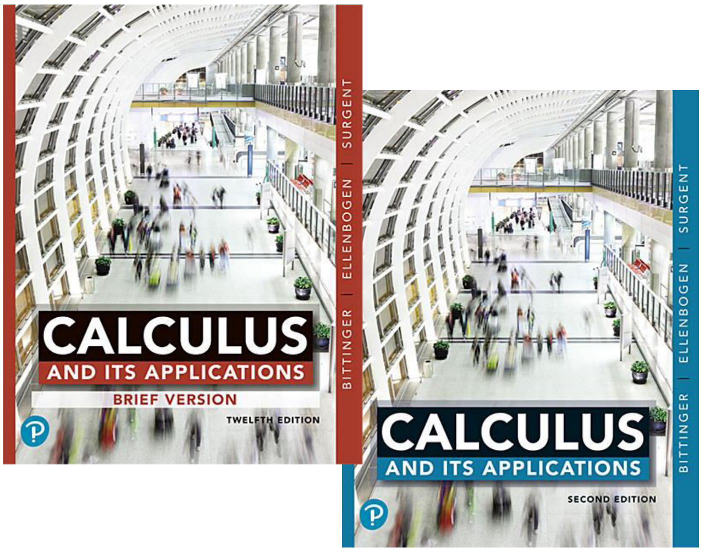
Chapter 1

Differentiation



1.5 Leibniz Notation and the Power and Sum-Difference Rules

OBJECTIVE

- Differentiate using the Power Rule or the Sum-Difference Rule.
- Differentiate a constant or a constant times a function.
- Determine points at which a tangent line has a specified slope.

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Leibniz's Notation:

When y is a function of x, we will also designate the derivative, $f'(x)$, as

$$\frac{dy}{dx},$$

which is read "the derivative of y with respect to x."

1.5 Leibniz Notation and the Power and Sum-Difference Rules

THEOREM 3: The Power Rule

For any real number k,

$$\frac{dy}{dx}x^k = k \cdot x^{k-1}$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 1: Differentiate each of the following:

a) $y = x^5$ b) $y = x$ c) $y = x^{-4}$

a) $\frac{d}{dx}x^5 = 5 \cdot x^{5-1}$

$\frac{d}{dx}x^5 = 5x^4$

b) $\frac{d}{dx}x = 1 \cdot x^{1-1}$

$\frac{d}{dx}x = 1$

c) $\frac{d}{dx}x^{-4} = -4x^{-4-1}$

$\frac{d}{dx}x^{-4} = -4x^{-5}$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Quick Check 1

a.) Differentiate:

(i) $y = x^{15}$; (ii) $y = x^{-7}$

b.) Explain why $\frac{d}{dx}(\pi^2) = 0$, not 2π.

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Quick Check 1 solution

a.) Use the Power Rule for both parts: $\frac{d}{dx}\left[x^k\right]=kx^{k-1}$

(i) $y=x^{15}, \quad \frac{dy}{dx}=15x^{14}$

(ii) $y=x^{-7}, \quad \frac{dy}{dx}=-7x^{-8}$

b.) The reason $\frac{d}{dx}(\pi^2)=0$ and not 2π, is because π is a constant, not a variable and the derivative of any constant is 0:

$$\frac{d}{dx}c=0$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 2: Differentiate:

a) $y=\sqrt{x}$ b) $y=x^{0.7}$

a)
$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}-1}$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2}x^{-\frac{1}{2}}, \; or$$

$$= \frac{1}{2x^{\frac{1}{2}}}, \; or$$

$$= \frac{1}{2\sqrt{x}}$$

b)
$$\frac{d}{dx}x^{0.7} = 0.7\cdot x^{0.7-1}$$

$$\frac{d}{dx}x^{0.7} = 0.7x^{-0.3}$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

THEOREM 4:

The derivative of a constant function is 0. That is,

$$\frac{d}{dx}c = 0.$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

THEOREM 5:

The derivative of a constant times a function is the constant times the derivative of the function. That is,

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx} f(x)$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 3: Find each of the following derivatives:

a) $\dfrac{d}{dx}7x^4$ b) $\dfrac{d}{dx}(-9x)$ c) $\dfrac{d}{dx}\left(\dfrac{1}{5x^2}\right)$

a)
$$\frac{d}{dx}7x^4 = 7\cdot\frac{d}{dx}x^4$$
$$= 7\cdot 4x^{4-1}$$
$$\frac{d}{dx}7x^4 = 28x^3$$

b)
$$\frac{d}{dx}(-9x) = -9\cdot\frac{d}{dx}x$$
$$= -9\cdot 1x^{1-1}$$
$$\frac{d}{dx}(-9x) = -9$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 3 (concluded):

c)
$$\frac{d}{dx}\left(\frac{1}{5x^2}\right) = \frac{1}{5}\cdot\frac{d}{dx}\left(\frac{1}{x^2}\right)$$
$$= \frac{1}{5}\cdot\frac{d}{dx}x^{-2}$$
$$= \frac{1}{5}\cdot -2x^{-2-1}$$
$$\frac{d}{dx}\left(\frac{1}{5x^2}\right) = -\frac{2}{5}x^{-3},\ \text{or} = -\frac{2}{5x^3}$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Quick Check 2

Differentiate each of the following:

a.) $y = 10x^9$; $\frac{dy}{dx} = 9 \cdot 10x^{9-1} = 90x^8$

b.) $y = \pi x^3$; $\frac{dy}{dx} = 3 \cdot \pi x^{3-1} = 3\pi x^2$

c.) $y = \frac{2}{3x^4} = \frac{2}{3}x^{-4}$; $\frac{dy}{dx} = -4\frac{2}{3}x^{-4-1} = -\frac{8}{3}x^{-5} = -\frac{8}{3x^5}$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

THEOREM 6: The Sum-Difference Rule

Sum: The derivative of a sum is the sum of the derivatives.

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

Difference: The derivative of a difference is the difference of the derivatives.

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 4: Find each of the following derivatives:

a) $\dfrac{d}{dx}(5x^3-7)$ b) $\dfrac{d}{dx}\left(24x - \sqrt{x} + \dfrac{5}{x}\right)$

a)
$$\frac{d}{dx}(5x^3-7) = \frac{d}{dx}(5x^3)-\frac{d}{dx}(7)$$
$$= 5\cdot\frac{d}{dx}x^3 - 0 = 5\cdot 3x^{3-1}$$
$$\frac{d}{dx}(5x^3-7) = 15x^2$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 4 (concluded):

b)
$$\frac{d}{dx}\left(24x - \sqrt{x} + \frac{5}{x}\right) = \frac{d}{dx}(24x)-\frac{d}{dx}\sqrt{x}+\frac{d}{dx}\left(\frac{5}{x}\right)$$
$$= 24\cdot\frac{d}{dx}x-\frac{d}{dx}x^{\frac{1}{2}}+5\cdot\frac{d}{dx}x^{-1}$$
$$= 24\cdot 1x^{1-1}-\frac{1}{2}x^{\frac{1}{2}-1}+5\cdot -1x^{-1-1}$$
$$= 24-\frac{1}{2}x^{-\frac{1}{2}}-5x^{-2}, \quad \text{or} \quad = 24-\frac{1}{2\sqrt{x}}-\frac{5}{x^2}$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Quick Check 3

Differentiate: $y = 3x^5 + 2\sqrt[3]{x} + \frac{1}{3x^2} + \sqrt{5}$

$$\frac{dy}{dx}\left(3x^5 + 2\sqrt[3]{x} + \frac{1}{3x^2} + \sqrt{5}\right) = \frac{dy}{dx}3x^5 + \frac{dy}{dx}2\sqrt[3]{x} + \frac{dy}{dx}\frac{1}{3x^2} + \frac{dy}{dx}\sqrt{5}$$

$$= \frac{dy}{dx}3x^5 + \frac{dy}{dx}2x^{\frac{1}{3}} + \frac{dy}{dx}\frac{1}{3}x^{-2} + \frac{dy}{dx}\sqrt{5} = 5\cdot 3x^{5-1} + \frac{2}{3}x^{\frac{1}{3}-1} - \frac{2}{3}x^{-2-1} + 0$$

$$= 15x^4 + \frac{2}{3}x^{-2/3} - \frac{2}{3}x^{-3} = 15x^4 + \frac{2}{3\sqrt[3]{x^2}} - \frac{2}{3x^3}$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 5: Find the points on the graph of $f(x) = -x^3 + 6x^2$ at which the tangent line is horizontal.

Recall that the derivative is the slope of the tangent line to a curve, and the slope of a horizontal line is 0. Therefore, we wish to find all the points on the graph of *f* where the derivative of *f* equals 0.

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 5 (continued):

So, for $f(x) = -x^3 + 6x^2$

$$f'(x) = -3 \cdot x^{3-1} + 6 \cdot 2x^{2-1}$$

$$f'(x) = -3x^2 + 12x$$

Setting $f'(x)$ equal to 0:

$$-3x^2 + 12x = 0$$

$$-3x(x-4) = 0$$

$$-3x = 0 \qquad x - 4 = 0$$

$$x = 0 \qquad x = 4$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 5 (continued):

To find the corresponding y-values for these x-values, substitute back into $f(x) = -x^3 + 6x^2$.

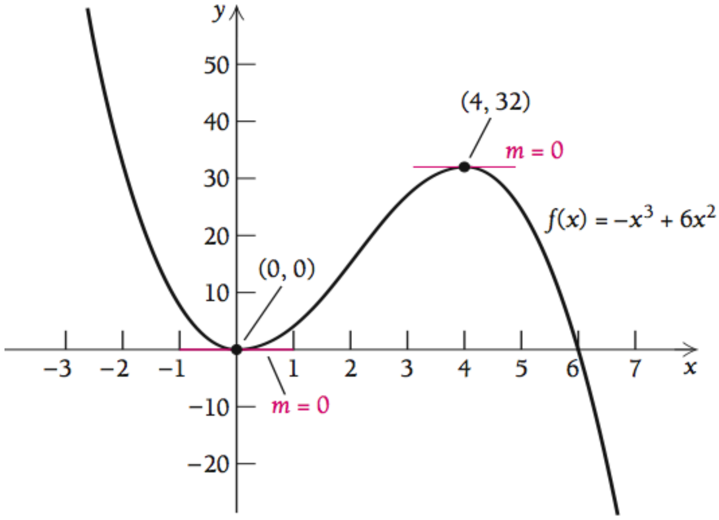
$$f(0) = -0^3 + 6 \cdot 0^2 \qquad f(4) = -4^3 + 6 \cdot 4^2$$

$$f(0) = 0 \qquad f(4) = 32$$

Thus, the tangent line to the graph of $f(x) = -x^3 + 6x^2$ is horizontal at the points (0, 0) and (4, 32).

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 5 (concluded):



1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 6: Find the points on the graph of $f(x) = -x^3 + 6x^2$ at which the tangent line has slope 6.

Here we will employ the same strategy as in Example 6, except that we are now concerned with where the derivative equals 6.

Recall that we already found that $f'(x) = -3x^2 + 12x$.

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 6 (continued):

Thus,

$$-3x^2+12x = 6$$

$$-3x^2+12x-6 = 0$$

$$\frac{-3x^2+12x-6}{-3} = \frac{0}{-3}$$

$$x^2-4x+2 = 0$$

$$x = \frac{-(-4)\pm\sqrt{(-4)^2-4\cdot 1\cdot 2}}{2\cdot 1} = \frac{4\pm\sqrt{16-8}}{2}$$

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 6 (continued):

$$x = \frac{4\pm\sqrt{8}}{2}$$

$$x = \frac{4\pm 2\sqrt{2}}{2}$$

$$x = 2+\sqrt{2} \text{ and } 2-\sqrt{2}$$

Again, to find the corresponding y-values, we will substitute these x-values into $f(x)=-x^3+6x^2$.

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 6 (continued):

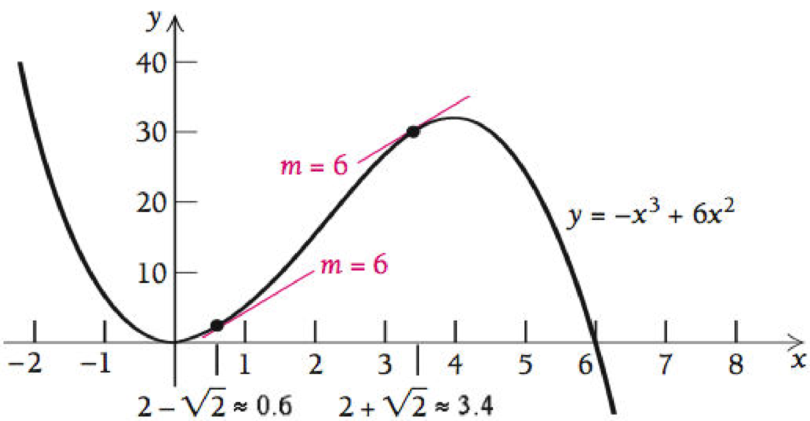
$$f\left(2+\sqrt{2}\right)=-\left(2+\sqrt{2}\right)^3+6\left(2+\sqrt{2}\right)^2$$
$$=-\left(8+12\sqrt{2}+12+2\sqrt{2}\right)+6\left(4+4\sqrt{2}+2\right)$$
$$=-20-14\sqrt{2}+36+24\sqrt{2}$$
$$=16+10\sqrt{2}$$

Similarly, $f\left(2-\sqrt{2}\right)=16-10\sqrt{2}.$

Thus, the tangent line to $f(x)=-x^3+6x^2$ has a slope of 6 at $\left(2+\sqrt{2},\ 16+10\sqrt{2}\right)$ and $\left(2-\sqrt{2},\ 16-10\sqrt{2}\right)$.

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Example 6 (concluded):



1.5 Leibniz Notation and the Power and Sum-Difference Rules

Section Summary

- Common forms of notation for the derivative of a function are

$$y' \qquad f'(x) \qquad \frac{dy}{dx} \qquad \frac{d}{dx}f(x)$$

- The *Power Rule* for differentiation is $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$, for all real numbers k.
- The derivative of a constant is zero: $\frac{d}{dx}c = 0$.

1.5 Leibniz Notation and the Power and Sum-Difference Rules

Section Summary Concluded

- The derivative of a constant times a function is the constant time the derivative of the function:

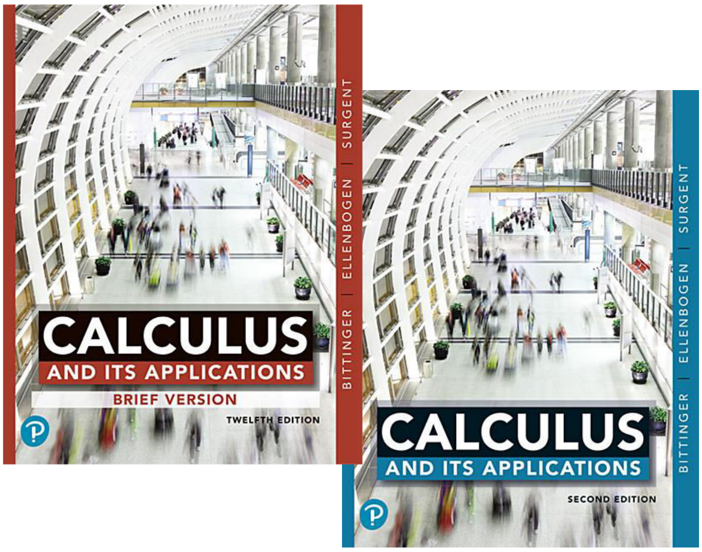
$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx} f(x)$$

- The derivative of a sum (or difference) is the sum (or difference) of the derivatives of the terms:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$$

Chapter 1

Differentiation



1.6 The Product and Quotient Rules

OBJECTIVE

- Differentiate using the Product and the Quotient Rules.
- Use the Quotient Rule to differentiate the average cost, revenue, and profit functions.

1.6 The Product and Quotient Rules

THEOREM 7: The Product Rule

Let $F(x) = f(x) \cdot g(x)$. Then,

$$F'(x) = \frac{d}{dx}[f(x) \cdot g(x)]$$

$$F'(x) = f(x) \cdot \frac{d}{dx} g(x) + g(x) \cdot \frac{d}{dx} f(x)$$

1.6 The Product and Quotient Rules

Example 1: Find $\frac{d}{dx}\left[\left(x^4 - 2x^3 - 7\right)\left(3x^2 - 5x\right)\right]$.

$$\frac{d}{dx}\left[\left(x^4 - 2x^3 - 7\right)\left(3x^2 - 5x\right)\right] =$$

$$\left(x^4 - 2x^3 - 7\right) \cdot (6x - 5) + \left(3x^2 - 5x\right) \cdot \left(4x^3 - 6x^2\right)$$

1.6 The Product and Quotient Rules

Quick Check 1

Use the Product Rule to differentiate each of the following functions. Do not simplify.

a.) $y=(2x^5+x-1)(3x-2)$

b.) $y=(\sqrt{x}+1)(\sqrt[5]{x}-x)$

1.6 The Product and Quotient Rules

Quick Check 1 Solution

a.) $y=(2x^5+x-1)(3x-2)$

Using the Product Rule: $\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f(x)\cdot\frac{d}{dx}\left[g(x)\right]+g(x)\cdot\frac{d}{dx}\left[f(x)\right]$

We get: $y'=(2x^5+x-1)(3x^{1-1}-0)+(3x-2)(5\cdot 2x^{5-1}+x^{1-1}-0)$

$$y'=3(2x^5+x-1)+(3x-2)(10x^4+1)$$

b.) $y=\left(\sqrt{x}+1\right)\left(\sqrt[5]{x}-x\right)$

Again, using the Product Rule, we get:

$$y'=(\sqrt{x}+1)(x^{\frac{1}{5}-1}-x^{1-1})+(\sqrt[5]{x}-x)(x^{\frac{1}{2}-1}+0)$$

$$y'=\left(\sqrt{x}+1\right)\left(\frac{1}{5\sqrt[5]{x^4}}-1\right)+\left(\sqrt[5]{x}-x\right)\left(\frac{1}{2\sqrt{x}}\right)$$

1.6 The Product and Quotient Rules

THEOREM 8: The Quotient Rule

If $Q(x) = \dfrac{N(x)}{D(x)}$, then,

$$Q'(x) = \frac{D(x) \cdot N'(x) - N(x) \cdot D'(x)}{\left[D(x)\right]^2}$$

1.6 The Product and Quotient Rules

Example 2: Differentiate $f(x) = \dfrac{x^2 - 3x}{x - 1}$.

$$f'(x) = \frac{(x-1)(2x-3) - (x^2 - 3x)(1)}{(x-1)^2}$$

$$f'(x) = \frac{2x^2 - 5x + 3 - x^2 + 3x}{(x-1)^2}$$

$$f'(x) = \frac{x^2 - 2x + 3}{(x-1)^2}$$

1.6 The Product and Quotient Rules

Quick Check 2

a.) Differentiate: $f(x)=\dfrac{1-3x}{x^2+2}$. Simplify your result.

b.) Show that

$$\frac{d}{dx}\left[\frac{ax+1}{bx+1}\right]=\frac{a-b}{(bx+1)^2}$$

1.6 The Product and Quotient Rules

Quick Check 2 Solution

a.) Using the Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{g(x)\cdot\dfrac{d}{dx}[f(x)]-f(x)\cdot\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$

We get: $f'(x)=\dfrac{(x^2+2)(0-3)-(1-3x)(2x+0)}{(x^2+2)^2}$

$$f'(x)=\frac{-3x^2-6-2x+6x^2}{x^4+4x^2+4}$$

$$f'(x)=\frac{3x^2-2x-6}{x^4+4x^2+4}$$

1.6 The Product and Quotient Rules

Quick Check 2 Solution Concluded

b.) Using the Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{g(x)\cdot\dfrac{d}{dx}[f(x)]-f(x)\cdot\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$

We know that: $\dfrac{d}{dx}\left[\dfrac{ax+1}{bx+1}\right]=\dfrac{(bx+1)(a)-(ax+1)(b)}{\left(bx+1\right)^2}$

$$=\frac{(abx+a)-(abx+b)}{(bx+1)^2}$$

$$=\frac{abx+a-abx-b}{(bx+1)^2}$$

$$=\frac{a-b}{(bx+1)^2}$$

1.6 The Product and Quotient Rules

DEFINITION:

If *C(x)* is the cost of producing *x* items, then the **average cost** of producing *x* items is $\dfrac{C(x)}{x}$.

If *R(x)* is the revenue from the sale of *x* items, then the **average revenue** from selling *x* items is $\dfrac{R(x)}{x}$.

If *P(x)* is the profit from the sale of *x* items, then the **average profit** from selling *x* items is $\dfrac{P(x)}{x}$.

1.6 The Product and Quotient Rules

Example 3: Paulsen's Greenhouse finds that the cost, in dollars, of growing x hundred geraniums is given by $C(x) = 200 + 100 \cdot \sqrt[4]{x}$. If the revenue from the sale of x hundred geraniums is given by $R(x) = 120 + 90 \cdot \sqrt{x}$, find each of the following.

a) The average cost, the average revenue, and the average profit when x hundred geraniums are grown and sold.

b) The rate at which average profit is changing when 300 geraniums are being grown.

1.6 The Product and Quotient Rules

Example 3 (continued):

a) We let A_C, A_R, and A_P represent average cost, average revenue, and average profit.

$$A_C(x) = \frac{C(x)}{x} = \frac{200 + 100 \cdot \sqrt[4]{x}}{x}$$

$$A_R(x) = \frac{R(x)}{x} = \frac{120 + 90 \cdot \sqrt{x}}{x}$$

$$A_P(x) = \frac{P(x)}{x} = \frac{R(x) - C(x)}{x} = \frac{120 + 90 \cdot \sqrt{x} - 200 - 100 \cdot \sqrt[4]{x}}{x}$$

$$A_P(x) = \frac{-80 + 90 \cdot \sqrt{x} - 100 \cdot \sqrt[4]{x}}{x}$$

1.6 The Product and Quotient Rules

Example 3 (continued):

b) First we must find $A_P'(x)$. Then we can substitute 3 (hundred) into $A_P'(x)$.

$$A_P'(x) =$$

$$= \frac{x\left(90 \cdot \frac{1}{2}x^{-\frac{1}{2}} - 100 \cdot \frac{1}{4}x^{-\frac{3}{4}}\right) - \left(-80 + 90 \cdot x^{\frac{1}{2}} - 100 \cdot x^{\frac{1}{4}}\right) \cdot 1}{x^2}$$

$$= \frac{45x^{\frac{1}{2}} - 25x^{\frac{1}{4}} + 80 - 90x^{\frac{1}{2}} + 100x^{\frac{1}{4}}}{x^2}$$

1.6 The Product and Quotient Rules

Example 3 (concluded):

$$A_P'(x) = \frac{80 - 45x^{\frac{1}{2}} + 75x^{\frac{1}{4}}}{x^2}$$

$$A_P'(3) = \frac{80 - 45(3)^{\frac{1}{2}} + 75(3)^{\frac{1}{4}}}{3^2}$$

$$A_P'(3) \approx 11.196$$

Thus, at 300 geraniums, Paulsen’s average profit is increasing by about \$11.20 per plant.

1.6 The Product and Quotient Rules

Section Summary

- The *Product Rule* is:

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f(x)\cdot\frac{d}{dx}[g(x)]+g(x)\cdot\frac{d}{dx}[f(x)]$$

- The *Quotient Rule* is:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\cdot\frac{d}{dx}[f(x)]-f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

- Be careful to note the order in which you write out the factors when using the Quotient Rule. Because the Quotient Rule involves subtraction and division, the order in which you perform the operations is important.