A Lecture on Topological Operators

Kantaro Ohmori

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What is this

vol This is a lecture note prepared for two sets of "intensive lectures":[1]

- at Tohoku University, Oct. 11-13, 2023, and
- at Yukawa Insititute for Theoretical Physics, Kyoto University, Nov. 29-1, 2023.

In this lecture I will try to explain the constructions of topological defects corresponding to generalized symmetries. Due to lack of time and (more significantly) my understanding, the lecture will focus on bosonic systems, and the generalization to fermionic systems is left for the readers/audiences.

> ⚠ Warning
>
> This note is **under construction**, and there are many missed equations, figures, explanations, sections, and *references*.

Prerequisite

- Basic knowledge about scalar field theory and (abelian) gauge theory in path-integral formalism, and
- Knowledge about renormalization group (RG) flows to understand motivations.
- Knowledge about differential form and Stokes's theorem in terms of it.

What is contained and what is not

Other Lectures/Reviews

Recently there has been a surge of lecture notes/ review articles on generalized symmetries. The ones I have noticed are [1–6]. Because this lecture will focus on the fundamental aspects of the topic and will not connect very well with the existent literature (so sorry about that),

[1] In Japan, an "intensive lecture" is a format of a lecture course where a lecturer (usually from another university) gives lectures in consecutive days filling 7-9 slots in usually 3 days.

readers/audiences are strongly encouraged to refer to at least one of them, or something similar.

Also, about conventional symmetries and their anomalies, there are nice old lectures. The one I would particularly recommend is [7].

1 Introduction

1.1 Symmetry

Symmetry plays a crucial role in theoretical physics. In this lecture, we will discuss its application in *quantum field theories* (QFTs). A fundamental aspect of symmetry in QFTs is its preservation along the renormalization group flow. More precisely, when an ultraviolet (UV) theory $\mathcal{T}_{\mathrm{UV}}$ transitions into an infrared theory $\mathcal{T}_{\mathrm{IR}}$, a canonical homomorphism f_{RG} exists from the UV symmetry group G_{UV} to the IR symmetry group G_{IR}:

Symmetry is a cornerstone concept in theoretical physics. Particularly within the context of *quantum field theories* (QFTs). A key principle in QFTs is the preservation of symmetry throughout the renormalization group flow. To elaborate, when an ultraviolet (UV) theory, denoted as $\mathcal{T}_{\mathrm{UV}}$, transitions into an infrared theory, represented as $\mathcal{T}_{\mathrm{IR}}$, a canonical homomorphism f_{RG} is established. This homomorphism maps the UV symmetry group G_{UV} to the IR symmetry group G_{IR}:[^SymRG]

! RG flow homomorphism from UV symmetry to IR symmetry

$$f_{\mathrm{RG}} : G_{\mathrm{UV}} \to G_{\mathrm{IR}}. \tag{1.1}$$

Given this relation, there are two ways of applying symmetry in QFT:]

! RG flow homomorphism from UV symmetry to IR symmetry

$$f_{\mathrm{RG}} : G_{\mathrm{UV}} \to G_{\mathrm{IR}}. \tag{1.2}$$

Here, if the UV theory is a fixed point, G_{UV} should be understood as the one preserved by the deformation triggering the RG flow.

i Property of f_{RG}

The property of f_{RG} depends on what exactly is meant by the IR theory $\mathcal{T}_{\mathrm{IR}}$. If the RG flow is to a lower nonzero energy, and the IR theory $\mathcal{T}_{\mathrm{IR}}$ retains all the (even very massive) degrees of freedom and all the irrelevant interactions, the map f_{RG} is an isomorphism. However, typically one integrates out massive degrees of freedom in the description of

$\mathcal{T}_{\mathrm{RG}}$, in which case some symmetry can decouple and thus f_{RG} can be non-surjective. Also, if one also drops some higher-order interaction terms, or runs the flow to zero energy, there can be an *emergent* symmetry, in which case f_{RG} can be non-injective.

Given this relationship, symmetry in QFT can be applied in two ways:

- UV to IR: Starting with a microscopic model (e.g., a model of elementary particles or electrons in matter), we can use symmetry to constrain or predict what happens on a macroscopic scale.
- IR to UV: Given certain macroscopic phenomena, we can use symmetry to constrain or infer the possible microscopic origins (e.g., inferring the QCD Lagrangian from the hadron spectrum).

1.2 Locality

One of the defining characteristics of symmetry in quantum field theories (QFTs) is its *preservation of locality*. In the context of classical symmetry in fields, this means that the symmetry transformation is local:

$$\phi(x) \mapsto F(\phi(x')), \tag{1.3}$$

In this equation, $F(\phi(x'))$ is a function that relies solely on the *local* value of a field (or a set of fields and its derivatives) at a point x'. When $x' \neq x$, the symmetry involves a *spacetime* symmetry, while when $x' = x$, it is an *internal* symmetry. The preservation of locality in a symmetry underpins the symmetry relation in Equation 1.2, which will be elaborated further in the lecture.

Note

Please note that a QFT, when quantized on a fixed space manifold M, does possess unitary operators that commute with its Hamiltonian, other than those originating from locality-preserving symmetry. In most cases, such unitary operators are irrelevant; an example is the one that multiplies a phase to a specific eigenstate. Therefore, in this lecture, when we refer to a symmetry, we assume that it preserves locality. In fact, we assert that this property is the *last* one to be discarded in the context of generalized symmetry, if ever, due to the invariance under the renormalization group flow. [1]

[1] Also, note that the Colman-Mandula theorem assumes a strong locality-preserving condition; that the symmetry generators act on multi-particle asymptotic states as tensor products.

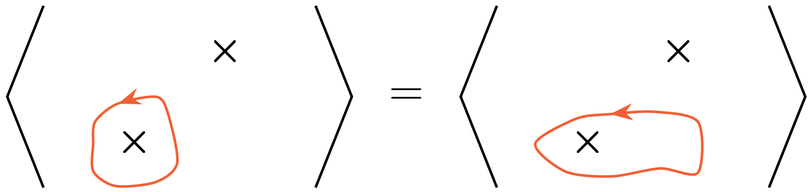

Figure 1.1: Topological operator.

Terminology (Locality-Preserving)

To avoid confusion, it's important to note that locality-preserving symmetry does *not* mean gauge redundancy, which is sometimes labeled as local symmetry. The global – spacetime, or internal – symmetries typically encountered in a QFT textbook all preserve locality.

However, not all symmetries in QFT that preserve locality take the form of Equation 1.3, i.e., a *classical* symmetry. Other types of symmetries exist, such as *topological symmetry* or *quantum symmetry*, which emerge from topologically nontrivial field configurations. Examples include the winding symmetry in 1+1d compact boson, and the monopole symmetries in 2+1d abelian gauge theories. In many cases, a topological/quantum symmetry is mapped to a symmetry of the type Equation 1.3 under a duality, and thus it should also be considered as preserving locality.

From a contemporary standpoint, the universal characterization of locality-preserving symmetries is their correspondence to **topological operators**. A topological operator $\mathcal{D}[W_n]$ in a QFT is an extended operator defined on an n-dimensional submanifold of the spacetime. The correlators that include it should remain invariant under the smooth deformation of the supporting manifold W_n (See Figure 1.1).

The first goal of this lecture is to understand the following correspondence:

Symmetry/Topological Operator Correspondence

$$\begin{aligned}&\text{(Conventional) locality-preserving symmetry}\\ \Longleftrightarrow\ &\text{invertible topological operator of codimension 1.}\end{aligned} \tag{1.4}$$

In this correspondence, the topological operator should be viewed as a generalization of the **Noether charge** for potentially discrete symmetry. More precisely, we consider this correspondence as the right-hand side *defining* the left-hand side. We will explicitly verify that this correspondence/definition reproduces the known symmetries in the case of a classical symmetry in a scalar field theory in Chapter 2, and in the case of abelian gauge theory in Chapter 4. The case of fermions is both intriguing and crucial, but it will be left as an exercise, or a work, for the audience/readers.

Terminology (Topological Defect)

There exists an unfortunate discrepancy in terminology. Outside the realm of generalized symmetry literature, a "topological defect" typically refers to a dynamical object, or its trajectory, viewed as an operator in the IR theory. As an operator in the IR theory, it is *not* necessarily topological in the sense of Figure 1.1. Conversely, within the generalized symmetry literature, "topological defect (operator)" often signifies an extended operator that is inherently topological. To mitigate potential confusion in this lecture, we will adopt the term "topological operator", even when the supporting submanifold of the operator is not within a time-slice.

1.3 Generalized Symmetry

The concept of **generalized (global) symmetry** is fundamentally based on the correspondence in Equation 1.4, as introduced by [8] [2]. This concept expands the traditional notion of symmetry by loosening the constraints on the right-hand side of Equation 1.4. Hence, we *define* generalized symmetry through the following correspondence, which extends Equation 1.4:

Generalized Symmetry/Topological Operator Correspondence

$$\begin{aligned}&\text{Generalized symmetry (in a "usual" QFT)}\\ \overset{\text{def}}{\Longleftrightarrow}\ &\text{General topological operator.}\end{aligned} \tag{1.5}$$

In more specific terms, a generalized symmetry that corresponds to an operator of codimension $p+1$ is referred to as a ***p*-form symmetry**. Meanwhile, a generalized symmetry that corresponds to an operator without an inverse is known as a **non-invertible symmetry** (also referred to as category symmetry or topological symmetry).

In the context of an "unusual" QFT, the topological constraint on the operator in the right-hand side of Equation 1.5 can be further relaxed. This leads to the concept of **subsystem** symmetry, which will be briefly mentioned in **?@sec-trivial-scalar**, but will not be discussed in detail in this lecture.

The table below summarizes the subclasses of generalized symmetry:

[2] The concept of global higher-form symmetry has been previously explored and investigated in the literature, for example, in [9, 10]. Its gauged version was essentially known from [11].

Table 1.1: Subclasses of generalized symmetry and defining properties of their corresponding topological operators.

	p-form	non-invertible	subsystem
codimension	$p+1$		
Invertible?		No	
Topological?			Partially

These subclasses are not mutually exclusive. Therefore, in theory, a 2-form non-invertible subsystem symmetry could exist, for example.

1.4 Contents of the Lecture

FIXME

2 Topological Operators for Classical Symmetry

This section explores topological operators in the context of classical symmetry in scalar field theory.

2.1 Set Up

For specificity, consider a complex scalar field theory with the following Lagrangian (density) on a spacetime M of dimension D:

$$\begin{aligned}\mathcal{L}(\phi) &= -\left(\frac{1}{2}\partial_\mu \phi(x)^* \partial^\nu \phi(x) + V(\phi(x))\right) \mathrm{vol} \\ &= \frac{1}{2}\, d\phi \wedge * \, d\phi - V(\phi(x))\, \mathrm{vol}\,.\end{aligned}$$

In this equation, $*$ represents the Hodge star, $\mathrm{vol} = *\,1 = \prod_{i=1}^{D} dx_i$ is the volume form for the flat space, and $V(\phi)$ is the potential. Then, the action is given by:

$$S[\phi] = \int_M \mathcal{L}(\phi).$$

Consider a symmetry transformation of the scalar field as follows:

$$\phi(x) \mapsto \phi^g(x). \tag{2.1}$$

We assume that this transformation is parametrized by an element g in a group G, which is constant over M and leaves the action invariant:

$$S[\phi] = S[\phi^g]. \tag{2.2}$$

This implies that the Lagrangian is invariant up to a total derivative:

$$\mathcal{L}(\phi^g) = \mathcal{L}(\phi) + ds(\phi, g). \tag{2.3}$$

In this equation, $s(\phi, g)$ is a $(D-1)$-form on M that depends on the constant g and the field ϕ. This $(D-1)$-form $s(\phi, g)$ is subject to the ambiguity coming from shifting by an exact term. Hereafter we use this ambiguity to set $s(\phi, \mathrm{id}) = 0$.

From the consecutive transformation with g_1 and g_2, we have

$$s(\phi^{g_1}, g_2) + s(\phi, g_1) = s(\phi, g_2 g_1) + ds^{(1)}(\phi, g_1, g_2), \tag{2.4}$$

for some $(D-2)$-form $s^{(1)}(\phi, g_1, g_2)$. In particular, setting $g_2 = g_1^{-1}$, we have

$$s(\phi^g, g^{-1}) + s(\phi, g) = ds^{(1)}(\phi, g, g^{-1}). \tag{2.5}$$

Here we list two basic examples of classical symmetries. The first one is the standard U(1) rotation corresponds to the transformation:

$$\phi^g(x) = g\,\phi(x),$$

where $g = e^{\mathrm{i}\,\alpha}$ represents a U(1) phase. The potential $V(\phi)$ may partially break the U(1) rotation into its subgroup $\mathbb{Z}_k$. For example, when $V(\phi) \propto \phi^k + (\phi^*)^k$, the parameter g takes *discrete* values: $g = e^{\mathrm{i}\,\frac{2\pi p}{k}}$, for an integer p.

Moreover, when $V(\phi) = 0$, the action $S[\phi]$ also admits the shift symmetry[1]:

$$\phi^\alpha(x) = \phi(x) + \alpha.$$

In the rest of this section, our goal is to construct the **topological operator** corresponding to these *classical* symmetries.

ℹ Note

This construction can be applied to various types of scalar field theory, such as real and/or multiple scalar fields, provided the kinetic term is standard enough (more details in Tip 2). Also note that the spacetime manifold M and its metric do not need to be flat. The metric's signature is not significant in this lecture, even though we use Euclidean notation.

[1] If we use the form of the Lagrangian $\mathcal{L}' = -\frac{1}{2}\phi\, d * d\phi$, this provides an example where the total derivative in Equation 2.3 is nonzero: $s = -\frac{1}{2}\alpha * d\phi$.

Note

In this lecture, we focus on constructing topological operators that correspond to the *finite* transformation Equation 2.1, as opposed to the conventional approach of considering infinitesimal transformations. This approach allows us to explicitly discuss *discrete* symmetries (and their anomalies) in terms of topological operators, and it also motivates us to consider generalized symmetries.

Tip 1: equivariant cohomology

As the reader might have expected, the structure regarding s and $s^{(1)}$ continues: there are $(D-3)$-forms $s^{(2)}(\phi, g_1, g_2, g_3)$ governing the "*higher associativity*" among $s^{(1)}$, and so on, until $s^{(D-1)}(\phi, g_1, g_2, \cdots, g_D)$ that are 0-forms. Although we consider the free scalar fields these higher surface terms can be taken to be zero and irrelevant, it might not be the case when we consider a sigma model with target X with a non-trivial topology.
Let $s^{(-1)}$ be the metric independent part of $\mathcal{L}$, and $s^{(0)} = s$. The tuple $(s^{(-1)}, s^{(0)}, \cdots, s^{(D-1)})$ defineds an degree-D **equivariant cohomology class s** $\in H^D_G(X)$. In other words G-symmetric topologicall terms in X-target sigma model is classified by the G-equivariant cohomology, when G is a classical symmetry acting on X. The model[2] of the equivariant cohomology realized by $(s^{(-1)}, s^{(0)}, \cdots)$ is called simplicial de Rham cohomology, see e.g. @. Note that this model works for discrete groups, as opposed to the Weyl model based on Lie derivatives that applies to continuous groups.

2.2 Construction of the Topological Operator

As a fundamental example of the correspondence Equation 1.4, we aim to construct the topological operator $U_\alpha[W]$ that corresponds to the transformation Equation 2.1. Specifically, we will construct an operator $U_\alpha[W]$, defined on a codimension-1 submanifold W of the spacetime M, that satisfies the following properties:

Properties of the Symmetry Topological Operator

1. Topological: $U_g[W] = U_g[W']$ if W can be continuously deformed into W' without intersecting other operators.
2. Symmetry action: When a deformation from W to W'' intersects a local operator $\mathcal{O}$, it undergoes the symmetry action specified by g, resulting in another operator $\mathcal{O}^g$.

[2] The word "model" in mathematics, in particular homotopy theory, is used to indecate a concrete construction of an abstract mathematical concept. This usage might go in the opposite direction to the physicists' one: an abstraction of the (quasi-)real world.

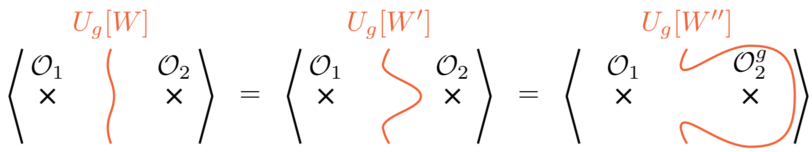

Figure 2.1: The topological operator $U_g[W]$ is designed to be invariant under a continuous deformation and to implement the symmetry action.

3. Noether: When the symmetry group is continuous, we can consider the group element as the infinitesimal deformation of id: $g = \mathrm{id} + \alpha + \mathcal{O}(\alpha^2)$. In this case, the operator $U_g[W]$ is approximated by the Noether charge

$$U_{1+\alpha+\mathcal{O}(\alpha^2)} = 1 + \alpha \int_W *j + \mathcal{O}(\alpha^2), \tag{2.6}$$

where $j = j_\mu \, dx^\mu$ is the Noether current one-form. **FIXME:sign is uncertain..**

$$*j = \left.\frac{\delta \mathcal{L}(\phi^{1+\alpha(x)})}{\delta d\alpha}\right|_{\alpha=0} + \left.\frac{\partial s(\phi, 1+\alpha)}{\partial \alpha}\right|_{\alpha=0} .$$

Note that when W is a time-slice $W = \{t = 0\}$,

$$\int_W *j = \int_{\{t=0\}} j^0 \, d^{D-1}x$$

is precisely the Noether charge as described in any QFT textbook.

Properties 1 and 2 are summarized in Figure 2.1.

The construction is based on the concept of "cutting-and-gluing-with-twist". Initially, we partition the spacetime M into two subregions: $M_D = M_L \cup_W M_R$ with a common boundary W (refer to Figure 2.2. We orient W such that $\partial M_L = -\partial M_R = W$.). We also divide the scalar field ϕ into two fields: $\phi_L(x)$ for $x \in M_L$ and $\phi_R(x)$ for $x \in M_R$. Subsequently, we reconnect the two regions and their respective fields, with a twisted identification:

$$\phi_L|_W = \phi_R^{g^{-1}}|_W.$$

In path-integral, this construction can be implemented as follows:

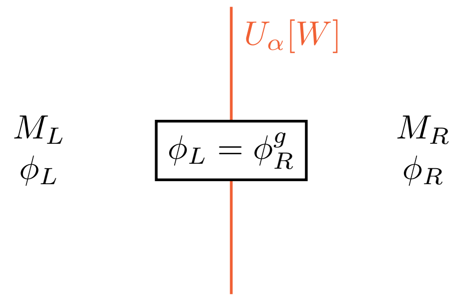

Figure 2.2: The cutting and twisted gluing, implementing the topological operator $U_g[W]$.

! Symmetry Topological Operator for a Classical Symmetry

$$\begin{aligned}\langle U_g[W]\cdots\rangle = &\int \mathcal{D}^{M_L}\phi_L\,\mathcal{D}^{M_R}\phi_R\,\mathcal{D}^W\lambda\cdots \\ &\times\exp\left(-S_L[\phi_L]-S_R[\phi_R]-G_W[\lambda,\phi_L,\phi_R,g]\right)\end{aligned} \tag{2.7}$$

Here, $\mathcal{D}^X$ denotes the measure for the path-integral for a field defined on a submanifold X of the spacetime M. The actions on the submanifold $M_{L,R}$ are written as $S_{L,R}=\int_{M_{L,R}}\mathcal{L}(\phi_{L,R})$, and "$\cdots$" denotes additional operator insertions. The "gluing" action G_W on the submanifold W is defined as

$$G_W[\lambda,\phi_L,\phi_R,g] = -\mathrm{i}\int_W \lambda(\phi_L-\phi_R^{g^{-1}})\,\mathrm{vol}_W + \int_W s(\phi_R^{g^{-1}},g). \tag{2.8}$$

The key point of the above expression is that integrating the Lagrange multiplier λ results in the "delta functional":

$$\int \mathcal{D}^W\lambda\exp\left(\mathrm{i}\int_W \lambda(\phi_L-\phi_R^{g^{-1}})\,\mathrm{vol}_W\right) = \prod_{x\in W}\delta(\phi_L(x)-\phi_R^{g^{-1}}(x)), \tag{2.9}$$

which should implement Figure 2.2. Before studying the operator $U_g[W]$, we should first examine the *trivial* case where the symmetry transformation g is the identity map $g=\mathrm{id}$.

$$\langle \mathrm{id}[W]\cdots\rangle = \langle\cdots\rangle. \tag{2.10}$$

We refer to the codimension-1 operator $\mathrm{id}[W]$ with this property as the **identity wall**. It can also be termed as the **transparent wall** or similar. Upon expanding Equation 2.10, the following equation should be satisfied:

$$\begin{aligned}\int\mathcal{D}^M\phi\exp(-S)\cdots = &\int \mathcal{D}^{M_L}\phi_L\,\mathcal{D}^{M_R}\phi_R\,\mathcal{D}^W\lambda \\ &\times\exp\left(-S_L-S_R+\mathrm{i}\int_W\lambda(\phi_L-\phi_R)\,\mathrm{vol}_W\right)\cdots.\end{aligned} \tag{2.11}$$

The key distinction between providing a field ϕ on M and providing a pair of fields (ϕ_L, ϕ_R) on (M_L, M_R) is that the latter is not required to be continuous across W. On the right-hand side, continuity is enforced by integrating out λ due to Equation 2.9.

Tip 2: Continuity of Fields in "Exotic" QFTs and Subsystem Symmetry

In this context, we assume that the path integral $\int \mathcal{D}^M \phi$ should be over the *continuous* fields. This is because the standard kinetic term would diverge when the field ϕ becomes discontinuous, and thus such configuration does not contribute to the path-integral.
However, this assumption does not hold for QFTs with higher-derivative kinetic terms. Examples of such exotic QFTs (without relativistic symmetry) include tensor gauge theories (refer to [12, 13] for more details). An example of such kinetic term is $(\partial_t \phi)^2 + (\partial_x \partial_y \phi)^2$. In these theories, a field *can* be discontinuous, but some higher derivatives are constrained to scale appropriately with the ratio of the lattice size to the system size. In such scenarios, the construction of the trivial operator should differ.
These QFTs describe what is known as the **fracton phases** of matter, which do not have emergent continuous rotational symmetry in the IR. Moreover, these models typically possess **subsystem symmetries**, whose corresponding operator is not entirely topological. The existence of this new type of symmetry, absent in standard relativistic systems, could be related to this subtlety regarding identity wall .

Let us study the equations of motion (EOMs) on the right-hand side of Equation 2.11. The EOM with respect to λ simply enforces $\phi_L(x) = \phi_R(x)$ for $x \in W$. On the other hand, the surface term of the Euler-Lagrange equation for ϕ_L and ϕ_R yields

$$\left.\frac{\delta \mathcal{L}[\phi_L]}{\delta\, d\phi_L}\right|_W = \lambda\, \mathrm{vol}_W = \left.\frac{\delta \mathcal{L}[\phi_R]}{\delta\, d\phi_R}\right|_W .$$

If W is spacelike, or if we interpret the direction perpendicular to W as the imaginary time in Euclidean signature, this condition ensures the continuity of the canonical momentum across W.

Locality

Equation 2.11 encapsulates the **locality** of the path-integral. We can employ the same procedure to decompose the path-integral $\int \mathcal{D}^M \phi$ on M into path-integrals on local patches, like $\int \prod_i \mathcal{D}^{V_i} \phi_i$ (accompanied by numerous Lagrange multipliers). Here, $\bigcup_i V_i = M$ and $V_i \cap V_j$ has codimension 1 in M if not empty.
In the realm of topological quantum field theory (TQFT), a similar **cutting-and-gluing** axiom is utilized in the Atiyah-Segal formulation of topological quantum field theory. Later, Lurie's cobordism hypothesis [14] established the relationship between this axiom and locality.

A note on fermions

While this lecture does not cover fermions, it's worth noting how they would differ in this context.
In scalar field theory, we enforce the continuity of the "position" variables (in the analytical-mechanical sense) ϕ, and the continuity of the momentum variables follows from the equations of motion (EOM).
However, in a chiral fermion theory, the kinetic term involves only one derivative, making it typically impossible to separate position and momentum variables while preserving Lorentz or global chiral symmetry. Consequently, the "cutting" process would seemingly violate invariance under Lorentz or another symmetry, which can be interpreted as a manifestation of the gravitational and global symmetry anomaly. A precise understanding of this perspective is beyond the scope of this lecture.

2.3 Topological-ness

Let us show the topological-ness of U_g, i.e. the first equation in Figure 2.1, or

$$U_g[W_1]\,\mathrm{id}[W_2] = \mathrm{id}[W_1]U_g[W_2]. \tag{2.12}$$

To show Equation 2.12, we divide the spacetime into three parts: M_L, M_M and M_L, where W separates M_L and M_M, while W' separates M_M and M_R. The relevant part of paht-integral regarding the left hand side of Equation 2.15 reads

$$\int \mathcal{D}^{M_M}\phi_M \mathcal{D}^{W_1}\lambda_1 \mathcal{D}^{W_2}\lambda_2\, e^{-S_M[\phi_M]-G_{W_1}[\phi_L,\phi_M,g]-G_{W_2}[\phi_M,\phi_R,\mathrm{id}]}.$$

Here $S_M = \int_{M_M}\mathcal{L}[\phi_M]$ and we omit the path-integral measures for $\phi_{L,R}$ and also the actions on M_L and M_R for the sake of readability. By changing the variable by $\tilde{\phi}^g_M = \phi_M$, we get the expression

$$\int \mathcal{D}^{M_M}\tilde{\phi}^g_M \mathcal{D}^{W_1}\lambda_1 \mathcal{D}^{W_2}\lambda_2\, e^{-S_M[\tilde{\phi}^g_M]-G_{W_1}[\lambda_1,\phi_L,\phi_M,\mathrm{id}]-\int_{W_1}s(\tilde{\phi}_M,g)-G_{W_2}[\lambda_2,\tilde{\phi}^g_M,\phi_R,\mathrm{id}]}.$$

As M_M is open, the action $S_M[\tilde{\phi}^g_M]$ catches up the surface term:

$$S_M[\tilde{\phi}^g_M] = S_M[\tilde{\phi}_M] - \int_{W_1} s(\tilde{\phi}_M,g) + \int_{W_2} s(\tilde{\phi}_M,g).$$

Using this equation combined with Equation 2.5, we get

$$\int \mathcal{D}^{M_M}\tilde{\phi}^g_M \mathcal{D}^{W_1}\lambda_1 \mathcal{D}^{W_2}\lambda_2\, e^{-S_M[\tilde{\phi}^g_M]-G_{W_1}[\lambda_1,\phi_L,\tilde{\phi}_M,\mathrm{id}]-G_{-W_2}[\lambda_2,\phi_R,\tilde{\phi}_M,g^{-1}]},$$

where $-W_2$ is the orientation reversal of W_2. Then we use the invariance of the path-integral measure of the sclar fields (up to the overall nomarlization):

$$\mathcal{D}^{M_M}\phi_M = \mathcal{D}^{M_M}\phi_M^g \tag{2.13}$$

to get

$$U_g[W_1] = U_{g^{-1}}[-W_2].$$

Lastly, we need to show

$$U_{g^{-1}}[-W] = U_g[W]. \tag{2.14}$$

To show this, we assume that there exists a transformation λ^g of λ that satisfies

$$\lambda(\phi_L - \phi_R) = \lambda^g(\phi_L^g - \phi_R^g)$$

and

$$\mathcal{D}^W\lambda^g = \mathcal{D}^W\lambda.$$

For example, when the action is a U(1) rotation λ^g is the rotation by the inverse, and when the action is a shift of ϕ, we can set $\lambda^g = \lambda$. Then,

$$\begin{aligned}
U_{g^{-1}}[-W] &= \int \mathcal{D}^W\lambda\, e^{-G_{-W}[\lambda^{g^{-1}},\phi_R,\phi_L,g^{-1}]} \\
&= \int \mathcal{D}^W\lambda\, e^{\mathrm{i}\int_{-W}\lambda(\phi_R-\phi_L^g)\,\mathrm{vol}_W - \int_{-W}s(\phi_L^g,g^{-1})} \\
&= \int \mathcal{D}^W\lambda\, e^{\mathrm{i}\int_W\lambda^{g^{-1}}(\phi_L-\phi_R^{g^{-1}})\,\mathrm{vol}_W - \int_W s(\phi_L,g)} \\
&= \int \mathcal{D}^W\lambda^{g^{-1}}\, e^{-G_W[\lambda^{g^{-1}},\phi_L,\phi_R,g]} \\
&= U_g[W].
\end{aligned}$$

2.4 Group Multiplication Law and Junction

For the conventional symmetry, we assume that transformations g forms a group. Correspondingly, the topological operators should satisfy the same multiplication law:

$$U_{g_1}[W_1]U_{g_2}[W_2] = U_{g_1g_2}[W_1] \tag{2.15}$$

where the codimension-1 submanifold W_2 is parralel to W_1 but slightly right to it. One can show this in the same manner we have done for Equation 2.12. In particular, from Equation 2.14 we have

$$U_g[W_1]U_g[-W_2] = U_g[W_1]U_{g^{-1}}[W_2] = U_{\mathrm{id}}[W_1] = \mathrm{id}[W_1]. \tag{2.16}$$

That is, the orientation reversal of U_g is its inverse. The properties Equation 2.15 and Equation 2.16 are the properties of a conventional symmetry and will be relatex in a generalized — specifically non-invertible — symmetry.

For later convenience, here we also introduce the **topological junctions** of topological operators $U_{g_1}, U_{g_2}, U_{g_3}$. A junction is the object connecting multiple extended operators. For the classical symmetry, such an junction is defined by putting three (or more) operators on submanifolds W_1, W_2 and W_3 sharing their boundary X:

$$U_{g_1}[W_1]U_{g_2}[W_2]U_{g_3}[W_3]J_{g_1,g_2,g_3}[X] = \int \prod_{i=1}^{3} \mathcal{D}^{W_i}\lambda_i \, e^{-\sum_i G_{W_i}[\lambda_i,\phi_i,\phi_{i+1},g_i] - \int_X s^{(1)}(\phi_1,g_1,g_2)},$$

which can be proven to be topological when $g_3 = g_1 g_2$. **FIGURE Example of nonzero $s^{(1)}$ or higher?**

2.5 Symmetry Action and Ward-Takahashi identity

Let us turn to the non-trivial operator Equation 2.7 and check the property depicted in Figure 2.1. In order to do it, start from the correlator where U_g is inserted along $W = W_1$ and the trivial operator id along $W'' = W_2$ in Figure 2.1. (Here we only talk about the equality connecting the leftmost and rightmost figures of Figure 2.1. For the middle one the argument is the same. We also renamed the manifolds for the later convenience.) Therefore we split the manifold M into three regions M_L, M_M, M_R, and also the field ϕ into ϕ_L, ϕ_M, ϕ_R. Then to show the equality in Figure 2.1, we perform the change of the variable $\tilde{\phi}_M = \phi_M^{g^{-1}}$:

$$
\begin{aligned}
&\langle \mathcal{O}_1 U_g[W_1] \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 U_g[W_1] \mathcal{O}_2 \, \mathrm{id}[W_2] \rangle \\
&= \int \prod_{i=L,M,R} \mathcal{D}^{M_i} \phi_i \prod_{a=1,2} \mathcal{D}^{W_a} \lambda_a \, e^{-S_L[\phi_L]-S_M[\phi_M]-S_R[\phi_R]} \mathcal{O}_1[\phi_L] \mathcal{O}_2[\phi_M] \\
&\times \exp\left(-G_{W_1}[\lambda_1, \phi_L, \phi_M, g] - G_{W_2}[\lambda_2, \phi_M, \phi_R, \mathrm{id}]\right) \\
&= \int \prod_{i=L,M,R} \mathcal{D}^{M_i} \phi_i \prod_{a=1,2} \mathcal{D}^{W_a} \lambda_a \, e^{-S_L[\phi_L]-S_M[\tilde{\phi}_M^g]-S_R[\phi_R]} \mathcal{O}_1[\phi_L] \mathcal{O}_2[\tilde{\phi}_M^g] \\
&\times \exp\left(-G_{W_1}[\lambda_1, \phi_L, \tilde{\phi}_M, \mathrm{id}] - G_{W_2}[\lambda_2, \tilde{\phi}_M^g, \phi_R, \mathrm{id}]\right) \\
&\times \exp\left(+\int_W s(\phi_L, g)\right) \\
&= \int \prod_{i=L,M,R} \mathcal{D}^{M_i} \phi_i \prod_{a=1,2} \mathcal{D}^{W_a} \lambda_a \, e^{-S_L[\phi_L]-S_M[\tilde{\phi}_M^g]-S_R[\phi_R]} \mathcal{O}_1[\phi_L] \mathcal{O}_2^g[\tilde{\phi}_M] \\
&\times \exp\left(-G_{W_1}[\lambda_1, \phi_L, \tilde{\phi}_M, \mathrm{id}] - G_{-W_2}[\lambda_2, \phi_R, \tilde{\phi}_M, g^{-1}]\right) \\
&\times \exp\left(+\int_{W_1} s(\phi_M, g) + \int_{W_2} s(\phi_M, g^{-1})\right) \\
&= \int \prod_{i=L,M,R} \mathcal{D}^{M_i} \phi_i \prod_{a=1,2} \mathcal{D}^{W_a} \lambda_a \, e^{-S_L[\phi_L]-S_M[\tilde{\phi}_M]-S_R[\phi_R]} \mathcal{O}_1[\phi_L] \mathcal{O}_2^g[\tilde{\phi}_M] \\
&\times \exp\left(-G_{W_1}[\lambda_1, \phi_L, \tilde{\phi}_M, \mathrm{id}] - G_{-W_2}[\lambda_2, \phi_R, \tilde{\phi}_M, g^{-1}]\right) \\
&= \langle \mathcal{O}_1 \, \mathrm{id}[W_1] \mathcal{O}_2 U_{g^{-1}}[-W_2] . \rangle
\end{aligned}
$$

In the second to last equality we used Equation 2.3 and in the last equality we assumed that the path-integral measure is invariant: Lastly, from Equation 2.9 we get $U_{g^{-1}}[-W] = U_g[W]$, so we obtain Figure 2.1, that is

$$
\langle \mathcal{O}_1 U_g[W_1] \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \mathcal{O}_2^g U_g[W_2] \rangle. \tag{2.17}
$$

We note that we can also get the following equation in the same way as above:

$$
\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \rangle = \langle U_g[W_0] \mathcal{O}_1 \mathcal{O}_2 \cdots \rangle, \tag{2.18}
$$

when W_0 encloses a compact region of M and contains no operator. Then, by repeatedly using these equations, we get the **Ward-Takahashi identity** :

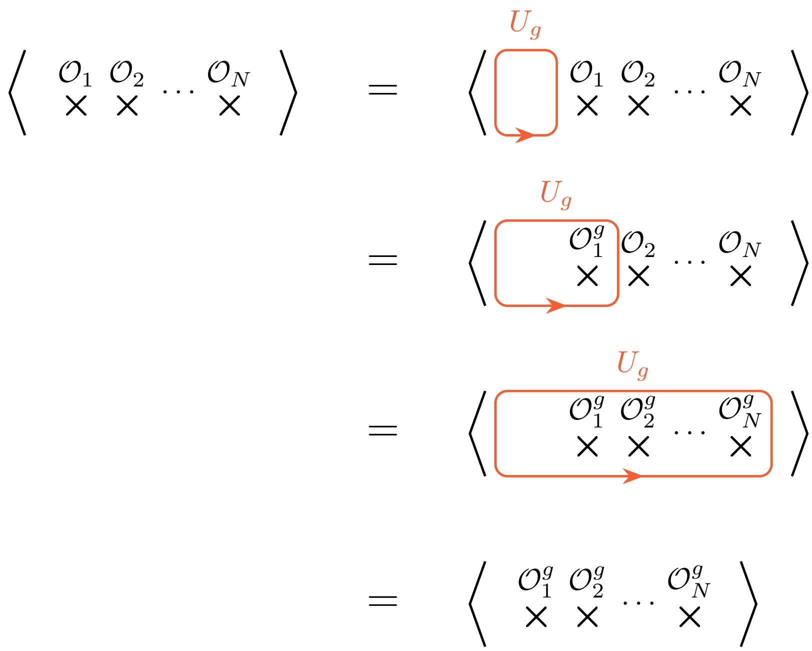

Figure 2.3: Derivation of Ward-Takahashi identity Equation 2.19.

❗ Ward-Takahashi Identity

$$\begin{aligned}\langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_N\rangle &= \langle U_g[W_0]\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_N\rangle \\ &= \langle \mathcal{O}_1^g U_g[W_1]\mathcal{O}_2\cdots\mathcal{O}_N\rangle \\ &= \langle \mathcal{O}_1^g\mathcal{O}_2^g\cdots\mathcal{O}_N^g U_g[W_N]\rangle \\ &= \langle \mathcal{O}_1^g\mathcal{O}_2^g\cdots\mathcal{O}_N^g\rangle,\end{aligned} \tag{2.19}$$

where W_i encloses the operators $\mathcal{O}_1\cdots\mathcal{O}_i$, and at the last step we collapsed $U_g[W_N]$ towards the infinity (or whatever point if M is compact). See also Figure 2.3.

We could get the same result by considering global change of variable $\tilde{\phi} = \phi^{g^{-1}}$ in the path-integral and could avoid topological operator to derive this identity. However, the deriviation in Figure 2.3 applies as long as we have a topological operator, and it acts on local operators as in Figure 2.1. Thus this derivation is applicable to quantum and duality symmetry we will encounter in Chapter 3, and also will be convenient starting point to generalizing the identity to non-invertible case; see Cordova and Ohmori [15] and Copetti et al. [16] for example.

Let us quickly recall that Ward-Takahashi identity means selection rules in correlation functions. For simplicity, let us consider the U(1) action:

$$\mathcal{O}_i^g = g^{q_i}\mathcal{O}_i \tag{2.20}$$

for a phase $g \in \mathrm{U}(1)$. The integer q_i is the U(1) globalcharge of $\mathcal{O}_i$. In this case, Equation 2.19 reads

$$\left(1 - g^{\sum_i q_i}\right) \left\langle \prod_i \mathcal{O}_i \right\rangle = 0,$$

for any phase $g \in \mathrm{U}(1)$. That is, either $\sum_i q_i = 0$ — the total U(1) charge is saturated — or that the correlation function vanishes. This is the familiar selection rule. If the U(1) symmetry is broken down to its $\mathbb{Z}_k$ subgroup, Equation 2.20 holds only for $g \in \mathbb{Z}_k$. This means the total charge $\sum_i q_i$ should vanish only modulo k for the correlator to be nonzero.

Mixed-gravitational anomaly

The symmetry we discuss here does not suffer from anomaly and the Ward-Takahashi idendity (Equation 2.19) follows. If we instead consider the topological operator corresponding to a symmetry with mixed-gravitational anomaly, the invariance of the measure (Equation 2.13) does not necessarily hold when M_M does not have the topology of a ball, and likewise Equation 2.18 can fail when W_0 is not a ball. Thus, on a spacetime with non-trivial topology (other than $\mathbb{R}^4$ or S^4), (Equation 2.19) might fail at the last equality, while (by choosing W_0 to be a ball) other steps goes through. The failure is by a phase depending on topology of the spacetime, and has an interesting consequences of this discussed in [17].

2.6 Relation to Noether Charge

Here we show the relation Equation 2.6 of the topological operator $U_g[W]$ to the conventional Noether charge. This can be done by applying the change of variable $\widetilde{\phi_R} = \phi_R^{\mathrm{id} - \alpha f(x_n)}$ to Equation 2.7, where $f(x_n, \delta)$ is a function of the coordinate x_n perpendicular to W and one positive parameter δ and satisfies $f(0) = 1$ and $f(x_n > \delta) = 0$. Then we take $\delta \to 0$ limit and compare. **FIXME:write equations**

3 Compact Boson

Here we consider *compact* boson, where the field ϕ is real and subject to the identification

$$\phi(x) \cong \phi(x) + 2\pi .$$

We take the Lagrangian to be

$$\mathcal{L} = \frac{R^2}{4\pi} \int_M d\phi \wedge * d\phi \, .$$

We can normalize the field ϕ so that it has the kinetic term with a fixed coefficient, in which case the normalized field has a periodicity radius proportional to R.

The theory has the shift (or "momentum") U(1) symmetry

$$\phi^\alpha = \phi + \alpha$$

with identification of the parameter $\alpha \cong \alpha + 2\pi$. One can add a periodic potential $V(\phi)$ and restrict oneself to a discrete symmetry preserving the potential.

In addition to the shift symmetry, the system has other generalized symmetries:

1. **winding** U(1) $(D-2)$-form symmetry [8], and
2. when $D = 2$ and R^2 is rational, there exists a **T-duality** symmetry that is in general *non-invertible* [18, 19, 20, 21].

The purpose of this section is to understand these generalized symmetries, but before that we review the shift symmetry.

3.1 Trivial Operator and Shift Symmetry Operator

We start from the identity operator $\mathrm{id}[W^{D-1}]$ that cuts and glues the path-integral. The construction is almost the same as before, but when we glue the fields ϕ_L and ϕ_R along W^{D-1}, the gluing can be up to a integer multiple of 2π:

$$\phi_L(x) = \phi_R(x) + 2\pi n$$

with an integer n (assuming W^{D-1} is connected). Therefore, the gluing part of the path-integral is

$$\mathrm{id}[W^{D-1}] = \sum_{n\in\mathbb{Z}} \int \mathcal{D}^{W^{D-1}}\lambda \exp\left(\mathrm{i}\int_{W^{D-1}} \lambda\,(\phi_L - \phi_R - 2\pi n)\right) \mathrm{vol}_{W^{D-1}}\,.$$

We can sum n out to restrict λ to satisfy

$$\int_{W^{D-1}} \lambda\, \mathrm{vol}_{W^{D-1}} \in \mathbb{Z}.$$

An integration over such lambda can be replaced by integration in terms of $(D-2)$-form U(1) gauge field V with

$$dV = 2\pi\lambda\, \mathrm{vol}_{W^{D-1}}\,.$$

Therefore the identity operator can be written as[1]

$$\mathrm{id}[W^{D-1}] = \int \mathcal{D}^{W^{D-1}} V \exp\left(\frac{\mathrm{i}}{2\pi}\int_{W^{D-1}} dV\,(\phi_L - \phi_R)\right). \tag{3.1}$$

p-form gauge field

A p-form gauge field V is *locally* (i.e. in a patch) p-form, but V is not necessarily a *global* p-form and dV rather satisfy $\int_{\Sigma_{p+1}} dV \in 2\pi\mathbb{Z}$ for any $p+1$ dimensional submanifold.
If a reader is not familiar with this concept, one can assume $D = 2, 3$. When $D = 3$, V is a usual (one-form) abelian gauge field, whose magnetic flux is quantized, while when $D = 2$, V is a periodic scalar field. About higher-form gauge field, a motivated reader can consult e.g. Hsieh et al. [22].

Now the topological operator for the shift symmetry is simply

$$U_\alpha^{\mathrm{shift}}[W^{D-1}] = \int \mathcal{D}^{W^{D-1}} V \exp\left(\frac{\mathrm{i}}{2\pi}\int_{W^{D-1}} dV\,(\phi_L - \phi_R + \alpha)\right). \tag{3.2}$$

3.2 Winding Symmetry

The compact boson theory has another topological operator of dimension 1 (codimension $D-1$), which is simply

$$U_\alpha^{\mathrm{winding}}[\gamma^1] = \exp\left(\mathrm{i}\,\alpha\int_{\gamma^1}\frac{d\phi}{2\pi}\right). \tag{3.3}$$

[1] We assume that the integration should be done over *gauge equivalent classes* of V. In other words the gauge fixing procedure is implicit in this lecture.

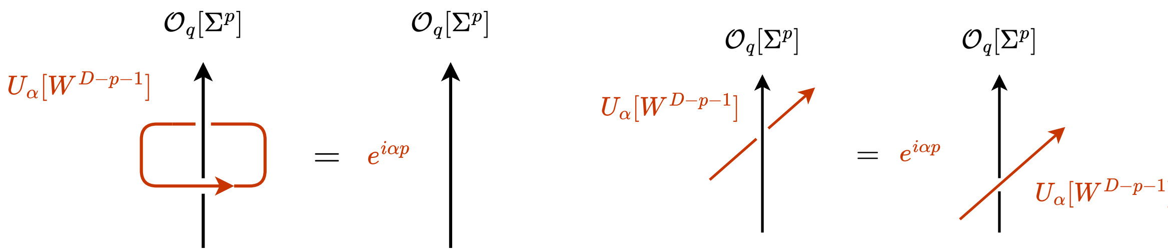

Figure 3.1: Action of p-form symmetry operator $U_\alpha[W^{D-p-1}]$ on a p-dimensional operator $\mathcal{O}_q[\Sigma^p]$ of p-form charge q. We can consider the two types of actions, one is *encircling* action (left), and the other is the *passing* action (right). From the latter we can close the loop on the right of the $\mathcal{O}_q$, and contract the loop on the right hand side, to get the former type of action.

Given the periodicity, the integral $\int_{\gamma^1} d\phi$ for a closed γ^1 takes a value in $2\pi\mathbb{Z}$, and therefore the operator is invariant against deformation of γ^1. This also indicates that the parameter α is 2π periodic. According to Table 1.1, this topological operator should define a U(1) p-form symmetry with $p = (D-2)$, called the **winding symmetry**.

What are the operators *charged* under the symmetry? When $p \geq 1$ ($D \geq 3$), the operator Equation 3.3 cannot act on a local (point) operator, because the one-dimensional submanifold γ^1 can always be deformed one configuration to another without colliding with a point. On the other hand, it can potentially act on a p-dimensional extended operator: Figure 3.1.

However, the construction of a charged operator is a bit tricky. A way of doing it is to first insert the identity operator Equation 3.1, then utilize the field V on the identity operator to defing an operator charged under Equation 3.3. Concretely, the (non-topological) operator with winding charge n can be constructed as

$$\langle \mathcal{O}_n^{\text{winding}}[\Sigma^{D-2}] \rangle = \int \mathcal{D}^{W^{D-1}} V \exp\left(\frac{\mathrm{i}}{2\pi} \int_{W^{D-1}} dV\,(\phi_L - \phi_R) + \mathrm{i}\, n \int_{\Sigma^{D-2}} V \right). \tag{3.4}$$

Here, we take arbitrary W^{D-1} that contains Σ^{D-2}, and the correlator is independent of the choice. The coefficient n has to be an integer for $\int_{\Sigma^{D-1}}$ to be invariant under global gauge transformations.

The operator Equation 3.4 is often defined as a "disorder" operator that enforces singular behavior. Here we see the explicit construction of such by *integrating in* the Lagrange multiplier V on W^{D-1}.

Note

Note that the topological operator Equation 2.7 is also of disorder-type; it enforces a jump of the field across W. It is curious that, for symmetry of field transformation, the charg*ed* operators are direct to construct, while the symmetry topological operator was somewhat tedious to do; and it is opposite for the winding symmetry, or more generally topological charges.

	Field transformation	Topological charge
Charged operator	not disorder	disorder
Topological operator	disorder	not disorder

One aim of this lecture is to demystify the "disorder" operators – they can be explicitly written in terms of correct set of Lagrange multipliers – so that one can talk about the two types of the symmetry in a unified way.

FIXME:derivation of the charge, from EOM of V

Now the Ward-Takahashi identity Equation 2.19 formally follows from the topological-ness of Equation 3.3. Explicitly, we have

$$\langle\prod_i \mathcal{O}^{\text{winding}}_{n_i}(x_i)\rangle = \langle\prod_i e^{\mathrm{i}\,\alpha n_i}\mathcal{O}^{\text{winding}}_{n_i}(x_i)\rangle$$

for any α. And thus the both sides vanish unless $\sum n_i = 0$.

3.3 Mixed Anomaly between Shift and Winding Symmetry

3.3.1 Intersection

Having explicit descriptions of topological operators enables us to directly compute **quantum anomaly** (often called 't Hooft anomaly) of the symmetries. This is because, from a modern perspective, the anomaly is a subtlety arises when symmetry operators collide. Here we observe one example of anomaly – the mixed anomaly between the shift and winding symmetry in the compact boson theory – explicitly from the topological operator perspective. For a general theory about anomaly and topological operator, readers can consult other resources, e.g. Tachikawa [7].

Let us study the intersection of $U^{\text{shift}}_\alpha[W^{D-1}]$ (Equation 3.2) and $U^{\text{winding}}_\beta[\gamma^1]$ (Equation 3.3). **FIXME:figure** The shift symmetry operator divides γ^1 into γ^1_L and γ^1_R, and the winding operator thus now, naively, look like

$$U_\alpha^{\text{shift}} U_\beta^{\text{winding}}[\gamma^1] \overset{\text{naive}}{=} U_\alpha^{\text{shift}} \exp\left(\mathrm{i}\,\beta\left(\int_{\gamma_L^1}\frac{d\phi_L}{2\pi}+\int_{\gamma_R^1}\frac{d\phi_R}{2\pi}\right)\right)$$
$$\sim U_\alpha^{\text{shift}} \exp\left(\mathrm{i}\,\beta/2\pi(\phi_L(x_0)-\phi_R(x_0))\right),$$

where in the second line, $\sim$ refers to the contribution local to the intersection point x_0 (i.e. we ignored the contribution from the other ends of γ_L^1 and γ_R^1 far from W^{D-1}). However, the shift symmetry defect enforces $\phi_L = \phi_R - \alpha \mod 2\pi$, but the local contribution at x_0 *depends* on $\phi_L - \phi_R \mod 2\pi$. Therefore the naive definition of intersected operator is not well-defined (or, it becomes zero if we average over the branches of $\phi_L(x_0) - \phi_R(x_0)$).

A way to define the intersection is to abandon the periodicity of either of α or β. If we regard α to be in $\mathbb{R}$ and not $\mathbb{R}/2\pi\mathbb{Z}$, we can modify the above naive definition to be

$$U_\alpha^{\text{shift}} U_\beta^{\text{winding}}[\gamma^1] \overset{\text{def1}}{=} U_\alpha^{\text{shift}} \exp\left(\mathrm{i}\,\beta\left(\int_{\gamma_L^1}\frac{d\phi_L}{2\pi}+\int_{\gamma_R^1}\frac{d\phi_R}{2\pi}+\{\alpha/2\pi\}\right)\right)$$
$$\sim U_\alpha^{\text{shift}} \exp\left(\mathrm{i}\,\beta[\alpha/2\pi]\right),$$

where $[r]$ is the integer part of a real number r, and $\{r\} = r - [r]$. With this definition, or regularization, of the intersection, α is no longer periodic, but β is kept periodic. One can do other regularizations where α is periodic but β is not, or just abandon both of periodicity, but cannot save both.

This incompatibility of periodicity, or the group multiplication law, when topological operators intersects is the hallmark of anomaly.

3.3.2 Group Cohomology

The incompatibility above is better characterized as a group cohomology (or its generalization to a higher-group). Here we see how to characterize the mixed anomaly of the compact boson in 1+1d as a group cohomology element. (Here we do not delve into the general theory of group cohomology. See e.g. Tachikawa [7]).

In 1+1d, both U_α^{shift} and U_β^{winding} are line operators. Both operators are 2π periodic in its parameters, when intersection between them are absent. A more precise statement that applies even with intersections is that U_α^{shift} and $U_{\alpha+2\pi}^{\text{shift}}$ can be connected with an invertible topological line-changing operator or a *line-isomrphism* operator for short[2] (See also Figure 3.2):

$$\exp\left(\frac{\mathrm{i}}{2\pi}\int^{\square} dV(\phi_L-\phi_R-\alpha)+\mathrm{i}\,V(\square)+\frac{\mathrm{i}}{2\pi}\int_{\square} dV(\phi_L-\phi_R-\alpha+2\pi)\right)$$

[2]In the language of category theory, such an invertible topological line-changing operator defines an *isomorphism* between the lines. In (higher)-category theory it is important to distinguish being isomorphic from being "the same", and in this context physically it means that we remember the total derivative terms on the line.

Figure 3.2: Line-isomorphism operators.

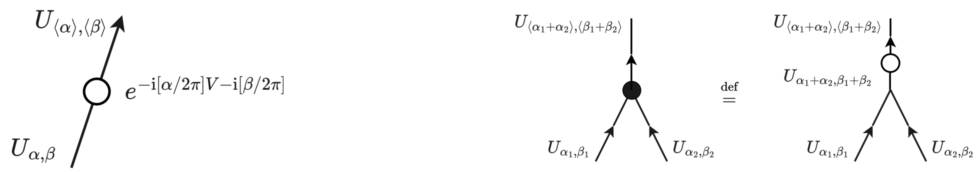

Figure 3.3: The definition of junction fusing two $U_{\alpha,\beta}$'s into one.

where $\square$ denotes the point connecting two line operators. Note that the winding operator $e^{\mathrm{i}\,V(\square)}$ is precisely cancelled when the integral in the last term is evaluated, and thus the junction operator is topological. The existence of its inverse is also manifest.

Likewise, the line isomorphism operator for the winding symmetry operator is

$$\exp\left(\mathrm{i}\,\alpha\int^{\triangle}\frac{d\phi}{2\pi}+\mathrm{i}\,\phi(\triangle)+\mathrm{i}(\alpha+2\pi)\int_{\triangle}\frac{d\phi}{2\pi}\right).$$

We let the composition of the two lines be denoted by $U_{\alpha,\beta}=U^{\text{shift}}_{\alpha}U^{\text{winding}}_{\beta}$. With it, we can define "naive" junction where two line operators U_{α_1,β_1} and U_{α_2,β_2} fusing into $U_{\alpha_1+\alpha_2,\beta_1+\beta_2}$. However, the fused operator in general has parameters in the fundamental domain other than those of the fused ones. Therefore, we normalize the junction by using the $\square$ and $\triangle$ isomorphisms so that the output is $U_{\langle\alpha_1+\alpha_2\rangle,\langle\beta_1+\beta_2\rangle}$, where $\langle\alpha\rangle=2\pi\{\alpha/2\pi\}$; see Figure 3.3. The point is that this isomorphism results in a phase when crossing an intersection of the lines.

Now, we consider the two consecutive junctions, which fuses three lines $U_{\langle\alpha_i\rangle,\langle\beta_i\rangle}$ $(i=1,2,3)$ into $U_{\langle\sum_i\alpha_i\rangle,\langle\sum_j\beta_j\rangle}$; see Figure 3.4.

There are two ways of such fusion as depicted in the figure, and the two are related by a nontrivial phase which is

$$f(g_1,g_2,g_3)=\alpha_1\left[\frac{\langle\beta_2\rangle}{2\pi}+\frac{\langle\beta_3\rangle}{2\pi}\right]+\beta_3\left[\frac{\langle\alpha_1\rangle}{2\pi}+\frac{\langle\alpha_2\rangle}{2\pi}\right], \tag{3.5}$$

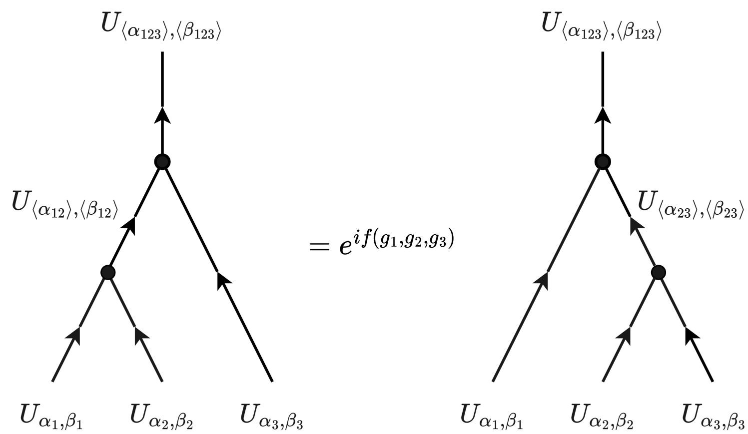

Figure 3.4: The anomalous phase from the F-move. We let $\alpha_{12} = \alpha_1 + \alpha_2$, etc.

where $g_i = (\alpha_i, \beta_i) \in U(1)^2$. The two configurations are gauge-equivalent, yet there is a non-trivial phase relating them. This is the *anomalous phase* of this anomalous $U(1)^2$ symmetry of 1+1d compact boson.

i Group cohomology

Mathematically speaking, the function $f : (U(1)^2)^3 \to \mathbb{R}/2\pi\mathbb{Z}$ is a 3-cocycle on the group $G = U(1)^2$; it satisfies the *pentagon identity*, depicted in **?@fig-pentagon**.
Futher, it is subject to an ambiguity; we can modify the junction by a phase $h(g_1, g_2)$.
This modifies the anomalous phase f as

$$f(g_1, g_2, g_3) \mapsto f(g_1, g_2, g_3) + (\delta h)(g_1, g_2, g_3)$$

where $\delta h : (U(1))^3 \to \mathbb{R}/2\pi\mathbb{Z}$ is the following function called a *coboundary* (FIXME:sign?):

$$(\delta h)(g_1, g_2, g_3) = h(g_1, g_2) + h(g_1 g_2, g_3) - h(g_1, g_2 g_3) - h(g_2, g_3).$$

The third **group cohomology** is an element of the quatient group

$$H^3 = \{f \in \mathrm{Map}(G^3, \mathbb{R}/2\pi\mathbb{Z}) \mid \text{pentagon eq.}\}/\delta(\mathrm{Map}(G^3, \mathbb{R}/2\pi\mathbb{Z})),$$

where $\mathrm{Map}(X, Y)$ is the abelian group of Y valued functions on X [3]. This group cohomology classifies the anomalous phases of 1+1-dimensional bosonic theories with G-symmetry.

[3] When G is continuous, the definition of $\mathrm{Map}(G^n, \mathbb{R}/2\pi\mathbb{Z})$ is subtle. The particular cocycle f in Equation 3.5 is not continuous, so we do not want $\mathrm{Map}(X, Y)$ to mean continuous maps. On the other hand arbitrary discontinuous maps would contain too wild cocycles to be realized in a QFT. In this particular case it

Exercise: $\mathbb{Z}_2$ anomaly

The diagonal $\mathbb{Z}_3$ subgroup of $U(1)^2$ generate by $U_{2\pi/3,2\pi/3}$ is anomalous. Confirm this fact by calculating $f(g_1, g_2, g_3)$ for g_i's in this subgroup, and also proving that no h can satisfy $f = -\delta(h)$. Optionally one might consider some other finite subgroups of $U(1)^2$.

Comparison to more conventional calculation.

The mixed anomaly between the shift and winding symmetry in 1+1d compact boson is usually computed from the two-point function of the corresponding current operators. Such computation corresopnds to *non-flat* and infinitesimal background for the symmetries. On the other hand, inserting topological operator corresponds to *flat* and finite background. The two computations results in the same result is quite non-trivial, and is explained by the Chern-Weil theory.
While the computation based on non-flat background is easier, it is only for continuous symmetries and not obvious how it restricts to a finite subgroup, for example.

3.4 T-duality

The compact boson in $D = 2$-dimensions famously has *T-duality*.

The T-duality is equivalence between the compact boson with radius R and radius $\frac{1}{R}$, and maps the shift (or momentum) symmetry of one side to the winding symmetry of the other side, and vise versa. Note that this makes sense because in $D = 2$ the winding symmetry is a 0-form symmetry.

When the radius is the self-dual radius $R = 1$, the duality becomes a (conventional) $\mathbb{Z}_2$ symmetry, which is a part of the larger emergent SU(2) symmetry.

In this section we study

1. the explicit construction of the T-duality topological interface connecting radius R and $1/R$ theories [23], and
2. its generalization giving **non-invertible** symmetry (i.e. self-dual topological interface) at $R = \sqrt{N}$ for an integer N.

The latter can further be generalized to the case of rational R^2 [19, 20].

seems considering *piecewise continuous* functions is necessary and sufficient. For general cases in general dimensions KO is not sure.

3.4.1 T-duality topological interface

Here we study the T-duality topological interface connecting two compact boson theories in $D = 2$-dimensions; with radius R and radius $1/R$. **FIXME:figure!** When $R = 1$, the interface is a topological operator within the same thoery, and thus defines a symmetry. In this case it turns out a $\mathbb{Z}_2$ symmetry, known to be contained in the larger SU(2) enhanced symmetry.

According to [23], the partition functino involving the interface is

$$\begin{aligned}\langle \mathcal{O}_L \mathcal{I}_1^{\mathrm{T}}[W] \mathcal{O}_R \rangle = & \int \mathcal{D}^{M_L}\phi_L\, \mathcal{D}^{M_R}\phi_R\, \mathcal{O}_L[\phi_L]\mathcal{O}_R[\phi_R] \\ & \exp\left(-\frac{R^2}{4\pi}\int_{M_L} d\phi_L * d\phi_L - \frac{\mathrm{i}}{2\pi}\int_W \phi_L\, d\phi_R - \frac{1}{4\pi R^2}\int_{M_R} d\phi_R * d\phi_R \right).\end{aligned} \tag{3.6}$$

For simplicity let us take $\mathcal{O}_R = 1$, and M_R be a compact region in the spacetime. In this case, we expect the supposed topological-ness implies

$$\langle \mathcal{O}_L \mathcal{I}_1^{\mathrm{T}}[W] \rangle = \langle \mathcal{O}_L \rangle.$$

To show this, we replace the variable ϕ_R with $F_R = d\phi_R$ by

$$\int \mathcal{D}\phi_R = \int \mathcal{D}F_R\, \mathcal{D}\tilde{\phi}_R \exp\left(\frac{\mathrm{i}}{2\pi}\int_{M_R} \tilde{\phi}_R\, dF_R \right),$$

where the periodic scalar $\tilde{\phi}_R$ is the Lagrange multiplier enforcing the closedness and the quantization of $F_R = d\phi_R$. Substituting this into Equation 3.6, the EOMs with respect to F_R are

$$\begin{aligned} \frac{1}{R} * d\phi_R(x) = \mathrm{i}\, d\tilde{\phi}_R(x) \quad & \text{for } x \in M_R \\ \phi_L(x) = \tilde{\phi}_R(x) \quad & \text{for } x \in W. \end{aligned} \tag{3.7}$$

Substituting the former equation to the Lagrangea of ϕ_R, we get

$$\frac{R^2}{4\pi}\int_{M_R} d\tilde{\phi}_R * d\tilde{\phi}_R$$

with is the same as for ϕ_L, while the latter of Equation 3.7 connects ϕ_L and $\tilde{\phi}_R$ along W. As a whole, we recover the path-integral over M resulting in $\langle \mathcal{O}_L \rangle$. For a more general topological-ness about local deformation of W can be derived in the same way but with more letters.

Now, let us set $\mathcal{O}_R = \mathcal{O}_n^{\text{winding}}(x)$ and see how the topological interface acts on the operator. As the operator $\mathcal{O}_n^{\text{winding}}$ in Equation 3.4 is defined on the trivial surface operator, we have to divide the manifold into three parts: M_L, M_M, m_R, separated by W_1 and W_2. As the M_L and field on it is not going to be relevant, we suppress them in the following equations. The calculation goes as:

$$
\begin{aligned}
\mathcal{I}_1^{\mathrm{T}} \mathcal{O}_n^{\text{winding}}(x) &= \int \mathcal{D}\phi_M \, \mathcal{D}\phi_R \, \mathcal{D}^{W_2} V \exp\left(-S_M^{1/R} - S_R^{1/R} - \frac{\mathrm{i}}{2\pi} \int_{W_1} \phi_L \, d\phi_M \right) \\
&\quad \times \exp\left(\frac{\mathrm{i}}{2\pi} \int_{W_2} dV(\phi_M - \phi_R) + \mathrm{i}\, n V(x) \right) \\
&= \int \mathcal{D}F_M \, \mathcal{D}F_R \, \mathcal{D}^{W_2} V \, \mathcal{D}\tilde{\phi}_M \, \mathcal{D}\tilde{\phi}_R \\
&\quad \times \exp\left(-S_M^{1/R} - S_R^{1/R} - \frac{\mathrm{i}}{2\pi} (\int_{W_1} \phi_L \, d\phi_M - \int_{M_M} \tilde{\phi}_M \, dF_M - \int_{M_R} \tilde{\phi}_R \, dF_R) \right) \\
&\quad \times \exp\left(+ \frac{\mathrm{i}}{2\pi} \int_{W_2} V(F_M - F_R) + \mathrm{i}\, n V(x) \right)
\end{aligned}
$$

Now, the EOMs in terms of F_M and F_R state

$$
\begin{aligned}
\frac{1}{R} * d\phi_{M,R} &= \mathrm{i}\, d\tilde{\phi}_{M,R}(x) \quad \text{on} \;\; M_{M,R}, \\
\phi_L &= \tilde{\phi}_M \quad \text{on} \;\; W_1, \\
\tilde{\phi}_M &= V = \tilde{\phi}_R \quad \text{on} \;\; W_2.
\end{aligned}
$$

Thus by substituting back we get **FIXME:check the sign!**

$$
\mathcal{I}_1^{\mathrm{T}} \cdots \mathcal{O}_n^{\text{winding}} = \mathcal{O}_{\pm ?n}^{\text{shift}}.
$$

In addition, let us calculate $(\mathcal{I}_1[W]^{\mathrm{T}})^2$. For this, we insert the defects along parallel submanifolds W_1 and W_2 and take the limit where the separation of the two shrinks. **FIXME:figure** This can be calculated as

$$
\begin{aligned}
\mathcal{I}_1^{\mathrm{T}}[W_1] \mathcal{I}_1^{\mathrm{T}}[W_2] &= \int \mathcal{D}^{M_M} \phi_M \exp(-\mathrm{i} \int_{W_1} \phi_L d\phi_M - \mathrm{i} \int_{W_2} \phi_M d\phi_R) \\
&= \int \mathcal{D}^{W} \phi_M \exp(-\mathrm{i} \int_{W} (\phi_L - \phi_R) d\phi_M) \\
&= \mathrm{id}[W],
\end{aligned}
$$

where in the second line we collide W_1 and W_2, and noted that only the mode of ϕ_M constant along the direction perpendicular to W_1 and W_2 contributes. Therefore, the T-duality interface squares to the identity. In particular, at $R = 1$, the self-interface defines an invertible $\mathbb{Z}_2$ symmetry.

3.4.2 Non-invertible symmetry from T-duality

Choi et al. [18] generalized the Kapustin-Tikhonov T-duality interface by Kapustin and Tikhonov [23] as

$$\begin{aligned}\langle \mathcal{O}_L \mathcal{I}^{\mathrm{T}}_N[W] \mathcal{O}_R \rangle = \int \mathcal{D}^{M_L}\phi_L\, \mathcal{D}^{M_R}\phi_R\, \mathcal{O}_L[\phi_L]\mathcal{O}_R[\phi_R] \\ \exp\left(-\frac{R^2}{4\pi}\int_{M_L} d\phi_L * d\phi_L - \frac{\mathrm{i}\,N}{2\pi}\int_W \phi_L\, d\phi_R - \frac{N^2}{4\pi R^2}\int_{M_R} d\phi_R * d\phi_R\right).\end{aligned} \tag{3.8}$$

This interface is a *self-interface* at $R = \sqrt{N}$. The same procedure as we did for $\mathcal{I}_1$ leads to

$$\begin{aligned}\langle \mathcal{O}_L \mathcal{I}^{\mathrm{T}}_N[W] \rangle = \int \mathcal{D}^{M_L}\phi_L\, \mathcal{D}^{M_R}\phi_R\, \mathcal{D}^W V'\, \mathcal{O}_L[\phi_L] \\ \exp\left(-\frac{R^2}{4\pi}\int_{M_L} d\phi_L * d\phi_L - \frac{\mathrm{i}}{2\pi}\int_W (N\phi_L - \tilde{\phi}_R)\, dV - \frac{R^2}{4\pi N^2}\int_{M_R} d\tilde{\phi}_R * d\tilde{\phi}_R\right).\end{aligned} \tag{3.9}$$

We further rescale the fields as

$$V' = NV$$
$$N\tilde{\phi}'_R = \tilde{\phi}_R \mod 2\pi$$

so that we have

$$\begin{aligned}\langle \mathcal{O}_L \mathcal{I}^{\mathrm{T}}_N[W] \rangle = \mathcal{N} \int \mathcal{D}^{M_L}\phi_L\, \mathcal{D}^{M_R}\phi_R\, \mathcal{D}^W V'\, \mathcal{O}_L[\phi_L] \\ \exp\left(-\frac{R^2}{4\pi}\int_{M_L} d\phi_L * d\phi_L - \frac{\mathrm{i}}{2\pi}\int_W (\phi_L - \tilde{\phi}'_R)\, dV' - \frac{R^2}{4\pi}\int_{M_R} d\tilde{\phi}'_R * d\tilde{\phi}'_R\right).\end{aligned} \tag{3.10}$$

To go from Equation 3.10 to Equation 3.9, we define a $\mathbb{Z}_N$ valued variable $n_{\widetilde{\phi'}_R} = [N\widetilde{\phi'}_R/2\pi]$ mod N so that $\widetilde{\phi'} = \frac{2\pi}{N}(\{\tilde{\phi}/2\pi\} + 2\pi n_{\widetilde{\phi'}_R})$. Then, after substituting it, the sum over $n_{\widetilde{\phi'}}$ enforces V' be divisible by N (that is, the winding of V along W is constrained to be a multiple of N). Thus, we have

$$\langle \mathcal{O}_L \mathcal{I}^{\mathrm{T}}_N[W] \rangle = \mathcal{N}\langle \mathcal{O}_L \rangle.$$

We are not caring enough about the absolute size of path-integral measure to determine the normalization constant $\mathcal{N}$, but will determined in by another mean later.

3.4.2.1 Fusion rule

The product, or *fusion* of the generalized operator is

$$
\begin{aligned}
(\mathcal{J}_N^{\mathrm{T}})^2 &= \int \mathcal{D}^W V \exp(\frac{\mathrm{i}}{2\pi} N \int_W dV(\phi_L - \phi_R)) \\
&= \sum_{n=0}^{N-1} \int \mathcal{D}^W V' \exp(\frac{\mathrm{i}}{2\pi} \int_W dV(\phi_L - \phi_R + 2\pi n/N)) \\
&= \sum_{n=0}^{N-1} U^{\mathrm{shift}}_{2\pi n/N},
\end{aligned} \tag{3.11}
$$

where in the second line we did the chage of variable $NV = V'$, and enforced the divisibility of V' by N by the sum over n. One can also see $\mathcal{J}_N^{\mathrm{T}}[-W] = \mathcal{J}_N^{\mathrm{T}}[W]$ (note that when we consider an operator on W the notion of "left" and "right" also flips), so

$$
\mathcal{J}_N^{\mathrm{T}}[W]\mathcal{J}_N^{\mathrm{T}}[-W] = \sum_{n=0}^{N-1} U^{\mathrm{shift}}_{2\pi n/N}.
$$

This contrasts to the case of conventional symmetry operator, where $U_g[W] = U_{g^{-1}}[W]$ and thus

$$
U_g[W]U_g[-W] = \mathrm{id}[W].
$$

In the case of $\mathcal{J}_N^{\mathrm{T}}$, $N \geq 2$, the fusion with its orientation reversal is not the trivial operator, but a sum. This is a one of hallmarks of **non-invertible** symmetry. $\mathcal{J}_N^{\mathrm{T}}[-W]$, called the **dual** of the original operator, is the closest possible thing to be the "inverse", but it fails to be so. Therefore, the compact boson theory in 1+1d at $R = \sqrt{N}$ has the non-invertible T-duality symmetry.

Here, we can determine that the coefficient in the above equations are correct. This is because that we can insert the one-dimensional operator $(\mathcal{J}_N^{\mathrm{T}})^2$ along the time direction, which should determine the *defect Hilbert* space. On the right hand side, we should have a direct sum of defect Hilbert spaces for the involved invertible symmetry operators. There is no way to take "average" over Hilbert spaces, or divide it by a number, so we can assume the minimal possible coefficient is realized, which is the one in Equation 3.11.

Given Equation 3.11, the coefficient $\mathcal{N}$ in Equation 3.10 is determined by

$$
\langle (\mathcal{J}_N^{\mathrm{T}})^2[W] \rangle = \sum_{n=0}^{N-1} \langle U^{\mathrm{shift}}_{2\pi n/N}[W] \rangle = \mathcal{N}^2,
$$

thus $\mathcal{N} = \sqrt{N}$. This quantity is called the **quantum dimension** of $\mathcal{J}_N^{\mathrm{T}}$.

3.4.2.2 Action on the local operators

What is the action of $\mathcal{I}_N^{\mathrm{T}}$ on the local operator $\mathcal{O}_n^{\text{winding}}$? Naively repeating the procedure in the previous section, one might think

$$\mathcal{I}_N^{\mathrm{T}} \cdot \mathcal{O}_n^{\text{winding}}(x) \stackrel{?}{=} \mathcal{O}_{n/N}^{\text{shift}}(x)$$

however the left hand side, $e^{\mathrm{i}\, n/N \phi}$, does not make sense when N does not divide n as it is incompatible with the periodicity of ϕ. Thus the right hand side vanishes unless $N|n$, and we have

$$\mathcal{I}_N^{\mathrm{T}} \cdot \mathcal{O}_n^{\text{winding}}(x) = \begin{cases} e^{\mathrm{i}\, \frac{n}{N} \phi} & N|n \\ 0 & \text{otherwise.} \end{cases}$$

Note that here we considered the *encircling* action (FIXME:Figure!!). We can instead consider the *passing* action, in which case we have

$$\mathcal{I}_N^{\mathrm{T}}[W_L] \mathcal{O}_n^{\text{winding}}(x) = e^{\mathrm{i}\, \frac{n}{N} \int_{\gamma^x} d\phi} \mathcal{I}_N^{\mathrm{T}}[W_R],$$

$W_{L,R}$ goes through the left/right side of the point x, and the path γ^x starts from a point on W_R and ends at x.

3.4.3 Non-invertible T-duality for a rational R

We can further generalize Equation 3.8 so that it is a self duality at $R^2 = p/q$ (p and q are taken to be coprime) [19, 20]. The construction is

$$\mathcal{I}_{p,q}^{\mathrm{T}}[W] = \int \mathcal{D}^W a\, \mathcal{D}^W b \exp\left(-\frac{\mathrm{i}}{2\pi} \int_W (q\, a\, db + p\, a\, d\phi_L - b\, d\phi_R) \right),$$

where a and b are periodic scalar fields on W. The fusion is

$$(\mathcal{I}_{p,q}^{\mathrm{T}}[W])^2 = \sum_{n_1=0}^{p-1} \sum_{n_2=0}^{q-1} U_{2\pi n_1/N}^{\text{shift}} U_{2\pi n_2/N}^{\text{winding}},$$

and the quantum dimension is

$$\langle \mathcal{I}_{p,q}^{\mathrm{T}}[W] \rangle = \sqrt{pq}.$$

4 Vector

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A Classical Symmetry in Sigma Model and equivariant cohomology

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