A Lecture on Topological Operators

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What is this

vol This is a lecture note prepared for two sets of "intensive lectures":[1]

- at Tohoku University, Oct. 11-13, 2023, and
- at Yukawa Insititute for Theoretical Physics, Kyoto University, Nov. 29-1, 2023.

In this lecture I will try to explain the constructions of topological defects corresponding to generalized symmetries. Due to lack of time and (more significantly) my understanding, the lecture will focus on bosonic systems, and the generalization to fermionic systems is left for the readers/audiences.

> Warning
>
> This note is **under construction**, and there are many missed equations, figures, explanations, sections, and *references*.

Prerequisite

- Basic knowledge about scalar field theory and (abelian) gauge theory in path-integral formalism, and
- Knowledge about renormalization group (RG) flows to understand motivations.
- Knowledge about differential form and Stokes's theorem in terms of it.

What is contained and what is not

Other Lectures/Reviews

Recently there has been a surge of lecture notes/ review articles on generalized symmetries. The ones I have noticed are [1–6]. Because this lecture will focus on the fundamental aspects of the topic and will not connect very well with the existent literature (so sorry about that),

[1]In Japan, an "intensive lecture" is a format of a lecture course where a lecturer (usually from another university) gives lectures in consecutive days filling 7-9 slots in usually 3 days.

readers/audiences are strongly encouraged to refer to at least one of them, or something similar.

Also, about conventional symmetries and their anomalies, there are nice old lectures. The one I would particularly recommend is [7].

1 Introduction

1.1 Symmetry

Symmetry plays a fundamental role in theoretical physics. In this lecture we consider them in *quantum field theories* (QFTs). The fundamental fact about symmetry in QFTs is that is it preserved along the renormalization group flow. More precisely, when an ultraviolet (UV) theory $\mathcal{T}_{\mathrm{UV}}$ flows into an infrared theory $\mathcal{T}_{\mathrm{IR}}$, there is a canonical homomorphism f_{RG} from the UV symmetry group G_{UV} to the IR symmetry group G_{IR}:[1]

! RG flow homomorphism from UV symmetry to IR symmetry

$$f_{\mathrm{RG}} : G_{\mathrm{UV}} \to G_{\mathrm{IR}}. \tag{1.1}$$

Given this relation, there are two ways of applying symmetry in QFT:

- UV to IR: Given a microscopic model (e.g. a model of elementary particles or electrons in a matter), constrain/guess what happens in the macroscopic scale.
- IR to UV: Given some macroscopic phenomena, constrain/guess what could be the microscopic origin of it (e.g. guessing QCD Lagrangian from hadron spectrum).

i Terminology (global symmetry)

In this lecture, "symmetry" means a *global* symmetry. Here *global* means that the symmetry operation acts on the entire space. In addition, in most contents we exclude symmetries acting on the spacetime out of consideration for simplicity.

[1] If the UV theory is a fixed point, G_{UV} should be understood as the one preserved by the perturbation triggering the RG flow. If the RG flow is to a lower nonzero energy, and if one retains all the (even very massive) degrees of freedom in the description of $\mathcal{T}_{\mathrm{IR}}$, the map f_{RG} is an isomorphism. However, typically one integrates out massive dofs in the description of $\mathcal{T}_{\mathrm{RG}}$, in which case some symmetry can decouple and thus f_{RG} can be non-surjective. Also, if one also drops some higher-order interaction terms, or runs the flow to the zero energy, there can be an *emergent* symmetry, in which case f_{RG} can be non-injective.

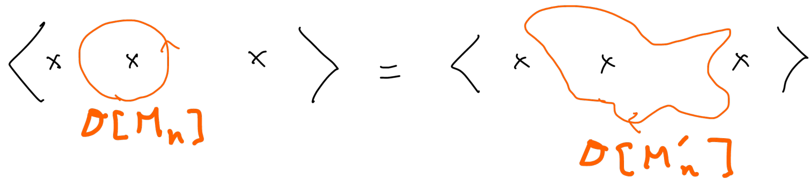

Figure 1.1: Topological operator.

1.2 Locality

The second important keyword in this lecture is **locality**. By the word quantum field theory, in most cases we implicitly mean *local* quantum field theory. This can roughly be explained by that the action is written as the integration of Lagrangian density, which is a local functional of fields over the spacetime.

Locality of QFT has a consequence with respect to symmetry: that is, in most situations we only consider symmetries that *preserves locality*. In terms of fields, this means that the symmetry transformation is local:

$$\phi(x) \mapsto F(\phi(x)), \tag{1.2}$$

where $F(\phi(x))$ is a function depends on the *local* value a field (or a collection of fields and its derivatives) at the point x. If the symmetry involves a spacetime transformation, the point x in the right-hand side should be replaced by the image of x under the symmetry. This preservation of locality of a symmetry is the reason for the symmetry relation in Equation 1.1. This will be made clear in the lecture.

However, not all the (locality-preserving) symmetry in QFT takes the form of Equation 1.2, which is called *classical symmetry*. There are types of symmetry called *topological symmetry* or *quantum symmetry*, which arises from topologically-nontrivial field configuration. Examples are the winding symmetry in 1+1d compact boson, and the monopole symmetries in 2+1d abelian gauge theories. In many occurrences a topological/quantum symmetry is mapped to a symmetry of type of Equation 1.2 under a duality, and thus it should also be considered as being locality-preserving.

From the modern perspective, the universal characterization of locality-preserving symmetries is its correspondence to **topological operators**. A topological operator $\mathcal{D}[W_n]$ in a QFT is an extended operator defined on a n-dimensional submanifold of the spacetime and the correlators containing it should be invariant under the smooth deformation of the supporting manifold W_n (See Figure 1.1).

The first aim of the lecture is to understand the correspondence, that is

Symmetry/Topological Operator Correspondence

$$\begin{aligned}&\text{(Conventional) locality-preserving symmetry}\\ \Longleftrightarrow\ &\text{invertible topological operator of codimension 1.}\end{aligned} \tag{1.3}$$

In this correspondence, the topological operator shoould be regarded as the generalization of the **Noether charge** into possibly discrete symmetry. To be precise, we regard this correspondence as the right-hand side *defining* the left-hand side. We will explicitly check this correspondence/definition reproduces the known symmetries in the case of a classical symmetry in a scalar field theory in Chapter 2, and in the case of abelian gauge theory in Chapter 4. The case of fermion is very interesting and crucial, but it will be remained to be worked out by audiences/readers.

Terminology (locality-preserving)

Again, we are *not* talking about gauge redundancy, which is sometimes called local symmetry. Global symmetries one encounters in a QFT textbook are all locality-preserving.

Terminology (topological defect)

Here is another unfortunate conflict of terminology. In the literature (outside generalized symmetry literature), a "topological defect" refers to a dynamical object, or its trajectory viewed as an operator in the IR theory, charg**ed** under a topological (higher) symmetry. As an operator in the IR theory, it is *not* a priori guaranteed to be topological in the sense of Figure 1.1 . On the other hand, in the generalized symmetry literature, "topological defect (operator)" often means an extended operator that is itself topological. In this lecture, in order to ease the confusion, we use the term "topological operator".

1.3 Generalized Symmetry

The correspondence in Equation 1.3 is the core in the notion of **generalized (global) symmetry**, coined by [8] [2]. That is, the notion of symmetry can be generalized by relaxing the adjective in the right-hand side of Equation 1.3. Therefore, we *define* generalized symmetry by the following correspondence generalizing Equation 1.3:

[2] The global higher-form symmetry itself had appeared and investigated in the literature, e.g. [9, 10], and its gauged version was essentially known from [11].

❗ Generalized Symmetry/Topological Operator Correspondence

$$\begin{aligned} &\text{Generalized symmetry (in a “usual” QFT)} \\ &\overset{\text{def}}{\Longleftrightarrow}\ \text{General topological operator.} \end{aligned} \tag{1.4}$$

More specifically, a generalized symmetry corresponding to an operator of codimension $p+1$ is called p**-form symmetry**, and a generalized symmetry corresponding to an operator without its inverse is called **non-invertible symmetry** (among other names like category symmetry and topological symmetry).

In an “unusual” QFT, we can even relax the topological-ness of the operator in the right-hand side of Equation 1.4, resulting in what is called **subsystem** symmetry. We will briefly make a comment on it in Section 2.3.

The subclasses of generalized symmetry is summarized in the table below:

Table 1.1: Subclasses of generalized symmetry and defining properties of their corresponding topological operators.

	p-form	non-invertible	subsystem
codimension	$p+1$		
Invertible?		No	
Topological?			Partically

The subclasses are not mutually exclusive, so, in principle, there can be a 2-form non-invertible subsystem symmetry.

1.4 Contents of the Lecture

FIXME

2 Topological Operators for Classical Symmetry

In this section we take the classical symmetry in scalar field theory as an example to study topological operators.

2.1 Set Up

To be concrete, let us consider the complex scalar field theory whose Lagrangian (density) on a spacetime of dimension D is given by

$$\begin{aligned}\mathcal{L}(\phi) &= -\left(\frac{1}{2}\partial_\mu\phi(x)^*\partial^\nu\phi(x) + V(\phi(x))\right)\mathrm{vol}\\ &= \frac{1}{2}\,d\phi\wedge * \,d\phi - V(\phi(x))\,\mathrm{vol},\end{aligned}$$

where $*$ is the Hodge star, $\mathrm{vol} = *1 = \prod_{i=1}^{D} dx_i$ is the volume form for the flat space, and $V(\phi)$ is the potential. The action is the integral over the spacetime M (without boundary):

$$S[\phi] = \int_M \mathcal{L}(\phi).$$

We consider a symmetry transformation of the scalar field

$$\phi(x) \mapsto \phi^g(x) \tag{2.1}$$

parametrized by a group element g (constant over M) that leaves the action invariant:

$$S[\phi] = S[\phi^g]. \tag{2.2}$$

This means that the Lagrangian is invariant up to a total derivative:

$$\mathcal{L}(\phi^g) = \mathcal{L}(\phi) + ds(\phi, g) \tag{2.3}$$

where $s(\phi, g)$ is a $(D-1)$-form on M depending on the constant g and the field ϕ. We set $s(\phi, g = \mathrm{id}) = 0$, where id is the unit of the symmetry group. Then, by $\phi = (\phi^g)^{g^{-1}}$, we can set that $s(\phi, g) = -s(\phi, g^{-1})$.

For example, the usual U(1) rotation corresponds to the transformation

$$\phi^g(x) = g\,\phi(x),$$

where $g = e^{\mathrm{i}\,\alpha}$ is a U(1) phase. The potential $V(\phi)$ might partially break the U(1) rotation into its subgroup $\mathbb{Z}_k$, e.g. $V(\phi) \propto \phi^k + (\phi^*)^k$. In such a case the parameter g takes *discrete* values: $g = e^{\mathrm{i}\,\frac{2\pi i}{k}}$, $i = 0 \cdots k-1$.

In addition, when $V(\phi) = 0$, the action $S[\phi]$ also admit the shift symmetry[1]

$$\phi^\alpha(x) = \phi(x) + \alpha.$$

In this section, we would like to construct the **topological operator** corresponding to these *classical* symmetry.

> **Note**
>
> The construction will apply to other types of scalar field theory, e.g. real and/or multiple scalar fieldsas long as the kinetic term is standard enough (more on this in Section 2.3). For non-linear sigma model with topologically non-trivial target, the Lagrange multiplier λ below should take values in the correct set. We will consider the case with S^1 target (compact boson) in Chapter 3. Also, the spacetime manifold M and the metric on it do not have to be flat. The signature of the metric is also insignificant in this lecture, although we use the Euclidean notation.

> **Note**
>
> In this lecture we directly construct the topological operators corresponding to the *finite* transformation Equation 2.1, rather than the conventional approach considerting infinitesimal transformation. This will enable us to explicitly talk about *finite* symmetries (and their anomalies) in terms of topological operators, and also motivate us to consider generalized symmetries.

2.2 Construction of Topological Operator

As a basic example of Equation 1.3, we would like to construct the topological operator $U_\alpha[W]$ corresponding to the transformation Equation 2.1. The topological operator $U_\alpha[W]$,defined with respect to a codimension-1 submanifold W of the spacetime M, should satisfy the following properties:

[1] If we use the form of Lagrangian $\mathcal{L}' = -\frac{1}{2}\phi\, d * d\phi$, this also gives an example where the total derivative in Equation 2.3 is nonzero: $s = -\frac{1}{2}\alpha * d\phi$.

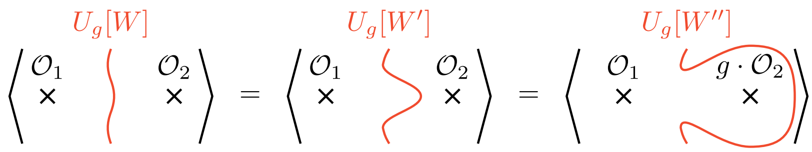

Figure 2.1: The topological operator $U_g[W]$ should be invariant under a continuous deformation and also implement the symmetry action.

❗ Properties of Symmetry Topological Operator

1. Topological: $U_g[W] = U_g[W']$ is W can be continuously defomed into W' without crossing other operators.
2. Symmetry action: when a deformation from W to W'' crosses an local operator $\mathcal{O}$, it gets the symmetry action specified by g, resulting in another operator $\mathcal{O}^g$.
3. Noether: when the symmetry group is continuous, we can take the group element to be the infinitsimal deformation of id: $g = \mathrm{id} + \alpha + \mathcal{O}(\alpha^2)$. Then, the operator $U_g[W]$ is approximated by the Noether charge

$$U_{1+\alpha+\mathcal{O}(\alpha^2)} = 1 + \alpha \int_W * j + \mathcal{O}(\alpha^2), \tag{2.4}$$

where $j = j_\mu \, dx^\mu$ is the Nother current one-form **FIXME:sign is uncertain..**

$$* j = \left. \frac{\delta \mathcal{L}(\phi^{1+\alpha(x)})}{\delta d\alpha} \right|_{\alpha=0} + \left. \frac{\partial s(\phi, 1+\alpha)}{\partial \alpha} \right|_{\alpha=0} .$$

Note that when W is a time-slice $W = \{t = 0\}$,

$$\int_W * j = \int_{\{t=0\}} j^0 \, d^{D-1} x$$

is exactly the Noether charge written in any QFT textbook.

The properties 1. and 2. are summarized in Figure 2.1.

The idea of the construction is "*cutting-and-gluing-with-twist*". That is, we first divide the spacetime M into two parts: $M_D = M_L \cup_W M_R$ with shared boundary W (see Figure 2.2. We take the orientation of W so that $\partial M_L = -\partial M_R = W$.). We also separate the scalar field ϕ into two sets of fields: $\phi_L(x)$ for $x \in M_L$ and $\phi_R(x)$ for $x \in M_R$. Then, we glue the two regions and fields on those back together, with the twisted identification:

$$\phi_L|_W = \phi_R^{g^{-1}}|_W.$$

In path-integral, this construction can be implemented as follows:

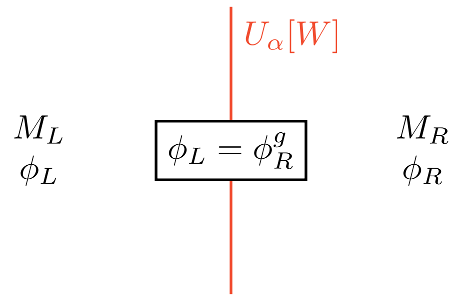

Figure 2.2: The cutting and twisted gluing, implementing the topological operator $U_g[W]$.

❗ Symmetry Topological Operator

$$\begin{aligned}\langle U_g[W]\cdots\rangle = \int \mathcal{D}^{M_L}\phi_L\,\mathcal{D}^{M_R}\phi_R\,\mathcal{D}^W\lambda\cdots \\ \times\exp\left(-S_L[\phi_L] - S_R[\phi_R] - G_W[\lambda,\phi_L,\phi_R,g]\right)\end{aligned} \tag{2.5}$$

Here, $\mathcal{D}^X$ denotes the measure for path-integral for a field defined on a submanifold X of the spacetime M, and $S_{L,R} = \int_{M_{L,R}} \mathcal{L}(\phi_{L,R})$ are actions on the submanifold $M_{L,R}$, and "$\cdots$" represents additional insertions of operators. In addition, the "gluing" action G_W on the submanifold W is

$$G_W[\lambda,\phi_L,\phi_R,g] = -\mathrm{i}\int_W \lambda(\phi_L - \phi_R^{g^{-1}})\,\mathrm{vol}_W + \int_W s(\phi_M, g). \tag{2.6}$$

The heart of the above expression is that integrating the Lagrange multiplier λ out gives the "delta functional":

$$\int \mathcal{D}^W\lambda\exp\left(\mathrm{i}\int_W \lambda(\phi_L - \phi_R^{g^{-1}})\,\mathrm{vol}_W\right) = \prod_{x\in W}\delta(\phi_L(x) - \phi_R^{g^{-1}}(x)), \tag{2.7}$$

which should implement Figure 2.2. Before studying the operator $U_g[W]$, we should study the *trivial* case where the symmetry transformation g is the identity map $g = \mathrm{id}$.

2.3 Identity Wall

When $g = \mathrm{id}$, the operator $U_{g=\mathrm{id}}[W]$ should be *trivial*. That is, we have

$$\langle \mathrm{id}[W]\cdots\rangle = \langle\cdots\rangle. \tag{2.8}$$

We call the codimension-1 operator $\mathrm{id}[W]$ with this property **identity wall**. It can also be called **transparent wall** or like that. Expanding Equation 2.8, the following equation should hold:

$$\int \mathcal{D}^M\phi \exp(-S)\cdots = \int \mathcal{D}^{M_L}\phi_L\, \mathcal{D}^{M_R}\phi_R\, \mathcal{D}^W\lambda$$
$$\times \exp\left(-S_L - S_R + \mathrm{i}\int_W \lambda(\phi_L - \phi_R)\,\mathrm{vol}_W\right)\cdots. \tag{2.9}$$

The point is that the difference between giving a field ϕ and giving a pair of fields (ϕ_L, ϕ_R) is that the latter is not constrained to be continuous across W. On the right-hand side it is rather enforced by integrating out λ because of Equation 2.7.

Continuity of fields in "exotic" QFTs and subsystem symmetry

Here we assumed that the path integral $\int \mathcal{D}^M\phi$ should be over the *continuous* fields. The reason for this is that the standard kinetic term would diverge in the limit where the filed ϕ becomes discontinuous.
This assumption, however, does not apply to QFTs with higher-derivative kinetic terms. Examples of such exotic QFTs (without relativistic symmetry) includes tensor gauge theories (see e.g. [12, 13]). In these theories a field *can* be discontinuous, but some of higher derivatives are constrained to scale correctly with respect to the ratio of the lattice size and the system size. In such cases the construction of the trivial operator should differ. *KO does not know how to describe it because of the UV/IR mixing and totally confused.*
These QFTs describes what is called the **fracton phases** of matter, which does not have emergent continuous rotational symmetry in the IR. And the models typically posses **subsytemsymmetries**, whose corresponding operator is not totally topological. Existence of the new kind of symmetry lacked in standard relativistic systems would be related to the fact that the identy wall looks different from the first place.

It is instructive to study the equation of motions (EOMs) on the right hand side of Equation 2.9. The EOM with respect to λ simply states $\phi_L(x) = \phi_R(x)$ for $x \in W$. The surface term of Euler-Lagrange equation for ϕ_L and ϕ_R gives

$$\left.\frac{\delta\mathcal{L}[\phi_L]}{\delta\, d\phi_L}\right|_W = \lambda\,\mathrm{vol}_W = \left.\frac{\delta\mathcal{L}[\phi_R]}{\delta\, d\phi_R}\right|_W.$$

If W is spacelike, or we regard the direction perpendicular to W as the imaginary time in Euclidean signature, this enforces that the canonical momentum also be continuous across W.

Locality

Equation 2.9 expresses the **locality** of the path-integral. We can use the same procedure to decompose the path-integral $\int \mathcal{D}^M\phi$ on M into path-integrals on local patches like $\int \prod_i \mathcal{D}^{V_i}\phi_i$ (comes with many Lagrange multipliers). Here $\bigcup_i V_i = M$ and $V_i \cap V_j$ has codimension 1 in M if not empty.
Indeed, in the context of topological quantum field theory (TQFT), a similar **cutting-and-gluing** axiom is hired by the Atiyah-Segal formulation of topological quantum field theory and later Lurie's cobordism hypothesis [14] established the relation between it and the locality.
Although in this lecture we satisfy ourselves with the formal non-rigorous treatment of path-integrals, deeper understanding of locality in non-topological QFT is strongly desired, and there are some promising results e.g. in [15].

A comment about fermion

Because we do not plan to talk about fermion in this lecture, we comment on what will differ in the case of fermions.
In the scalar field theory, we impose the continuity of the "position" variables (in analytical-mechanical sense) ϕ, then the continuity of the momentum variables follows by EOM.
In a chiral fermion theory, as its kinetic term involves only one derivative, momentum and position variables cannot typically be possible in a way preserving Lorentz or global chiral symmetry. Thus, the "cutting" have to induce an apparent violation of invariance under the Lorentz or the other symmetry, which is a way of seeing the gravitational and global symmetry anomaly. The precise understanding of this perspective is remained to be open in this lecture (and not in the literature as far as the author knows).

2.4 Symmetry Action and Ward-Takahashi identity

Let us turn to the non-trivial operator Equation 2.5 and check the property depicted in Figure 2.1. In order to do it, start from the correlator where U_g is inserted along $W = W_1$ and the trivial operator id along $W'' = W_2$ in Figure 2.1. (Here we only talk about the equality connecting the leftmost and rightmost figures of Figure 2.1. For the middle one the argument is the same. We also renamed the manifolds for the later convenience.) Therefore we split the manifold M into three regions M_L, M_M, M_R, and also the field ϕ into ϕ_L, ϕ_M, ϕ_R. Then to show the equality in Figure 2.1, we perform the change of the variable $\tilde{\phi}_M = \phi_M^{g^{-1}}$:

$$
\begin{aligned}
&\langle \mathcal{O}_1 U_g[W_1] \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 U_g[W_1] \mathcal{O}_2 \,\mathrm{id}[W_2] \rangle \\
&= \int \prod_{i=L,M,R} \mathcal{D}^{M_i} \phi_i \prod_{a=1,2} \mathcal{D}^{W_a} \lambda_a \, e^{-S_L[\phi_L] - S_M[\phi_M] - S_R[\phi_R]} \mathcal{O}_1[\phi_L] \mathcal{O}_2[\phi_M] \\
&\times \exp\left(-G_{W_1}[\lambda_1, \phi_L, \phi_M, g] - G_{W_2}[\lambda_2, \phi_M, \phi_R, \mathrm{id}] \right) \\
&= \int \prod_{i=L,M,R} \mathcal{D}^{M_i} \phi_i \prod_{a=1,2} \mathcal{D}^{W_a} \lambda_a \, e^{-S_L[\phi_L] - S_M[\tilde{\phi}_M^g] - S_R[\phi_R]} \mathcal{O}_1[\phi_L] \mathcal{O}_2[\tilde{\phi}_M^g] \\
&\times \exp\left(-G_{W_1}[\lambda_1, \phi_L, \tilde{\phi}_M, \mathrm{id}] - G_{W_2}[\lambda_2, \tilde{\phi}_M^g, \phi_R, \mathrm{id}] \right) \\
&\times \exp\left(+ \int_W s(\phi_L, g) \right) \\
&= \int \prod_{i=L,M,R} \mathcal{D}^{M_i} \phi_i \prod_{a=1,2} \mathcal{D}^{W_a} \lambda_a \, e^{-S_L[\phi_L] - S_M[\tilde{\phi}_M^g] - S_R[\phi_R]} \mathcal{O}_1[\phi_L] \mathcal{O}_2^g[\tilde{\phi}_M] \\
&\times \exp\left(-G_{W_1}[\lambda_1, \phi_L, \tilde{\phi}_M, \mathrm{id}] - G_{-W_2}[\lambda_2, \phi_R, \tilde{\phi}_M, g^{-1}] \right) \\
&\times \exp\left(+ \int_{W_1} s(\phi_M, g) + \int_{W_2} s(\phi_M, g^{-1}) \right) \\
&= \int \prod_{i=L,M,R} \mathcal{D}^{M_i} \phi_i \prod_{a=1,2} \mathcal{D}^{W_a} \lambda_a \, e^{-S_L[\phi_L] - S_M[\tilde{\phi}_M] - S_R[\phi_R]} \mathcal{O}_1[\phi_L] \mathcal{O}_2^g[\tilde{\phi}_M] \\
&\times \exp\left(-G_{W_1}[\lambda_1, \phi_L, \tilde{\phi}_M, \mathrm{id}] - G_{-W_2}[\lambda_2, \phi_R, \tilde{\phi}_M, g^{-1}] \right) \\
&= \langle \mathcal{O}_1 \,\mathrm{id}[W_1] \mathcal{O}_2 U_{g^{-1}}[-W_2]. \rangle
\end{aligned}
$$

In the second to last equality we used Equation 2.3 and in the last equality we assumed that the path-integral measure is invariant:

$$\mathcal{D}\phi_M = \mathcal{D}\phi_M^g. \tag{2.10}$$

Lastly, from Equation 2.7 we get $U_{g^{-1}}[-W] = U_g[W]$, so we obtain Figure 2.1, that is

$$\langle \mathcal{O}_1 U_g[W_1] \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \mathcal{O}_2^g U_g[W_2] \rangle. \tag{2.11}$$

We note that we can also get the following equation in the same way as above:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \rangle = \langle U_g[W_0] \mathcal{O}_1 \mathcal{O}_2^g \cdots \rangle, \tag{2.12}$$

when W_0 encloses a compact region of M and contains no operator. Then, by repeatedly using these equations, we get the **Ward-Takahashi identity** (**FIXME:figure**)

Ward-Takahashi Identity

$$\begin{aligned}\langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_N\rangle &= \langle U_g[W_0]\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_N\rangle \\ &= \langle \mathcal{O}_1^g U_g[W_1]\mathcal{O}_2\cdots\mathcal{O}_N\rangle \\ &= \langle \mathcal{O}_1^g\mathcal{O}_2^g\cdots\mathcal{O}_N^g U_g[W_N]\rangle \\ &= \langle \mathcal{O}_1^g\mathcal{O}_2^g\cdots\mathcal{O}_N^g\rangle,\end{aligned} \tag{2.13}$$

where W_i encloses the operators $\mathcal{O}_1\cdots\mathcal{O}_i$, and at the last step we collapsed $U_g[W_N]$ towards the infinity (or whatever point if M is compact).

Of course we could get the same result by considering global change of variable $\tilde{\phi} = \phi^{g^{-1}}$, the point is that, once the existence of topological operator is established, the derivation of the Ward-Takahashi identity can be done in the same way, even if the symmetry does not come from field transformation.

FIXME:explain about the selection rule

Mixed-gravitational anomaly

The symmetry we discuss here does not suffer from anomaly and the Ward-Takahashi idendity (Equation 2.13) follows. If we instead consider the topological operator corresponding to a symmetry with mixed-gravitational anomaly, the invariance of the measure (Equation 2.10) does not necessarily hold when M_M does not have the topology of a ball, and likewise Equation 2.12 can fail when W_0 is not a ball. Thus, on a spacetime with non-trivial topology (other than $\mathbb{R}^4$ or S^4), (Equation 2.13) might fail at the last equality, while (by choosing W_0 to be a ball) other steps goes through. The failure is by a phase depending on topology of the spacetime, and has an interesting consequences of this discussed in [16].

2.5 Relation to Noether Charge

Here we show the relation Equation 2.4 of the topological operator $U_g[W]$ to the conventional Noether charge. This can be done by applying the change of variable $\widetilde{\phi_R} = \phi_R^{\mathrm{id}-\alpha f(x_n)}$ to Equation 2.5, where $f(x_n, \delta)$ is a function of the coordinate x_n perpendicular to W and one positive parameter δ and satisfies $f(0) = 1$ and $f(x_n > \delta) = 0$. Then we take $\delta \to 0$ limit and compare. **FIXME:write equations**

3 Compact Boson

Here we consider *compact* boson, where the field ϕ is real and subject to the identification

$$\phi(x) \cong \phi(x) + 2\pi.$$

We take the Lagrangian to be

$$\mathcal{L} = \frac{R^2}{4\pi} \int_M d\phi \wedge * d\phi \,.$$

We can normalize the field ϕ so that it has the kinetic term with a fixed coefficient, in which case the normalized field has a periodicity radius proportional to R.

The theory has the shift (or "momentum") U(1) symmetry

$$\phi^\alpha = \phi + \alpha$$

with identification of the parameter $\alpha \cong \alpha + 2\pi$. One can add a periodic potential $V(\phi)$ and restrict oneself to a discrete symmetry preserving the potential.

In addition to the shift symmetry, the system has other generalized symmetries:

1. **winding** U(1) $(D-2)$-form symmetry [8], and
2. when $D = 2$ and R^2 is rational, there exists a **T-duality** symmetry that is in general *non-invertible* [17, 18, 19, 20].

The purpose of this section is to understand these generalized symmetries, but before that we review the shift symmetry.

3.1 Trivial Operator and Shift Symmetry Operator

We start from the identity operator $\mathrm{id}[W^{D-1}]$ that cuts and glues the path-integral. The construction is almost the same as before, but when we glue the fields ϕ_L and ϕ_R along W^{D-1}, the gluing can be up to a integer multiple of 2π:

$$\phi_L(x) = \phi_R(x) + 2\pi n$$

with an integer n (assuming W^{D-1} is connected). Therefore, the gluing part of the path-integral is

$$\mathrm{id}[W^{D-1}] = \sum_{n\in\mathbb{Z}} \int \mathcal{D}^{W^{D-1}}\lambda \exp\left(\mathrm{i}\int_{W^{D-1}} \lambda\left(\phi_L - \phi_R - 2\pi n\right)\right) \mathrm{vol}_{W^{D-1}}\,.$$

We can sum n out to restrict λ to satisfy

$$\int_{W^{D-1}} \lambda\, \mathrm{vol}_{W^{D-1}} \in \mathbb{Z}.$$

An integration over such lambda can be replaced by integration in terms of $(D-2)$-form U(1) gauge field V with

$$dV = 2\pi\lambda\, \mathrm{vol}_{W^{D-1}}\,.$$

Therefore the identity operator can be written as[1]

$$\mathrm{id}[W^{D-1}] = \int \mathcal{D}^{W^{D-1}} V \exp\left(\frac{\mathrm{i}}{2\pi}\int_{W^{D-1}} dV\left(\phi_L - \phi_R\right)\right). \tag{3.1}$$

p-form gauge field

A p-form gauge field V is *locally* (i.e. in a patch) p-form, but V is not necessarily a *global* p-form and dV rather satisfy $\int_{\Sigma_{p+1}} dV \in 2\pi\mathbb{Z}$ for any $p+1$ dimensional submanifold. If a reader is not familiar with this concept, one can assume $D = 2, 3$. When $D = 3$, V is a usual (one-form) abelian gauge field, whose magnetic flux is quantized, while when $D = 2$, V is a periodic scalar field. About higher-form gauge field, a motivated reader can consult e.g. Hsieh et al. [21].

Now the topological operator for the shift symmetry is simply

$$U^{\mathrm{shift}}_\alpha[W^{D-1}] = \int \mathcal{D}^{W^{D-1}} V \exp\left(\frac{\mathrm{i}}{2\pi}\int_{W^{D-1}} dV\left(\phi_L - \phi_R + \alpha\right)\right). \tag{3.2}$$

3.2 Winding Symmetry

The compact boson theory has another topological operator of dimension 1 (codimension $D-1$), which is simply

$$U^{\mathrm{winding}}_\alpha[\gamma^1] = \exp\left(\mathrm{i}\,\alpha\int_{\gamma^1}\frac{d\phi}{2\pi}\right). \tag{3.3}$$

[1] We assume that the integration should be done over *gauge equivalent classes* of V. In other words the gauge fixing procedure is implicit in this lecture.

Given the periodicity, the integral $\int_{\gamma^1} d\phi$ for a closed γ^1 takes a value in $2\pi\mathbb{Z}$, and therefore the operator is invariant against deformation of γ^1. This also indicates that the parameter α is 2π periodic. According to Table 1.1, this topological operator should define a U(1) p-form symmetry with $p = (D-2)$, called the **winding symmetry**.

What are the operators *charged* under the symmetry? When $p \geq 1$ ($D \geq 3$), the operator Equation 3.3 cannot act on a local (point) operator, because the one-dimensional submanifold γ^1 can always be deformed one configuration to another without colliding with a point. On the other hand, it can potentially act on a p-dimensional extended operator: Figure 3.1.

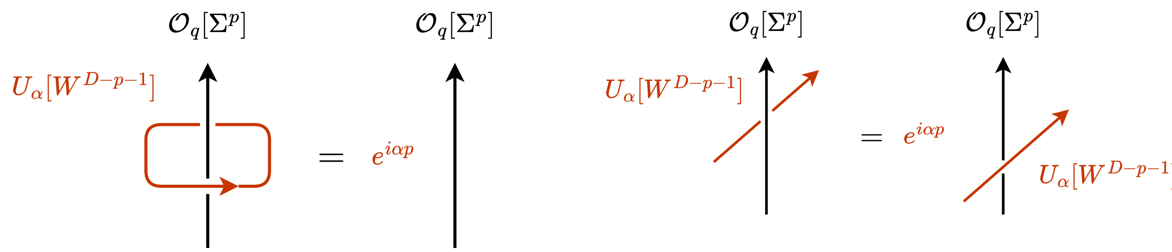

Figure 3.1: Action of p-form symmetry operator $U_\alpha[W^{D-p-1}]$ on a p-dimensional operator $\mathcal{O}_q[\Sigma^p]$ of p-form charge q. We can consider the two types of actions, one is *encircling* action (left), and the other is the *passing* action (right). From the latter we can close the loop on the right of the $\mathcal{O}_q$, and contract the loop on the right hand side, to get the former type of action.

However, the construction of a charged operator is a bit tricky. A way of doing it is to first insert the identity operator Equation 3.1, then utilize the field V on the identity operator to defing an operator charged under Equation 3.3. Concretely, the (non-topological) operator with winding charge n can be constructed as

$$\langle \mathcal{O}_n^{\text{winding}}[\Sigma^{D-2}]\rangle = \int \mathcal{D}^{W^{D-1}} V \exp\left(\frac{\mathrm{i}}{2\pi}\int_{W^{D-1}} dV\,(\phi_L - \phi_R) + \mathrm{i}\,n\int_{\Sigma^{D-2}} V\right). \tag{3.4}$$

Here, we take arbitrary W^{D-1} that contains Σ^{D-2}, and the correlator is independent of the choice. The coefficient n has to be an integer for $\int_{\Sigma^{D-1}}$ to be invariant under global gauge transformations.

The operator Equation 3.4 is often defined as a "disorder" operator that enforces singular behavior. Here we see the explicit construction of such by *integrating in* the Lagrange multiplier V on W^{D-1}.

i Note

Note that the topological operator Equation 2.5 is also of disorder-type; it enforces a jump of the field across W. It is curious that, for symmetry of field transformation, the charg*ed* operators are direct to construct, while the symmetry topological operator was somewhat tedious to do; and it is opposite for the winding symmetry, or more generally topological charges.

	Field transformation	Topological charge
Charged operator	not disorder	disorder
Topological operator	disorder	not disorder

One aim of this lecture is to demystify the "disorder" operators – they can be explicitly written in terms of correct set of Lagrange multipliers – so that one can talk about the two types of the symmetry in a unified way.

FIXME:derivation of the charge, from EOM of V

Now the Ward-Takahashi identity Equation 2.13 formally follows from the topological-ness of Equation 3.3. Explicitly, we have

$$\langle \prod_i \mathcal{O}^{\text{winding}}_{n_i}(x_i) \rangle = \langle \prod_i e^{\mathrm{i}\,\alpha n_i} \mathcal{O}^{\text{winding}}_{n_i}(x_i) \rangle$$

for any α. And thus the both sides vanish unless $\sum n_i = 0$.

3.3 Mixed Anomaly between Shift and Winding Symmetry

3.3.1 Intersection

Having explicit descriptions of topological operators enables us to directly compute **quantum anomaly** (often called 't Hooft anomaly) of the symmetries. This is because, from a modern perspective, the anomaly is a subtlety arises when symmetry operators collide. Here we observe one example of anomaly – the mixed anomaly between the shift and winding symmetry in the compact boson theory – explicitly from the topological operator perspective. For a general theory about anomaly and topological operator, readers can consult other resources, e.g. Tachikawa [7].

Let us study the intersection of $U^{\text{shift}}_{\alpha}[W^{D-1}]$ (Equation 3.2) and $U^{\text{winding}}_{\beta}[\gamma^1]$ (Equation 3.3). **FIXME:figure** The shift symmetry operator divides γ^1 into γ^1_L and γ^1_R, and the winding operator thus now, naively, look like

$$U^{\text{shift}}_\alpha U^{\text{winding}}_\beta[\gamma^1] \overset{\text{naive}}{=} U^{\text{shift}}_\alpha \exp\left(\mathrm{i}\,\beta\left(\int_{\gamma^1_L}\frac{d\phi_L}{2\pi}+\int_{\gamma^1_R}\frac{d\phi_R}{2\pi}\right)\right)$$
$$\sim U^{\text{shift}}_\alpha \exp\left(\mathrm{i}\,\beta/2\pi(\phi_L(x_0)-\phi_R(x_0))\right),$$

where in the second line, $\sim$ refers to the contribution local to the intersection point x_0 (i.e. we ignored the contribution from the other ends of γ^1_L and γ^1_R far from W^{D-1}). However, the shift symmetry defect enforces $\phi_L = \phi_R - \alpha \mod 2\pi$, but the local contribution at x_0 *depends* on $\phi_L - \phi_R \mod 2\pi$. Therefore the naive definition of intersected operator is not well-defined (or, it becomes zero if we average over the branches of $\phi_L(x_0) - \phi_R(x_0)$).

A way to define the intersection is to abandon the periodicity of either of α or β. If we regard α to be in $\mathbb{R}$ and not $\mathbb{R}/2\pi\mathbb{Z}$, we can modify the above naive definition to be

$$U^{\text{shift}}_\alpha U^{\text{winding}}_\beta[\gamma^1] \overset{\text{def1}}{=} U^{\text{shift}}_\alpha \exp\left(\mathrm{i}\,\beta\left(\int_{\gamma^1_L}\frac{d\phi_L}{2\pi}+\int_{\gamma^1_R}\frac{d\phi_R}{2\pi}+\{\alpha/2\pi\}\right)\right)$$
$$\sim U^{\text{shift}}_\alpha \exp\left(\mathrm{i}\,\beta[\alpha/2\pi]\right),$$

where $[r]$ is the integer part of a real number r, and $\{r\} = r - [r]$. With this definition, or regularization, of the intersection, α is no longer periodic, but β is kept periodic. One can do other regularizations where α is periodic but β is not, or just abandon both of periodicity, but cannot save both.

This incompatibility of periodicity, or the group multiplication law, when topological operators intersects is the hallmark of anomaly.

3.3.2 Group Cohomology

The incompatibility above is better characterized as a group cohomology (or its generalization to a higher-group). Here we see how to characterize the mixed anomaly of the compact boson in 1+1d as a group cohomology element. (Here we do not delve into the general theory of group cohomology. See e.g. Tachikawa [7]).

In 1+1d, both U^{shift}_α and U^{winding}_β are line operators. Both operators are 2π periodic in its parameters, when intersection between them are absent. A more precise statement that applies even with intersections is that U^{shift}_α and $U^{\text{shift}}_{\alpha+2\pi}$ can be connected with an invertible topological line-changing operator or a *line-isomrphism* operator for short[2] (See also Figure 3.2):

$$\exp\left(\frac{\mathrm{i}}{2\pi}\int^{\square} dV(\phi_L-\phi_R-\alpha)+\mathrm{i}\,V(\square)+\frac{\mathrm{i}}{2\pi}\int_{\square} dV(\phi_L-\phi_R-\alpha+2\pi)\right)$$

[2] In the language of category theory, such an invertible topological line-changing operator defines an *isomorphism* between the lines. In (higher)-category theory it is important to distinguish being isomorphic from being "the same", and in this context physically it means that we remember the total derivative terms on the line.

where $\square$ denotes the point connecting two line operators. Note that the winding operator $e^{\mathrm{i}\,V(\square)}$ is precisely cancelled when the integral in the last term is evaluated, and thus the junction operator is topological. The existence of its inverse is also manifest.

Figure 3.2: Line-isomorphism operators.

Likewise, the line isomorphism operator for the winding symmetry operator is

$$\exp\left(\mathrm{i}\,\alpha\int^{\triangle}\frac{d\phi}{2\pi}+\mathrm{i}\,\phi(\triangle)+\mathrm{i}(\alpha+2\pi)\int_{\triangle}\frac{d\phi}{2\pi}\right).$$

We let the composition of the two lines be denoted by $U_{\alpha,\beta}=U^{\text{shift}}_{\alpha}U^{\text{winding}}_{\beta}$. With it, we can define "naive" junction where two line operators U_{α_1,β_1} and U_{α_2,β_2} fusing into $U_{\alpha_1+\alpha_2,\beta_1+\beta_2}$. However, the fused operator in general has parameters in the fundamental domain other than those of the fused ones. Therefore, we normalize the junction by using the $\square$ and $\triangle$ isomorphisms so that the output is $U_{\langle\alpha_1+\alpha_2\rangle,\langle\beta_1+\beta_2\rangle}$, where $\langle\alpha\rangle=2\pi\{\alpha/2\pi\}$; see Figure 3.3. The point is that this isomorphism results in a phase when crossing an intersection of the lines.

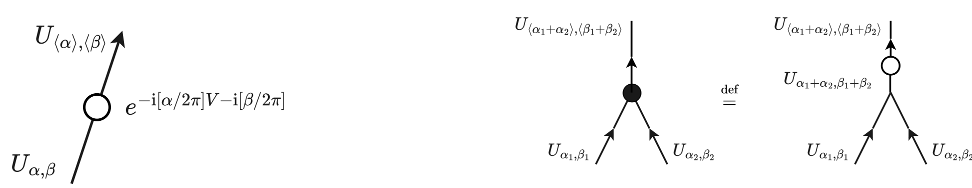

Figure 3.3: The definition of junction fusing two $U_{\alpha,\beta}$'s into one.

Now, we consider the two consecutive junctions, which fuses three lines $U_{\langle\alpha_i\rangle,\langle\beta_i\rangle}$ $(i=1,2,3)$ into $U_{\langle\sum_i\alpha_i\rangle,\langle\sum_j\beta_j\rangle}$; see Figure 3.4.

There are two ways of such fusion as depicted in the figure, and the two are related by a nontrivial phase which is

$$f(g_1,g_2,g_3)=\alpha_1\left[\frac{\langle\beta_2\rangle}{2\pi}+\frac{\langle\beta_3\rangle}{2\pi}\right]+\beta_3\left[\frac{\langle\alpha_1\rangle}{2\pi}+\frac{\langle\alpha_2\rangle}{2\pi}\right], \tag{3.5}$$

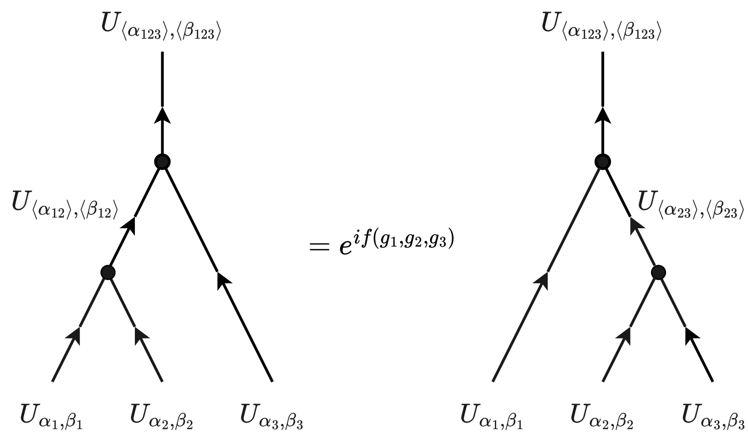

Figure 3.4: The anomalous phase from the F-move. We let $\alpha_{12} = \alpha_1 + \alpha_2$, etc.

where $g_i = (\alpha_i, \beta_i) \in U(1)^2$. The two configurations are gauge-equivalent, yet there is a non-trivial phase relating them. This is the *anomalous phase* of this anomalous $U(1)^2$ symmetry of 1+1d compact boson.

Group cohomology

Mathematically speaking, the function $f : (U(1)^2)^3 \to \mathbb{R}/2\pi\mathbb{Z}$ is a 3-cocycle on the group $G = U(1)^2$; it satisfies the *pentagon identity*, depicted in **?@fig-pentagon**.
Futher, it is subject to an ambiguity; we can modify the junction by a phase $h(g_1, g_2)$. This modifies the anomalous phase f as

$$f(g_1, g_2, g_3) \mapsto f(g_1, g_2, g_3) + (\delta h)(g_1, g_2, g_3)$$

where $\delta h : (U(1))^3 \to \mathbb{R}/2\pi\mathbb{Z}$ is the following function called a *coboundary* (FIXME:sign?):

$$(\delta h)(g_1, g_2, g_3) = h(g_1, g_2) + h(g_1 g_2, g_3) - h(g_1, g_2 g_3) - h(g_2, g_3).$$

The third **group cohomology** is an element of the quatient group

$$H^3 = \{f \in \mathrm{Map}(G^3, \mathbb{R}/2\pi\mathbb{Z}) \mid \text{pentagon eq.}\}/\delta(\mathrm{Map}(G^3, \mathbb{R}/2\pi\mathbb{Z})),$$

where $\mathrm{Map}(X, Y)$ is the abelian group of Y valued functions on X [3]. This group cohomology classifies the anomalous phases of 1+1-dimensional bosonic theories with G-symmetry.

[3]When G is continuous, the definition of $\mathrm{Map}(G^n, \mathbb{R}/2\pi\mathbb{Z})$ is subtle. The particular cocycle f in Equation 3.5

Exercise: $\mathbb{Z}_2$ anomaly

The diagonal $\mathbb{Z}_3$ subgroup of $U(1)^2$ generate by $U_{2\pi/3,2\pi/3}$ is anomalous. Confirm this fact by calculating $f(g_1, g_2, g_3)$ for g_i's in this subgroup, and also proving that no h can satisfy $f = -\delta(h)$. Optionally one might consider some other finite subgroups of $U(1)^2$.

Comparison to more conventional calculation.

The mixed anomaly between the shift and winding symmetry in 1+1d compact boson is usually computed from the two-point function of the corresponding current operators. Such computation corresopnds to *non-flat* and infinitesimal background for the symmetries. On the other hand, inserting topological operator corresponds to *flat* and finite background. The two computations results in the same result is quite non-trivial, and is explained by the Chern-Weil theory.
While the computation based on non-flat background is easier, it is only for continuous symmetries and not obvious how it restricts to a finite subgroup, for example.

3.4 T-duality

The compact boson in $D = 2$-dimensions famously has *T-duality*.

The T-duality is equivalence between the compact boson with radius R and radius $\frac{1}{R}$, and maps the shift (or momentum) symmetry of one side to the winding symmetry of the other side, and vise versa. Note that this makes sense because in $D = 2$ the winding symmetry is a 0-form symmetry.

When the radius is the self-dual radius $R = 1$, the duality becomes a (conventional) $\mathbb{Z}_2$ symmetry, which is a part of the larger emergent SU(2) symmetry.

In this section we study

1. the explicit construction of the T-duality topological interface connecting radius R and $1/R$ theories [22], and
2. its generalization giving **non-invertible** symmetry (i.e. self-dual topological interface) at $R = \sqrt{N}$ for an integer N.

The latter can further be generalized to the case of rational R^2 [18, 19].

is not continuous, so we do not want $\mathrm{Map}(X, Y)$ to mean continuous maps. On the other hand arbitrary discontinuous maps would contain too wild cocycles to be realized in a QFT. In this particular case it seems considering *piecewise continuous* functions is necessary and sufficient. For general cases in general dimensions KO is not sure.

3.4.1 T-duality topological interface

Here we study the T-duality topological interface connecting two compact boson theories in $D = 2$-dimensions; with radius R and radius $1/R$. **FIXME:figure!** When $R = 1$, the interface is a topological operator within the same thoery, and thus defines a symmetry. In this case it turns out a $\mathbb{Z}_2$ symmetry, known to be contained in the larger SU(2) enhanced symmetry.

According to [22], the partition functino involving the interface is

$$\begin{aligned}\langle \mathcal{O}_L \mathcal{I}^{\mathrm{T}}_1[W] \mathcal{O}_R \rangle = &\int \mathcal{D}^{M_L}\phi_L\, \mathcal{D}^{M_R}\phi_R\, \mathcal{O}_L[\phi_L] \mathcal{O}_R[\phi_R] \\ &\exp\left(-\frac{R^2}{4\pi}\int_{M_L} d\phi_L * d\phi_L - \frac{\mathrm{i}}{2\pi}\int_W \phi_L\, d\phi_R - \frac{1}{4\pi R^2}\int_{M_R} d\phi_R * d\phi_R \right).\end{aligned} \tag{3.6}$$

For simplicity let us take $\mathcal{O}_R = 1$, and M_R be a compact region in the spacetime. In this case, we expect the supposed topological-ness implies

$$\langle \mathcal{O}_L \mathcal{I}^{\mathrm{T}}_1[W] \rangle = \langle \mathcal{O}_L \rangle.$$

To show this, we replace the variable ϕ_R with $F_R = d\phi_R$ by

$$\int \mathcal{D}\phi_R = \int \mathcal{D}F_R\, \mathcal{D}\tilde{\phi}_R \exp\left(\frac{\mathrm{i}}{2\pi}\int_{M_R} \tilde{\phi}_R\, dF_R \right),$$

where the periodic scalar $\tilde{\phi}_R$ is the Lagrange multiplier enforcing the closedness and the quantization of $F_R = d\phi_R$. Substituting this into Equation 3.6, the EOMs with respect to F_R are

$$\begin{aligned}\frac{1}{R} * d\phi_R(x) &= \mathrm{i}\, d\tilde{\phi}_R(x) \quad \text{for } x \in M_R \\ \phi_L(x) &= \tilde{\phi}_R(x) \quad \text{for } x \in W.\end{aligned} \tag{3.7}$$

Substituting the former equation to the Lagrangea of ϕ_R, we get

$$\frac{R^2}{4\pi}\int_{M_R} d\tilde{\phi}_R * d\tilde{\phi}_R$$

with is the same as for ϕ_L, while the latter of Equation 3.7 connects ϕ_L and $\tilde{\phi}_R$ along W. As a whole, we recover the path-integral over M resulting in $\langle \mathcal{O}_L \rangle$. For a more general topological-ness about local deformation of W can be derived in the same way but with more letters.

Now, let us set $\mathcal{O}_R = \mathcal{O}^{\text{winding}}_n(x)$ and see how the topological interface acts on the operator. As the operator $\mathcal{O}^{\text{winding}}_n$ in Equation 3.4 is defined on the trivial surface operator, we have to divide the manifold into three parts: M_L, M_M, m_R, separated by W_1 and W_2. As the M_L and field on it is not going to be relevant, we suppress them in the following equations. The calculation goes as:

$$
\begin{aligned}
\mathcal{I}_1^{\mathrm{T}}\mathcal{O}_n^{\mathrm{winding}}(x) &= \int \mathcal{D}\phi_M\,\mathcal{D}\phi_R\,\mathcal{D}^{W_2}V \exp\left(-S_M^{1/R} - S_R^{1/R} - \frac{\mathrm{i}}{2\pi}\int_{W_1}\phi_L\,d\phi_M\right)\\
&\times \exp\left(\frac{\mathrm{i}}{2\pi}\int_{W_2} dV(\phi_M-\phi_R) + \mathrm{i}\,nV(x)\right)\\
&= \int \mathcal{D}F_M\,\mathcal{D}F_R\,\mathcal{D}^{W_2}V\,\mathcal{D}\tilde{\phi}_M\,\mathcal{D}\tilde{\phi}_R\\
&\times \exp\left(-S_M^{1/R} - S_R^{1/R} - \frac{\mathrm{i}}{2\pi}(\int_{W_1}\phi_L\,d\phi_M - \int_{M_M}\tilde{\phi}_M\,dF_M - \int_{M_R}\tilde{\phi}_R\,dF_R)\right)\\
&\times \exp\left(+\frac{\mathrm{i}}{2\pi}\int_{W_2} V(F_M-F_R) + \mathrm{i}\,nV(x)\right)
\end{aligned}
$$

Now, the EOMs in terms of F_M and F_R state

$$
\begin{aligned}
\frac{1}{R} * d\phi_{M,R} &= \mathrm{i}\,d\tilde{\phi}_{M,R}(x) \quad \text{on} \;\; M_{M,R},\\
\phi_L &= \tilde{\phi}_M \quad \text{on} \;\; W_1,\\
\tilde{\phi}_M &= V = \tilde{\phi}_R \quad \text{on} \;\; W_2.
\end{aligned}
$$

Thus by substituting back we get **FIXME:check the sign!**

$$
\mathcal{I}_1^{\mathrm{T}} \cdots \mathcal{O}_n^{\mathrm{winding}} = \mathcal{O}_{\pm?n}^{\mathrm{shift}}.
$$

In addition, let us calculate $(\mathcal{I}_1[W]^{\mathrm{T}})^2$. For this, we insert the defects along parallel submanifolds W_1 and W_2 and take the limit where the separation of the two shrinks. **FIXME:figure** This can be calculated as

$$
\begin{aligned}
\mathcal{I}_1^{\mathrm{T}}[W_1]\mathcal{I}_1^{\mathrm{T}}[W_2] &= \int \mathcal{D}^{M_M}\phi_M \exp(-\mathrm{i}\int_{W_1}\phi_L d\phi_M - \mathrm{i}\int_{W_2}\phi_M d\phi_R)\\
&= \int \mathcal{D}^{W}\phi_M \exp(-\mathrm{i}\int_{W}(\phi_L-\phi_R)d\phi_M)\\
&= \mathrm{id}[W],
\end{aligned}
$$

where in the second line we collide W_1 and W_2, and noted that only the mode of ϕ_M constant along the direction perpendicular to W_1 and W_2 contributes. Therefore, the T-duality interface squares to the identity. In particular, at $R=1$, the self-interface defines an invertible $\mathbb{Z}_2$ symmetry.

3.4.2 Non-invertible symmetry from T-duality

Choi et al. [17] generalized the Kapustin-Tikhonov T-duality interface by Kapustin and Tikhonov [22] as

$$\begin{aligned}\langle \mathcal{O}_L \mathcal{I}_N^{\mathrm{T}}[W] \mathcal{O}_R \rangle = \int \mathcal{D}^{M_L}\phi_L\, \mathcal{D}^{M_R}\phi_R\, \mathcal{O}_L[\phi_L]\mathcal{O}_R[\phi_R] \\ \exp\left(-\frac{R^2}{4\pi}\int_{M_L} d\phi_L * d\phi_L - \frac{\mathrm{i}\,N}{2\pi}\int_W \phi_L\, d\phi_R - \frac{N^2}{4\pi R^2}\int_{M_R} d\phi_R * d\phi_R\right).\end{aligned} \tag{3.8}$$

This interface is a *self-interface* at $R = \sqrt{N}$. The same procedure as we did for $\mathcal{I}_1$ leads to

$$\begin{aligned}\langle \mathcal{O}_L \mathcal{I}_N^{\mathrm{T}}[W] \rangle = \int \mathcal{D}^{M_L}\phi_L\, \mathcal{D}^{M_R}\phi_R\, \mathcal{D}^W V'\, \mathcal{O}_L[\phi_L] \\ \exp\left(-\frac{R^2}{4\pi}\int_{M_L} d\phi_L * d\phi_L - \frac{\mathrm{i}}{2\pi}\int_W (N\phi_L - \tilde{\phi}_R)\, dV - \frac{R^2}{4\pi N^2}\int_{M_R} d\tilde{\phi}_R * d\tilde{\phi}_R\right).\end{aligned} \tag{3.9}$$

We further rescale the fields as

$$\begin{aligned} V' &= NV \\ N\tilde{\phi}'_R &= \tilde{\phi}_R \mod 2\pi \end{aligned}$$

so that we have

$$\begin{aligned}\langle \mathcal{O}_L \mathcal{I}_N^{\mathrm{T}}[W] \rangle = \mathcal{N}\int \mathcal{D}^{M_L}\phi_L\, \mathcal{D}^{M_R}\phi_R\, \mathcal{D}^W V'\, \mathcal{O}_L[\phi_L] \\ \exp\left(-\frac{R^2}{4\pi}\int_{M_L} d\phi_L * d\phi_L - \frac{\mathrm{i}}{2\pi}\int_W (\phi_L - \tilde{\phi}'_R)\, dV' - \frac{R^2}{4\pi}\int_{M_R} d\tilde{\phi}'_R * d\tilde{\phi}'_R\right).\end{aligned} \tag{3.10}$$

To go from Equation 3.10 to Equation 3.9, we define a $\mathbb{Z}_N$ valued variable $n_{\widetilde{\phi'}_R} = [N\widetilde{\phi'}_R/2\pi]$ mod N so that $\widetilde{\phi'} = \frac{2\pi}{N}(\{\tilde{\phi}/2\pi\} + 2\pi n_{\widetilde{\phi'}_R})$. Then, after substituting it, the sum over $n_{\widetilde{\phi'}}$ enforces V' be divisible by N (that is, the winding of V along W is constrained to be a multiple of N). Thus, we have

$$\langle \mathcal{O}_L \mathcal{I}_N^{\mathrm{T}}[W] \rangle = \mathcal{N}\langle \mathcal{O}_L \rangle.$$

We are not caring enough about the absolute size of path-integral measure to determine the normalization constant $\mathcal{N}$, but will determined in by another mean later.

3.4.2.1 Fusion rule

The product, or *fusion* of the generalized operator is

$$
\begin{aligned}
(\mathcal{J}_N^{\mathrm{T}})^2 &= \int \mathcal{D}^W V \exp(\frac{\mathrm{i}}{2\pi} N \int_W dV(\phi_L - \phi_R)) \\
&= \sum_{n=0}^{N-1} \int \mathcal{D}^W V' \exp(\frac{\mathrm{i}}{2\pi} \int_W dV(\phi_L - \phi_R + 2\pi n/N)) \\
&= \sum_{n=0}^{N-1} U^{\mathrm{shift}}_{2\pi n/N},
\end{aligned}
\tag{3.11}
$$

where in the second line we did the chage of variable $NV = V'$, and enforced the divisibility of V' by N by the sum over n. One can also see $\mathcal{J}_N^{\mathrm{T}}[-W] = \mathcal{J}_N^{\mathrm{T}}[W]$ (note that when we consider an operator on W the notion of "left" and "right" also flips), so

$$
\mathcal{J}_N^{\mathrm{T}}[W]\mathcal{J}_N^{\mathrm{T}}[-W] = \sum_{n=0}^{N-1} U^{\mathrm{shift}}_{2\pi n/N}.
$$

This contrasts to the case of conventional symmetry operator, where $U_g[W] = U_{g^{-1}}[W]$ and thus

$$
U_g[W]U_g[-W] = \mathrm{id}[W].
$$

In the case of $\mathcal{J}_N^{\mathrm{T}}$, $N \geq 2$, the fusion with its orientation reversal is not the trivial operator, but a sum. This is a one of hallmarks of **non-invertible** symmetry. $\mathcal{J}_N^{\mathrm{T}}[-W]$, called the **dual** of the original operator, is the closest possible thing to be the "inverse", but it fails to be so. Therefore, the compact boson theory in 1+1d at $R = \sqrt{N}$ has the non-invertible T-duality symmetry.

Here, we can determine that the coefficient in the above equations are correct. This is because that we can insert the one-dimensional operator $(\mathcal{J}_N^{\mathrm{T}})^2$ along the time direction, which should determine the *defect Hilbert* space. On the right hand side, we should have a direct sum of defect Hilbert spaces for the involved invertible symmetry operators. There is no way to take "average" over Hilbert spaces, or divide it by a number, so we can assume the minimal possible coefficient is realized, which is the one in Equation 3.11.

Given Equation 3.11, the coefficient $\mathcal{N}$ in Equation 3.10 is determined by

$$
\langle (\mathcal{J}_N^{\mathrm{T}})^2[W] \rangle = \sum_{n=0}^{N-1} \langle U^{\mathrm{shift}}_{2\pi n/N}[W] \rangle = \mathcal{N}^2,
$$

thus $\mathcal{N} = \sqrt{N}$. This quantity is called the **quantum dimension** of $\mathcal{J}_N^{\mathrm{T}}$.

3.4.2.2 Action on the local operators

What is the action of $\mathcal{J}_N^{\mathrm{T}}$ on the local operator $\mathcal{O}_n^{\text{winding}}$? Naively repeating the procedure in the previous section, one might think

$$\mathcal{J}_N^{\mathrm{T}} \cdot \mathcal{O}_n^{\text{winding}}(x) \stackrel{?}{=} \mathcal{O}_{n/N}^{\text{shift}}(x)$$

however the left hand side, $e^{\mathrm{i}\, n/N\phi}$, does not make sense when N does not divide n as it is incompatible with the periodicity of ϕ. Thus the right hand side vanishes unless $N|n$, and we have

$$\mathcal{J}_N^{\mathrm{T}} \cdot \mathcal{O}_n^{\text{winding}}(x) = \begin{cases} e^{\mathrm{i}\,\frac{n}{N}\phi} & N|n \\ 0 & \text{otherwise.} \end{cases}$$

Note that here we considered the *encircling* action (FIXME:Figure!!). We can instead consider the *passing* action, in which case we have

$$\mathcal{J}_N^{\mathrm{T}}[W_L]\mathcal{O}_n^{\text{winding}}(x) = e^{\mathrm{i}\,\frac{n}{N}\int_{\gamma^x} d\phi}\mathcal{J}_N^{\mathrm{T}}[W_R],$$

$W_{L,R}$ goes through the left/right side of the point x, and the path γ^x starts from a point on W_R and ends at x.

3.4.3 Non-invertible T-duality for a rational R

We can further generalize Equation 3.8 so that it is a self duality at $R^2 = p/q$ (p and q are taken to be coprime) [18, 19]. The construction is

$$\mathcal{J}_{p,q}^{\mathrm{T}}[W] = \int \mathcal{D}^W a\, \mathcal{D}^W b \exp\left(-\frac{\mathrm{i}}{2\pi}\int_W (q\, a\, db + p\, a\, d\phi_L - b\, d\phi_R)\right),$$

where a and b are periodic scalar fields on W. The fusion is

$$(\mathcal{J}_{p,q}^{\mathrm{T}}[W])^2 = \sum_{n_1=0}^{p-1}\sum_{n_2=0}^{q-1} U_{2\pi n_1/N}^{\text{shift}} U_{2\pi n_2/N}^{\text{winding}},$$

and the quantum dimension is

$$\langle \mathcal{J}_{p,q}^{\mathrm{T}}[W]\rangle = \sqrt{pq}.$$

4 Vector

test

References

[1] J. McGreevy, "Generalized Symmetries in Condensed Matter", 10 . 1146 / annurev - conmatphys-040721-021029 (2022), arXiv:2204.03045 [cond-mat.str-el].

[2] S. Schafer-Nameki, "ICTP Lectures on (Non-)Invertible Generalized Symmetries", (2023), arXiv:2305.18296 [hep-th].

[3] P. R. S. Gomes, "An introduction to higher-form symmetries", SciPost Phys. Lect. Notes **74**, 1 (2023), arXiv:2303.01817 [hep-th].

[4] L. Bhardwaj, L. E. Bottini, L. Fraser-Taliente, L. Gladden, D. S. W. Gould, A. Platschorre, and H. Tillim, "Lectures on Generalized Symmetries", (2023), arXiv:2307.07547 [hep-th].

[5] R. Luo, Q.-R. Wang, and Y.-N. Wang, "Lecture Notes on Generalized Symmetries and Applications", in (July 2023), arXiv:2307.09215 [hep-th].

[6] S.-H. Shao, "What's Done Cannot Be Undone: TASI Lectures on Non-Invertible Symmetry", (2023), arXiv:2308.00747 [hep-th].

[7] Y. Tachikawa, "Lecture on anomalies and topological phases", (2019), https://member.ipmu.jp/yuji.tachikawa/lectures/2019-top-anom/.

[8] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, "Generalized Global Symmetries", JHEP **02**, 172 (2015), arXiv:1412.5148 [hep-th].

[9] A. Kapustin and R. Thorngren, "Higher symmetry and gapped phases of gauge theories", (2013), arXiv:1309.4721 [hep-th].

[10] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, "Symmetry Fractionalization, Defects, and Gauging of Topological Phases", Phys. Rev. B **100**, 115147 (2019), arXiv:1410.4540 [cond-mat.str-el].

[11] M. Kalb and P. Ramond, "Classical direct interstring action", Phys. Rev. D **9**, 2273–2284 (1974), https://link.aps.org/doi/10.1103/PhysRevD.9.2273.

[12] M. Pretko, X. Chen, and Y. You, "Fracton Phases of Matter", Int. J. Mod. Phys. A **35**, 2030003 (2020), arXiv:2001.01722 [cond-mat.str-el].

[13] N. Seiberg and S.-H. Shao, "Exotic Symmetries, Duality, and Fractons in 2+1-Dimensional Quantum Field Theory", SciPost Phys. **10**, 027 (2021), arXiv:2003.10466 [cond-mat.str-el].

[14] J. Lurie, "On the classification of topological field theories", Current developments in mathematics **2008**, 129–280 (2008).

[15] D. Grady and D. Pavlov, "Extended field theories are local and have classifying spaces", arXiv preprint arXiv:2011.01208 (2020).

[16] C. Córdova and K. Ohmori, "Anomaly Constraints on Gapped Phases with Discrete Chiral Symmetry", Phys. Rev. D **102**, 025011 (2020), arXiv:1912.13069 [hep-th].

[17] Y. Choi, C. Cordova, P.-S. Hsin, H. T. Lam, and S.-H. Shao, "Noninvertible duality defects in 3+1 dimensions", Phys. Rev. D **105**, 125016 (2022), arXiv:2111.01139 [hep-th].

[18] P. Niro, K. Roumpedakis, and O. Sela, "Exploring non-invertible symmetries in free theories", JHEP **03**, 005 (2023), arXiv:2209.11166 [hep-th].

[19] C. Cordova and K. Ohmori, "Quantum Duality in Electromagnetism and the Fine-Structure Constant", (2023), arXiv:2307.12927 [hep-th].

[20] Y. Nagoya and S. Shimamori, "Non-invertible duality defect and non-commutative fusion algebra", (2023), arXiv:2309.05294 [hep-th].

[21] C.-T. Hsieh, Y. Tachikawa, and K. Yonekura, "Anomaly Inflow and p-Form Gauge Theories", Commun. Math. Phys. **391**, 495–608 (2022), arXiv:2003.11550 [hep-th].

[22] A. Kapustin and M. Tikhonov, "Abelian duality, walls and boundary conditions in diverse dimensions", JHEP **11**, 006 (2009), arXiv:0904.0840 [hep-th].