<u>Department of Physics, IITM</u>

PH1020 Physics II Problem set 3

1. A spherical conductor *A* of radius *R* contains two spherical cavities with radii *a*, *b* respectively as shown in the figure. The total charge on the conductor itself is zero. At the center of each cavity a point charges $+q_a$ and $+q_b$ are placed. (i) Find the surface charges σ_a, σ_b and σ_R. (ii) What is the field outside the conductor? (iii) What is the field within each cavity? (iv) What is the force on $+q_a$ and $+q_b$? (v) What is the force on a third charge $+q_c$ placed at a large distance? (vi) Which of these answers would change if q_c were brought near the conductor?

2. Two parallel infinite conducting plates at $x = 0$ and $x = L$ have potentials Φ_0 and 0 respectively. Using Poisson's equation with the appropriate boundary conditions, find (i) the electric field between the plates and (ii) the surface charge densities on the plates, when the free volume charge density between the plates is equal to a constant k ($k \neq 0$).

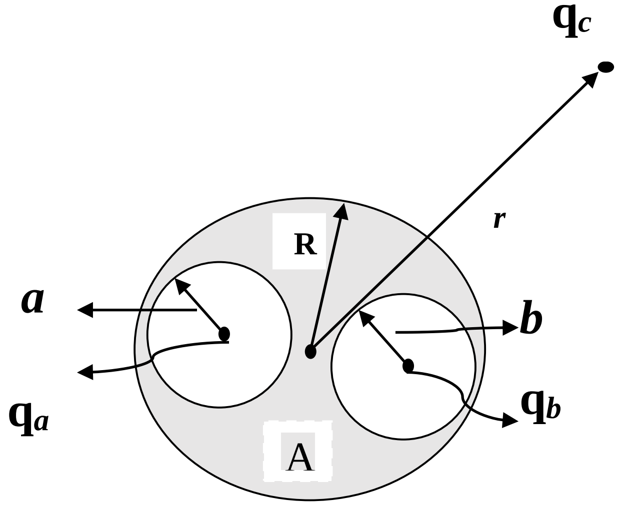
3. Find the capacitance per unit length of two coaxial metallic cylindrical tubes of radii *a* and *b*, where $b > a$.

4. A point dipole of moment $\vec{p} = p_0\left(\hat{e}_x + 2\hat{e}_y + 3\hat{e}_z\right)$ is placed in an electrostatic potential Φ given by
$$\emptyset(x,y,z) = \emptyset_0\left[1 + \frac{x^2+y^2+z^2}{a^2} + \frac{(x^4+y^4+z^4)}{a^4}\right]$$
Where Φ_0 and *a* are appropriate constants. Find the force and the couple acting on the dipole when it is located at the point (*a*, *a*, *a*). Find also the torque of the force about the origin.

5. A sphere of radius *R* carries a charge density $\rho(r) = kr$ (where *k* is a constant). Find the energy of the configuration.

6. Find the monopole, dipole and quadrupole moments of the following charge distributions about the origin. (i) a line charge of constant line charge density λ_o and of length *L* lying in the first quadrant of the *xoy* plane with one end at the origin making an angle α with the positive *x*-axis. (ii) A spherical shell of radius *R* with surface charge density $\sigma_c = \sigma_o \cos\theta$, where σ_o is a constant with its center at the origin.

7. Show that the interaction energy of two dipoles separated by a displacement *r* is
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}[\vec{p}_1\,.\,\vec{p}_2 - 3(\vec{p}_1\,.\,\hat{r})(\vec{p}_2\,.\,\hat{r})]$$



SOLUTIONS | T-3

soln ① (i) $\sigma_a = -\dfrac{q_a}{4\pi a^2}$; $\sigma_b = -\dfrac{q_b}{4\pi b^2}$; $\sigma_R = \dfrac{q_a + q_b}{4\pi R^2}$

(ii) The Electric field outside the conductor

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_a + q_b}{r^2}\right)\hat{r}$$

(iii) Field within the cavity is

$$\vec{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r_a^2}\hat{r}_a \quad , \quad \vec{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r_b^2}\hat{r}_b$$

(IV) On account of spherical symmetry the forces acting on the point charges q_a and q_b at the center of the cavities are equal to zero

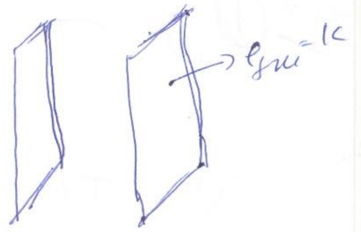
(V) As r is very large, we can approximate the interaction between sphere A and point charge q_c by electrostatic force between point charges $q_a + q_b$ (at the center) and q_c,

$$F = \frac{q_c(q_a + q_b)}{4\pi\epsilon_0 r^2}$$

(VI) The charge distribution over the surface of each cavity is always uniform and independent of the magnitude of r. However, charge distribution over the surface of the sphere A will not be uniform and thus non-uniformity will become more and more evident as r decreases. Therefore Electric field outside changes but not E_a and E_B

Soln (2) Since plates are conducting the $E = 0$ inside the plates.

$\rho_{free} = k$ is given.



In the region between the plates

$$\frac{d^2\phi}{dx^2} = -\frac{k}{\epsilon_0} \quad \text{(Poisson's eq}^n\text{)}$$

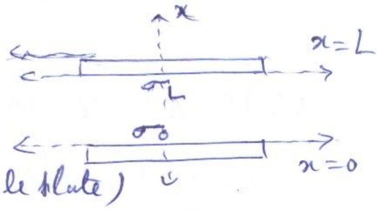
$$\therefore \phi = -\frac{k}{2\epsilon_0}x^2 + Ax + B$$

This gives $B = \phi_0$ and $A = \frac{kL}{2\epsilon_0} - \frac{\phi_0}{L}$ $\left(\because \begin{array}{l} \phi = \phi_0 \text{ at } x = 0 \\ \phi = 0 \text{ at } x = L \end{array}\right)$

$$\therefore \phi = -\frac{k}{2\epsilon_0}\left(x^2 - Lx\right) + \phi_0\left(1 - \frac{x}{L}\right)$$

$$E = -\nabla\phi = \left(\frac{k}{2\epsilon_0}(2x - L) + \frac{\phi_0}{L}\right)\hat{e}_x$$

At $x = \frac{L}{2}$ $\quad E_{x=L/2} = \left(\frac{k}{2\epsilon_0}(L - L) + \frac{\phi_0}{L}\right)\hat{e}_x = \frac{\phi_0}{L}\hat{e}_x$



To find the surface charge density σ_0 at $x = 0$, we note that this surface is an interface of two regions i.e. $E = 0$ (inside plate) and the other with $E \neq 0$ (region between two plates)

The discontinuity $(E_{above} - E_{below})$ at $x = 0$ equals $\frac{\sigma_0}{\epsilon_0}$. Thus

$$\frac{\sigma_0}{\epsilon_0} = -\frac{kL}{2\epsilon_0} + \frac{\phi_0}{L}$$

$$\boxed{\therefore \sigma_0 = -\frac{kL}{2} + \frac{\epsilon_0\phi_0}{L}}$$

IIIy the discontinuity at $x = L$ ~~determines~~ $\frac{\sigma_L}{\epsilon_0}$:

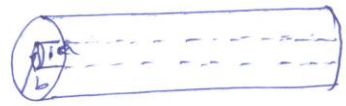
$$\frac{\sigma_L}{\epsilon_0} = 0 - \left(\frac{kL}{2\epsilon_0} + \frac{\phi_0}{L}\right)$$

$$\boxed{\sigma_L = -\frac{kL}{2} - \frac{\epsilon_0\phi_0}{L}}$$

Problems & solutions of sheet-2

(3-111)

Solutn

Charge per unit length λ on inner 

$-\lambda$ on outer

$$\bar{E}(a<\rho<b) = \frac{\lambda}{2\pi\epsilon_0 \rho}\,\hat{e}_\rho$$

Use $\int \bar{E}\cdot d\bar{l} = V$ to find potential difference

$$V = -\int_b^a E_\rho\, d\rho = -\int_b^a \frac{\lambda}{2\pi\epsilon_0 \rho}\, d\rho$$

$$V = \frac{-\lambda}{2\pi\epsilon_0} \ln\rho \Big|_b^a = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

$\therefore V \propto \lambda$; ~~C per~~

$$C \text{ per unit length} = \frac{\lambda}{V} = \frac{2\pi\epsilon_0}{\ln(b/a)}$$

Problems & solutions of sheet-2

(3-1v)

Soln 4

Electric field $\bar{E} = -\bar{\nabla}\phi$ and $\phi = \phi_0\left[1+\frac{(x^2+y^2+z^2)}{a^2}+\frac{(x^4+y^4+z^4)}{a^4}\right]$

$$\bar{E} = -\frac{\phi_0}{a^2}\left[\left(2x+\frac{4x^3}{a^2}\right)\hat{e}_x+\left(2y+\frac{4y^3}{a^2}\right)\hat{e}_y+\left(2z+\frac{4z^3}{a^2}\right)\hat{e}_z\right]$$

$$\bar{E}\Big|_{(a,a,a)} = \frac{6\phi_0}{a}\left[\hat{e}_x+\hat{e}_y+\hat{e}_z\right]$$

Force on a point dipole in a spacially inhomogenous electrostatic field is given by. $\bar{F} = (\bar{p}\cdot\bar{\nabla})\bar{E}$

$$\therefore \bar{F} = -\frac{p_0\phi_0}{a^2}\left[\left(2+\frac{12x^2}{a^2}\right)\hat{e}_x+2\left(2+\frac{12y^2}{a^2}\right)\hat{e}_y+3\left(2+\frac{12z^2}{a^2}\right)\hat{e}_z\right]$$

$$\bar{F}\Big|_{a,a,a} = -\frac{p_0\phi_0}{a^2}\left[14\hat{e}_x+28\hat{e}_y+42\hat{e}_z\right] = -\frac{14p_0\phi_0}{a^2}\left[\hat{e}_x+2\hat{e}_y+3\hat{e}_z\right]$$

The torque, τ, on a point dipole in an electric field is

$$\bar{\tau} = (\bar{p}\times\bar{E})$$

$$\therefore \bar{\tau}\Big|_{a,a,a} = \frac{6p_0\phi_0}{a}\left(\hat{e}_x-2\hat{e}_y+\hat{e}_z\right)$$

τ about the origin is $\bar{\tau}\Big|_{(a,a,a)} + a(\hat{e}_x+\hat{e}_y+\hat{e}_z)\times\bar{F}$

$$= -\frac{8p_0\phi_0}{a}\left(\hat{e}_x-2\hat{e}_y+\hat{e}_z\right)$$

Problems & solutions of sheet-2

3-V

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Soln The electric field inside & outside the sphere.

GAUSS LAW

$$\epsilon_0 \oint \bar{E}\cdot d\bar{a} = \epsilon_0 4\pi r^2 E = Q_{enc} = \int \rho \, d\tau$$

$$= \int (k\bar{r})\, r^2 \sin\theta \, d\bar{r}\, d\theta\, d\phi$$

$$= 4\pi k \int_0^r r^3 dr = \begin{cases} \pi k r^4 & (r<R) \\ \pi k R^4 & (r>R) \end{cases}$$

So $E = \dfrac{k}{4\epsilon_0} r^2 \hat{r} \quad (r<R)$

$E = \dfrac{kR^4}{4\epsilon_0 r^2} \quad (r>R)$

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\int_0^R \left(\frac{kr^2}{4\epsilon_0}\right)^2 4\pi r^2 dr$$

$$+ \frac{\epsilon_0}{2}\int_R^\infty \left(\frac{kR^4}{4\epsilon_0 r^2}\right) 4\pi r^2 dr$$

$$= 4\pi \frac{\epsilon_0}{2}\left(\frac{k}{4\epsilon_0}\right)^2 \left\{ \int_0^R r^6 dr + R^8 \int_R^\infty \frac{1}{r^2} dr \right.$$

$$= \frac{\pi k^2}{8\epsilon_0}\left\{ \frac{R^7}{7} + R^8 \left(\frac{-1}{r}\right)\Big|_R^\infty \right\}$$

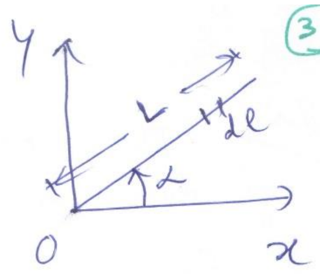
$$= \frac{\pi k^2}{8\epsilon_0}\left(\frac{R^7}{7} + R^7\right) = \frac{\pi k^2 R^7}{7\epsilon_0}$$

Attention you can use

$W = \frac{1}{2}\int \rho V d\tau$

Soln (3-VI)

(6) (i) (a) Monopole moment $= \lambda_0 L$



(b) Dipole moment about the origin.

$$\bar{p} = \int dq\, \bar{r} = \int_0^L \lambda_0 dl\, \bar{r}$$

$$= \int_0^{L\cos\alpha} \lambda_0 (x\,\hat{e}_x + y\,\hat{e}_y)\sec\alpha\, dx$$

$y = x\tan\alpha$

$dy = dx\tan\alpha$

$dl = dx\sec\alpha$

$$= \lambda_0 \int_0^{L\cos\alpha} x(\hat{e}_x + \tan\alpha\,\hat{e}_y)\sec\alpha\, dx$$

$$= \frac{\lambda_0 L^2}{2}\left[\cos\alpha\,\hat{e}_x + \sin\alpha\,\hat{e}_y\right]$$

(c) As the line charge lies on xy plane,

$$Q_{xz} = Q_{yz} = 0$$

$$Q_{xy} = \int_0^L \lambda_0 dl\, 3xy = \int_0^{L\cos\alpha} \lambda_0 \sec\alpha\, 3x \cdot x\tan\alpha\, dx$$

$$= \lambda_0 \sec\alpha\tan\alpha \frac{L^3\cos^3\alpha}{3} = \lambda_0 L^3 \sin\alpha\cos\alpha$$

$$Q_{xx} = \int_0^L \lambda_0 dl(3x^2 - r^2) = \int_0^{L\cos\alpha} \lambda_0 \sec\alpha(2x^2 - x^2\tan^2\alpha)\, dx$$

$$= \lambda_0 \sec\alpha(2 - \tan^2\alpha)\cdot\frac{L^3\cos^3\alpha}{3} = \frac{\lambda_0 L^3}{3}(2\cos^2\alpha - \sin^2\alpha)$$

$$Q_{yy} = \int_0^L \lambda_0 dl(3y^2 - r^2) = \int_0^{L\cos\alpha} \lambda_0 \sec\alpha(2x^2\tan^2\alpha - x^2)\, dx$$

$$= \lambda_0(2\tan^2\alpha - 1)\sec\alpha\cdot\frac{L^3\cos^3\alpha}{3} = \frac{\lambda_0 L^3}{3}(2\sin^2\alpha - \cos^2\alpha)$$

$$Q_{zz} = -(Q_{xx} + Q_{yy}) = -\frac{\lambda_0 L^3}{3}(2-1) = \frac{\lambda_0 L^3}{3}$$

or

$$Q_{zz} = \int_0^L \lambda_0 dl(-r^2) = -\int_0^{L\cos\alpha} \lambda_0 \sec\alpha\, x^2\sec^2\alpha\, dx$$

$$= -\frac{\lambda_0 \sec^3\alpha}{3} L^3\cos^3\alpha = -\frac{\lambda_0 L^3}{3}$$

Problems & solutions of sheet-2

(3-VII)

6(ii)

$\sigma = \sigma_0 \cos\theta$ in a spherical annular region of radius R.

$$\text{Monopole moment} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \sigma_0 \cos\theta\, R^2 \sin\theta\, d\theta\, d\phi = 0$$

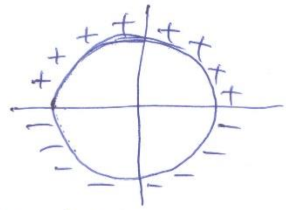
$$\text{Dipole moment} = \bar{p} = \int \sigma_0 \cos\theta\, R^2 \sin\theta\, d\theta\, d\phi\, R\, \hat{e}_r$$

The nature of the charge distribution suggests that the dipole moment contribution is along $\hat{e}_z$

$$\therefore \bar{p} = \int_0^{\pi} \sigma_0\, 2\pi R^3 . \cos^2\theta \sin\theta\, d\theta\, \hat{e}_z$$

$$= 2\pi \sigma_0 R^3 \left(-\frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} = \frac{4\pi\sigma_0 R^3}{3}$$

By symmetry $Q_{xx} = Q_{yy}$



$$\therefore Q_{zz} = -2Q_{xx}$$

$$= \int_s (3z^2 - r^2)\, \sigma_0 \cos\theta\, R^2 \sin\theta\, d\theta\, d\phi$$

$$= \int_s (3R^2\cos^2\theta - R^2)\, \sigma_0 R^2 \sin\theta \cos\theta\, d\theta\, d\phi$$

$$= 2\pi\sigma_0 R^4 \int_0^{\pi} (3\cos^2\theta - 1)\sin\theta\cos\theta\, d\theta = 0$$

$$\therefore Q_{xx} = Q_{yy} = 0$$

$$Q_{xy} = \int 3xy\, \sigma_0 \cos\theta\, R^2 \sin\theta\, d\theta\, d\phi$$

$$= 3\sigma_0 \cos\theta\, (R\sin\theta\cos\phi)(R\sin\theta\sin\phi) R^2 \sin\theta\, d\theta\, d\phi$$

$$= 0$$

|||ly $Q_{yz} = Q_{xz} = 0$

This charge distribution has zero quadrupole moment.

Problems & solutions of sheet-2

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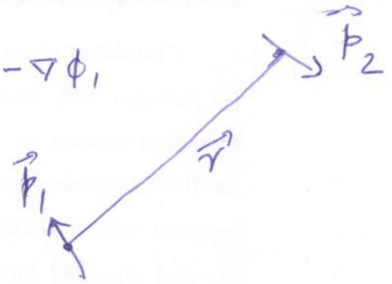

Electric field E_1 due to p_1 ~~in terms~~ is $E_1 = -\nabla \phi_1$

$$\phi_1 = \frac{p_1 \vec{r}}{4\pi\epsilon_0 r^3}$$

$$\vec{E}_1 = -\frac{1}{4\pi\epsilon_0} \vec{\nabla}\left[(\vec{p}_1 \cdot \vec{r})/r^3\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^3}\vec{\nabla}(\vec{p}_1 \cdot \vec{r}) + (\vec{p}_1 \cdot \vec{r})\nabla \frac{1}{r^3}\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}_1}{r^3} - (\vec{p}_1 \cdot \vec{r})\frac{3\hat{r}}{r^4}\right]$$

$$= +\frac{1}{4\pi\epsilon_0 r^3}\left[3(\vec{p}_1 \cdot \hat{r})\hat{r} - \vec{p}_1\right]$$

The potential energy of $\vec{p}_2$ in field $\vec{E}_1$ is

$$U = -\vec{p}_2 \cdot \vec{E}_1$$

$$U = \frac{1}{4\pi\epsilon_0 r^3}\left[\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})\right]$$

The right hand side of the above eq^n is often referred to as the dipole-dipole interaction term.