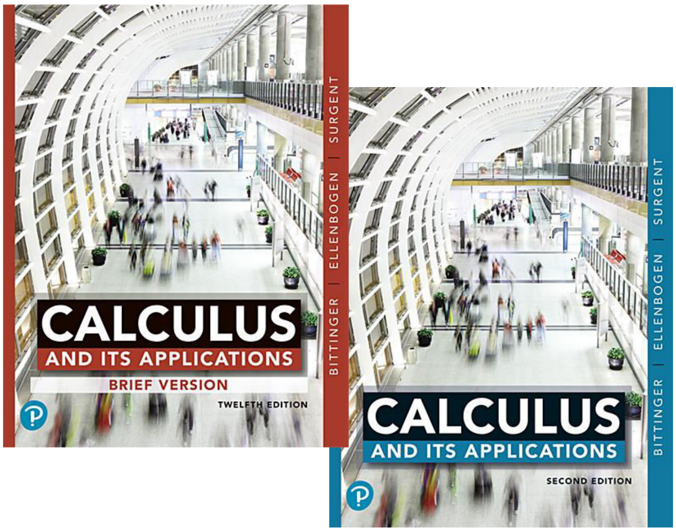
Chapter R

Functions, Graphs, and Models



R.1 Graphs and Equations

OBJECTIVE

- Graph equations.
- Use the graphs as mathematical models to make predictions.
- Carry out calculations involving compound interest.

R.1 Graphs and Equations

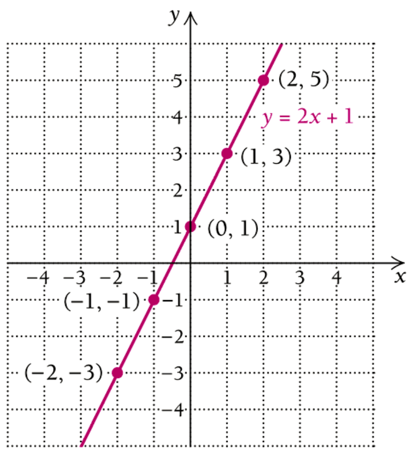
DEFINITION:

The **graph** of an equation is a drawing that represents all ordered pairs that are solutions of the equation.

R.1 Graphs and Equations

Example 1: Graph $y = 2x + 1$. We first find some ordered pairs that are solutions and arrange them in a table.

x	y	(x, y)
-2	-3	$(-2, -3)$
-1	-1	$(-1, -1)$
0	1	$(0, 1)$
1	3	$(1, 3)$
2	5	$(2, 5)$



(1) Choose any x.
(2) Compute y.
(3) Form the pair (x, y).
(4) Plot the points.

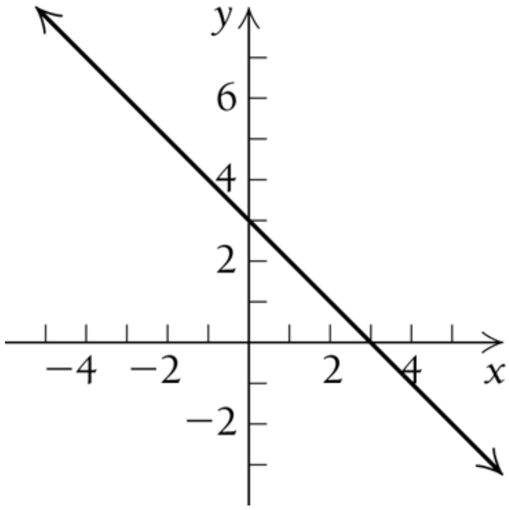
R.1 Graphs and Equations

Quick Check 1

Graph: $y = 3 - x.$

First let's create a table of ordered pairs, then graph the equation:

x	y	(x, y)
–2	5	$(-2,5)$
–1	4	$(-1,4)$
0	3	$(0,3)$
1	2	$(1,2)$
2	1	$(2,1)$



R.1 Graphs and Equations

Example 2: Graph $3x + 5y = 10$.

First solve this equation for y.

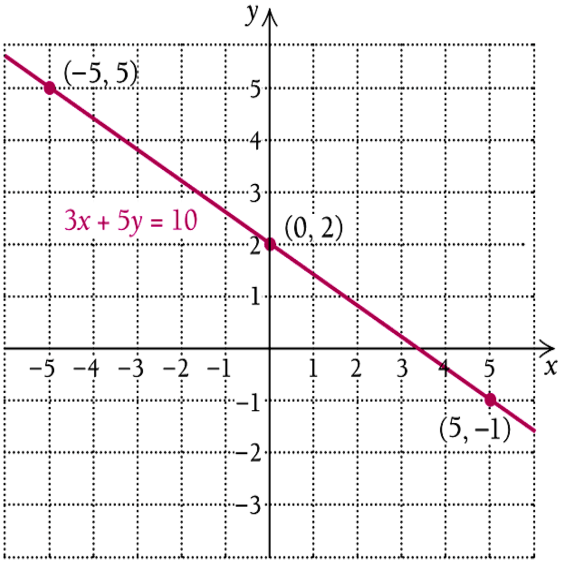
$$\begin{aligned} 3x+5y &= 10 \\ 5y &= -3x+10 \\ y &= -\frac{3}{5}x+2 \end{aligned}$$

R.1 Graphs and Equations

Example 2 (concluded):

Then, we will find three ordered pairs (choosing multiples of 5 to avoid fractions) and use them to sketch the graph.

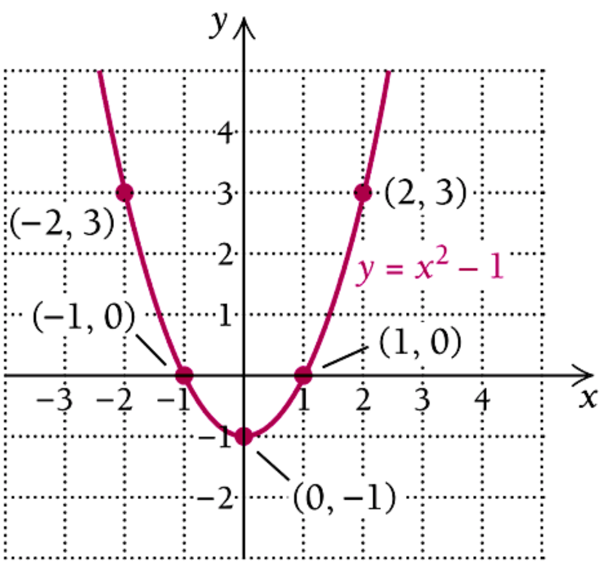
x	y	(x, y)
0	2	(0, 2)
5	−1	(5, −1)
−5	5	(−5, 5)



R.1 Graphs and Equations

Example 3: Graph $y = x^2 - 1$.

x	y	(x, y)
−2	3	(−2, 3)
−1	0	(−1, 0)
0	−1	(0, −1)
1	0	(1, 0)
2	3	(2, 3)



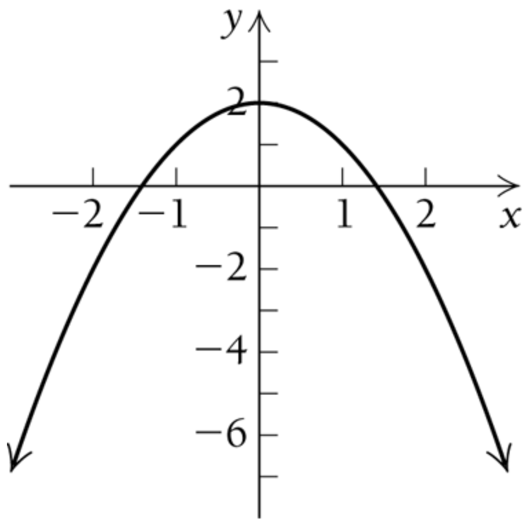
R.1 Graphs and Equations

Quick Check 2

Graph: $y = 2 - x^2$.

First create a table of values, then graph.

x	y	(x, y)
-2	-2	$(-2,-2)$
-1	1	$(-1,1)$
0	2	$(0,2)$
1	1	$(1,1)$
2	-2	$(2,-2)$

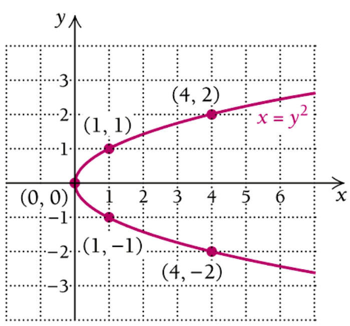


R.1 Graphs and Equations

Example 4: Graph $x = y^2$.

In this case, x is expressed in terms of the variable y. Thus, we first choose numbers for y and then compute x.

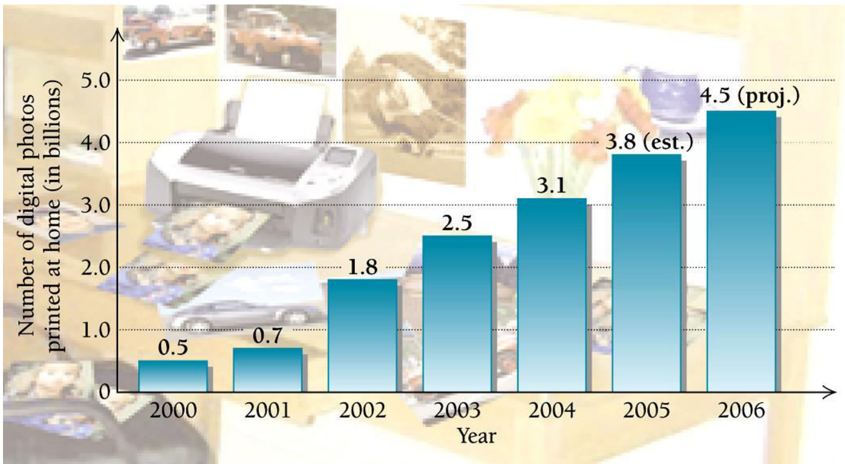
x	y	(x, y)
4	-2	$(4, -2)$
1	-1	$(1, -1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
4	2	$(4, 2)$



(1) Choose any y.
(2) Compute x.
(3) Form the pair (x, y).
(4) Plot the points.

R.1 Graphs and Equations

Example 5: The graph below shows the numbers of digital photos printed at home from 2000 to 2006.



(*Source*: Photo Marketing Association International.)

R.1 Graphs and Equations

Example 5 (continued): Use the model $h = 0.7t + 0.3$, where t is the number of years after 2000 and h is the number of digital photos printed at home, in billions, to predict the number of digital photos printed at home in 2008.

Since 2008 is 8 years after 2000, we substitute, using $t = 8$.

$$h = 0.7 \cdot 8 + 0.3 = 5.9 \text{ billion digital photos}$$

R.1 Graphs and Equations

THEOREM 1

If an amount P is invested at interest rate r, expressed as a decimal, and compounded annually, then in t years it will grow to an amount A given by

$$A = P(1 + r)^t.$$

R.1 Graphs and Equations

Example 6: Suppose that $1000 is invested at 5%, compounded annually. How much is in the account at the end of the second year?

$$\begin{aligned} A &= 1000(1+0.05)^2 \\ &= 1000(1.05)^2 \\ &= \$1102.50 \end{aligned}$$

There is $1102.50 in the account after 2 years.

R.1 Graphs and Equations

Quick Check 3

Repeat Example 6 for an interest rate of 6%.

$$\begin{aligned} A &= 1000(1+0.06)^2 \\ &= 1000(1.06)^2 \\ &= \$1123.60 \end{aligned}$$

There is \$1123.60 in the account after 2 years.

R.1 Graphs and Equations

THEOREM 2

If a principal P is invested at interest rate r, expressed as a decimal, and compounded n times a year, in t years it will grow to an amount A given by

$$A = P\left(1+\frac{r}{n}\right)^{nt}.$$

R.1 Graphs and Equations

Example 7: Suppose that $1000 is invested at 8%, compounded quarterly. How much is in the account at the end of 3 years?

$$A = 1000\left(1 + \frac{0.08}{4}\right)^{4\cdot 3}$$
$$= 1000(1.02)^{12}$$
$$= \$1268.24$$

There is $1268.24 in the account after 3 years.

R.1 Graphs and Equations

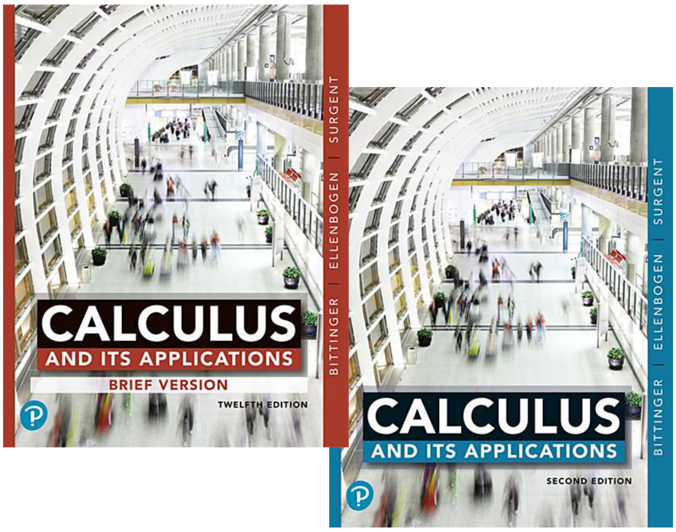
Section Summary

- Most graphs can be created by plotting points and looking for patterns. A graphing calculator can create graphs rapidly.
- Mathematical equations can serve as models in a wide variety of applications.
- An example of mathematical model is the formula for compound interest. If P dollars are invested at interest rate r, compounded n times a year for t years, then the amount A at the end of the t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Chapter R

Functions, Graphs, and Models



R.2 Functions and Models

OBJECTIVE

- Determine whether or not a correspondence is a function.
- Find function values.
- Graph functions and determine whether or not a graph is that of a function.
- Graph functions that are piecewise defined.

R.2 Functions and Models

DEFINITION:

A **function** is a correspondence between a first set, called the **domain**, and a second set, called the **range**, such that each member of the domain corresponds to *exactly one* member of the range.

R.2 Functions and Models

Example 1: Determine whether or not each correspondence is a function.

a) *Microsoft Stock*

Domain		Range
March 14, 2006	$\longrightarrow$	\$27.23
March 15, 2006	$\longrightarrow$	\$27.36
March 16, 2006	$\longrightarrow$	\$27.27
March 17, 2006	$\longrightarrow$	\$27.50

This relationship is a function because each member of the domain corresponds to only one member of the range.

R.2 Functions and Models

Example 1 (continued):

b) *Squaring*

Domain		Range
3	→	9
4	→	16
5	→	25
–5	→	25

This relationship is a function because each member of the domain corresponds to only one member of the range, even though two members of the domain correspond to 25.

R.2 Functions and Models

Example 1 (continued):

c) *Baseball Teams*

Domain		Range
Arizona	→	Diamondbacks
Chicago	→	Cubs
Chicago	→	White Sox
Baltimore	→	Orioles

This relationship is *not* a function because one member of the domain, Chicago, corresponds to two members of the range, Cubs and White Sox.

R.2 Functions and Models

Example 1 (continued):

d) *Baseball Teams*

Domain		Range
Diamondbacks	$\longrightarrow$	Arizona
Cubs	$\longrightarrow$	Chicago
White Sox	$\nearrow$	
Orioles	$\longrightarrow$	Baltimore

This relationship is a function because each member of the domain corresponds to only one member of the range, even though two members of the domain correspond to Chicago.

R.2 Functions and Models

Example 2: The squaring function, f, is given by $f(x) = x^2$.

Find $f(-3)$, $f(1)$, $f(k)$, $f(\sqrt{k})$, $f(1+t)$, and $f(x+h)$.

$$f(-3) = (-3)^2 = 9$$
$$f(1) = 1^2 = 1$$
$$f(k) = k^2$$
$$f\left(\sqrt{k}\right) = \left(\sqrt{k}\right)^2 = k$$
$$f(1+t) = (1+t)^2 = 1 + 2t + t^2$$
$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$$

R.2 Functions and Models

Quick Check 1

A function f is given by $f\left(x\right)=3x+5$.

Find: $f\left(4\right),\ f\left(-5\right),\ f\left(0\right),\ f(a),$ and $f\left(a+h\right)$.

$$f\left(4\right)=3\left(4\right)+5=12+5=17$$
$$f\left(-5\right)=3\left(-5\right)+5=-15+5=-10$$
$$f\left(0\right)=3\left(0\right)+5=0+5=5$$
$$f\left(a\right)=3\left(a\right)+5=3a+5$$
$$f\left(a+h\right)=3\left(a+h\right)+5=3a+3h+5$$

R.2 Functions and Models

Example 3: A function subtracts the square of an input from the input. A description of f is given by $f(x)=x-x^2$.

Find $f(4),\ f(x+h),$ and $\dfrac{f(x+h)-f(x)}{h}$.

$$\begin{aligned} f(4) &= 4-4^2 \\ &= -12 \end{aligned}$$

R.2 Functions and Models

Example 3 (concluded):

$$
\begin{aligned}
f(x+h) &= x+h-(x+h)^2 \\
&= x+h-(x^2+2xh+h^2) \\
&= x+h-x^2-2xh-h^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{x+h-x^2-2xh-h^2-(x-x^2)}{h} \\
&= \frac{x+h-x^2-2xh-h^2-x+x^2}{h} \\
&= \frac{h(1-2x-h)}{h} \\
&= 1-2x-h, \quad \text{for } h \neq 0
\end{aligned}
$$

R.2 Functions and Models

Quick Check 2

A function f is given by $f(x)=2x-x^2$.

Find: $f(4)$, $f(x+h)$ and $\dfrac{f(x+h)=f(x)}{h}$

$$f(4)=2(4)-(4)^2=8-16=-8$$

$$f(x+h)=2(x+h)-(x+h)^2=2x+2h-x^2-2xh-h^2$$

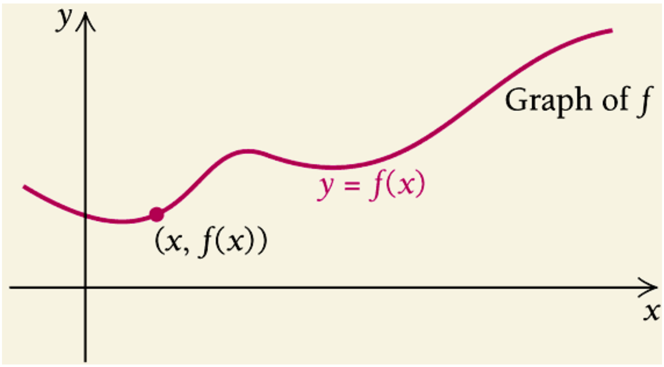
$$\frac{f(x+h)-f(x)}{h}=\frac{2x+2h-x^2-2xh-h^2-2x+x^2}{h}$$

$$=\frac{2h-2xh-h^2}{h}=\frac{h(2-2x-h)}{h}=2-2x-h$$

R.2 Functions and Models

Definition

The **graph** of a function f is a drawing that represents all the input-output pairs, $(x, f(x))$. When the function is given by an equation, the graph of a function is the graph of the equation, $y = f(x)$.

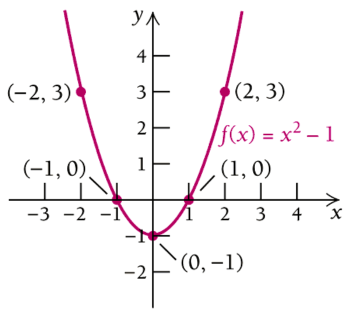


R.2 Functions and Models

Example 4: Graph $f(x) = x^2 - 1$.

x	$f(x)$	$(x, f(x))$
-2	3	$(-2, 3)$
-1	0	$(-1, 0)$
0	-1	$(0, -1)$
1	0	$(1, 0)$
2	3	$(2, 3)$

(1) Choose any x.
(2) Compute y.
(3) Form the pair (x, y).
(4) Plot the points.



R.2 Functions and Models

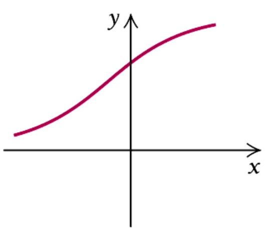
The Vertical Line Test

A graph represents a function if it is impossible to draw a vertical line that intersects the graph more than once.

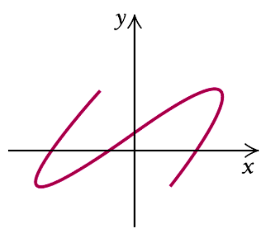
R.2 Functions and Models

Example 4: Determine whether each of the following is the graph of a function.

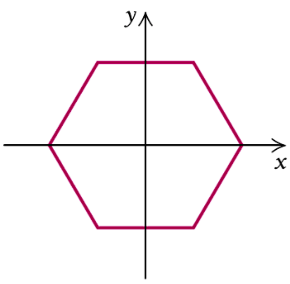
a)



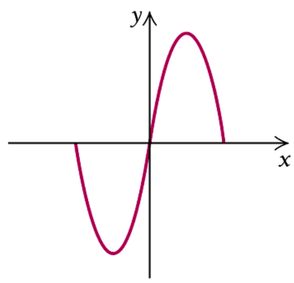
b)



c)



d)



R.2 Functions and Models

Example 4 (concluded):

a) The graph is that of a function. It impossible to draw a vertical line that intersects the graph more than once.

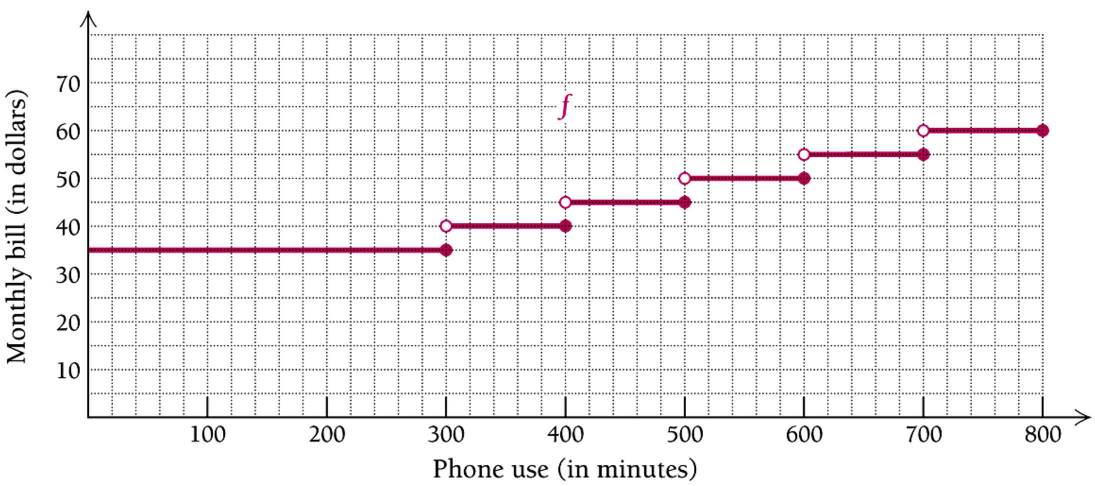
b) The graph is not that of a function. A vertical line (in fact many) can intersect the graph more than once.

c) The graph is not that of a function.

d) The graph is that of a function.

R.2 Functions and Models

Example 5: In 2005, Sprint® offered a cellphone calling plan in which a customer's monthly bill can be modeled by the graph below. The amount of the bill is a function f of the number of minutes of phone use.



R.2 Functions and Models

Example 5 (continued):

Each open dot on the graph indicates that the point at that location is not included in the graph.

a) Under this plan, if a customer uses the phone for 360 min, what is his or her monthly bill?

b) If a monthly bill is \$55, for how many minutes did the customer use the phone?

R.2 Functions and Models

Example 5 (concluded):

a) To find the bill for 360 min of use, we locate 360 on the horizontal axis and move directly up to the graph. We then move across to the vertical axis. Thus, the bill is \$40.

b) To find the number of minutes of use when a monthly bill is \$55, we locate 55 on the vertical axis, move horizontally to the graph, and note that many inputs correspond to 55. If t represents the number of minutes of use, we must have $600 < t \leq 700$.

R.2 Functions and Models

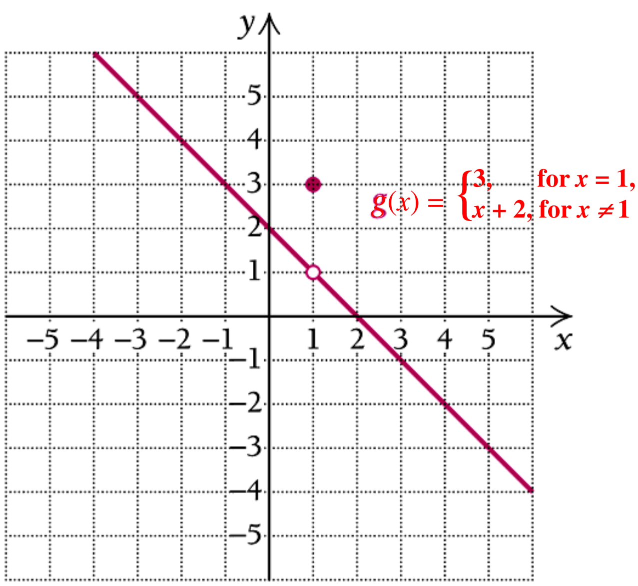
Example 6: Graph the function defined as follows:

$$g(x) = \begin{cases} 3, & \text{for } x = 1 \\ -x + 2, & \text{for } x \neq 1 \end{cases}$$

The function is defined such that $g(1) = 3$ and for all other x-values (that is, for $x \neq 1$), we have $g(x) = -x + 2$. Thus, to graph this function, we graph the line given by $g(x) = -x + 2$, but with an open dot at the point above $x = 1$. To complete the graph, we plot the point $(1, 3)$ since $g(1) = 3$.

R.2 Functions and Models

Example 6 (concluded):



x	$g(x)$	$(x, g(x))$
-3	$-(-3) + 2$	$(-3, 5)$
0	$-0 + 2$	$(0, 2)$
1	3	$(1, 3)$
2	$-2 + 2$	$(2, 0)$
3	$-3 + 2$	$(3, -1)$

R.2 Functions and Models

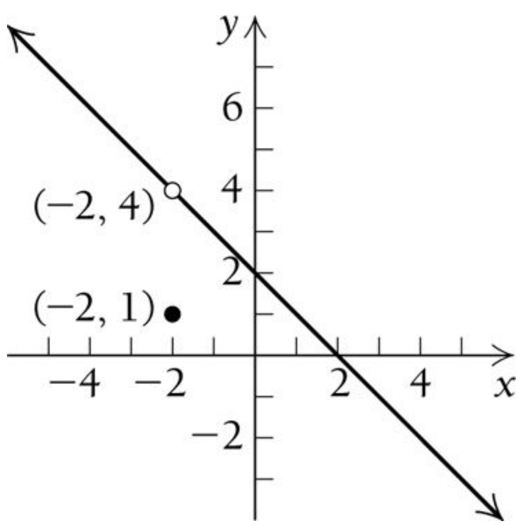
Quick Check 3

Graph the function defined as follows: $f(x)=\begin{cases}1, & \text{for } x=-2,\\ 2-x, & \text{for } x\neq -2.\end{cases}$

The function is defined such that $f(-2)=1$ and for all other x-values (that is, for $x\neq -2$), we have $f(x)=2-x$. Thus, to graph this function, we graph the line given by $f(x)=2-x$, but with an open dot at the point above $x=-2$. To complete the graph, we plot the point $(-2,1)$ since $f(-2)=1$.

R.2 Functions and Models

Quick Check 3 Concluded



x	y	(x,y)
–3	$2-(-3)$	$(-3,5)$
–2	1	$(-2,1)$
0	$2-(0)$	$(0,2)$
1	$2-(1)$	$(1,1)$
2	$2-(2)$	$(2,0)$

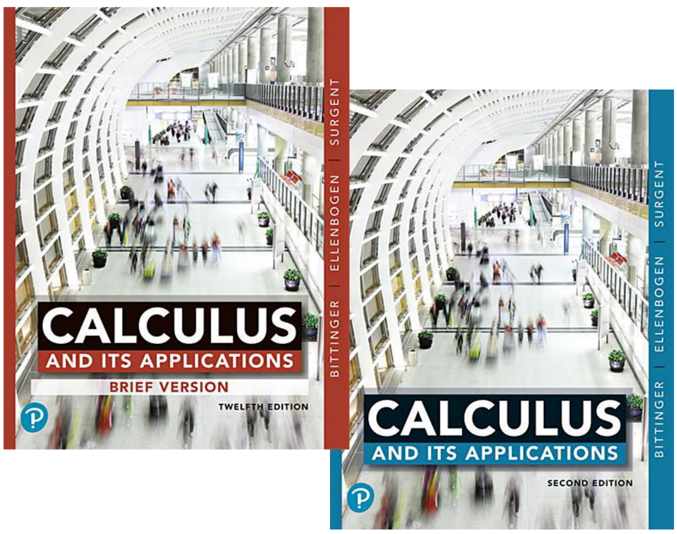
R.2 Functions and Models

Section Summary

- *Functions* are a key concept in mathematics.
- The essential trait of a function is that to each number in the *domain* there corresponds one and only one number in the *range*.

Chapter R

Functions, Graphs, and Models



R.3 Finding Domain and Range

OBJECTIVE

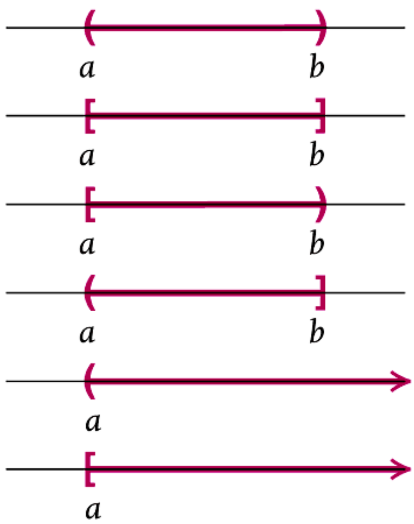
- Write interval notation for a set of points.
- Find the domain and range of a function

R.3 Finding Domain and Range

Intervals: Notation and Graphs

Interval Notation	Set Notation
(a, b)	$\{x \mid a < x < b\}$
$[a, b]$	$\{x \mid a \leq x \leq b\}$
$[a, b)$	$\{x \mid a \leq x < b\}$
$(a, b]$	$\{x \mid a < x \leq b\}$
(a, ∞)	$\{x \mid x > a\}$
$[a, \infty)$	$\{x \mid x \geq a\}$

Graph



R.3 Finding Domain and Range

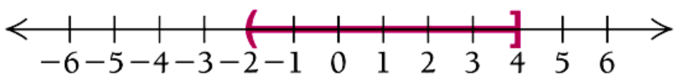
Intervals: Notation and Graphs (concluded)

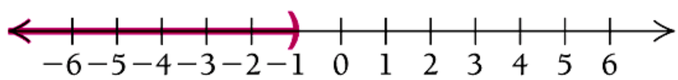
Interval Notation	Set Notation	Graph
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leq b\}$	
$(-\infty, \infty)$	$\{x \mid x \text{ is a real number}\}$	

R.3 Finding Domain and Range

Example 1: Write interval notation for each set or graph.

a) $\{x \mid -4 < x < 5\}$ b) $\{x \mid x \geq -2\}$

c)


d)


e) $\{x \mid -3 < x < 1 \text{ } or \text{ } x \geq 2\}$

f) $\{x \mid x \leq 2 \text{ } and \text{ } x > 0\}$

a) $(-4, 5)$ b) $[-2, \infty)$

c) $(-2, 4]$ d) $(-\infty, -1)$

e) $(-3,1) \cup [2,\infty)$ f) $(-\infty, 2] \cap (0,\infty) = (0,2]$

R.3 Finding Domain and Range

Quick Check 1

Write interval notation for each set:

a.) $\{x | -2 \le x \le 5\}$; $[-2,5]$

b.) $\{x | -2 \le x < 5\}$; $[-2,5)$

c.) $\{x | -2 < x \le 5\}$; $(-2,5]$

d.) $\{x | -2 < x < 5\}$; $(-2,5)$

e.) $\{x \mid x < 0 \; or \; 2 \le x < 5\}$; $(-\infty,0) \cup [2,5)$

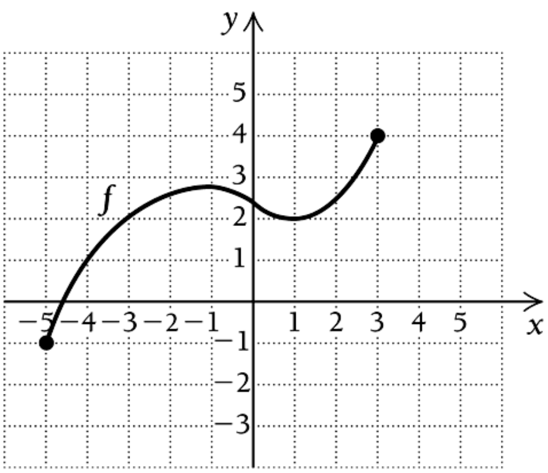
f.) $\{x \mid 2 < x < 5 \; and \; 4 \le x\}$; $[4,5)$

R.3 Finding Domain and Range

Example 2: For the function *f* whose graph is shown below, determine each of the following.

a) The number in the range that is paired with 1 (from the domain). That is, find $f(1)$.

b) The domain of *f*.

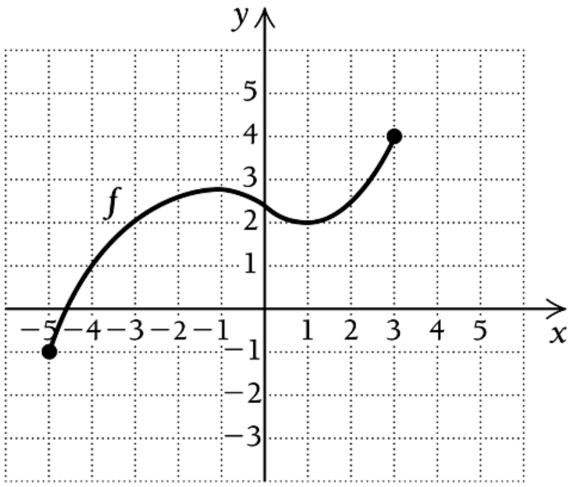


R.3 Finding Domain and Range

Example 2 (continued):

c) The number(s) in the domain that is (are) paired with 1 (from the range).
That is, find all x-values for which $f(x) = 1$.

d) The range of f.



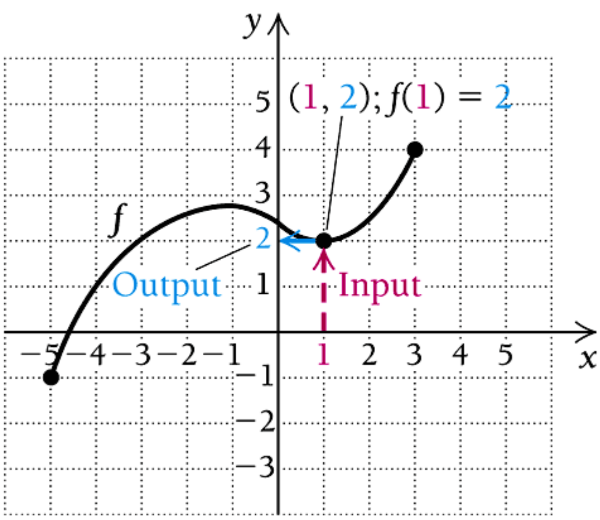
R.3 Finding Domain and Range

Example 2 (continued):

a) To determine which number in the range is paired with 1 in the domain, we locate 1 on the horizontal axis.

Next, we find the point on the graph of f for which 1 is the first coordinate. From that point, we can look to the vertical axis to find the corresponding y-coordinate, 2.

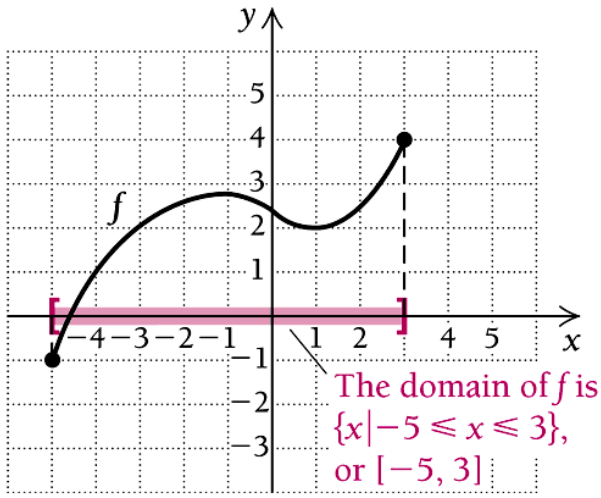
The input 1 has the output 2 – that is, $f(1) = 2$.



R.3 Finding Domain and Range

Example 2 (continued):

b) The domain of the function is the set of all *x*-values, or inputs, of the points on the graph.
These extend from –5 to 3 and can be viewed as the curve's shadow onto the *x*-axis. Thus, the domain is the set $\{x \mid -5 \leq x \leq 3\}$, or, in interval notation, [–5, 3].



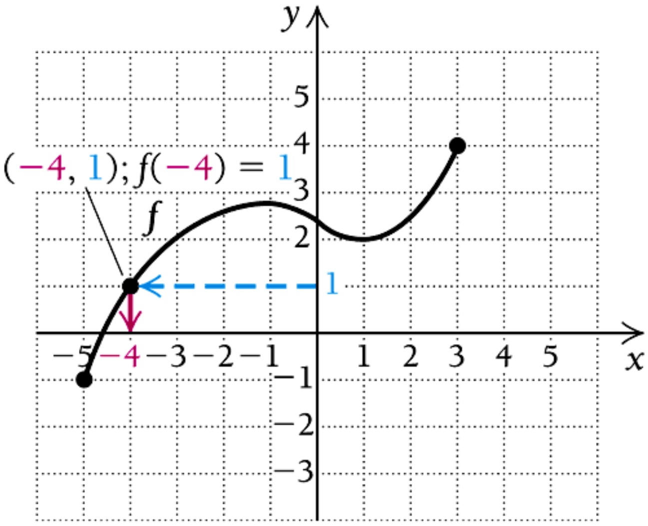
R.3 Finding Domain and Range

Example 2 (continued):

c) To determine which number(s) in the domain is (are) paired with 1 in the range, we locate 1 on the vertical axis. From there, we look left and right to the graph of *f* to find any points for which 1 is the second coordinate. One such point exists: (–4, 1). For this function, we note that $x = -4$ is the only member of the domain paired with 1. For other functions, there might be more than one member of the domain paired with a member of the range.

R.3 Finding Domain and Range

Example 2 (continued):



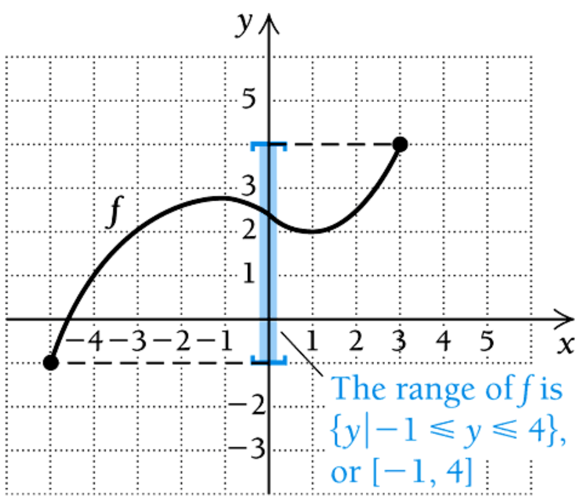
R.3 Finding Domain and Range

Example 2 (concluded):

d) The range of the function is the set of all y-values, or outputs, of the points on the graph.

These extend from –1 to 4 and can be viewed as the curve's shadow, or projection, onto the y-axis.

Thus, the range is the set $\{y \mid -1 \leq y \leq 4\}$, or, in interval notation, $[-1, 4]$.



R.3 Finding Domain and Range

Quick Check 2

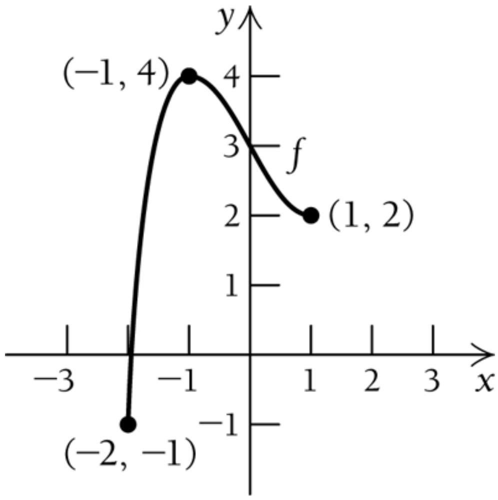
For the function f whose graph follows, determine each of the following:

a.) $f(-1)$

b.) $f(1)$

c.) the domain

d.) the range

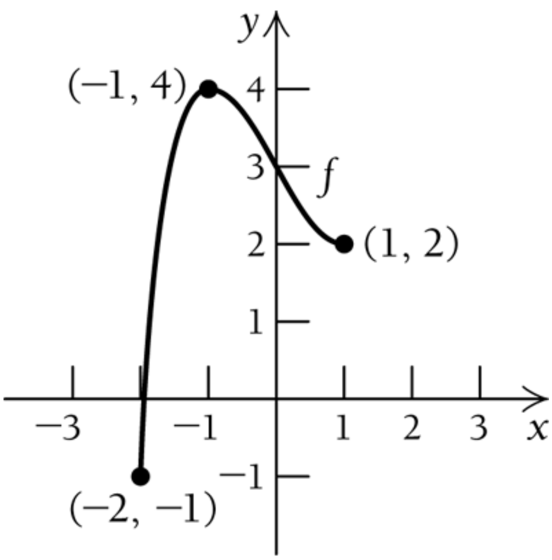


R.3 Finding Domain and Range

Quick Check 2 Continued

a.) To find $f(-1)$, we must use the graph to find what the corresponding y-value is when $x = -1$. Using the graph, we can see that when $x = -1$, $y = 4$. Thus $f(-1) = 4$.

b.) To find $f(1)$, we must use the graph to find what the corresponding y-value is when $x = 1$. Using the graph, we can see that when $x = 1$, $y = 2$. Thus $f(1) = 2$.

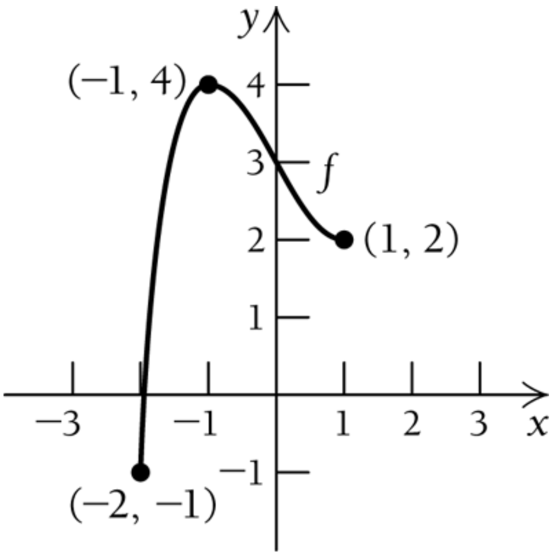


R.3 Finding Domain and Range

Quick Check 2 Continued

c.) The domain of the function is the set of all x-values, or inputs, of the points on the graph. These extend from –2 to 2 and can be viewed as the curve's shadow onto the x-axis.

Thus the range is the set $\{x | -2 \le x \le 2\}$ or, in interval notation, $[-2, 2]$.

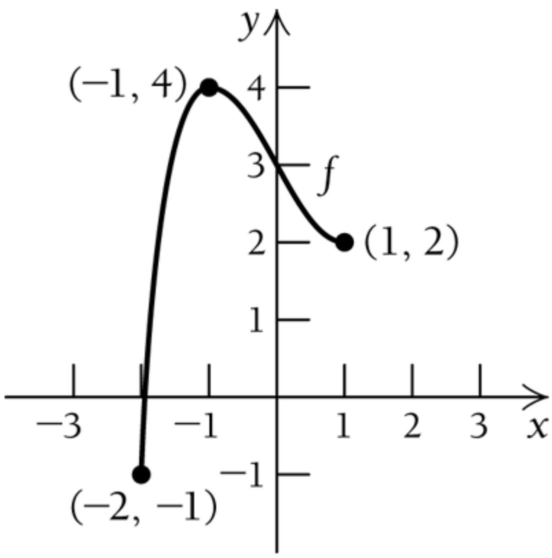


R.3 Finding Domain and Range

Quick Check 2 Concluded

d.) The range of the function is the set of all y-values, or outputs, of the points on the graph. These extend from –1 to 4 and can be viewed as the curve's shadow onto the y-axis.

Thus the range is the set $\{y | -1 \le y \le 4\}$ or, in interval notation, $[-1, 4]$.



R.3 Finding Domain and Range

Example 3: Find the domain: $f(x) = |x|$.

We ask, "What can we substitute?" Is there any number x for which we cannot calculate $|x|$? The answer is no. Thus, the domain of f is the set of all real numbers.

R.3 Finding Domain and Range

Example 4: Find the domain: $f(x) = \dfrac{3}{2x-5}$.

We ask, "What can we substitute?" Is there any number x for which we cannot calculate f? Since f cannot be calculated when the denominator is 0, we set the denominator equal to 0 and solve to find those real numbers that must be excluded from the domain of f.

R.3 Finding Domain and Range

Example 4 (concluded):

$$\begin{aligned} 2x-5 &= 0 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

Thus, 5/2 is not in the domain, whereas all other real numbers are. The domain of f is $\{x|\ x$ is a real number *and* $x \neq 5/2\}$, or in interval notation, $(-\infty, 5/2) \cup (5/2, \infty)$. The symbol $\cup$ indicates the *union* of two sets and means that all elements in both sets are included in the domain.

R.3 Finding Domain and Range

Example 5: Find the domain: $f(x) = \sqrt{4+3x}$.

We ask, "What can we substitute?" Is there any number x for which we cannot calculate f? Since f is not a real number when the radicand $4 + 3x$ is negative, the domain is all real numbers for which $4 + 3x \geq 0$. We find them by solving the inequality.

$$\begin{aligned} 4+3x &\geq 0 \\ 3x &\geq -4 \\ x &\geq -\frac{4}{3} \end{aligned}$$

Thus, the domain is $[-4/3, \infty)$.

R.3 Finding Domain and Range

Quick Check 3

Find the domain of each function. Express your answers in interval notation.

a.) $f(x) = \dfrac{5}{x-8}$

b.) $f(x) = x^3 + |2x|$

c.) $f(x) = \sqrt{2x-8}$

R.3 Finding Domain and Range

Quick Check 3 Continued

a.) $f(x) = \dfrac{5}{x-8}$: Since we know that 0 cannot be in the denominator of a fraction, we can find where we cannot calculate f by solving the equation $x-8=0$.

$$x - 8 = 0$$
$$x = 8$$

We know that we can not calculate f when $x = 8$. So the domain in interval notation would be $(-\infty, 8) \cup (8, \infty)$.

R.3 Finding Domain and Range

Quick Check 3 Continued

b.) $f(x) = x^3 + |2x|$:

Since there is no number for which we cannot calculate $f(x)$, the domain of f is the set of all real numbers.

In interval notation, this is $(-\infty, \infty)$.

R.3 Finding Domain and Range

Quick Check 3 Concluded

c.) $f(x) = \sqrt{2x-8}$:

Since we know that $\sqrt{x}$ cannot be calculated when $x < 0$, to find the domain we need to solve the inequality $2x - 8 \geq 0$:

$$2x - 8 \geq 0$$
$$2x \geq 8$$
$$x \geq 4$$

Thus the domain is $[4, \infty)$.

R.3 Finding Domain and Range

Example 6: Find the domain of $h(x) = \sqrt{x} + \sqrt{3-x}$.

The radicands of both square root terms must be nonnegative.

The domain of $\sqrt{x}$ is $x \geq 0$, or in the interval $[0, \infty)$.
The domain of $\sqrt{3-x}$ is $x \leq 3$, or in the interval $(-\infty, 3]$.
The domain of $h(x)$ is the intersection of $[0, \infty)$ and $(-\infty, 3]$.
Thus, the domain of $h(x)$ is $[0, 3]$.

R.3 Finding Domain and Range

Quick Check 4

Find the domain of $g(x) = \sqrt{x+2} + \sqrt{5-x}$.

The radicands of both square root terms must be nonnegative.

The domain of $\sqrt{x+2}$ is $x \geq -2$, or the interval $[-2, \infty)$.
The domain of $\sqrt{5-x}$, is $x \leq 5$, or the interval $(-\infty, 5]$.
The domain of $g(x)$ is the intersection of $[-2, \infty)$ and $(-\infty, 5]$.
Thus, the domain of $g(x)$ is $[-2, 5]$.

R.3 Finding Domain and Range

Example 7: Suppose that \$500 is invested at 6%, compounded quarterly for *t* years. From Theorem 2 in Section R.1, we know that the amount in the account is given by

$$A(t) = 500\left(1 + \frac{0.06}{4}\right)^{4t}$$
$$= 500(1.015)^{4t}.$$

The amount *A* is a function of the number of years for which the money is invested. Determine the domain.

R.3 Finding Domain and Range

Example 7 (concluded):

We can substitute any real number for *t* into the formula, but a negative number of years is not meaningful. The context of the application excludes negative numbers. Thus, the domain is the set of all nonnegative numbers, $[0, \infty)$.

R.3 Finding Domain and Range

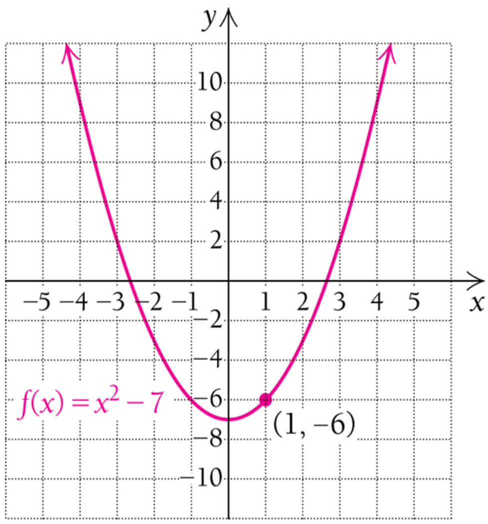
Section Summary

The following is a review of the function concepts considered in Sections R.1–R.3.

Function Concepts

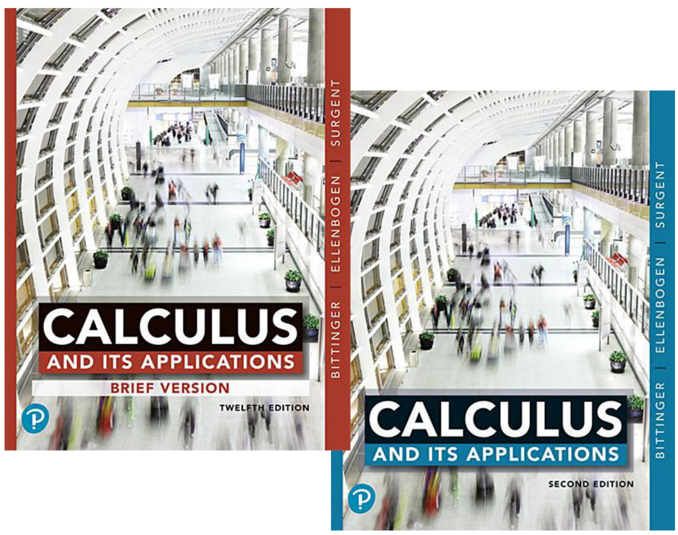
- Formula for f : $f(x) = x^2 - 7$
- For every input of f, there is exactly one output.
- For the input 1, -6 is the output; $f(1) = -6$
- For the output -3, the inputs are -2 and 2; $f(-2) = -3$ and $f(2) = -3$.
- Domain = The set of all inputs
  = The set of all real numbers, $\mathbb{R}$
- Range = The set of all outputs
  $= [-7, \infty)$

Graph



Chapter R

Functions, Graphs, and Models



R.4 Slope and Linear Functions

OBJECTIVES

- Find and interpret the slope of a line.
- Graph equations of the form $y=c$ and $x=a$.
- Graph linear functions.
- Find an equation of a line when given the slope and one point on the line and when given two points on the line.
- Solve applied problems involving slope and linear functions.

R.4 Slope and Linear Functions

Definition

The slope of line containing points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$

R.4 Slope and Linear Functions

Example 1: Find the slope of the line containing the points (–2, 6) and (–4, 9).

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{9-6}{-4-(-2)}$$

$$m = \frac{3}{-4+2}$$

$$m = -\frac{3}{2}$$

R.4 Slope and Linear Functions

Quick Check 1

Find the slope of the line containing the points $(2,3)$ and $(1,-4)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-4-3}{1-2}$$

$$m = \frac{-7}{-1}$$

$$m = 7$$

R.4 Slope and Linear Functions

Example 2:

a) Graph $y = 4$.

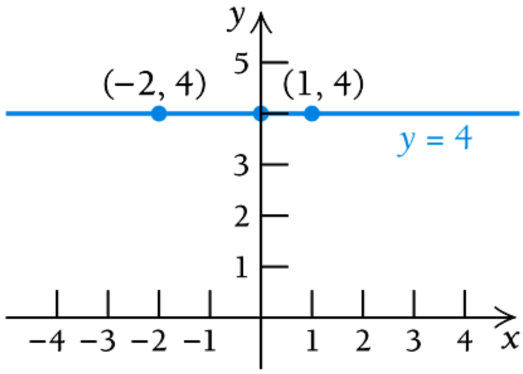
b) Decide whether the graph represents a function.

c) What is the line's slope?

a) The graph consists of all ordered pairs whose second coordinate is 4.

b) The vertical line test holds. Thus, the graph represents a function.

c) Since there is no change in y, the line's slope is 0.



R.4 Slope and Linear Functions

Example 3:

a) Graph $x = –3$.

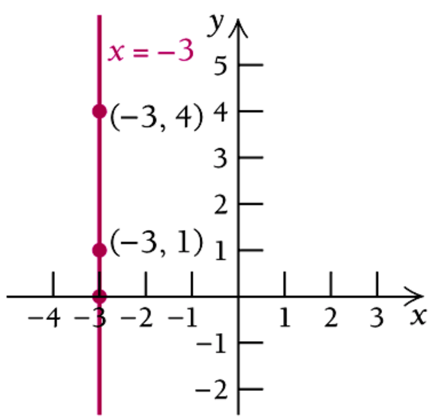
b) Decide whether the graph represents a function.

c) What is the line's slope?

a) The graph consists of all ordered pairs whose first coordinate is –3.

b) This graph does not represent a function because it does not pass the vertical line test.

c) Since there is no change in x, this line's slope is undefined.

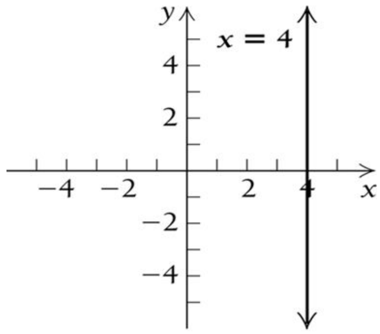


R.4 Slope and Linear Functions

Quick Check 2

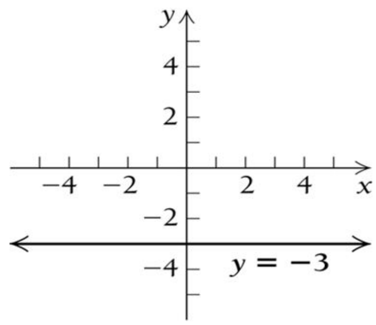
Graph each equation:

a.) $x = 4$



The graph consists of all ordered pairs whose first coordinate is 4.

b.) $y = -3$



The graph consists of all ordered pairs whose second coordinate is -3.

R.4 Slope and Linear Functions

Definition

The graph of $y = c$, or $f(x) = c$, a horizontal line, is the graph of a function. Such a function is referred to as a **constant function**.

The graph of $x = a$ is a vertical line and does not represent a function.

R.4 Slope and Linear Functions

Definition

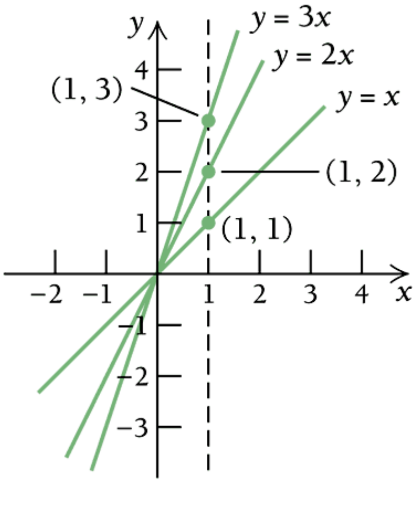
The graph of a function given by

$$y = mx \quad \text{or} \quad f(x) = mx$$

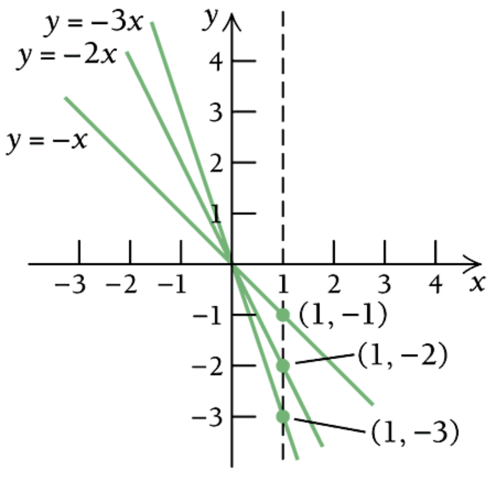
is the straight line through the origin (0, 0) and the point (1, *m*). The constant *m* is called the **slope** of the line.

R.4 Slope and Linear Functions

Here are various graphs of $y = mx$ for positive values of *m*.



Here are various graphs of $y = mx$ for negative values of *m*.



R.4 Slope and Linear Functions

DEFINITION

The variable y **varies directly** as x if there is some positive constant m such that $y = mx$. We also say that y is **directly proportional** to x.

R.4 Slope and Linear Functions

Example 4: The weight M, in pounds, of an object on the moon is directly proportional to the weight E of that object on Earth. An astronaut who weighs 180 lb on Earth will weigh 28.8 lb on the moon.

a) Find an equation of variation.
b) An astronaut weighs 120 lb on Earth. How much will the astronaut weigh on the moon?

R.4 Slope and Linear Functions

Example 4 (continued):

a) The equation has the form $M = mE$. To find m, we substitute.

$$28.8 = m \cdot 180$$
$$\frac{28.8}{180} = m$$
$$0.16 = m$$

Thus, $M = 0.16E$ is the equation of variation.

R.4 Slope and Linear Functions

Example 4 (concluded):

b) To find the weight on the moon of an astronaut who weighs 120 lb on Earth, we substitute 120 for E in the equation of variation.

$$M = 0.16 \cdot 120$$
$$M = 19.2$$

Thus, an astronaut who weighs 120 lb on Earth weighs 19.2 lb on the moon.

R.4 Slope and Linear Functions

DEFINITION:

A **linear function** is given by

$$y = mx + b \qquad \text{or} \qquad f(x) = mx + b$$

and has a graph that is the straight line parallel to the graph of $y = mx$ and crossing the x-axis at $(0, b)$. The point $(0, b)$ is called the ***y*-intercept**.

R.4 Slope and Linear Functions

DEFINITION:

$y = mx + b$ is called the **slope-intercept equation** of a line.

R.4 Slope and Linear Functions

Quick Check 3

Find the slope and the y-intercept of the graph of $3x-6y-7=0$.

We solve for y:

$$\begin{aligned} 3x-6y-7&=0 \\ -6y&=-3x+7 \\ y&=\frac{3}{6}x-\frac{7}{6} \\ y&=\frac{1}{2}x-\frac{7}{6} \end{aligned}$$

So the slope of the graph is $\frac{1}{2}$ and the y-intercept is $\left(0,-\frac{7}{6}\right)$.

R.4 Slope and Linear Functions

Example 5: Find an equation of the line with slope 3 containing the point (–1, –5).

From the slope-intercept equation, we have

$$y = 3x + b.$$

So, we must substitute to find b.

$$\begin{aligned} -5&=3(-1)+b \\ -2&=b \end{aligned}$$

Thus, the equation is $y = 3x - 2$.

R.4 Slope and Linear Functions

DEFINITION:

$y - y_1 = m(x - x_1)$ is called the **point-slope equation** of a line.

R.4 Slope and Linear Functions

Example 6: Find an equation of the line with slope 2/3 containing the point (–1, –5).

Substituting, we get

$$y - y_1 = m(x - x_1)$$

$$y - (-5) = \frac{2}{3}(x - (-1))$$

$$y + 5 = \frac{2}{3}(x + 1)$$

$$y + 5 = \frac{2}{3}x + \frac{2}{3}$$

$$y = \frac{2}{3}x + \frac{2}{3} - 5$$

$$y = \frac{2}{3}x - \frac{13}{3}$$

R.4 Slope and Linear Functions

Example 7: Raggs, Ltd., a clothing firm, has **fixed costs** of \$10,000 a year. These costs, such as rent, maintenance, and so on, must be paid no matter how much the company produces. To produce x units of a certain kind of suit, it costs \$20 per suit (unit) in addition to the fixed costs. That is, the **variable costs** for producing x of these suits are $20x$ dollars. These costs are due to the amount produced and stem from items such as material, wages, fuel, and so on.

R.4 Slope and Linear Functions

Example 7 (continued):

The **total cost** $C(x)$ of producing x suits a year is given by a function C.

$$C(x) = (\text{Variable costs}) + (\text{Fixed costs}) = 20x + 10{,}000.$$

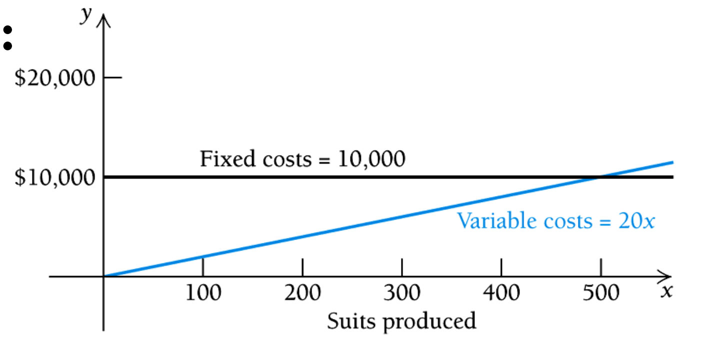
a) Graph the variable-cost, the fixed-cost, and the total cost functions.

b) What is the total cost of producing 100 suits? 400 suits?

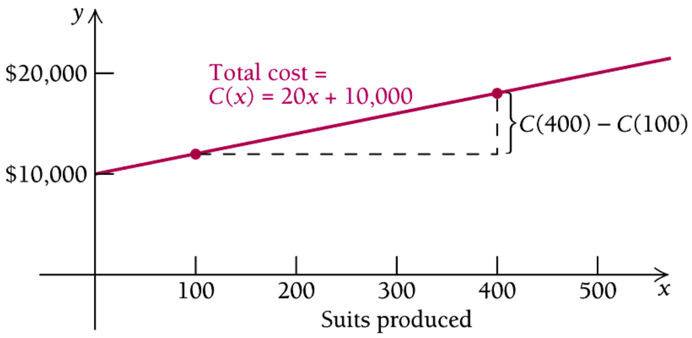
R.4 Slope and Linear Functions

Example 7 (continued):

a) The variable-cost and fixed-cost functions appear in the graph on the top.



The total-cost function is shown in the graph on the bottom.



R.4 Slope and Linear Functions

Example 7 (concluded):

b) The total cost of producing 100 suits is

$$C(100) = 20 \cdot 100 + 10{,}000 = \$12{,}000.$$

The total cost of producing 400 suits is

$$C(400) = 20 \cdot 400 + 10{,}000 = \$18{,}000.$$

R.4 Slope and Linear Functions

Example 8: When a business sells an item, it receives the *price* paid by the consumer (this is normally greater than the *cost* to the business of producing the item).

a) The **total revenue** that a business receives is the product of the number of items sold and the price paid per item. Thus, if Riggs, Ltd., sells x suits at \$80 per suit, the total revenue $R(x)$, in dollars is given by

$$R(x) = \text{Unit price} \cdot \text{Quantity sold} = 80x.$$

If $C(x) = 20x + 10{,}000$, graph R and C using the same set of axes.

R.4 Slope and Linear Functions

Example 8 (continued):

b) The **total profit** that a business receives is the amount left after all costs have been subtracted from the total revenue. Thus, if $P(x)$ represents the total profit when x items are produced and sold, we have

$$P(x) = (\text{Total Revenue}) - (\text{Total Costs}) = R(x) - C(x).$$

Determine $P(x)$ and draw its graph using the same set of axes as was used for the graph in part (a).

R.4 Slope and Linear Functions

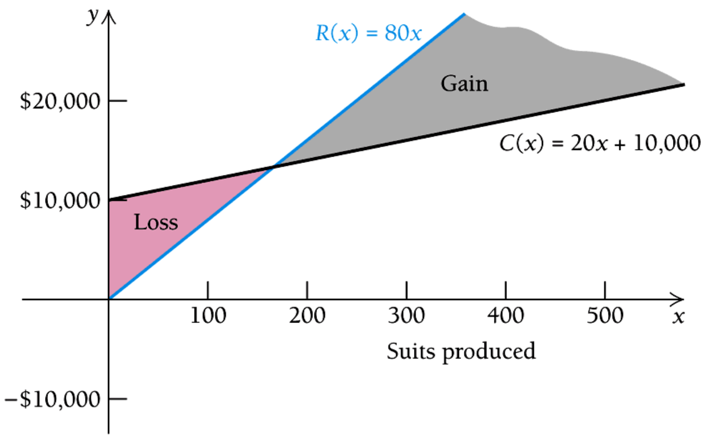
Example 8 (continued):

c) The company will *break even* at that value of x for which $P(x) = 0$ (that is, no profit and no loss). This is the point at which $R(x) = C(x)$. Find the **break-even** value of x.

R.4 Slope and Linear Functions

Example 8 (continued):

a) The graphs of $R(x)$ and $C(x)$ are shown below. When $C(x)$ is above $R(x)$, a loss will occur. This is shown by the region shaded red. When $R(x)$ is above $C(x)$, a gain will occur. This is shown by the region shaded in gray.



R.4 Slope and Linear Functions

Example 8 (continued):

b) To find P, the profit function, we have

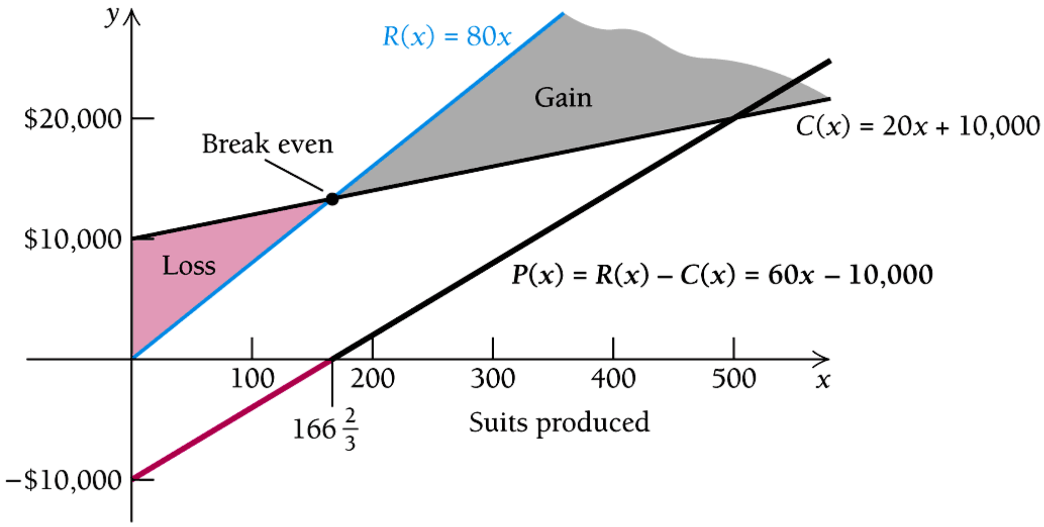
$$P(x) = R(x) - C(x) = 80x - (20x + 10{,}000)$$
$$= 60x - 10{,}000.$$

The graph is shown on the next slide. The graph of $P(x)$ is shown by the heavy line. The red portion of the line shows a "negative" profit, or loss. The black portion of the heavy line shows a "positive" profit, or gain.

R.4 Slope and Linear Functions

Example 8 (continued):

b)



R.4 Slope and Linear Functions

Example 8 (concluded):

c) To find the break-even value, we solve $R(x) = C(x)$:

$$R(x) = C(x)$$
$$80x = 20x + 10{,}000$$
$$60x = 10{,}000$$
$$x = 166\frac{2}{3} \approx 167 \text{ suits}$$

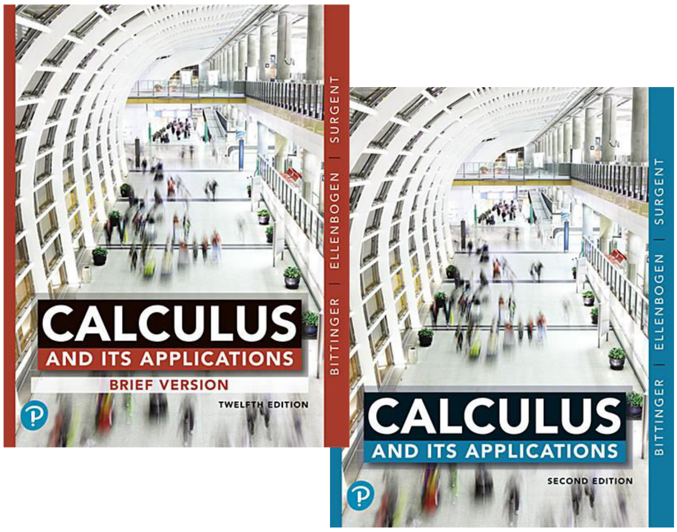
R.4 Slope and Linear Functions

Section Summary

- The slope m of a line containing the points (x_1, y_1) and (x_2, y_2) is given by
  $m = \dfrac{y_2 - y_1}{x_2 - x_1}.$
- The slope of a line can be interpreted $\dfrac{\text{change in } y}{\text{change in } x}.$
- A horizontal line has slope $m = 0$, and a vertical line has an undefined slope.
- Graphs of functions that are straight lines (*linear functions*) are characterized by an equation of the type $f(x) = mx + b$, where m is the slope and $(0, b)$ is the y-intercept, the point at which the graph crosses the y-axis.
- The form $y = mx + b$ is called the *slope–intercept equation of a line*.
- The *point–slope equation* of a line is $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a point on the line and m is the slope.

Chapter R

Functions, Graphs, and Models



R.5 Nonlinear Functions and Models

OBJECTIVE

- Graph functions and solve applied problems.
- Manipulate radical expressions and rational exponents.
- Determine the domain of a rational function and graph certain rational functions.
- Find the equilibrium point given a supply function and a demand function.

R.5 Nonlinear Functions and Models

DEFINITION:

A **quadratic function** f is given by

$$f(x) = ax^2 + bx + c, \quad \text{where } a \neq 0.$$

R.5 Nonlinear Functions and Models

The graph of a **quadratic function** $f(x) = ax^2 + bx + c$, where $a \neq 0$, is called a **parabola**.

a) It is always a cup-shaped curve.

b) It opens upward if $a > 0$ and opens downward if $a < 0$.

c) It has a turning point, or **vertex**, at a point with first coordinate $x = -\dfrac{b}{2a}$.

d) The vertical line $x = -b/(2a)$ serves as the line of symmetry.

R.5 Nonlinear Functions and Models

Example 1: Graph $f(x) = x^2 - 2x - 3$.

Note that for $f(x)$ we have $a = 1$, $b = -2$, and $c = -3$. Since $a > 0$, the graph opens upward. Next, let's find the vertex. The x-coordinate of the vertex is

$$x = -\frac{-2}{2(1)} = 1.$$

To find the second coordinate, we find $f(1)$.

$$f(1) = (1)^2 - 2(1) - 3 = -4.$$

R.5 Nonlinear Functions and Models

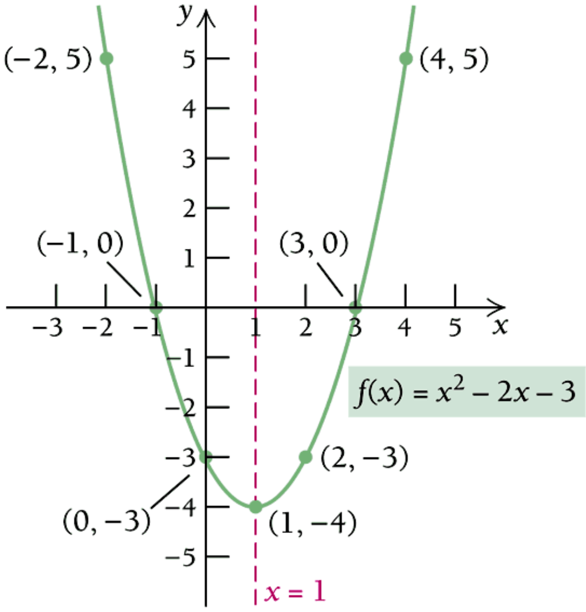
Example 1 (continued):
Thus, the vertex is $(1, -4)$. The vertical line $x = 1$ is the line of symmetry of the graph. We choose some x-values on each side of the vertex, compute some y-values, plot the points, and graph the parabola.

R.5 Nonlinear Functions and Models

Example 1 (concluded):

x	$f(x)$
1	−4
0	−3
2	−3
3	0
4	5
−1	0
−2	5

← Vertex



R.5 Nonlinear Functions and Models

Quick Check 1

Graph each function:

a.) $f(x) = x^2 + 2x - 3$

b.) $f(x) = -2x^2 - 10x - 5$

a.) Note that for $f(x)$, we have $a = 1,\ b = 2$, and $c = -3$.

Let's find the vertex: $x = -\dfrac{b}{2a}$

$$x = -\frac{2}{2(1)} = -1$$

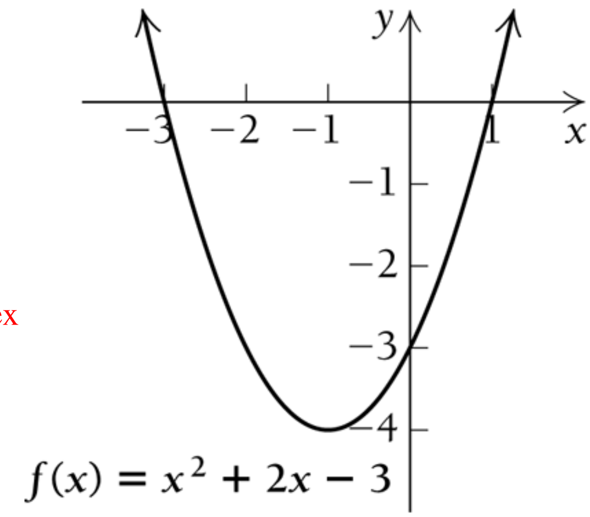
So the vertex is: $(-1, -4)$

R.5 Nonlinear Functions and Models

Quick Check 1 Continued

Next we choose some points on either side of the vertex, and then graph the function:

x	y	(x, y)
–3	0	$(-3,0)$
–2	–3	$(-2,-3)$
–1	–4	$(-1,-4)$ ← vertex
0	–3	$(0,-3)$
1	0	$(1,0)$



R.5 Nonlinear Functions and Models

Quick Check 1 Continued

b.) $f(x) = -2x^2 - 10x - 5$:

For $f(x)$, we have $a = -2,\ b = -10,$ and $c = -5.$

Let's find the vertex:

$$x = -\frac{b}{2a} = -\frac{-10}{2(-2)} = -\frac{-10}{-4} = -\frac{5}{2}$$

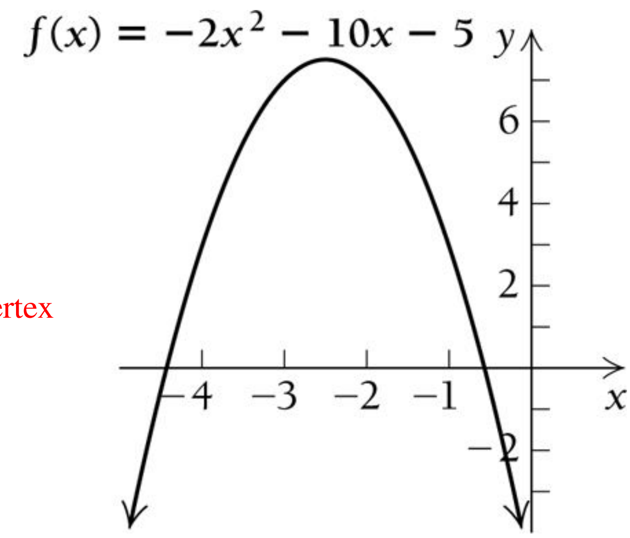
Thus the vertex is: $\left(-\frac{5}{2}, \frac{15}{2}\right)$

R.5 Nonlinear Functions and Models

Quick Check 1 Concluded

Next we choose some points on either side of the vertex, then graph the function:

x	y	(x, y)
-4	3	$(-4,3)$
-3	7	$(-3,7)$
$-\frac{5}{2}$	$\frac{15}{2}$	$\left(-\frac{5}{2},\frac{15}{2}\right)$ ← vertex
-2	7	$(-2,7)$
-1	3	$(-1,3)$



R.5 Nonlinear Functions and Models

THEOREM 3: The Quadratic Formula

The solutions of any quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0,$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

R.5 Nonlinear Functions and Models

Example 2: Solve: $3x^2 - 4x = 2$.

We first find the standard form of this quadratic function, and then determine a, b, and c.

$$3x^2 - 4x - 2 = 0$$

$$a = 3,\ b = -4, \text{ and } c = -2.$$

Then, we use the quadratic formula.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-2)}}{2 \cdot 3}$$

R.5 Nonlinear Functions and Models

Example 2 (concluded):

$$
\begin{aligned}
x &= \frac{4 \pm \sqrt{16 + 24}}{6} = \frac{4 \pm \sqrt{40}}{6} \\
&= \frac{4 \pm \sqrt{4 \cdot 10}}{6} = \frac{4 \pm 2\sqrt{10}}{6} \\
&= \frac{2\left(2 \pm \sqrt{10}\right)}{2 \cdot 3} \\
&= \frac{2 \pm \sqrt{10}}{3}
\end{aligned}
$$

The solutions are $\frac{2 + \sqrt{10}}{3} \approx 1.721$ and $\frac{2 - \sqrt{10}}{3} \approx -0.387$.

R.5 Nonlinear Functions and Models

Quick Check 2

Solve: $3x^2+2x=7.$

First we must find the standard form for this equation, then find a, b, and c.

$$3x^2+2x-7=0$$
$$a=3,\ b=2,\ \text{and } c=-7.$$

Then we can use the quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

R.5 Nonlinear Functions and Models

Quick Check 2 Concluded

$$x=\frac{-(2)\pm\sqrt{(2)^2-4(3)(-7)}}{2(3)}$$
$$x=\frac{-2\pm\sqrt{4+84}}{6}$$
$$x=\frac{-2\pm\sqrt{88}}{6}$$
$$x=\frac{-2\pm2\sqrt{22}}{6}$$
$$x=\frac{-1\pm\sqrt{22}}{3}\qquad \text{or } x\approx1.230 \text{ and } x\approx-1.897$$

R.5 Nonlinear Functions and Models

DEFINITION:

A **polynomial function** f is given by

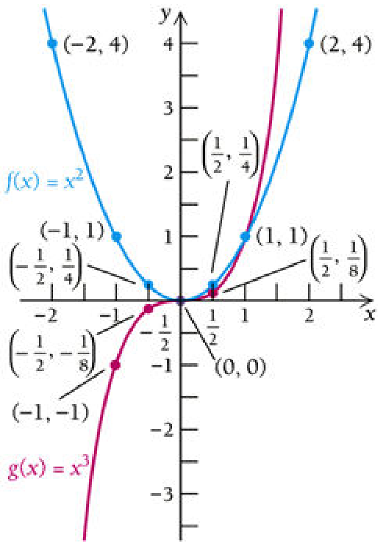
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0.$$

where n is a nonnegative integer and $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers, called **coefficients**.

R.5 Nonlinear Functions and Models

Example 3: Using the same set of axes, graph $f(x) = x^2$ and $g(x) = x^3$.

x	x^2	x^3
-2	4	-8
-1	1	-1
$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{8}$
0	0	0
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
1	1	1
2	4	8



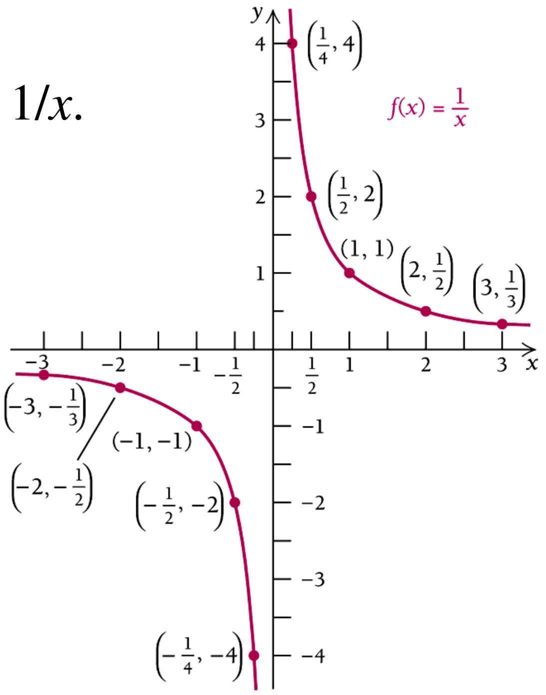
R.5 Nonlinear Functions and Models

DEFINITION:

Functions given by the quotient, or ratio, of two polynomials are called **rational functions**.

R.5 Nonlinear Functions and Models

Example 4:
Graph $f(x) = 1/x$.



x	$f(x)$
-3	$-\frac{1}{3}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{4}$	-4
$\frac{1}{4}$	4
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$

R.5 Nonlinear Functions and Models

DEFINITION:

y **varies inversely** as x if there is some positive number k such that $y = k/x$. We also say that y is **inversely proportional** to x.

R.5 Nonlinear Functions and Models

Example 5: Certain economists theorize that stock prices are inversely proportional to the price of gold. That is, when the price of gold goes up, the prices of stocks go down; and when the price of gold goes down, the prices of stocks go up. Let's assume that the Dow Jones Industrial Average D, an index of the overall price of stock, is inversely proportional to the price of gold G, in dollars per ounce. One day the Dow Jones was 9177 and the price of gold was \$364 per ounce. What will the Dow Jones Average be if the price of gold drops to \$300?

R.5 Nonlinear Functions and Models

Example 5 (concluded):

We know that $D = k/G$, so $9177 = k/364$ and $k = 3{,}340{,}428$. Thus,

$$D = \frac{3{,}340{,}428}{G}.$$

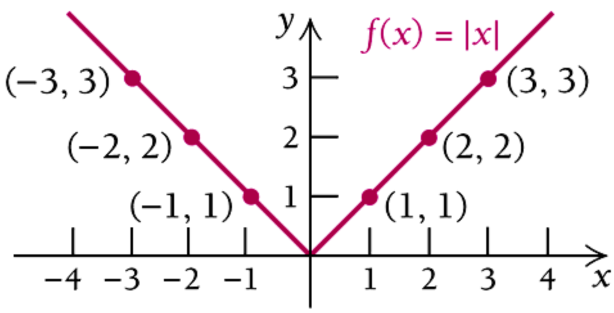
We substitute 300 for G and compute D.

$$D = \frac{3{,}340{,}428}{300} = 11{,}134.76$$

R.5 Nonlinear Functions and Models

Example 6: The Absolute Value Function

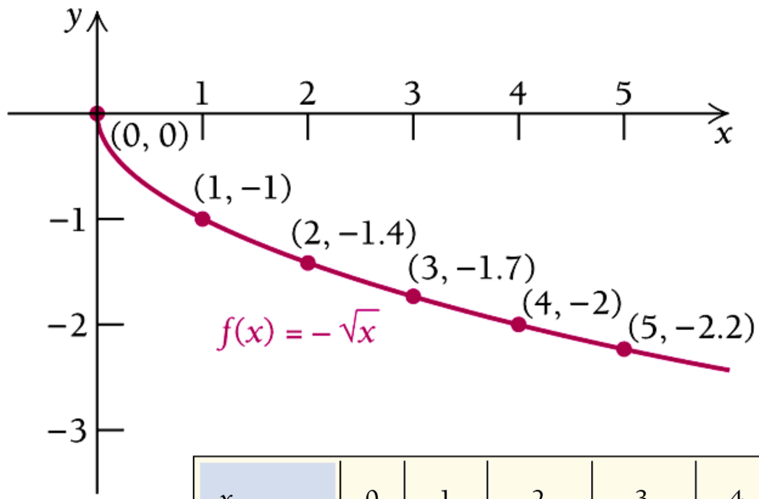
Graph $f(x) = |x|$.



x	$f(x)$
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

R.5 Nonlinear Functions and Models

Example 7: The Square Root Function Graph $f(x) = -\sqrt{x}$.



x	0	1	2	3	4	5
$f(x)$, or $-\sqrt{x}$	0	−1	−1.4	−1.7	−2	−2.2

R.5 Nonlinear Functions and Models

For any nonzero real number a and any integers n and m,

$$a^n \cdot a^m = a^{n+m};$$

$$\frac{a^n}{a^m} = a^{n-m};$$

$$\left(a^n\right)^m = a^{n \cdot m};$$

$$a^{-m} = \frac{1}{a^m}.$$

R.5 Nonlinear Functions and Models

Example 8: Rewrite each of the following as an equivalent expression with rational exponents:

a) $\sqrt[4]{x}$, for $x \geq 0$

b) $\sqrt[3]{r^2}$

c) $\sqrt{x^{10}}$, for $x \geq 0$

d) $\dfrac{1}{\sqrt[3]{b^5}}$, $b \neq 0$

R.5 Nonlinear Functions and Models

Example 8 (concluded):

a) $\sqrt[4]{x} = x^{\frac{1}{4}}$, for $x \geq 0$

b) $\sqrt[3]{r^2} = r^{\frac{2}{3}}$

c) $\sqrt{x^{10}} = x^{\frac{10}{2}} = x^5$, for $x \geq 0$

d) $\dfrac{1}{\sqrt[3]{b^5}} = \dfrac{1}{b^{\frac{5}{3}}} = b^{-\frac{5}{3}}$, for $b \neq 0$

R.5 Nonlinear Functions and Models

Example 9: Simplify: a) $8^{\frac{5}{3}}$; b) $81^{\frac{3}{4}}$.

$$\text{a) } 8^{\frac{5}{3}} = \left(8^{\frac{1}{3}}\right)^5 = \left(\sqrt[3]{8}\right)^5 = 2^5 = 32$$

$$\text{b) } 81^{\frac{3}{4}} = \left(81^{\frac{1}{4}}\right)^3 = \left(\sqrt[4]{81}\right)^3 = 3^3 = 27$$

R.5 Nonlinear Functions and Models

Quick Check 3

Find the domain of the function given by $f(x) = \sqrt{x+3}$.

We know that $f(x) = \sqrt{x}$ cannot be a real number if $x < 0$. So we need to solve the inequality $x + 3 \geq 0$.

$$x + 3 \geq 0$$
$$x \geq -3$$

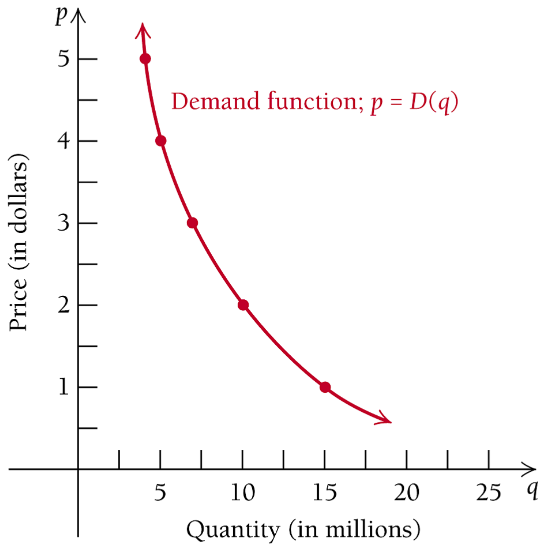
So the domain of the function is $\{x \mid x \geq -3\}$ or $[-3, \infty)$.

R.5 Nonlinear Functions and Models

Demand Function

The graph shows the relationship between price and quantity that consumers will demand at that price. The quantity consumers demand is inversely proportional to the price.

Price, x, per 5-lb Bag	Quantity, q, of 5-lb Bags (in millions)
$5	4
4	5
3	7
2	10
1	15

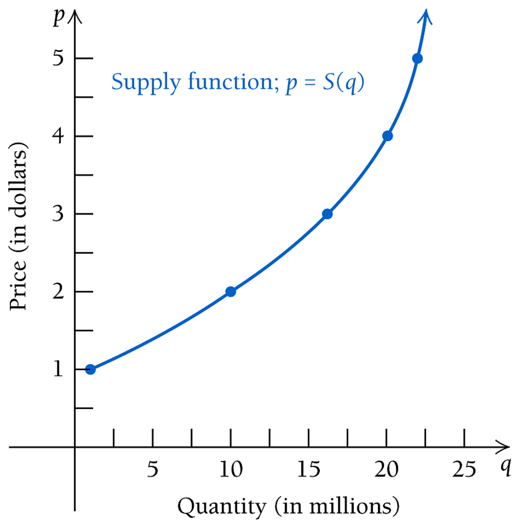


R.5 Nonlinear Functions and Models

Supply Function

The graph shows the relationship between price and quantity that sellers are willing to supply at that price. At higher prices suppliers are willing to supply greater quantities than they are at lower prices.

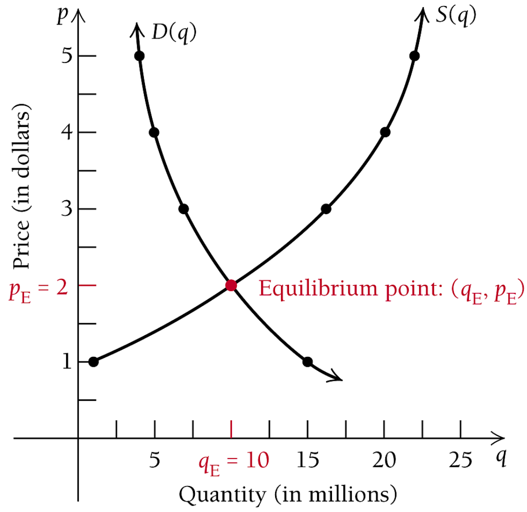
Price, x, per 5-lb Bag	Quantity, q, of 5-lb Bags (in millions)
$1	0
2	10
3	16
4	20
5	22



R.5 Nonlinear Functions and Models

Supply and Demand - Equilibrium Point

The equilibrium point is the price and quantity for which suppliers are willing to supply and consumers are willing to buy. It is the intersection of the two functions: supply = demand.



R.5 Nonlinear Functions and Models

Example 10: Find the equilibrium point for the demand and supply functions for the Ultra-Fine coffee maker. Here q represents the number of coffee makers produced, in hundreds, and x is the price, in dollars.

Demand: $q = 50 - \frac{1}{4}x$

Supply: $q = x - 25$

R.5 Nonlinear Functions and Models

Example 10 (continued): To find the equilibrium point, the quantity demanded must match the quantity produced.

$$
\begin{aligned}
50-\frac{1}{4}x &= x-25 \\
75 &= 1\frac{1}{4}x \\
75 &= \frac{5}{4}x \\
60 &= x
\end{aligned}
$$

R.5 Nonlinear Functions and Models

Example 10 (concluded):

To find q_E (the y-coordinate), we substitute x_E into either function.

$$q_E = x_E - 25 = 60 - 25 = 35$$

Thus, the equilibrium quantity is 3500 and the equilibrium point is ($60, 3500).

R.5 Nonlinear Functions and Models

Section Summary

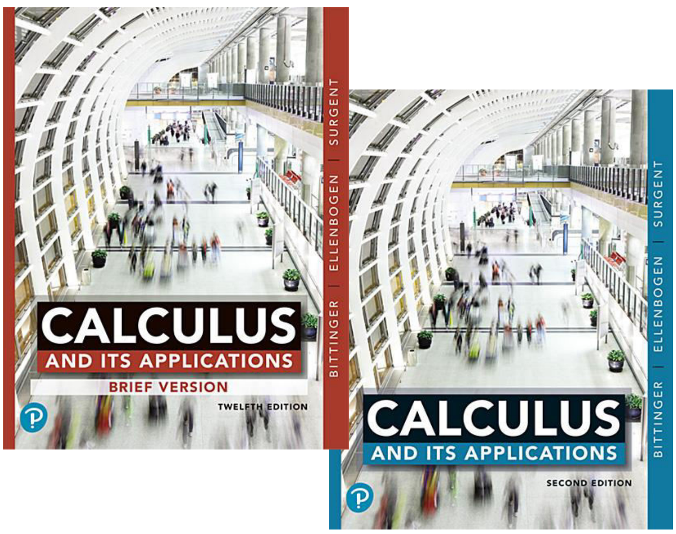
• Many types of functions have graphs that are not straight lines; among these are *quadratic functions, polynomial functions, power functions, rational functions, absolute-value functions,* and *square-root functions.*

• Demand is modeled by a decreasing function, that is, as the demand q gets larger, the price p gets smaller.

• Supply is modeled by an increasing function, that is, as the quantity supplied gets larger, so does the price p. The point of intersection of graphs of demand and supply functions for the same product is called the *equilibrium point.*

Chapter R

Functions, Graphs, and Models



R.6 Exponential and Logarithmic Functions

Objective

- Graph exponential functions.
- Graph logarithmic functions.
- Convert an exponential equation into an equivalent logarithmic equation.
- Convert a logarithmic equation into an equivalent exponential equation.
- Find the domain of a logarithmic function.
- Use exponential and logarithmic functions to solve certain applied problems.

R.6 Exponential and Logarithmic Functions

DEFINITION:

An **exponential function** f is given by $f(x) = a_0 \cdot a^x$, where x is any real number and $a > 0$ and $a \neq 1$. The number a is the **base**, and the y-intercept is $(0, a_0)$.

R.6 Exponential and Logarithmic Functions

Example 1: Graph $y = f(x) = \left(\frac{5}{4}\right)^x = 1.25^x$, and state the percentage change in y per unit increase in x.

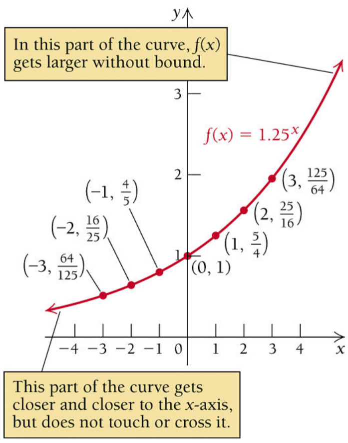
First, let's create a table of function values:

	$y = f(x) = \left(\frac{5}{4}\right)^x = 1.25^x$
-3	$\frac{64}{125}$
-2	$\frac{16}{25}$
-1	$\frac{4}{5}$
0	1
1	$\frac{5}{4}$
2	$\frac{25}{16}$
3	$\frac{125}{64}$

R.6 Exponential and Logarithmic Functions

Example 1 (concluded):

Plotting these points we get the smooth curve shown below:



Here, we have $a = 1 + r = 1.25$, so $r = 0.25$, or 25%. The value of y is increasing by 25% for every unit increase in x.

R.6 Exponential and Logarithmic Functions

Example 2: Population Decay. The city of Cuprite had a population of 15,000 people in 2012, when the closing of a mine initiated a decrease in population that persists to the present. Assume that the population decreases by the same percentage each year and that in 2017 the population was 11,250.

a) Find $P(t)$, the population of Cuprite t years after 2012.

b) Estimate Cuprite's population in 2022.

c) Graph $P(t)$ for $0 \le t \le 10$.

d) Find the annual percent change in Cuprite's population.

R.6 Exponential and Logarithmic Functions

Example 2 (continued):

a) Here $P(t) = P_0 \cdot a^t$, where $P_0 = 15,000$ is the population at t=0 or the population in 2012. To find a, we use the fact that the population in 2017 (t = 5) is 11,250.

Substituting, we get: $11,250 = 15,000 \cdot a^5$

$$\frac{11,250}{15,000} = a^5$$

$$a = \left(\frac{11,250}{15,000}\right)^{1/5} \approx 0.944$$

Thus, the function that models Cuprite's population t years after 2012 is given by $P(t) = 15,000(0.944)^t$

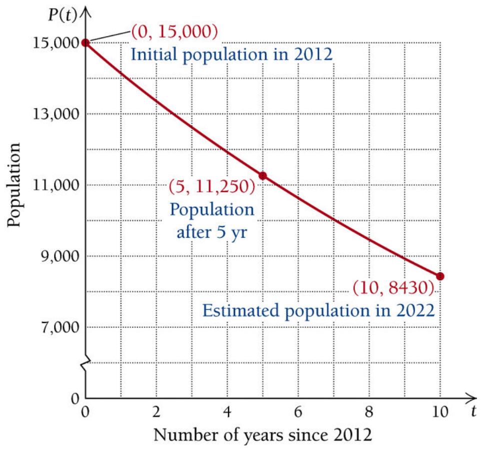
R.6 Exponential and Logarithmic Functions

Example 2 (concluded):

b) In 2022 (t = 10), Cuprite's estimated population will be:

$$P(10) = 15000(0.944)^{10} \approx 8430$$

c) Graph:



d) Since $a = 1 + r$, we have $1 + r = 0.944$, so $r = 0.056$. Cuprite's population is *decreasing* at a rate of 5.6% per year.

R.6 Exponential and Logarithmic Functions

Definition

The **logarithm**, base a, of y is the number x for which $a^x = y$. Equivalently, $a^x = y$, if and only if $\log_a y = x$. The base a must be positive and not equal to 1, that is, $a > 0$ and $a \neq 1$.

R.6 Exponential and Logarithmic Functions

Example 3: Rewrite each exponential equation as an equivalent logarithmic equation:

Using the definition of a logarithm, we get:

a) $2^5 = 32 \quad \Rightarrow \quad \log_2 32 = 5$

b) $4^2 = 16 \quad \Rightarrow \quad \log_4 16 = 2$

c) $6^1 = 6 \quad \Rightarrow \quad \log_6 6 = 1$

d) $3^{-2} = \dfrac{1}{9} \quad \Rightarrow \quad \log_3\left(\dfrac{1}{9}\right) = -2$

e) $7^0 = 1 \quad \Rightarrow \quad \log_7 1 = 0$

R.6 Exponential and Logarithmic Functions

Quick Check 1

Rewrite each exponential equation as an equivalent logarithmic equation:
Using the definition of logarithms, we get:

a) $5^3 = 125 \quad \Rightarrow \quad \log_5 125 = 3$

b) $9^{-3} = \dfrac{1}{729} \quad \Rightarrow \quad \log_9\left(\dfrac{1}{729}\right) = -3$

c) $15^0 = 1 \quad \Rightarrow \quad \log_{15} 1 = 0$

R.6 Exponential and Logarithmic Functions

Example 4: Solve for x in each logarithmic equation.

a) $\log_2 x = 6$

Rewriting this in exponential form, we see that the base is 2 and the exponent is 6. Thus, we have $2^6 = x$. So, $x = 64$.

b) $\log_3\left(\frac{1}{3}\right) = x$

Here, the base is 3 and the exponent is x. Thus, we get: $3^x = \frac{1}{3}$.

Thus, $x = -1$.

c) $\log_x 49 = 2$

The base is x and the exponent is 2. Thus, rewriting in exponential form, we get: $x^2 = 49$. So, x is either -7 or 7. But since the base must be positive, we conclude that $x = 7$.

Example 4 (concluded):

d) $\log_{16} x = -\frac{1}{2}$

Here we have the base is 16 and the exponent is $-\frac{1}{2}$.

So, $16^{-\frac{1}{2}} = x.$

Thus, $x = \dfrac{1}{16^{\frac{1}{2}}} = \dfrac{1}{\sqrt{16}} = \dfrac{1}{4}.$

R.6 Exponential and Logarithmic Functions

Theorem 4: Properties of Logarithms

P1. $\log_a(MN) = \log_a M + \log_a N$

P2. $\log_a\left(\dfrac{M}{N}\right) = \log_a M - \log_a N$

P3. $\log_a(M^b) = b\log_a M$

P4. $\log_a a = 1$

Note: P1, P2, and P3 require that M and N are positive.

R.6 Exponential and Logarithmic Functions

Theorem 4 (concluded):

P5. $\log_a 1 = 0$

P6. $\log_a M = \dfrac{\log_b M}{\log_b a}$ (change of base formula)

P7. $\log_a a^k = k$

P8. For any $x > 0$, $a^{\log_a x} = x$

R.6 Exponential and Logarithmic Functions

Example 5: Given $\log_a 2 = 0.301$ and $\log_a 3 = 0.477$, find the following:

a) $\log_a 6$

$$\log_a 6 = \log_a (2 \cdot 3)$$

$$= \log_a 2 + \log_a 3 \qquad \text{Using Property P1}$$

$$= 0.301 + 0.477$$

$$= 0.778$$

R.6 Exponential and Logarithmic Functions

Example 5 (continued) :

b) $\log_a \left(\frac{2}{3} \right)$

$$\log_a \left(\frac{2}{3} \right) = \log_a 2 - \log_a 3$$

$$= 0.301 - 0.477 \qquad \text{Using Property P2}$$

$$= -0.176$$

R.6 Exponential and Logarithmic Functions

Example 5 (continued) :

c) $\log_a 81$

$$\log_a 81 = \log_a 3^4$$

$$= 4\log_a 3 \quad \text{Using Property P3}$$

$$= 4(0.477)$$

$$= 1.908$$

R.6 Exponential and Logarithmic Functions

Example 5 (continued) :

d) $\log_a\left(\frac{1}{3}\right)$

$$\log_a\left(\frac{1}{3}\right) = \log_a 1 - \log_a 3 \qquad \text{Using Property P2}$$

$$= 0 - 0.477 \qquad \text{Using Property P5}$$

$$= -0.477$$

R.6 Exponential and Logarithmic Functions

Example 5 (continued) :

e) $\log_a \sqrt{a}$

$$\log_a \sqrt{a} = \log_a a^{1/2}$$
$$= \frac{1}{2} \qquad \text{Using Property P7}$$

f) $\log_a (2a)$

$$\log_a (2a) = \log_a 2 + \log_a a \qquad \text{Using Property P1}$$
$$= 0.301 + 1 \qquad \text{Using Property P4}$$
$$= 1.301$$

R.6 Exponential and Logarithmic Functions

Example 5 (concluded) :

g) $\dfrac{\log_a 3}{\log_a 2}$

$$\frac{\log_a 3}{\log_a 2} = \frac{0.477}{0.301} \square 1.58 \qquad \text{By substitution and division}$$

h) $\log_a 5$

$\log_a 5$ is not possible by the property of logs since $\log_a 5 \neq \log_a 2 + \log_a 3.$

R.6 Exponential and Logarithmic Functions

Definition

The number $\log_{10} x$ is called the **common logarithm of** x and denoted $\log x$.

R.6 Exponential and Logarithmic Functions

Example 6: Use common logarithms and the change of base formula to find the following:

a) $\log 45$

$\log 45 \approx 1.65321$ using a calculator since it is already base 10.

b) $\log_5 12$

$$\log_5 12 = \frac{\log 12}{\log 5} \approx 1.54396$$ by the change of base formula

c) $\log_6 27$

$$\log_6 27 = \frac{\log 27}{\log 6} \approx 1.83944$$ by the change of base formula

R.6 Exponential and Logarithmic Functions

Quick Check 2

Find $\log_7 21$ and $\log_{30} 5$.

$\log_7 21 = \dfrac{\log 21}{\log 7}$ Using change of base formula

≈ 1.56458

$\log_{30} 5 = \dfrac{\log 5}{\log 30}$ Using change of base formula

≈ 0.47320

R.6 Exponential and Logarithmic Functions

Example 7: Graph $y = \log_5(x+2)$ and state its domain.

Rewrite $y = \log_5(x+2)$ as an equation in exponential form: $5^y = x+2$

Thus, $x = 5^y - 2.$ Now, choose convenient y values and find x.

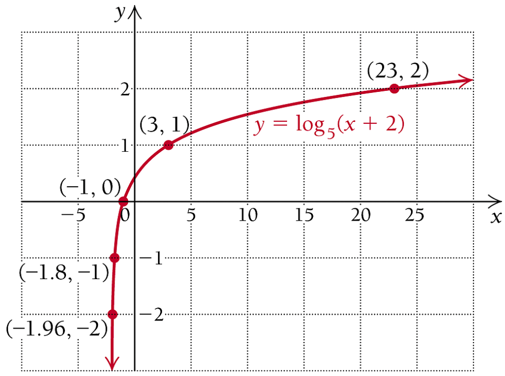
y	$x = 5^y - 2.$
-2	$5^{-2} - 2 = -1.96$
-1	$5^{-1} - 2 = -1.8$
0	$5^0 - 2 = -1$
1	$5^1 - 2 = 3$
2	$5^2 - 2 = 23$

Plot these ordered pairs (x,y) and draw a smooth curve joining them.

R.6 Exponential and Logarithmic Functions

Example 7 (concluded):

When we plot the ordered pairs (x,y) and draw a smooth curve joining them, we get the graph of $y = \log_5(x+2)$ below:



Note the domain of $y = \log_5(x+2)$ is $\{x \mid x > -2\}$ or $(-2, \infty)$. We can see this from the graph above, or algebraically, we know that the argument of the logarithm function must be positive. That is, $x + 2 > 0$. So, $x > -2$.

R.6 Exponential and Logarithmic Functions

Example 8: Business: Compound Interest. Bert deposits \$10,000 in a savings account that earns 3.25% interest per year. When will his account have a value of \$20,000?

Substituting these values into the formula $A(t) = P(1+r)^t$, we get:

$$20,000 = 10,000(1 + .0325)^t$$

$$20,000 = 10000(1.0325)^t$$

$$2 = 1.0325^t$$

Rewriting as an equivalent logarithmic equation, we get:

$$t = \log_{1.0325} 2$$

$$t = \frac{\log 2}{\log 1.0325} \quad \text{Using the change of base formula}$$

$$t \approx 21.67 \text{ years.}$$

R.6 Exponential and Logarithmic Functions

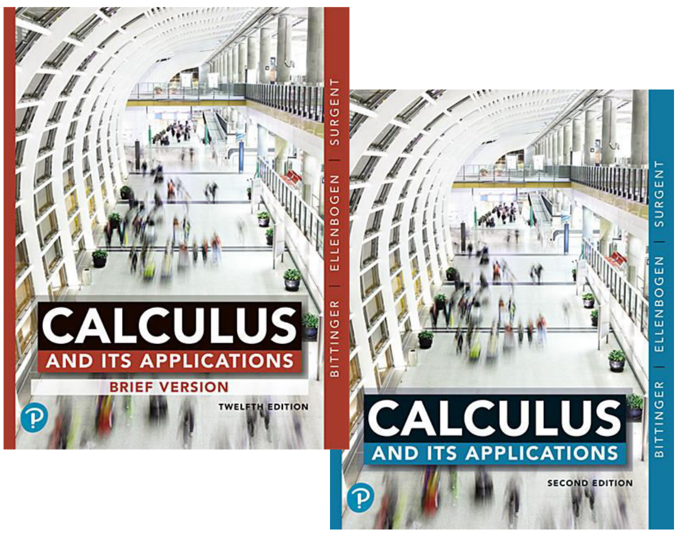
Section Summary

- An exponential function is of the form $f(x) = a_0 \cdot a^x$, where $a > 0$ and $a \neq 1$. The y-intercept is $(0, a_0)$; there are no x-intercepts; the domain is all of the real numbers, and the range is $\{y \mid y > 0\}$.

- An logarithmic function is of the form $f(x) = \log_a x$, where $a > 0$ and $a \neq 1$. The x-intercept is $(1, 0)$; there are no y-intercepts; the domain is $\{x \mid x > 0\}$ and the range is all real numbers.

Chapter R

Functions, Graphs, and Models



R.7 Mathematical Modeling and Curve Fitting

OBJECTIVE

- Use curve fitting to find a mathematical model for a set of data and use the model to make predictions.

R.7 Mathematical Modeling and Curve Fitting

Example 1: For the scatterplots and graphs on the following slides, determine which, if any, of the following functions might be used as a model for the data.

Linear, $f(x) = mx + b$

Quadratic, $f(x) = ax^2 + bx + c$, $a > 0$

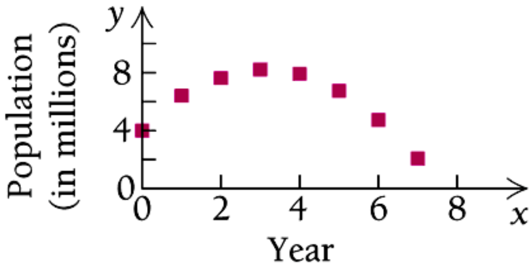
Quadratic, $f(x) = ax^2 + bx + c$, $a < 0$

Polynomial, neither linear nor quadratic

R.7 Mathematical Modeling and Curve Fitting

Example 1 (continued):

a)



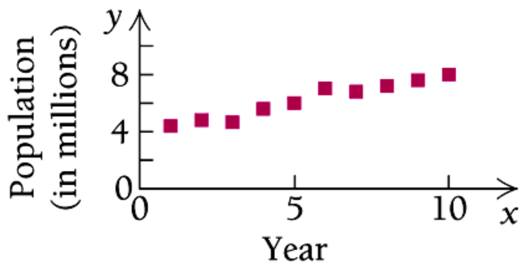
The data rise and then fall in a curved manner, fitting a quadratic function,

$$f(x) = ax^2 + bx + c,\ a < 0.$$

R.7 Mathematical Modeling and Curve Fitting

Example 1 (continued):

b)



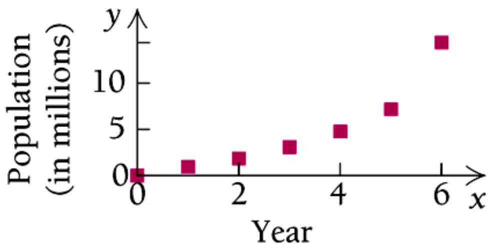
The data seem to fit a linear function,

$$f(x) = mx + b.$$

R.7 Mathematical Modeling and Curve Fitting

Example 1 (continued):

c)



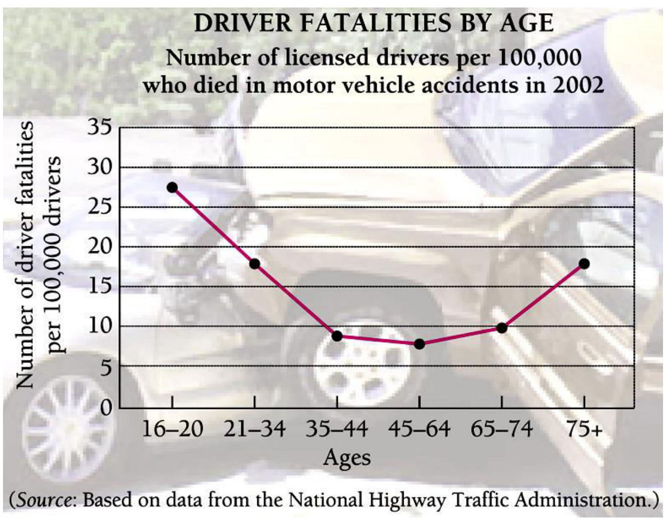
The data rise in a manner fitting the right-hand side of a quadratic function,

$$f(x) = ax^2 + bx + c,\ a > 0.$$

R.7 Mathematical Modeling and Curve Fitting

Example 1 (continued):

d)



The data fall and then rise in a curved manner, fitting a quadratic function,

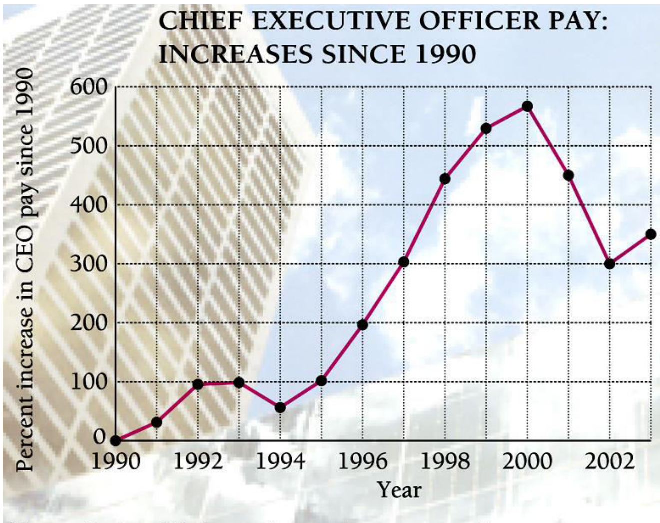
$$f(x) = ax^2 + bx + c,\ a > 0.$$

R.7 Mathematical Modeling and Curve Fitting

Example 1 (continued):

e)

The data rise and fall more than once, so they do not fit a linear or quadratic function but might fit a polynomial function that is neither quadratic nor linear.



(*Source*: *Business Week* annual executive pay surveys.)

R.7 Mathematical Modeling and Curve Fitting

Example 2: The following table shows the annual percent increases in pay since 1996 for a U.S. production worker.

Number of years, *x*, since 1996	1	2	3	4	5	6	7
Percent increase since 1996, *P*	1.9	7.4	11.7	19.5	28.2	29.7	31.3

R.7 Mathematical Modeling and Curve Fitting

Example 2 (continued):

a) Make a scatterplot of the data and determine whether the data seem to fit a linear function.

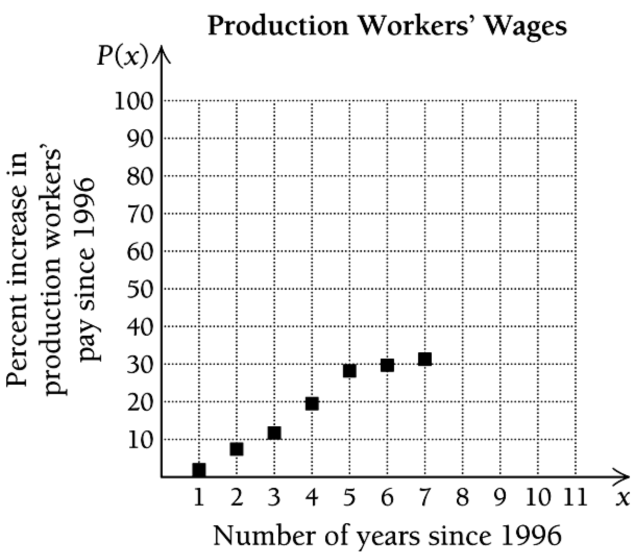
b) Find a linear function that (approximately) fits the data.

c) Use the model to predict the percentage by which 2010 wages will exceed 1996 wages.

R.7 Mathematical Modeling and Curve Fitting

Example 2 (continued):

a) The scatterplot is shown below. The data tend to follow a straight line, although a "perfect" straight line cannot be drawn through the data points.



R.7 Mathematical Modeling and Curve Fitting

Example 2 (continued):

b) We consider the function $P(x) = mx + b$, where $P(x)$ is the percentage by which the wages x years after 1996 exceed the wages in 1996. To derive the constants (or parameters) m and b, we choose two data points. Although this procedure is somewhat arbitrary, we try to choose two points that follow the general linear pattern. In this case, we pick (1, 1.9) and (4, 19.5).

R.7 Mathematical Modeling and Curve Fitting

Example 2 (continued):

Since the points are to be solutions of the linear equation, it follows that

$$\begin{aligned} 1.9 &= m \cdot 1 + b \\ 19.5 &= m \cdot 4 + b \end{aligned} \quad \text{or} \quad \begin{aligned} 1.9 &= m + b \\ 19.5 &= 4m + b. \end{aligned}$$

We now have a system of equations. We solve by subtracting each side of the top equation from each side of the bottom equation to eliminate b.

$$\begin{aligned} 17.6 &= 3m \\ 5.87 &\approx m \end{aligned}$$

R.7 Mathematical Modeling and Curve Fitting

Example 2 (continued):

Substituting 5.87 for m into either equation, we can solve for b.

$$\begin{aligned} 1.9 &= 5.87 + b \\ -3.97 &= b \end{aligned}$$

Thus, we get the equation (model)

$$P(x) = 5.87x - 3.97.$$

R.7 Mathematical Modeling and Curve Fitting

Example 2 (concluded):

c) The percentage by which U.S. production workers' wages will have increased in 2010, a total of 14 years after 1996, is

$$P(14) = 5.87 \cdot 14 - 3.97 = 78.21\%.$$

R.7 Mathematical Modeling and Curve Fitting

Example 3: In a study by Dr. Harold J. Morowitz of Yale University, data were gathered that showed the relationship between the death rate of men and the average number of hours per day that the men slept. These data are listed in the table on the next slide.

a) Make a scatterplot of the data, and determine whether the data seem to fit a quadratic function.

b) Find a quadratic function that fits the data.

c) Use the model to find the death rate for males who sleep 2 hr, 8 hr, and 10 hr.

R.7 Mathematical Modeling and Curve Fitting

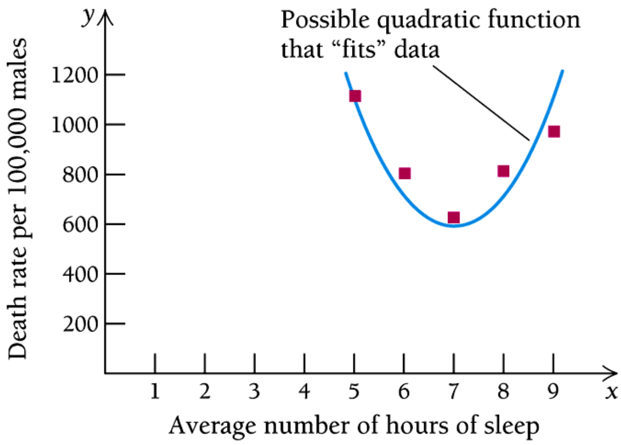
Example 3 (continued):

Average Number of Hours of Sleep, x	Death Rate per 100,000 Males, y
5	1121
6	805
7	626
8	813
9	967

R.7 Mathematical Modeling and Curve Fitting

Example 3 (continued):

a) The scatterplot is shown below. Note that the rate drops and then rises, which suggests that a quadratic function might fit the data.



R.7 Mathematical Modeling and Curve Fitting

Example 3 (continued):

b) We consider the quadratic model,

$$y = ax^2 + bx + c.$$

To derive the constants (or parameters) a, b, and c, we use the three data points (5, 1121), (7, 626), and (9, 967). Since these points are to be solutions of a quadratic equation, it follows that

$$\begin{aligned} 1121 &= a\cdot 5^2 + b\cdot 5 + c \\ 626 &= a\cdot 7^2 + b\cdot 7 + c \\ 967 &= a\cdot 9^2 + b\cdot 9 + c \end{aligned} \quad \text{or} \quad \begin{aligned} 1121 &= 25a + 5b + c \\ 626 &= 49a + 7b + c \\ 967 &= 81a + 9b + c. \end{aligned}$$

R.7 Mathematical Modeling and Curve Fitting

Example 3 (continued):

We solve this system of three equations in three variables using procedures of algebra and get

$$a = 104.5, \quad b = -1501.5, \quad \text{and } c = 601.6,$$

Thus, the quadratic model for this data is given by

$$y = 104.5x^2 - 1501.5x + 601.6.$$

c) The death rate for males who sleep 2 hr is given by

$$y = 104.5 \cdot 2^2 - 1501.5 \cdot 2 + 601.6 = 3431.$$

R.7 Mathematical Modeling and Curve Fitting

Example 3 (concluded):

The death rate for males who sleep 8 hr is given by

$$y = 104.5 \cdot 8^2 - 1501.5 \cdot 8 + 601.6 = 692.$$

The death rate for males who sleep 10 hr is given by

$$y = 104.5 \cdot 10^2 - 1501.5 \cdot 10 + 601.6 = 1451.$$

R.7 Mathematical Modeling and Curve Fitting

Quick Check 1

From the data below, use the data points $(16,34)$, $(27,113.9)$, and $(37,35.4)$ to find a quadratic function that fits the data. Use the model to predict the average number of live births to women age 20.

Age, x	Average Number of Live Births per 1000 women
16	34
18.5	86.5
22	111.1
27	113.9
32	84.5
37	35.4
42	6.8

R.7 Mathematical Modeling and Curve Fitting

Quick Check 1 Continued

We consider the quadratic model,

$$y = ax^2 + bx + c$$

To derive the constants (or parameters) a, b, and c, we use the three data points $(16,34)$, $(27,113.9)$, and $(37,35.4)$. Since these points are to be solutions of a quadratic equation, it follows that

$$\begin{aligned} 34 &= a\cdot 16^2 + b\cdot 16 + c \\ 113.9 &= a\cdot 27^2 + b\cdot 27 + c \\ 35.4 &= a\cdot 37^2 + b\cdot 37 + c \end{aligned} \quad \text{or} \quad \begin{aligned} 34 &= 256a + 16b + c \\ 113.9 &= 729a + 27b + c \\ 35.4 &= 1369a + 37b + c \end{aligned}$$

R.7 Mathematical Modeling and Curve Fitting

Quick Check 1 Concluded

We solve this system of three equations in three variables using procedures of algebra and get

$$a = -0.7197,\ b = 38.2106,\ \text{and } c = -393.1272.$$

Thus the quadratic model for this data is given by

$$y = -0.7197x^2 + 38.2106x - 393.1272.$$

To predict the average number of live births to women age 20, we simply solve for y where $x = 20$:

$$y = -0.7197(20)^2 + 38.2106(20) - 393.1272$$

$$y = -287.88 + 764.212 - 393.1272 = 83.2048$$

So the model predicts that there will be 83.2 live births per 1000 women age 20.