A Lecture on Topological Operators

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What is this

This is a lecture note prepared for two sets of “intensive lectures”:[1]

- at Tohoku University, Oct. 11-13, 2023, and
- at Yukawa Insititute for Theoretical Physics, Kyoto University, Nov. 29-1, 2023.

In this lecture I will try to explain the constructions of topological defects corresponding to generalized symmetries. Due to lack of time and (more significantly) my understanding, the lecture will focus on bosonic systems, and the generalization to fermionic systems is left for the readers/audiences.

Prerequisite

- Basic knowledge about scalar field theory and (abelian) gauge theory in path-integral formalism, and
- Knowledge about renoarmalization group (RG) flows to understand motivations.

What is contained and what is not

Other Lectures/Reviews

Recently there has been a surge of lecture notes/ review articles on generalized symmetries. The ones I have noticed are [**McGreevy:2022oyu**, **Schafer-Nameki:2023jdn**, **Gomes:2023ahz**, **Bhardwaj:2023kri**, **Luo:2023ive**, **Shao:2023gho**]. Because this lecture will focus on the fundamental aspects of the topic and will not connect very well with the existent literature (so sorry about that), readers/audiences are strongly encouraged to refer to at least one of them, or something similar.

Also, about conventional symmetries and their anomalies, there are nice old lectures. The one I would particularly recommend is [**TachikawaTasi**].

[1] In Japan, an “intensive lecture” is a format of a lecture course where a lecturer (usually from another university) gives lectures in consecutive days filling 7-9 slots in usually 3 days.

1 Introduction

1.1 Symmetry

Symmetry plays a fundamental role in theoretical physics. In this lecture we consider them in *quantum field theories* (QFTs). The fundamental fact about symmetry in QFTs is that is it preserved along the renormalization group flow. More precisely, when an ultraviolet (UV) theory $\mathcal{T}_{\mathrm{UV}}$ flows into an infrared theory $\mathcal{T}_{\mathrm{IR}}$, there is a canonical homomorphism f_{RG} from the UV symmetry group G_{UV} to the IR symmetry group G_{IR}:[1]

! RG flow homomorphism from UV symmetry to IR symmetry

$$f_{\mathrm{RG}} : G_{\mathrm{UV}} \to G_{\mathrm{IR}}. \tag{1.1}$$

Given this relation, there are two ways of applying symmetry in QFT:

- UV to IR: Given a microscopic model (e.g. a model of elementary particles or electrons in a matter), constrain/guess what happens in the macroscopic scale.
- IR to UV: Given some macroscopic phenomena, constrain/guess what could be the microscopic origin of it (e.g. guessing QCD lagrangian from hadron spectrum).

i Terminology

In this lecture, "symmetry" means a *global* symmetry. Here *global* means that the symmetry operation acts on the entire space. In addition, in most contents we exclude symmetries acting on the spacetime out of consideration for simplicity.

[1]If the UV theory is a fixed point, G_{UV} should be understood as the one preserved by the perturbation triggering the RG flow. If the RG flow is to a lower nonzero energy, and if one retains all the (even very massive) degrees of freedom in the description of $\mathcal{T}_{\mathrm{IR}}$, the map f_{RG} is an isomorphism. However, typically one integrates out massive dofs in the description of $\mathcal{T}_{\mathrm{RG}}$, in which case some symmetry can decouple and thus f_{RG} can be non-surjective. Also, if one also drops some higher-order interaction terms, or runs the flow to the zero energy, there can be an *emergent* symmetry, in which case f_{RG} can be non-injective.

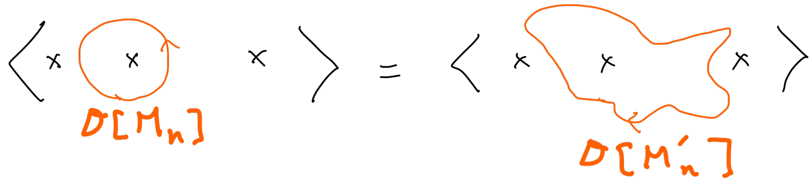

Figure 1.1: Topological operator.

1.2 Locality

The second important keyword in this lecture is **locality**. By the word quantum field theory, in most cases we implicitly mean *local* quantum field theory. This can roughly be explain that the action is written as the integration of Lagrangian density, which is a local functional of fields over the spacetime.

Locality of QFT has a concequence with respect to symmetry: that is, in most situations we only consider symmetries that *preserves locality*. In terms of fields, this means that the symmetry transformation is local:

$$\phi(x) \mapsto F(\phi(x)), \tag{1.2}$$

where $F(\phi(x))$ is a function depends on the *local* value a field (or a collection of fields and its derivatives) at the point x. If the symmetry involves a spacetime transformation, the point x in the right hand side should be replaced by the image of x under the symmetry. This preservation of locality of a symmetry is the reason for the symmetry relation in Equation **??**. This will be made clear in the lecture.

However, not all the (locality-preserving) symmetry in QFT takes the form of Equation **??**. There are types of symmetry called "topological symmetry", which arises from topologically-nontrivial field configuration. Examples are the winding symmetry in 1+1d compact boson, and the monopole symmetries in 2+1d abelian gauge theories. In many occurrences a topological symmetry is mapped to a symmetry of type of Equation **??** under a duality, and thus it should also be considered as being locality-preserving.

From the modern perspective, the universal characterization of locality-preserving symmetries is its correspondence to **topological operators**. A topological operator $\mathcal{D}[M_n]$ in a QFT is an extended operator defined on a n-dimensional submanifold of the spacetime and the correlators containing it should be invariant under the smooth deformation of the supporting manifold M_n (See Figure **??**).

The first aim of the lecture is to understand the correspondence, that is

! Symmetry/Topological Operator Correspondence

$$\begin{aligned}&\text{(Conventional) locality-preserving symmetry}\\ \iff\ &\text{invertible topological operator of codimension 1.}\end{aligned} \tag{1.3}$$

We will explicitly study this correspondence in the case of scalar field theory in Chapter **??** and in the case of abelian gauge theory in Chapter **??**. The case of fermion is very interesting and crucial, but it will be remained to be worked out by audiences/readers.

i Terminology

Again, we are *not* talking about gauge redundancy, which is sometimes called local symmetry. Global symmetries one encounters in a QFT textbook are all locality preserving.

i Terminology

Here is another unfortunate conflict of terminology. In the literature (outside generalized symmetry literature), a "topological defect" refers to a dynamical object, or its trajectory viewed as an operator in the IR theory, charg**ed** under a topological (higher) symmetry. As an operator in the IR theory, it is *not* a priori guaranteed to be topological in the sense of Figure **??** . On the other hand, in the generalized symmetry literature, "topological defect (operator)" often means an extended operator that is itself topological. In this lecture, in order to ease the confusion, we use the term "topological operator".

1.3 Generalized Symmetry

The correspondence in Equation **??** is the core in the notion of **generalized (global) symmetry**, coined by [**Gaiotto:2014kfa**] [2]. That is, the notion of symmetry can be generalized by relaxing the adjective in the right hand side of Equation **??**. Therefore, we *define* generalized symmetry by the following correspondence generalizing Equation **??**:

! Generalized Symmetry/Topological Operator Correspondence

$$\begin{aligned}&\text{Generalized symmetry (in a ``usual'' QFT)}\\ \overset{\text{def}}{\iff}\ &\text{General topological operator.}\end{aligned} \tag{1.4}$$

[2] The global higher-form symmetry itself had appeared and investigated in the literature, e.g. [**Kapustin:2013uxa**, **Barkeshli:2014cna**], and its gauged version was essentially known from [**KalbRamond**].

More specifically, a generalized symmetry corresponding to a operator of codimension $p+1$ is called **p-form symmetry**, and a generalized symmetry corresponding to a operator without its inverse is called **non-invertible symmetry** (among other names like category symmetry and topological symmetry).

In an "unsual" QFT, we can even relax the topological-ness of the operator in the right hand side of Equation **??**, resulting in what is called **subsystem** symmetry. We will briefly make a comment on it.

The sub-classes of generalized symmetry is summarized in the table below:

	p-form	non-invertible	subsystem
codimension	$p+1$		
Invertible?		No	
Topological?			Partically

The sub-classes are not mutually exclusive, so, in principle, there can be a 2-form non-invertible subsystem symmetry.

2 Topological Operators in Scalar Field Theory

2.1 Scalar Transformations

2.2 Widing Symmetry in 1+1d Compact Boson

2.3 T-duality

3 Vector

4 Summary

In summary, this book has no content whatsoever.

References