cdot(8-2x)\cdot x \\ V(x) &= (64-32x+4x^2)\cdot x \\ V(x) &= 4x^3-32x^2+64x \end{aligned}$$

Note that x must be between 0 and 4. So, we need to maximize the volume equation on the interval (0, 4).

3.5 Optimization: Business, Economic and General Applications

Example 1 (continued):

$$V' = 12x^2-64x+64 = 0$$

$$3x^2-16x+16 = 0$$

$$(3x-4)(x-4) = 0$$

$$3x-4=0 \text{ or } x-4=0$$

$$x=\frac{4}{3} \text{ or } x=4$$

$\frac{4}{3}$ is the only critical value in (0, 4). So, now we find the second derivative.

3.5 Optimization: Business, Economic and General Applications

Example 1 (concluded):

$$V''(x) = 24x - 64$$

$$V''\left(\frac{4}{3}\right) = 24\left(\frac{4}{3}\right) - 64$$

$$V''\left(\frac{4}{3}\right) = -32 \quad < 0$$

Thus, the volume is maximized when $\frac{4}{3}$ inches are cut away from each corner. The maximum volume is

$$V\left(\frac{4}{3}\right) = 4\left(\frac{4}{3}\right)^3 - 32\left(\frac{4}{3}\right)^2 + 64\left(\frac{4}{3}\right)$$

$$V\left(\frac{4}{3}\right) = 37\frac{25}{27} \text{ in}^3$$

3.5 Optimization: Business, Economic and General Applications

Quick Check 1

Repeat Example 1 with a sheet of cardboard measuring 8.5 in. by 11 in. (the size of a typical sheet of paper). Will this box hold 1 liter (L) of liquid? (Hint: $1 \text{ L} = 1000 \text{ cm}^3$ and $1 \text{ in}^3 = 16.38 \text{ cm}^3$).

Going from Example 1, we can visualize that the dimensions of the box will be $8.5 - 2x, 11 - 2x, x$

3.5 Optimization: Business, Economic and General Applications

Quick Check 1 Continued

Next lets find the equation for volume:

$$V = l \cdot w \cdot h$$

$$V = (11-2x)(8.5-2x)x$$

$$V = 4x^3 - 39x^2 + 93.5x$$

Since, both $11-2x>0$ and $8.5-2x>0$, $0<x<5.5$ and $0<x<4.25$

So, x must be between 0 and 4.25. So we must maximize the volume equation on the interval $(0, 4.25)$.

3.5 Optimization: Business, Economic and General Applications

Quick Check 1 Continued

$$V' = 12x^2 - 78x + 93.5 \quad = 0$$

Using the quadratic equation: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-78) \pm \sqrt{(-78)^2 - 4(12)(93.5)}}{2(12)}$$

$$x \approx \frac{39 \pm \sqrt{399}}{12} \;, \qquad x \approx 4.917 \quad \text{or} \quad x \approx 1.585$$

Since 4.917 does not fall into the interval (0, 4.25), we use $x \approx 1.585$.

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3.5 Optimization: Business, Economic and General Applications

Quick Check 1 Concluded

$$V'' = 24x - 78$$

$$V''(1.585) = 24(1.585) - 78$$
$$V''(1.585) = 38.04 - 78$$
$$V''(1.585) = -39.96 < 0$$

Therefore there is a maximum at $x \approx 1.585$.

Thus the dimensions are approximately 1.585 in. by 5.33 in. by 7.83 in., giving us a volume of 66.15 in.^3 or 1084.0 cm.^3, which is greater than 1 Liter. Thus the box can hold 1 Liter of water.

3.5 Optimization: Business, Economic and General Applications

Example 2: A stereo manufacturer determines that in order to sell x units of a new stereo, the price per unit, in dollars, must be $p(x) = 1000 - x$. The manufacturer also determines that the total cost of producing x units is given by $C(x) = 3000 + 2x$.

a) Find the total revenue $R(x)$.

b) Find the total profit $P(x)$.

c) How many units must the company produce and sell in order to maximize profit?

d) What is the maximum profit?

e) What price per unit must be charged in order to make this maximum profit?

3.5 Optimization: Business, Economic and General Applications

Example 2 (continued):

a) Revenue = quantity · price

$$R(x) = x \cdot p$$
$$R(x) = x(1000 - x)$$
$$R(x) = 1000x - x^2$$

b) Profit = Total Revenue – Total Cost

$$P(x) = R(x) - C(x)$$
$$P(x) = 1000x - x^2 - (3000 + 20x)$$
$$P(x) = -x^2 + 980x - 3000$$

3.5 Optimization: Business, Economic and General Applications

Example 2 (continued):

c) $P'(x) = -2x + 980 = 0$

$$-2x = -980$$
$$x = 490$$

Since there is only one critical value, we can use the second derivative to determine whether or not it yields a maximum or minimum.

$$P''(x) = -2$$

Since $P''(x)$ is negative, $x = 490$ yields a maximum. Thus, profit is maximized when 490 units are produced and sold.

3.5 Optimization: Business, Economic and General Applications

Example 2 (concluded):

d) The maximum profit is given by

$$\begin{aligned} P(490) &= -(490)^2 + 980(490) - 3000 \\ P(490) &= \$237{,}100. \end{aligned}$$

Thus, the stereo manufacturer makes a maximum profit of \$237,100 when 490 units are bought and sold.

e) The price per unit to achieve this maximum profit is

$$\begin{aligned} p(490) &= 1000 - 490 \\ p(490) &= \$510. \end{aligned}$$

3.5 Optimization: Business, Economic and General Applications

Quick Check 2

Repeat Example 2 with the price function $p(x) = 1750 - 2x$ and the cost function $C(x) = 2250 + 15x$. Round your answers when necessary.

a.) Find the total revenue $R(x)$

$$\begin{aligned} \text{Revenue} &= \text{quantity} \cdot \text{price} \\ R(x) &= x \cdot p \\ R(x) &= x(1750 - 2x) \\ R(x) &= 1750x - 2x^2 \end{aligned}$$

3.5 Optimization: Business, Economic and General Applications

Quick Check 2 Continued

b.) Find the total Profit:

$$\begin{aligned} \text{Profit} &= \text{Total Revenue} - \text{Total Cost} \\ P(x) &= R(x) - C(x) \\ P(x) &= 1750x - 2x^2 - (2250 + 15x) \\ P(x) &= -2x^2 + 1735x - 2250 \end{aligned}$$

c.) Find the number of units to produce by using the First Derivative Test to find a critical value: $P'(x) = -4x + 1735 = 0, \quad x = 433.75$

We can use the second derivative to see if this is a maximum or minimum. Since $P''(x) = -4 < 0$, we know profits are maximized when 434 units are sold.

3.5 Optimization: Business, Economic and General Applications

Quick Check 2 concluded

d.) Find the Maximum Profit:

The maximum profit is given by:

$$P(434) = -2(434)^2 + 1735(434) - 2250$$

$$P(434) = 374{,}028$$

Thus the maximum profit is $374,028.

e.) Find the price per unit.

The price per unit to achieve this maximum profit is:

$$p(434) = 1750 - 2(434), \; p(434) = 1750 - 868, \quad p(434) = \$882$$

3.5 Optimization: Business, Economic and General Applications

Example 3: Life Science: Concentration of a Drug in the Bloodstream. A drug is administered into the bloodstream of a patient. Let $C(t) = 25te^{-.04t}$ represent the concentration, $C(t)$, in nanograms per milliliter (ng/mL), of the drug t minutes after it is administered. What is the peak concentration of the drug in the bloodstream, and when does this peak occur?

Solution: First, find the critical values by differentiating:

$$C'(t) = 25t \cdot (-0.04)e^{-0.04t} + 25e^{-0.04t} \qquad \text{By Product Rule}$$

$$C'(t) = 25e^{-0.04t}(1 - 0.04t) \qquad \text{Factoring}$$

3.5 Optimization: Business, Economic and General Applications

Example 3 continued:

Setting $C'(t) = 0$ and solving, we get:

$$25e^{-0.04t}(1 - 0.04t) = 0$$

$$1 - 0.04t = 0$$

$$0.04t = 1$$

$$t = 25$$

Thus, $t = 25$ is a critical value. Evaluating C(25), we get:

$$C(25) = 25(25)e^{-.04(25)}$$

$$C(25) = 625e^{-1} \approx 229.92 \frac{ng}{mL}$$

3.5 Optimization: Business, Economic and General Applications

Example 3 continued:

Using the second derivative test:

$$C''(t) = 25e^{-0.04t}(-0.04) + (1 - 0.04t)(-e^{-0.04t})$$

$$C''(t) = -e^{-0.04t} - e^{-0.04t}(1 - 0.04t)$$

$$C''(t) = e^{-0.04t}(-1 - 1 + 0.04t)$$

$$C''(t) = e^{-0.04t}(0.04t - 2)$$

Evaluating at t = 25, we get:

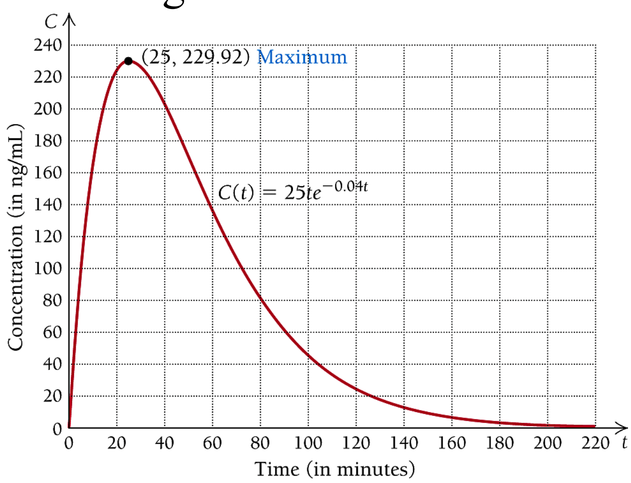
$$C''(25) = e^{-.04(25)}(.04(25) - 2)$$

$$C''(25) = e^{-1}(1 - 2) = -\frac{1}{e} < 0$$

3.5 Optimization: Business, Economic and General Applications

Example 3 concluded:

Since $C''(25) < 0$, the graph is concave down, and the critical point (25, 229.92) is a local maximum point. The drug reaches its highest concentration of 229.92 ng/mL 25 min after it was administered.



3.5 Optimization: Business, Economic and General Applications

THEOREM 10

Maximum profit occurs at those x-values for which

$$R'(x) = C'(x) \quad \text{and} \quad R''(x) < C''(x).$$

3.5 Optimization: Business, Economic and General Applications

Example 4: Promoters of international fund-raising concerts must walk a fine line between profit and loss, especially when determining the price to charge for admission to closed-circuit TV showings in local theaters. By keeping records, a theater determines that, at an admission price of \$26, it averages 1000 people in attendance. For every drop in price of \$1, it gains 50 customers. Each customer spends an average of \$4 on concessions. What admission price should the theater charge in order to maximize total revenue?

3.5 Optimization: Business, Economic and General Applications

Example 4 (continued):

Let x = the number of dollars by which the price of \$26 should be decreased (if x is negative, the price should be increased).

$$\text{Revenue} = (\text{Rev. from tickets}) + (\text{Rev. from concessions})$$

$$R(x) = (\#\text{ of people}) \cdot (\text{ticket price}) + (\#\text{ of people}) \cdot 4$$

$$R(x) = (1000+50x)(26-x) + (1000+50x) \cdot 4$$

$$R(x) = 26{,}000 - 1000x + 1300x - 50x^2 + 4000 + 200x$$

$$R(x) = -50x^2 + 500x + 30{,}000$$

3.5 Optimization: Business, Economic and General Applications

Example 4 (continued):

To maximize $R(x)$, we find $R'(x)$ and solve for critical values.

$$\begin{aligned} R'(x) = -100x + 500 &= 0 \\ -100x &= -500 \\ x &= 5 \end{aligned}$$

Since there is only one critical value, we can use the second derivative to determine if it yields a maximum or minimum.

3.5 Optimization: Business, Economic and General Applications

Example 4 (concluded):

$$R''(x) = -100$$
$$R''(5) = -100$$

Thus, $x = 5$ yields a maximum revenue. So, the theater should charge

$26 – $5 = $21 per ticket.

3.5 Optimization: Business, Economic and General Applications

Quick Check 3

A baseball team charges $30 per ticket and averages 20,000 people in attendance per game. Each person spends an average of $8 on concessions. For every drop of $1 in the ticket price, the attendance rises by 800 people. What ticket price should the team charge to maximize total revenue?

From Example 3, we can use the Revenue formula to find the ticket price to maximize income:

$$R(x) = (20{,}000 + 800x)(30 - x) + (20{,}000 + 800x)\cdot 8$$
$$R(x) = 600{,}000 + 4{,}000x - 800x^2 + 160{,}000 + 6{,}400x$$
$$R(x) = -800x^2 + 10{,}400x + 760{,}000$$

3.5 Optimization: Business, Economic and General Applications

Quick Check 3 Concluded

Next we find $R'(x)$: $R'(x) = -1,600x + 10,400$

Then find where $R'(x) = 0$:

$$R'(x) = -1,600x + 10,400 = 0$$
$$-1,600x = -10,400$$
$$x = 6.5$$

To check to make sure this is a maximum, we need to use the second derivative. Since $R''(x) = -1,600 < 0,$ we know that this is in fact a maximum of the revenue equation.

Substitute x back into the formula for ticket price we get:

$$\$30 - \$6.50 = \$23.50$$

3.5 Optimization: Business, Economic and General Applications

Section Summary

- In many real-life applications, we wish to determine the minimum or maximum value of some function modeling a situation.
- Identify a realistic interval for the domain of the input variable. If it is closed interval, its endpoints should be considered as possible critical values.
- Maximum profit occurs at those x-values for which $R'(x) = C'(x)$ and $R''(x) < C''(x)$, where $R(x)$ is the total revenue and $C(x)$ is the total cost when x units are produced.