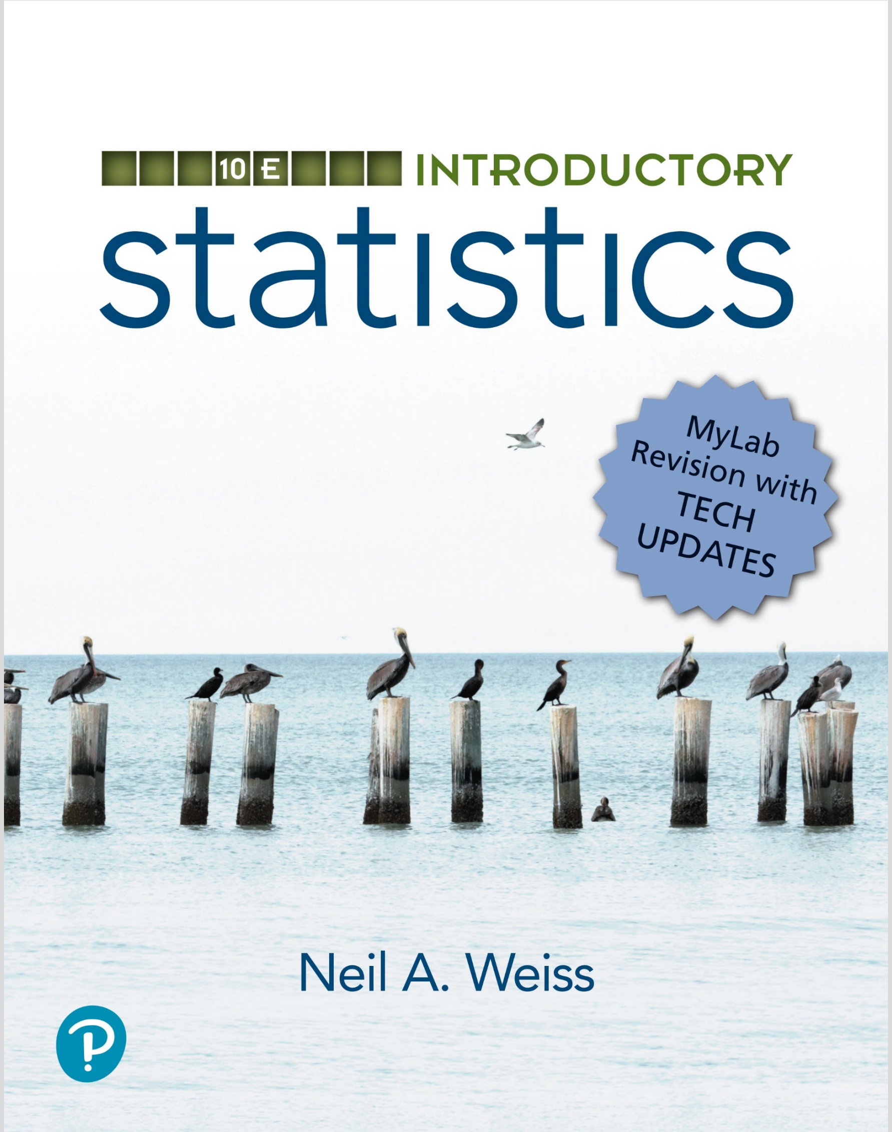
Chapter 9

Hypothesis Tests for One Population Mean



Section 9.1

The Nature of Hypothesis Testing

Definition 9.1

Null and Alternative Hypotheses; Hypothesis Test

Null hypothesis: A hypothesis to be tested. We use the symbol H_0 to represent the null hypothesis.

Alternative hypothesis: A hypothesis to be considered as an alternative to the null hypothesis. We use the symbol H_a to represent the alternative hypothesis.

Hypothesis test: The problem in a hypothesis test is to decide whether the null hypothesis should be rejected in favor of the alternative hypothesis.

Definition 9.2

Type I and Type II Errors

Type I error: Rejecting the null hypothesis when it is in fact true.

Type II error: Not rejecting the null hypothesis when it is in fact false.

Definition 9.3

Significance Level

The probability of making a Type I error, that is, of rejecting a true null hypothesis, is called the **significance level,** α, of a hypothesis test.

Key Fact 9.1

Relation between Type I and Type II Error Probabilities

For a fixed sample size, the smaller we specify the significance level, α, the larger will be the probability, β, of not rejecting a false null hypothesis.

Key Fact 9.2

Possible Conclusions for a Hypothesis Test

Suppose that a hypothesis test is conducted at a small significance level.

- If the null hypothesis is rejected, we conclude that the data provide sufficient evidence to support the alternative hypothesis.
- If the null hypothesis is not rejected, we conclude that the data do not provide sufficient evidence to support the alternative hypothesis.

Section 9.2

Critical-Value Approach to Hypothesis Testing

Figure 9.2

Criterion for deciding whether to reject the null hypothesis

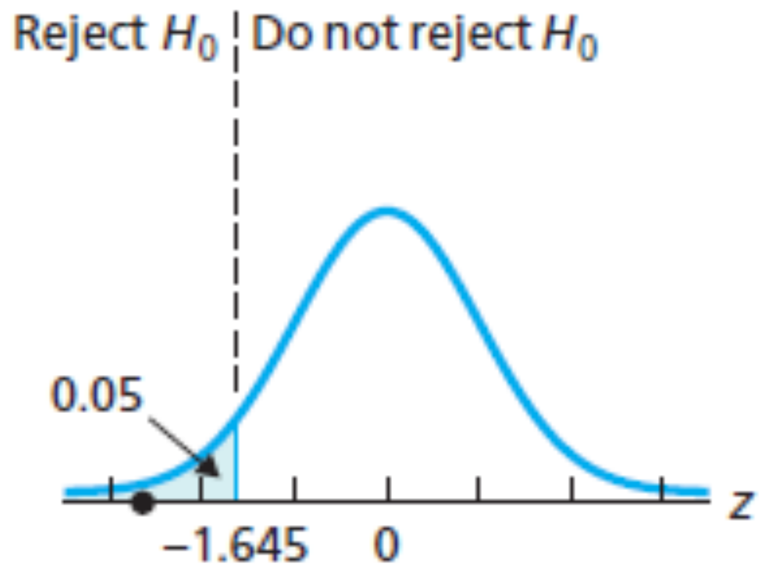
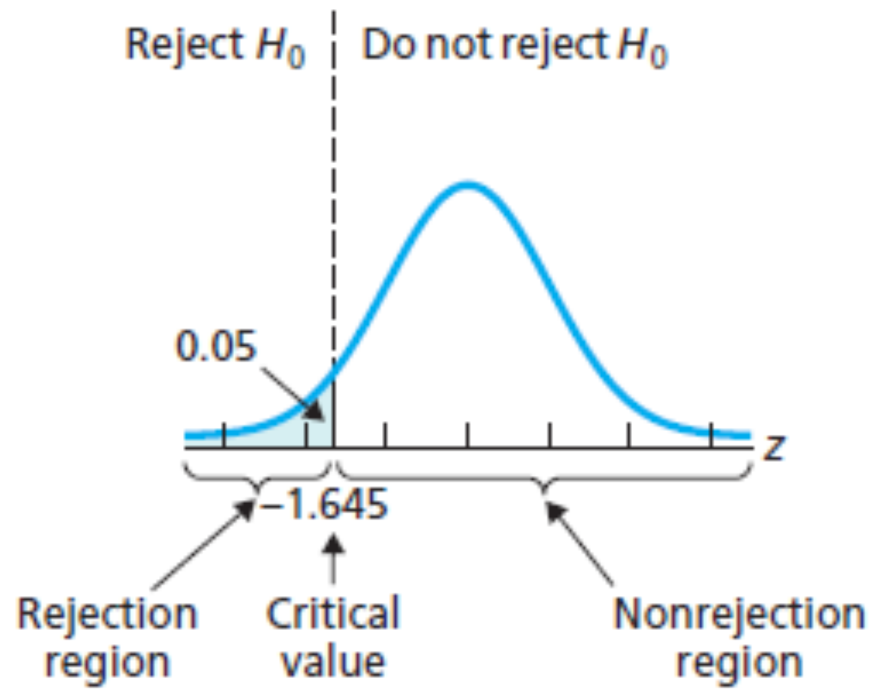


Figure 9.3

Rejection region, nonrejection region, and critical value for the golf-driving-distances hypothesis test



Definition 9.4

Rejection Region, Nonrejection Region, and Critical Values

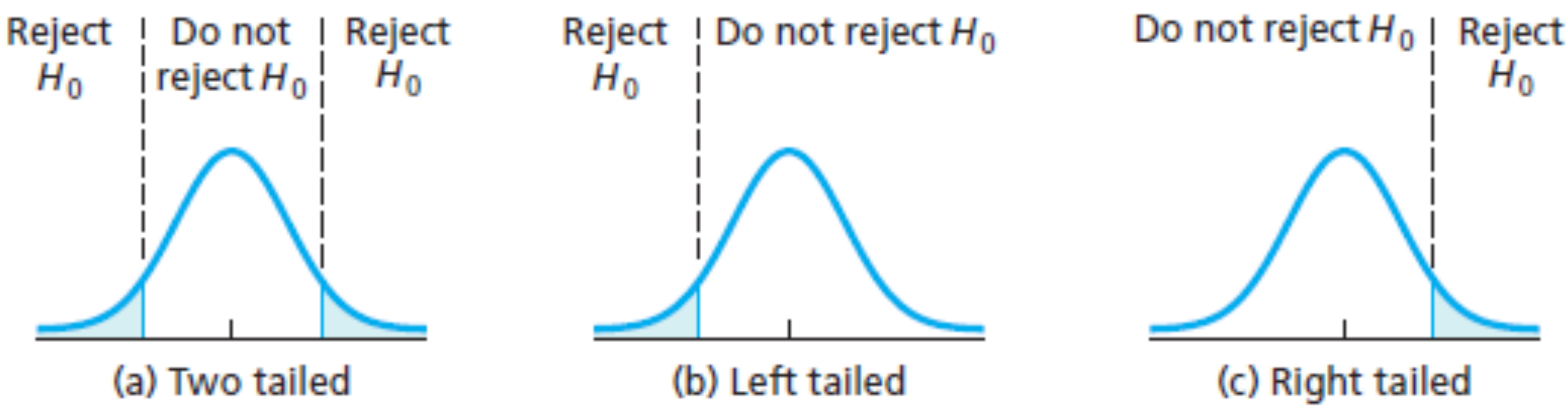
Rejection region: The set of values for the test statistic that leads to rejection of the null hypothesis.

Nonrejection region: The set of values for the test statistic that leads to nonrejection of the null hypothesis.

Critical value(s): The value or values of the test statistic that separate the rejection and nonrejection regions. A critical value is considered part of the rejection region.

Figure 9.4

Graphical display of rejection regions for two-tailed, left-tailed, and right-tailed tests



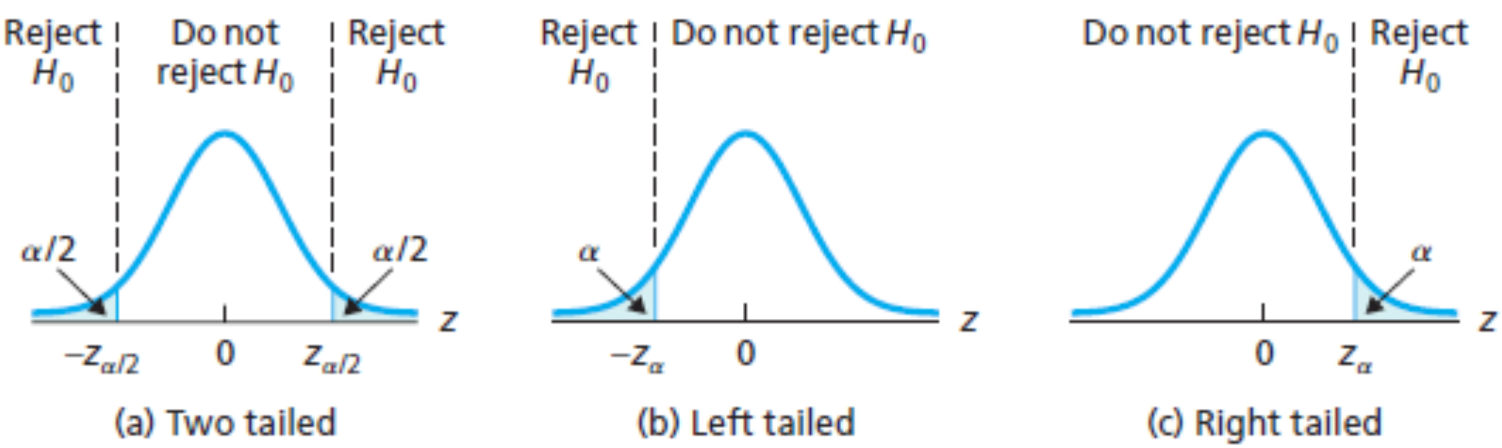
Definition 9.3

Obtaining Critical Values

Suppose that a hypothesis test is to be performed at the significance level α. Then the critical value(s) must be chosen so that, if the null hypothesis is true, the probability is α that the test statistic will fall in the rejection region.

Figure 9.5

Critical value(s) for a one-mean z-test at the significance level α if the test is (a) two tailed, (b) left tailed, or (c) right tailed

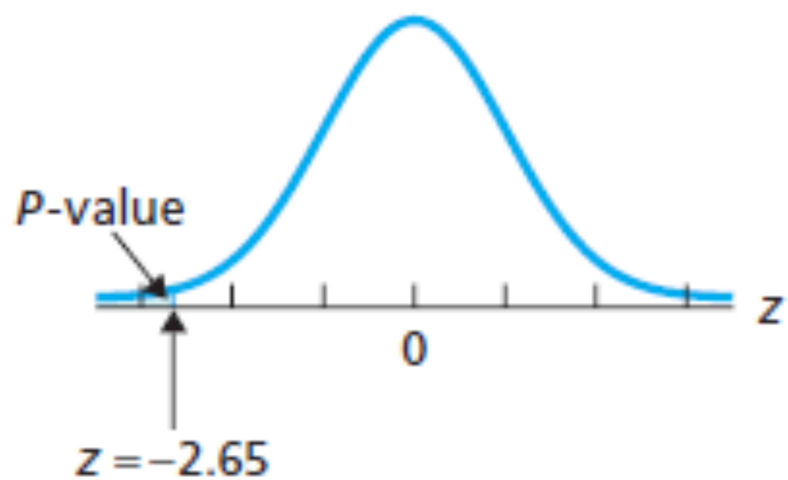


Section 9.3

P-Value Approach to Hypothesis Testing

Figure 9.7

P-value for golf-driving-distances hypothesis test



Definition 9.5

P-Value

The ***P*-value** of a hypothesis test is the probability of getting sample data at least as inconsistent with the null hypothesis (and supportive of the alternative hypothesis) as the sample data actually obtained. We use the letter ***P*** to denote the *P*-value.

Key Fact 9.4

Decision Criterion for a Hypothesis Test Using the *P*-Value

If the *P*-value is less than or equal to the specified significance level, reject the null hypothesis; otherwise, do not reject the null hypothesis. In other words, if $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Key Fact 9.5

P-Value as the Observed Significance Level

The *P*-value of a hypothesis test equals the smallest significance level at which the null hypothesis can be rejected, that is, the smallest significance level for which the observed sample data results in rejection of H_0.

Key Fact 9.6

Determining a *P*-Value

To determine the *P*-value of a hypothesis test, we assume that the null hypothesis is true and compute the probability of observing a value of the test statistic as extreme as or more extreme than that observed. By *extreme* we mean “far from what we would expect to observe if the null hypothesis is true.”

Figure 9.8

P-value for a one-mean *z*-test if the test is (a) two tailed, (b) left tailed, or (c) right tailed

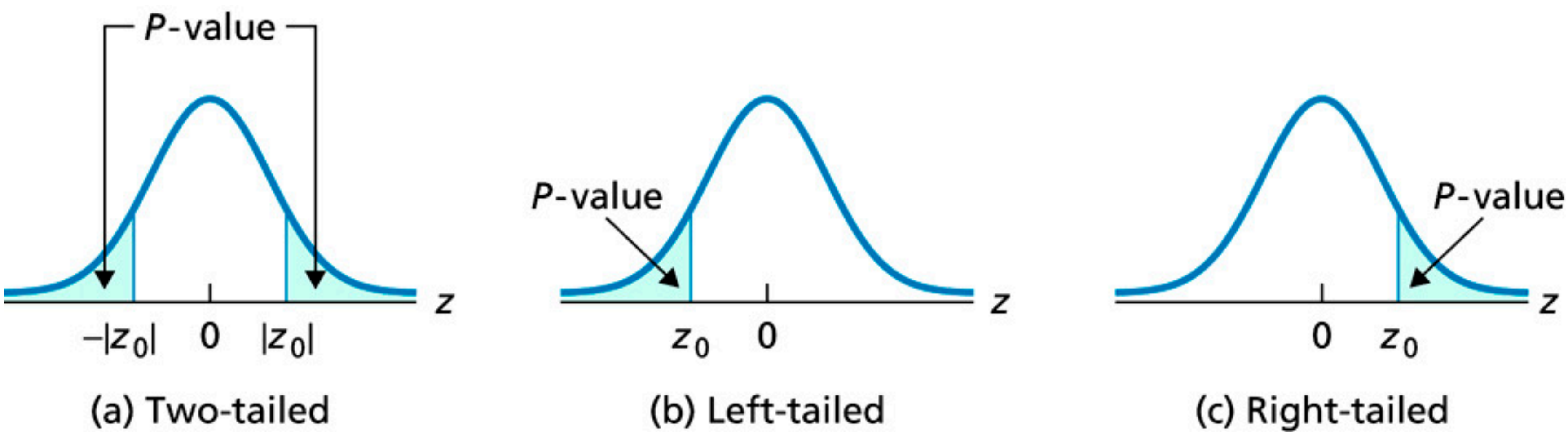


Figure 9.9

Value of the test statistic and the *P*-value

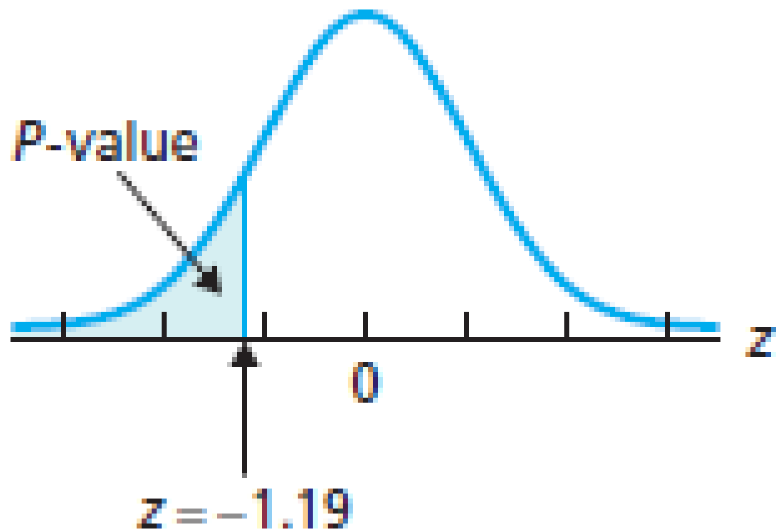


Table 9.7

General steps for the *P*-value approach to hypothesis testing

***P*-VALUE APPROACH TO HYPOTHESIS TESTING**	
Step 1	State the null and alternative hypotheses.
Step 2	Decide on the significance level, α.
Step 3	Compute the value of the test statistic.
Step 4	Determine the P-value, P.
Step 5	If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.
Step 6	Interpret the result of the hypothesis test.

Table 9.8

Guidelines for using the *P*-value to assess the evidence against the null hypothesis

***P*-value**	**Evidence against H_0**
$P > 0.10$	Weak or none
$0.05 < P \leq 0.10$	Moderate
$0.01 < P \leq 0.05$	Strong
$P \leq 0.01$	Very strong

Hypothesis Tests Without Significance Levels: Many researchers do not explicitly refer to significance levels. Instead, they simply obtain the P-value and use it (or let the reader use it) to assess the strength of the evidence against the null hypothesis.

Section 9.4

Hypothesis Tests for One Population Mean When σ Is Known

Procedure 9.1

One-Mean *z*-Test

Purpose To perform a hypothesis test for a population mean, μ

Assumptions

1. Simple random sample
2. Normal population or large sample
3. σ known

Step 1 **The null hypothesis is H_0: $\mu = \mu_0$, and the alternative hypothesis is**

$$H_a: \mu \neq \mu_0 \quad \text{or} \quad H_a: \mu < \mu_0 \quad \text{or} \quad H_a: \mu > \mu_0$$

(Two tailed) or (Left tailed) or (Right tailed)

Step 2 **Decide on the significance level, α.**

Step 3 **Compute the value of the test statistic**

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

and denote that value z_0.

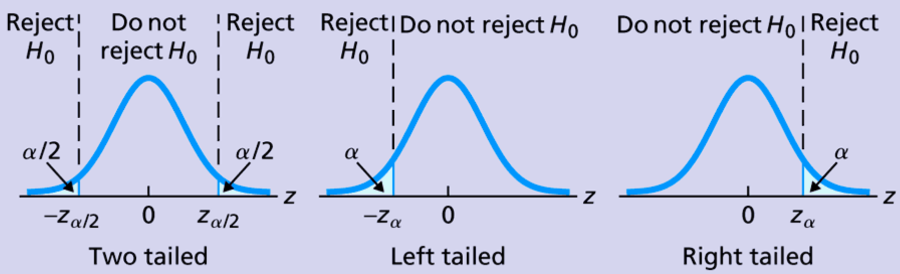
Procedure 9.1 (cont.)

CRITICAL-VALUE APPROACH	OR	*P*-VALUE APPROACH

Step 4 The critical value(s) are

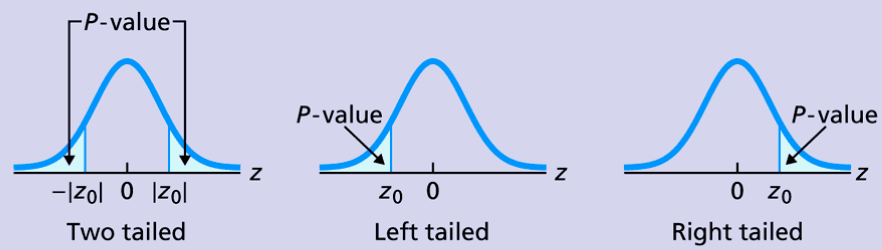
$\pm z_{\alpha/2}$ (Two tailed) or $-z_{\alpha}$ (Left tailed) or z_{α} (Right tailed)

Use Table II to find the critical value(s).



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

Step 4 Use Table II to obtain the P-value.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Note: The hypothesis test is exact for normal populations and is approximately correct for large samples from nonnormal populations.

Key Fact 9.7

When to Use the One-Mean z-Test

•For small samples—say, of size less than 15—the *z*-test should be used only when the variable under consideration is normally distributed or very close to being so.

•For samples of moderate size—say, between 15 and 30—the *z*-test can be used unless the data contain outliers or the variable under consideration is far from being normally distributed.

•For large samples—say, of size 30 or more—the *z*-test can be used essentially without restriction. However, if outliers are present and their removal is not justified, you should perform the hypothesis test once with the outliers and once without them to see what effect the outliers have. If the conclusion is affected, use a different procedure or take another sample, if possible.

•If outliers are present but their removal is justified and results in a data set for which the *z*-test is appropriate (as previously stated), the procedure can be used.

Section 9.5

Hypothesis Tests for One Population Mean When σ Is Unknown

Figure 9.19

Estimating the *P*-value of a left-tailed *t*-test with a sample size of 12 and test statistic $t = -1.938$

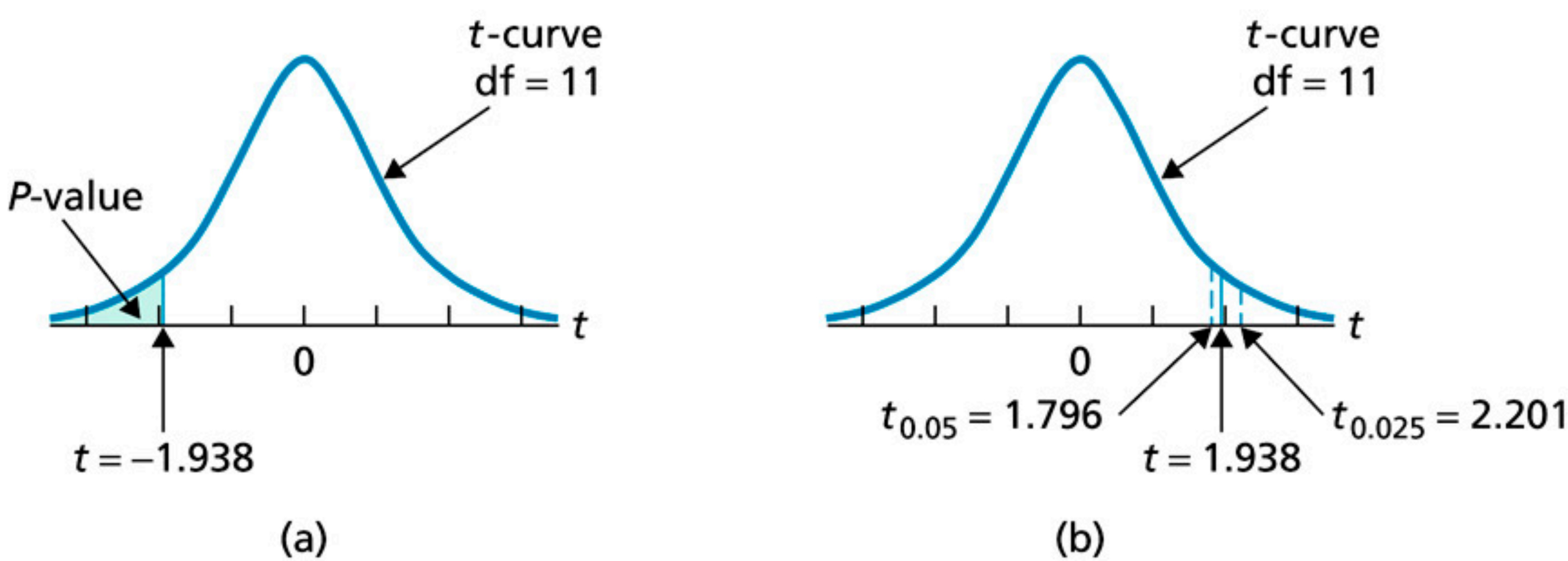
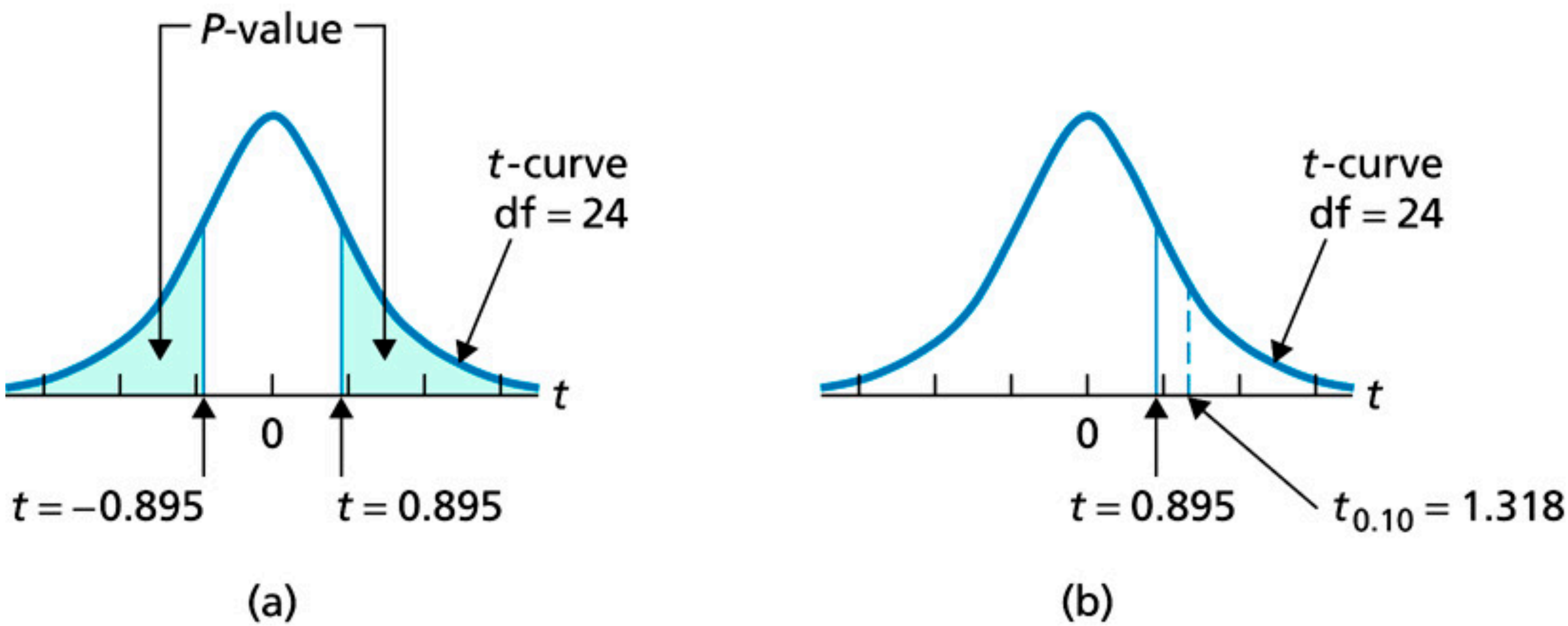


Figure 9.20

Estimating the *P*-value of a two-tailed *t*-test with a sample size of 25 and test statistic $t = -0.895$



Procedure 9.2

One-Mean *t*-Test

Purpose To perform a hypothesis test for a population mean, μ

Assumptions

1. Simple random sample
2. Normal population or large sample
3. σ unknown

Step 1 **The null hypothesis is H_0: $\mu = \mu_0$, and the alternative hypothesis is**

H_a: $\mu \neq \mu_0$ **(Two tailed)**	**or**	H_a: $\mu < \mu_0$ **(Left tailed)**	**or**	H_a: $\mu > \mu_0$ **(Right tailed)**

Step 2 **Decide on the significance level, α.**

Step 3 **Compute the value of the test statistic**

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

and denote that value t_0.

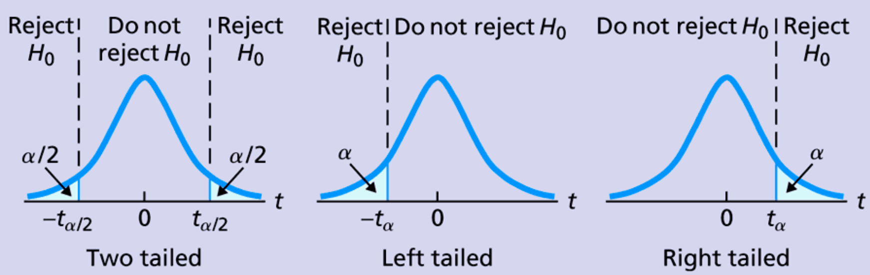
Procedure 9.2 (cont.)

CRITICAL-VALUE APPROACH OR ***P*-VALUE APPROACH**

Step 4 The critical value(s) are

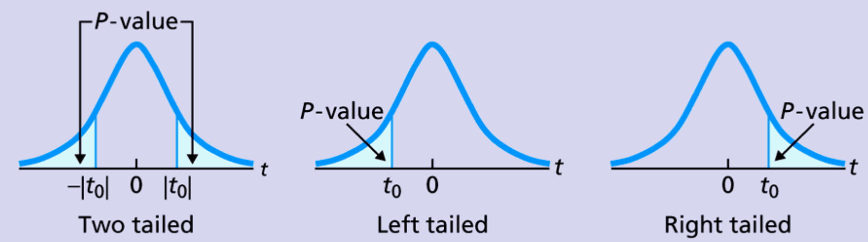
$\pm t_{\alpha/2}$ (Two tailed) or $-t_\alpha$ (Left tailed) or t_α (Right tailed)

with df $= n - 1$. Use Table IV to find the critical value(s).



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

Step 4 The t-statistic has df $= n - 1$. Use Table IV to estimate the P-value, or obtain it exactly by using technology.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Note: The hypothesis test is exact for normal populations and is approximately correct for large samples from nonnormal populations.

Section 9.6

The Wilcoxon Signed-Rank Test

Table 9.13

Sample of weekly food costs ($)

143	169	149	135	161
138	152	150	141	159

Table 9.14

Steps for ranking the data in Table 9.13 according to distance and direction from the null hypothesis mean

Cost ($) x	Difference $D = x - 157$	$\lvert D \rvert$	Rank of $\lvert D \rvert$	Signed rank R
143	−14	14	7	−7
138	−19	19	9	−9
169	12	12	6	6
152	−5	5	3	−3
149	−8	8	5	−5
150	−7	7	4	−4
135	−22	22	10	−10
141	−16	16	8	−8
161	4	4	2	2
159	2	2	1	1

Step 1 *Subtract μ_0 from x.*

Step 2 *Make each difference positive by taking absolute values.*

Step 3 *Rank the absolute differences in order from smallest (1) to largest (10).*

Step 4 *Give each rank the same sign as the sign in the Difference column.*

Procedure 9.3

Wilcoxon Signed-Rank Test

Purpose To perform a hypothesis test for a population mean, μ

Assumptions

1. Simple random sample
2. Symmetric population

Step 1 **The null hypothesis is H_0: $\mu = \mu_0$, and the alternative hypothesis is**

H_a: $\mu \neq \mu_0$ (Two tailed)	or	H_a: $\mu < \mu_0$ (Left tailed)	or	H_a: $\mu > \mu_0$ (Right tailed)

Step 2 **Decide on the significance level, α.**

Step 3 **Compute the value of the test statistic**

$$W = \text{sum of the positive ranks}$$

and denote that value W_0. To do so, construct a work table of the following form.

Observation x	Difference $D = x - \mu_0$	$\lvert D \rvert$	Rank of $\lvert D \rvert$	Signed rank R
⋮	⋮	⋮	⋮	⋮

Procedure 9.3 (cont.)

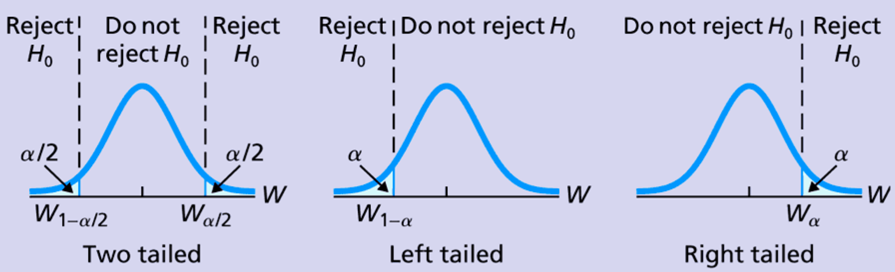
CRITICAL-VALUE APPROACH	OR	*P*-VALUE APPROACH

Critical-value approach

Step 4 The critical value(s) are

$W_{1-\alpha/2}$ and $W_{\alpha/2}$ (Two tailed) or $W_{1-\alpha}$ (Left tailed) or W_{α} (Right tailed)

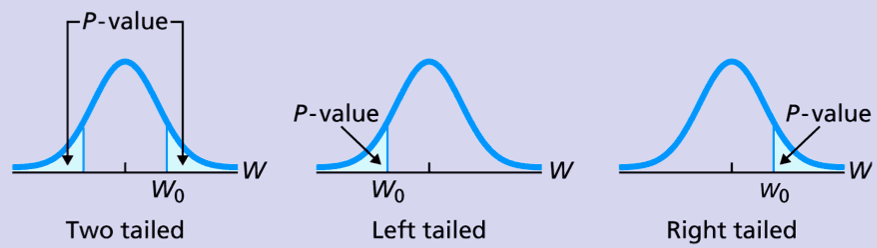
Use Table V to find the critical value(s). For a left-tailed or two-tailed test, you will also need the relation $W_{1-A} = n(n+1)/2 - W_A$.



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

P-value approach

Step 4 Obtain the P-value by using technology.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Key Fact 9.8

Wilcoxon Signed-Rank Test Versus the *t*-Test

Suppose that you want to perform a hypothesis test for a population mean. When deciding between the *t*-test and the Wilcoxon signed-rank test, follow these guidelines:

- If you are reasonably sure that the variable under consideration is normally distributed, use the *t*-test.
- If you are not reasonably sure that the variable under consideration is normally distributed but are reasonably sure that it has a symmetric distribution, use the Wilcoxon signed-rank test.

Section 9.7

Type II Error Probabilities; Power

Figure 9.30

Decision criterion for the gas mileage illustration (α = 0.05, n = 30)

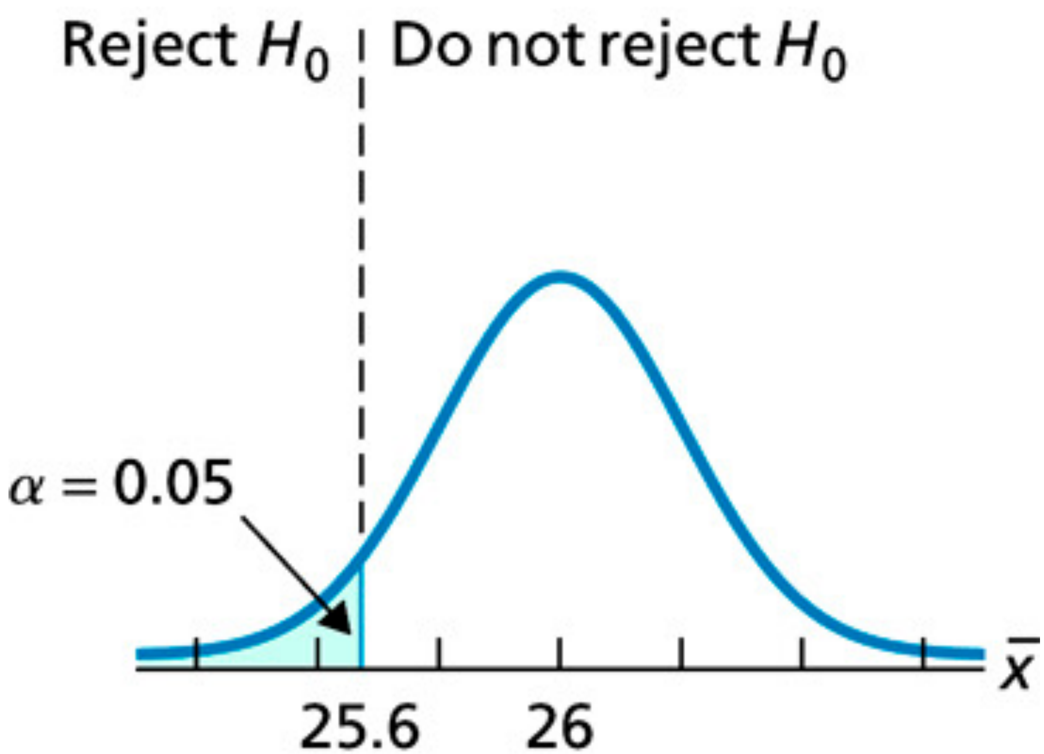
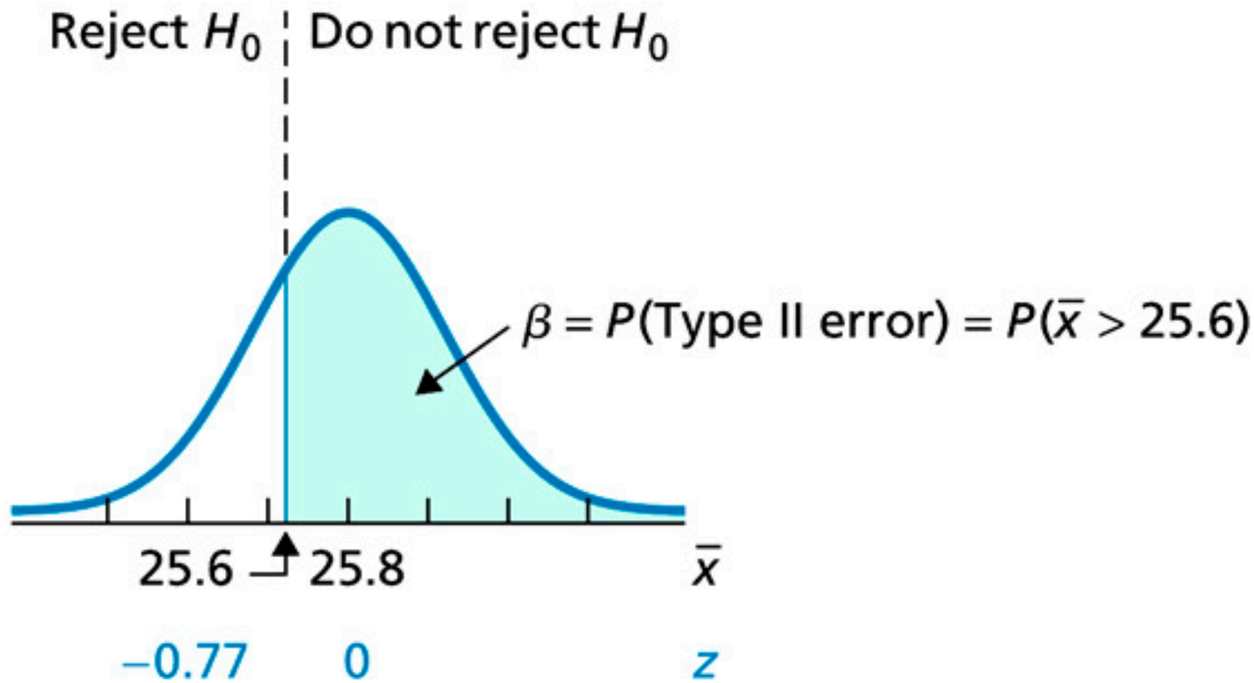


Figure 9.31

Determining the probability of a Type II error if μ = 25.8 mpg



z-score computation: Area to the left of z:

$\bar{x} = 25.6 \longrightarrow z = \dfrac{25.6 - 25.8}{0.26} = -0.77$ 0.2206

Shaded area = 1 − 0.2206 = 0.7794

Procedure 9.4

Type II Error Probabilities for a One-Mean z-Test

Purpose To find Type II error probabilities for a one-mean z-test when the true population mean equals μ_a

Assumptions

1. Simple random sample
2. Normal population or large sample
3. σ known

Step 1 The $\bar{x}$ critical value(s) are

$$\underset{\text{(Two tailed)}}{\mu_0 \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}} \quad \text{or} \quad \underset{\text{(Left tailed)}}{\mu_0 - z_{\alpha} \cdot \sigma/\sqrt{n}} \quad \text{or} \quad \underset{\text{(Right tailed)}}{\mu_0 + z_{\alpha} \cdot \sigma/\sqrt{n}}.$$

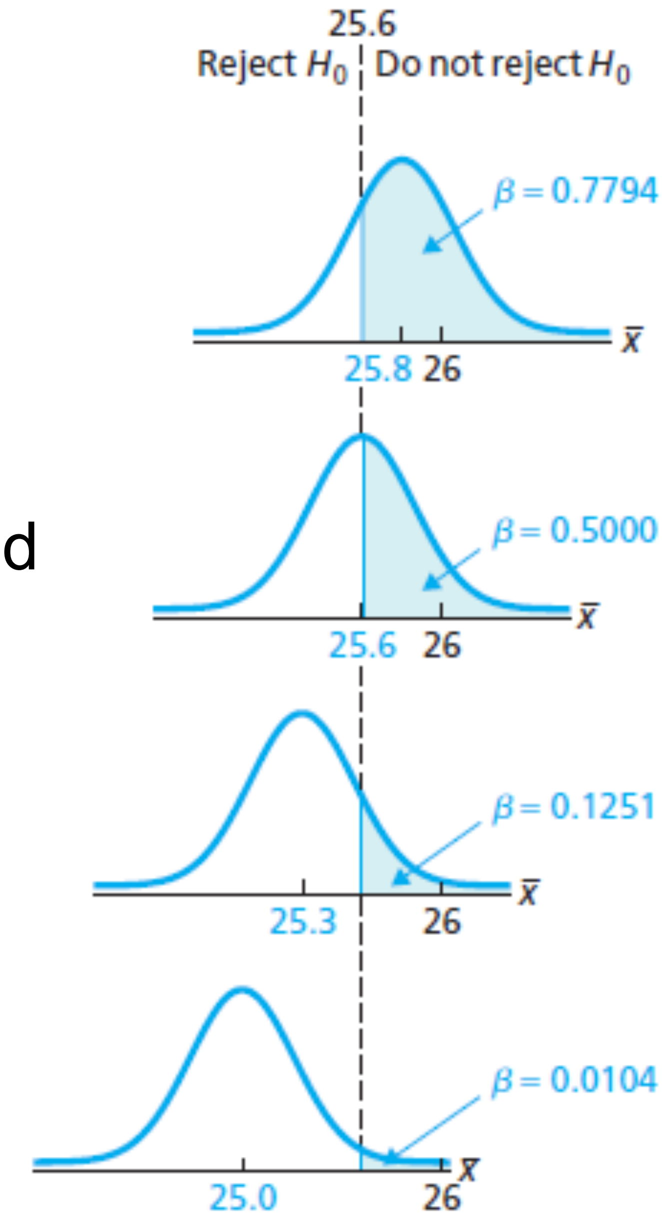
where μ_0 is the null-hypothesis value of the mean, α is the significance level, σ is the population standard deviation, and n is the sample size.

Step 2 The probability of a Type II error, β, equals the area under the normal curve with parameters μ_a and $\sigma/\sqrt{n}$ that lies

between the two $\bar{x}$ critical values (Two tailed)	or	to the right of the $\bar{x}$ critical value (Left tailed)	or	to the left of the $\bar{x}$ critical value (Right tailed).

Figure 9.36

Type II error probabilities for μ = 25.8, 25.6, 25.3, and 25.0 (α = 0.05, n = 30)



Definition 9.6

Power

The **power** of a hypothesis test is the probability of not making a Type II error, that is, the probability of rejecting a false null hypothesis. We have

$$\text{Power} = 1 - P(\text{Type II error}) = 1 - \beta.$$

Procedure 9.5

Power Curve for a One-Mean z-Test

Purpose To construct a power curve for a one-mean z-test

Assumptions

1. Simple random sample
2. Normal population or large sample
3. σ known

Step 1 Decide on equidistant values of μ_a to be used in plotting the power curve. Choose values of μ_a

on both sides of μ_0 (Two tailed) or to the left of μ_0 (Left tailed) or to the right of μ_0 (Right tailed).

Step 2 Construct a work table of the form

True mean μ	P (Type II error) β	Power $1 - \beta$
.	.	.
.	.	.
.	.	.

Fill in the first column of the work table with the values of μ_a chosen in Step 1.

Procedure 9.5 (cont.)

Step 3 For each value of μ_a chosen in Step 1, use Procedure 9.4 (page 425) to obtain the probability of a Type II error. Fill in the second column of the work table from Step 2.

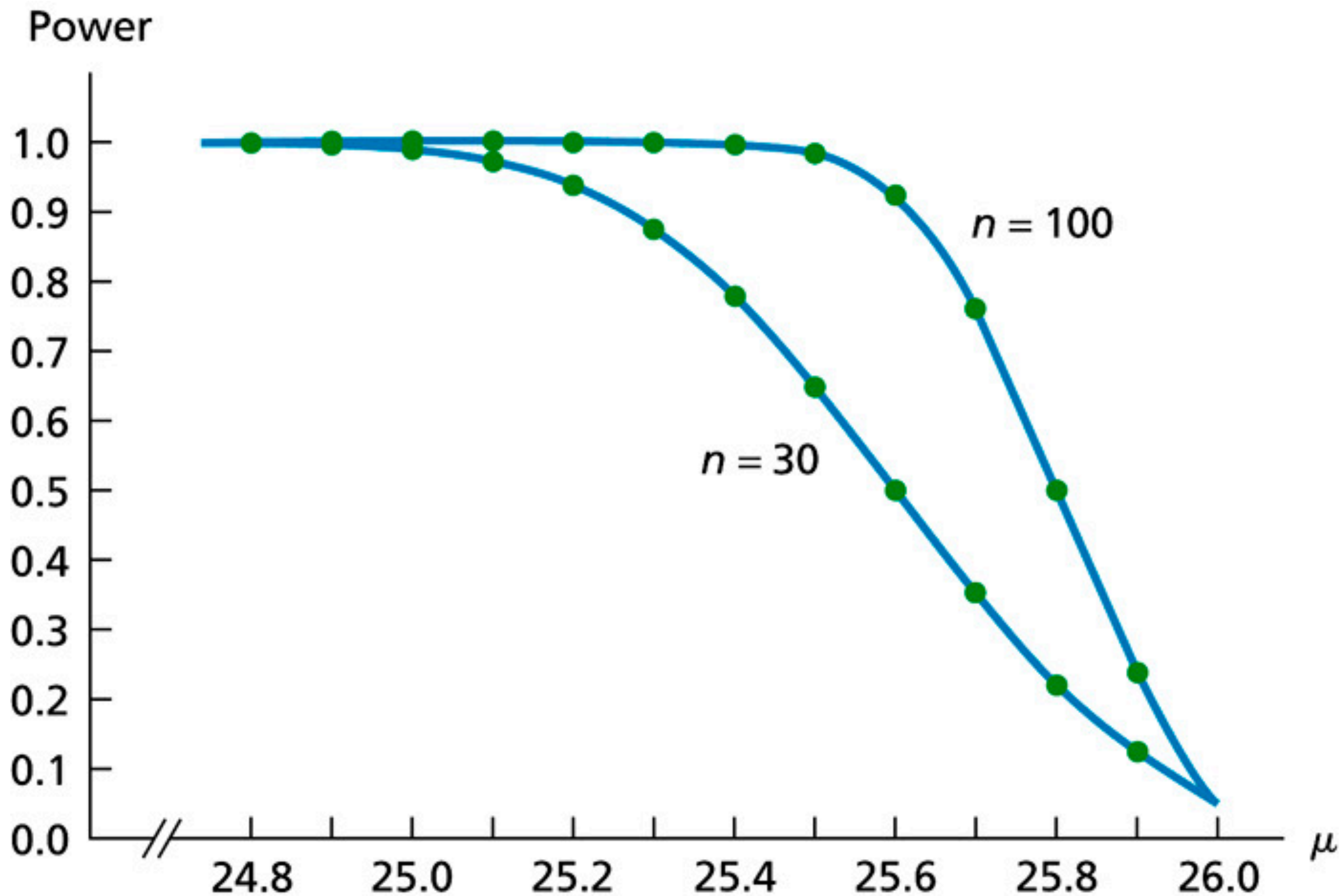
Step 4 Use the results of Step 3 and the relation Power $= 1 - \beta$ to fill in the third column of the work table from Step 2.

Step 5 Plot the values in the first and third columns of the work table from Step 2 as points on a graph with μ on the horizontal axis and power on the vertical axis. Include on the graph a hollow point for the null-hypothesis mean (μ_0) and the significance level (α).

Step 6 Connect the points in the graph from Step 5 with a smooth curve.

Figure 9.38

Power curves for the gas mileage illustration when $n = 30$ and $n = 100$ ($\alpha = 0.05$)



Key Fact 9.9

Sample Size and Power

For a fixed significance level, increasing the sample size increases the power.

Section 9.8

Which Procedure Should Be Used?

Figure 9.39

Flowchart for choosing the correct hypothesis testing procedure for a population mean