T.S $\quad d(S-\{x\}) = d(S)$

Proof $\quad S-\{x\} \subseteq S$

$d(S-\{x\}) \subseteq d(S)$ — ①

Let $l \in d(S)$

$\Rightarrow I_\varepsilon(l) \cap (S)$ is inf

$\Rightarrow I_\varepsilon(l) \cap (S-\{x\})$ will be inf

$\Rightarrow l \in d(S-\{x\})$

$\Rightarrow d(S) \subseteq d(S-\{x\})$ — ②

by ① & ② $\quad d(S) = d(S-\{x\})$

Date: / /

let $l \in d(d(S))$

$\therefore \forall \varepsilon > 0$, $I_\varepsilon(l) \cap d(S)$ is inf

$\exists x_1 \in d(S)$, $x_1 \in I_\varepsilon(l)$, $x_1 \neq l$

let $\varepsilon_1 = |x_1 - l|$, $\varepsilon_1 < \varepsilon$, $\varepsilon_1 > 0$

$\because x_1 \in d(S)$

$\therefore$ in particular for ε_1, $I_{\varepsilon_1}(x_1) \cap S$ is inf

$\because x_1 \in I_\varepsilon(l)$ & $\varepsilon_1 < \varepsilon$

$\therefore I_{\varepsilon_1}(x) \subseteq I_\varepsilon(l)$

$\therefore I_{\varepsilon_1}(x) \cap S \subseteq I_\varepsilon(l)$

$\therefore \exists x_2 \in S$ s.t $x_2 \in I_\varepsilon(l)$

$\therefore I_\varepsilon(l) \cap S - \{l\} \neq \phi$

$\therefore l \in d(S)$

$\therefore d(d(S)) \subseteq d(S)$

We claim

$d(S) \not\subseteq d(d(S))$

Ex let $S = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$

$d(S) = \{1\}$

$d(d(S)) = \phi$

$\therefore \{1\} \not\subseteq \phi$

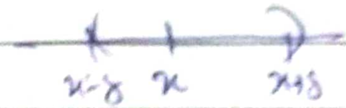
$\therefore d(S) \not\subseteq d(d(S))$

Let if possible $N = \bigcap_{i\in\Lambda} N_i(x) \neq \{x\}$

$\Rightarrow N = \bigcap_{i\in\Lambda} N_i(x) = \phi$ or $\exists a \neq x$ and $a \in \bigcap_{i\in\Lambda} N_i(x)$

$N = \bigcap_{i\in\Lambda} N_i$

$\therefore \exists \delta > 0 \ \forall i \in \Lambda$ s.t $I_\delta(x) \subseteq N_i$



$\therefore x \in N_i \ \forall i$

$\therefore x \in \bigcap_{i\in\Lambda} N_i$

$\therefore N \neq \phi$

Now, let $\exists a \neq x$ st $a \in \bigcap_{i\in\Lambda} N(x)$

let $\delta_1 = |a-x| > 0$

Consider $I_\delta = (x-\delta, x+\delta)$

I_δ is nbd of x $\forall \delta$

$\therefore$ in particular for $\delta_1 = |a-x| > 0$, I_{δ_1} is also nbd of x

and $a \in I_{\delta_1}$

but it contradicts the fact $a \in \bigcap_{i\in\Lambda} N(x)$

Hence our assumption was wrong

$\therefore \bigcap_{i\in\Lambda} N(x) = \{x\}$