Topological Aspects of Symmetry in Low Dimensions

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In this lecture we will study the symmetry and its anomaly in low-dimensional, i.e. 0+1d and 1+1d, quantum field theories. In 0+1-dimensional quantum field theory, a.k.a quantum mechanics, the Wigner's theorem tells that a global symmetry forms a group and acts on the Hilbert (state) space as a projective representation. We will see example with non-trivial projective phases and how it can be related to symmetry protected topological phases in 1+1-dimensions. We then see how the story are generalized/changed in 1+1-dimensional (relativistic) quantum field theory, where the locality of the theory plays an important role. Time permits, we also see how the inclusion of fermions affects the story.

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1 Introduction

Symmetry is a guiding principle in physics. In many case, given a system, you first analyze its symmetry. Or, to model a given phenomena, the symmetry is often be the first clue. Therefore, there have been numerous research on the topic. What is surprising is that, still, in 2022, it is a hot area of research and there are many things to be understood.

1.1 Lecture guide

1.1.1 Usage of this note

This note is provided primarily as a website. The usage is self-explaining but you might find a useful tips if you click the “i” mark in the top navigation bar. Also from the navigation bar one can download the pdf version. (If you want a epub file, I can easily generate it too, so let me know.) **However, the equation number does not much currently between the html and pdf format.**

The parts having * at the tail of its title will probably be skipped in the lecture.

1.1.2 Objective

This lecture aims to be an introduction to the field of symmetry and its anomaly in quantum field theory (QFT). In the first part of the lecture topological aspects of symmetry in quantum mechanics are reviewed, then in the latter part of the lecture we proceed to symmetry in 1+1-dimensional quantum field theory.

1.1.3 Prerequisite

Proficiency in the undergraduate level quantum mechanics and some basic knowledge about quantum field theory and group theory (e.g. what are $SO(3)$, $SU(2)$, $\mathbb{Z}_2$, and so on) are assumed, but (hopefully) not much more. Especially, the first half will focus on quantum mechanics so it is hopefully understandable to even advanced undergraduates.

1.1.4 Useful references

1. Tachikawa [1]: The first half of Yuji’s lecture is about the big framework the most of researchers assume (but not necessarily proven), which I will omit. The second half of Yuji’s will serve as an advanced version of this lecture.

2. Witten [2], Witten [3]: While there are not so much overlap between this lecture by me and these lecture note and paper by E. Witten, and Witten's is a bit more advanced, they are undoubtedly ones of the best entry points to the field.

1.2 Motivation

Why do we care about symmetry and its anomaly in quantum field theory? There are two (closely related) uses cases of symmetry and its anomaly in the study of QFT and its applications:

1. to construction of a model, given observed spectrum or other desired properties, and
2. to constraining possible long range physics from 't Hooft anomaly matching.

An example for the case one is the Standard Model (SM) of the particle physics. The spectrum of the fermions fits into a representation of $SU(3) \times SU(2) \times U(1)$ nicely, and gauging of the symmetry, after including the Higgs boson, magically explains almost all physics that occur in a collider. (The finding of the quark model involving the color symmetry was also mainly from group theory: it was to reproduce the observed $SU(3)$ symmetry among hadrons.)

The quantum anomaly is also very important in construction of the SM: a single family of fermions in SM has chiral spectrum but seemingly miraculous cancellation of the quantum anomaly, which makes the model consistent. This cancellation also leads to the idea of grand unification.

The second case is about the quantum anomaly for *global* symmetry.[1] 't Hooft anomaly matching states that the anomaly should match between the UV and IR of an renormalization group (RG) flow (see Fig. 1). Let G_{UV} and G_{IR} be the symmetry group for the UV and IR theory, respectively. The existence of the RG flow between them in particular means that an isomorphism between them:

$$\phi : G_{\text{UV}} \to G_{\text{IR}}. \tag{1.1}$$

The RG flow also assigns a map between the quantum anomalies, which are a property of symmetries in a local quantum system, for the symmetries G_{UV} and G_{IR}; but this time it is backwards:

$$\phi^* : \mathcal{A}_{\text{IR}} \to \mathcal{A}_{\text{UV}}. \tag{1.2}$$

This map is in a sense linear, in particular $\phi^*(0) = 0$, where $\mathcal{A} = 0$ means that there is no anomaly. Therefore, if you know that $\mathcal{A}_{\text{UV}} \neq 0$, which immediately means $\mathcal{A}_{\text{IR}} \neq 0$.

[1] Some authors, including the author of this lecture note, sometimes use the term "'t Hooft anomaly", to mean a quantum anomaly for a *global* symmetry, to distinguish it from a quantum anomaly involving *gauge* symmetry. However this terminology does not seem to have a historical root (while the term "'t Hooft anomaly matching" is easily justified), so in this lecture KO tries to avoid the terminology to avoid any possible confusion.

In turn, you also conclude that the IR *theory* it self is *not* trivial: you need some degrees of freedom to **match** the anomaly.

This is the power of 't Hooft anomaly matching: that you can say something without analyzing the dynamics about where the theory can flow into. This can be done even if the theory is very hard to analyze, i.e. *strongly coupled*, for example asymptotically free theories. For such theories the anomaly matching (and some generalization) are sometimes the only, or one of the few, analytical tools that one can apply. Thus the 't Hooft anomaly matching is a part of the foundation in the research of strongly coupled quantum systems.

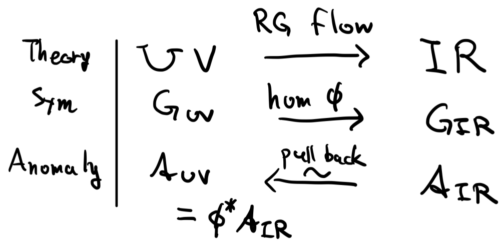

Figure 1: 't Hooft anomaly matching

2 Quantum Anomaly in Quantum Mechanics

In this chapter we learn about quantum anomaly of a symmetry in quantum mechanics, without assuming any kind of locality. Here, locality roughly means that we have a notion of "observables localized in a subregion of the space (or spacetime)." Both quantum field theory (QFT) and a quantum system on lattice are local in this sense, which greatly affect the possible behavior of a symmetry. Here we will see what kind of "topological" phenomena are possible regarding symmetry when the locality is absent. In such case, we often regard that all of the observables are associated to a single point consisting the entire "space", justifying call such a system "0+1-dimensional."

2.1 Basics about quantum mechanics

Let us recall the basics. Given a quantum system, we have a Hilbert space $\mathcal{H}$, in which a state lives. A unit state $|\psi\rangle$ and another $|\psi'\rangle$ describes the same state, which we write as $|\psi\rangle \sim |\psi'\rangle$, if (and only if)

$$|\psi\rangle = e^{\mathrm{i}\alpha}\,|\psi'\rangle \tag{2.1}$$

for some phase α. Then, we define the *ray space* $\mathbb{P}\mathcal{H}$ as the projective space of $\mathcal{H}$:

$$\mathbb{P}\mathcal{H} := S\mathcal{H}/\sim, \tag{2.2}$$

where $S\mathcal{H}$ denotes the space of unit states. By definition, elements in the ray space are one-to-one corresponded to physical states of the system. Given two states $[|\psi\rangle], [|\phi\rangle] \in \mathbb{P}\mathcal{H}$, where $[,]$ denotes the equivalence class in the definition (2.2), the transition probability of the two states are

$$|\langle\psi|\phi\rangle|^2. \tag{2.3}$$

Note that this definition does not depends on the choice of the representatives ψ, ϕ in each class. We put the structure of abelian group to $C^2(G, M)$ induced by the abelian group structure of M.

Sometimes it is convenient to focus on *operators* acting on the states, rather than the states itself. We let the algebra of the (bounded) operators be denoted by $\mathcal{A}$. If the Hilbert space is finite dimensional, i.e. $\mathcal{H} = \mathbb{C}^n$ for some integer n, the algebra is simply the matrix algebra:

$$\mathcal{A} \cong \mathrm{Mat}(\mathbb{C}^n). \tag{2.4}$$

In this lecture we consider the dynamics by a constant Hamiltonian[2]. We use the Heisenberg picture, so the state $|\psi\rangle$ does not develop but an operator $\mathcal{O}$ develops in time as

$$\mathcal{O}(t) = e^{\mathrm{i}Ht}\mathcal{O}(0)e^{-\mathrm{i}Ht}. \tag{2.5}$$

Here we have set the plank constant $\hbar = 1$.

2.2 Wigner's theorem

According to E. Wigner, a **symmetry transformation** U acting on a quantum system is a bijection

$$T : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}, \tag{2.6}$$

that preserves the transition probability (2.3). We call a bijection U on $\mathcal{H}$ is compatible with T if U induces the same action on $\mathbb{P}\mathcal{H}$ as T.

Then, Wigner's theorem states:

Theorem 2.1 (Wigner 1931)**.** *Given a symmetry transformation T, there exists a bijection U_T on $\mathcal{H}$ compatible with T. This U_T is either linear (over $\mathbb{C}$) and unitary, or anti-linear and anti-unitary. If* $\dim\mathcal{H} \geq 2$*, this U is unique up to a overall phase redefinition $U_T \mapsto e^{\mathrm{i}\alpha}U_T$. (When* $\dim\mathcal{H} = 1$*, T is unique, and U_T can be taken either of unitary or anti-unitary one. Once the choice is fixed, it is up to the phase.)*

[2]Instead, one might consider a discrete (finite time) evolution by a unitary operator. Such a system is often called Floquet system. The term originally means that you have an external field periodic in the time. The discrepancy between the continuous and discrete evolutions is an interesting ongoing research topic.

This is why we usually care about unitary operator (like Pauli matrices). The proof can be found somewhere, e.g. Weinberg [4]. We will see examples soon.

We also have an action of T on the algebra of operators $\mathcal{A}$ through this theorem:

$$T \curvearrowright \mathcal{A} \to \mathcal{A} \tag{2.7}$$

$$\mathcal{O} \mapsto U_T \mathcal{O} U_T^\dagger. \tag{2.8}$$

Note that this action is independent of the phase freedom of U_T and thus uniquely defined. (2.5) is a special case of this.

Finally, we call a symmetry T is **preserved** by the Hamiltonian if

$$[U_T, H] = 0. \tag{2.9}$$

2.3 Bloch sphere and projective representation

Let us take a close look at the case of a qubit, i.e. $\mathcal{H} = \mathbb{C}^2$. Pick a basis $|0\rangle$ and $|1\rangle$ and expand a unit state $|\psi\rangle$ as

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + \sin(\theta/2)e^{\mathrm{i}\phi}\,|1\rangle\,, \tag{2.10}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. One can always bring a unit state to this form using the phase ambiguity $|\psi\rangle \sim e^{\mathrm{i}\alpha}\,|\psi\rangle$. Note that when $\theta = 0$ and $\theta = \pi$, the the correspoinding point in $\mathbb{P}\mathcal{H}$ is independent of ϕ. Therefore (θ, ϕ) is the coordinates (polar and azimuthal angles) on S^2, called the Bloch sphere:

$$\mathbb{P}\mathcal{H} \cong \mathbb{CP}^1 \cong S^2. \tag{2.11}$$

The relationship between the polar coordinates on S^2 and the Cartesian coordinates on $\mathbb{R}^3$ is $(x, y, z) = (\sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta)$.

Given two states $|\psi_1\rangle$ and $|\psi_2\rangle$, the transition probability is $|\,\langle\psi_2|\psi_1\rangle\,|^2 = \cos^2(\theta_{12}/2)$, where θ_{12} is the angle between the two corresponding point in the Bloch sphere. Symmetry transformation T is a bijection on Bloch sphere preserving this quantity. Such transformations are one-to-one corresponded to the orthogonal group

$$\{\text{Symmetry transformations on a qubit}\} \cong O(3). \tag{2.12}$$

We can also identify the composition of symmetry transformations as the multiplication of the group $O(3)$. What are the corresponding (anti)unitaries?

2.3.1 Projective phase of $SO(3)$

Let us first study the part $SO(3) \subset O(3)$ that preserves the orientation of the sphere. In particular, the rotation $R_z(\lambda)$ around the z ($\theta = 0$ direction) axis sends

$$R_z(\lambda) : (\theta, \phi) \mapsto (\theta, \phi + \lambda). \tag{2.13}$$

A unitary compatible with this transformation is

$$U'_{R_z(\lambda)} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\lambda} \end{pmatrix}. \tag{2.14}$$

Or, if we can demand that the unitary is also special ($\det U = 1$) by using the phase ambiguity, achieving

$$U_{R_z(\lambda)} = \begin{pmatrix} e^{\mathrm{i}\lambda/2} & 0 \\ 0 & e^{\mathrm{i}\lambda/2} \end{pmatrix}. \tag{2.15}$$

However, this expression has a peculiar feature: that the 2π rotation is mapped to $U_{R_z(\lambda=2\pi)} = -\mathbf{I}_2$, not to the identity matrix $\mathbf{I}_2$! In other words, if we use the special unitary $U_{R_z(\lambda)}$ ($0 \le \lambda < 2\pi$), the composition law of the unitary operators is

$$U_{R_z(\lambda)} U_{R_z(\lambda')} = e^{\mathrm{i}\alpha(\lambda,\lambda')} U_{R_z(\lambda+\lambda' \bmod 2\pi)} \tag{2.16}$$

which is not quite straightforward. The function $\alpha(\lambda, \lambda')$ is called the **projective phase** and in this case it is

$$\alpha(\lambda, \lambda') = \begin{cases} 0 & (\lambda + \lambda' < 2\pi) \\ \pi & (\lambda + \lambda' \ge 2\pi) \end{cases}. \tag{2.17}$$

If one only focus on $R_z(\lambda)$, one can use U' in (2.14), which has the trivial projective phase. However, it is known that we cannot find a good unitaries U_g for all of the group $g \in SO(3)$ acting on the Block sphere. One choice of the general expression is

$$U_{R_{\mathbf{n}}(\lambda)} = \exp(-i\lambda \mathbf{n} \cdot \vec{\sigma}/2). \tag{2.18}$$

The claim is that you cannot come up with a function $\beta : SO(3) \to \mathbb{R}/2\pi\mathbb{Z}$ such that $e^{\mathrm{i}\beta(g)} U_{R_{\mathbf{n}}(\lambda)}$ exactly satisfies the group multiplication law of $SO(3)$ without projective phase. We will prove this in the next section.

When a group G acts on a vector space, but the represented matrices U_g cannot avoid the projective phase, the pair of the vector space and the action is called a **projective representation**. The qubit space $\mathbb{C}^2$ is a projective representation of $SO(3)$. The general presentation mat

2.3.2 $SO(3)$ or $SU(2)$?

You might have confused; you might been taught that a qubit, or a spin, is acted by $SU(2)$, and wondering why I am emphasizing $SO(3)$ instead. The Lie groups $SU(2)$ and $SO(3)$ are closely related: $SU(2)$ is a double cover of $SO(3)$. In other words,

$$SO(3) = SU(2)/Z(SU(2)), \tag{2.19}$$

where $Z(SU(2)) \cong \mathbb{Z}_2$ is the center of $SU(2)$ generated by $-\mathbf{I}_2$. The $SU(2)$ can naturally act on $\mathbb{C}^2$. However, $-\mathbf{I}_2 \in SU(2)$ does not change the *ray* of the state, which is the physical entity. Therefore, more precise statement is that a symmetry of a qubit/spin is $SO(3)$, as Wigner defined it, but to describe the Hilbert space $SU(2)$, that does not suffer from the projective phase, is more convenient. And the projective phase is the **quantum anomaly**: something is not quite right about the symmetry.

Another way of saying the same this is to focus on the algebra $\mathcal{A}$ of observables. The symmetry acts on the algebra of operators by conjugation ((2.7)), and thus the center $-\mathbf{I}_2$ of $SU(2)$ acts trivially. So the symmetry acting on the *observables* is $SO(3)$, not $SU(2)$. And the projective phase, or the double cover $SU(2)$ arises only when we consider the vector space of the states.

2.3.3 Anti-Unitary symmetry*

What happens for the orientation reversing map in $O(3)$, specifically for the reflection

$$\mathrm{Ref} : (\theta, \phi) \mapsto (\theta, -\phi). \tag{2.20}$$

For this we have to assign an *anti*-unitary operator, which is

$$U_{\mathrm{Ref}} = K\mathbf{I}_2, \tag{2.21}$$

where K is the complex conjugation map acting on $\mathbb{C}$. A general element in $O(3)$ $SO(3)$ can be obtained by multiplying a rotation to the reflection. There is no projective phase regarding Ref.

2.3.4 General finite states model*

For general n state system, $\mathcal{H} \cong \mathbb{C}^n$, the projective space is the complex projective space $\mathbb{CP}^n$. The symmetry transformations are

$$\{\text{Symmetry transformations on } \mathbb{C}^n\} \cong PSU(n) \rtimes \mathbb{Z}_2, \tag{2.22}$$

where $PSU(n) = SU(n)/\mathbb{Z}_n$ and $\mathbb{Z}_2$ acts as the charge conjugation. $\mathbb{C}^n$ is a projective representation of $PSU(n)$, while the $\mathbb{Z}_2$ part is realized as an anti-unitary.

2.4 $\mathbb{Z}_2 \times \mathbb{Z}_2$ projective representation

Here we introduce a systematic method in studying the projective phase, using the same example of qubit. For simplicity, we concentrate on the following $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup inside $SO(3)$:

$$\{\mathrm{Id}, R_x(\pi), R_y(\pi), R_z(\pi)\} \subset SO(3). \tag{2.23}$$

Note that $R_x(\pi) = R_y(\pi)R_z(\pi) = R_z(\pi)R_y(\pi)$, so this subset is closed under the multiplication forming the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. To be concise, we rename the generators of this group as $a := R_z(\pi)$ and $b := R_y(\pi)$, then $R_x(\pi) = ab = ba$. The explicit actions of these generators on the coordinates are

$$a \curvearrowright (\theta, \phi) \mapsto (\theta, \phi + \pi) \tag{2.24}$$
$$b \curvearrowright (\theta, \phi) \mapsto (\pi - \theta, \pi - \phi) \tag{2.25}$$

From this and (2.10), we can read the action of a and b on the states as

$$a \curvearrowright [\cos(\theta/2)\,|0\rangle + \sin(\theta/2)e^{\mathrm{i}\phi}\,|1\rangle\,] \mapsto [\cos(\theta/2)\,|0\rangle - \sin(\theta/2)e^{\mathrm{i}\phi}\,|1\rangle\,] \tag{2.26}$$
$$b \curvearrowright [\cos(\theta/2)\,|0\rangle + \sin(\theta/2)e^{\mathrm{i}\phi}\,|1\rangle\,] \mapsto [\sin(\theta/2)\,|0\rangle - \cos(\theta/2)e^{-\mathrm{i}\phi}\,|1\rangle\,] \tag{2.27}$$
$$\sim [\sin(\theta/2)e^{\mathrm{i}\phi}\,|0\rangle - \cos(\theta/2)\,|1\rangle\,]. \tag{2.28}$$

The last line is the equivalence between the states by the overall phase $e^{\mathrm{i}\phi}$. These transformations can be easily realized by the unitary matrices

$$U_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3, \quad U_b' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma_2. \tag{2.29}$$

However, the latter matrix squares to $-\mathbf{I}_2$, although $b^2 = 1$. To fix this, one can instead use

$$U_b = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \sigma_2. \tag{2.30}$$

However, an issue remains: although $ab = ba = R_x(\pi)$, we have $U_aU_b = -U_bU_a$. This discrepancy between the multiplication law among group elements and their corresponding unitaries is the projective phase. If we set $U_{ab} = U_bU_a = \sigma_1$, the multiplication law among U_g, $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ is

$$U_gU_{g'} = e^{\mathrm{i}\alpha(g,g')}U_{gg'}, \tag{2.31}$$

where in this example the projective phase is

$$\alpha(a, b) = \pi, \quad \alpha(g, g') = 0 \text{ (otherwise)}. \tag{2.32}$$

Recall that we have the room to redefine each U_g by a phase. Can we use this freedom to eliminate the projective phase? Let us say the we redefine U_g by phase $\beta(g)$:

$$U_g \mapsto e^{\mathrm{i}\beta(g)}U_g. \tag{2.33}$$

However, such phase rotation cannot eliminate the commutator $[U_a, U_b]$, which has to be zero if $\alpha = 0$.

When a projective phase cannot be removed by the phase rotation of the unitaries, the phase is called (cohomologically) nontrivial. The phase α in (2.32) is nontrivial.

One significance of the nontrivial projective phase is that it prohibits a one-dimensional representatoin: in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ example, since $[U_a, U_b] \neq 0$ (even though the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is abelian), the such operators cannot act on a one-dimensional space. Note that for a linear representation one always has the *trivial* representation; with a nontrivial projective phase, there is no such thing.

This in turn put a restriction on the possible Hamiltonian. Assume that the Hamiltonian on the qubit preserves the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. If the two energy were distinct, each state should consist a one-dimensional representation with the projective phase, which contradicts with the above observation. Hence the projective action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is enough to ensure that the two states are degenerate.

Now, we also see that the entire group $SO(3)$ acting on the qubit also has a nontrivial projective phase. This is because other elements of $SO(3)$ does not help to resolve the projective phase that $\mathbb{Z}_2 \times \mathbb{Z}_2$ has.

2.5 Charged particle on the Aharonov-Bohm ring

What is the use of the projective phase? It provides a way to find a qubit : if you can cook up a system with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ with the nontrivial projective phase — or **quantum anomaly** —, you are guaranteed to have (at least) 2-fold degeneracy.

Let us see how this happens in an example with a infinite dimensional Hilbert space. Here we consider a charged free particle confined in a ring S^1 with radius R. We put the magnetic flux Φ through the ring. The Hamiltonian is

$$H = \frac{1}{2m}(p_x + \frac{e}{c}A_x)^2, \tag{2.34}$$

where x is the coordinate on S^1 with the identification $x \sim x + 2\pi R$, $p_x = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$, and $A_x = \Phi/(2\pi R)$ is the vector potential along the ring. We rewrite this as

$$H = \frac{\hbar^2}{2mR^2}\Big(\hat{n} + \frac{\theta}{2\pi}\Big)^2, \tag{2.35}$$

were $\hat{n} = \frac{Rp_x}{\hbar}$ and $\theta = \frac{e\Phi}{\hbar c}$.

A wave function $\psi(x)$ can be expanded by the periodic exponential function $\psi_n(x) \propto e^{\mathrm{i}n\,x/R}$, $n \in \mathbb{Z}$. The corresponding state $|n\rangle$ is the eigenstate of the operators $\hat{n}$ and

H:[3]

$$\hat{n}\,|n\rangle = n\,|n\rangle\,, \quad H\,|n\rangle = E_n(\theta)\,|n\rangle = \frac{\hbar^2}{2mR^2}(n+\frac{\theta}{2\pi})^2\,|n\rangle\,. \tag{2.36}$$

This system has the symmetry shifting x as $x \mapsto x + \alpha R$ with a 2π-periodic parameter α. The unitary corresponding to this symmetry can be directly read off from the wave function as[4]

$$U_\lambda\,|n\rangle = e^{\mathrm{i}\lambda n}\,|n\rangle\,. \tag{2.37}$$

There are other symmetries, depending on θ. For example, when $\theta = 0$, there is a $\mathbb{Z}_2$ flipping x: $x \mapsto -x$, $|n\rangle \mapsto |-n\rangle$. A more interesting case is $\theta = \pi$, then the $|n\rangle \mapsto |n\rangle$ action is not a symmetry of the Hamiltonian, but the following is preserved:

$$U_{\mathbb{Z}_2}\,|n\rangle = |-n-1\rangle\,. \tag{2.38}$$

Now the multiplication between U_λ and $\mathbb{Z}_2$ is

$$U_\lambda U_{\mathbb{Z}_2} = e^{-\mathrm{i}\lambda} U_{\mathbb{Z}_2} U_{-\lambda}. \tag{2.39}$$

Note that in right hand side the parameter λ of U_λ is flipped, so the two symmetry forms the semidirect product $U(1) \rtimes \mathbb{Z}_2$. On top to that, we oberserve the projective phase $e^{-\mathrm{i}\lambda}$. If we focus on the subgroup $\mathbb{Z}_2^{\text{shift}}$ $(\lambda = \pi)$ subgroup of the $U(1)$, we have the subgroup $\mathbb{Z}_2^{\text{shift}} \times \mathbb{Z}_2$ of $U(1) \rtimes \mathbb{Z}_2$ and the projective phase $e^{-\mathrm{i}\pi} = -1$ is the same phase we observed in Section 2.4!

Recall that, in Section 2.4, we have proven that there is no one-dimensional representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ with the nontrivial projective phase. Therefore *every* eigenspace of H at $\theta = \pi$ has dimension at least two. This can be easily verified because $E_n(\pi) = E_{-n-1}(\pi)$. Therefore, in particular, the ground state has exact two-fold degeneracy that you might use as a qubit.[5]

Ok, but we could find the degeneracy without discussing the projective phase. The real advantage of the quantum anomaly is that it shows the degeneracy is *robust*; this is the result of the symmetry and its quantum anomaly, so, as long as they are preserved, the degeneracy cannot be lifted. Also, the quantum anomaly is a *discrete* thing and therefore it cannot be changed by continuous deformation. (We will discuss about his a bit more in the next section.) So, even if we deform the Hamiltonian as

$$H' = \frac{\hbar^2}{2mR^2}(\hat{n}+\frac{1}{2})^2 + \mathfrak{R}(e^{\mathrm{i}mx/R} + e^{\mathrm{i}(-m-1)x/R}), \tag{2.40}$$

[3] $E_n(\theta+2\pi) = E_{n+1}(\theta)$ and thus the spectrum is unchanged by the 2π-shift of θ. This is closely related to the Dirac quantization of magnetic charge; $\theta = 2\pi$ is the flux created by the Dirac monopole.

[4] If one does the 2π-shift of theta adiabatically, the spectrum is unchanged as noticed in the above footnote, but the $U(1)$ eigenvalue n carried by the state is shifted by 1. This (or something similar) is called Thouless pump. Indeed this is yet *another quantum anomaly*, "anomaly involving parameter space". See Córdova et al. [5] for more detail.

[5] I believe this is called the flux qubit, but do not trust me, I'm not an expert on quantum computing.

we know that there is a degeneracy, although in this system cannot be easily solved. This is the power of quantum anomaly: it tells us something nontrivial *without solving the system*! [6]

2.6 Group cohomology

Let us formalize what we learned above for a general group G. We assume that for each $g \in G$, we have a symmetry transformation T_g acting on $\mathbb{P}\mathcal{H}$ as symmetry transformations satisfying the multiplication law:

$$T_g T_{g'} = T_{gg'} \tag{2.41}$$

By Wigner's theorem, for each g, we also have U_g. The multiplication law of T_g only guarantees

$$U_g U_{g'} = e^{\mathrm{i}\alpha(g,g')} U_{gg'}, \tag{2.42}$$

for some phase $\alpha(g, g')$, because the correspondence between T_g and U_g is only up to a phase. Here we introduce the terminology from algebraic topology. A map from n-th power of G to some abelian group M is called M-valued n-**cochain**, denoted by

$$C^n(G, M) := Map(G^n, M). \tag{2.43}$$

Note that a cochain is not demanded to be a group homomorphism; it is just a map. α is a $\mathbb{R}/2\pi\mathbb{Z}$ ($\cong U(1)$) valued 2-cochain. In this lecture, M is almost always $\mathbb{R}/2\pi\mathbb{Z}$, so we will not explicitly declare the value domain onwards.

The 2-cochain α is actually not completely arbitrary. The constraint comes form the associativity of the unitary operators: $(U_{g_1} U_{g_2}) U_{g_3} = U_{g_1}(U_{g_2} U_{g_3})$, which leads

$$\delta_3\alpha(g_1, g_2, g_3) := -\alpha(g_1, g_2) - \alpha(g_1 g_2, g_3) + \alpha(g_1, g_2 g_3) + \alpha(g_2, g_3) = 0. \tag{2.44}$$

Here we have defined the derivative $\delta_3 : C^2(G) \to C^3(G)$ on 2-cochains. A 2-cochain satisfies this condition is called a cocycle, and the set of cocycle is denoted by[7]

$$Z^2(G) := \mathrm{Ker}(\delta_3) = \{\alpha \in C^2(G) \mid \delta_3\alpha = 0\}. \tag{2.45}$$

Exercise 2.1. Show that the particular cochain $\alpha \in C^2(\mathbb{Z}_2 \times \mathbb{Z}_2)$ defined in (2.32) is indeed a cocycle.

[6] Probably what is more useful in practice (i.e. when you devise a quantum computer) is the "anomaly involving parameter space", briefly discussed in the footnote above. It can show that , with *arbitrary* potential $V(x)$, you can find a degeneracy if you tune θ. (It does not tell which θ you have to tune in, though.)

[7] The derivative δ_{n+1} for general n-cochain can be defined, but we will postpone this until next chapter, other than δ_2 which is defined just below.

Not all the (nonzero) elements of $C^2(G)$ are interesting: some of them might be eliminated by redefinition of the unitary operators U_g:

$$U_g \mapsto e^{\mathrm{i}\beta(g)} U_g, \tag{2.46}$$

with $\beta \in C^1(G)$. The phase $\alpha(g, g')$ gets shifted by this redefinition as:

$$\alpha(g_1, g_2) \mapsto \alpha(g_1, g_2) - \delta_2\beta(g_1, g_2) := \alpha(g_1, g_2) + \beta(g_1 g_2) - \beta(g_1) - \beta(g_2). \tag{2.47}$$

The map $\delta_2 : C^1(G) \to C^2(G)$ is also called the derivative. An image of δ_2 is called a coboudnary, denoted by

$$B^2(G) := \mathrm{Im}(\delta_2) = \{\delta_2\beta \mid \beta \in C^1(G)\}. \tag{2.48}$$

The crucial feature of the derivative is that it vanishes when composed:

$$\delta_3 \circ \delta_2 = 0. \tag{2.49}$$

Therefore we have $B^2(G) \subset Z^2(G)$. Now, interesting projective phases are those not in $B^2(G)$. This motivates us the define the *group cohomology group*[8]

$$H^2(G) := Z^2(G)/B^2(G). \tag{2.50}$$

In this lecture we do not have time to learn how to compute the cohomology group, but the examples we have seen are

$$H^2(SO(3)) \cong \mathbb{Z}_2, \tag{2.51}$$

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_2. \tag{2.52}$$

Exercise 2.2. Prove (2.49).

Exercise 2.3. Show that $H^2(\mathbb{Z}_2) = 0$. (If you want, also calculate $H^2(\mathbb{Z}_2, \mathbb{Z}_2)$.)

2.7 Anomalous symmetry and 1D SPT

2.8 Anomalies for fermions *?

2.9 Kitaev Chain *?

References

[1] Yuji Tachikawa. *Lecture on anomalies and topological phases*. 2019. URL: https://member.ipmu.jp/yuji.tachikawa/lectures/2019-top-anom/.

[8] Do not confuse this with the cohomology group of the group as a topological space. The two, the group cohomology of a group and the geometric cohomology of the same, can be denoted by the same symbol $H(G)$, but they are different. They are confusing especially when G is continuous. Rather, the group cohomology is identified with the (simplicial/cellular) cohomology of an infinite-dimensional topological space called the classifying space BG.

[2] Edward Witten. “Three lectures on topological phases of matter”. In: *Riv. Nuovo Cim.* 39.7 (2016), pp. 313–370. DOI: 10.1393/ncr/i2016-10125-3. arXiv: 1510.07698 [cond-mat.mes-hall].

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