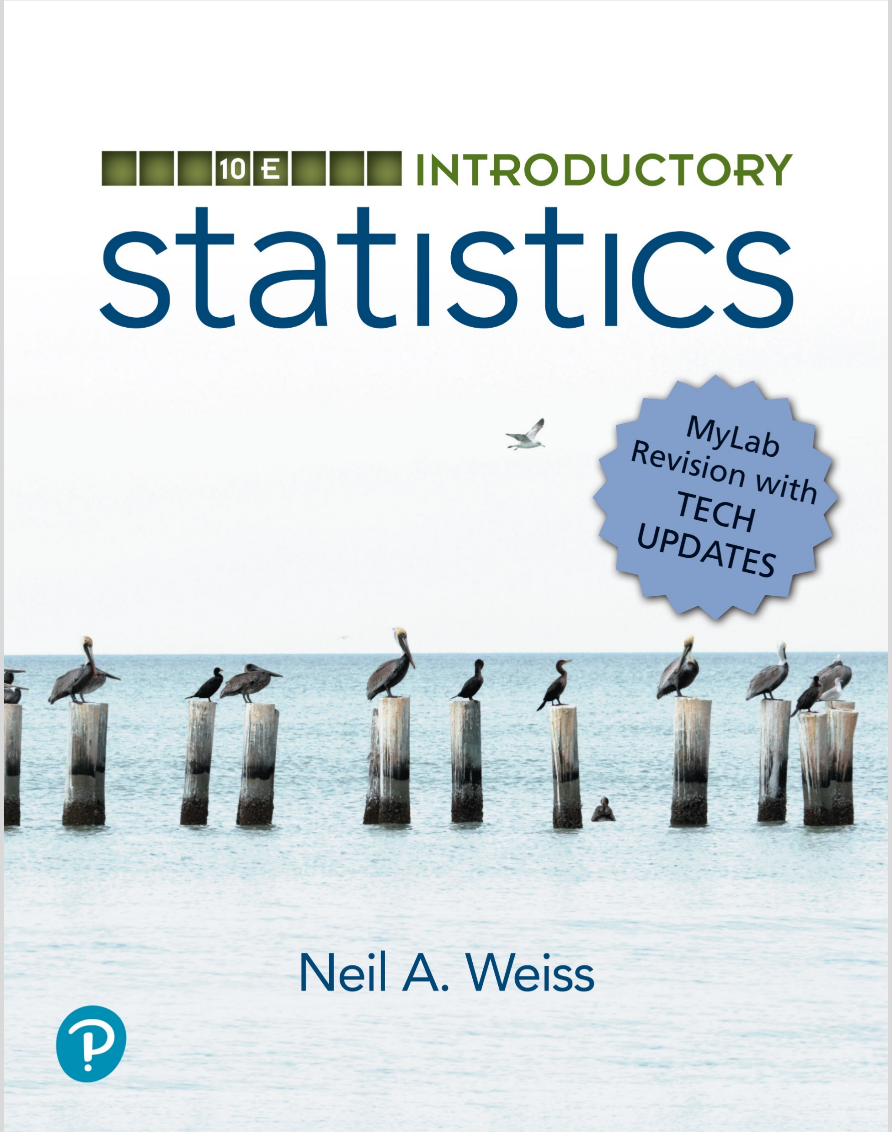
Chapter 16

Analysis of Variance (ANOVA)

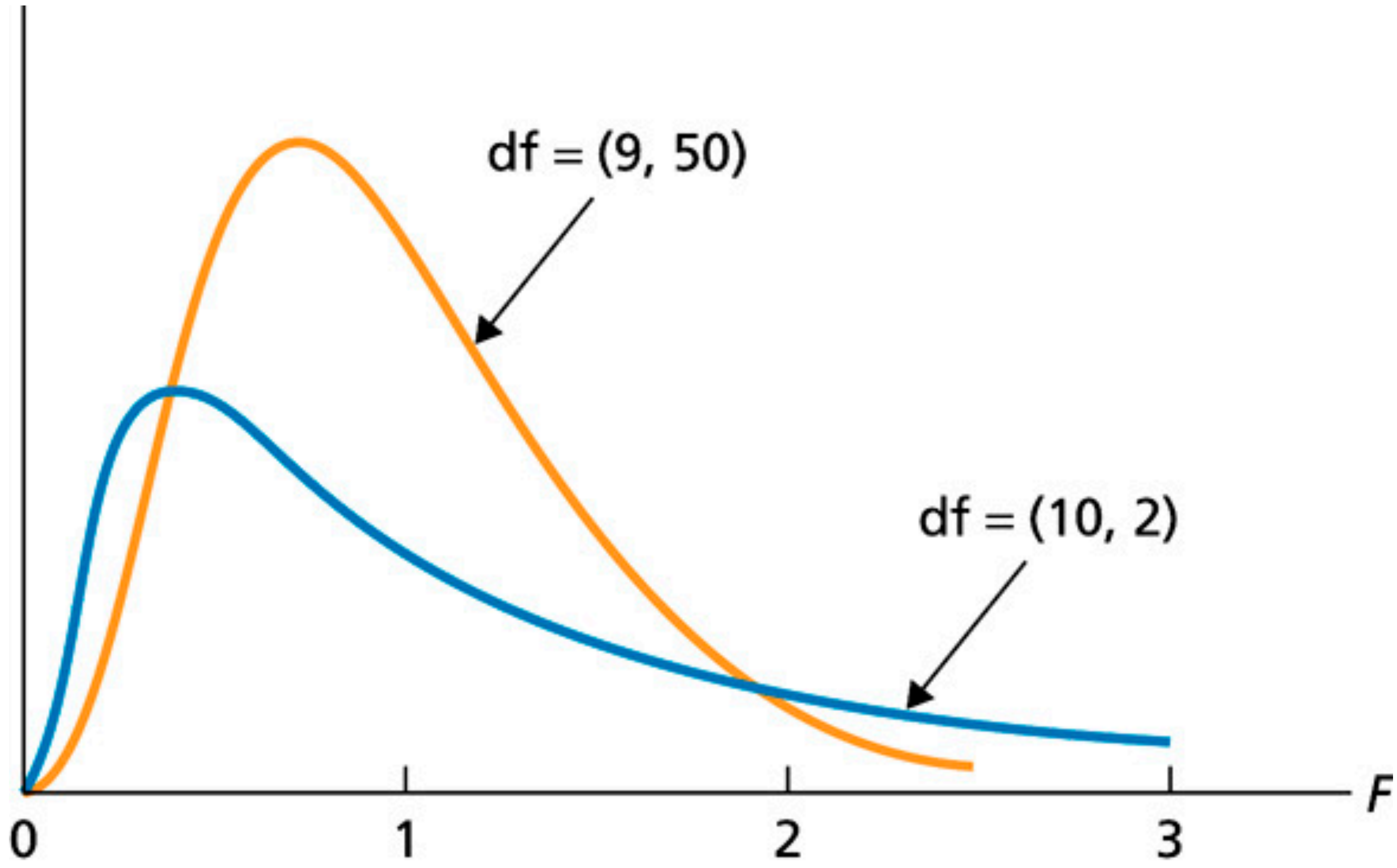


Section 16.1

The *F*-Distribution

Figure 16.1

Two different *F*-curves



Key Fact 16.1

Basic Properties of *F*-Curves

Property 1: The total area under an *F*-curve equals 1.

Property 2: An *F*-curve starts at 0 on the horizontal axis and extends indefinitely to the right, approaching, but never touching, the horizontal axis as it does so.

Property 3: An *F*-curve is right skewed.

Section 16.2

One-Way ANOVA: The Logic

Key Fact 16.2

Assumptions (Conditions) for One-Way ANOVA

1. **Simple random samples:** The samples taken from the populations under consideration are simple random samples.
2. **Independent samples:** The samples taken from the populations under consideration are independent of one another.
3. **Normal populations:** For each population, the variable under consideration is normally distributed.
4. **Equal standard deviations:** The standard deviations of the variable under consideration are the same for all the populations.

Definition 16.1

Mean Squares and *F*-Statistic in One-Way ANOVA

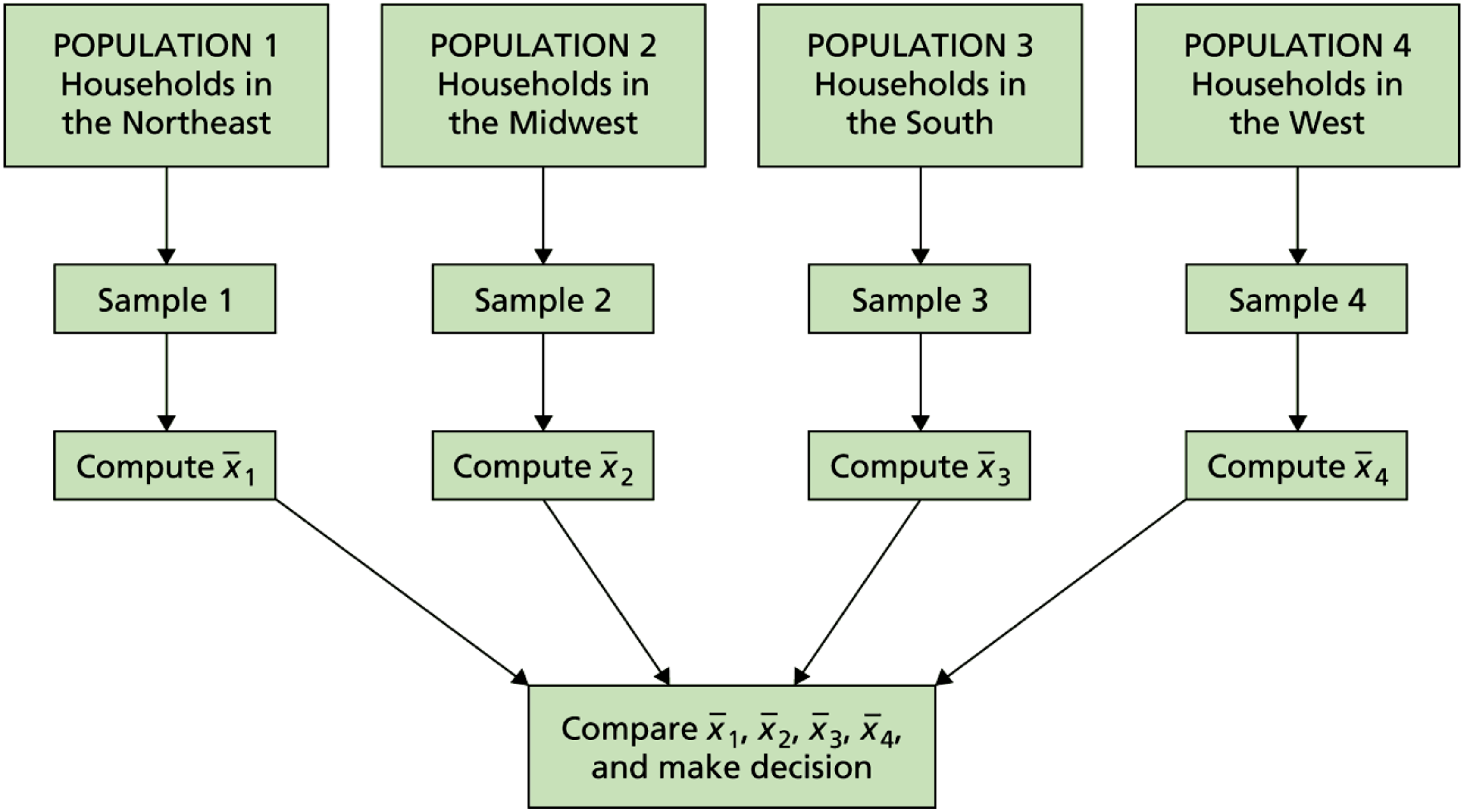
Treatment mean square, *MSTR*: The variation among the sample means: $MSTR = SSTR/(k - 1)$, where $SSTR$ is the treatment sum of squares and k is the number of populations under consideration.

Error mean square, *MSE*: The variation within the samples: $MSE = SSE/(n - k)$, where SSE is the error sum of squares and n is the total number of observations.

***F*-statistic, *F*:** The ratio of the variation among the sample means to the variation within the samples: $F = MSTR/MSE$.

Figure 16.5

Process for comparing four population means



Section 16.3

One-Way ANOVA: The Procedure

Key Fact 16.3

Distribution of the *F*-Statistic for One-Way ANOVA

Suppose that the variable under consideration is normally distributed on each of k populations and that the population standard deviations are equal. Then, for independent samples from the k populations, the variable

$$F = \frac{MSTR}{MSE}$$

has the F-distribution with $\mathrm{df} = (k - 1, n - k)$ if the null hypothesis of equal population means is true. Here, n denotes the total number of observations.

Key Fact 16.4

One-Way ANOVA Identity

The total sum of squares equals the treatment sum of squares plus the error sum of squares: $SST = SSTR + SSE$.

Table 16.6

ANOVA table format for a one-way analysis of variance

Source	df	SS	$MS = SS/\text{df}$	F-statistic
Treatment	$k - 1$	$SSTR$	$MSTR = \dfrac{SSTR}{k-1}$	$F = \dfrac{MSTR}{MSE}$
Error	$n - k$	SSE	$MSE = \dfrac{SSE}{n-k}$	
Total	$n - 1$	SST		

Table 16.7

One-way ANOVA table for the energy consumption data

Source	df	*SS*	*MS* = *SS*/df	*F*-statistic
Treatment	3	105.9	35.3	6.32
Error	16	89.3	5.581	
Total	19	195.2		

Formula 16.1

Sums of Squares in One-Way ANOVA

For a one-way ANOVA of k population means, the defining and computing formulas for the three sums of squares are as follows.

Sum of squares	Defining formula	Computing formula
Total, SST	$\Sigma(x_i - \bar{x})^2$	$\Sigma x_i^2 - (\Sigma x_i)^2/n$
Treatment, $SSTR$	$\Sigma n_j(\bar{x}_j - \bar{x})^2$	$\Sigma(T_j^2/n_j) - (\Sigma x_i)^2/n$
Error, SSE	$\Sigma(n_j - 1)s_j^2$	$SST - SSTR$

In this table, we used the notation

$$n = \text{total number of observations}$$
$$\bar{x} = \text{mean of all } n \text{ observations;}$$

and, for $j = 1, 2, \ldots, k$,

$$n_j = \text{size of sample from Population } j$$
$$\bar{x}_j = \text{mean of sample from Population } j$$
$$s_j^2 = \text{variance of sample from Population } j$$
$$T_j = \text{sum of sample data from Population } j.$$

Note that a summation involving a subscript i is over all n observations; one involving a subscript j is over the k populations.

Procedure 16.1

One-Way ANOVA Test

Purpose To perform a hypothesis test to compare k population means, $\mu_1, \mu_2, \ldots, \mu_k$

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations
4. Equal population standard deviations

Step 1 The null and alternative hypotheses are, respectively,

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k$$
$$H_a: \text{Not all the means are equal.}$$

Step 2 Decide on the significance level, α.

Step 3 Compute the value of the test statistic

$$F = \frac{MSTR}{MSE}$$

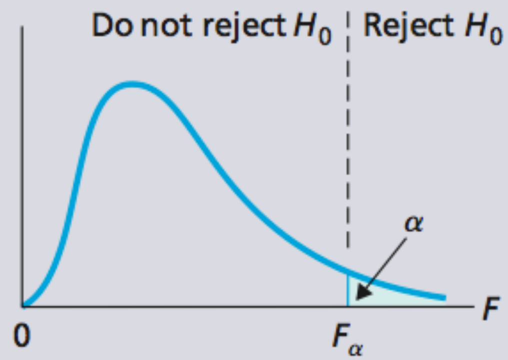
and denote that value F_0. To do so, construct a one-way ANOVA table:

Source	df	SS	$MS = SS/\text{df}$	F-statistic
Treatment	$k - 1$	$SSTR$	$MSTR = \dfrac{SSTR}{k-1}$	$F = \dfrac{MSTR}{MSE}$
Error	$n - k$	SSE	$MSE = \dfrac{SSE}{n-k}$	
Total	$n - 1$	SST		

Procedure 16.1 (cont.)

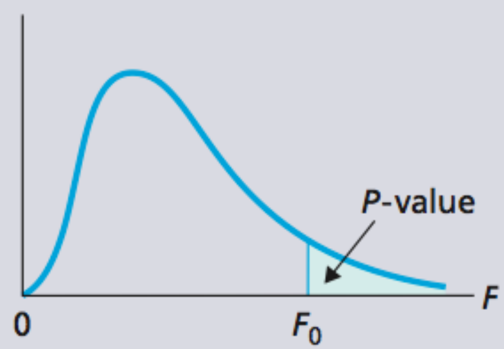
CRITICAL-VALUE APPROACH	OR	P-VALUE APPROACH

Step 4 The critical value is F_α with df $= (k-1, n-k)$. Use Table VIII to find the critical value.



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

Step 4 The F-statistic has df $= (k-1, n-k)$. Use Table VIII to estimate the P-value or obtain it exactly by using technology.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Section 16.4

Multiple Comparisons

Procedure 16.2

Tukey Multiple-Comparison Method

Purpose To determine the relationship among k population means $\mu_1, \mu_2, \ldots, \mu_k$

Assumptions

1. Simple random samples
2. Independent samples
3. Normal populations
4. Equal population standard deviations

Step 1 **Decide on the family confidence level, $1 - \alpha$.**

Step 2 **Find q_α for the q-curve with parameters $\kappa = k$ and $\nu = n - k$, where n is the total number of observations.**

Step 3 **Obtain the endpoints of the confidence interval for $\mu_i - \mu_j$:**

$$(\bar{x}_i - \bar{x}_j) \pm \frac{q_\alpha}{\sqrt{2}} \cdot s\sqrt{(1/n_i) + (1/n_j)},$$

where $s = \sqrt{MSE}$. Do so for all possible pairs of means with $i < j$.

Step 4 **Declare two population means different if the confidence interval for their difference does not contain 0; otherwise, do not declare the two population means different.**

Step 5 **Summarize the results in Step 4 by ranking the sample means from smallest to largest and by connecting with lines those whose population means were not declared different.**

Step 6 **Interpret the results of the multiple comparison.**

Section 16.5

The Kruskal-Wallis Test

Table 16.12

Number of miles driven (1000s) last year for independent samples of cars, buses, and trucks

Cars	Buses	Trucks
19.9	1.8	24.6
15.3	7.2	37.0
2.2	7.2	21.2
6.8	6.5	23.6
34.2	13.3	23.0
8.3	25.4	15.3
12.0		57.1
7.0		14.5
9.5		26.0
1.1		

Table 16.11

Results of ranking the combined data from Table 16.10

Cars	Rank	Buses	Rank	Trucks	Rank
19.9	16	1.8	2	24.6	20
15.3	14.5	7.2	7.5	37.0	24
2.2	3	7.2	7.5	21.2	17
6.8	5	6.5	4	23.6	19
34.2	23	13.3	12	23.0	18
8.3	9	25.4	21	15.3	14.5
12.0	11			57.1	25
7.0	6			14.5	13
9.5	10			26.0	22
1.1	1				
	9.850		9.000		19.167 ← *Mean ranks*

Key Fact 16.5

Distribution of the *K*-Statistic for Kruskal-Wallis Test

Suppose that the *k* distributions (one for each population) of the variable under consideration have the same shape. Then, for independent samples from the *k* populations, the variable

$$K = \frac{SSTR}{SST/(n-1)}$$

has approximately a chi-square distribution with $df = k - 1$ if the null hypothesis of equal population means is true. Here, *n* denotes the total number of observations.

Procedure 16.3

Kruskal–Wallis Test

Purpose To perform a hypothesis test to compare k population means, $\mu_1, \mu_2, \ldots, \mu_k$

Assumptions

1. Simple random samples
2. Independent samples
3. Same-shape populations
4. All sample sizes are 5 or greater

Step 1 **The null and alternative hypotheses are, respectively,**

$$H_0\text{: } \mu_1 = \mu_2 = \cdots = \mu_k$$

$$H_a\text{: Not all the means are equal.}$$

Step 2 **Decide on the significance level, α.**

Step 3 **Compute the value of the test statistic**

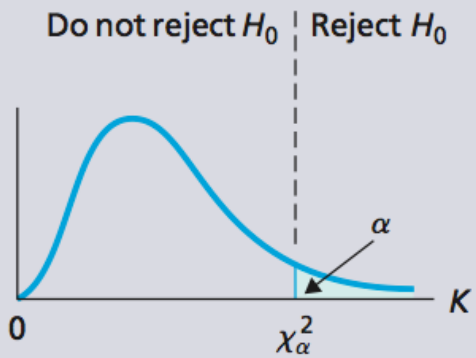
$$K = \frac{12}{n(n+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} - 3(n+1)$$

and denote that value K_0. Here, n is the total number of observations and $R_1, R_2, \ldots, R_k$ denote the sums of the ranks for the sample data from Populations 1, 2, …, k, respectively. To obtain K, first construct a work table to rank the data from all the samples combined.

Procedure 16.3 (cont.)

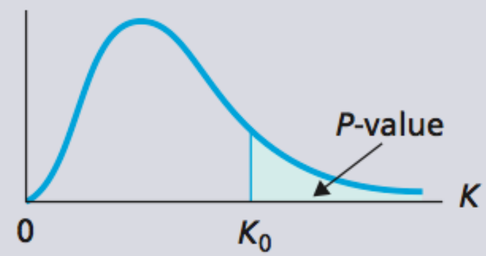
CRITICAL-VALUE APPROACH	OR	P-VALUE APPROACH

Step 4 The critical value is χ^2_α with df $= k - 1$. Use Table VII to find the critical value.



Step 5 If the value of the test statistic falls in the rejection region, reject H_0; otherwise, do not reject H_0.

Step 4 The K-statistic has df $= k - 1$. Use Table VII to estimate the P-value or obtain it exactly by using technology.



Step 5 If $P \leq \alpha$, reject H_0; otherwise, do not reject H_0.

Step 6 Interpret the results of the hypothesis test.

Key Fact 16.6

The Kruskal-Wallis Test Versus the One-Way ANOVA Test

Suppose that the distributions of a variable of several populations have the same shape and that you want to compare the population means, using independent simple random samples. When deciding between the one-way ANOVA test and the Kruskal-Wallis test, follow these guidelines: If you are reasonably sure that the distributions are normal, use the one-way ANOVA test; otherwise, use the Kruskal-Wallis test.