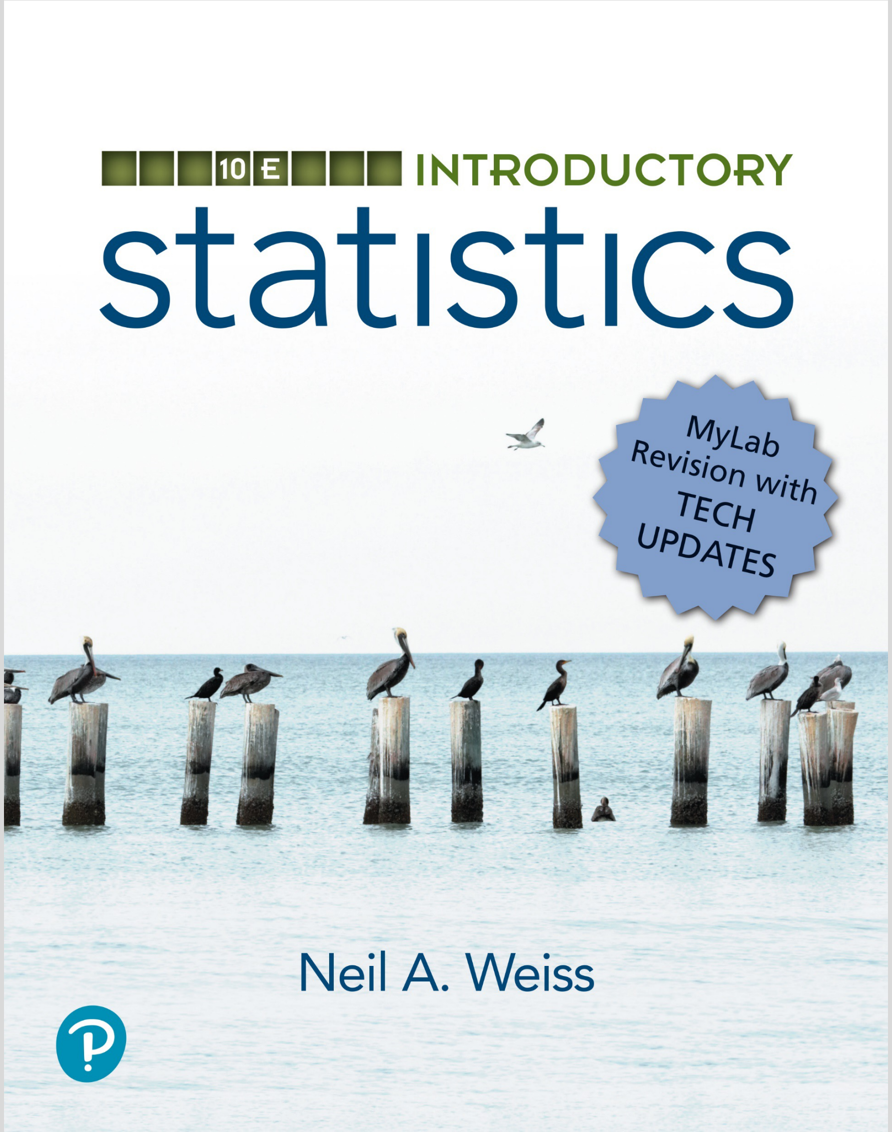
Chapter 4

Probability Concepts



Chapter 4

Probability Concepts

Section 4.1

Probability Basics

Definition 4.1

Probability for Equally Likely Outcomes (*f*/*N* Rule)

Suppose an experiment has N possible outcomes, all equally likely. An event that can occur in f ways has probability f/N of occurring:

$$\text{Probability of an event} = \frac{\overset{\swarrow \text{ Number of ways event can occur}}{f}}{\underset{\nwarrow \text{ Total number of possible outcomes}}{N}}.$$

Figure 4.1

Possible outcomes for rolling a pair of dice

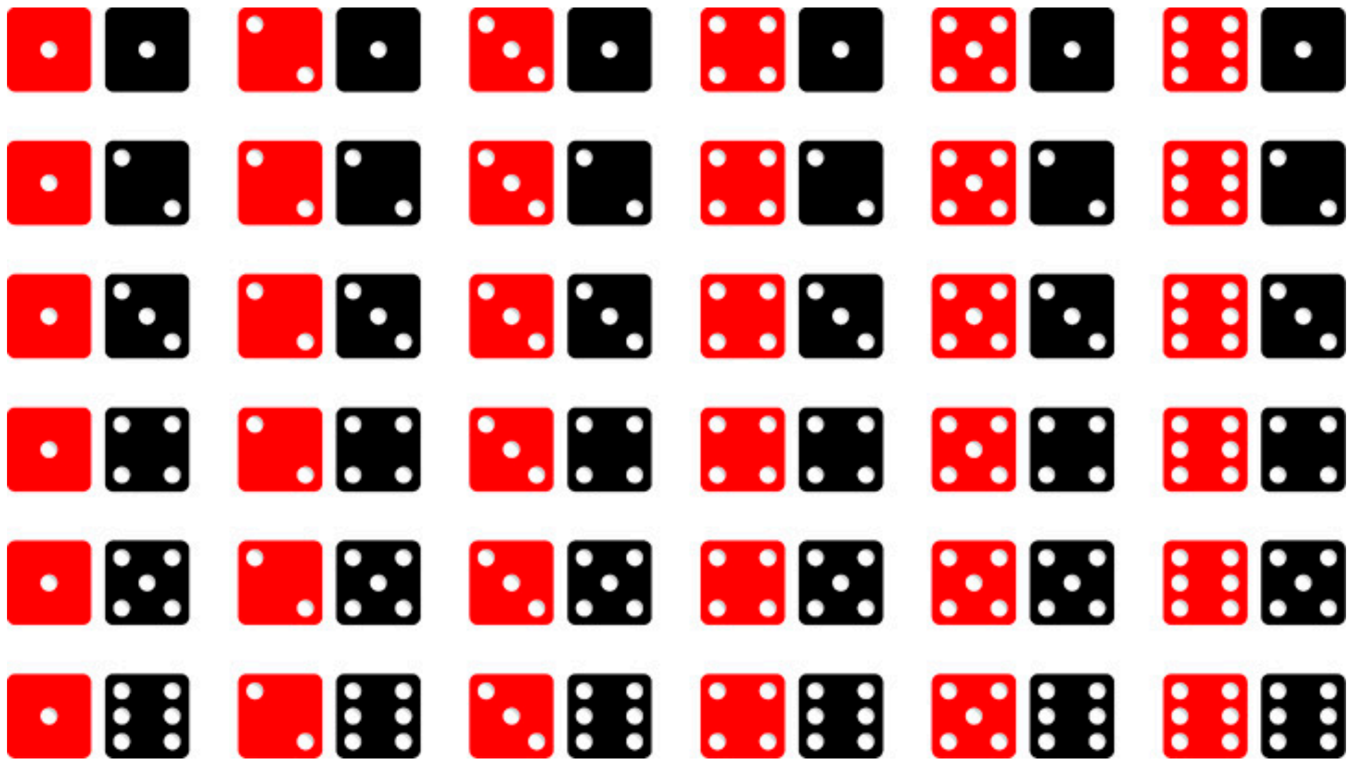
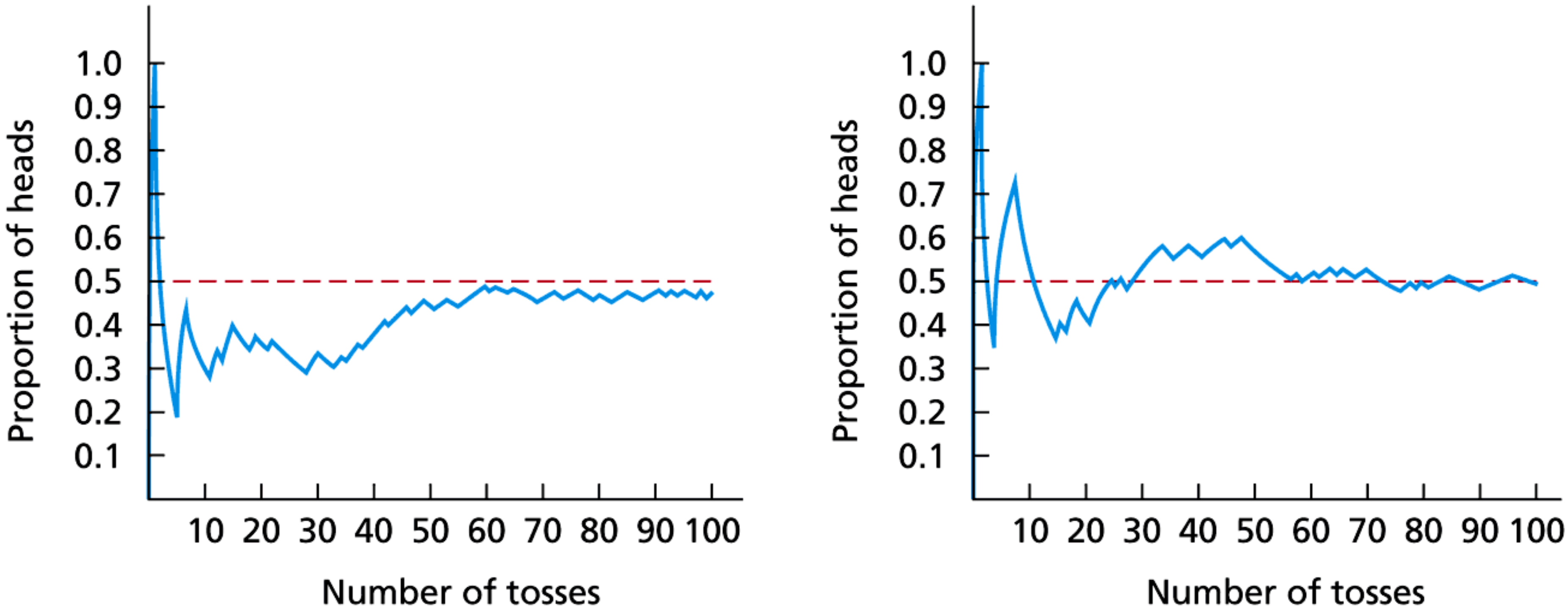


Figure 4.2

Two computer simulations of tossing a balanced coin 100 times



Key Fact 4.1

Basic Properties of Probabilities

Property 1: The probability of an event is always between 0 and 1, inclusive.

Property 2: The probability of an event that cannot occur is 0. (An event that cannot occur is called an **impossible event.**)

Property 3: The probability of an event that must occur is 1. (An event that must occur is called a **certain event.**)

Section 4.2

Events

Definition 4.2

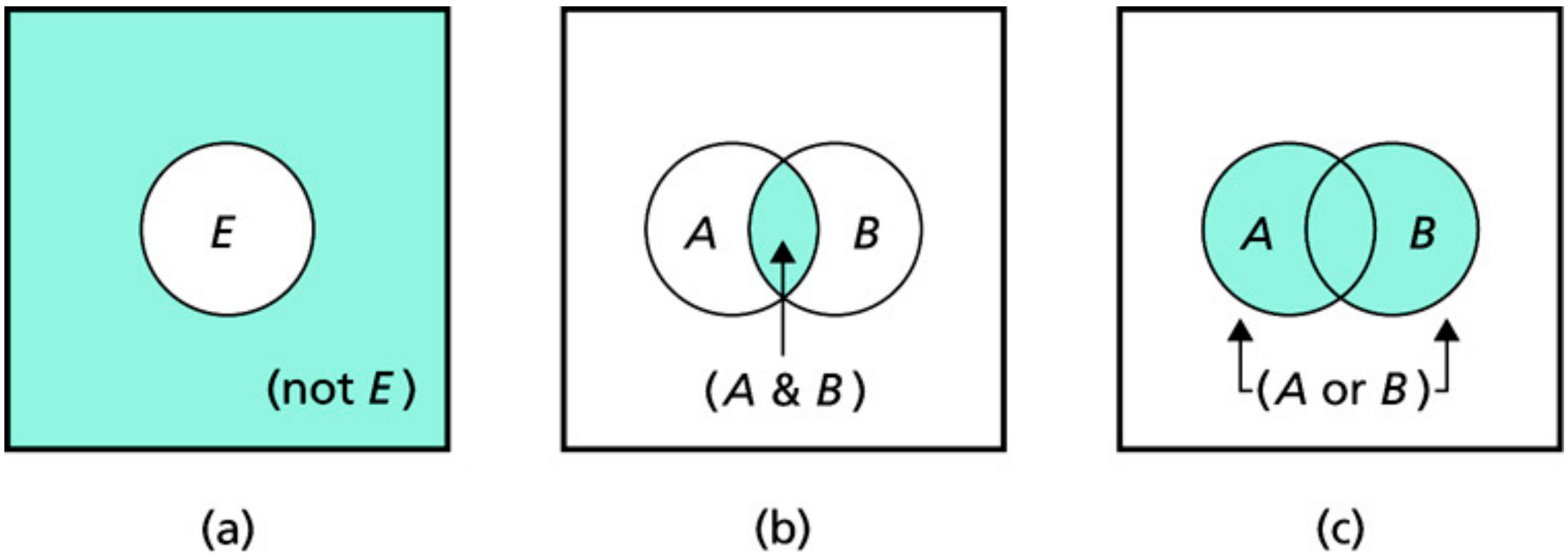
Sample Space and Event

Sample space: The collection of all possible outcomes for an experiment.

Event: A collection of outcomes for the experiment, that is, any subset of the sample space. An event **occurs** if and only if the outcome of the experiment is a member of the event.

Figure 4.9

Venn diagrams for (a) event (not E), (b) event (A & B), and (c) event (A or B)



Definition 4.3

Relationships Among Events

(not E): The event "E does not occur"

(A & B): The event "both A and B occur"

(A or B): The event "either A or B or both occur"

Definition 4.4

Mutually Exclusive Events

Two or more events are **mutually exclusive events** if no two of them have outcomes in common.

Figure 4.14

(a) Two mutually exclusive events;
(b) two non–mutually exclusive events

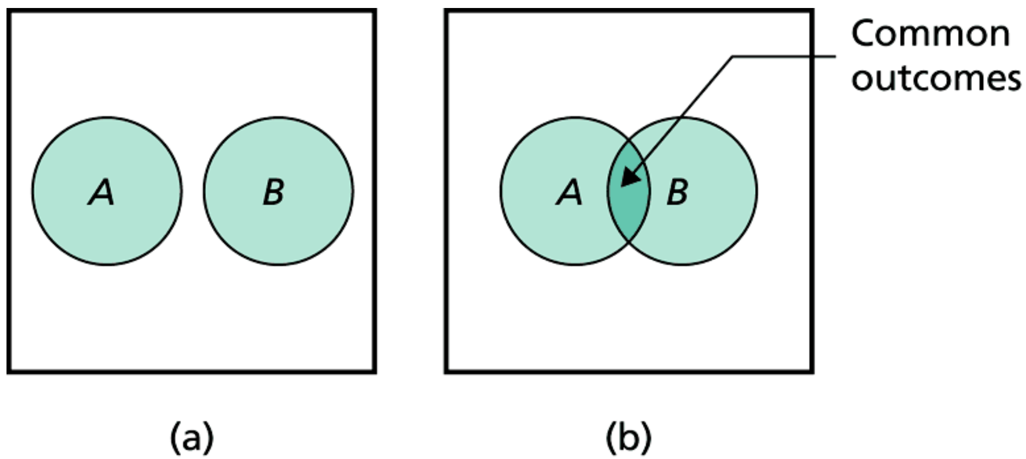
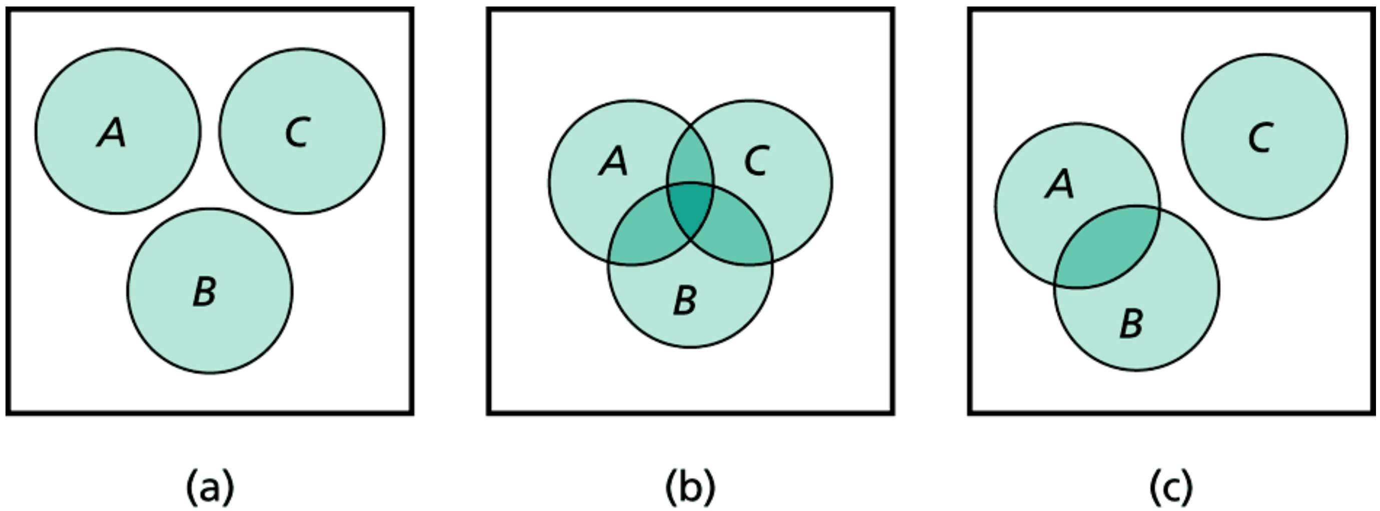


Figure 4.15

(a) Three mutually exclusive events;
(b) three non–mutually exclusive events;
(c) three non–mutually exclusive events



Section 4.3

Some Rules of Probability

Definition 4.5

Probability Notation

If E is an event, then $\boldsymbol{P(E)}$ represents the probability that event E occurs. It is read “the probability of E.”

Formula 4.1

The Special Addition Rule

If event A and event B are mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B).$$

More generally, if events A, B, C, ... are mutually exclusive, then

$$P(A \text{ or } B \text{ or } C \text{ or } \cdots) = P(A) + P(B) + P(C) + \cdots .$$

Formula 4.2

The Complementation Rule

For any event E,

$$P(E) = 1 - P(\text{not } E).$$

Formula 4.3

The General Addition Rule

If A and B are any two events, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \;\&\; B).$$

Section 4.4

Contingency Tables; Joint and Marginal Probabilities

Table 4.6

Contingency table for age and rank of faculty members

Age (yr)		Rank				
		Full professor R_1	Associate professor R_2	Assistant professor R_3	Instructor R_4	**Total**
Age (yr)	Under 30 A_1	2	3	57	6	68
	30–39 A_2	52	170	163	17	402
	40–49 A_3	156	125	61	6	348
	50–59 A_4	145	68	36	4	253
	60 & over A_5	75	15	3	0	93
	Total	430	381	320	33	1164

Table 4.7

Joint probability distribution corresponding to Table 4.6

Age (yr)	Rank: Full professor R_1	Associate professor R_2	Assistant professor R_3	Instructor R_4	$P(A_i)$
Under 30 A_1	0.002	0.003	0.049	0.005	0.058
30–39 A_2	0.045	0.146	0.140	0.015	0.345
40–49 A_3	0.134	0.107	0.052	0.005	0.299
50–59 A_4	0.125	0.058	0.031	0.003	0.217
60 & over A_5	0.064	0.013	0.003	0.000	0.080
$P(R_j)$	0.369	0.327	0.275	0.028	1.000

Section 4.5

Conditional Probability

Definition 4.6

Conditional Probability

The probability that event *B* occurs given that event *A* occurs is called a **conditional probability**. It is denoted ***P*(*B* | *A*),** which is read "the probability of *B* given *A*." We call *A* the **given event.**

Formula 4.4

The Conditional Probability Rule

If A and B are any two events with $P(A) > 0$, then

$$P(B \mid A) = \frac{P(A \,\&\, B)}{P(A)}.$$

Table 4.9

Joint probability distribution of marital status and gender

		Marital status				
		Single M_1	Married M_2	Widowed M_3	Divorced M_4	$P(S_i)$
Gender	Male S_1	0.147	0.281	0.013	0.044	0.485
	Female S_2	0.121	0.284	0.050	0.060	0.515
	$P(M_j)$	0.268	0.565	0.063	0.104	1.000

Section 4.6

The Multiplication Rule; Independence

Formula 4.5

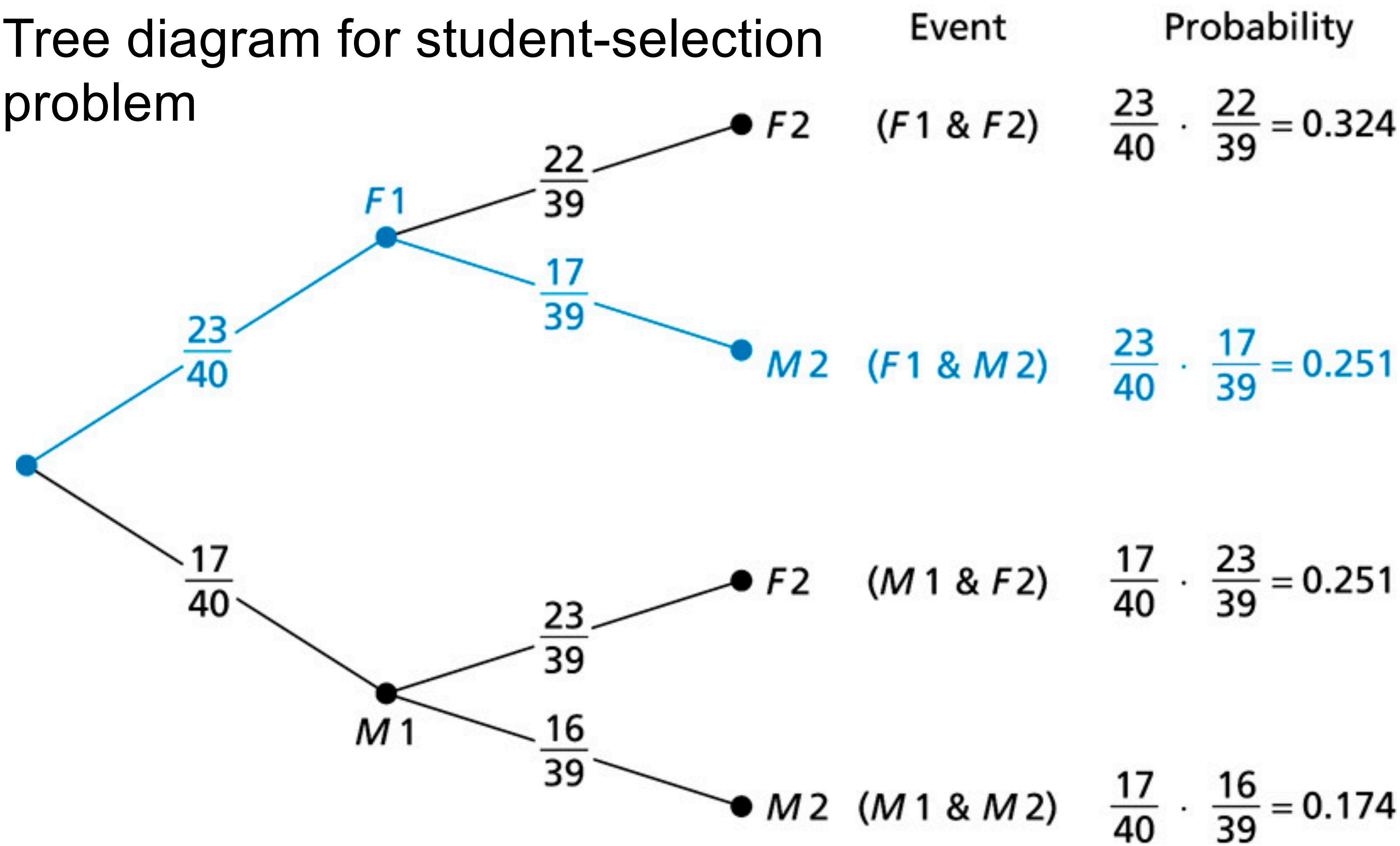
The General Multiplication Rule

If A and B are any two events, then

$$P(A \,\&\, B) = P(A) \cdot P(B \mid A).$$

Figure 4.25

Tree diagram for student-selection problem



Definition 4.7

Independent Events

Event B is said to be **independent** of event A if $P(B \mid A) = P(B)$.

Formula 4.6

The Special Multiplication Rule (for Two Independent Events)

If A and B are independent events, then

$$P(A \,\&\, B) = P(A) \cdot P(B),$$

and conversely, if $P(A \,\&\, B) = P(A) \cdot P(B)$, then A and B are independent events.

Formula 4.7

The Special Multiplication Rule

If events A, B, C, . . . are independent, then

$$P(A \,\&\, B \,\&\, C \,\&\, \cdots) = P(A) \cdot P(B) \cdot P(C) \cdots.$$

Section 4.7

Bayes's Rule

Formula 4.8

The Rule of Total Probability

Suppose that events $A_1, A_2, \ldots, A_k$ are mutually exclusive and exhaustive; that is, exactly one of the events must occur. Then for any event B,

$$P(B) = \sum_{j=1}^{k} P(A_j) \cdot P(B \mid A_j).$$

Table 4.11 & 4.12

Percentage distribution for region of residence and percentage of seniors in each region

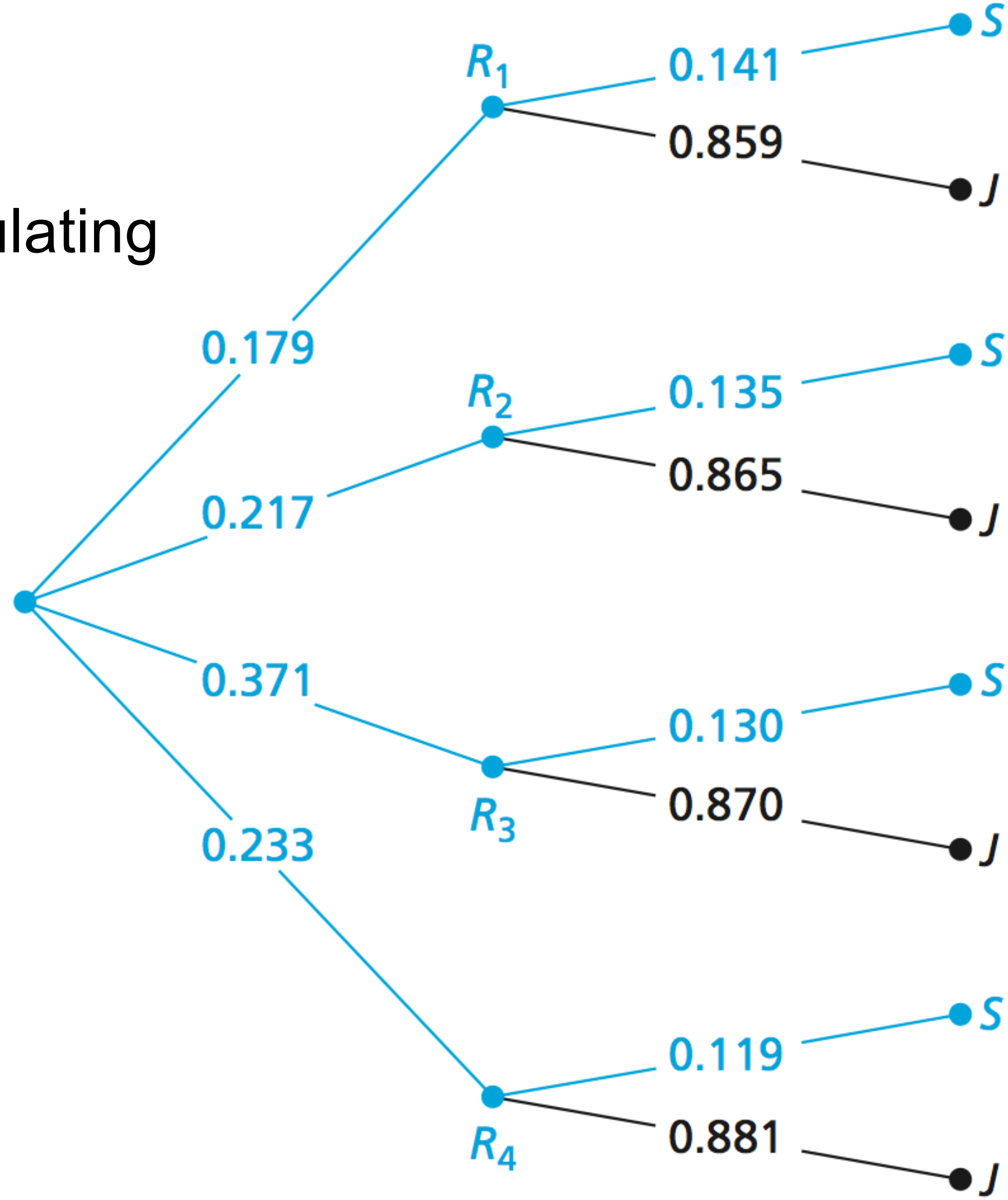
Region	Percentage of U.S. population	Percentage seniors
Northeast	17.9	14.1
Midwest	21.7	13.5
South	37.1	13.0
West	23.3	11.9
	100.0	

Probabilities derived from Table 4.11

$$P(R_1) = 0.179 \quad P(S \mid R_1) = 0.141$$
$$P(R_2) = 0.217 \quad P(S \mid R_2) = 0.135$$
$$P(R_3) = 0.371 \quad P(S \mid R_3) = 0.130$$
$$P(R_4) = 0.233 \quad P(S \mid R_4) = 0.119$$

Figure 4.27

Tree diagram for calculating $P(S)$, using the rule of total probability



Formula 4.9

Bayes's Rule

Suppose that events $A_1, A_2, \ldots, A_k$ are mutually exclusive and exhaustive. Then for any event B,

$$P(A_i \mid B) = \frac{P(A_i) \cdot P(B \mid A_i)}{\sum_{j=1}^{k} P(A_j) \cdot P(B \mid A_j)},$$

where A_i can be any one of events $A_1, A_2, \ldots, A_k$.

Section 4.8

Counting Rules

Key Fact 4.2

The Basic Counting Rule (BCR)

Suppose that r actions are to be performed in a definite order. Further suppose that there are m_1 possibilities for the first action and that corresponding to each of these possibilities are m_2 possibilities for the second action, and so on. Then there are $m_1 \cdot m_2 \cdots m_r$ possibilities altogether for the r actions.

Definition 4.8

Factorials

The product of the first k positive integers (counting numbers) is called ***k* factorial** and is denoted $\boldsymbol{k!}$. In symbols,

$$k! = k(k-1)\cdots 2\cdot 1.$$

We also define $0! = 1$.

Table 4.14

Possible permutations of three letters from the collection of five letters

abc	*abd*	*abe*	*acd*	*ace*	*ade*	*bcd*	*bce*	*bde*	*cde*
acb	*adb*	*aeb*	*adc*	*aec*	*aed*	*bdc*	*bec*	*bed*	*ced*
bac	*bad*	*bae*	*cad*	*cae*	*dae*	*cbd*	*cbe*	*dbe*	*dce*
bca	*bda*	*bea*	*cda*	*cea*	*dea*	*cdb*	*ceb*	*deb*	*dec*
cab	*dab*	*eab*	*dac*	*eac*	*ead*	*dbc*	*ebc*	*ebd*	*ecd*
cba	*dba*	*eba*	*dca*	*eca*	*eda*	*dcb*	*ecb*	*edb*	*edc*

Formula 4.10

The Permutations Rule

The number of possible permutations of r objects from a collection of m objects is given by the formula

$$_{m}P_{r} = \frac{m!}{(m-r)!}.$$

Formula 4.11

The Special Permutations Rule

The number of possible permutations of m objects among themselves is $m!$.

Formula 4.12

The Combinations Rule

The number of possible combinations of r objects from a collection of m objects is given by the formula

$$_{m}C_{r} = \frac{m!}{r!\,(m-r)!}.$$

Formula 4.13

Number of Possible Samples

The number of possible samples of size n from a population of size N is ${}_{N}C_{n}$.

Figure 4.29

Calculating the number of outcomes in which exactly 2 of the 5 TVs selected are defective