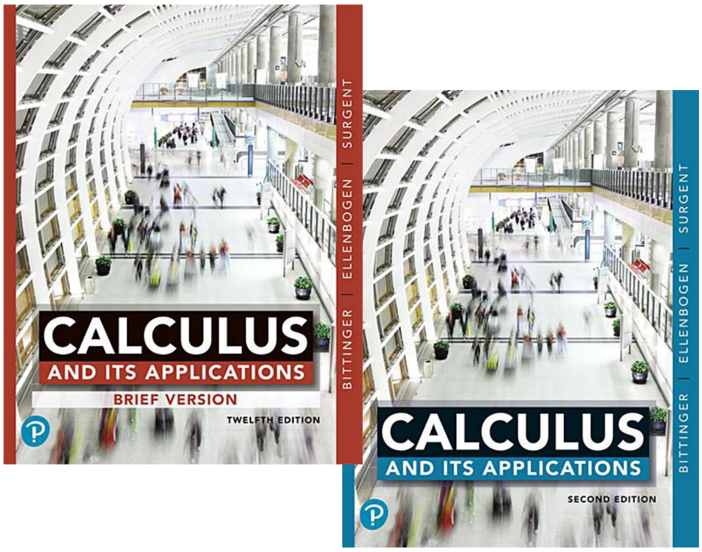
Chapter 1

Differentiation



1.7 The Chain Rule

OBJECTIVE

- Find the composition of two functions.
- Differentiate using the Chain Rule.

1.7 The Chain Rule

DEFINITION:

The **composed** function $f \circ g$, the **composition** of f and g, is defined as

$$f \circ g = f(g(x)).$$

1.7 The Chain Rule

Example 1: For $f(x) = x^3$ and $g(x) = 1 + x^2$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(1+x^2) \\ &= (1+x^2)^3 \\ &= 1 + 3x^2 + 3x^4 + x^6\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^3) \\ &= 1 + (x^3)^2 \\ &= 1 + x^6\end{aligned}$$

1.7 The Chain Rule

Example 2: For $f(x)=\sqrt{x}$ and $g(x) = x-1$, find $(f\circ g)(x)$ and $(g\circ f)(x)$.

$$(f\circ g)(x)=f(g(x))$$
$$=f(x-1)$$
$$(f\circ g)(x)=\sqrt{x-1}$$

$$(g\circ f)(x)=g(f(x))$$
$$=g(\sqrt{x})$$
$$(g\circ f)(x)=\sqrt{x}-1$$

1.7 The Chain Rule

Quick Check 1

For the functions in Example 2, find:

a.) $(f\circ f)(x)$

$$(f\circ f)(x)=f(f(x))=f(\sqrt{x})=\sqrt{\sqrt{x}}=\sqrt[4]{x}$$

b.) $(g\circ g)(x)$

$$(g\circ g)(x)=g(g(x))=g(x-1)=(x-1)-1=x-2$$

1.7 The Chain Rule

THEOREM 9: The Chain Rule

The derivative of the composition $f \circ g$ is given by

$$\frac{d}{dx}[(f \circ g)(x)] = \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x).$$

1.7 The Chain Rule

Example 3: For $y = 2 + \sqrt{u}$ and $u = x^3 + 1$,

find $\frac{dy}{du}$, $\frac{du}{dx}$, and $\frac{dy}{dx}$.

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} \quad \text{and} \quad \frac{du}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot 3x^2 = \frac{3x^2}{2\sqrt{x^3 + 1}}$$

1.7 The Chain Rule

Quick Check 2

If $y = u^2 + u$ and $u = x^2 + x$, find $\dfrac{dy}{dx}$.

We will start by finding $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$:

$$\frac{dy}{du} = 2u + 1 \qquad \frac{du}{dx} = 2x + 1$$

Next we find $\dfrac{dy}{dx}$, remembering to substitute $x^2 + x$ for u when appropriate.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (2u+1)(2x+1) = (2(x^2+x)+1)(2x+1)$$

$$= (2x^2 + 2x + 1)(2x+1)$$

1.7 The Chain Rule

Example 4: For $y = u^2 - 3u$ and $u = 5t - 1$,

find $\dfrac{dy}{dt}$.

$$\frac{dy}{du} = 2u - 3 \quad \text{and} \quad \frac{du}{dt} = 5$$

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = (2u-3)(5)$$

$$= 10u - 15 = 10(5t-1) - 15$$

$$= 50t - 10 - 15 = 50t - 25$$

1.7 The Chain Rule

THEOREM 10: The Extended Power Rule

Suppose that $g(x)$ is a differentiable function of x. Then, for any real number k,

$$\frac{d}{dx}\left[g(x)\right]^{k} = k\left[g(x)\right]^{k-1} \cdot \frac{d}{dx}g(x)$$

1.7 The Chain Rule

Example 5: Differentiate $f(x) = \left(1+x^3\right)^{\frac{1}{2}}$.

$$\frac{d}{dx}\left(1+x^3\right)^{\frac{1}{2}} = \frac{1}{2}\left(1+x^3\right)^{\frac{1}{2}-1} \cdot 3x^2$$

$$= \frac{3x^2}{2}\left(1+x^3\right)^{-\frac{1}{2}}$$

$$= \frac{3x^2}{2\sqrt{1+x^3}}$$

1.7 The Chain Rule

Example 6:
Differentiate $f(x)=(3x-5)^4(7-x)^{10}$.

Combine Product Rule and Extended Power Rule

$$f'(x)=(3x-5)^4 10(7-x)^9(-1)+$$
$$4(3x-5)^3(7-x)^{10}(3)$$

Simplified:

$$f'(x)=2(3x-5)^3(7-x)^9(67-21x)$$

1.7 The Chain Rule

Quick Check 3

Differentiate: $f(x)=\dfrac{(2x^2-1)}{(3x^4+2)^2}$

We will combine both the quotient rule and the chain rule:

$$f'(x)=\frac{(3x^4+2)^2\cdot\frac{d}{dx}(2x^2-1)-(2x^2-1)\cdot\frac{d}{dx}((3x^4+2)^2)}{[(3x^4+2)^2]^2}$$

$$f'(x)=\frac{(3x^4+2)^2\cdot(4x)-(2x^2-1)\cdot(2(3x^4+2)(12x^3)}{(3x^4+2)^4}$$

$$f'(x)=\frac{4x(3x^4+2)^2-(2x^2-1)(72x^7+48x^3)}{(3x^4+2)^4}$$

$$f'(x)=\frac{-36x^5+24x^3+8x}{(3x^4+2)^3}$$

1.7 The Chain Rule

Section Summary

- The *composition* of $f(x)$ with $g(x)$ is written $(f \circ g)(x)$ and is defined as $(f \circ g)(x) = f(g(x))$.
- In general, $(f \circ g)(x) \neq (g \circ f)(x)$.
- The *Chain Rule* is used to differentiate a composition of functions.

If $\quad F(x) = (f \circ g)(x) = f(g(x))$

Then $\quad F'(x) = \dfrac{d}{dx}[(f \circ g)(x)] = f'\left(g\left(x\right)\right) \cdot g'\left(x\right).$

1.7 The Chain Rule

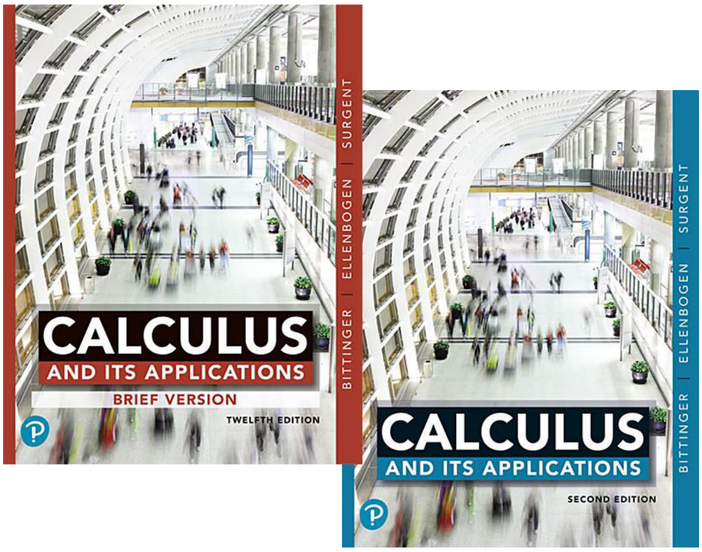
Section Summary Concluded

- The *Extended Power Rule* tells us that if $y = \left[f(x)\right]^k$, then

$$y' = \frac{d}{dx}[f(x)]^k = k[f(x)]^{k-1} \cdot f'(x).$$

Chapter 1

Differentiation



1.8 Higher Order Derivatives

OBJECTIVE

- Find derivatives of higher order.
- Given a formula for distance, find velocity and acceleration.

1.8 Higher Order Derivatives

Higher-Order Derivatives:

Consider the function given by

$$y = f(x) = x^5 - 3x^4 + x.$$

Its derivative f' is given by

$$y' = f'(x) = 5x^4 - 12x^3 + 1.$$

The derivative function f' can also be differentiated. We can think of the derivative f' as the rate of change of the slope of the tangent lines of f. It can also be regarded as the rate at which $f'(x)$ is changing.

1.8 Higher Order Derivatives

Higher-Order Derivatives (continued):

We use the notation f'' for the derivative $(f')'$.
That is,

$$f''(x) = \frac{d}{dx} f'(x)$$

We call f'' the *second derivative* of f. For

$$y = f(x) = x^5 - 3x^4 + x,$$

the second derivative is given by

$$y'' = f''(x) = 20x^3 - 36x^2.$$

1.8 Higher Order Derivatives

Higher-Order Derivatives (continued):

For higher-order derivatives, we use the notation $f^{(n)}(x)$ to express the n^{th} derivative of f.

Continuing in this manner, we have

$f^{(3)}(x) = 60x^2 - 72x,$ the third derivative of f,

$f^{(4)}(x) = 120x - 72,$ the fourth derivative of f,

$f^{(5)}(x) = 120,$ the fifth derivative of f.

1.8 Higher Order Derivatives

Higher-Order Derivatives (continued):

For $y = f(x) = x^5 - 3x^4 + x,$ we have

$$\begin{aligned} f^{(3)}(x) &= 60x^2 - 72x, \\ f^{(4)}(x) &= 120x - 72, \\ f^{(5)}(x) &= 120, \\ f^{(6)}(x) &= 0, \text{ and} \\ f^{(n)}(x) &= 0, \text{ for any integer } n \geq 6. \end{aligned}$$

1.8 Higher Order Derivatives

Higher-Order Derivatives (continued):

Leibniz's notation for the second derivative of a function given by $y = f(x)$ is

$$\frac{d^2y}{dx^2}, \ or \ \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

read "the second derivative of y with respect to x." The 2's in this notation are NOT exponents.

1.8 Higher Order Derivatives

Higher-Order Derivatives (concluded):

If $y = x^5 - 3x^4 + x$, then

$$\frac{dy}{dx} = 5x^4 - 12x^3 + 1, \qquad \frac{d^4y}{dx^4} = 120x - 72,$$

$$\frac{d^2y}{dx^2} = 20x^3 - 36x^2, \qquad \frac{d^5y}{dx^5} = 120.$$

$$\frac{d^3y}{dx^3} = 60x^2 - 72x,$$

1.8 Higher Order Derivatives

Example 1: For $y = \frac{1}{x}$, find $\frac{d^2y}{dx^2}$.

$$y = x^{-1}$$

$$\frac{dy}{dx} = -x^{-2}$$

$$\frac{d^2y}{dx^2} = 2x^{-3}, \text{ or } \frac{2}{x^3}$$

1.8 Higher Order Derivatives

Example 2: For $y = (x^2 + 10x)^{20}$, find y' and y''.

By the Extended Chain Rule, $y' = 20(x^2 + 10x)^{19}(2x + 10)$.

Using the Product Rule and Extended Chain Rule,

$$y'' = 20(x^2 + 10x)^{19} \cdot 2 + 20(2x + 10) \cdot 19(x^2 + 10x)^{18}(2x + 10)$$

$$= 40(x^2 + 10x)^{18}\left((x^2 + 10x) + 19(x + 5)(2x + 10)\right)$$

$$= 40(x^2 + 10x)^{18}\left(x^2 + 10x + 19(2x^2 + 20x + 50)\right)$$

$$= 40(x^2 + 10x)^{18}\left(x^2 + 10x + 38x^2 + 380x + 950)\right)$$

$$y'' = 40(x^2 + 10x)^{18}\left(39x^2 + 390x + 950\right).$$

1.8 Higher Order Derivatives

Quick Check 1

a.) Find y'':

(i) $y = -6x^4 + 3x^2$

(ii) $y = \dfrac{2}{x^3}$

(iii) $y = (3x^2+1)^2$

b.) Find

$$\frac{d^4}{dx^4}\left[\frac{1}{x}\right]$$

1.8 Higher Order Derivatives

Quick Check 1 Solution

a.) For the following problems, remember that $y'' = (y')'$

(i) $y = -6x^4 + 3x^2$

$$y' = -24x^3 + 6x, \quad y'' = -72x^2 + 6$$

(ii) $y = \dfrac{2}{x^3}$

$$y' = -\frac{6}{x^4}, \qquad y'' = \frac{24}{x^5}$$

(iii) $y = (3x^2+1)^2$

$$y' = 2\left(3x^2+1\right)\left(6x\right) \quad = 36x^3 + 12x, \quad y'' = 108x^2 + 12$$

1.8 Higher Order Derivatives

Quick Check 1 Solution Concluded

b.) Find $\frac{d^4}{dx^4}\left[\frac{1}{x}\right]$

$$\frac{d^4}{dx^4}\left[\frac{1}{x}\right] = \frac{d^3}{dx^3}\left[-\frac{1}{x^2}\right] = \frac{d^2}{dx^2}\left[\frac{2}{x^3}\right] = \frac{d}{dx}\left[-\frac{6}{x^4}\right]$$

$$= \frac{24}{x^5}$$

1.8 Higher Order Derivatives

DEFINITION:

The **velocity** of an object that is $s(t)$ units from a starting point at time t is given by

$$\text{Velocity} = v(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

1.8 Higher Order Derivatives

DEFINITION:

$$\text{Acceleration} = a(t) = v'(t) = s''(t).$$

1.8 Higher Order Derivatives

Example 3: For $s(t) = 10t^2$ find $v(t)$ and $a(t)$, where s is the distance from the starting point, in miles, and t is in hours. Then, find the distance, velocity, and acceleration when $t = 4$ hr.

$$v(t) = s'(t) = 20t$$

$$a(t) = v'(t) = s''(t) = 20$$

$$s(4) = 10(4)^2 = 160 \text{ mi}$$

$$v(4) = 20(4) = 80 \text{ mi/hr}$$

$$a(4) = 20 \text{ mi/hr}^2$$

Pearson
Copyright © 2020, 2016, 2012 Pearson Education, Inc.

1.8 Higher Order Derivatives

Quick Check 2

A pebble is dropped from a hot-air balloon. Find how far it has fallen, how fast it is falling, and its acceleration after 3.5 seconds. Let $s(t)=16t^2$, where t is in seconds, and s is in feet.

Distance: $s(3.5) = 16(3.5)^2 = 16(12.25) = 196$ feet

Velocity: $v(t) = s'(t) = 32t$

$v(3.5) = 32(3.5) = 112$ feet/second

Acceleration: $a(t) = v'(t) = s''(t) = 32$

$a(3.5) = 32$ feet/second^2

1.8 Higher Order Derivatives

Section Summary

- The *second derivative* is the derivative of the first derivative of a function. In symbols, $f''(x) = \frac{d}{dx}\left[f'(x)\right]$.
- The second derivative describes the rate of change of the rate of change. In other words, it describes the rate of change of the first derivative.

1.8 Higher Order Derivatives

Section Summary Concluded

- A real-life example of a second derivative is *acceleration*. If $s(t)$ represents distance as a function of time of a moving object, then $v(t) = s'(t)$ describes the speed (velocity) of the object. Any change in the speed of the object is acceleration: $a(t) = v'(t) = s''(t)$
- The common notation for the nth derivative of a function is

$$f^{(n)}(x) \quad \text{or} \quad \frac{d^n}{dx^n} f(x).$$