Euclidean Vs Non-Euclidean Geometry

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Euclidean Geometry

1. **Euclidean geometry** is a mathematical system created by the ancient Greek mathematician Euclid, who documented it in his textbook called **the Elements**.
2. Euclid's method involves starting with a few simple and intuitive axioms (or postulates) and then using logical deduction to prove many other propositions (or theorems) based on these axioms.
3. He was the first to systematically organize these propositions into a logical system, where each result is derived from axioms and previously proven theorems.

History [1]

- In the early 19th century, significant progress was made in the development of non-Euclidean geometry. Russian mathematician Nikolai Ivanovich Lobachevsky and Hungarian mathematician János Bolyai independently published treatises on hyperbolic geometry in 1829-1830 and 1832 respectively.
- Lobachevsky's approach involved negating the parallel postulate, while Bolyai developed a geometry where both Euclidean and hyperbolic geometries were possible depending on a parameter "k".

History

- In 1854, Bernhard Riemann, in a famous lecture, established the field of Riemannian geometry. He discussed concepts such as manifolds, Riemannian metric, and curvature, which are fundamental to understanding non-Euclidean geometries.
- Riemann also constructed an infinite family of non-Euclidean geometries by providing a formula for a family of Riemannian metrics on the unit ball in Euclidean space, expanding the understanding and possibilities of non-Euclidean geometries.

Euclid's Postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All Right Angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two Right Angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the Parallel Postulate.

Playfair's axiom states that given a line and a point not on that line, there is exactly one line through the point that is parallel to the given line. In other words, the axiom asserts that parallel lines exist and are unique. This statement is equivalent to Euclid's parallel postulate and can be used as a substitute for it in many geometric proofs.

Various attempts to prove the 5th postulate

- **Proclus' attempt (400s CE):** He tried to prove the parallel postulate by assuming its negation and showing that it leads to a contradiction. However, his proof was incomplete and contained some errors.
- **Ibn al-Haytham's attempt (1000s CE):** Ibn al-Haytham attempted to prove the parallel postulate by using a visual method called "alhazen's problem" which involves drawing lines tangent to circles. While his approach was geometrically sound, it did not actually prove the parallel postulate.
- **John Wallis' attempt (1600s):** John Wallis tried to prove the parallel postulate by assuming a different postulate known as the "obtuse angle postulate" which states that given any triangle, there exists an obtuse angle. However, this approach was not widely accepted as a valid proof.
- **Giordano Vitale's attempt (1700s):** He attempted to prove the parallel postulate by using the method of proof by contradiction. However, his proof was flawed and contained circular reasoning.

Various attempts to prove the 5th postulate

- **Johann Lambert's attempt (1700s):** Johann Lambert tried to prove the parallel postulate by using a non-Euclidean geometry, which assumes that the sum of the angles in a triangle is less than two right angles. While his approach was valid, it did not actually prove the parallel postulate within Euclidean geometry.
- **Nikolai Lobachevsky's attempt (1800s):** Nikolai Lobachevsky tried to prove the parallel postulate by assuming its negation and constructing a consistent non-Euclidean geometry where the postulate does not hold. This was a breakthrough in the development of non-Euclidean geometries and led to a new branch of mathematics.
- **János Bolyai's attempt (1800s):** János Bolyai independently arrived at a similar non-Euclidean geometry as Lobachevsky and also demonstrated that the parallel postulate is not a necessary assumption for geometry.

Axiomatic Basis Of Non-Euclidean Geometry [2]

To obtain a non-Euclidean geometry, the parallel postulate (or its equivalent) must be replaced by its negation. Negating Playfair's axiom form, since it is a compound statement (... there exists one and only one ...), can be done in two ways:

- In the first case, replacing the parallel postulate (or its equivalent) with the statement "In a plane, given a point P and a line l not passing through P, there exists at least two lines through P, which do not meet l" and keeping all the other axioms, yields hyperbolic geometry.

Axiomatic Basis Of Non-Euclidean Geometry

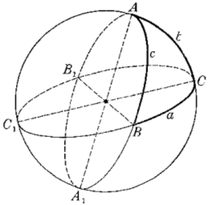
- The second case is not dealt with as easily. Simply replacing the parallel postulate with the statement, "In a plane, given a point P and a line l not passing through P, all the lines through P meet l", does not give a consistent set of axioms. This follows since parallel lines exist in absolute geometry, but this statement says that there are no parallel lines. This problem was known (in a different guise) to Khayyam, Saccheri and Lambert and was the basis for their rejecting what was known as the "obtuse angle case". To obtain a consistent set of axioms that includes this axiom about having no parallel lines, some other axioms must be tweaked. These adjustments depend upon the axiom system used. Among others, these tweaks have the effect of modifying Euclid's second postulate from the statement that line segments can be extended indefinitely to the statement that lines are unbounded. Riemann's elliptic geometry emerges as the most natural geometry satisfying this axiom. In this paper, we will be discussing about Spherical geometry, which is closely related to Elliptical geometry.

Spherical Geometry [3]

1. In Euclidean geometry, it is well-known that the shortest distance between two points is a straight line. However, in more complex two-dimensional surfaces, if the surface is represented by parameters, the theory of differential geometry can be used to derive a complex differential equation, whose solution represents the shortest path between two points on the surface.
2. Such a curve is called a **geodesic**. As **"geo"** means earth in Greek, the concept of a geodetic refers to the shortest path between two points on the surface of the earth.

Geodesic in Spherical Geometry

- When a sphere is cut with a plane passing through the center of the sphere, the intersection forms a great circle, which has a diameter equal to the diameter of the sphere.
- In the field of differential geometry, it can be proven that the shortest path between two points on a sphere is a segment of the great circle that passes through those two points.



Angles in Spherical Geometry

- The points where two different great circles intersect are located exactly opposite to each other along a diameter, which also serves as the line where the planes that generate the two great circles intersect. The angle between two great circles is determined by the angle between their corresponding planes.
- The axis, or diameter, that is perpendicular to a plane belonging to a great circle intersects the sphere at two opposite points known as the poles of the great circle.
- When one great circle intersects the other at its poles, the two great circles are perpendicular to each other, as shown in the diagram below.

Spherical Geometry

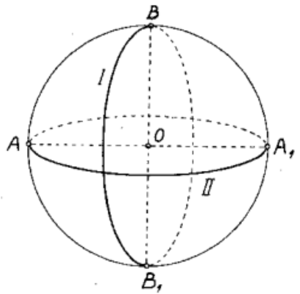


Fig. 49.

Spherical Triangles

- A spherical triangle is a portion of a sphere that is enclosed by the arcs of three great circles, which are referred to as the sides of the spherical triangle. These sides are typically measured in degrees or radians.
- The angles of the spherical triangle are determined by the intersection angles between the great circles that form it.
- The concepts of isosceles triangles, height, bisecting lines, and bisector normal are similar to those in plane geometry.
- However, it's worth noting that the sum of the three angles in a spherical triangle is always greater than π.

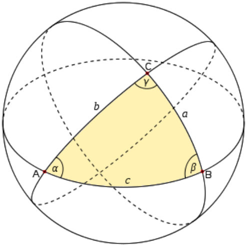
Area of Spherical Triangle [4]

The surface area of any spherical triangle is given by :

$$A = R^2 E$$

where R is the radius of the sphere and E is the angular excess given by

$$E = A + B + C - \pi$$

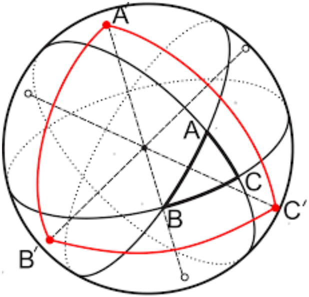


Perpendicularity of lines in Spherical triangles

- Every great circle has a pair of poles : meets of sphere with altitude line to the plane of the great circle through the center.
- **Perpendicularity**: Two lines are perpendicular iff one passes through the poles of the other.
- **Polar of a Point:** For a point A on the sphere, its polar is the line *greatcircle* perpendicular to the diameter *axis* through A. So, every line has two poles, and every point has one polar line.
- **Perpendicularity of points:** Two points are perpendicular iff one lies on the polar of the other.

Spherical Geometry

Polar Triangle: If ABC is a spherical triangle , choose poles A', B', C' of lines BC, AC, AB respectively which are on the same sides as A,B,C. Then $A'B'C'$ is the polar triangle of ABC.



Theorem : [5] Let $\triangle$ ABC be a spherical triangle on the surface of a sphere whose center is O. Let the sides a,b,c of $\triangle$ ABC be measured by the angles subtended at O, where a,b,c are opposite A,B,C respectively. Let $\triangle A'B'C'$ be the polar triangle of $\triangle$ ABC. Then A' is the supplement of a.That is:

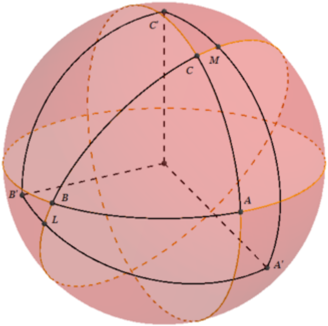
$$A' = \pi - a$$

and it follows by symmetry that:

$$B' = \pi - b$$

$$C' = \pi - c$$

Spherical Geometry



Applying the above theorem about the sides and the angles in the polar triangle to a spherical triangle, we can prove that the sum of the three angles in a spherical triangle is always bigger than 180 °and less than 540°.

We assume that we have constructed the polar triangle to a spherical triangle ABC. According to the above theorem the sides in the polar triangle are $180^\circ - A$, $180^\circ - B$, $180^\circ - C$

Since the sum of the three sides must be less than 360 $^\circ$, the following inequality is valid:

$$180^\circ - A + 180^\circ - B + 180^\circ - C < 360^\circ$$

$$\iff A + B + C > 180^\circ$$

Furthermore, the sum of the sides in the polar triangle must be greater than 0

$$180^\circ - A + 180^\circ - B + 180^\circ - C > 0^\circ$$

$$\iff A + B + C < 540^\circ$$

Spherical Geometry

Similarities/Differences with the Euclidean postulates: [10]

1. A line segment can be drawn joining any two points(although not unique).
2. Any line that is extended will eventually loop back on itself to form a closed curve.
3. Given any line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All Right Angles are congruent.
5. There is no point through which a line can be drawn that never intersects a given line.

Hyperbolic Geometry [6]

There are various methods to create hyperbolic geometry, referred to as "models". Among these models, we will focus on one that is straightforward and practical, known as the upper half-plane model.

- The upper half-plane $\mathbb{H}$ is the set of complex numbers z with positive imaginary part:

$$\mathbb{H} = \{z \in \mathbb{C} | Im(z) > 0\}$$

- **Definition :** The circle at infinity or boundary of $\mathbb{H}$ is defined to be the set

$$\partial\mathbb{H} = \{z \in \mathbb{C} |\ \mathrm{Im(z)} = 0\} \bigcup \{\infty\}$$

. That is, $\partial\mathbb{H}$ is the real axis together with the point ∞.

Hyperbolic Geometry

Remarks:

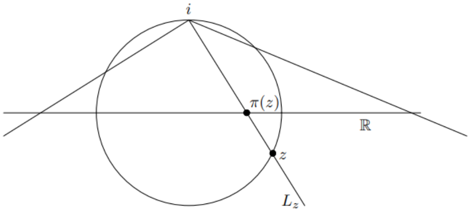
1. What does ∞ mean? It's just a point that we have 'invented' so that it makes sense to write things like $1/x \to \infty$ as $x \to 0$
2. We call $\partial\mathbb{H}$ the circle at infinity because topologically, it is a circle. We can see this using a process known as stereographic projection.

Let

$$K = \{z \in \mathbb{C} |\ |z| = 1\}$$

denote the unit circle in the complex plane $\mathbb{C}$.

$$\pi : K \to \mathbb{R} \bigcup \infty$$



The map π is a homeomorphism from K to $\mathbb{R} \bigcup \infty$.

Path Integrals

Prior to establishing distances in $\mathbb{H}$, it is necessary to review the methodology for computing path integrals in $\mathbb{C}$ (or equivalently, in $\mathbb{R}^2$). A path σ in the complex plane $\mathbb{C}$ refers to the representation of a continuous function,

$$\sigma(.) : [a, b] \to \mathbb{C}$$

where $[a, b] \subset \mathbb{R}$ is an interval. It is assumed that σ is differentiable and its derivative $\sigma'(t)$ is continuous. The points $\sigma(a)$ and $\sigma(b)$ are referred to as the end-points of the path σ. A function $\sigma : [a, b] \to \mathbb{C}$ that maps the interval to a given path is called a parametrization of that path. It is important to note that a path can have multiple parametrizations.

Hyperbolic Geometry

Let $f : \mathbb{C} \to \mathbb{R}$ be a continuous function. Then the integral of f along a path σ is defined to be:

$$\int_\sigma f = \int_a^b f(\sigma(t))\,|\sigma|'(t)\,dt$$

here $|\,.\,|$ denotes the usual modulus of a complex number, i.e.

$$|\sigma'(t)| = \sqrt{(Re(\sigma'(t)))^2 + (Im(\sigma'(t)))^2}$$

Distance in Hyperbolic Geometry

To define the hyperbolic metric in the upper half-plane model of hyperbolic space , we first define the length of an arbitrary piecewise continuously differentiable path in $\mathbb{H}$.
Let $\sigma : [a, b] \to \mathbb{H}$ be a path in the upper half-plane $\mathbb{H} = z \in \mathbb{C} | Im(z) > 0$.
Then the hyperbolic length of σ is obtained by integrating the function f(z) = 1/ Im(z) along σ i.e.

$$length_{\mathbb{H}}(\sigma) = \int_{\sigma} \frac{1}{Im(z)} = \int_{a}^{b} \frac{|\sigma'(t)|}{Im(\sigma(t))}\, dt$$

We are now in a position to define the hyperbolic distance between two points in $\mathbb{H}$.

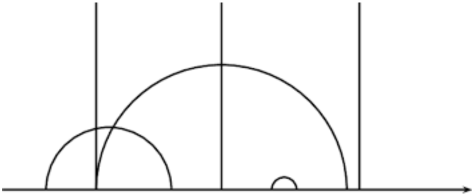
<u>**Definition :**</u> Let $z, z' \in \mathbb{H}$. We define the hyperbolic distance $d_{\mathbb{H}}(z, z')$ between z and z' as

$$d_{\mathbb{H}}(z,z') = \inf\{length_{\mathbb{H}}(\sigma) | \sigma \text{is a piecewise continuously differentiable path with end-points z and z}'\}$$

Remark: The smallest possible length (infimum) is attained by a specific path called a geodesic, and this geodesic is the only one that achieves this minimum length.

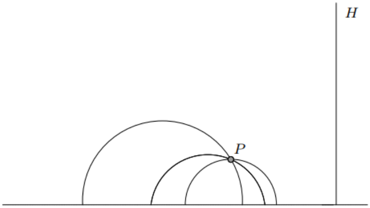
Geodesics in $\mathbb{H}$

The geodesics, or the shortest paths, in the half-plane model (denoted as $\mathbb{H}$) of hyperbolic geometry are either semi-circles that are perpendicular to the real axis, or vertical straight lines. Additionally, for any two points in $\mathbb{H}$, there is a single geodesic that passes through them, and this geodesic is unique.



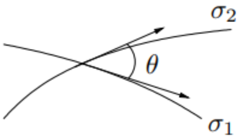
Failure of Euclid's parallel postulate

We can observe that Euclid's parallel postulate does not hold in the half-plane model of hyperbolic geometry. Specifically, if we have a geodesic (shortest path) and a point not on that geodesic, there are infinitely many other geodesics that pass through that point but do not intersect the given geodesic.



Angles

Assume that we have two paths, denoted as σ_1 and σ_2, which intersect at a point z in the half-plane model denoted as $\mathbb{H}$. By selecting suitable parameterizations for the paths, we can assume that z is the initial point of both paths, i.e., $z = \sigma_1(0) = \sigma_2(0)$. The angle between σ_1 and σ_2 at z is defined as the angle between their tangent vectors at the point of intersection.



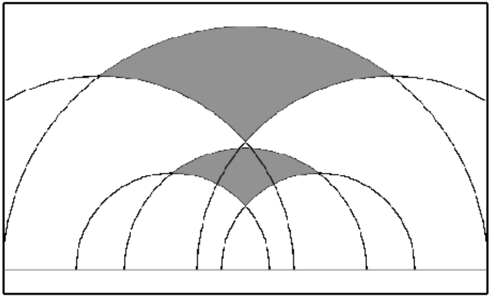
Area

Let $A \subset \mathbb{H}$ be a subset of the upper half-plane. The hyperbolic area of A is defined to be the double integral

$$\text{Area}_{\mathbb{H}}(A) = \int\int_A \frac{1}{y^2} dx\, dy = \int\int_A \frac{1}{Im(z)^2} dz$$

Triangle

Sum of angles of triangles in hyperbolic geometry is less than 180°



Similarities/Differences with the Euclidean postulates:[11]

1. A line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All Right Angles are congruent.
5. Given any straight line segment and a point not on the line, there are at least two parallel lines to the given line passing through the point.

Applications of Non-Euclidean Geometry

Hyperbolic Deep Neural Networks [7]

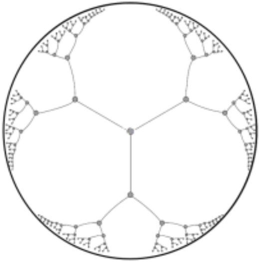
- In most of the current deep learning applications, the representation learning is conducted in the Euclidean space, which makes sense as the Euclidean space is the natural generalization of the visual three-dimensional space. However, recent research shows that many types of complex data exhibit a highly non-Euclidean latent anatomy.Also, it appears in several applications that the dissimilarity measures constructed by experts tend to have non-Euclidean behavior. In such cases, the Euclidean space does not provide the most powerful or meaningful geometrical representations.
- In many domains, data is with a tree-like structure or can be represented hierarchically. For example, social networks, human skeletons, sentences in natural language, and evolutionary relationships between biological entities in phylogenetics.

Applications of Non-Euclidean Geometry

- Recently, hyperbolic spaces have been proposed as an alternative continuous approach to learn hierarchical representations from textual and graph-structured data. The negative-curvature of the hyperbolic space results in very different geometric properties, which makes it widely employed in many such areas. In the hyperbolic space, circle circumference ($2sinh(r)$) and disc area ($2\pi(cosh(r)-1)$ grow exponentially with radius r, unlike the Euclidean space where they only grow linearly and quadratically. The exponential growth of the Poincare surface area with respect to its radius is analogous to the exponential growth of the number of leaves in a tree with respect to its depth, rather than polynomially as in the Euclidean case.

Applications of Non-Euclidean Geometry

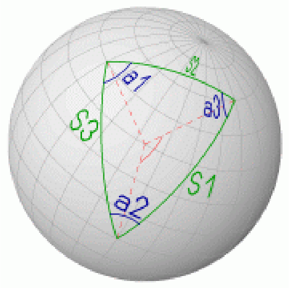
- Even when using an infinite number of dimensions, Euclidean space is unable to achieve comparable low distortion for tree data. Additionally, the smoothness of the hyperbolic spaces makes it possible to employ deep learning strategies that depend on differentiability. Hence, hyperbolic spaces have recently gained popularity in the context of deep neural networks to model embedded data into the space.



Spherical Geometry and Navigation [8]

A spherical triangle is defined by three sides with length S1, S2 and S3 and three including angles a1, a2 and a3.

The sides are segments of great circles and the length of each sides is defined by an angle. The angles of the sides are measured at the center of the sphere between the starting and ending "legs" of the great circle segment (shown red in the picture below).



Applications of Non-Euclidean Geometry

Law of Sines:

$$sin(a1)/sin(S1) = sin(a2)/sin(S2) = sin(a3)/sin(S3)$$

Law of Cosines of Sides:

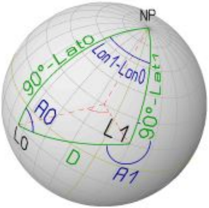
1. $cos(S1) = cos(S2) \cdot cos(S3) + sin(S2) \cdot sin(S3) \cdot cos(a1)$
2. $cos(S2) = cos(S3) \cdot cos(S1) + sin(S3) \cdot sin(S1) \cdot cos(a2)$
3. $cos(S3) = cos(S1) \cdot cos(S2) + sin(S1) \cdot sin(S2) \cdot cos(a3)$

Law of Cosines of Angles:

1. $cos(a1) = -cos(a2) \cdot cos(a3) + sin(a2) \cdot sin(a3) \cdot cos(S1)$
2. $cos(a2) = -cos(a3) \cdot cos(a1) + sin(a3) \cdot sin(a1) \cdot cos(S2)$
3. $cos(a3) = -cos(a1) \cdot cos(a2) + sin(a1) \cdot sin(a2) \cdot cos(S3)$

With a combination of the "Law of Cosines for Angles" and the "Law of Sines" the following identities can be deduced:

$$tan(a1) = sin(S1) \cdot sin(a3)/[cos(S1) \cdot sin(S2) - cos(a3) \cdot sin(S1) \cdot cos(S2)]$$
$$tan(a2) = sin(S2) \cdot sin(a1)/[cos(S2) \cdot sin(S3) - cos(a1) \cdot sin(S2) \cdot cos(S3)]$$
$$tan(a3) = sin(S3) \cdot sin(a2)/[cos(S3) \cdot sin(S1) - cos(a2) \cdot sin(S3) \cdot cos(S1)]$$



<u>**The Distance Problem:**</u> With the "Law of Cosines for Sides" applied for side D the following result is obtained:

$$cos(D) = cos(90\,\text{deg} - Lat0)cos(90\,\text{deg} - Lat1) +$$

$$sin(90\,\text{deg} - Lat0)sin(90\,\text{deg} - Lat1)cos(Lon1 - Lon0)$$

This can be reduced to:

$$cos(D) = sin(Lat0)sin(Lat1) + cos(Lat0)cos(Lat1)cos(Lon1 - Lon0)$$
$$D[\text{deg}] = cos^{-1}[sin(Lat0)sin(Lat1) + cos(Lat0)cos(Lat1)cos(Lon1 - Lon0)]$$

This identity gives the "angular" distance D between L0 and L1 in degrees.

Applications of Non-Euclidean Geometry

The Destination Problem: If a great-circle journey is initiated from a location L0 in an initial direction A0 and the distance travelled is D, the coordinates of the destination location L1 can be found by solving the spherical triangle for the side "90°-Lat1" and the angle "Lon1-Lon0". Giving us the coordinates of a destination (Lat1,Lon1) for a given departure (Lat0,Lon0), initial bearing (A0) and distance (D):

$$Lat1 = sin^{-1}[sin(Lat0) \cdot cos(D) + cos(Lat0) \cdot sin(D) \cdot cos(A0)]$$

$$Lon1 = Lon0 + tan^{-1} \frac{sin(D) \cdot sin(A0)}{[cos(D) \cdot cos(Lat0) cos(A0) \cdot sin(D) \cdot sin(Lat0)]}$$

Cosmology

- The metric for the universe is given by

$$\mathrm{d}s^2 = c^2\ \mathrm{d}t^2 - S^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\ \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right)$$

- Here, $S(t)$ is the scale factor of the universe, which can be seen as the "radius of the universe".
- The value of k determines the geometry of the universe
- For $k = 0$ the metric reduces to $ds^2 = c^2 dt^2 - S^2(t)\left(\mathrm{d}r^2 + r^2\left(\ \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right)$ hence we have flat universe.

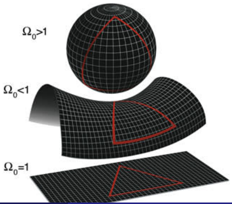
Applications of Non-Euclidean Geometry

- For $k = 1$, metric becomes $\mathrm{d}s^2 = c^2\ \mathrm{d}t^2 - S^2(t)\left(\frac{\mathrm{d}r^2}{1-r^2} + r^2\left(\ \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right)$ we have spherical universe. For $r = sin\chi$, $\mathrm{d}\sigma^2 = S_0^2\left[\ \mathrm{d}\chi^2 + \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right]$. which describes a spherical universe.
- For $k = -1$, metric becomes $\mathrm{d}\sigma^2 = S_0^2\left[\ \mathrm{d}\chi^2 + \sin^2\chi\left(\mathrm{d}\theta^2 + \sinh^2\theta \mathrm{d}\phi^2\right)\right]$ which describes a hyperbolic universe.

Applications of Non-Euclidean Geometry

How matter affects the geometry of the universe [9]

- Einstein's Field equation: $G_{ab} = \frac{8\pi G}{c^4} T_{ab}$ determines the geometry of the universe depending on the matter/energy density in it.
- Friedmann equations are derived from the field equations
- First Friedmann equation: $\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G \rho_0}{3} \frac{S_0^3}{S^3}$
- Second Friedmann equation: $2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = 0$
- For $k = 0$, $\rho_0 = \frac{3H_0^2}{8\pi G} \equiv \rho_{\mathrm{c}}$. Hence the universe is flat.
- Likewise $k = 1$(Spherical), we $\rho > \rho_c$ and for $k = -1$(Hyperbolic), we have $\rho < \rho_c$
- $\Omega_0 = \frac{\rho_0}{\rho_c}$



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