EC1101 : Signals and Systems

Tutorial 3 Solutions

1. The fundamental period of $x(t) = \sin(\frac{3\pi}{5}t)$ is,

$$T = \frac{2\pi}{(3\pi/5)} = \frac{10}{3}.$$

For $x[n] = \sin(\frac{3\pi}{5}T_s n)$ to be a periodic signal, there must be an integer m such that,

$$\frac{(3\pi/5)T_s}{2\pi} = \frac{m}{N}$$

And the periodicity N of $x[n]$ is given by,

$$N = m.\ \frac{2\pi}{(3\pi/5)T_s}\ , \qquad m \in Z.$$

(a) For $T_s = 1\ sec$,

$$x[n] = \sin\left(\frac{3\pi}{5}n\right).$$

Since

$$\frac{(3\pi/5)}{2\pi} = \frac{3}{10} = \frac{m}{N},$$

the signal $x[n]$ is periodic with fundamental period $N = 10$.

(b) For $T_s = 5\ sec$,

$$x[n] = \sin(3\pi n).$$

Since

$$\frac{(3\pi)}{2\pi} = \frac{3}{2} = \frac{m}{N},$$

the signal $x[n]$ is periodic with fundamental period $N = 2$.

(c) For $T_s = 1/\pi\ sec$,

$$x[n] = \sin\left(\frac{3}{5}n\right).$$

Since

$$\frac{(3/5)}{2\pi} = \frac{3}{10\pi} = \frac{m}{N}$$

is an irrational number, the signal $x[n]$ is not periodic.

2. (a) Given: $x[n]$ is periodic and $y_1[n] = x[2n]$. Let the fundamental period of $x[n]$ be N. Then, $x[n] = x[n + kN]$, for any integer $k \neq 0$. Now,

$$y_1[n] = x[2n] = x[2n + kN].$$

For some integer M_1, we have,

$$y_1[n + M_1] = x[2(n + M_1)] = x[2n + 2M_1].$$

The signal $y_1[n]$ will be periodic only if $y_1[n] = y_1[n + M_1]$, for some non-zero integer M_1. In other words, $x[2n + 2M_1] = x[2n + kN]$, for some integer $M_1 \neq 0$. Consider the following cases:

Case 1: Suppose N is even. Then, $x[2n+2M_1] = x[2n+kN] \Longrightarrow 2M_1 = kN \Longrightarrow M_1 = \frac{kN}{2}$. Now, since N is even, $\frac{N}{2}$ is an integer (not a fraction). Hence, $y_1[n]$ repeats itself after every $\frac{kN}{2}$ samples, where k is any non-zero integer. This implies that $y_1[n]$ is periodic. The fundamental period of $y_1[n]$ is obtained by assigning the smallest positive integer value for k such that $\frac{kN}{2}$ is an integer. According to that, the smallest value of k is 1.

Thus, the fundamental period of $y_1[n]$ is $\frac{N}{2}$.

Case 2: Suppose N is odd. Then, $x[2n + 2M_1] = x[2n + kN] \Longrightarrow M_1 = \frac{kN}{2}$. Here, $\frac{N}{2}$ is not an integer as N is odd. Hence, M_1 will be an integer only for even integer values of k, i.e., $k = 2l$, for any integer $l \neq 0$. This results in $M_1 = lN$. Thus, $y_1[n]$ is periodic. Again, the fundamental period is obtained by assigning the smallest positive integer value to l in the expression $M_1 = lN$ to make M_1 an integer. And, that value is 1.

Therefore, $y_1[n]$ is periodic with fundamental period N.

On the whole, if $x[n]$ is periodic, then so is $y_1[n] = x[2n]$.

(b) Given: $x[n]$ is periodic and $y_2[n] = x[n/2]$, for n-even and zero otherwise. Since, $y_2[n]$ contains all the samples of $x[n]$ each separated by a single zero sample, $y_2[n]$ retains the periodic structure of $x[n]$ in it. Hence, $y_2[n]$ is periodic with fundamental period equal to twice that of $x[n]$.

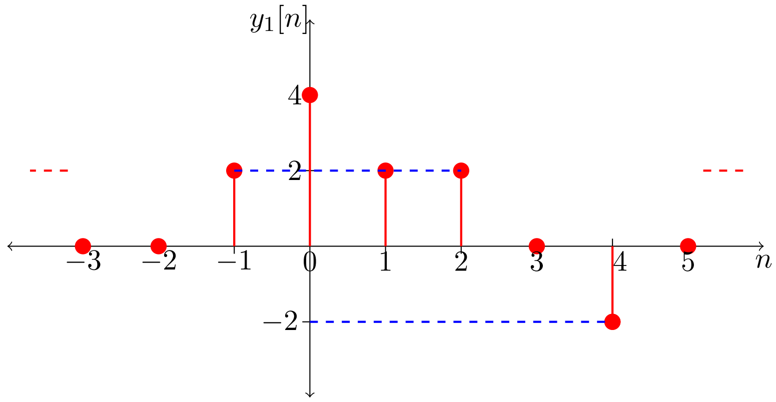
3. To compute the given convolutions, we first compute $x[n] * \delta[n-a]$ (a is an integer) and then use linearity and time invariance of convolution operation.

$$x[n] * \delta[n-a] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k-a] = x[n-a]$$

.

(a)

$$\begin{aligned} y_1[n] &= x[n] * (2\delta[n+1] + 2\delta[n-1]) = 2(x[n+1] + x[n-1]) \\ &= 2\delta[n+1] + 4\delta[n] + 2\delta[n-2] + 2\delta[n-1] - 2\delta[n-4]. \end{aligned}$$

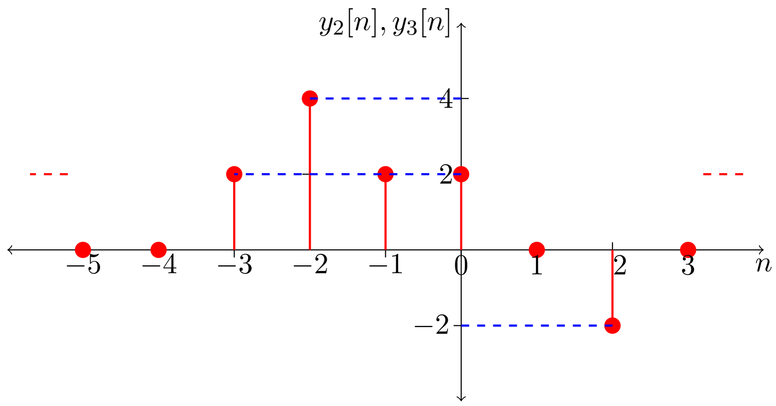


(b) Here, we use commutative and associative properties of convolution operator and get,

$$\begin{aligned} y_2[n] &= x[n+2] * h[n] = (x[n] * \delta[n+2]) * h[n] = x[n] * h[n] * \delta[n+2] \\ &= y_1[n] * \delta[n+2] = y_1[n+2]. \end{aligned}$$

(c) In this case, we employ associative property of convolution operator and obtain,

$$\begin{aligned} y_3[n] &= x[n] * h[n+2] = x[n] * (h[n] * \delta[n+2]) = (x[n] * h[n]) * \delta[n+2] \\ &= y_1[n] * \delta[n+2] = y_1[n+2] = y_2[n]. \end{aligned}$$



4. Given, $x[n] = 0$, outside $0 \leq n \leq N-1$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=0}^{N-1} x[k]h[n-k] \end{aligned}$$

Now, substitute diiferent values for 'n' and expand the sumation,

$$\begin{aligned} y[0] &= x[0]h[0] + x[1]h[-1] + \cdots + x[N-1]h[-(N-1)] \\ y[1] &= x[0]h[1] + x[1]h[0] + \cdots + x[N-1]h[-N+2] \\ &\vdots \\ y[N-1] &= x[0]h[N-1] + x[1]h[N-2] + \cdots + x[N-1]h[0] \end{aligned}$$

We can write the above equations in matrix form ($\mathbf{y}$=$\mathbf{Hx}$) as follows ,

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[-1] & \cdots & h[-N+1] \\ h[1] & h[0] & \cdots & h[-N+2] \\ \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

5. Let

$$x[n] = u[n]$$
$$h[n] = a^n u[-n-1], |a| > 1$$

$$\text{Now,} \quad y[n] = x[n] * h[n]$$
$$y[n] = \sum_{k=-\infty}^{\infty} a^k u[-k-1]u[n-k]$$
$$y[n] = \sum_{k=-\infty}^{n} a^k u[-k-1]$$

if $n > -1$,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{-1} a^k = a^{-1} + a^{-2} + \cdots \\ &= a^{-1}[1 + \frac{1}{a} + \frac{1}{a^2} + \cdots] \\ &= \frac{1}{a-1}. \end{aligned}$$

if $n \le -1$,

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n} a^k = a^n + a^{n-1} + a^{n-2} + \cdots \\ &= a^n(1 + \frac{1}{a} + \frac{1}{a^2} + \cdots) = a^n\Big(\frac{1}{1-\frac{1}{a}}\Big) \\ &= \frac{a^{n+1}}{a-1} \end{aligned}$$
$$\therefore\ y[n] = \begin{cases} \frac{a^{n+1}}{a-1}, & n \le -1 \\ \frac{1}{a-1}, & n > -1 \end{cases}$$

6. The signal y[n] is

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

In this case, the summation reduces to

$$y[n] = \sum_{k=0}^{9} x[k]h[n-k] = \sum_{k=0}^{9} h[n-k]$$

$$y[4]=\sum_{k=0}^{9} h[4-k]$$
$$\Rightarrow 5=h[4]+h[3]+h[2]+h[1]+h[0]+h[-1]+h[-2]+h[-3]+h[-4]+h[-5]$$
$$\Rightarrow 5=h[4]+h[3]+h[2]+h[1]+h[0] \ (\because h[n]=0 \ \forall n<0)$$

$\therefore N \geq 4$

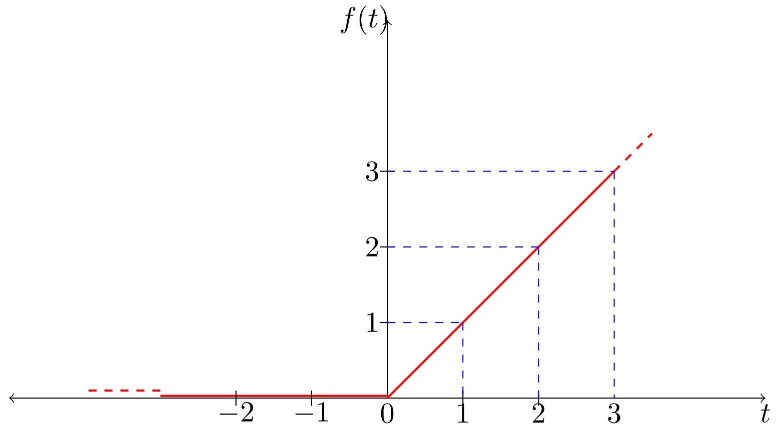
$$y[14]=\sum_{k=0}^{9} h[14-k]$$
$$\Rightarrow 0=h[14]+h[13]+h[12]+h[11]+h[10]+h[9]+h[8]+h[7]+h[6]+h[5]$$

As value of h[n] is either 0 or 1, in order to satisfy the above condition we need $h[14] = h[13] = h[12] = h[11] = h[10] = h[9] = h[8] = h[7] = h[6] = h[5] = 0$. Therefore $N = 4$

7. (a) We know that $y(t) = \int\limits_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$. Here, $h(\tau)$ is non-zero only in $(3,4)$, then the above integral becomes $y(t) = \int\limits_{3}^{4} x(t-\tau)d\tau$. Further, $x(t)$ is non-negative only in $(2,3)$, and zero elsewhere. Eventually, $x(t-\tau)$ will be non-zero only between $t-3$ and $t-2$. $y(t)$ will be zero $t-3>4 \Rightarrow t>7$ and $t-2<3 \Rightarrow t<5$. This implies that the above integral is non-zero for $5 \leq t \leq 7$. Hence, y(t) is non-zero for $t \in (5,7)$.

   (b) $y(t) = \int\limits_{3}^{4} x(t-\tau)d\tau = \int\limits_{t-4}^{t-3} x(\tau)d\tau$. Again, $x(\tau)$ is **non-negative** only in $\tau \in (2,3)$, with symmetry around $\tau = \frac{5}{2}$. The integral computes the complete area occupied by $x(\tau)$ only when $t = 6$, as only at $t = 6$ the limits of the integral is 2 to 3. For other values of t, the integration will either be equal to area of a part of $x(\tau)$ or zero. Therefore, $y(t)$ will have maximum value at $t = 6$.

8. (a) $f(t) = u(t) * u(t) = \int\limits_{-\infty}^{\infty} u(t-\tau)u(\tau)d\tau = \int\limits_{0}^{t} 1\, d\tau = t$, (for $t \geq 0$). Thus $f(t) = tu(t)$.



(b)

$$f(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} (-e^{-\tau} + 2e^{-2\tau})u(\tau)10e^{-3(t-\tau)}u(t-\tau)d\tau$$

$$= \int_{0}^{t} 10(-e^{-\tau} + 2e^{-2\tau})e^{-3(t-\tau)}d\tau = 10\int_{0}^{t}(-e^{2\tau-3t} + 2e^{\tau-3t})d\tau = -5e^{-t} + 20e^{-2t} - 15e^{-3t}.$$

Hence, $f(t) = -5e^{-t} + 20e^{-2t} - 15e^{-3t}$ for $t \geq 0$ and zero elsewhere.

(c) Given: $h(t) = 2e^{-2t}u(t)$ and $x(t) = 1, \forall 2 \leq t \leq 4$ and zero otherwise. Let $y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$. Now, $x(t-\tau)$, as a function of τ, will be 1 from $\tau = t-4$ to $\tau = t-2$, for any given t, and zero outside this range. Consider the following cases:

Case 1: When $t - 2 < 0 \Rightarrow t < 2$.

The product $h(\tau)x(t-\tau) = 0$, as there is no common overlap between the non-zero regions of these two signals. Hence, $y(t) = 0, \forall t < 2$.

Case 2: Suppose $t - 2 \geq 0$ and $t - 4 < 0$, i.e., $2 \leq t < 4$.

Then, $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{t-2} 2e^{-2\tau}d\tau = 1 - e^{-2t+4}, \forall 2 \leq t < 4$.

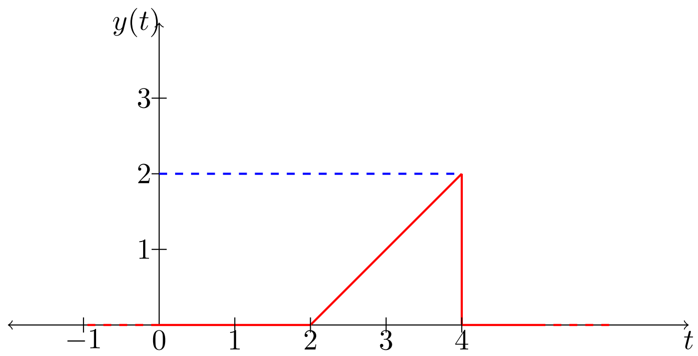
Case 3: Finally, $t - 4 \geq 0$, i.e., $t \geq 4$.

Now, $y(t) = \int_{t-4}^{t-2} 2e^{-2\tau}d\tau = -(e^{-2t+4} - e^{-2t+8})$. This value of $y(t)$ is for the range $t \geq 4$.

Hence, the signal $y(t)$ is given by,

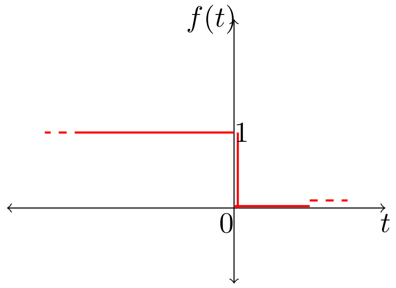
$$y(t) = \begin{cases} 0, & t < 2 \\ 1 - e^{-2t+4}, & 2 \leq t < 4 \\ e^{-2t+8} - e^{-2t+4}, & t \geq 4 \end{cases}$$

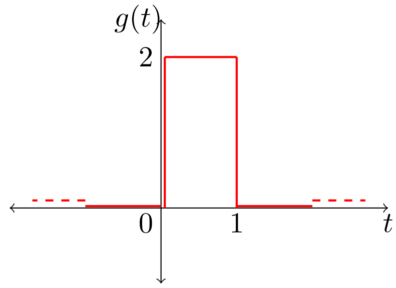
(d) $y(t) = [u(t) * u(t-2)]u(4-t) = r(t-2)u(4-t)$.



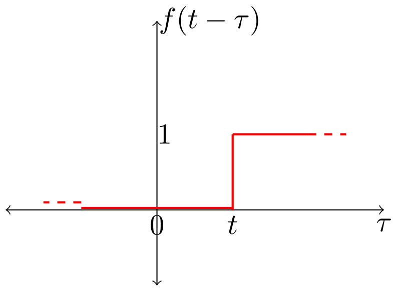
(e) **i)**

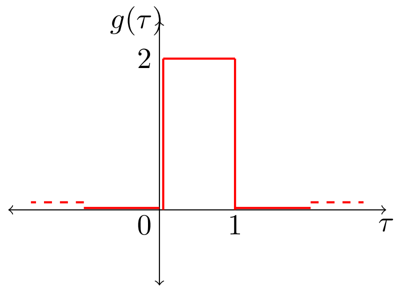
Given: $f(t) = u(-t)$ and $g(t) = 2\Big(u(t) - u(t-1)\Big)$. The signals look like,





Now, $h(t) = f(t) * g(t) = \int\limits_{-\infty}^{\infty} f(t-\tau)g(\tau)\, d\tau$.

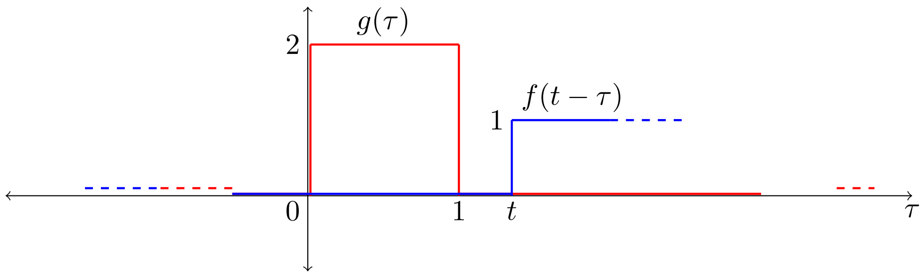




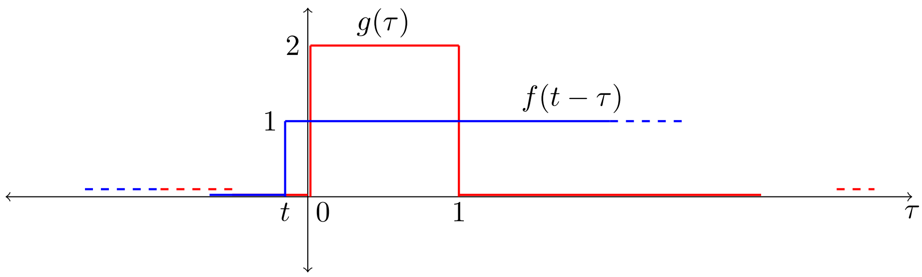
Consider the following cases:

Case 1: $t > 1$.

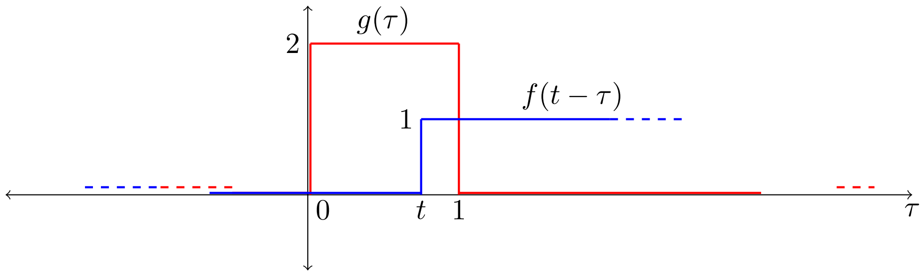
In this case, there is no overlap between $g(\tau)$ and $f(t-\tau)$. Thus, $h(t) = 0, \forall t > 1$.



Case 2: $t \leq 0$. Here, we get, $h(t) = \int\limits_{0}^{1} 2\, d\tau = 2$.

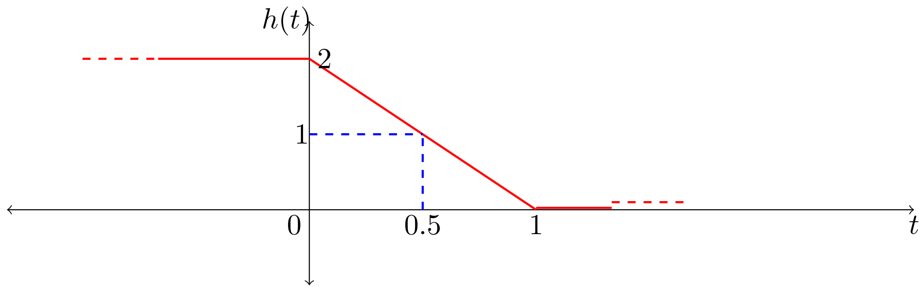


Case 3: $0 < t \leq 1$. Then, $h(t) = \int_t^1 2\,d\tau = 2(1-t)$.



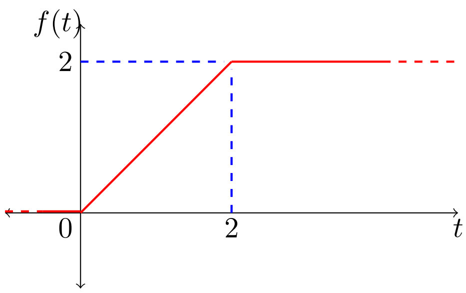
The final expression for the signal $h(t)$ is given by,

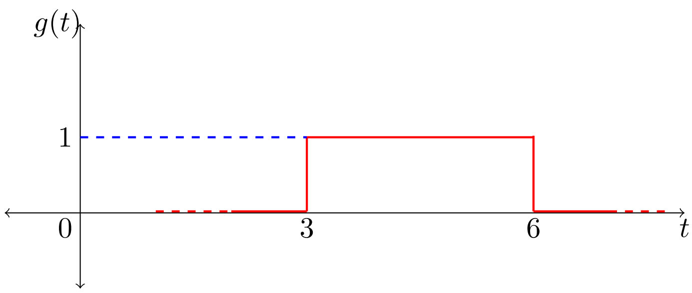
$$h(t) = \begin{cases} 2 & t < 0 \\ 2(1-t) & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}$$



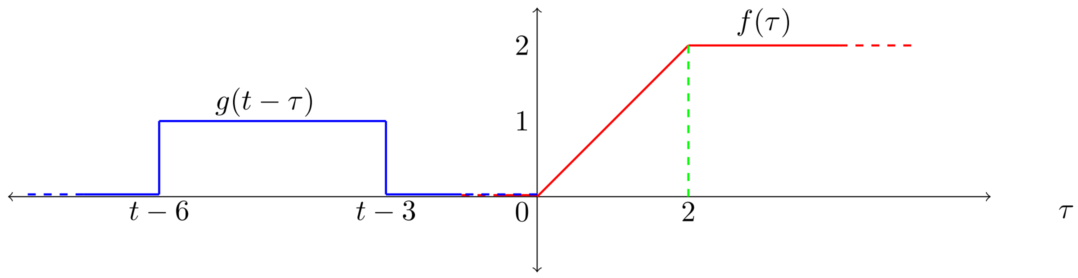
ii)

The signals $f(t)$ and $g(t)$ are as given below,



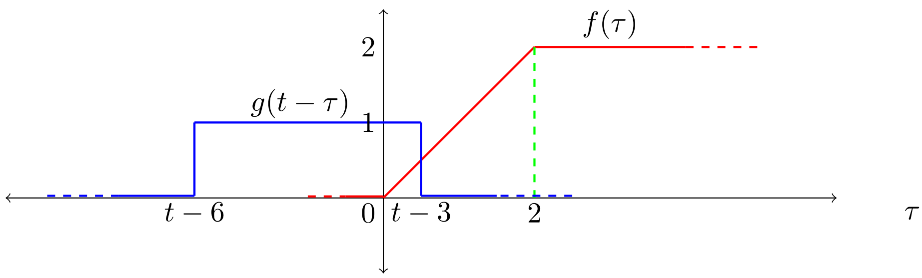


Case 1: For $t < 3$, we have the following:



Hence, $h(t)$ will be zero for $t < 3$, as there is no overlap between $g(t-\tau)$ and $f(\tau)$.

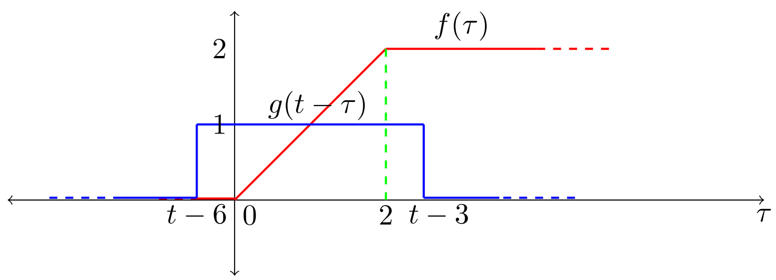
Case 2: For $3 \leq t \leq 5$,



Here, $h(t)$ will be,

$$h(t) = \int_0^{t-3} \tau d\tau = \frac{(t-3)^2}{2}.$$

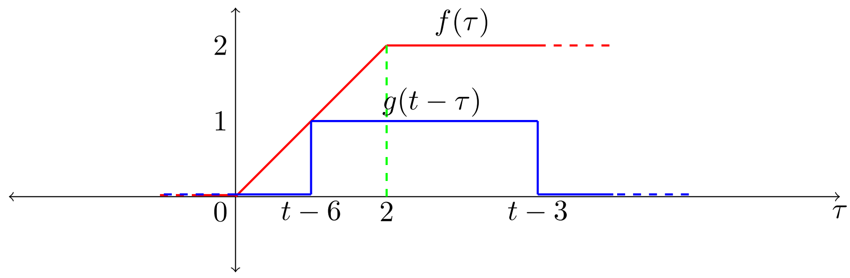
Case 3: For $5 \leq t \leq 6$,



In this case, we get,

$$h(t) = \int_0^2 \tau d\tau + \int_2^{t-3} 2d\tau = 2(t-4).$$

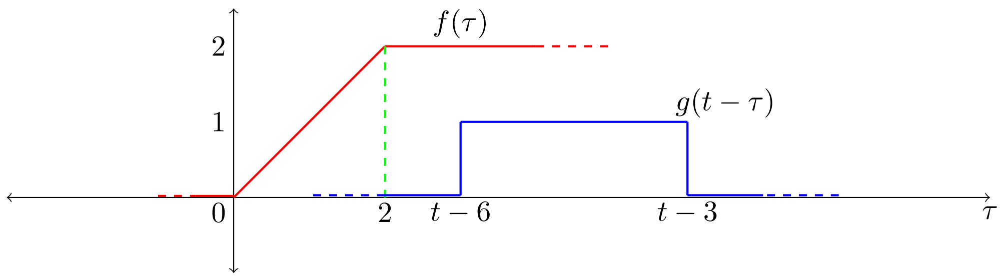
Case 4: For $6 \leq t \leq 8$,



Due to the above overlapping fashion, $h(t)$ for $6 \leq t \leq 8$ will be,

$$h(t) = \int_{t-6}^{2} \tau d\tau + \int_{2}^{t-3} 2d\tau = -\frac{t^2}{2} + 8t - 26.$$

Case 5: For $t > 8$, we obtain,



$$h(t) = \int_{t-6}^{t-3} 2d\tau = 6.$$

Finally, the signal $h(t)$ is given by, $h(t) = \begin{cases} 0 & t \leq 3 \\ \frac{(t-3)^2}{2}, & 3 < t < 5, \\ 2(t-4), & 5 \leq t < 6 \\ \frac{-t^2}{2} + 8t - 26, & 6 \leq t < 8, \\ 6, & t \geq 8 \end{cases}$

9. Given: $y(t) = \int\limits_{-\infty}^{t+1} \sin(t-\tau)x(\tau)d\tau$.

   (a) The response to a delayed input will be,

$$y_1(t) = \int\limits_{-\infty}^{t+1} \sin(t-\tau)x(\tau - t_o)d\tau = \int\limits_{-\infty}^{t-t_o+1} \sin(t-\tau^{'} - t_o)x(\tau^{'})d\tau^{'}$$

However, the delayed response of the system is given by,

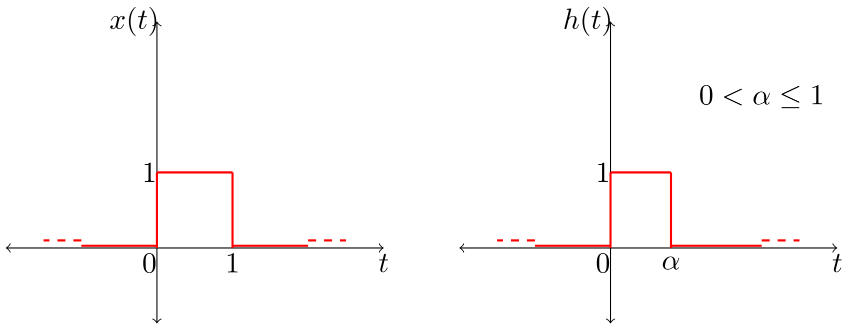
$$y_2(t) = \int_{-\infty}^{t-t_o+1} \sin(t - t_o - \tau)x(\tau)d\tau.$$

Since, $y_1(t) = y_2(t)$, the given system is time-invariant.

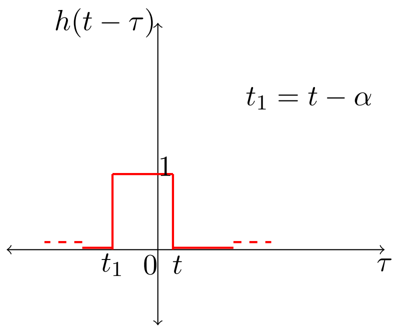
(b) Now, $y(t) = \int_{-\infty}^{t+1} \sin(t-\tau)x(\tau)d\tau = \int_{-\infty}^{\infty} \sin(t-\tau)u(t+1-\tau)x(\tau)d\tau$. Hence, the impulse response of the system is given by, $h(t) = \sin(t)u(t+1)$.

(c) The system given is non-causal since the output depends on future values of the input.

10. The given signals are plotted below.



Now, $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$. Consider the signal $h(t-\tau)$ shown below: Hence, we have the following cases:



Case 1: $t < 0$.

Under this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$, which implies $y(t) = 0, \forall t < 0$.

Case 2: $t \geq 0$ and $t - \alpha < 0$, i.e., $0 \leq t < \alpha$.

Now, $y(t) = \int_0^t 1\, d\tau = t$. So, $y(t) = t, \forall 0 \leq t < \alpha$.

Case 3: $t < 1$ and $t - \alpha \geq 0$, i.e., $\alpha \leq t < 1$.

Here, $y(t) = \int_{t-\alpha}^{t} 1\, d\tau = \alpha$. Therefore, $y(t) = \alpha, \forall t \in [\alpha, 1)$.

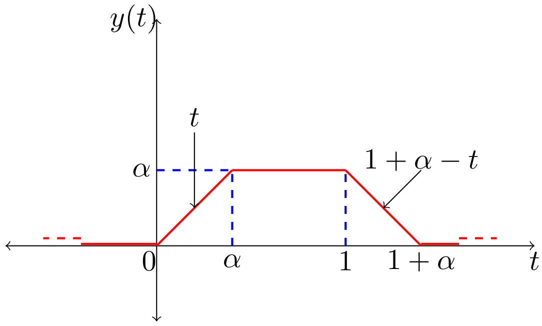
Case 4: $t - \alpha < 1$ and $t \geq 1$, i.e., $1 \leq t < 1 + \alpha$.

In this case, we get, $y(t) = \int_{t-\alpha}^{1} 1\, d\tau = 1 + \alpha - t$. This is true for every $t \in [1, 1+\alpha)$.

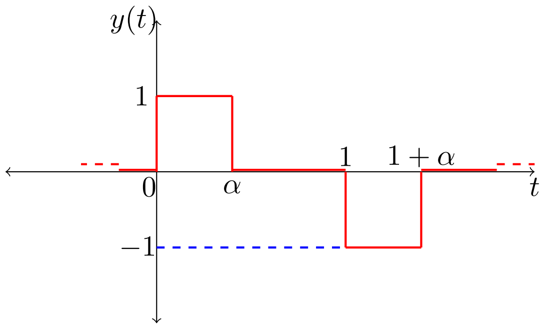
Case 5: Finally, $t - \alpha \geq 1$, i.e., $t \geq 1 + \alpha$.

There is no overlap between $x(\tau)$ and $h(t-\tau)$, and hence, $y(t) = 0$ in this range.

(a) The signal $y(t)$, therefore, looks like:



(b) The first derivative of $y(t)$ will be,



(c) It is evident from the plot of $\frac{dy(t)}{dt}$, that α must be equal to 1 for it to have exactly three discontinuities.

11. Suppose $x(t)$ and $y(t)$ be the input and output of a time-invariant system respectively, i.e., $x(t) \rightarrow y(t)$. Given that $x(t)$ is periodic with period T. Hence,

$$x(t+T) = x(t) \Longrightarrow x(t+T) \rightarrow y(t). \tag{1}$$

Further, from time-invariant property, we get

$$x(t) \rightarrow y(t) \Longrightarrow x(t+T) \rightarrow y(t+T). \tag{2}$$

From (1) and (2), we observe that, $y(t) = y(t+T)$. Therefore, y(t) is also periodic.