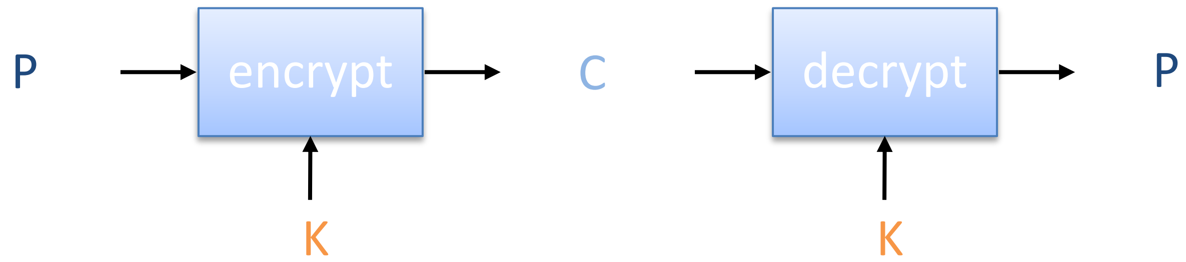
Cryptography

Symmetric Cryptosystem

- Scenario
  - Alice wants to send a message (plaintext P) to Bob.
  - The communication channel is insecure and can be eavesdropped
  - If Alice and Bob have previously agreed on a symmetric encryption scheme and a secret key K, the message can be sent encrypted (ciphertext C)
- Issues
  - What is a good symmetric encryption scheme?
  - What is the complexity of encrypting/decrypting?
  - What is the size of the ciphertext, relative to the plaintext?

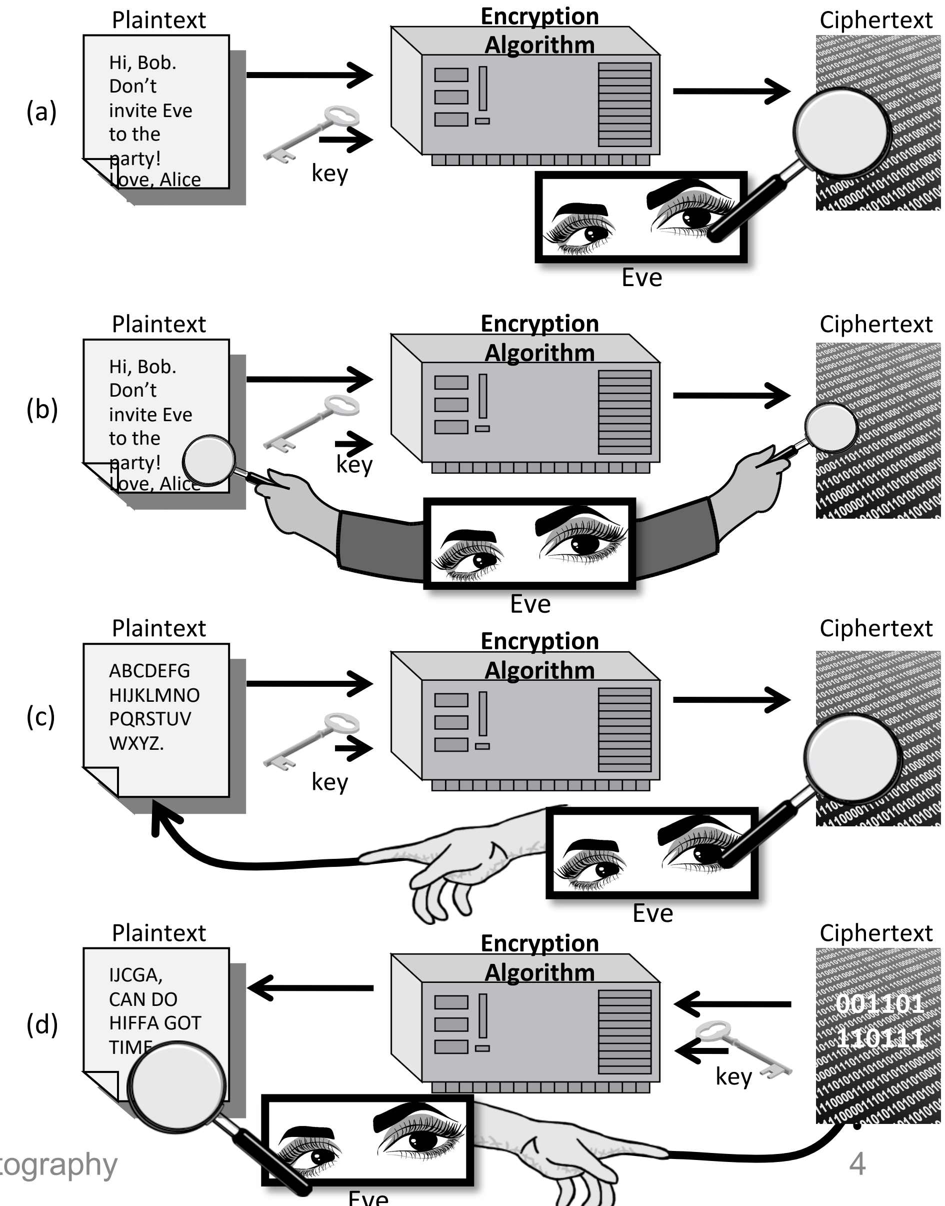


Basics

- Notation
  - Secret key K
  - Encryption function $E_K(P)$
  - Decryption function $D_K(C)$
  - Plaintext length typically the same as ciphertext length
  - Encryption and decryption are permutation functions (bijections) on the set of all n-bit arrays
- Efficiency
  - functions E_K and D_K should have efficient algorithms
- Consistency
  - Decrypting the ciphertext yields the plaintext
  - $D_K(E_K(P)) = P$

Attacks

- Attacker may have
  a) collection of ciphertexts (ciphertext only attack)
  b) collection of plaintext/ciphertext pairs (known plaintext attack)
  c) collection of plaintext/ciphertext pairs for plaintexts selected by the attacker (chosen plaintext attack)
  d) collection of plaintext/ciphertext pairs for ciphertexts selected by the attacker (chosen ciphertext attack)



Brute-Force Attack

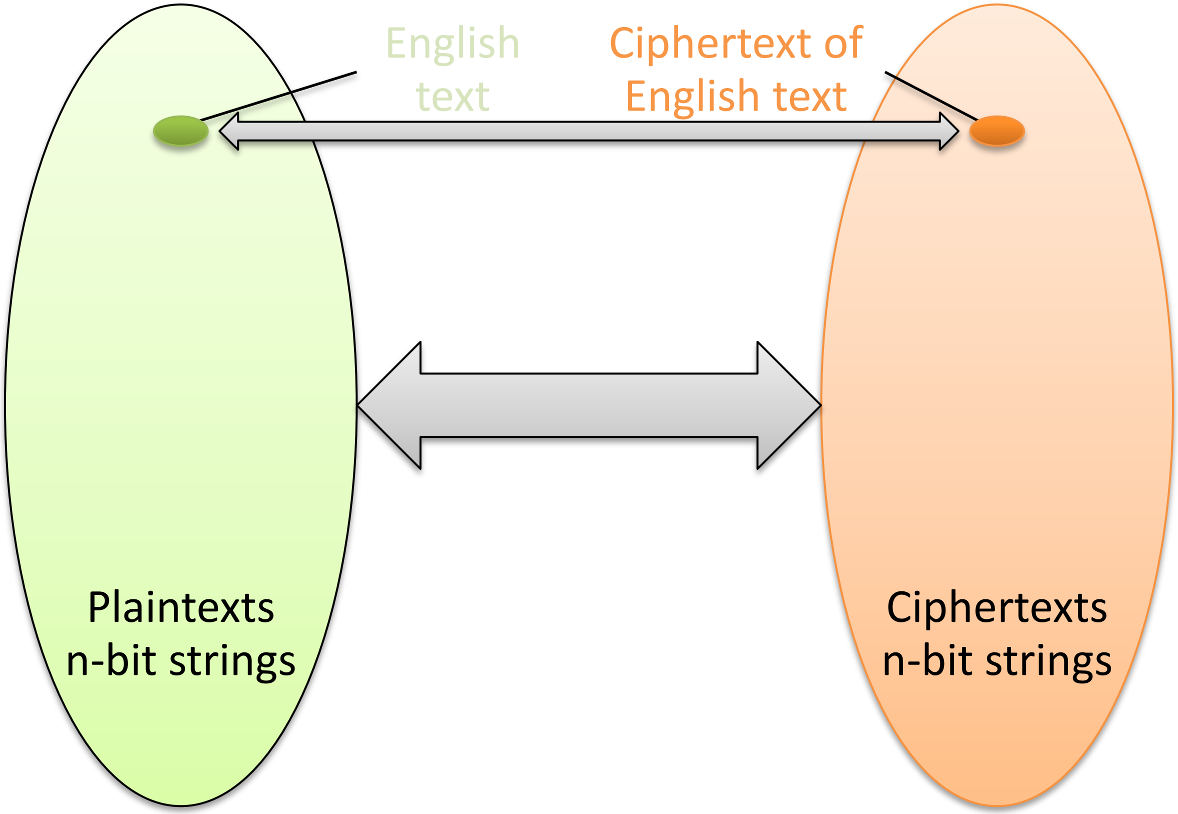
- Try all possible keys K and determine if $D_K(C)$ is a likely plaintext
  - Requires some knowledge of the structure of the plaintext (e.g., PDF file or email message)
- Key should be a sufficiently long random value to make exhaustive search attacks unfeasible

Image by Michael Cote from http://commons.wikimedia.org/wiki/File:Bingo_cards.jpg

Encrypting English Text

- English text typically represented with 8-bit ASCII encoding
- A message with t characters corresponds to an n-bit array, with n = 8t

- Redundancy due to repeated words and patterns
  - E.g., “th”, “ing”
- English plaintexts are a very small subset of all n-bit arrays



Entropy of Natural Language

- Information content (entropy) of English: 1.25 bits per character
- t-character arrays that are English text:

$$(2^{1.25})^t = 2^{1.25\,t}$$

- n-bit arrays that are English text:

$$2^{1.25\,n/8} \approx 2^{0.16\,n}$$

- For a natural language, constant $\alpha < 1$ such that there are $2^{\alpha n}$ messages among all n-bit arrays
- Fraction (probability) of valid messages

$$2^{\alpha n} / 2^n = 1 / 2^{(1-\alpha)n}$$

- Brute-force decryption
  - Try all possible 2^k decryption keys
  - Stop when valid plaintext recognized
- Given a ciphertext, there are 2^k possible plaintexts
- Expected number of valid plaintexts

$$2^k / 2^{(1-\alpha)n}$$

- Expected unique valid plaintext , (no spurious keys) achieved at unicity distance

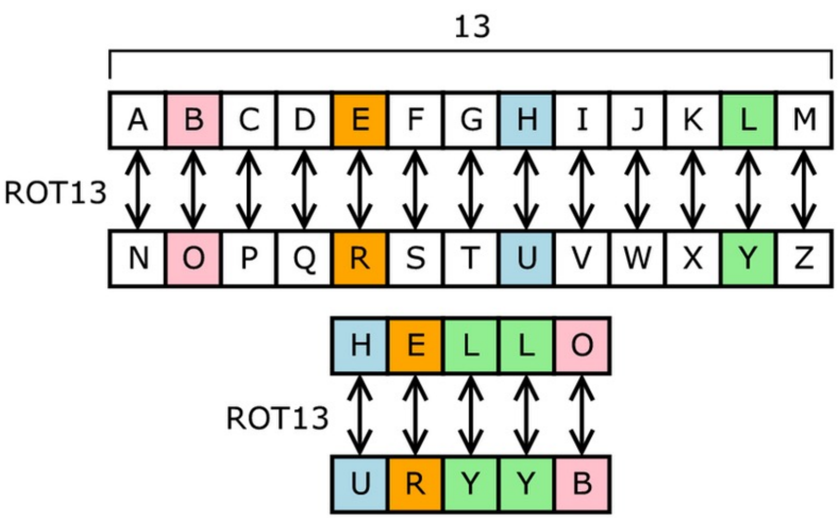
$$n = k / (1-\alpha)$$

- For English text and 256-bit keys, unicity distance is 304 bits

Substitution Ciphers

- Each letter is uniquely replaced by another.
- There are 26! possible substitution ciphers.
- There are more than 4.03×10^{26} such ciphers.

- One popular substitution "cipher" for some Internet posts is ROT13.



Public domain image from http://en.wikipedia.org/wiki/File:ROT13.png

Frequency Analysis

- Letters in a natural language, like English, are not uniformly distributed.
- Knowledge of letter frequencies, including pairs and triples can be used in cryptologic attacks against substitution ciphers.

a:	8.05%	b:	1.67%	c:	2.23%	d:	5.10%
e:	12.22%	f:	2.14%	g:	2.30%	h:	6.62%
i:	6.28%	j:	0.19%	k:	0.95%	l:	4.08%
m:	2.33%	n:	6.95%	o:	7.63%	p:	1.66%
q:	0.06%	r:	5.29%	s:	6.02%	t:	9.67%
u:	2.92%	v:	0.82%	w:	2.60%	x:	0.11%
y:	2.04%	z:	0.06%				

Letter frequencies in the book *The Adventures of Tom Sawyer*, by Twain.

Substitution Boxes

- Substitution can also be done on binary numbers.
- Such substitutions are usually described by substitution boxes, or S-boxes.

	00	01	10	11
00	0011	0100	1111	0001
01	1010	0110	0101	1011
10	1110	1101	0100	0010
11	0111	0000	1001	1100

(a)

	0	1	2	3
0	3	8	15	1
1	10	6	5	11
2	14	13	4	2
3	7	0	9	12

(b)

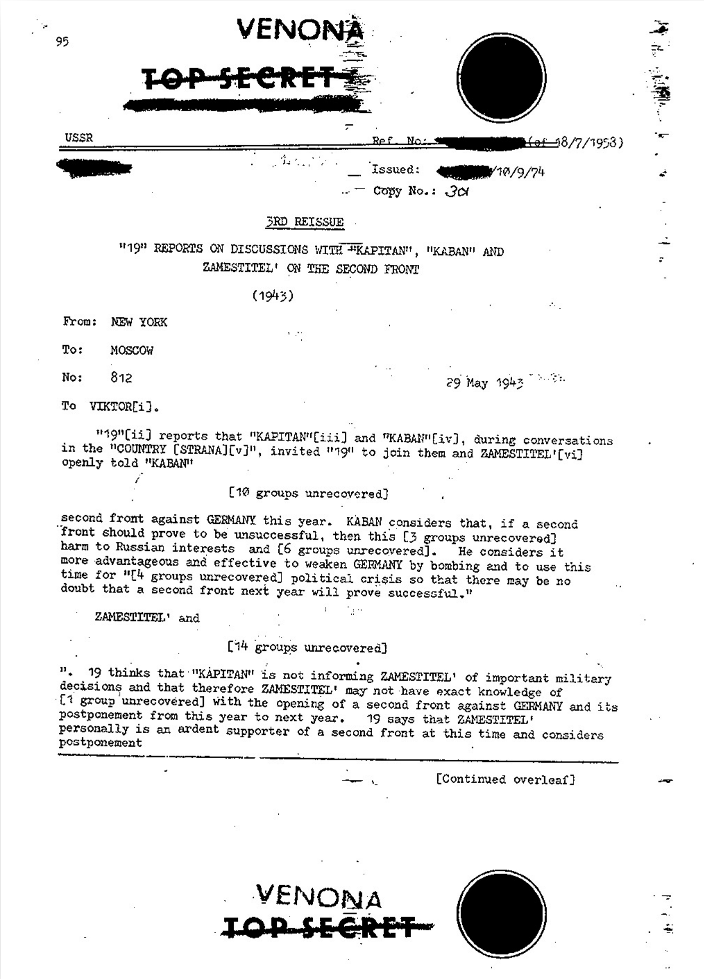
Figure 8.3: A 4-bit S-box (a) An S-box in binary. (b) The same S-box in decimal.

One-Time Pads

- There is one type of substitution cipher that is absolutely unbreakable.
  - The **one-time pad** was invented in 1917 by Joseph Mauborgne and Gilbert Vernam
  - We use a block of shift keys, $(k_1, k_2, \ldots, k_n)$, to encrypt a plaintext, M, of length n, with each shift key being chosen uniformly at random.
- Since each shift is random, every ciphertext is equally likely for any plaintext.

Weaknesses of the One-Time Pad

- In spite of their perfect security, one-time pads have some weaknesses
- The key has to be as long as the plaintext
- Keys can never be reused
  - Repeated use of one-time pads allowed the U.S. to break some of the communications of Soviet spies during the Cold War.



95

VENONA

~~TOP SECRET~~

[redacted]

[redacted]

USSR Ref. No: [redacted] (of 18/7/1953)

[redacted]

Issued: [redacted]/10/9/74

Copy No.: 3a

3RD REISSUE

"19" REPORTS ON DISCUSSIONS WITH "KAPITAN", "KABAN" AND ZAMESTITEL' ON THE SECOND FRONT

(1943)

From: NEW YORK

To: MOSCOW

No: 812 29 May 1943

To VIKTOR[i].

"19"[ii] reports that "KAPITAN"[iii] and "KABAN"[iv], during conversations in the "COUNTRY [STRANA][v]", invited "19" to join them and ZAMESTITEL'[vi] openly told "KABAN"

[10 groups unrecovered]

second front against GERMANY this year. KABAN considers that, if a second front should prove to be unsuccessful, then this [3 groups unrecovered] harm to Russian interests and [6 groups unrecovered]. He considers it more advantageous and effective to weaken GERMANY by bombing and to use this time for "[4 groups unrecovered] political crisis so that there may be no doubt that a second front next year will prove successful."

ZAMESTITEL' and

[14 groups unrecovered]

". 19 thinks that "KAPITAN" is not informing ZAMESTITEL' of important military decisions and that therefore ZAMESTITEL' may not have exact knowledge of [1 group unrecovered] with the opening of a second front against GERMANY and its postponement from this year to next year. 19 says that ZAMESTITEL' personally is an ardent supporter of a second front at this time and considers postponement

[Continued overleaf]

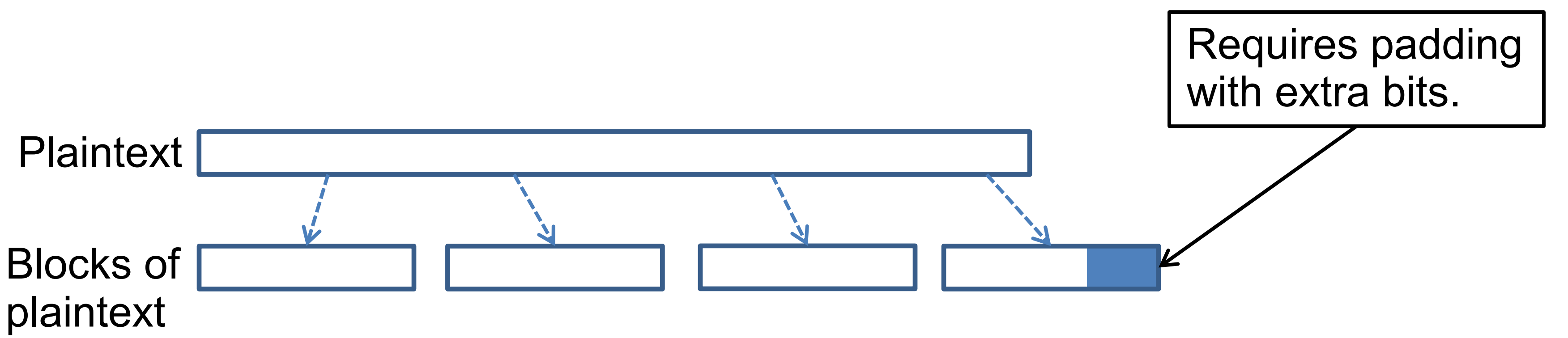
VENONA

~~TOP SECRET~~

[redacted]

Block Ciphers

- In a **block cipher:**
  - Plaintext and ciphertext have fixed length b (e.g., 128 bits)
  - A plaintext of length n is partitioned into a sequence of m **blocks**, P[0], …, P[m–1], where $n \leq bm < n + b$
- Each message is divided into a sequence of blocks and encrypted or decrypted in terms of its blocks.



Padding

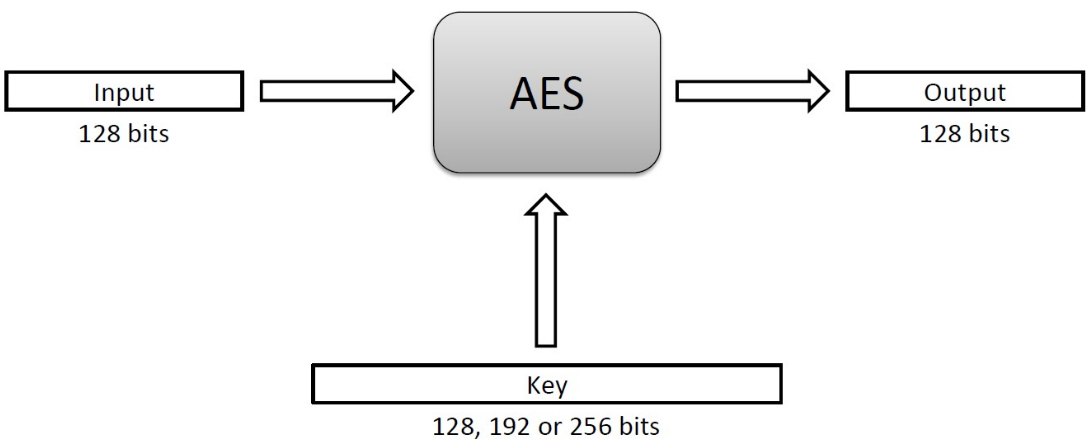
- Block ciphers require the length n of the plaintext to be a multiple of the block size b
- Padding the last block needs to be unambiguous (cannot just add zeroes)
- When the block size and plaintext length are a multiple of 8, a common padding method (PKCS5) is a sequence of identical bytes, each indicating the length (in bytes) of the padding
- Example for b = 128 (16 bytes)
  - Plaintext: “Roberto” (7 bytes)
  - Padded plaintext: “Roberto999999999” (16 bytes), where 9 denotes the number and not the character
- We need to always pad the last block, which may consist only of padding

Block Ciphers in Practice

- Data Encryption Standard (DES)
  - Developed by IBM and adopted by NIST in 1977
  - 64-bit blocks and 56-bit keys
  - Small key space makes exhaustive search attack feasible since late 90s
- Triple DES (3DES)
  - Nested application of DES with three different keys KA, KB, and KC
  - Effective key length is 168 bits, making exhaustive search attacks unfeasible
  - $C = E_{KC}(D_{KB}(E_{KA}(P)))$; $P = D_{KA}(E_{KB}(D_{KC}(C)))$
  - Equivalent to DES when KA=KB=KC (backward compatible)
- Advanced Encryption Standard (AES)
  - Selected by NIST in 2001 through open international competition and public discussion
  - 128-bit blocks and several possible key lengths: 128, 192 and 256 bits
  - Exhaustive search attack not currently possible
  - AES-256 is the symmetric encryption algorithm of choice

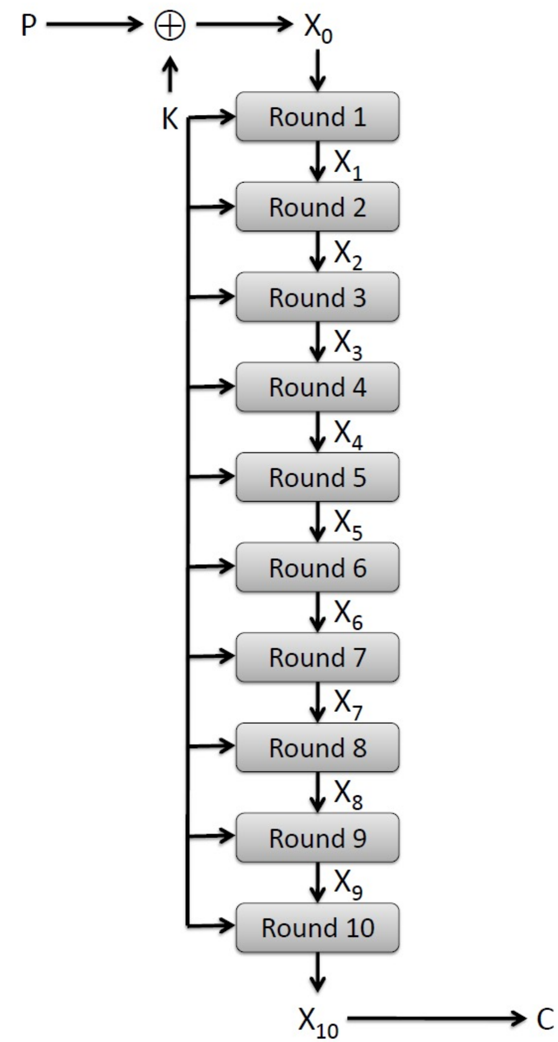
The Advanced Encryption Standard (AES)

- In 1997, the U.S. National Institute for Standards and Technology (NIST) put out a public call for a replacement to DES.
- It narrowed down the list of submissions to five finalists, and ultimately chose an algorithm that is now known as the **Advanced Encryption Standard** (**AES**).
- AES is a block cipher that operates on 128-bit blocks. It is designed to be used with keys that are 128, 192, or 256 bits long, yielding ciphers known as AES-128, AES-192, and AES-256.



AES Round Structure

- The 128-bit version of the AES encryption algorithm proceeds in ten rounds.
- Each round performs an invertible transformation on a 128-bit array, called **state**.
- The initial state X_0 is the XOR of the plaintext P with the key K:
- X_0 = P XOR K.
- Round i (i = 1, …, 10) receives state X_{i-1} as input and produces state X_i.
- The ciphertext C is the output of the final round: C = X_{10}.



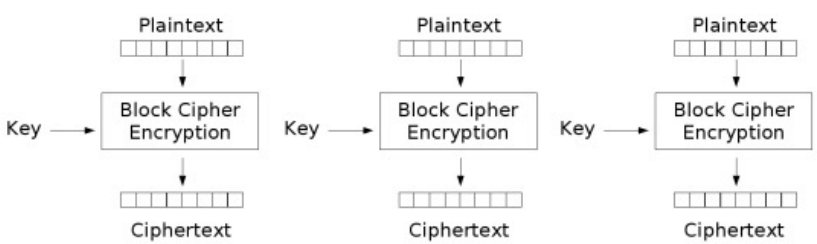
AES Rounds

- Each round is built from four basic steps:

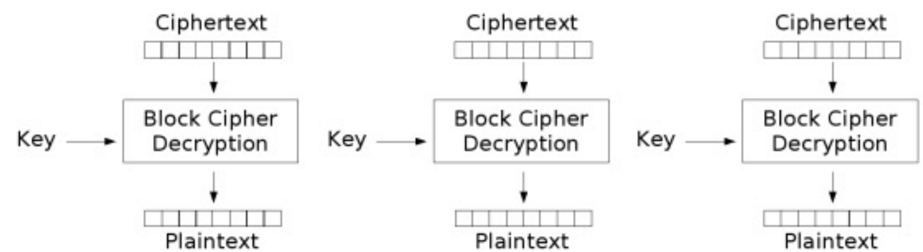
1. **SubBytes step**: an S-box substitution step
2. **ShiftRows step**: a permutation step
3. **MixColumns step**: a matrix multiplication step
4. **AddRoundKey step**: an XOR step with a **round key** derived from the 128-bit encryption key

Block Cipher Modes

- A block cipher mode describes the way a block cipher encrypts and decrypts a sequence of message blocks.
- Electronic Code Book (ECB) Mode (is the simplest):
  - Block P[i] encrypted into ciphertext block C[i] = E_K(P[i])
  - Block C[i] decrypted into plaintext block M[i] = D_K(C[i])



Electronic Codebook (ECB) mode encryption

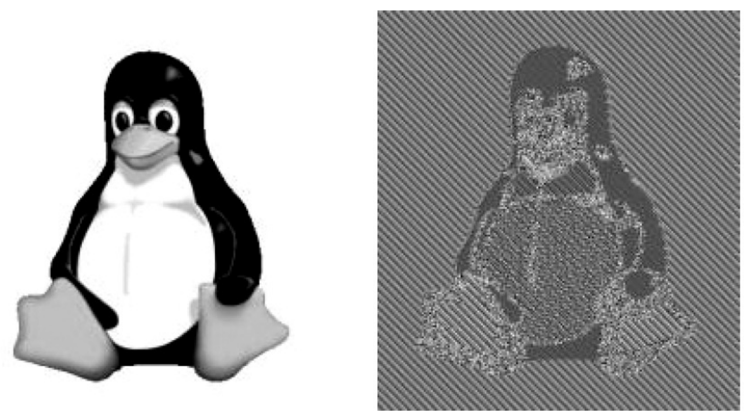


Electronic Codebook (ECB) mode decryption

Strengths and Weaknesses of ECB

- Strengths:
  – Is very simple
  – Allows for parallel encryptions of the blocks of a plaintext
  – Can tolerate the loss or damage of a block

- Weakness:
  – Documents and images are not suitable for ECB encryption since patters in the plaintext are repeated in the ciphertext:

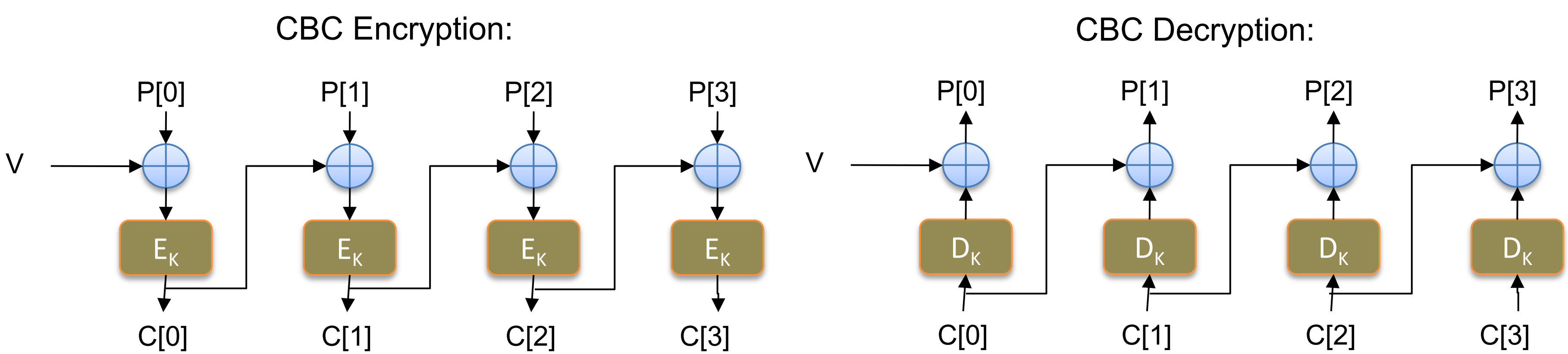


(a) (b)

Figure 8.6: How ECB mode can leave identifiable patterns in a sequence of blocks: (a) An image of Tux the penguin, the Linux mascot. (b) An encryption of the Tux image using ECB mode. (The image in (a) is by Larry Ewing, lewing@isc.tamu.edu, using The Gimp; the image in (b) is by Dr. Juzam. Both are used with permission via attribution.)

Cipher Block Chaining (CBC) Mode

- In Cipher Block Chaining (CBC) Mode
  - The previous ciphertext block is combined with the current plaintext block $C[i] = E_K(C[i-1] \oplus P[i])$
  - $C[-1] = V$, a random block separately transmitted encrypted (known as the initialization vector)
  - Decryption: $P[i] = C[i-1] \oplus D_K(C[i])$



Strengths and Weaknesses of CBC

- Strengths:
  - Doesn’t show patterns in the plaintext
  - Is the most common mode
  - Is fast and relatively simple

- Weaknesses:
  - CBC requires the reliable transmission of all the blocks sequentially
  - CBC is not suitable for applications that allow packet losses (e.g., music and video streaming)

Java AES Encryption Example

- Source

  http://java.sun.com/javase/6/docs/technotes/guides/security/crypto/CryptoSpec.html

- Generate an AES key

```
KeyGenerator keygen = KeyGenerator.getInstance("AES");
SecretKey aesKey = keygen.generateKey();
```

- Create a cipher object for AES in ECB mode and PKCS5 padding

```
Cipher aesCipher;
aesCipher = Cipher.getInstance("AES/ECB/PKCS5Padding");
```

- Encrypt

```
aesCipher.init(Cipher.ENCRYPT_MODE, aesKey);
byte[] plaintext = "My secret message".getBytes();
byte[] ciphertext = aesCipher.doFinal(plaintext);
```

- Decrypt

```
aesCipher.init(Cipher.DECRYPT_MODE, aesKey);
byte[] plaintext1 = aesCipher.doFinal(ciphertext);
```

Stream Cipher

- Key stream
  - Pseudo-random sequence of bits S = S[0], S[1], S[2], …
  - Can be generated on-line one bit (or byte) at the time
- Stream cipher
  - XOR the plaintext with the key stream $C[i] = S[i] \oplus P[i]$
  - Suitable for plaintext of arbitrary length generated on the fly, e.g., media stream
- Synchronous stream cipher
  - Key stream obtained only from the secret key K
  - Works for unreliable channels if plaintext has packets with sequence numbers
- Self-synchronizing stream cipher
  - Key stream obtained from the secret key and q previous ciphertexts
  - Lost packets cause a delay of q steps before decryption resumes

Key Stream Generation

- RC4
  - Designed in 1987 by Ron Rivest for RSA Security
  - Trade secret until 1994
  - Uses keys with up to 2,048 bits
  - Simple algorithm
- Block cipher in counter mode (CTR)
  - Use a block cipher with block size b
  - The secret key is a pair (K,t), where K a is key and t (counter) is a b-bit value
  - The key stream is the concatenation of ciphertexts

$$E_K(t),\ E_K(t+1),\ E_K(t+2),\ \ldots$$

  - Can use a shorter counter concatenated with a random value
  - Synchronous stream cipher

Attacks on Stream Ciphers

- Repetition attack
  - if key stream reused, attacker obtains XOR of two plaintexts
- Insertion attack [Bayer Metzger, TODS 1976]
  - retransmission of the plaintext with
    - a chosen byte inserted by attacker
    - using the same key stream
  - e.g., email message resent with new message number

Original

P	P[i]	P[i+1]	P[i+2]	P[i+3]
S	S[i]	S[i+1]	S[i+2]	S[i+3]
C	C[i]	C[i+1]	C[i+2]	C[i+3]

Retransmission

P	P[i]	X	P[i+1]	P[i+2]
S	S[i]	S[i+1]	S[i+2]	S[i+3]
C	C[i]	C′[i+1]	C′[i+2]	C′[i+3]

Public Key Encryption

Facts About Numbers

- Prime number p:
  - p is an integer
  - $p \geq 2$
  - The only divisors of p are 1 and p
- Examples
  - $2, 7, 19$ are primes
  - $-3, 0, 1, 6$ are not primes
- Prime decomposition of a positive integer n:

$$n = p_1^{e_1} \times \ldots \times p_k^{e_k}$$

- Example:
  - $200 = 2^3 \times 5^2$

Fundamental Theorem of Arithmetic

The prime decomposition of a positive integer is unique

Greatest Common Divisor

- The greatest common divisor (GCD) of two positive integers $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted $\gcd(\boldsymbol{a}, \boldsymbol{b})$, is the largest positive integer that divides both $\boldsymbol{a}$ and $\boldsymbol{b}$
- The above definition is extended to arbitrary integers
- Examples:

  $\gcd(18, 30) = 6$ $\quad\quad$ $\gcd(0, 20) = 20$

  $\gcd(-21, 49) = 7$

- Two integers a and b are said to be relatively prime if

$$\gcd(\boldsymbol{a}, \boldsymbol{b}) = 1$$

- Example:
  - Integers 15 and 28 are relatively prime

Modular Arithmetic

- Modulo operator for a positive integer $\boldsymbol{n}$

  $$\boldsymbol{r} = \boldsymbol{a} \bmod \boldsymbol{n}$$

  equivalent to

  $$\boldsymbol{a} = \boldsymbol{r} + \boldsymbol{kn}$$

  and

  $$\boldsymbol{r} = \boldsymbol{a} - \lfloor \boldsymbol{a}/\boldsymbol{n} \rfloor \boldsymbol{n}$$

- Example:

  $$29 \bmod 13 = 3 \qquad 13 \bmod 13 = 0 \qquad -1 \bmod 13 = 12$$
  $$29 = 3 + 2\times 13 \qquad 13 = 0 + 1\times 13 \qquad 12 = -1 + 1\times 13$$

- Modulo and GCD:

  $$\gcd(\boldsymbol{a}, \boldsymbol{b}) = \gcd(\boldsymbol{b}, \boldsymbol{a} \bmod \boldsymbol{b})$$

- Example:

  $$\gcd(21, 12) = 3 \qquad \gcd(12, 21 \bmod 12) = \gcd(12, 9) = 3$$

Euclid’s GCD Algorithm

- Euclid’s algorithm for computing the GCD repeatedly applies the formula

$$\gcd(\boldsymbol{a}, \boldsymbol{b}) = \gcd(\boldsymbol{b}, \boldsymbol{a} \bmod \boldsymbol{b})$$

- Example
  - $\gcd(412, 260) = 4$

Algorithm *EuclidGCD*($\boldsymbol{a}$, $\boldsymbol{b}$)
  Input integers $\boldsymbol{a}$ and $\boldsymbol{b}$
  Output $\gcd(\boldsymbol{a}, \boldsymbol{b})$

  if $\boldsymbol{b} = 0$
    return $\boldsymbol{a}$
  else
    return *EuclidGCD*($\boldsymbol{b}$, $\boldsymbol{a}$ mod $\boldsymbol{b}$)

a	412	260	152	108	44	20	4
b	260	152	108	44	20	4	0

Analysis

- Let a_i and b_i be the arguments of the i-th recursive call of algorithm ***EuclidGCD***
- We have

$$a_{i+2} = b_{i+1} = a_i \bmod a_{i+1} < a_{i+1}$$

- Sequence $a_1, a_2, \ldots, a_n$ decreases exponentially, namely

$$a_{i+2} \le \tfrac{1}{2} a_i \text{ for } i > 1$$

Case 1 $a_{i+1} \le \frac{1}{2} a_i$ $\quad a_{i+2} < a_{i+1} \le \frac{1}{2} a_i$

Case 2 $a_{i+1} > \frac{1}{2} a_i$ $\quad a_{i+2} = a_i \bmod a_{i+1} = a_i - a_{i+1} \le \frac{1}{2} a_i$

- Thus, the maximum number of recursive calls of algorithm $\boldsymbol{EuclidGCD}(a. b)$ is

$$1 + 2 \log \max(a. b)$$

- Algorithm $\boldsymbol{EuclidGCD}(a, b)$ executes $O(\log \max(a, b))$ arithmetic operations
- The running time can also be expressed as $O(\log \min(a, b))$

Multiplicative Inverses (1)

- The residues modulo a positive integer n are the set

  $$Z_n = \{0, 1, 2, \ldots, (n-1)\}$$

- Let x and y be two elements of Z_n such that

  $$xy \bmod n = 1$$

  We say that y is the multiplicative inverse of x in Z_n and we write $y = x^{-1}$

- Example:
  - Multiplicative inverses of the residues modulo 11

x	0	1	2	3	4	5	6	7	8	9	10
x^{-1}		1	6	4	3	9	2	8	7	5	10

Multiplicative Inverses (2)

Theorem

An element x of Z_n has a multiplicative inverse if and only if x and n are relatively prime

- Example
  - The elements of Z_{10} with a multiplicative inverse are $1, 3, 7, 9$

Corollary

If is p is prime, every nonzero residue in Z_p has a multiplicative inverse

Theorem

A variation of Euclid's GCD algorithm computes the multiplicative inverse of an element x of Z_n or determines that it does not exist

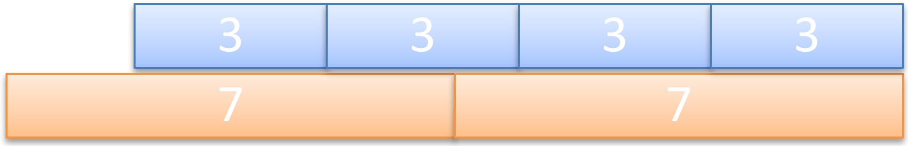
x	0	1	2	3	4	5	6	7	8	9
x^{-1}		1		7				3		9

Example: Measuring Lengths

- Consider a stick of length a and a stick of length b such that a and b are relatively prime
- Given two integers i and j, we can measure length

$$n = ia + jb$$

- We show that any integer n can be written as $n = ia + jb$ for some integers i and j
  - Let s be the inverse of a in Z_b We have $sa \bmod b = 1$
  - There exists integer t such that $sa + tb = 1$
  - Pick $i = ns$ and $j = nt$
- Thus, given two sticks of relatively prime integer lengths, we can measure any integer length
- Example, measure length 2 with sticks of length 3 and 7



Example: Double Hashing

- Consider a hash table whose size $\boldsymbol{n}$ is a prime
- In open addressing with double hashing, an operation on key $\boldsymbol{x}$ probes the following locations modulo $\boldsymbol{n}$

$$\boldsymbol{i},\ \boldsymbol{i}+\boldsymbol{d},\ \boldsymbol{i}+2\boldsymbol{d},\ \boldsymbol{i}+3\boldsymbol{d},\ \ldots,\ \boldsymbol{i}+(\boldsymbol{n}-1)\boldsymbol{d}$$

  where $\boldsymbol{i} = \boldsymbol{h}_1(\boldsymbol{x})$ and $\boldsymbol{d} = \boldsymbol{h}_2(\boldsymbol{x})$
- We show that each table location is probed by this sequence once
  - Suppose $(\boldsymbol{i}+\boldsymbol{jd}) \bmod \boldsymbol{n} = (\boldsymbol{i}+\boldsymbol{kd}) \bmod \boldsymbol{n}$ for some integers $\boldsymbol{j}$ and $\boldsymbol{k}$ in the range $[0, \boldsymbol{n}-1]$
  - We have $(\boldsymbol{j}-\boldsymbol{k})\boldsymbol{d} \bmod \boldsymbol{n} = 0$
  - Since $\boldsymbol{n}$ is prime, we have that $\boldsymbol{n}$ and $\boldsymbol{d}$ are relatively prime
  - Thus, $\boldsymbol{d}$ has an inverse $\boldsymbol{d}^{-1}$ in $\boldsymbol{Z}_{\boldsymbol{n}}$
  - Multiplying each side by $\boldsymbol{d}^{-1}$, we obtain $(\boldsymbol{j}-\boldsymbol{k}) \bmod \boldsymbol{n} = 0$
  - We conclude that $\boldsymbol{j} = \boldsymbol{k}$

Powers

- Let p be a prime
- The sequences of successive powers of the elements of Z_p exhibit repeating subsequences
- The sizes of the repeating subsequences and the number of their repetitions are the divisors of $p - 1$
- Example ($p = 7$)

x	x^2	x^3	x^4	x^5	x^6
1	1	1	1	1	1
2	4	1	2	4	1
3	2	6	4	5	1
4	2	1	4	2	1
5	4	6	2	3	1
6	1	6	1	6	1

Fermat's Little Theorem

Theorem

Let $\boldsymbol{p}$ be a prime. For each nonzero residue $\boldsymbol{x}$ of $\boldsymbol{Z_p}$, we have $x^{p-1} \bmod p = 1$

- Example ($p = 5$):

$1^4 \bmod 5 = 1$

$2^4 \bmod 5 = 16 \bmod 5 = 1$

$3^4 \bmod 5 = 81 \bmod 5 = 1$

$4^4 \bmod 5 = 256 \bmod 5 = 1$

Corollary

Let $\boldsymbol{p}$ be a prime. For each nonzero residue $\boldsymbol{x}$ of $\boldsymbol{Z_p}$, the multiplicative inverse of $\boldsymbol{x}$ is $x^{p-2} \bmod p$

Proof

$$x(x^{p-2} \bmod p) \bmod p = xx^{p-2} \bmod p = x^{p-1} \bmod p = 1$$

Euler's Theorem

- The multiplicative group for $\boldsymbol{Z_n}$, denoted with $\boldsymbol{Z^*_n}$, is the subset of elements of $\boldsymbol{Z_n}$ relatively prime with $\boldsymbol{n}$
- The totient function of $\boldsymbol{n}$, denoted with $\phi(\boldsymbol{n})$, is the size of $\boldsymbol{Z^*_n}$
- Example

  $\boldsymbol{Z^*}_{10} = \{ 1, 3, 7, 9 \}$ $\qquad$ $\phi(10) = 4$
- If $\boldsymbol{p}$ is prime, we have

  $\boldsymbol{Z^*_p} = \{1, 2, \ldots, (\boldsymbol{p} - 1)\}$ $\qquad$ $\phi(\boldsymbol{p}) = \boldsymbol{p} - 1$

Euler's Theorem

For each element $\boldsymbol{x}$ of $\boldsymbol{Z^*_n}$, we have $\boldsymbol{x}^{\phi(\boldsymbol{n})} \bmod \boldsymbol{n} = 1$

- Example ($\boldsymbol{n} = 10$)

$$3^{\phi(10)} \bmod 10 = 3^4 \bmod 10 = 81 \bmod 10 = 1$$
$$7^{\phi(10)} \bmod 10 = 7^4 \bmod 10 = 2401 \bmod 10 = 1$$
$$9^{\phi(10)} \bmod 10 = 9^4 \bmod 10 = 6561 \bmod 10 = 1$$

RSA Cryptosystem

- Setup:
  - $\boldsymbol{n} = \boldsymbol{pq}$, with $\boldsymbol{p}$ and $\boldsymbol{q}$ primes
  - $\boldsymbol{e}$ relatively prime to $\phi(n) = (\boldsymbol{p} - 1)\ (\boldsymbol{q} - 1)$
  - $\boldsymbol{d}$ inverse of $\boldsymbol{e}$ in $\boldsymbol{Z}_{\phi(\boldsymbol{n})}$
- Keys:
  - Public key: $\boldsymbol{K_E} = (\boldsymbol{n}, \boldsymbol{e})$
  - Private key: $\boldsymbol{K_D} = \boldsymbol{d}$
- Encryption:
  - Plaintext $\boldsymbol{M}$ in $\boldsymbol{Z_n}$
  - $\boldsymbol{C} = \boldsymbol{M}^{\boldsymbol{e}} \bmod \boldsymbol{n}$
- Decryption:
  - $\boldsymbol{M} = \boldsymbol{C}^{\boldsymbol{d}} \bmod \boldsymbol{n}$

- Example
  - Setup:
    - $\boldsymbol{p} = 7,\ \ \boldsymbol{q} = 17$
    - $\boldsymbol{n} = 7 \cdot 17 = 119$
    - $\phi(\boldsymbol{n}) = 6 \cdot 16 = 96$
    - $\boldsymbol{e} = 5$
    - $\boldsymbol{d} = 77$
  - Keys:
    - public key: $(119, 5)$
    - private key: 77
  - Encryption:
    - $\boldsymbol{M} = 19$
    - $\boldsymbol{C} = 19^5 \bmod 119 = 66$
  - Decryption:
    - $\boldsymbol{C} = 66^{77} \bmod 119 = 19$

Complete RSA Example

- Setup:
  - $p = 5, q = 11$
  - $n = 5 \cdot 11 = 55$
  - $\phi(n) = 4 \cdot 10 = 40$
  - $e = 3$
  - $d = 27$ $(3 \cdot 27 = 81 = 2 \cdot 40 + 1)$
- Encryption
  - $C = M^3 \bmod 55$
- Decryption
  - $M = C^{27} \bmod 55$

M	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
C	1	8	27	9	15	51	13	17	14	10	11	23	52	49	20	26	18	2
M	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
C	39	25	21	33	12	19	5	31	48	7	24	50	36	43	22	34	30	16
M	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54
C	53	37	29	35	6	3	32	44	45	41	38	42	4	40	46	28	47	54

Security

- Security of RSA based on difficulty of factoring
  - Widely believed
  - Best known algorithm takes exponential time
- RSA Security factoring challenge (discontinued)
- In 1999, 512-bit challenge factored in 4 months using 35.7 CPU-years
  - 160 175-400 MHz SGI and Sun
  - 8 250 MHz SGI Origin
  - 120 300-450 MHz Pentium II
  - 4 500 MHz Digital/Compaq
- In 2005, a team of researchers factored the RSA-640 challenge number using 30 2.2GHz CPU years
- In 2004, the prize for factoring RSA-2048 was $200,000
- Current practice is 2,048-bit keys
- Estimated resources needed to factor a number within one year

Length (bits)	PCs	Memory
430	1	128MB
760	215,000	4GB
1,020	342×10^6	170GB
1,620	1.6×10^{15}	120TB

Correctness

- We show the correctness of the RSA cryptosystem for the case when the plaintext $\boldsymbol{M}$ does not divide $\boldsymbol{n}$
- Namely, we show that

$$(\boldsymbol{M}^{\boldsymbol{e}})^{\boldsymbol{d}} \bmod \boldsymbol{n} = \boldsymbol{M}$$

- Since $\boldsymbol{ed} \bmod \phi(\boldsymbol{n}) = 1$, there is an integer $\boldsymbol{k}$ such that

$$\boldsymbol{ed} = \boldsymbol{k}\phi(\boldsymbol{n}) + 1$$

- Since $\boldsymbol{M}$ does not divide $\boldsymbol{n}$, by Euler's theorem we have

$$\boldsymbol{M}^{\phi(\boldsymbol{n})} \bmod \boldsymbol{n} = 1$$

- Thus, we obtain

$$\begin{aligned}
(\boldsymbol{M}^{\boldsymbol{e}})^{\boldsymbol{d}} \bmod \boldsymbol{n} &= \\
\boldsymbol{M}^{\boldsymbol{ed}} \bmod \boldsymbol{n} &= \\
\boldsymbol{M}^{\boldsymbol{k}\phi(\boldsymbol{n})+1} \bmod \boldsymbol{n} &= \\
\boldsymbol{M}\boldsymbol{M}^{\boldsymbol{k}\phi(\boldsymbol{n})} \bmod \boldsymbol{n} &= \\
\boldsymbol{M}\,(\boldsymbol{M}^{\phi(\boldsymbol{n})})^{\boldsymbol{k}} \bmod \boldsymbol{n} &= \\
\boldsymbol{M}\,(\boldsymbol{M}^{\phi(\boldsymbol{n})} \bmod \boldsymbol{n})^{\boldsymbol{k}} \bmod \boldsymbol{n} &= \\
\boldsymbol{M}\,(1)^{\boldsymbol{k}} \bmod \boldsymbol{n} &= \\
\boldsymbol{M} \bmod \boldsymbol{n} &= \\
\boldsymbol{M}
\end{aligned}$$

- Proof of correctness can be extended to the case when the plaintext $\boldsymbol{M}$ divides $\boldsymbol{n}$

Algorithmic Issues

- The implementation of the RSA cryptosystem requires various algorithms
- Overall
  - Representation of integers of arbitrarily large size and arithmetic operations on them
- Encryption
  - Modular power
- Decryption
  - Modular power
- Setup
  - Generation of random numbers with a given number of bits (to generate candidates p and q)
  - Primality testing (to check that candidates p and q are prime)
  - Computation of the GCD (to verify that e and $\phi(n)$ are relatively prime)
  - Computation of the multiplicative inverse (to compute d from e)

Modular Power

- The repeated squaring algorithm speeds up the computation of a modular power $a^p \bmod n$
- Write the exponent p in binary

  $p = p_{b-1} p_{b-2} \cdots p_1 p_0$
- Start with

  $Q_1 = a^{p_{b-1}} \bmod n$
- Repeatedly compute

  $Q_i = ((Q_{i-1})^2 \bmod n) a^{p_{b-i}} \bmod n$
- We obtain

  $Q_b = a^p \bmod n$
- The repeated squaring algorithm performs $O(\log p)$ arithmetic operations

- Example
  - $3^{18} \bmod 19$ (18 = 10010)
  - $Q_1 = 3^1 \bmod 19 = 3$
  - $Q_2 = (3^2 \bmod 19)3^0 \bmod 19 = 9$
  - $Q_3 = (9^2 \bmod 19)3^0 \bmod 19 =$ $81 \bmod 19 = 5$
  - $Q_4 = (5^2 \bmod 19)3^1 \bmod 19 =$ $(25 \bmod 19)3 \bmod 19 =$ $18 \bmod 19 = 18$
  - $Q_5 = (18^2 \bmod 19)3^0 \bmod 19 =$ $(324 \bmod 19) \bmod 19 =$ $17 \cdot 19 + 1 \bmod 19 = 1$

p_{5-i}	1	0	0	1	0
$2^{p_{5-i}}$	3	1	1	3	1
Q_i	3	9	5	18	1

Modular Inverse

Theorem

Given positive integers $\boldsymbol{a}$ and $\boldsymbol{b}$, let $\boldsymbol{d}$ be the smallest positive integer such that

$$\boldsymbol{d} = \boldsymbol{ia} + \boldsymbol{jb}$$

for some integers $\boldsymbol{i}$ and $\boldsymbol{j}$.

We have

$$\boldsymbol{d} = \gcd(\boldsymbol{a},\boldsymbol{b})$$

- Example
  - $\boldsymbol{a} = 21$
  - $\boldsymbol{b} = 15$
  - $\boldsymbol{d} = 3$
  - $\boldsymbol{i} = 3, \boldsymbol{j} = -4$
  - $3 = 3 \cdot 21 + (-4) \cdot 15 = 63 - 60 = 3$

- Given positive integers $\boldsymbol{a}$ and $\boldsymbol{b}$, the extended Euclid's algorithm computes a triplet $(\boldsymbol{d},\boldsymbol{i},\boldsymbol{j})$ such that
  - $\boldsymbol{d} = \gcd(\boldsymbol{a},\boldsymbol{b})$
  - $\boldsymbol{d} = \boldsymbol{ia} + \boldsymbol{jb}$
- To test the existence of and compute the inverse of $\boldsymbol{x} \in \boldsymbol{Z_n}$, we execute the extended Euclid's algorithm on the input pair $(\boldsymbol{x},\boldsymbol{n})$
- Let $(\boldsymbol{d},\boldsymbol{i},\boldsymbol{j})$ be the triplet returned
  - $\boldsymbol{d} = \boldsymbol{ix} + \boldsymbol{jn}$

Case 1: $\boldsymbol{d} = 1$

$\boldsymbol{i}$ is the inverse of $\boldsymbol{x}$ in $\boldsymbol{Z_n}$

Case 2: $\boldsymbol{d} > 1$

$\boldsymbol{x}$ has no inverse in $\boldsymbol{Z_n}$

Pseudoprimality Testing

- The number of primes less than or equal to n is about $n / \ln n$
- Thus, we expect to find a prime among $O(b)$ randomly generated numbers with b bits each
- Testing whether a number is prime (primality testing) is a difficult problem, though polynomial-time algorithms exist
- An integer $n \geq 2$ is said to be a base-x pseudoprime if
  - $x^{n-1} \bmod n = 1$ (Fermat's little theorem)
- Composite base-x pseudoprimes are rare:
  - A random 100-bit integer is a composite base-2 pseudoprime with probability less than 10^{-13}
  - The smallest composite base-2 pseudoprime is 341
- Base-x pseudoprimality testing for an integer n:
  - Check whether $x^{n-1} \bmod n = 1$
  - Can be performed efficiently with the repeated squaring algorithm

Randomized Primality Testing

- Compositeness witness function $\textit{witness}(x, n)$ with error probability q for a random variable x

  Case 1: n is prime

  $\textit{witness}(x, n) = \textit{false}$ always

  Case 2: n is composite

  $\textit{witness}(x, n) = \textit{true}$ in most cases, $\textit{false}$ with small probability $q < 1$

- Algorithm ***RandPrimeTest*** tests whether n is prime by repeatedly evaluating $\textit{witness}(x, n)$
- A variation of base- x pseudoprimality provides a suitable compositeness witness function for randomized primality testing (Rabin-Miller algorithm)

Algorithm ***RandPrimeTest***(n, k)

Input integer n,confidence parameter k and composite witness function $\textit{witness}(x,n)$ with error probability q

Output an indication of whether n is composite or prime with probability 2^{-k}

$t \leftarrow k/\log_2(1/q)$
for $i \leftarrow 1$ **to** t
  $x \leftarrow \textit{random}()$
  if $\textit{witness}(x, n) = \textit{true}$
    return "n is composite"
return "n is prime"

Cryptographic Hash Functions

Hash Functions

- A hash function h maps a plaintext x to a fixed-length value x = h(P) called hash value or digest of P
  - A collision is a pair of plaintexts P and Q that map to the same hash value, h(P) = h(Q)
  - Collisions are unavoidable
  - For efficiency, the computation of the hash function should take time proportional to the length of the input plaintext
- Hash table
  - Search data structure based on storing items in locations associated with their hash value
  - Chaining or open addressing deal with collisions
  - Domain of hash values proportional to the expected number of items to be stored
  - The hash function should spread plaintexts uniformly over the possible hash values to achieve constant expected search time

Cryptographic Hash Functions

- A cryptographic hash function satisfies additional properties
  - Preimage resistance (aka one-way)
    - Given a hash value x, it is hard to find a plaintext P such that h(P) = x
  - Second preimage resistance (aka weak collision resistance)
    - Given a plaintext P, it is hard to find a plaintext Q such that h(Q) = h(P)
  - Collision resistance (aka strong collision resistance)
    - It is hard to find a pair of plaintexts P and Q such that h(Q) = h(P)
- Collision resistance implies second preimage resistance
- Hash values of at least 256 bits recommended to defend against brute-force attacks
- A random oracle is a theoretical model for a cryptographic hash function from a finite input domain $\mathcal{P}$ to a finite output domain $\mathcal{X}$
  - Pick randomly and uniformly a function h: $\mathcal{P} \rightarrow \mathcal{X}$ over all possible such functions
  - Provide only oracle access to h: one can obtain hash values for given plaintexts, but no other information about the function h itself

Birthday Attack

- The brute-force birthday attack aims at finding a collision for a hash function h
  - Randomly generate a sequence of plaintexts $X_1, X_2, X_3, \ldots$
  - For each X_i compute $y_i = h(X_i)$ and test whether $y_i = y_j$ for some $j < i$
  - Stop as soon as a collision has been found
- If there are m possible hash values, the probability that the i-th plaintext does not collide with any of the previous $i - 1$ plaintexts is $1 - (i - 1)/m$
- The probability F_k that the attack fails (no collisions) after k plaintexts is

$$F_k = (1 - 1/m)\,(1 - 2/m)\,(1 - 3/m)\,\ldots\,(1 - (k - 1)/m)$$

- Using the standard approximation $1 - x \approx e^{-x}$

$$F_k \approx e^{-(1/m + 2/m + 3/m + \ldots + (k-1)/m)} = e^{-k(k-1)/2m}$$

- The attack succeeds/fails with probability ½ when $F_k = ½$, that is,

$$e^{-k(k-1)/2m} = ½$$

$$k \approx 1.17\, m^{½}$$

- We conclude that a hash function with b-bit values provides about b/2 bits of security

Message-Digest Algorithm 5 (MD5)

- Developed by Ron Rivest in 1991
- Uses 128-bit hash values
- Still widely used in legacy applications although considered insecure
- Various severe vulnerabilities discovered
- Chosen-prefix collisions attacks found by Marc Stevens, Arjen Lenstra and Benne de Weger
  - Start with two arbitrary plaintexts P and Q
  - One can compute suffixes S1 and S2 such that P||S1 and Q||S2 collide under MD5 by making 250 hash evaluations
  - Using this approach, a pair of different executable files or PDF documents with the same MD5 hash can be computed

Secure Hash Algorithm (SHA)

- Developed by NSA and approved as a federal standard by NIST
- SHA-0 and SHA-1 (1993)
  - 160-bits
  - Considered insecure
  - Still found in legacy applications
  - Vulnerabilities less severe than those of MD5
- SHA-2 family (2002)
  - 256 bits (SHA-256) or 512 bits (SHA-512)
  - Still considered secure despite published attack techniques
- Public competition for SHA-3 announced in 2007

Iterated Hash Function

- A compression function works on input values of fixed length
- An iterated hash function extends a compression function to inputs of arbitrary length
  - padding, initialization vector, and chain of compression functions
  - inherits collision resistance of compression function
- MD5 and SHA are iterated hash functions