

Quantum Anomalies as Projective Phases

Kantaro Ohmori, University of Tokyo kant.ohmori_at_gmail.com

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In this lecture we will study the symmetry and its anomaly in low-dimensional, i.e. 0+1d and 1+1d, quantum field theories. In 0+1-dimensional quantum field theory, a.k.a quantum mechanics, the Wigner’s theorem tells that a global symmetry forms a group and acts on the Hilbert (state) space as a projective representation. We will see example with non-trivial projective phases and how it can be related to symmetry protected topological phases in 1+1-dimensions. We then see how the story are generalized/changed in 1+1-dimensional (relativistic) quantum field theory, where the locality of the theory plays an important role. This lecture aims to formalize quantum anomalies from Hamiltonian perspective, while a conventional approach usually heavily relies on path-integral perspectives.

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1 Introduction

Symmetry is a guiding principle in physics. In many case, given a system, you first analyze its symmetry. Or, to model a given phenomena, the symmetry is often be the first clue. Therefore, there have been numerous research on the topic. What is surprising is that, still, in 2022, it is a hot area of research and there are many things to be understood.

1.1 Lecture guide

1.1.1 Usage of this note

This note is provided primarily as a website. The usage is self-explaining but you might find a useful tips if you click the “i” mark in the top navigation bar. Also from the navigation bar one can download the pdf version. (If you want a epub file, I can easily generate it too, so let me know.) **However, the equation number does not much currently between the html and pdf format.**

The parts having * at the tail of its title will probably be skipped in the lecture.

1.1.2 Objective

This lecture aims to be an introduction to the field of symmetry and its anomaly in quantum field theory (QFT). In the first part of the lecture topological aspects of symmetry in quantum mechanics are reviewed, then in the latter part of the lecture we proceed to symmetry in 1+1-dimensional quantum field theory.

1.1.3 Prerequisite

Proficiency in the undergraduate level quantum mechanics and some basic knowledge about quantum field theory and group theory (e.g. what are $SO(3)$, $SU(2)$, \mathbb{Z}_2 , and so on) are assumed, but (hopefully) not much more. Especially, the first half will focus on quantum mechanics so it is hopefully understandable to even advanced undergraduates.

1.1.4 Useful references

1. Tachikawa [1]: The first half of Yuji’s lecture is about the big framework the most of researchers assume (but not necessarily proven), which I will omit. The second half of Yuji’s will serve as an advanced version of this lecture.

2. Witten [2], Witten [3]: While there are not so much overlap between this lecture by me and these lecture note and paper by E. Witten, and Witten's is a bit more advanced, they are undoubtedly ones of the best entry points to the field.

1.2 Motivation

Why do we care about symmetry and its anomaly in quantum field theory? There are two (closely related) uses cases of symmetry and its anomaly in the study of QFT and its applications:

1. to construction of a model, given observed spectrum or other desired properties, and
2. to constraining possible long range physics from 't Hooft anomaly matching.

An example for the case one is the Standard Model (SM) of the particle physics. The spectrum of the fermions fits into a representation of $SU(3) \times SU(2) \times U(1)$ nicely, and gauging of the symmetry, after including the Higgs boson, magically explains almost all physics that occur in a collider. (The finding of the quark model involving the color symmetry was also mainly from group theory: it was to reproduce the observed $SU(3)$ symmetry among hadrons.)

The quantum anomaly is also very important in construction of the SM: a single family of fermions in SM has chiral spectrum but seemingly miraculous cancellation of the quantum anomaly, which makes the model consistent. This cancellation also leads to the idea of grand unification.

The second case is about the quantum anomaly for *global* symmetry.¹ 't Hooft anomaly matching states that the anomaly should match between the UV and IR of an renormalization group (RG) flow (see Fig. 1). Let G_{UV} and G_{IR} be the symmetry group for the UV and IR theory, respectively. The existence of the RG flow between them in particular means that an isomorphism between them:

$$\phi : G_{UV} \rightarrow G_{IR}. \quad (1.1)$$

The RG flow also assigns a map between the quantum anomalies, which are a property of symmetries in a local quantum system, for the symmetries G_{UV} and G_{IR} ; but this time it is backwards:

$$\phi^* : \mathcal{A}_{IR} \rightarrow \mathcal{A}_{UV}. \quad (1.2)$$

This map is in a sense linear, in particular $\phi^*(0) = 0$, where $\mathcal{A} = 0$ means that there is no anomaly. Therefore, if you know that $\mathcal{A}_{UV} \neq 0$, which immediately means $\mathcal{A}_{IR} \neq 0$.

¹Some authors, including the author of this lecture note, sometimes use the term “t Hooft anomaly”, to mean a quantum anomaly for a *global* symmetry, to distinguish it from a quantum anomaly involving *gauge* symmetry. However this terminology does not seem to have a historical root (while the term “t Hooft anomaly matching” is easily justified), so in this lecture KO tries to avoid the terminology to avoid any possible confusion.

In turn, you also conclude that the IR *theory* it self is *not* trivial: you need some degrees of freedom to **match** the anomaly.

This is the power of 't Hooft anomaly matching: that you can say something without analyzing the dynamics about where the theory can flow into. This can be done even if the theory is very hard to analyze, i.e. *strongly coupled*, for example asymptotically free theories. For such theories the anomaly matching (and some generalization) are sometimes the only, or one of the few, analytical tools that one can apply. Thus the 't Hooft anomaly matching is a part of the foundation in the research of strongly coupled quantum systems.

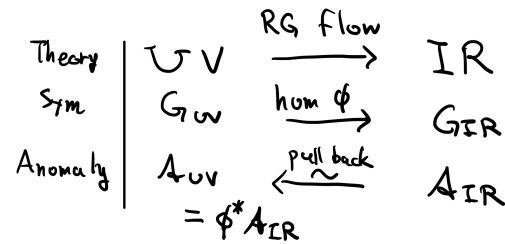


Figure 1: 't Hooft anomaly matching

2 Quantum Anomaly in Quantum Mechanics

In this chapter we learn about quantum anomaly of a symmetry in quantum mechanics, without assuming any kind of locality. Here, locality roughly means that we have a notion of “observables localized in a subregion of the space (or spacetime).” Both quantum field theory (QFT) and a quantum system on lattice are local in this sense, which greatly affect the possible behavior of a symmetry. Here we will see what kind of “topological” phenomena are possible regarding symmetry when the locality is absent. In such case, we often regard that all of the observables are associated to a single point consisting the entire “space”, justifying call such a system “0+1-dimensional.”

2.1 Basics about quantum mechanics

Let us recall the basics. Given a quantum system, we have a Hilbert space \mathcal{H} , in which a state lives. A unit state $|\psi\rangle$ and another $|\psi'\rangle$ describes the same state, which we write as $|\psi\rangle \sim |\psi'\rangle$, if (and only if)

$$|\psi\rangle = e^{i\alpha} |\psi'\rangle \quad (2.1)$$

for some phase α . Then, we define the *ray space* $\mathbb{P}\mathcal{H}$ as the projective space of \mathcal{H} :

$$\mathbb{P}\mathcal{H} := S\mathcal{H} / \sim, \quad (2.2)$$

where $S\mathcal{H}$ denotes the space of unit states. By definition, elements in the ray space are one-to-one corresponded to physical states of the system. Given two states $[|\psi\rangle], [|\phi\rangle] \in \mathbb{P}\mathcal{H}$, where $[,]$ denotes the equivalence class in the definition (2.2), the transition probability of the two states are

$$|\langle\psi|\phi\rangle|^2. \quad (2.3)$$

Note that this definition does not depends on the choice of the representatives ψ, ϕ in each class. We put the structure of abelian group to $C^2(G, M)$ induced by the abelian group structure of M .

Sometimes it is convenient to focus on *operators* acting on the states, rather than the states itself. We let the algebra of the (bounded) operators be denoted by \mathcal{A} . If the Hilbert space is finite dimensional, i.e. $\mathcal{H} = \mathbb{C}^n$ for some integer n , the algebra is simply the matrix algebra:

$$\mathcal{A} \cong \text{Mat}(\mathbb{C}^n). \quad (2.4)$$

In this lecture we consider the dynamics by a constant Hamiltonian². We use the Heisenberg picture, so the state $|\psi\rangle$ does not develop but an operator \mathcal{O} develops in time as

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}. \quad (2.5)$$

Here we have set the plank constant $\hbar = 1$.

²Instead, one might consider a discrete (finite time) evolution by a unitary operator. Such a system is often called Floquet system. The term originally means that you have an external field periodic in the time. The discrepancy between the continuous and discrete evolutions is an interesting ongoing research topic.

2.2 Wigner's theorem

According to E. Wigner, a **symmetry transformation** T acting on a quantum system is a bijection

$$T : \mathbb{P}\mathcal{H} \rightarrow \mathbb{P}\mathcal{H}, \quad (2.6)$$

that preserves the transition probability (2.3). We call a bijection U on \mathcal{H} is compatible with T if U induces the same action on $\mathbb{P}\mathcal{H}$ as T .

Then, Wigner's theorem states:

Theorem 2.1 (Wigner 1931). *Given a symmetry transformation T , there exists a bijection U_T on \mathcal{H} compatible with T . This U_T is either linear (over \mathbb{C}) and unitary, or anti-linear and anti-unitary. If $\dim\mathcal{H} \geq 2$, U_T is unique up to a overall phase redefinition $U_T \mapsto e^{i\alpha}U_T$. (When $\dim\mathcal{H} = 1$, T is unique, and U_T can be taken either of unitary or anti-unitary one. Once the choice is fixed, it is up to the phase.)*

This is why we usually care about unitary operator (like Pauli matrices). The proof can be found somewhere, e.g. Weinberg [4]. We will see examples soon.

We also have an action of T on the algebra of operators \mathcal{A} through this theorem:

$$\begin{aligned} T \curvearrowright \mathcal{A} &\rightarrow \mathcal{A} \\ \mathcal{O} &\mapsto U_T \mathcal{O} U_T^\dagger. \end{aligned} \quad (2.7)$$

Note that this action is independent of the phase freedom of U_T and thus uniquely defined. (2.5) is a special case of this.

Finally, we call a symmetry T is **preserved** by the Hamiltonian if

$$[U_T, H] = 0. \quad (2.8)$$

2.3 Bloch sphere and projective representation

Let us take a close look at the case of a qubit, i.e. $\mathcal{H} = \mathbb{C}^2$. Pick a basis $|0\rangle$ and $|1\rangle$ and expand a unit state $|\psi\rangle$ as

$$|\psi\rangle = \cos(\theta/2) |0\rangle + \sin(\theta/2) e^{i\phi} |1\rangle, \quad (2.9)$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. One can always bring a unit state to this form using the phase ambiguity $|\psi\rangle \sim e^{i\alpha} |\psi\rangle$. Note that when $\theta = 0$ and $\theta = \pi$, the corresponding point in $\mathbb{P}\mathcal{H}$ is independent of ϕ . Therefore (θ, ϕ) is the coordinates (polar and azimuthal angles) on S^2 , called the Bloch sphere:

$$\mathbb{P}\mathcal{H} \cong \mathbb{CP}^1 \cong S^2. \quad (2.10)$$

The relationship between the polar coordinates on S^2 and the Cartesian coordinates on \mathbb{R}^3 is $(x, y, z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$.

Given two states $|\psi_1\rangle$ and $|\psi_2\rangle$, the transition probability is $|\langle\psi_2|\psi_1\rangle|^2 = \cos^2(\theta_{12}/2)$, where θ_{12} is the angle between the two corresponding point in the Bloch sphere. Symmetry transformation T is a bijection on Bloch sphere preserving this quantity. Such transformations are one-to-one corresponded to the orthogonal group

$$\{\text{Symmetry transformations on a qubit}\} \cong O(3). \quad (2.11)$$

We can also identify the composition of symmetry transformations as the multiplication of the group $O(3)$. What are the corresponding (anti)unitaries?

2.3.1 Projective phase of $SO(3)$

Let us first study the part $SO(3) \subset O(3)$ that preserves the orientation of the sphere. In particular, the rotation $R_z(\lambda)$ around the z ($\theta = 0$ direction) axis sends

$$R_z(\lambda) : (\theta, \phi) \mapsto (\theta, \phi + \lambda). \quad (2.12)$$

A unitary compatible with this transformation is

$$U'_{R_z(\lambda)} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\lambda} \end{pmatrix}. \quad (2.13)$$

Or, if we can demand that the unitary is also special ($\det U = 1$) by using the phase ambiguity, achieving

$$U_{R_z(\lambda)} = \begin{pmatrix} e^{i\lambda/2} & 0 \\ 0 & e^{i\lambda/2} \end{pmatrix}. \quad (2.14)$$

However, this expression has a peculiar feature: that the 2π rotation is mapped to $U_{R_z(\lambda=2\pi)} = -\mathbf{I}_2$, not to the identity matrix \mathbf{I}_2 ! In other words, if we use the special unitary $U_{R_z(\lambda)}$ ($0 \leq \lambda < 2\pi$), the composition law of the unitary operators is

$$U_{R_z(\lambda)} U_{R_z(\lambda')} = e^{i\alpha(\lambda, \lambda')} U_{R_z(\lambda + \lambda' \bmod 2\pi)} \quad (2.15)$$

which is not quite straightforward. The function $\alpha(\lambda, \lambda')$ is called the **projective phase** and in this case it is

$$\alpha(\lambda, \lambda') = \begin{cases} 0 & (\lambda + \lambda' < 2\pi) \\ \pi & (\lambda + \lambda' \geq 2\pi) \end{cases}. \quad (2.16)$$

If one only focus on $R_z(\lambda)$, one can use U' in (2.13), which has the trivial projective phase. However, it is known that we cannot find a good unitaries U_g for all of the group $g \in SO(3)$ acting on the Bloch sphere. One choice of the general expression is

$$U_{R_{\mathbf{n}}(\lambda)} = \exp(-i\lambda \mathbf{n} \cdot \vec{\sigma}/2). \quad (2.17)$$

The claim is that you cannot come up with a function $\beta : SO(3) \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ such that $e^{i\beta(g)}U_{R_n(\lambda)}$ exactly satisfies the group multiplication law of $SO(3)$ without projective phase. We will prove this in the next section.

When a group G acts on a vector space, but the represented matrices U_g cannot avoid the projective phase, the pair of the vector space and the action is called a **projective representation**. The qubit space \mathbb{C}^2 is a projective representation of $SO(3)$. The general presentation mat

2.3.2 $SO(3)$ or $SU(2)$?

You might have confused; you might have been taught that a qubit, or a spin, is acted by $SU(2)$, and wondering why I am emphasizing $SO(3)$ instead. The Lie groups $SU(2)$ and $SO(3)$ are closely related: $SU(2)$ is a double cover of $SO(3)$. In other words,

$$SO(3) = SU(2)/Z(SU(2)), \quad (2.18)$$

where $Z(SU(2)) \cong \mathbb{Z}_2$ is the center of $SU(2)$ generated by $-\mathbf{I}_2$. The $SU(2)$ can naturally act on \mathbb{C}^2 . However, $-\mathbf{I}_2 \in SU(2)$ does not change the *ray* of the state, which is the physical entity. Therefore, more precise statement is that a symmetry of a qubit/spin is $SO(3)$, as Wigner defined it, but to describe the Hilbert space $SU(2)$, that does not suffer from the projective phase, is more convenient. And the projective phase is the **quantum anomaly**: something is not quite right about the symmetry.

Another way of saying the same this is to focus on the algebra \mathcal{A} of observables. The symmetry acts on the algebra of operators by conjugation ((2.7)), and thus the center $-\mathbf{I}_2$ of $SU(2)$ acts trivially. So the symmetry acting on the *observables* is $SO(3)$, not $SU(2)$. And the projective phase, or the double cover $SU(2)$ arises only when we consider the vector space of the states.

2.3.3 Anti-Unitary symmetry*

What happens for the orientation reversing map in $O(3)$, specifically for the reflection

$$\text{Ref} : (\theta, \phi) \mapsto (\theta, -\phi). \quad (2.19)$$

For this we have to assign an *anti*-unitary operator, which is

$$U_{\text{Ref}} = K\mathbf{I}_2, \quad (2.20)$$

where K is the complex conjugation map acting on \mathbb{C} . A general element in $O(3)$ $SO(3)$ can be obtained by multiplying a rotation to the reflection. There is no projective phase regarding Ref.

2.3.4 General finite states model*

For general n state system, $\mathcal{H} \cong \mathbb{C}^n$, the projective space is the complex projective space \mathbb{CP}^n . The symmetry transformations are

$$\{\text{Symmetry transformations on } \mathbb{C}^n\} \cong PSU(n) \rtimes \mathbb{Z}_2, \quad (2.21)$$

where $PSU(n) = SU(n)/\mathbb{Z}_n$ and \mathbb{Z}_2 acts as the charge conjugation. \mathbb{C}^n is a projective representation of $PSU(n)$, while the \mathbb{Z}_2 part is realized as an anti-unitary.

2.4 $\mathbb{Z}_2 \times \mathbb{Z}_2$ projective representation

Here we introduce a systematic method in studying the projective phase, using the same example of qubit. For simplicity, we concentrate on the following $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup inside $SO(3)$:

$$\{\text{Id}, R_x(\pi), R_y(\pi), R_z(\pi)\} \subset SO(3). \quad (2.22)$$

Note that $R_x(\pi) = R_y(\pi)R_z(\pi) = R_z(\pi)R_y(\pi)$, so this subset is closed under the multiplication forming the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. To be concise, we rename the generators of this group as $a := R_z(\pi)$ and $b := R_y(\pi)$, then $R_x(\pi) = ab = ba$. The explicit actions of these generators on the coordinates are

$$\begin{aligned} a &\curvearrowright (\theta, \phi) \mapsto (\theta, \phi + \pi) \\ b &\curvearrowright (\theta, \phi) \mapsto (\pi - \theta, \pi - \phi) \end{aligned} \quad (2.23)$$

From this and (2.9), we can read the action of a and b on the states as

$$\begin{aligned} a &\curvearrowright [\cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle] \mapsto [\cos(\theta/2)|0\rangle - \sin(\theta/2)e^{i\phi}|1\rangle] \\ b &\curvearrowright [\cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle] \mapsto [\sin(\theta/2)|0\rangle - \cos(\theta/2)e^{-i\phi}|1\rangle] \\ &\quad \sim [\sin(\theta/2)e^{i\phi}|0\rangle - \cos(\theta/2)|1\rangle]. \end{aligned} \quad (2.24)$$

The last line is the equivalence between the states by the overall phase $e^{i\phi}$. These transformations can be easily realized by the unitary matrices

$$U_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3, \quad U'_b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2. \quad (2.25)$$

However, the latter matrix squares to $-\mathbf{I}_2$, although $b^2 = 1$. To fix this, one can instead use

$$U_b = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2. \quad (2.26)$$

However, an issue remains: although $ab = ba = R_x(\pi)$, we have $U_a U_b = -U_b U_a$. This discrepancy between the multiplication law among group elements and their corresponding unitaries is the projective phase. If we set $U_{ab} = U_b U_a = i\sigma_1$, the multiplication law among U_g , $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ is

$$U_g U_{g'} = e^{i\alpha(g, g')} U_{gg'}, \quad (2.27)$$

where in this example the projective phase is

$$\alpha(a, b) = \alpha(ab, ab) = \pi, \quad \alpha(g, g') = 0 \text{ (otherwise)}. \quad (2.28)$$

Recall that we have the room to redefine each U_g by a phase. Can we use this freedom to eliminate the projective phase? Let us say the we redefine U_g by phase $\beta(g)$:

$$U_g \mapsto e^{i\beta(g)} U_g. \quad (2.29)$$

However, such phase rotation cannot eliminate the commutator $[U_a, U_b]$, which has to be zero if $\alpha = 0$.

When a projective phase cannot be removed by the phase rotation of the unitaries, the phase is called (cohomologically) nontrivial. The phase α in (2.28) is nontrivial.

One significance of the nontrivial projective phase is that it prohibits a one-dimensional representatoin: in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ example, since $[U_a, U_b] \neq 0$ (even though the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is abelian), the such operators cannot act on a one-dimensional space. Note that for a linear representation one always has the *trivial* representation; with a nontrivial projective phase, there is no such thing.

This in turn put a restriction on the possible Hamiltonian. Assume that the Hamiltonian on the qubit preserves the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. If the two energy were distinct, each state should consist a one-dimensional representation with the projective phase, which contradicts with the above observation. Hence the projective action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is enough to ensure that the two states are degenerate.

Now, we also see that the entire group $SO(3)$ acting on the qubit also has a nontrivial projective phase. This is because other elements of $SO(3)$ does not help to resolve the projective phase that $\mathbb{Z}_2 \times \mathbb{Z}_2$ has.

2.5 Charged particle on the Aharonov-Bohm ring

What is the use of the projective phase? It provides a way to find a qubit : if you can cook up a system with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ with the nontrivial projective phase — or **quantum anomaly** —, you are guaranteed to have (at least) 2-fold degeneracy.

Let us see how this happens in an example with a infinite dimensional Hilbert space. Here we consider a charged free particle confined in a ring S^1 with radius R . We put the magnetic flux Φ through the ring. The Hamiltonian is

$$H = \frac{1}{2m} (p_x + \frac{e}{c} A_x)^2, \quad (2.30)$$

where x is the coordinate on S^1 with the identification $x \sim x + 2\pi R$, $p_x = -i\hbar \frac{d}{dx}$, and $A_x = \Phi/(2\pi R)$ is the vector potential along the ring. We rewrite this as

$$H = \frac{\hbar^2}{2mR^2} (\hat{n} + \frac{\theta}{2\pi})^2, \quad (2.31)$$

were $\hat{n} = \frac{Rp_x}{\hbar}$ and $\theta = \frac{e\Phi}{\hbar c}$.

A wave function $\psi(x)$ can be expanded by the periodic exponential function $\psi_n(x) \propto e^{inx/R}$, $n \in \mathbb{Z}$. The corresponding state $|n\rangle$ is the eigenstate of the operators \hat{n} and H :³

$$\hat{n}|n\rangle = n|n\rangle, \quad H|n\rangle = E_n(\theta)|n\rangle = \frac{\hbar^2}{2mR^2}\left(n + \frac{\theta}{2\pi}\right)^2|n\rangle. \quad (2.32)$$

This system has the symmetry shifting x as $x \mapsto x + \alpha R$ with a 2π -periodic parameter α . The unitary corresponding to this symmetry can be directly read off from the wave function as⁴

$$U_\lambda|n\rangle = e^{i\lambda n}|n\rangle. \quad (2.33)$$

There are other symmetries, depending on θ . For example, when $\theta = 0$, there is a \mathbb{Z}_2 flipping x : $x \mapsto -x$, $|n\rangle \mapsto |-n\rangle$. A more interesting case is $\theta = \pi$, then the $|n\rangle \mapsto |-n\rangle$ action is not a symmetry of the Hamiltonian, but the following is preserved:

$$U_{\mathbb{Z}_2}|n\rangle = |-n-1\rangle. \quad (2.34)$$

Note that this is also the \mathbb{Z}_2 flipping symmetry, in the sense of the following:

$$U_{\mathbb{Z}_2}^\dagger f(x) U_{\mathbb{Z}_2} = f(-x), \quad (2.35)$$

for any periodic function f of x . Now the multiplication between U_λ and \mathbb{Z}_2 is

$$U_\lambda U_{\mathbb{Z}_2} = e^{-i\lambda} U_{\mathbb{Z}_2} U_{-\lambda}. \quad (2.36)$$

Note that in right hand side the parameter λ of U_λ is flipped, so the two symmetry forms the semidirect product $U(1) \rtimes \mathbb{Z}_2$. On top to that, we observe the projective phase $e^{-i\lambda}$. If we focus on the subgroup $\mathbb{Z}_2^{\text{shift}}$ ($\lambda = \pi$) subgroup of the $U(1)$, we have the subgroup $\mathbb{Z}_2^{\text{shift}} \times \mathbb{Z}_2$ of $U(1) \rtimes \mathbb{Z}_2$ and the projective phase $e^{-i\pi} = -1$ is the same phase we observed in Section 2.4!

Recall that, in Section 2.4, we have proven that there is no one-dimensional representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ with the nontrivial projective phase. Therefore *every* eigenspace of H at $\theta = \pi$ has dimension at least two. This can be easily verified because $E_n(\pi) = E_{-n-1}(\pi)$. Therefore, in particular, the ground state has exact two-fold degeneracy that you might use as a qubit.⁵

Ok, but we could find the degeneracy without discussing the projective phase. The real advantage of the quantum anomaly is that it shows the degeneracy is *robust*; this is

³ $E_n(\theta + 2\pi) = E_{n+1}(\theta)$ and thus the spectrum is unchanged by the 2π -shift of θ . This is closely related to the Dirac quantization of magnetic charge; $\theta = 2\pi$ is the flux created by the Dirac monopole.

⁴If one does the 2π -shift of theta adiabatically, the spectrum is unchanged as noticed in the above footnote, but the $U(1)$ eigenvalue n carried by the state is shifted by 1. This (or something similar) is called Thouless pump. Indeed this is yet *another quantum anomaly*, “anomaly involving parameter space”. See Córdova et al. [5] for more detail.

⁵I believe this is called the flux qubit, but do not trust me, I’m not an expert on quantum computing.

the result of the symmetry and its quantum anomaly, so, as long as they are preserved, the degeneracy cannot be lifted. Also, the quantum anomaly is a *discrete* thing and therefore it cannot be changed by continuous deformation. (We will discuss about this a bit more in the next section.) So, even if we deform the Hamiltonian as

$$H' = \frac{\hbar^2}{2mR^2} \left(\hat{n} + \frac{1}{2} \right)^2 + V(x), \quad (2.37)$$

with an arbitrary (periodic) potential $V(x)$ that is even : $V(x) = V(-x)$, we know that there is a degeneracy, although in this system cannot be easily solved. This is the power of quantum anomaly: it tells us something nontrivial *without solving the system!*

2.6 Second group cohomology

Let us formalize what we have learned above for a general group G . We assume that for each $g \in G$, we have a symmetry transformation T_g acting on $\mathbb{P}\mathcal{H}$ as symmetry transformations satisfying the multiplication law:

$$T_g T_{g'} = T_{gg'} \quad (2.38)$$

By Wigner's theorem, for each g , we also have U_g . The multiplication law of T_g only guarantees

$$U_g U_{g'} = e^{i\alpha(g,g')} U_{gg'}, \quad (2.39)$$

for some phase $\alpha(g, g')$, because the correspondence between T_g and U_g is only up to a phase. Here we introduce the terminology from algebraic topology. A map from n -th power of G to some abelian group M is called M -valued n -**cochain**, denoted by

$$C^n(G, M) := \text{Map}(G^n, M). \quad (2.40)$$

Note that a cochain is not demanded to be a group homomorphism; it is just a map. α is a $\mathbb{R}/2\pi\mathbb{Z}$ ($\cong U(1)$) valued 2-cochain. In this lecture, M is almost always $\mathbb{R}/2\pi\mathbb{Z}$, so we will not explicitly declare the value domain onwards.

The 2-cochain α is actually not completely arbitrary. The constraint comes from the associativity of the unitary operators: $(U_{g_1} U_{g_2}) U_{g_3} = U_{g_1} (U_{g_2} U_{g_3})$, which leads

$$\delta_3 \alpha(g_1, g_2, g_3) := -\alpha(g_1, g_2) - \alpha(g_1 g_2, g_3) + \alpha(g_1, g_2 g_3) + \alpha(g_2, g_3) = 0. \quad (2.41)$$

Here we have defined the derivative $\delta_3 : C^2(G) \rightarrow C^3(G)$ on 2-cochains. A 2-cochain satisfies this condition is called a cocycle, and the set of cocycle is denoted by⁶

$$Z^2(G) := \text{Ker}(\delta_3) = \{\alpha \in C^2(G) \mid \delta_3 \alpha = 0\}. \quad (2.42)$$

⁶The derivative δ_{n+1} for general n -cochain can be defined, but we will postpone this until next chapter, other than δ_2 which is defined just below.

Exercise 2.1. Show that the particular cochain $\alpha \in C^2(\mathbb{Z}_2 \times \mathbb{Z}_2)$ defined in (2.28) is indeed a cocycle.

Not all the (nonzero) elements of $C^2(G)$ are interesting: some of them might be eliminated by redefinition of the unitary operators U_g :

$$U_g \mapsto e^{i\beta(g)} U_g, \quad (2.43)$$

with $\beta \in C^1(G)$. The phase $\alpha(g, g')$ gets shifted by this redefinition as:

$$\alpha(g_1, g_2) \mapsto \alpha(g_1, g_2) - \delta_2 \beta(g_1, g_2) := \alpha(g_1, g_2) + \beta(g_1 g_2) - \beta(g_1) - \beta(g_2). \quad (2.44)$$

The map $\delta_2 : C^1(G) \rightarrow C^2(G)$ is also called the derivative. An image of δ_2 is called a coboundary, denoted by

$$B^2(G) := \text{Im}(\delta_2) = \{\delta_2 \beta \mid \beta \in C^1(G)\}. \quad (2.45)$$

The crucial feature of the derivative is that it vanishes when composed:

$$\delta_3 \circ \delta_2 = 0. \quad (2.46)$$

Therefore we have $B^2(G) \subset Z^2(G)$. Now, interesting projective phases are those not in $B^2(G)$. This motivates us to define the *group cohomology group*⁷

$$H^2(G) := Z^2(G)/B^2(G). \quad (2.47)$$

In this lecture we do not have time to learn how to compute the cohomology group, but the examples we have seen are

$$\begin{aligned} H^2(SO(3)) &\cong \mathbb{Z}_2, \\ H^2(\mathbb{Z}_2 \times \mathbb{Z}_2) &\cong \mathbb{Z}_2. \end{aligned} \quad (2.48)$$

Exercise 2.2. Prove (2.46).

Exercise 2.3. Show that $H^2(\mathbb{Z}_2) = 0$. (If you want, also calculate $H^2(\mathbb{Z}_2, \mathbb{Z}_2)$.)

⁷Do not confuse this with the cohomology group of the group as a topological space. The two, the group cohomology of a group and the geometric cohomology of the same, can be denoted by the same symbol $H(G)$, but they are different. They are confusing especially when G is continuous. Rather, the group cohomology is identified with the (simplicial/cellular) cohomology of an infinite-dimensional topological space called the classifying space BG .

2.7 Anomalous symmetry and 1D SPT

2.7.1 Symmetry Protected Topological Phase

Before moving to the quantum anomaly in QFT, here let us see the crucial connection between quantum anomaly and **symmetry protected topological phase**(SPT), introduced in Chen et al. [6], a topic in condensed-matter physics. The relationship is summarized as the following hypothesis ⁸

Hypothesis 2.1 (SPT-inflow hypothesis). A $d + 1$ -dimensional QFT with an anomalous global symmetry G (and non-anomalous Lorentz symmetry) can be realized as a boundary of a $(d + 1) + 1$ -dimensional SPT phase with (non-anomalous) symmetry G .

This is a refinement of the Callan-Harvey anomaly-inflow principle Callan and Harvey [7]. Callan-Harvey anomaly-inflow asserts that a quantum anomaly is related to a *classical action* of backgrounds in $(d + 1) + 1$ dimension, but what kind of backgrounds/action can consist an “inflow action” was unclear; now it is replaced by “SPT”, which has more clear definition and physical interpretation.

We define the SPT on a lattice. To define, first we define the notion of “short range entangled” state.^{9 10}

Definition 2.1 (Short Range Entangled (SRE)). Consider a d (space) dimensional lattice system without boundary, whose Hilbert space is $\mathcal{H} = \bigotimes_i \mathcal{H}_i$, where \mathcal{H}_i is the local Hilbert space associated to the site i . A state $|\psi\rangle \in \mathcal{H}$ is short range entangled when you can make $|\psi\rangle$ a tensor product state by the following operation:

1. Enlarge the local Hilbert space $\mathcal{H}_i^{\text{enlarged}} = \mathcal{H}_i \otimes \mathcal{H}_i^{\text{ext}}$ by some Hilbert space $\mathcal{H}_i^{\text{ext}}$. Also pick a state $|\phi_i\rangle \in \mathcal{H}_i^{\text{ext}}$ from each $\mathcal{H}_i^{\text{ext}}$ and set $|\hat{\psi}\rangle \in \bigotimes_i \mathcal{H}_i^{\text{enlarged}}$ to be $|\hat{\psi}\rangle = |\psi\rangle \otimes_i |\phi_i\rangle$.
2. Then pick a unitary U_i acting on each of the enlarged local Hilbert spaces $\mathcal{H}_i^{\text{enlarge}}$.

⁸The name of the hypothesis is cooked up by KO. Also the correct reference is not clear to KO (other than having a very long list); it feels more like a theme that gradually arose over a time (through the decade of 2010).

⁹This definition is by the original paper. A. Kitaev, and therefore many other people, uses the same words (SRE/SPT) for a non-equivalent thing. Let us call a state $|\psi\rangle$ is **invertible** if $|\psi\rangle \otimes |\psi^\vee\rangle \in \mathcal{H} \otimes \mathcal{H}^\vee$ is SRE (as defined in the main text). Here \mathcal{H}^\vee is the dual of \mathcal{H} and $|\psi^\vee\rangle$ is the dual of $|\psi\rangle$. What Kitaev calls SRE is actually this invertibility. Note that while there is no nontrivial SPT when G is trivial by definition, but there can potentially be nontrivial invertible state even when G is trivial. An invertible state can actually captures a pure-gravitational anomaly on the boundary which improves Hypothesis 2.1.

¹⁰About continuum, see Yuji’s note Tachikawa [1]. In short an *invertible QFT*, the continuum version of invertible state defined in the above footnote, is defined (and classified!) in Freed and Hopkins [8] Yonekura [9].

The resulting state is $|\tilde{\psi}\rangle = \prod_i U_i |\hat{\psi}\rangle$, and $|\psi\rangle$ is SRE iff you can pick $\mathcal{H}_i^{\text{ext}}, |\phi_i\rangle$ and U_i so that $|\tilde{\psi}\rangle = \otimes_i |i\rangle$ for some local states $|i\rangle \in \mathcal{H}_i^{\text{enlarged}}$.

In short, a state is SRE when you can add additional degrees of freedom then resolve all the entanglement between the sites by local unitary. We also define G -SRE:

Definition 2.2 (G-SRE). Consider a d (space) dimensional lattice system without boundary with on-site symmetry G . A G -invariant state $|\psi\rangle$ is G -SRE iff it is SRE and all of the unitaries U_i in Definition 2.1 can be taken to be the ones preserving G .

Then an SPT is defined by

Definition 2.3 (Symmetry Protected Topological phase). A G -invariant state $|\psi\rangle$ is in a (non-trivial) symmetry protected topological phase iff $|\psi\rangle$ is SRE but not G -SRE.

In other words a SPT state is SRE but not in a way respecting the on-site G symmetry.

2.7.2 Example of 1+1d SO(3) SPT

Let us construct an example of SPT on 1+1d lattice for symmetry $SO(3)$. We put two qubits, labeled by A and B , on each site. So $\mathcal{H}_i = \mathcal{H}_i^A \otimes \mathcal{H}_i^B$, with $\mathcal{H}_i^A \cong \mathcal{H}_i^B \cong \mathbb{C}^2$. As a on-site Let $|s_i\rangle$ be the Bell state between the A -qubit at site i ($i \in \mathbb{Z}$) and the B -qubit at site $i+1$: $|s_i\rangle = |\uparrow_{i,B}\rangle |\uparrow_{i+1,A}\rangle + |\downarrow_{i,B}\rangle |\downarrow_{i+1,A}\rangle \in \mathcal{H}_i^A \otimes \mathcal{H}_{i+1}^B$. Then we define the total state $|\psi\rangle$ by¹¹

$$|\psi\rangle = \bigotimes_i |s_i\rangle. \quad (2.49)$$

The entanglement structure of this state is depicted in Figure 2. Note that, although (2.49) is defined by the tensor product, each $|s_i\rangle$ is not contained in a single site Hilbert space \mathcal{H}_i , and thus $|\psi\rangle$ is not a tensor product of local states, each in \mathcal{H}_i .

¹¹This state looks very artificial. Instead, one can consider the state

$$|\text{AKLT}\rangle = \prod_i P_i^{\text{triplet}} |\psi\rangle$$

where P_i^{triplet} , acting on \mathcal{H}_i , is the projection operator onto the triplet states consists of the A - and B - qubits on the *same* site i . This state, in the same SPT as $|\psi\rangle$, is the ground state of the AKLT Hamiltonian defined with spin-1 operators \vec{S}_i Affleck et al. [10]:

$$H_{\text{AKLT}} = \sum_i \left(\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} (\vec{S}_i \cdot \vec{S}_{i+1})^2 \right),$$

which was studied way before the invention the notion of SPT.

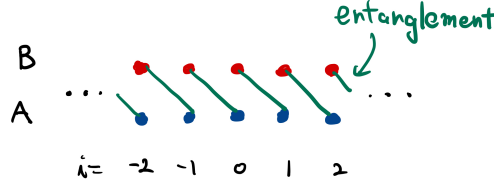


Figure 2: The entanglement structure of the state $|\psi\rangle$ defined in (2.49). The B -qubit at site i is entangled with the A -qubit at site $i + 1$, as indicated by the green line.

We let the group $SO(3)$ acts on the (projective space of) total states as follows. Let $T_g^{(i,A)}$ and $T_g^{(i,B)}$ be the $SO(3)$ symmetry transformation, defined in Section @ref{Bloch}, on the A - or B -qubit at the i -th site. Then the $SO(3)$ action for which $|\psi\rangle$ is in SPT is

$$T_g = \prod_i T_g^{(i,A)} T_{g^{-1}}^{(i,B)}. \quad (2.50)$$

The state $|\psi\rangle$ is invariant under this transformation, because $T_g = \prod_i T_{g^{-1}}^{(i,B)} T_g^{(i+1,A)}$ and $T_{g^{-1}}^{(i,B)} T_g^{(i+1,A)}$ makes the singlet state $|s_i\rangle$ invariant. Note that this symmetry transformation does not have the projective phase: it is cancelled between the ones coming from $U_g^{(i,A)}$ and $U_{g^{-1}}^{(i,B)}$ within each site.

The direct exposition that $|\psi\rangle$ is in a SPT can be found in Prakash, Wang, and Wei [11]. Here instead we observe how the Hypothesis 2.1 works in this example. The state $|\psi\rangle$ has no problem to be defined on a periodic lattice. However, when there is a boundary, say that only $i \in \mathbb{Z}_{\geq 0}$ participate in the lattice, the A -qubit at site $i = 0$ does not have its partner to create an entanglement. Thus, there are equally good/bad extensions of the state $|\psi\rangle$ to open boundary:

$$|\uparrow, \psi\rangle = |\uparrow_{0,A}\rangle \otimes \bigotimes_{i \geq 0} |s_i\rangle, \quad |\downarrow, \psi\rangle = |\downarrow_{0,A}\rangle \otimes \bigotimes_{i \geq 0} |s_i\rangle. \quad (2.51)$$

We call the degrees of freedom $|\uparrow_{0,A}\rangle$ the **edge modes**, as it localizes at the boundary. On this edge mode, the $SO(3)$ symmetry acts by T_g , which is exactly the projective representation the qubit has. This edge mode is *robust* – if you force the $(i = 0, A)$ -qubit be entangled with another qubit (in a finite distance) by some unitary, you get another unentangled qubit, as you cannot erase/create the entanglement by unitaries. In summary, if we cut the chain in the SPT, on the edge we have the qubit protected by the projective action of $SO(3)$.

2.7.3 SPT for a general projective representation*

It is easy to generalize the above example into a general projective representation R of a group G . (Here we assume that G acts by unitary operations.) For each site we

prepare the local Hilbert space as $\mathcal{H}_i = R_i \otimes R_i^\vee$, where R_i is the i -th copy of R and R_i^\vee is the dual of R . Pick a basis $|i\rangle$ in R , let $|i^\vee\rangle \in R^\vee$ denote the dual basis, and then define the singlet state $|s_i\rangle$ by

$$|s_i\rangle = \sum_i^{\dim R} |i^\vee\rangle \otimes |i\rangle \in R_i^\vee \otimes R_{i+1}. \quad (2.52)$$

This state is invariant under $T_{g^{-1}}^{(i,B)} T_g^{(i+1,A)}$ (and nonzero). Now the rest of the construction is the same as above, and we see the edge mode in R when the lattice has a boundary. Thus we have seen how the (one direction of) Hypothesis 2.1 works in the 1+1d bulk/0+1d boundary case.

3 Quantum Anomaly in 1+1d QFT

In this chapter we learn about quantum anomaly of symmetry in 1+1-dimensional QFT. A QFT is a quantum mechanics equipped with a continuous locality (while a lattice system possess a discrete locality).¹² Here, the locality means that we have the notion of “space”, and the observables are contained in the space. This locality (plus some assumption) prohibits the projective phase that we discussed so far, but then there is possibly more subtle anomaly about how much local the composition law of the symmetry is. It is not a very traditional way but we see this phenomenon through the action of the symmetry on the Hilbert spaces of QFT on *an interval* with different boundary conditions. The quantum anomaly in this dimensions for a finite group G can be captured by the 3rd group cohomology.

3.1 QFT on a manifold and symmetry

As emphasized, the locality plays the essential role in (topological properties of) QFT. One consequence of the locality is that, given a $d + 1$ -dimensional QFT, we can quantize the theory on a spacial closed manifold M_d to get \mathcal{H}_{M_d} .¹³ In other words a QFT is not just the Hilbert space and the Hamiltonian, but a rule to assign them when a closed manifold M_d is given. It is more precise to express the same thing in terms of the algebra of operators. For a QFT, given a region $R \in \mathbb{R}^{d,1}$, there is the algebra of local operators contained in R : $\mathcal{A}(R)$. Then we “glue” the algebra $\mathcal{A}(B^{d+1})$ on a ball to get $\mathcal{A}(R)$ for an arbitrary manifold R . In particular we can get $\mathcal{A}(M_d \times \mathbb{R})$.¹⁴

A symmetry also is expected to be local: that if there is a symmetry T in a QFT, that means, for arbitrary M_d , there should exists a symmetry transformation T_{M_d} on $\mathbb{P}\mathcal{H}_{M_d}$. This is manifest in the case of continuous symmetry. For example, the unitary operator U_λ for a $U(1)$ rotation by angle λ is, from the Noether’s theorem,

$$U_\alpha = e^{i\lambda \int_{M_d} j^0 d^d x}, \quad (3.1)$$

where j^μ is the Noether current. In other words the locality of the unitary operator is manifest in the integral in the expression. For a discrete symmetry, the Noether current is not available. However, we can define that a symmetry transformation is a automorphism of $\mathcal{A}(R)$:

$$T \curvearrowright \mathcal{A}(R) \rightarrow \mathcal{A}(R), \quad (3.2)$$

¹²Most things in this chapter are supposed to be applied to both of QFTs and lattice systems. It is just that KO is more familiar with QFTs and also the lecture is for high-energy students.

¹³We often need an additional structure to M_d . For example, if the theory includes fermionic fields, the manifold M_d has to be equipped with the spin structure. In this lecture we do not care about this subtle but interesting and important subtlety and assume that the theory is *bosonic*, i.e. does not contain fermionic fields. This amounts to assume only the orientation of M_d . (To put a QFT on an unoriented manifold, one needs an anti-unitary symmetry which also be ignored in this lecture.) The references mentioned in 1.1.4 have contents about fermionic theories.

¹⁴This explanation is awfully rough. For a better discussion the reader might refer to Witten [12]

and then T also naturally (projectively) acts on \mathcal{H}_{M_d} which is a representation of $\mathcal{A}(R)$. In a more explicit term, if the QFT is written in terms of fundamental fields and symmetry acts on these fields, the Hilbert space \mathcal{H}_{M_d} is the Fock space constructed from creation/annihilation operators for each modes of the fields on M_d , and the action of a symmetry on the fields naturally defines the action on the annihilation operators, in turn defining the action on \mathcal{H}_{M_d} .

For a local symmetry transformation T , defined in the above sense, to be the symmetry of the theory, the corresponding unitary has to commute with the Hamiltonian. We also assume that the Hamiltonian is local:

$$H = \int T_{00} d^d x, \quad (3.3)$$

where $T_{\mu\nu}$ is the energy-momentum tensor. For a symmetry in QFT, we demand that $[U, T_{\mu\nu}] = 0$. Otherwise U does not commute with the spacetime symmetry.

3.2 No projective phase on S^1

Now specialize to 1+1d QFT, and consider $M_1 = S^1$, which is the only connected closed 1-dimensional manifold. A reader might have guessed that, as a generalization of what we have learned in the case of QM, the unitary operator realizing the symmetry on \mathcal{H}_{S^1} might have projective phase in there multiplication rule. This very natural guess, however, is not correct.

For a general QFT, the Hamiltonian depends on the radius R of S^1 . When R is very large compared to any scale in the theory, we regard the QFT on $S^1 \times \mathbb{R}_{\text{time}}$ as the infrared-regulation of QFT on $\mathbb{R}^1 \times \mathbb{R}_{\text{time}}$.

Now, if the QFT has single vacuum, the cluster decomposition principle (which is another manifestation of locality) that the symmetry should preserve it. So S^1 with large enough R has single vacuum, which contradicts with a nontrivial projective phase. More generally, if a discrete symmetry is spontaneously broken, there can be multiple vacua in the $R \rightarrow \infty$ limit. However, if there is a domain wall connecting the two vacua, there is an exponentially suppressed amplitude between the two states with finite but large R . Therefore, we expect that, without additional mechanism prohibiting domain walls¹⁵, there is a single vacuum on S^1 with large but finite R . As the projective phase is discrete, captured by $H^2(G)$, it has to be constant of R . Therefore we do not expect any nontrivial projective phase of U acting on \mathcal{H}_{S^1} for any radius R of S^1 .

Then, is there any projective-ness in the symmetry in 2d QFT? To explicitly see that, we need to think about *manifolds with boundaries*.¹⁶

¹⁵The mechanism is the “one-form symmetry” [13]. An older reference is [14].

¹⁶Instead one can see it from closed manifold with symmetry twist. Actually this is more conventional way. However KO finds that the manifolds with boundary is more “clean” in a sense.

3.3 1+1-dimensional periodic scalar theory

Rather than explaining the general theory of a QFT on a open manifold, let us jump to a concrete example: 1+1-dimensional (relativistic) massless periodic scalar ϕ . The theory should be explained in any good textbooks on string theory, e.g. Polchinski [15]. Here we give a lightening review.

3.3.1 The model

The action of the free massless periodic scalar theory (on the Minkowski flat space) is (in Polchinski's normalization, in particular $\hbar = 1$ as usual)

$$S = -\frac{1}{4\pi\alpha'} \int_{M_2} d\tau d\sigma \eta^{ab} \partial_a X \partial_b X. \quad (3.4)$$

The scalar field X is periodic and we set the periodicity to be $2\pi R$: $X \sim X + 2\pi R$. The equation of motion is the massless Klein-Gordon one

$$\partial^2 X = 0. \quad (3.5)$$

The canonical momentum $\Pi(\sigma, \tau)$ is

$$\Pi = \frac{1}{2\pi\alpha'} \partial_\tau X, \quad (3.6)$$

and the Hamiltonian is

$$H = \frac{1}{2\pi\alpha'} \int d\sigma \delta^{ab} \partial_a X \partial_b X. \quad (3.7)$$

3.3.2 Quantization on S^1 and $U(1)^2$ symmetry

Let us quantize the theory on S^1 , where we set the periodicity of the spacial coordinate $\sigma \sim \sigma + 2\pi$. We write the general solution to the EOM (3.5) as

$$X^w(\sigma, \tau) = x + R w \sigma + 2\pi\alpha' p \tau + i \left(\frac{\alpha'}{2} \right) \sum_{m=-\infty}^{\infty} \frac{1}{m} \left(\frac{\alpha_m}{z_+^m} + \frac{\tilde{\alpha}_m}{z_-^m} \right), \quad (3.8)$$

where $z = \exp(-i(\tau \pm \sigma))$. The second term is possible because both σ and X is periodic, and the winding number w is quantized (discretized): $w \in \mathbb{Z}$. When there is such topologically distinct classical solutions to the EOM exists, we build the Hilbert space \mathcal{H}_w around each solution labeled by w , and the take the direct sum $\mathcal{H} = \bigotimes_w \mathcal{H}_w$ as the total Hilbert space. The canonical commutation relation gives

$$[x, p] = i, \quad [\alpha_m, \alpha_n] = [\tilde{\alpha}_m, \tilde{\alpha}_n] = m \delta_{m, -n}, \quad (3.9)$$

and the Hamiltonian is (up to a divergent constant coming from commuting α 's)

$$H = \frac{\alpha'}{4} \left(\frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} \right) + \sum_{n=1}^{\infty} (\alpha_{-n} \alpha_n + \tilde{\alpha}_{-n} \tilde{\alpha}_n), \quad (3.10)$$

where $n = Rp$ is the quantized momentum whose eigenvalues takes in \mathbb{Z} , as in the case of the particle on S^1 . A general eigenstate can be constructed by acting finite number of operators α_{-i} and $\tilde{\alpha}_{-j}$ on a state $|n, w\rangle$:

$$\alpha_{-i_1}^{t_1} \alpha_{-i_2}^{t_2} \cdots \tilde{\alpha}_{-j_1}^{s_1} \tilde{\alpha}_{-j_2}^{s_2} \cdots |n, w\rangle, \quad (3.11)$$

where i_a, j_b, t_c, s_d are positive integers, and the state $|n, w\rangle$ satisfies

$$\alpha_i |n, w\rangle = \tilde{\alpha}_j |n, w\rangle = 0, \quad (3.12)$$

and has the corresponding quantum number n and w .

This system famously have to $U(1)$ symmetries, whose charges are n and w respectively. The former is called the momentum $U(1)$, and the latter called the winding $U(1)$. We write the corresponding unitaries as U_λ^p and U_λ^w , where λ is the parameter of the symmetry transformation, so that

$$U_\lambda^p |n, w\rangle = e^{i\lambda n} |n, w\rangle, \quad U_\lambda^w |n, w\rangle = e^{i\lambda w} |n, w\rangle. \quad (3.13)$$

Also, the momentum $U(1)$ is basically the canonical momentum p to the average coordinate $x = \int X d\sigma$, it causes the translation in the space of X :

$$(U_\lambda^p)^\dagger f(X(\sigma, \tau)) U_\lambda^p = f(X(\sigma, \tau) + \lambda R), \quad (3.14)$$

where f is a periodic function. With this boundary condition,

3.3.3 Projective phases on the Hilbert spaces on an interval

Now let us see how the two $U(1)$ symmetries acts on the Hilbert spaces defined on a manifold with boudnaries. In 1d, there is only one (topology type) of a compact manifold with boundaries: the interval $I = [0, 2\pi]$. To define the theory on the interval, we have to pick boundary conditions for each of the boundaries: $\sigma = 0$ (left boundary) and $\sigma = 2\pi$ (right boundary). Here we pick the Dirichlet boundary condition $\mathcal{D}(\theta_L)$ and $\mathcal{D}(\theta_R)$ for the left and right boundaries, defined by

$$X(\sigma = 0, t) = R\theta_L, \quad X(\sigma = 2\pi, t) = R\theta_R. \quad (3.15)$$

The expansion of the periodic field X in this case is

$$X = R\theta_L + R w \sigma + i \left(\frac{\alpha'}{2} \right) e^{-i\tau} \sum_{m=-\infty}^{\infty} \frac{1}{m} \alpha_m \sin(m\sigma/2), \quad (3.16)$$

where the winding w now has to takes values in $w \in (\theta_R - \theta_L)/2\pi + \mathbb{Z}$. We write the Hilbert space of states on the interval with the boundary condition $\mathcal{D}(\theta_L, \theta_R)$ as $\mathcal{H}_{\theta_L, \theta_R}$, and the state with winding w in it as $|w\rangle_{\theta_L, \theta_R}$. We only consider the ground states for the oscillation modes but the story can be applied to the states with these modes without any change.

How does the $U(1)^2$ symmetries acts on the states? The momentum symmetry translate X , and breaks the boundary condition (3.15). But it still defines a linear map *between the Hilbert spaces*:

$$\begin{aligned} U_\lambda^p : \mathcal{H}_{\theta_L, \theta_R} &\rightarrow \mathcal{H}_{\theta_L + \lambda, \theta_R + \lambda} \\ |w\rangle_{\theta_L, \theta_R} &\mapsto |w\rangle_{\theta_L + \lambda, \theta_R + \lambda}. \end{aligned} \quad (3.17)$$

On the other hand, the winding $U(1)$ symmetry preserves the Dirichlet boundary condition and the naive extension of the action (3.13) to this case is

$$\begin{aligned} \widetilde{U}_\lambda^w : \mathcal{H}_{\theta_L, \theta_R} &\rightarrow \mathcal{H}_{\theta_L, \theta_R} \\ |w\rangle_{\theta_L, \theta_R} &\mapsto e^{i\lambda w} |w\rangle_{\theta_L, \theta_R}. \end{aligned} \quad (3.18)$$

Here we put the tilde on U_λ^w because there is one problem with this action, i.e. $\widetilde{U}_\lambda^w = e^{i(\theta_R - \theta_L)} \widetilde{U}_{\lambda+2\pi}^w$. This projective phase can be easily removed by redefining the unitary as

$$\begin{aligned} U_\lambda^w : \mathcal{H}_{\theta_L, \theta_R} &\rightarrow \mathcal{H}_{\theta_L, \theta_R} \\ |w\rangle_{\theta_L, \theta_R} &\mapsto e^{i\lambda(w - \{\theta_R\})} |w\rangle_{\theta_L, \theta_R}, \end{aligned} \quad (3.19)$$

where $\{\theta\}$ is the number in $[0, 2\pi)$ and $\{\theta\} = \theta \bmod 2\pi$. Note that the modification is not $\{\theta_R - \theta_L\}$, so that the modification is defined *locally* – it is a sum of independent contributions from the left and right boundaries. However, this redefinition produces a projective phase in the product of $U_{\lambda_p}^p$ and $U_{\lambda_w}^w$:

$$U_{\lambda_w}^w U_{\lambda_p}^p |_{\theta_L, \theta_R} = e^{i\lambda_w(-[\lambda_p + \theta_R] + [\lambda_p + \theta_L])/2\pi} U_{\lambda_p}^p U_{\lambda_w}^w |_{\theta_L, \theta_R}, \quad (3.20)$$

where $[\theta] = \theta - \{\theta\} \in 2\pi\mathbb{Z}$. The $|_{\theta_1, \theta_2}$ denotes the boundary conditions on which the operator just left of the vertical bar acts. For the other operator composed with this one, the domain is chosen so that the composition makes sense. This is the projective phase characterizing the mixed anomaly between the two $U(1)$ symmetry (or, the anomaly for the (anti-)diagonal of $U(1)$'s). This characterization is utterly different from what you find in Polchinski [15], so

Exercise 3.1. Find the relation ship between the characterization of the $U(1)^2$ anomaly here and more traditional one, defined by the 2-pt function of the currents. (KO does not know the answer, and he will be very pleased if you send an answer to him.)

An advantage of this way of stating the anomaly is

1. it is clear that this is some kind of generalization of Wigner's projective phase, and
2. applicable to discrete case.

For example, we can take the \mathbb{Z}_2 subgroup of the diagonal of $U(1)$'s whose unitary is $U_{\mathbb{Z}_2} = U_\pi^w U_\pi^p$ and then the projective phase is

$$U_{\mathbb{Z}_2}^2|_{\theta_L, \theta_R} = e^{i(-[\pi+\theta_R]+[\pi+\theta_L])/2} \text{id}|_{\theta_L, \theta_R}. \quad (3.21)$$

Next we see that this projective phase can be captured by the third group cohomology.

3.4 Third group cohomology

3.4.1 Projective phases on Hilbert spaces on intervals

Consider a 1+1-dimensional theory with boundary conditions \mathcal{B}_i and symmetry G . For each pair of boundary conditions $(\mathcal{B}_L, \mathcal{B}_R)$, we have the Hilbert space $\mathcal{H}_{\mathcal{B}_L, \mathcal{B}_R}$ (for a fixed length of the interval). A boundary condition \mathcal{B} does not necessarily fixed by the symmetry action, but rather mapped to another boundary condition $g\mathcal{B}$. Accordingly, the Unitary in general maps a Hilbert space to another Hilbert space¹⁷¹⁸:

$$U_g|_{\mathcal{B}_L, \mathcal{B}_R} : \mathcal{H}_{\mathcal{B}_L, \mathcal{B}_R} \rightarrow \mathcal{H}_{g\mathcal{B}_L, g\mathcal{B}_R}. \quad (3.22)$$

Then, the composition rule of U_g 's might involve projective phases:

$$U_{g_1} U_{g_2}|_{\mathcal{B}_L, \mathcal{B}_R} = e^{i\tilde{\alpha}(g_1, g_2, \mathcal{B}_L, \mathcal{B}_R)} U_{g_1 g_2}|_{\mathcal{B}_L, \mathcal{B}_R}. \quad (3.23)$$

We further assume that the phase is *local*; it should be a sum of contributions from left and right boundaries, and the two contributions should obey the same rule:

$$\tilde{\alpha}(g_1, g_2, \mathcal{B}_L, \mathcal{B}_R) = \hat{\alpha}(g_1, g_2, \mathcal{B}_L) - \hat{\alpha}(g_1, g_2, \mathcal{B}_R) \quad (3.24)$$

for some function $\hat{\alpha}$. The negative sign is because the orientation is reversed between the left and right boundaries; in other words the contribution from the right boundary should

¹⁷Of course all the separable Hilbert spaces are isomorphic but identifying all of the Hilbert spaces $\mathcal{H}_{\mathcal{B}_L, \mathcal{B}_R}$ by this universal isomorphism is just confusing and of no use.

¹⁸When the theory is a TQFT, there is a multiplication between the state in $\mathcal{H}_{\mathcal{B}_L, \mathcal{B}_M}$ and $\mathcal{H}_{\mathcal{B}_M, \mathcal{B}_R}$ valued in $\mathcal{H}_{\mathcal{B}_L, \mathcal{B}_R}$. You can regard this structure as a **category**, whose objects are the boundary conditions, and morphisms are states on intervals (equivalently, topological boundary-changing operators). Then U_g is a self-functor on this category, defining **categorical representation** of the (suspension of the) group G . However the composition rule of the functors is up to natural transformations, which are the projective phases. When the theory is a general QFT there is no canonical multiplication among the states but there is a *family* of multiplications. In such cases KO imagines that the boundary conditions would define a category over some operad (something like the swiss-cheese one), and we should be dealing with the representation theory valued in such categories. As far as the anomaly is concerned, the only thing for which we need the operad-category structure is to show the splitting (3.24). In higher dimensional QFTs, the category also has to be higher.

be the time-reversal of the one from the left. In particular, we assume $\tilde{\alpha}(g_1, g_2, \mathcal{B}, \mathcal{B}) = 0$. As this can also be considered as the IR cutoff of \mathbb{R} when the interval is long, this should indeed vanish according to the assumptions we made in Section 3.2.

We also consider the ambiguity of the unitary. In this case we also demand that the phase redefinition of the unitary is *local* and should look like

$$U_g|_{\mathcal{B}_L, \mathcal{B}_R} \mapsto e^{i(\hat{\beta}(g, \mathcal{B}_L) - \hat{\beta}(g, \mathcal{B}_R))} U_g|_{\mathcal{B}_L, \mathcal{B}_R} \quad (3.25)$$

for some function $\hat{\beta}$.

3.4.2 Group cohomology

Now let us see that the projective phase (3.23), subject to the splitting (3.24) and the ambiguity (3.25), defines a group cohomology. First, we fix a boundary condition \mathcal{B}_0 , define a 3-cochain $\alpha \in C^3(G)$ by

$$\alpha(g_1, g_2, g_3) = \tilde{\alpha}(g_1, g_2, g_3 \mathcal{B}_0, \mathcal{B}_0) = \hat{\alpha}(g_1, g_2, g_3 \mathcal{B}_0) - \hat{\alpha}(g_1, g_2, \mathcal{B}_0). \quad (3.26)$$

From this definition, we have

$$\tilde{\alpha}(g_1, g_2, g_3 \mathcal{B}_0, g_4 \mathcal{B}_0) = \alpha(g_1, g_2, g_3) - \alpha(g_1, g_2, g_4). \quad (3.27)$$

Now the associativity of $U_{g_1} U_{g_2} U_{g_3}|_{g_4 \mathcal{B}_0, \mathcal{B}_0}$ demands

$$\begin{aligned} 0 = (\delta_4 \alpha)(g_1, g_2, g_3, g_4) &= \alpha(g_2, g_3, g_4) + \alpha(g_1, g_2 g_3, g_4) \\ &\quad - \alpha(g_1, g_2, g_3 g_4) - \alpha(g_1, g_2, g_3) + \alpha(g_1 g_2, g_3, g_4). \end{aligned} \quad (3.28)$$

Also, the ambiguity (3.25) causes the shift to α by,

$$\alpha(g_1, g_2, g_3) \rightarrow \alpha(g_1, g_2, g_3) + (\delta_3 \beta)(g_1, g_2, g_3) \quad (3.29)$$

where $\beta(g_1, g_2) = \hat{\beta}(g_1, g_2 \mathcal{B}_0) - \hat{\beta}(g_1, \mathcal{B}_0)$ and δ_3 is the one we defined before in (2.41).

Exercise 3.2. Check that $\delta_4 \circ \delta_3 = 0$.

Now we define the 3rd cohomology in the same way we did for the second cohomology:

$$H^3(G) = \text{Ker}(\delta_4) / \text{Im}(\delta_3). \quad (3.30)$$

This is the invariant information from the projective phases associated to the Hilbert spaces on intervals.

Exercise 3.3. Calculate $H^3(\mathbb{Z}_2)$.

3.4.3 Anomaly-BC relationship

In the above definition (3.28), α depends on the choice of \mathcal{B}_0 . So is this projective phase a property of the boundary condition, rather than a theory? The answer is no, the cohomology class of the projective phase should be uniform over the boundary conditions of a fixed theory. To see this, we assume that the boundary conditions admits a sum so that $\mathcal{H}_{\mathcal{B}_1 \oplus \mathcal{B}_2, \mathcal{B}_R} = \mathcal{H}_{\mathcal{B}_1, \mathcal{B}_R} \oplus \mathcal{H}_{\mathcal{B}_2, \mathcal{B}_R}$. The boundary condition $\mathcal{B}_1 \oplus \mathcal{B}_2$ is like putting a qubit entangled with the boundary condition (e.g. the value of $X(0)$ in the periodic scalar example). Then, we also want the symmetry U_g acts on $\mathcal{H}_{\mathcal{B}_1 \oplus \mathcal{B}_2, \mathcal{B}_R}$. However if the cohomology classes associated to \mathcal{B}_1 and \mathcal{B}_2 were different, the projective “phase” on this direct sum space is not a overall phase, but just a diagonal matrix not proportional to the identity. This breaks the assumption that the symmetry is G . Therefore, the cohomology class is the **property of the theory**, not the boundary condition, and you can calculate it by picking any boundary condition you like.

This leads to the following theorem, first claimed by Thorngren and Wang [16]¹⁹:

Theorem 3.1. *If there is a simple boundary condition \mathcal{B} such that $g\mathcal{B} = \mathcal{B}$ for elements of a symmetry group $g \in G$, this symmetry G is non-anomalous, i.e. the cohomology class is trivial. (This hold for a QFT in a general dimensions.)*

Indeed by the definition (3.26) $\alpha = 0$ by choosing $\mathcal{B}_0 = \mathcal{B}$ in such a case.²⁰

3.5 Prospects

3.5.1 Background formalism

3.5.2 Higher diemnsions

3.5.3 Fermions

¹⁹See also Hellerman, Orlando, and Watanabe [17].

²⁰The work by Thorngren and Wang [16] is not this trivial, as their definition of the anomaly is different from us.

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