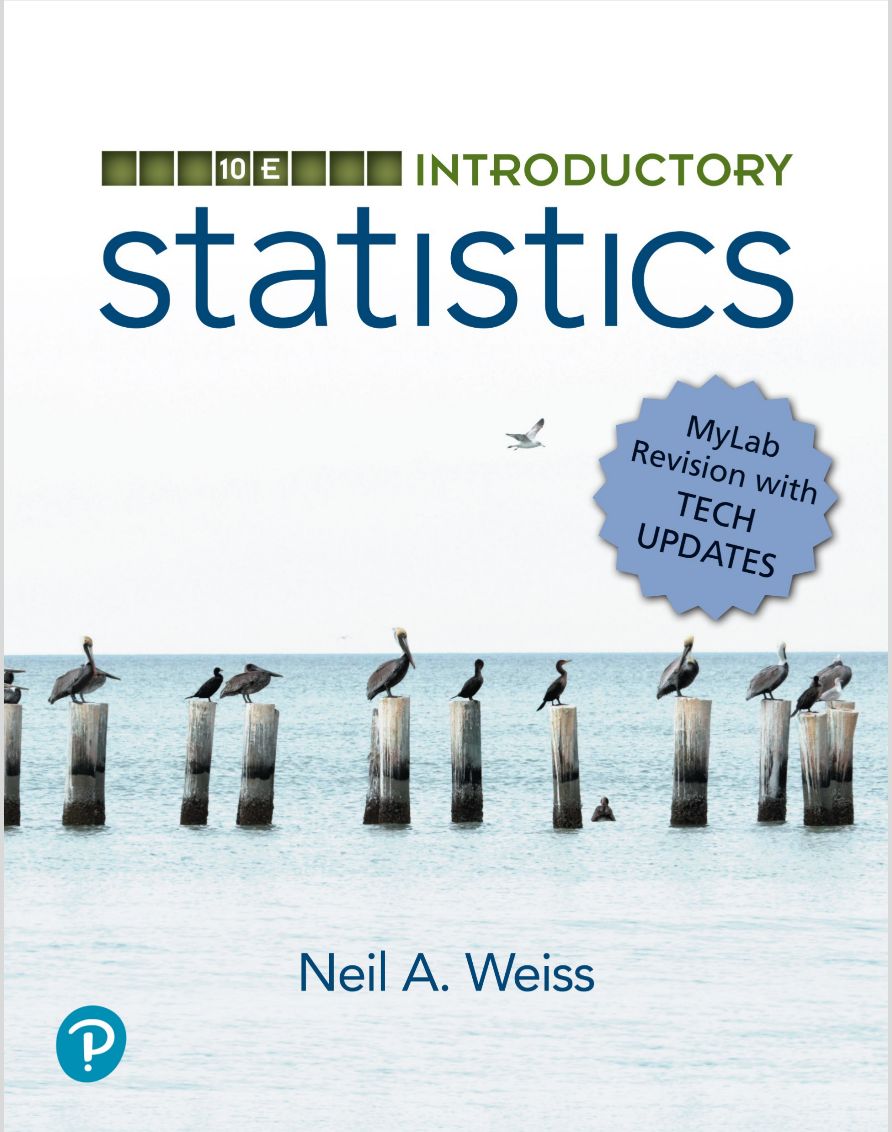
Chapter 5

Discrete Random Variables



Chapter 5

Discrete Random Variables

Section 5.1

Discrete Random Variables and Probability Distributions

Definitions 5.1 & 5.2

Random Variable

A **random variable** is a quantitative variable whose value depends on chance.

Discrete Random Variable

A **discrete random variable** is a random variable whose possible values can be listed. In particular, a random variable with only a finite number of possible values is a discrete random variable.

Definition 5.3

Probability Distribution and Probability Histogram

Probability distribution: A listing of the possible values and corresponding probabilities of a discrete random variable, or a formula for the probabilities.

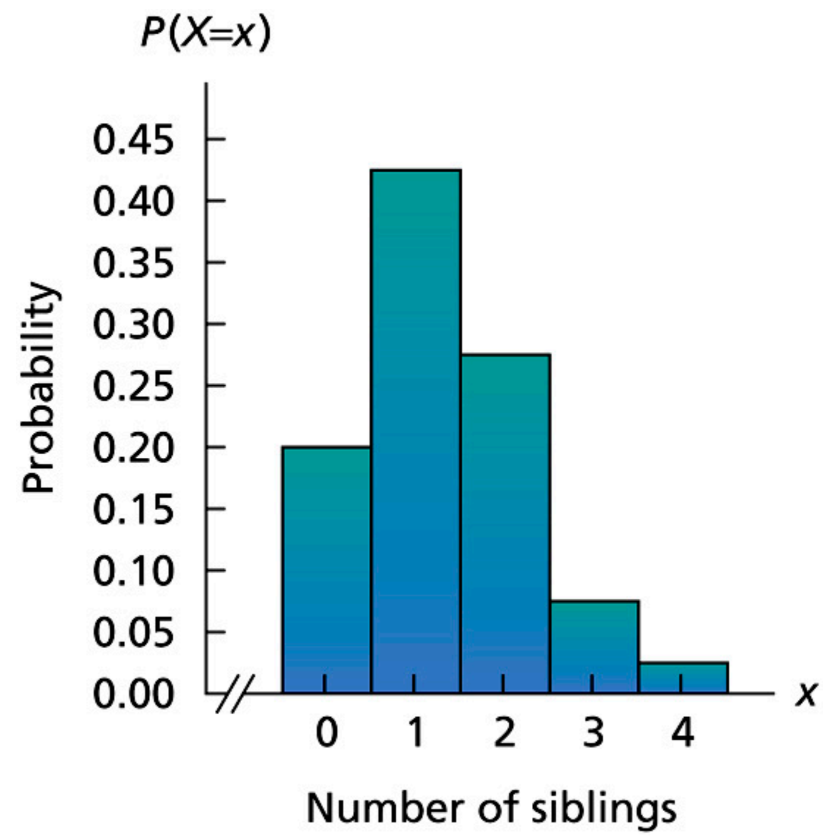
Probability histogram: A graph of the probability distribution that displays the possible values of a discrete random variable on the horizontal axis and the probabilities of those values on the vertical axis. The probability of each value is represented by a vertical bar whose height equals the probability.

Table 5.2 & Figure 5.1

Probability distribution of the random variable *X*, the number of siblings of a randomly selected student

Siblings x	Probability $P(X = x)$
0	0.200
1	0.425
2	0.275
3	0.075
4	0.025
	1.000

Probability histogram for the random variable *X*, the number of siblings of a randomly selected student



Key Fact 5.1

Sum of the Probabilities of a Discrete Random Variable

For any discrete random variable X, we have $\sum P(X = x) = 1$.

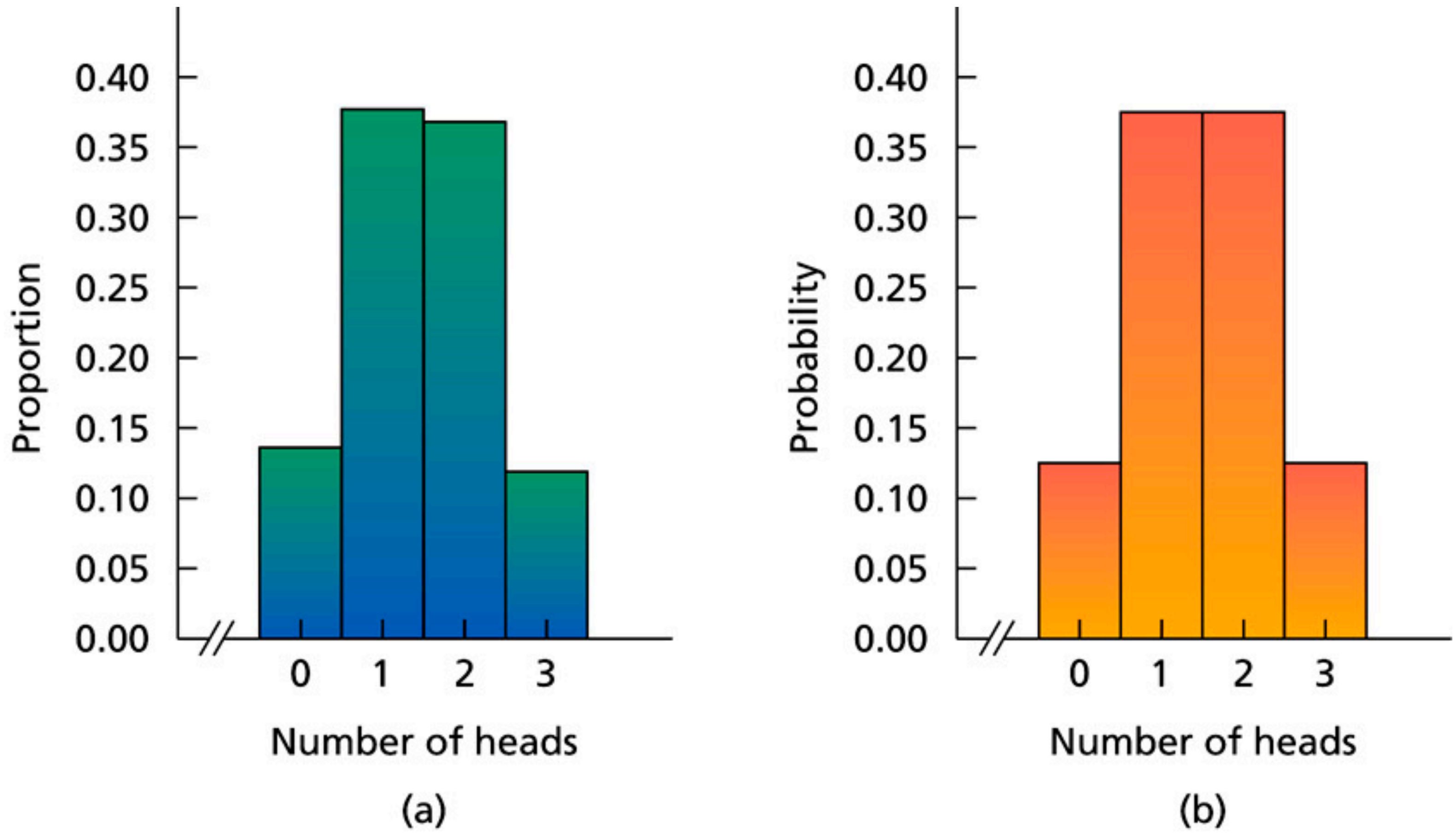
Key Fact 5.2

Interpretation of a Probability Distribution

In a large number of independent observations of a random variable *X*, the proportion of times each possible value occurs will approximate the probability distribution of *X*; or, equivalently, the proportion histogram will approximate the probability histogram for *X*.

Figure 5.2

(a) Histogram of proportions for the numbers of heads obtained in three tosses of a balanced dime for 1000 observations; (b) probability histogram for the number of heads obtained in three tosses of a balanced dime



Section 5.2

The Mean and Standard Deviation of a Discrete Random Variable

Definition 5.4

Mean of a Discrete Random Variable

The **mean of a discrete random variable *X*** is denoted $\boldsymbol{\mu_X}$ or, when no confusion will arise, simply $\boldsymbol{\mu}$. It is defined by

$$\mu = \Sigma x P(X = x).$$

The terms **expected value** and **expectation** are commonly used in place of the term *mean*.†

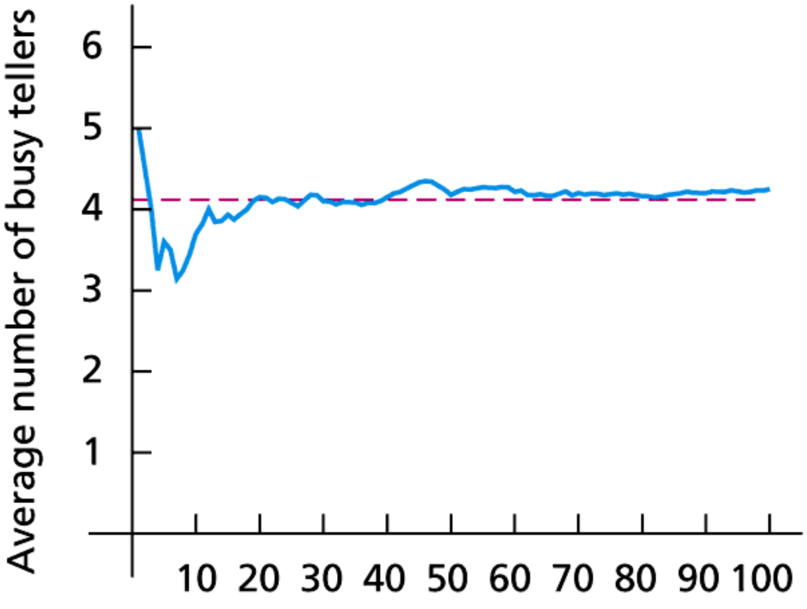
Key Fact 5.3

Interpretation of the Mean of a Random Variable

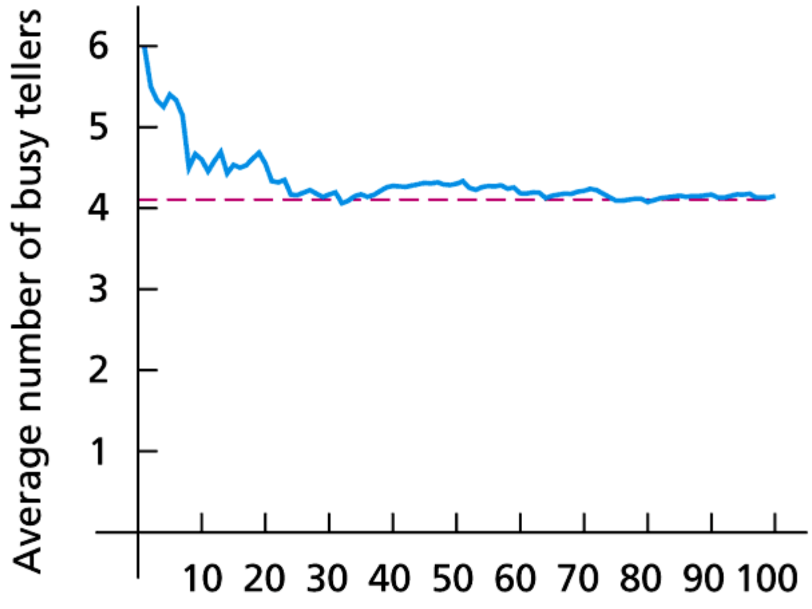
In a large number of independent observations of a random variable X, the average value of those observations will approximately equal the mean, μ, of X. The larger the number of observations, the closer the average tends to be to μ.

Figure 5.3

Graphs showing the average number of busy tellers versus the number of observations for two simulations of 100 observations each



(a)



(b)

Definition 5.5

Standard Deviation of a Discrete Random Variable

The **standard deviation of a discrete random variable *X*** is denoted σ_X or, when no confusion will arise, simply σ. It is defined as

$$\sigma = \sqrt{\Sigma(x-\mu)^2 P(X=x)}.$$

The standard deviation of a discrete random variable can also be obtained from the computing formula

$$\sigma = \sqrt{\Sigma x^2 P(X=x) - \mu^2}.$$

Section 5.3

The Binomial Distribution

Definition 5.6

Factorials

The product of the first k positive integers (counting numbers) is called ***k* factorial** and is denoted ***k*!**. In symbols,

$$k! = k(k-1)\cdots 2 \cdot 1.$$

We also define $0! = 1$.

Definition 5.7

Binomial Coefficients

If n is a positive integer and x is a nonnegative integer less than or equal to n, then the **binomial coefficient** $\binom{n}{x}$ is defined as

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}.^{\dagger}$$

Definition 5.8

Bernoulli Trials

Repeated trials of an experiment are called **Bernoulli trials** if the following three conditions are satisfied:

1. The experiment (each trial) has two possible outcomes, denoted generically ***s***, for **success,** and ***f***, for **failure.**
2. The trials are independent, meaning that the outcome on one trial in no way affects the outcome on other trials.
3. The probability of a success, called the **success probability** and denoted ***p***, remains the same from trial to trial.

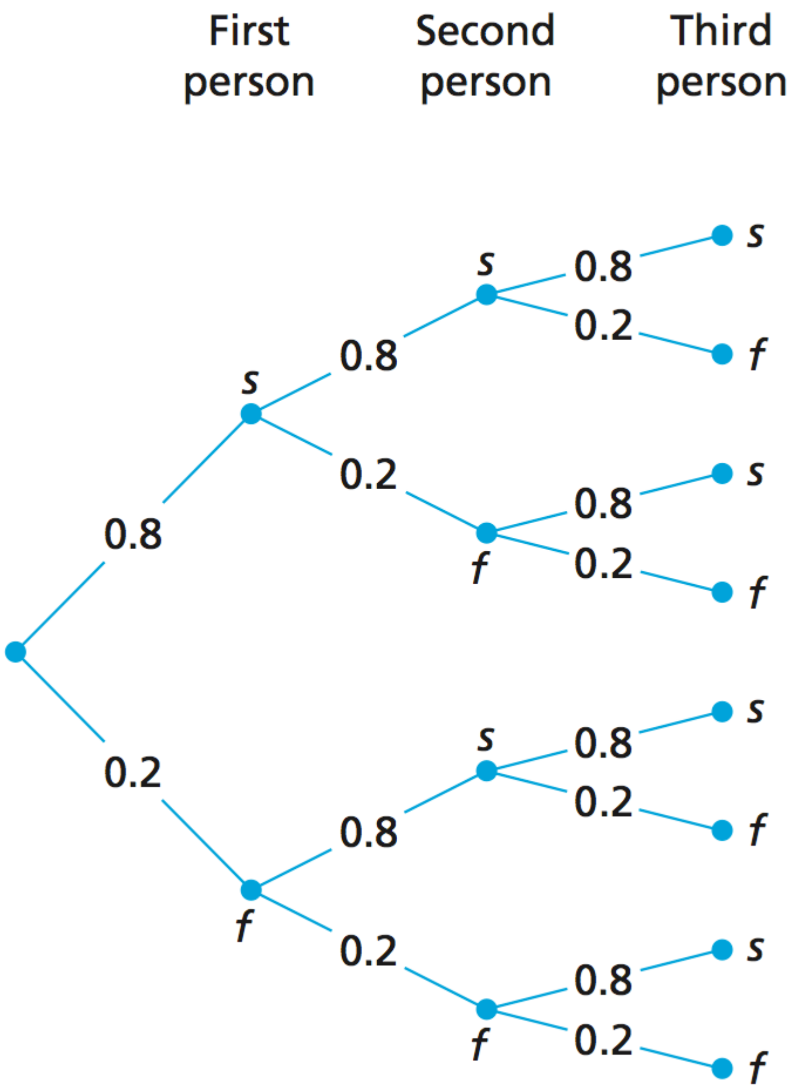
Table 5.14

Outcomes and probabilities for observing whether each of three people is alive at age 65

Outcome	Probability
sss	$(0.8)(0.8)(0.8) = 0.512$
ssf	$(0.8)(0.8)(0.2) = 0.128$
sfs	$(0.8)(0.2)(0.8) = 0.128$
sff	$(0.8)(0.2)(0.2) = 0.032$
fss	$(0.2)(0.8)(0.8) = 0.128$
fsf	$(0.2)(0.8)(0.2) = 0.032$
ffs	$(0.2)(0.2)(0.8) = 0.032$
fff	$(0.2)(0.2)(0.2) = 0.008$

Figure 5.4

Tree diagram corresponding to Table 5.14



Outcome	Probability
sss	$(0.8)(0.8)(0.8) = 0.512$
ssf	$(0.8)(0.8)(0.2) = 0.128$
sfs	$(0.8)(0.2)(0.8) = 0.128$
sff	$(0.8)(0.2)(0.2) = 0.032$
fss	$(0.2)(0.8)(0.8) = 0.128$
fsf	$(0.2)(0.8)(0.2) = 0.032$
ffs	$(0.2)(0.2)(0.8) = 0.032$
fff	$(0.2)(0.2)(0.2) = 0.008$

Key Fact 5.4

Number of Outcomes Containing a Specified Number of Successes

In n Bernoulli trials, the number of outcomes that contain exactly x successes equals the binomial coefficient $\binom{n}{x}$.

Formula 5.1

Binomial Probability Formula

Let X denote the total number of successes in n Bernoulli trials with success probability p. Then the probability distribution of the random variable X is given by

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \qquad x = 0, 1, 2, \ldots, n.$$

The random variable X is called a **binomial random variable** and is said to have the **binomial distribution** with parameters n and p.

Procedure 5.1

To Find a Binomial Probability Formula

Assumptions

1. n trials are to be performed.
2. Two outcomes, success or failure, are possible for each trial.
3. The trials are independent.
4. The success probability, p, remains the same from trial to trial.

Step 1 **Identify a success.**

Step 2 **Determine p, the success probability.**

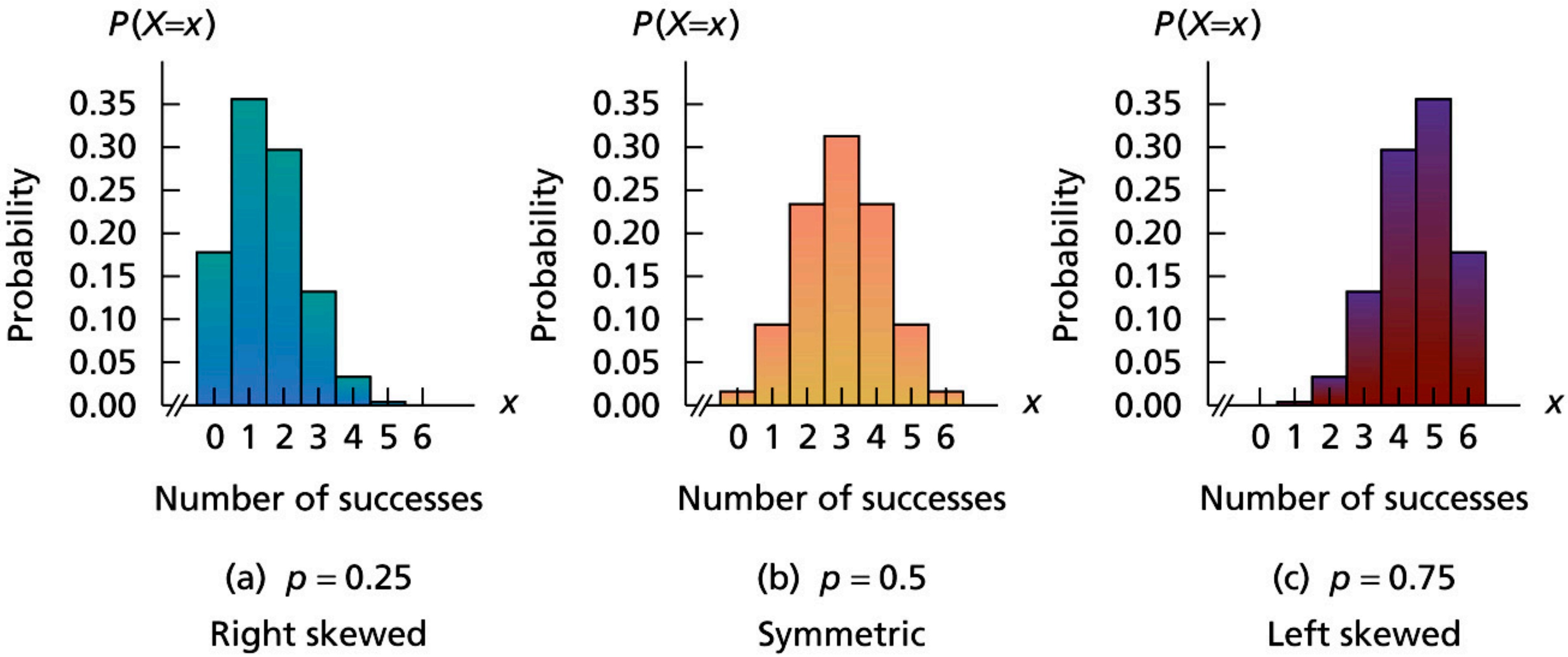
Step 3 **Determine n, the number of trials.**

Step 4 **The binomial probability formula for the number of successes, X, is**

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}.$$

Figure 5.6

Probability histograms for three different binomial distributions with parameter n = 6



Formula 5.2

Mean and Standard Deviation of a Binomial Random Variable

The mean and standard deviation of a binomial random variable with parameters n and p are

$$\mu = np \quad \text{and} \quad \sigma = \sqrt{np(1-p)},$$

respectively.

Key Fact 5.5

Sampling and the Binomial Distribution

Suppose that a simple random sample of size n is taken from a finite population in which the proportion of members that have a specified attribute is p. Then the number of members sampled that have the specified attribute

- has exactly a binomial distribution with parameters n and p if the sampling is done with replacement and
- has approximately a binomial distribution with parameters n and p if the sampling is done without replacement and the sample size does not exceed 5% of the population size.

Section 5.4

The Poisson Distribution

Formula 5.3

Poisson Probability Formula

Probabilities for a random variable X that has a Poisson distribution are given by the formula

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots,$$

where λ is a positive real number and $e \approx 2.718$. (Most calculators have an e key.) The random variable X is called a **Poisson random variable** and is said to have the **Poisson distribution** with parameter λ.

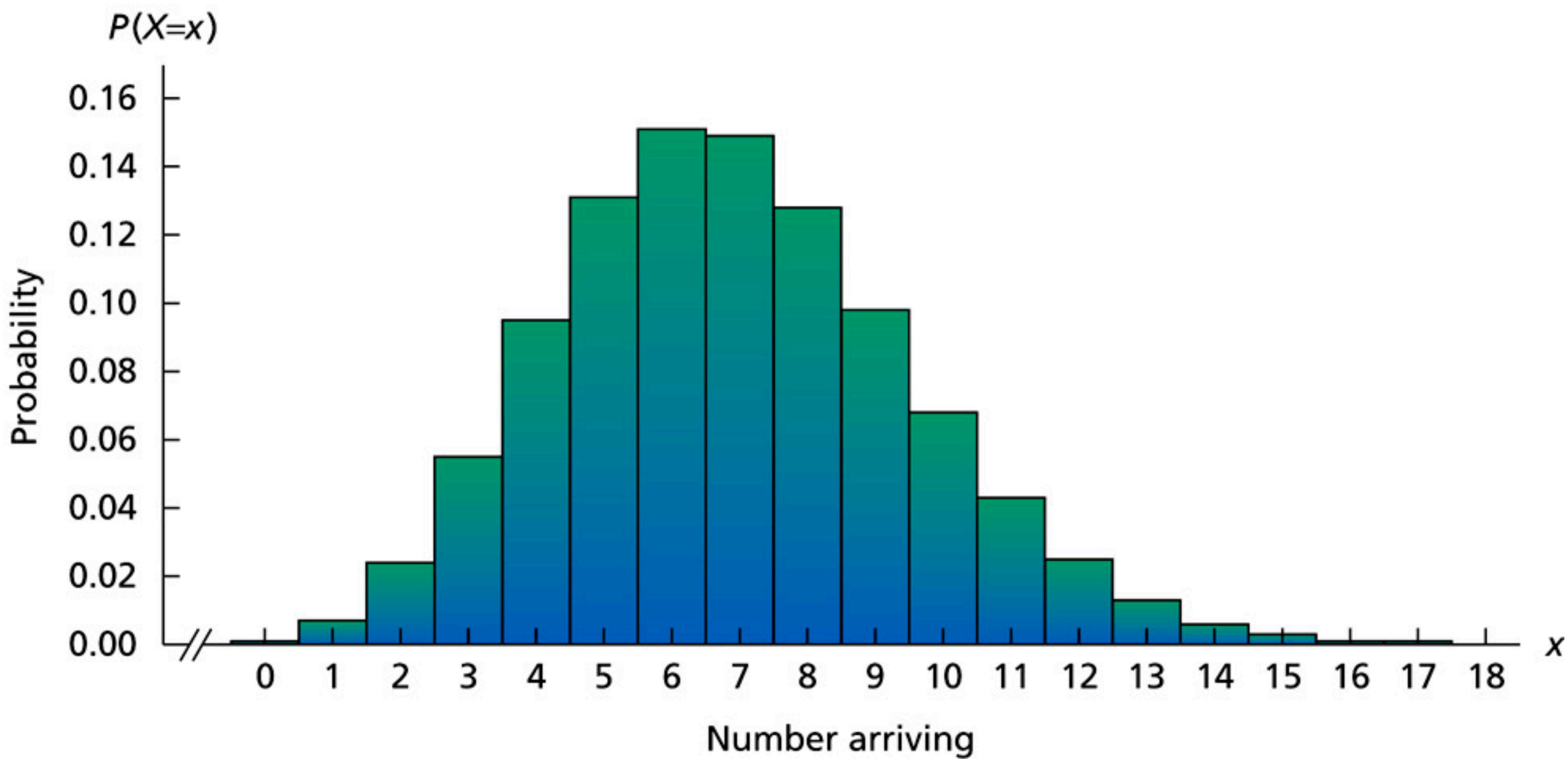
Table 5.16

Partial probability distribution of the random variable X, the number of patients arriving at the emergency room between 6:00 P.M. and 7:00 P.M.

Number arriving x	Probability $P(X = x)$	Number arriving x	Probability $P(X = x)$
0	0.001	10	0.068
1	0.007	11	0.043
2	0.024	12	0.025
3	0.055	13	0.013
4	0.095	14	0.006
5	0.131	15	0.003
6	0.151	16	0.001
7	0.149	17	0.001
8	0.128	18	0.000
9	0.098		

Figure 5.7

Partial probability histogram for the random variable X, the number of patients arriving at the emergency room between 6:00 P.M. and 7:00 P.M.



Formula 5.4

Mean and Standard Deviation of a Poisson Random Variable

The mean and standard deviation of a Poisson random variable with parameter λ are

$$\mu = \lambda \quad \text{and} \quad \sigma = \sqrt{\lambda},$$

respectively.

Procedure 5.2

To Approximate Binomial Probabilities by Using a Poisson Probability Formula

Step 1 Find n, the number of trials, and p, the success probability.

Step 2 Continue only if $n \geq 100$ and $np \leq 10$.

Step 3 Approximate the binomial probabilities by using the Poisson probability formula

$$P(X = x) = e^{-np}\frac{(np)^x}{x!}.$$