DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II

Tutorial 2 (29.1.2018)

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1. Consider a thick metallic spherical shell with inner and outer radii a and b, respectively, carrying a charge Q on it. A point charge q is fixed at the center of the shell. Calculate the charge on each surface of the shell, electric potential and field everywhere. Plot the variation of both the electric field and potential as a function of r, the distance from the center of the shell.
Solution: The spherical symmetry of the problem leads to uniform

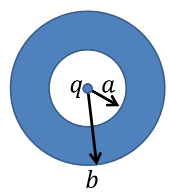

Figure 1: Charge q is at the center of the metallic shell.

charge densities on the inner and outer surfaces of the shell. Since the field inside a conductor should be zero, a Gaussian spherical shell centered at O, with radius $a < r < b$, shows that the charge on the inner surface should be $-q$ and hence the surface charge density

$$\sigma_a = \frac{-q}{4\pi a^2}$$

The total charge on the shell is given to be Q and hence the charge smeared on the outer surface is $(Q + q)$. Hence the charge density on the outer surface,

$$\sigma_b = \frac{Q + q}{4\pi b^2}$$

Using Gaussian shells centered at the origin and of appropriate radii we get

$$\vec{E}(r \geq b) = \frac{(Q+q)}{4\pi\epsilon_0 r^2}\hat{r}$$

$$\vec{E}(a < r < b) = 0$$

$$\vec{E}(r < a) = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

The potential at external points ($r \geq b$) is the work done in bringing a unit positive charge from infinity upto r, which is given by,

$$V(r \geq b) = \frac{(Q+q)}{4\pi\epsilon_0 r}$$

The potential is uniform in the conducting volume of the shell and is given by

$$V(a < r < b) = \frac{(Q+q)}{4\pi\epsilon_0 b}$$

The potential for $r < a$ has three contributions, from $\sigma_a, \sigma_b, \&\ q$ and is given by

$$V(r < a) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q+q}{b} - \frac{q}{a} + \frac{q}{r}\right)$$

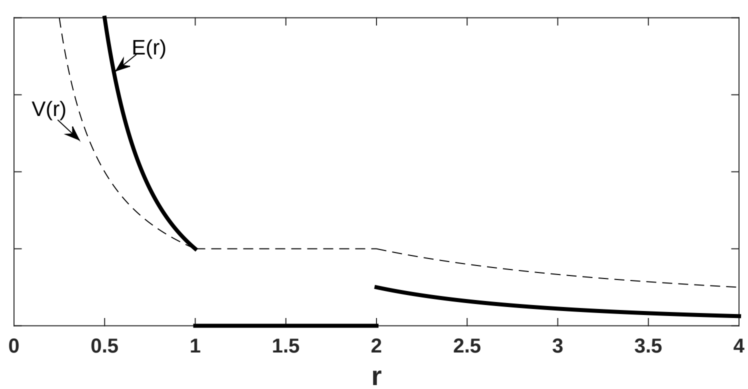

Figure 2: Electric field $E(\vec{r})$ and potential V(r). a & b were taken to be 1 and 2 units, respectively.

The surface charge densities σ_a and σ_b cause the discontinuities in $E(\vec{r})$ at these surfaces.

2. Show that the dipole moment of any arbitrary charge distribution $\rho(\vec{r'})$ depends on the choice of the origin. Determine the condition in which the dipole moment will be independent of the origin.

Solution: Consider two coordinate systems as shown in the diagram. $\vec{a}$ is the relative position vector between the two origins. The dipole

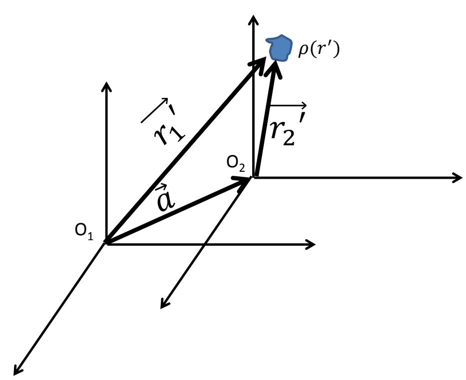

Figure 3:

moment in coordinate system 1 is

$$\vec{p_1} = \int \vec{r_1'}\rho(\vec{r_1'})d\tau_1' = \int (\vec{r_2'} + \vec{a})\rho(\vec{r_2'})d\tau_2' = \vec{p_2} + \vec{a}\int \rho(\vec{r_2'})d\tau_2' = \vec{p_2} + \vec{a}Q$$

where Q is the total charge of the distribution.

Thus the dipole moment depends on the choice of the origin except when the total charge of the distribution is zero.

3. A dipole $\vec{p_1}$ is fixed at the origin and aligned along the x axis. Another dipole $\vec{p_2}$ is at the point $B(a, a)$ as shown in the figure. Find the orientation $\vec{p_2}$ will take if it is free to rotate in the xoy plane at B. After $\vec{p_2}$ has taken its equilibrium position, it is fixed in that position and $\vec{p_1}$ is allowed to rotate in the xoy plane at the origin. What orientation will $\vec{p_1}$ take? If $\vec{p_2}$ and $\vec{p_1}$ are both free to rotate in their places, what orientations will they take?

Solution: The electric field at the point B due to $\vec{p_1}$ is given by

$$\vec{E_1} = \frac{1}{4\pi\epsilon_0 r^3}[3(\vec{p_1}\cdot\hat{r})\hat{r} - \vec{p_1}]$$

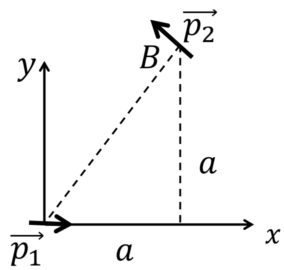

Figure 4:

For the point B,

$$\hat{r} = \frac{1}{\sqrt{2}}[\hat{e_x} + \hat{e_y}]$$
$$\vec{p_1} = p_1\hat{e_x}$$

hence,

$$\vec{E_1} = \frac{|\vec{p_1}|}{4\pi\epsilon_0 r^3}\left[\frac{1}{2}\hat{e_x} + \frac{3}{2}\hat{e_y}\right]$$

The dipole $\vec{p_2}$ will align itself along $\vec{E_1}$ to reduce its potential energy. Therefore $\vec{p_2}$ will be inclined at 71.56^0 to the $x-$axis. The unit vector along $\vec{E_1}$ is given by $\sqrt{\frac{2}{5}}\left[\frac{1}{2}\hat{e_x} + \frac{3}{2}\hat{e_y}\right]$. Now

$$\vec{p_2} = |\vec{p_2}|\sqrt{\frac{2}{5}}\left[\frac{1}{2}\hat{e_x} + \frac{3}{2}\hat{e_y}\right]$$

and its field at the origin is given by

$$\vec{E_2} = \frac{1}{4\pi\epsilon_0 r^3}[3(\vec{p_2}\cdot\hat{r})\hat{r} - \vec{p_2}]$$

where

$$\vec{p_2} = |\vec{p_2}|\sqrt{\frac{2}{5}}\left[\frac{1}{2}\hat{e_x} + \frac{3}{2}\hat{e_y}\right] \qquad \& \qquad \hat{r} = -\frac{1}{\sqrt{2}}[\hat{e_x} + \hat{e_y}]$$

Thus

$$\vec{E_2} = \frac{|\vec{p_2}|}{4\pi\epsilon_0 r^3}\sqrt{\frac{2}{5}}\left[\frac{5}{2}\hat{e_x} + \frac{3}{2}\hat{e_y}\right]$$

Therefore $\vec{p_1}$ will align itself inclined at 31^0 about $x-$axis. When both the dipoles are fixed at their respective positions and are free to rotate, then they will align parallel to OB.

4. Consider the following distribution of three point charges: $2q$ at $(0, a, a)$; q at $(0, -a, a)$ and $-q$ at $(0, 0, -a)$.
(a) Determine the dipole moment about the origin for this distribution and the dipole potential at $P(0, 0, z)$, where $z >> a$.
(b) Calculate the dipole field at $P(0, 0, z)$. Why is $\vec{E}_{dipole}(0, 0, z) \neq -\nabla_z V_{dipole}(0, 0, z)$?
Solution:

i	q_i	x_i	y_i	z_i
$i = 1$	$+2q$	0	a	a
$i = 2$	$+q$	0	-a	a
$i = 3$	$-q$	0	0	-a

It is easy to calculate the following:

$p_x = \sum_i q_i x_i = 0$; $p_y = \sum_i q_i y_i = qa$; and $p_z = \sum_i q_i z_i = 4qa$
This gives $\vec{p} = p_x \hat{e_x} + p_y \hat{e_y} + p_z \hat{e_z} = qa(\hat{e_y} + 4\hat{e_z})$

Thus the resultant $\vec{p}$ is in the yoz plane, inclined at an angle $\arctan(\frac{1}{4})$ to the z-axis. The potential at $P(0, 0, z)$ due to this $\vec{p}$, for $z \gg a$, is

$$V(0, 0, z) = \frac{\vec{p} \cdot z\hat{e_z}}{4\pi\epsilon_0 z^3} = \frac{p_z}{4\pi\epsilon_0 z^2} = \frac{4aq}{4\pi\epsilon_0 z^2} = \frac{aq}{\pi\epsilon_0 z^2}$$

The electric field at any point with position vector $\vec{r}$, due to a dipole at the origin, is given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}[3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}]$$

For points along z-axis, $\vec{r} = z\vec{e_z}$, therefore

$$\vec{E}(0, 0, z) = \frac{aq}{4\pi\epsilon_0}\left[\frac{-\hat{e_y} + 8\hat{e_z}}{z^3}\right]$$

The electric field at $P(0, 0, z)$ is not given in this case, by

$$\vec{E} = -\frac{d}{dz}V(0, 0, z)\hat{e_z}$$

because of lack of azimuthal symmetry.

5. A line charge on the z-axis extends from $z = -a$ to $z = +a$ and has linear charge density varying as

$$\lambda(z) = \begin{cases} \lambda_0 z^\alpha, & \text{when } 0 < z \le a \\ -\lambda_0 |z|^\alpha, & \text{when } -a \le z < 0 \end{cases}$$

where α is a positive constant and λ_0 is a positive constant of appropriate dimensions. Find the potential at any point (r, θ, ϕ) with $r > a$, up to the dipole term.

Solution: The total positive charge

$$Q = \int_0^a \lambda_0 z^\alpha dz = \lambda_0 . \frac{a^{\alpha+1}}{\alpha+1}$$

Thus

$$\lambda_0 = \frac{Q(\alpha+1)}{a^{\alpha+1}}$$

The total charge is zero and hence there is no contribution from the monopole term to the potential. The dipole potential, at the field point (r, θ, ϕ), due to positive line charge is

$$V^+_{dipole}(r,\theta) = \frac{1}{4\pi\epsilon_0 r^2} \hat{r} \cdot \vec{p}$$

where $\vec{p}$ is the dipole moment about the origin due to the positive line charge.

$$V^+_{dipole}(r,\theta) = \frac{1}{4\pi\epsilon_0 r^2} \hat{r} \cdot \int_0^a \vec{r'} \lambda_0 r'^\alpha dr'$$

(Note: $z' = r'$ since $\theta' = 0$)

$$\hat{r} \cdot \vec{r'} = \cos\theta$$

$$\begin{aligned} V^+_{dipole}(r,\theta) &= \frac{1}{4\pi\epsilon_0 r^2} \cos\theta \int_0^a \lambda_0 r'^{\alpha+1} dr' \\ &= \frac{1}{4\pi\epsilon_0 r^2} \cos\theta . \lambda_0 . \frac{a^{\alpha+2}}{\alpha+2} \\ &= \frac{Qa}{4\pi\epsilon_0 r^2} \cos\theta \left(\frac{\alpha+1}{\alpha+2} \right) \end{aligned}$$

In the contribution of the negative charges to the dipole term, the sign of the charge is -ve but $\hat{r} \cdot \hat{r'} = \cos(\pi - \theta)$. So $V^+_{dipole} = V^-_{dipole}$ and hence the total potential at the field point, up to the dipole term,

$$V(r,\theta,\phi) = \frac{2Qa}{4\pi\epsilon_0 r^2} \cos\theta \left(\frac{\alpha+1}{\alpha+2} \right)$$