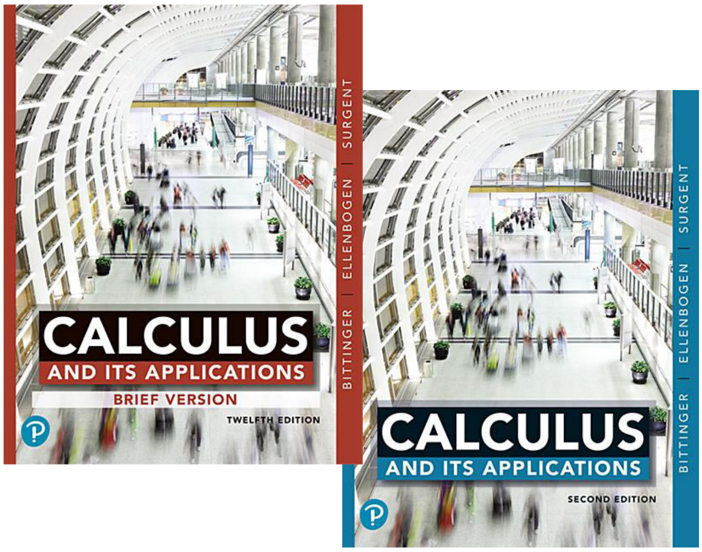
Chapter 2

Exponential and Logarithmic Functions



2.1 Exponential and Logarithmic Functions of Natural Base, e

OBJECTIVE

- Graph and solve exponential functions of the natural base, *e*.
- Graph and solve logarithmic functions of the natural base, *e*.

2.1 Exponential and Logarithmic Functions of Natural Base, e

Definition

The natural base, denoted e, is the value given by

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828182845...$$

2.1 Exponential and Logarithmic Functions of Natural Base, e

THEOREM 1: Continuous Exponential Growth

A quantity P, growing continuously at annual percentage rate r, expressed as a decimal, has a future value after t years given by $A = Pe^{rt}$.

2.1 Exponential and Logarithmic Functions of Natural Base, e

Example 1: Continuous Growth.
Luis invests \$5000 in an account that earns interest at an annual rate of 3.25%. Find the future value of Luis's account after 5 years if interest is compounded continuously.

Solution: We have $P = \$5000$, $r = 0.0325$, and $t = 5$. Since interest is compounded continuously, after 5 years, Luis's account will be worth:

$$\begin{aligned} A &= 5000e^{.0325(5)} \\ &= \$5882.24 \end{aligned}$$

2.1 Exponential and Logarithmic Functions of Natural Base, e

Quick Check 1

Rachel invests \$20,000 in an account that earns interest at an annual rate of 4%. Find the future value of her account after 3 years if interest is compounded continuously.

Solution: We have $P = \$20{,}000$, $r = 0.04$, and $t = 3$. Since interest is compounded continuously, after 3 years, Rachel's account will be worth:

$$\begin{aligned} A &= 20{,}000e^{.04(3)} \\ &= \$22{,}549.94 \end{aligned}$$

2.1 Exponential and Logarithmic Functions of Natural Base, e

DEFINITION

For any positive number x, the **natural logarithm**, or **logarithm, base *e***, of x, is given by $\ln x = \log_e x$.

The equation $y = \ln x$ is equivalent to $e^y = x$.

2.1 Exponential and Logarithmic Functions of Natural Base, e

Example 2: Find each of the following. If necessary, use a calculator to approximate values to three decimal places.

a) $\ln e^4$

b) $\ln 1$

c) $\ln e^{-1}$

d) $\ln 20$

2.1 Exponential and Logarithmic Functions of Natural Base, e

Example 2 Solution:

a) $\ln e^4$: The equation $y = \ln e^4$ is equivalent to $e^y = e^4$.
Thus, $y = 4$.

b) $\ln 1$: The equation $y = \ln 1$ is equivalent to $e^y = 1$.
Thus, $y = 0$.

c) $\ln e^{-1}$: The equation $y = \ln e^{-1}$ is equivalent to $e^y = e^{-1}$.
Thus, $y = -1$.

d) $\ln 20$: Using a calculator, we get $\ln 20 \approx 2.996$.

2.1 Exponential and Logarithmic Functions of Natural Base, e

Theorem 2: Properties of Natural Logarithms

P1. $\ln(MN) = \ln M + \ln N$

P2. $\ln\left(\dfrac{M}{N}\right) = \ln M - \ln N$

P3. $\ln(M^k) = k \ln M$

P4. $\ln e = 1$

Note: P1, P2, and P3 require that M and N are positive.

2.1 Exponential and Logarithmic Functions of Natural Base, e

Theorem 2 (concluded):

P5. $\ln 1 = 0$

P6. $\log_b M = \dfrac{\ln M}{\ln b}$ and $\ln M = \dfrac{\log M}{\log b}$

P7. $\ln e^x = x$, for all real numbers x

P8. $e^{\ln x} = x$, for all $x > 0$

2.1 Exponential and Logarithmic Functions of Natural Base, e

Example 3: Given ln 2 = 0.6931 and ln 3 = 1.0986, use the properties of natural logarithms to find each of the following:

a) $\ln 6$

b) $\ln 81$

c) $\ln \frac{1}{3}$

d) $\ln\left(2e^5\right)$

e) $\log_2 3$

2.1 Exponential and Logarithmic Functions of Natural Base, e

Example 3 solution:

a) $\ln 6 = \ln(2 \cdot 3)$

$= \ln 2 + \ln 3$ By Property P1

$= 0.6931 + 1.0986$

$= 1.7917$

b) $\ln 81 = \ln(3^4)$

$= 4\ln 3$ By Property P3

$= 4(1.0986)$

$= 4.3944$

2.1 Exponential and Logarithmic Functions of Natural Base, e

Example 3 solution concluded:

c) $\ln \frac{1}{3} = \ln 1 - \ln 3$ By Property P2

$= 0 - 1.0986 = -1.0986$ By Property P5

d) $\ln(2e^5) = \ln 2 + \ln e^5$ By Property P1

$= \ln 2 + 5$ By Property P7

$= 0.6931 + 5$

$= 5.6931$

e) $\log_2 3 = \frac{\ln 3}{\ln 2} = \frac{1.0986}{0.6931} \approx 1.5851$

By Property 6

2.1 Exponential and Logarithmic Functions of Natural Base, e

THEOREM 3:

$\ln x$ exists only for positive numbers x.

The domain is $(0, \infty)$.

$\ln x < 0$, for $0 < x < 1$;

$\ln x = 0$, for $x = 1$;

$\ln x > 0$, for $x > 1$.

The function $f(x) = \ln x$, is always increasing.

The range is $(-\infty, \infty)$.

2.1 Exponential and Logarithmic Functions of Natural Base, e

Example 4: Find the domain of $f(x) = \ln(5 - 2x)$.

Solution:

$$5 - 2x > 0$$
$$-2x > -5$$
$$x < \frac{5}{2}$$

The domain of f is $\left(-\infty, \frac{5}{2}\right)$.

2.1 Exponential and Logarithmic Functions of Natural Base, e

Quick Check 2:

Find the domain of $f(x) = \ln(7 - 3x)$.

Solution:

$$7 - 3x > 0$$
$$-3x > -7$$
$$x < \frac{7}{3}$$

The domain of f is $\left(-\infty, \frac{7}{3}\right)$.

Example 5: Solve the following exponential equations using logarithms. Give answers to three decimal places.

a) $e^{3x} = 2$
b) $250e^{0.015t} = 750$

2.1 Exponential and Logarithmic Functions of Natural Base, e

Example 5 continued:

Solution:

a) $e^{3x} = 2$

$\ln e^{3x} = \ln 2$ Take Natural logarithms of each side

$3x = \ln 2$ By Property P7

$x = \dfrac{\ln 2}{3}$ Divide both sides by 3

$x \approx 0.231$ Using a calculator

2.1 Exponential and Logarithmic Functions of Natural Base, e

Example 5 concluded:

Solution:

b) $250e^{0.015t} = 750$

$e^{0.015t} = 3$ Isolate the exponential expression

$\ln e^{0.015t} = \ln 3$ Take Natural logarithms of each side

$0.015t = \ln 3$ By Property P7

$t = \dfrac{\ln 3}{0.015}$ Divide both sides by 0.015

$t \approx 73.24$ Using a calculator

2.1 Exponential and Logarithmic Functions of Natural Base, e

Theorem 4

The **exponential growth rate** r (expressed as a decimal) and the **doubling time** T are related by

$$rT = \ln 2, \text{ or } r = \frac{\ln 2}{T}, \text{ and } T = \frac{\ln 2}{r}.$$

2.1 Exponential and Logarithmic Functions of Natural Base, e

Example 6: Business: Facebook Membership. Facebook connects people with other members they designate as friends. During its period of heaviest growth, membership in Facebook was doubling every 6 months. What was the exponential growth rate of Facebook membership, as a percentage?

Solution: We have $r = \dfrac{\ln 2}{6} \approx 0.116$

Thus, the exponential growth rate of Facebook membership was 11.6% per month.

2.1 Exponential and Logarithmic Functions of Natural Base, e

Example 7: Solve $5+7\ln(x+3)=19.$

Solution:

$5+7\ln(x+3)=19$	Start by isolating the logarithmic expression
$7\ln(x+3)=14$	By subtracting 5 from both sides
$\ln(x+3)=2$	By dividing both sides by 7
$x+3=e^2$	By writing equivalent exponential equation
$x=e^2-3$	By subtracting 3 from both sides
$x\approx 4.389$	

2.1 Exponential and Logarithmic Functions of Natural Base, e

Section Summary

- The natural base, denoted e, is the value given by

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=2.71828182845...$$

- Continuous Exponential Growth: A quantity P, growing continuously at annual percentage rate r, expressed as a decimal, has a future value after t years given by $A=Pe^{rt}$.

2.1 Exponential and Logarithmic Functions of Natural Base, e

Section Summary Continued

- The exponential growth rate r and the doubling time T are related by $rT = \ln 2$, So, $r = \dfrac{\ln 2}{T}$, and $T = \dfrac{\ln 2}{r}$.
- The natural logarithm of x, is given by $\ln x = \log_e x$.
- The equation $y = \ln x$ is equivalent to $e^y = x$.
- The function $f(x) = \ln x$, is always increasing with domain $(-\infty, \infty)$ and range $(0, \infty)$.

2.1 Exponential and Logarithmic Functions of Natural Base, e

Section Summary Continued

Properties of Natural Logarithms

P1. $\ln(MN) = \ln M + \ln N$

P2. $\ln\left(\dfrac{M}{N}\right) = \ln M - \ln N$

P3. $\ln(M^k) = k \ln M$

P4. $\ln e = 1$

P1, P2, and P3 require that M and N are positive.

2.1 Exponential and Logarithmic Functions of Natural Base, e

Section Summary Concluded

Properties of Natural Logarithms

P5. $\ln 1 = 0$

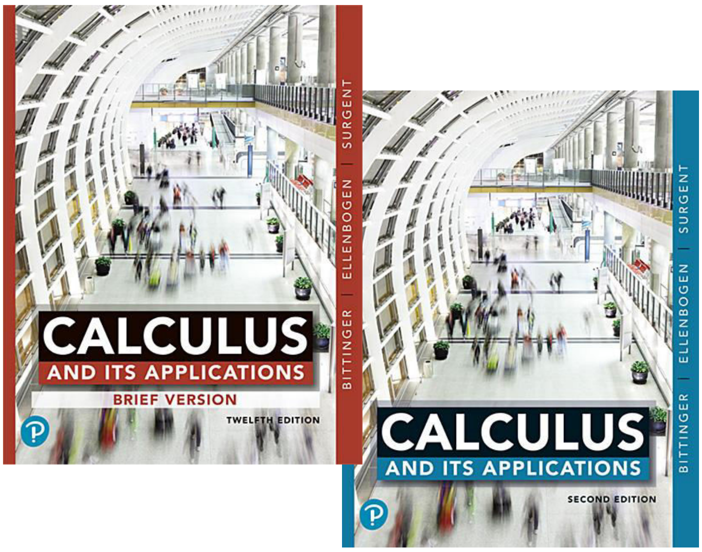
P6. $\log_b M = \dfrac{\ln M}{\ln b}$ and $\ln M = \dfrac{\log M}{\log b}$

P7. $\ln e^x = x$, for all real numbers x

P8. $e^{\ln x} = x$, for all $x > 0$

Chapter 2

Exponential and Logarithmic Functions



2.2 Derivatives of Exponential (Base-e) Functions

OBJECTIVE

- Differentiate exponential (base-e) functions.
- Solve applied problems involving exponential (base-*e*) functions and their derivatives.

2.2 Derivatives of Exponential (Base-e) Functions

THEOREM 5

The derivative of the function *f* given by $f(x) = e^x$ is itself:

$$f'(x) = f(x), \quad \text{or} \quad \frac{d}{dx}e^x = e^x$$

2.2 Derivatives of Exponential (Base-e) Functions

Example 1: Find dy/dx:

a) $y = 3e^x$; b) $y = x^2 e^x$; c) $y = \dfrac{e^x}{x^3}$.

$$\text{a) } \frac{dy}{dx}\left(3e^x\right) = 3\frac{d}{dx}e^x = 3e^x$$

$$\text{b) } \frac{d}{dx}\left(x^2 e^x\right) = x^2 \cdot e^x + e^x \cdot 2x = e^x\left(x^2 + 2x\right)$$

2.2 Derivatives of Exponential (Base-e) Functions

Example 2 (concluded):

$$\text{c) } \frac{d}{dx}\left(\frac{e^x}{x^3}\right) = \frac{x^3 \cdot e^x - e^x \cdot 3x^2}{\left(x^3\right)^2} = \frac{x^2 e^x\left(x-3\right)}{x^6} = \frac{e^x(x-3)}{x^4}$$

2.2 Derivatives of Exponential (Base-e) Functions

Quick Check 1

Differentiate:

a.) $y = 6e^x$, $\frac{dy}{dx}\left(6e^x\right) = 6e^x$

b.) $y = x^3e^x$, $\frac{dy}{dx}\left(x^3e^x\right) = 3x^2e^x + x^3e^x = x^2e^x(x+3)$

c.) $y = \frac{e^x}{x^2}$, $\frac{dy}{dx}\left(\frac{e^x}{x^2}\right) = \frac{x^2e^x - e^x(2x)}{x^4} = \frac{xe^x(x-2)}{x^4} = \frac{e^x(x-2)}{x^3}$

2.2 Derivatives of Exponential (Base-e) Functions

THEOREM 6

$$\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot f'(x)$$

or

$$\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$$

The derivative of *e* to some power is the product of *e* to that power and the derivative of the power.

2.2 Derivatives of Exponential (Base-e) Functions

Example 3: Differentiate each of the following with respect to x:

a) $y = e^{8x}$; b) $y = e^{-x^2+4x-7}$; c) $y = e^{\sqrt{x^2-3}}$.

$$\text{a)}\ \frac{d}{dx}e^{8x} = e^{8x} \cdot 8 = 8e^{8x}$$

$$\text{b)}\ \frac{d}{dx}e^{-x^2+4x-7} = e^{-x^2+4x-7} \cdot (-2x+4)$$

2.2 Derivatives of Exponential (Base-e) Functions

Example 3 (concluded):

$$\text{c)}\ \frac{d}{dx}e^{\sqrt{x^2-3}} = \frac{d}{dx}e^{\left(x^2-3\right)^{\frac{1}{2}}}$$

$$= e^{\left(x^2-3\right)^{\frac{1}{2}}} \cdot \frac{1}{2}\left(x^2-3\right)^{-\frac{1}{2}} \cdot 2x$$

$$= \frac{xe^{\sqrt{x^2-3}}}{\sqrt{x^2-3}}$$

2.2 Derivatives of Exponential (Base-e) Functions

Quick Check 3

Differentiate:

a.) $f(x)=e^{-4x}, \quad f'(x)=e^{-4x}\cdot -4 \;=\; -4e^{-4x}$

b.) $g(x)=e^{x^3+8x}, \quad g'(x)=e^{x^3+8x}(3x^2+8)$

c.) $h(x)=e^{\sqrt{x^2+5}}, \quad h'(x)=\frac{1}{2}\left(x^2+5\right)^{-\frac{1}{2}}(2x)e^{\sqrt{x^2+5}}$

$$h'(x)=\frac{2xe^{\sqrt{x^2+5}}}{2\sqrt{x^2+5}} \;=\; \frac{xe^{\sqrt{x^2+5}}}{\sqrt{x^2+5}}$$

2.2 Derivatives of Exponential (Base-e) Functions

Example 4: Find $\frac{d^2y}{dx^2}$ for $y=e^{-5x^2}$.

Solution: Using the Chain Rule, we first find $\frac{dy}{dx}$.

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{-5x^2}\right)$$

$$=e^{-5x^2}\left(-10x\right)$$

$$=-10xe^{-5x^2}.$$

Example 4 concluded:

Thus, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-10xe^{-5x^2}\right)$$
$$= (-10x)\cdot(-10xe^{-5x^2}) + (e^{-5x^2})\cdot(-10)$$
$$= 10e^{-5x^2}(10x^2 - 1).$$

2.2 Derivatives of Exponential (Base-e) Functions

Example 5: Business: Growth of an Account. Franco's Fishing Emporium invested \$50,000 in an account that earns 1.25% annual interest, compounded continuously. The value of the account after *t* years is given by $A(t) = 50{,}000e^{0.0125t}$.

Find $A(5)$ and $A'(5)$, and interpret their meanings.

Solution: $$A(5) = 50{,}000e^{0.0125(5)}$$
$$\approx \$53{,}224.72$$

2.2 Derivatives of Exponential (Base-e) Functions

Example 5 concluded:

To find $A'(5)$, we first find $A'(t)$.

$$A'(t) = \frac{d}{dt}\left(50{,}000e^{0.0125t}\right)$$

$$= 50{,}000\left(e^{0.0125t}\right)\left(0.0125\right) \quad \text{By Chain Rule}$$

$$= 625e^{0.0125t}$$

$$A'(5) = 625e^{0.0125(5)}$$

$$\approx 665.31$$

After exactly 5 years, the value of Franco's Fishing Emporium's account is \$53,224.72, and at that instant, the value is growing at the rate of \$665.31 per year.

2.2 Derivatives of Exponential (Base-e) Functions

Section Summary

- The derivative of $f(x) = e^x$ is itself: $\frac{d}{dx}\left(e^x\right) = e^x$.
- For functions of the form $y = e^{f(x)}$,

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x).$$

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