Department of Mathematics, Indian Institute of Technology Madras

MA1102 Series and Matrices Quiz-1 Solution

February 13, 2018 Tuesday 8:00-8:50 Maximum Marks: 20

Answer all the seven questions.

Question 3 carries 2 marks, and other questions carry 3 marks each.

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1. Does the improper integral $\int_0^3 \frac{dx}{x-1}$ converge? Justify your answer.

   $\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$.

   $\int_0^1 \frac{dx}{x-1} = \lim_{b\to 1-} \int_0^b \frac{dx}{x-1} = \lim_{b\to 1-} |x-1|\Big]_0^b = \lim_{b\to 1-} \ln(1-b) = -\infty$.

   Hence the integral $\int_0^3 \frac{dx}{x-1}$ does not converge.

2. Determine whether the improper integral $\int_1^\infty \frac{dx}{(1+x)^{1/2}(1+x^2)^{1/3}}$ converges.

   Consider $g(x) = \frac{1}{x^{1/2}x^{2/3}} = \frac{1}{x^{7/6}}$

   Now, $\frac{1}{(1+x)^{1/2}(1+x^2)^{1/3}} \Big/ g(x) = \frac{1}{(1+\frac{1}{x})^{1/2}(1+\frac{1}{x^2})^{1/3}} \to 1$ as $x \to \infty$.

   Since $\frac{7}{6} > 1$, the improper integral $\int_1^\infty g(x)dx$ converges.

   By comparison test, the given integral converges.

3. Test for convergence of the series $\sum_{n=1}^{\infty} \frac{(n-1)^n}{n^{2n}}$.

   Let $a_n = \frac{(n-1)^n}{n^{2n}}$. Now, $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{1-1/n}{n} = 0$.

   By Cauchy's root test, the series converges.

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ is convergent.

   Let $a_n = \frac{(\ln n)^2}{n^3}, \quad b_n = \frac{1}{n^2}$.

   $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{n\to\infty} \frac{2\ln n}{n} = \lim_{n\to\infty} \frac{2}{n} = 0$.

   Since $\sum \frac{1}{n^2}$ converges, by Limit Comparison test, the given series converges.

5. Does the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$ converge? Justify your answer.

   Let $a_n = (-1)^{n+1}\frac{2^n}{n!}$. Now, $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2}{n+1} = 0$.

   By D' Alembert's ratio test, the series $\sum |a_n|$ is convergent.

   Hence, the given series is convergent.

6. Determine whether the series $\sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(2n)!}$ converges or diverges to ∞.

   Let $a_n = \frac{4^n (n!)^2}{(2n)!}$. Now, $\frac{a_{n+1}}{a_n} = \frac{4^{n+1}((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{4^n (n!)^2} = \frac{2(n+1)}{2n+1} > 1.$
   Also, $a_1 = 2$. Hence $\sum a_n$ is a series of positive and increasing terms.

   Therefore, the series diverges to ∞.

7. Let (b_n) be a sequence, where $0 < b_n < 1$ for each $n \in \mathbb{N}$. Define

$$a_n = \frac{b_n}{1 - b_n} \quad \text{for } n \in \mathbb{N}.$$

   Show that if the series $\sum_{n=1}^{\infty} b_n$ is convergent, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.
   As $\sum b_n$ converges, $\lim_{n\to\infty} b_n = 0$.
   So, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{b_n}{1 - b_n} \cdot \frac{1}{b_n} \right) = \lim_{n\to\infty} \frac{1}{1 - b_n} = 1.$
   As $\sum b_n$ converges, by Limit Comparison test, $\sum a_n$ converges.
