Finding Eigen values and Eigen vectors of a matrix

Find Eigen values and Eigen vectors of

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We are looking for λ, a scalar &

v, a vector so that , λ?, v?

$$\boxed{A \cdot v = \lambda \cdot v}$$

If $Av = \lambda v$

$-\lambda v \quad -\lambda v$

$Av - \lambda v = 0$, v is common.

$(A - \lambda I) \cdot v = 0$, I is an identity matrix.

Since A is a 2×2 matrix

I will be 2×2 and it will be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(A - \lambda I) \cdot v = 0$, we don't want $v = 0$.

$|A - \lambda I| = 0$

As λ is a scalar & A & I are matrices we take the determinant to find λ.

$$\Rightarrow \left| \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} \right| = 0$$

$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

det is indicated by $| \ |$

$$(2-\lambda)(2-\lambda) - (-1)(-1) = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$+1 \quad +1$ adding one on both sides

$$(2-\lambda)^2 = 1$$

applying square root on both sides

$$\sqrt{(2-\lambda)^2} = \sqrt{1}$$

$$2-\lambda = \pm 1$$

$2-\lambda = +1$ or

$+\lambda \quad +\lambda$ adding λ on both sides

$2 = 1 + \lambda$

$-1 \quad -1$ taking away -1 from both sides

$$\boxed{1 = \lambda}$$

now let's substitute $\lambda = 1$ in

$$(A - \lambda I) \cdot v = 0$$

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$v_1 - v_2 = 0$ and $-v_1 + v_2 = 0$ these two are same Eq $v_1 - v_2 = 0 \Rightarrow \boxed{v_1 = v_2}$

if $v_2 = 1$, $v_1 = 1$ $\quad \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$2-\lambda = -1$

$+\lambda \quad +\lambda$ adding λ on both sides

$2 = -1 + \lambda$

$+1 \quad +1$ adding 1 to both sides

$$\boxed{\lambda = 3}$$

now let's substitute $\lambda = 3$ in $(A - \lambda I) v = 0$

$$\begin{pmatrix} 2-3 & -1 \\ -1 & 2-3 \end{pmatrix} \cdot \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = 0$$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = 0$$

$$-v_3 - v_4 = 0$$

$$\boxed{v_4 = -v_3}$$

if $v_3 = -1$, $v_4 = +1$ $\quad \begin{pmatrix} -1 \\ +1 \end{pmatrix}$

for $\overset{\text{Eigen value}}{\lambda = 1}$, the Eigen vector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

for the Eigen value $\lambda = 3$, the Eigen vector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Sometimes we might have to convert the Eigen vectors in unit vectors.

$\begin{pmatrix} a \\ b \end{pmatrix} = w$ is a unit vector if $\boxed{a^2 + b^2 = 1}$

Now let's convert Eigen vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ into unit vectors.

Consider $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ let's square each value & add them up.

$1^2 + 1^2 = 2$, so we have to divide $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by $\sqrt{2}$.

$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ is a unit vector b/c $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

Consider $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ let's square each value add them up.

$(-1)^2 + 1^2 = 2$.

So we have to divide $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ by $\sqrt{2}$

 $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ is a unit vector b/c $\left(\frac{-1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

So the Eigen values for $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

are $\lambda = 1$ and $\lambda = 3$

and the Eigen vectors are $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.