Applications of Baire's Category Theorem

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Basics(Metric Space)

definition

Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is said to be a *metric* or a distance function on X if d satisfies the following properties:

- $d(x, y) \geq 0 \; \forall x, y \in X$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) = d(y, x) \; \forall x, y \in X$
- $d(x, z) \leq d(x, y) + d(y, z) \in x, y, z \in X$ (Triangle Inequality)

The pair (X,d) consisting of set X together with distance function d defined on it is called metric space.

Example

Take $X = \mathbb{R}$. Then the following function define metric on X:
$d(x, y) = |x - y| \quad \forall x, y \in \mathbb{R}$

Metric Space(Balls and Bounded set)

Balls

Let (X, d) be the metric space. if $a \in X$ and $\varepsilon > 0$ then:
The *open ball* of radius ε centered at a is defined as

$$B(a, \varepsilon) = \{x \in X : d(x, a) < \varepsilon\}$$

Similarly, Closed ball of radius ε centered at a can be defined as:

$$\bar{B}(a, \varepsilon) = \{x \in X : d(x, a) \leq \varepsilon\}$$

Bounded set

Let (X, d) be a metric space, let $Y \subseteq X$.Then we say that Y is bounded if Y is contained in some open ball.

Open & Closed Sets

Open set

If (X, d) is a metric space, then we say that a subset $U \subseteq X$ is open if for each $y \in U$ there exists some $\delta > 0$ such that the open ball $B(y, \delta)$ is contained in U.

Closed Set

If (X, d) is a metric space, then we say that a subset $F \subseteq X$ is closed set in X if and only if its compliment $F^c = X \backslash F$ is an open subset.

Example

- $\mathbb{R}$ is open and closed in $\mathbb{R}$
- (0,1) is open and not closed & $[0, 1]$ is closed and not open in $\mathbb{R}$.
- $(0, 1]$ is neither open nor closed in $\mathbb{R}$.
- The rationals $\mathbb{Q}$ are not open in $\mathbb{R}$

Dense & Nowhere Dense

Dense set

We say a subset $D \subset X$ of a metric space is dense in X if for any given $x \in X$ and $r > 0$, we have $B(x, r) \cap D \neq \phi$. In other words, any non-empty open set in X must contain a point of D.

Nowhere Dense

A subset $A \subset X$ of a metric space is said to be nowhere dense in X, if given any nonempty open set U, we can find a nonempty open subset $V \subset U$ such that $A \cap V = \phi$.

Baire's Category Theorem

Let (X,d) be a complete metric space. Then

1. Let $\forall n$ U_n be open and dense in X. Then $\bigcap\limits_{n\in\mathbb{N}} U_n \neq \phi$
2. Let $\forall n$ F_n be closed set with empty interior i.e. F_n is closed and nowhere dense. Then $\bigcup\limits_{n\in\mathbb{N}} F_n \neq X$

Proof

Proof: As we can observe that both sentence in some sense equivalent. As for, G is open and dense *iff* its complement $F := X \setminus G$ is closed and nowhere dense. Hence any one of them follows from the other by taking compliment of each other. So, we confine our self to proving just one.

- Let $U := \bigcap_{n \in \mathbb{N}} U_n$. We have to prove that U is dense in X.

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- Let $U := \bigcap_{n \in \mathbb{N}} U_n$. We have to prove that U is dense in X.
- Let $x \in X$ and $r > 0$ be given. We need to show that $B(x, r) \cap U \neq \phi$.

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- Let $x \in X$ and $r > 0$ be given. We need to show that $B(x, r) \cap U \neq \phi$.
- Since U_1 is dense and $B(x, r)$ is open there exists $x_1 \in B(x, r) \cap U_1$.

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- Since U_1 is dense and $B(x,r)$ is open there exists $x_1 \in B(x,r) \cap U_1$.
- Since $B(x,r) \cap U_1$ is open, there exists r_1 such that $0 < r_1 < 1/2$ and $\bar{B}(x_1, r_1) \subset B(x,r) \cap U_1$.

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- Since $B(x, r) \cap U_1$ is open, there exists r_1 such that $0 < r_1 < 1/2$ and $\bar{B}(x_1, r_1) \subset B(x, r) \cap U_1$.
- We repeat this argument for the open set $B(x_1, r_1)$ and the dense set U_2 to get $x_2 \in B(x_1, r_1) \cap U_2$. Again, we can find r_2 such that $0 < r_2 < 2^{-2}$ and $\bar{B}(x2, r2) \subset B(x_1, r_1) \cap U_2$.

Proof Cont.

- Proceeding this way, we get for each $n \in \mathbb{N}$, x_n and r_n with the properties

$$\bar{B}(x_n, r_n) \subset B(x_{n-1}, r_{n-1}) \cap U_n \text{ and } 0 < r_n < 2^{-n}$$

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$$\bar{B}(x_n, r_n) \subset B(x_{n-1}, r_{n-1}) \cap U_n \text{ and } 0 < r_n < 2^{-n}$$

- Clearly, the sequence (x_n) is Cauchy: if $m \leq n$,

$$d(x_m, x_n) \leq d(x_n, x_{n-1}) + \ldots + d(x_{m+1}, x_m) \leq \sum_{k=m}^{n} 2^{-k}$$

Since $\sum_k 2^{-k}$ convergent, it follows that (x_n) is Cauchy.

Proof Cont.

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  Since $\sum_k 2^{-k}$ convergent, it follows that (x_n) is Cauchy.
- Since X is complete, there exists $x_0 \in X$ such that $x_n \to x_0$. Since x_0 is the limit of the sequence $(x_n)_{n \geq k}$ in the closed set $\bar{B}(x_k, r_k)$. we deduce that $x_0 \in \bar{B}(x_k, r_k) \subset B(x_{k-1}, r_{k-1}) \cap U_k$ for all k. In particular, $x_0 \in B(x, r) \cap U_k$

Applications of Baire's Category Theorem

1. There exists a continuous function on $[0, 1]$ which is not differentiable at any point.

Proof: Let f $\in$ C[0,1] be differentiable at t.

$$\text{Define } g : [-t, 1-t] \to \mathbb{R}$$

$$g(h) = \begin{cases} \frac{f(t+h)-f(t)}{h} & h \neq 0 \\ f'(t) & h = 0 \end{cases}$$

g is continuous. Hence, g is bounded (Since, it is defined on closed and bounded interval)

So, by APR, $\exists n \in \mathbb{N}$ such that

$$\left| \frac{f(t+h) - f(t)}{h} \right| \leq n$$

provided t+h $\in$ [0,1]

Now, for each positive integer n define

$$C_n \quad = \quad \{f \in C[0,1] : \left|\frac{f(t+h) - f(t)}{h}\right| \leq n, \\ \text{for some t and all h with t+h} \in [0,1]\}$$

We have already proved that if f is differentiable a even one point, then f $\in C_n$ for some n.
So, we are seeking a f $\notin \bigcup_{n\in\mathbb{N}} C_n$.
Consider the sup metric

$$d(f,g) = sup_{x\in[0,1]}|f(x) - g(x)|$$

We need to show

$$\bigcup_{n\in\mathbb{N}} C_n \neq C[0,1]$$

We aim to prove that each C_n is closed and nowhere dense.

Each C_n is closed:-
Since C[0,1] is a metric space, so it is enough to show that if $\{f_k\} \subseteq C_n$ with $f_k \to f$, then f $\in C_n$
Now, $f_k \in C_n$ implies $\exists t_k \in$ [0,1] such that

$$\left| \frac{f_k(t_k + h) - f_k(t_k)}{h} \right| \leq n \quad \forall h$$

t_k is a sequence in [0,1], by Bolzano–Weierstrass theorem, it has a convergent subsequence. We call this subsequence again by t_k and let $t_k \to t_0$

Now,

$$\begin{aligned}\left|\frac{f(t_0+h)-f(t_0)}{h}\right| &\leq \left|\frac{f(t_0+h)-f(t_k+h)}{h}\right| + \left|\frac{f(t_k+h)-f_k(t_k+h)}{h}\right| \\ &+ \left|\frac{f_k(t_k+h)-f_k(t_k)}{h}\right| + \left|\frac{f_k(t_k)-f(t_k)}{h}\right| \\ &+ \left|\frac{f(t_k)-f(t_0)}{h}\right| \\ &= (1)+(2)+(3)+(4)+(5)\end{aligned}$$

For fix h, for any $\epsilon > 0$, if k is large enough, (1) and (5) are smaller than ϵ, since f is continuous and $t_k \to t_0$
(2) and (4) are smaller than ϵ, since f_k is uniformly convergent to f
and (3) $\leq$ n.

Hence, we get

$$\left|\frac{f(t_0+h)-f(t_0)}{h}\right| \leq n+4\epsilon \text{ for any } \epsilon > 0$$

so that

$$\left|\frac{f(t_0+h)-f(t_0)}{h}\right| \leq n$$

and hence, f $\in C_n$. Thus, each C_n is closed

C_n is nowhere dense:-

We now show that C_n is nowhere dense; that is, given any $g \in C_n$ and any $\epsilon > 0$, there exists $f \in$ C[0, 1] such that d(f,g) $< \epsilon$ and f $\notin C_n$.

Now a typical example of a function in C[0, 1] which is not in C_n is the "sawtooth" function. For any n, we can find such a function, whose norm is less than or equal to any prescribed $\epsilon > 0$, and where the slope of each line segment is greater than n in absolute value. To find such a function f $\notin C_n$, with an ϵ distance from g, we need only construct a sawtooth function close to g, as in Figure 1

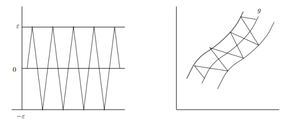

Figure 1: "Saw-tooth" Function

To construct such a function f, we use the uniform continuity of g to get a piecewise linear function g1 with d(g,g1) $< \frac{\epsilon}{2}$
For each linear piece of g1 we construct a saw-tooth function with slope greater than n in absolute value and hence C_n is nowhere dense.
So, by Baire's Category Theorem $\bigcup_{n \in \mathbb{N}} C_n \neq C[0,1]$

1. There exists a continuous function on $[0, 1]$ which is not differentiable at any point.
2. There exists no function $f : (0, 1) \to \mathbb{R}$ which is continuous at rationals and discontinuous at irrationals.

First, we will prove the following lemma

Lemma

$\mathbb{Q} \cap [0,1]$ can't be written as a countable intersection of open and dense subsets of $\mathbb{R}$

Proof: Assume

$$\mathbb{Q} \cap [0,1] = \bigcap_{n \in \mathbb{N}} U_n$$

where U_n open dense in [0,1], and $\mathbb{Q} \cap [0,1] = \{q_n\}_{n \in \mathbb{N}}$

Now define

$$V_n = U_n \backslash \{q_n\}$$

Then clearly, V_n open and dense in [0,1] Consider all $U_n's$ and $V_n's$ together to get a countable collection of open and dense sets

$$\left(\bigcap_{n \in \mathbb{N}} U_n \right) \cap \left(\bigcap_{n \in \mathbb{N}} V_n \right) = \mathbb{Q} \cap \mathbb{Q}' = \phi$$

which is a contradiction to BCT !

To Prove : There exists no function $f : [0,1] \to \mathbb{R}$ which is continuous at rationals and discontinuous at irrationals.
Let if possible there exist such a function f s.t.

$$f : [0,1] \to \mathbb{R}$$

We know f is continuous at a
$\iff \forall \epsilon > 0\ \exists \delta > 0$ s.t. $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$
We know, by Archimedean property of $\mathbb{R}$, we can find $n > \frac{1}{\epsilon}$ i.e. $\frac{1}{n} < \epsilon$ for every $\epsilon > 0$
So, δ, that will work for $\frac{1}{n}$ will also work for ϵ

Therefore, we can say f is continuous

$$\iff \forall \epsilon > 0 \; \exists \; \delta \text{ s.t. } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

$$\iff \forall n \in \mathbb{N} \; \exists \text{ open interval I} \ni a \text{ s.t. diameter}(f(I)) < 1/n$$

$$U_n = \cup\{I \mid \text{I is an open interval in [0,1] and diameter(f(I))} < \frac{1}{n}\}$$

$$\text{f is continuous at a} \iff a \in U_n \forall n \in \mathbb{N}$$

Therefore, points of continuity $= \bigcap_{n\in\mathbb{N}} U_n$
i.e. Points of continuity $\neq \mathbb{Q} \cap [0, 1]$
(since, we proved $\mathbb{Q} \cap [0, 1]$ can't be written as a countable intersection of open and dense subsets of $[0, 1]$)
So, there exists no function which is continuous at rationals and discontinuous at irrationals.

Application of Baire's Category Theorem

1. There exists a continuous function on $[0, 1]$ which is not differentiable at any point.
2. There exists no function $f : (0, 1) \to \mathbb{R}$ which is continuous at rationals and discontinuous at irrationals.
3. Non-empty complete metric space without isolated points is uncountable.

Proof

Let (X, d) is a countable complete metric space without isolated points. Then, we can write $X = \{x_n : n \geq 1\}$(as we assume it is countable). Now, consider the Sets $U_n = X \backslash \{x_n\}$. Since $\{x_n\}$ is closed, U_n is open. Also because $\{x_n\}$ is not isolated(as given), for all $\varepsilon > 0$, $B(x_n, \varepsilon) \cap U_n \neq \phi$, Hence U_n is dense in X.
Now, by Baire's Category Theorem $\bigcap_{i=1}^{\infty} U_n$ is dense in X. But since, $\bigcap_{i=1}^{\infty} U_n \neq \phi$ is a contradiction so our assumption that X is countable is not true hence, X must by uncountable.

Application of Baire's Category Theorem(Cont.)

1. There exists a continuous function on [0, 1] which is not differentiable at any point.
2. There exists no function f : $(0, 1) \rightarrow \mathbb{R}$ which is continuous at rationals and is continuous at irrationals.
3. non-empty complete metric space without isolated points is uncountable.
4. If F is infinitely differentiable and suppose that for each x there is an integer $n \in \mathbb{N}$ such that $f^{(n)}(x) = 0$. Then f is a polynomial.

Proof

1. Let $A_n = \{x \in R | f^{(n)}(x) = 0\}.E_n.$ the interior of A_n. Clearly $E_n \subset E_m$ for $n < m$, and by Baire E_n is eventually not empty.

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2. Each E_n is a countable union of open segments. It is easy to see that in passing from E_n to E_{n+1} new segments can appear, but those already in E_n remain unchanged. Moreover two such segments are never adjacent.

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3. By this remark is it enough to prove that $\bigcup E_n = \mathbb{R}$. Indeed if this holds and $E_n \neq \phi$, then $E_n = \mathbb{R}$, which implies the thesis. Otherwise the points in the boundary of E_n don't appear in the union.

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4. Let $E = \cup E_n$, B its complementary set, and assume by contradiction $B \neq \phi$. B is itself a complete metric space, hence can apply Baire to it. So for some k we find that $A_k \cap B$ has non-empty interior in B. This means that there is an interval I such that $B \cap I \subset A_k$ (and $B \cap I \neq \Phi$).

Proof Cont.

1. From 2, B has no isolated points. The contradiction that we want to find is that $I \backslash B \subset A_k$. Indeed from this it follows that $I \subset A_k$, hence $E_k \cap B \neq \phi$.

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2. By construction $I \backslash B$ is a union of intervals which appear in some E_n. Take such an interval J, say $J \subset E_N$ (where N is minimal), and let x be one end point of J (which is not on the boundary of I). Then $x \in I \cap B \subset A_k$, so $f^{(k)}(x) = 0$. Moreover x is not isolated in B, so it is the limit of a sequence x_i of points in B.

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3. By the same argument, $f^{(k)}(x_i) = 0$. Between two point where the $k-$th derivative vanish lies a point where the $k+1$-th does, so by continuity we find $f^{(k+1)}(x) = 0$. Similarly we find $f^m(x) = 0$ for all $m \geq k$. On J f is a polynomial of degree N; it follows that $N \leq k$, and we conclude that $J \subset E_k$. Since J was arbitrary we conclude that $I \backslash B \subset E_k$, which we have shown to be a contradiction.

Other Applications Baire's Category Theorem

- Infinite dimensional Banach space has no countable basis.
- There exist 2π -periodic continuous functions whose Fourier series diverge on an uncountable set.
- Every residual subset of $\mathbb{R}$ is dense.
- The rationals are not completely metrizable.

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