EE1101 Signals and Systems Jan-May 2018

Tutorial 2 Solutions

Solution 1

(a) Let $x_1(t)$ and $x_2(t)$ be periodic with periods T_1 and T_2 respectively. such that

$$x_1(t+T_1) = x_1(t)$$

$$x_2(t+T_2) = x_2(t)$$

For $x(t) = x_1(t) + x_2(t)$ to be periodic,

$$x(t+T) = x_1(t+T) + x_2(t+T)$$

$$\implies x(t+T) = x_1(t+mT_1) + x_2(t+kT_2)$$

So,

$$mT_1 = kT_2 = T$$

$$\frac{T_1}{T_2} = \frac{k}{m} = \text{rational number}$$

In other words, the sum of two periodic signals is periodic only if the ratio of their respective periods can be expressed as a rational number. Then the fundamental period is the least common multiple of T_1 and T_2 i.e. $T = $ L.C.M $\{T_1, T_2\}$

(b) (a) Let $x(t) = x_1(t) + x_2(t)$ where $x_1(t) = 2cos(10t+1)$ and $x_2(t) = -sin(4t-1)$. $x_1(t)$ and $x_2(t)$ are peridic with periods given by $T_1 = \dfrac{2\pi}{10}$ and $T_2 = \dfrac{2\pi}{4}$ respectively.
Since $\dfrac{T_1}{T_2} = \dfrac{2}{5}$ is a rational number, $x(t)$ is periodic.
The fundamental period is $T = $ L.C.M $\{\dfrac{2\pi}{10}, \dfrac{2\pi}{4}\} = \pi$

(b) For a discrete time signal, the same concept as in Q1 can be applied.
Each of the complex exponentials are periodic with periods $N_1 = \dfrac{2\pi k}{\frac{4\pi}{7}} = 7$ (for $k = 2$) and $N_2 = \dfrac{2\pi k}{\frac{2\pi}{5}} = 5$ (for $k = 1$) respectively. Since $\dfrac{N_1}{N_2} = \dfrac{7}{5}$ is a rational number, $x[n]$ is periodic.
The fundamental period would be $N = $ L.C.M$(1, 7, 5) = 35$

Solution 2

To prove $\delta(at) = \dfrac{1}{|a|}\delta(t)$
Let $g_1(t)$ and $g_2(t)$ be generalized functions. Then the equivalence property states that

$$g_1(t) = g_2(t) \iff \int_{-\infty}^{\infty} \phi(t)g_1(t)dt = \int_{-\infty}^{\infty} \phi(t)g_2(t)dt$$

for all suitably defined testing functions $\phi(t)$.
Consider the integral$\int_{-\infty}^{\infty} \phi(t)\delta(at)dt$
Let $\tau = at$. Consider the case when $a > 0 \Rightarrow \tau = at = |a|t \Rightarrow d\tau = |a|dt$

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(t)\delta(at)dt &= \frac{1}{|a|}\int_{-\infty}^{\infty} \phi(\frac{\tau}{a})\delta(\tau)d\tau \\
&= \frac{1}{|a|}\phi(\frac{\tau}{a})\bigg|_{\tau=0} \\
&= \frac{1}{|a|}\phi(0)
\end{aligned}$$

Now, consider the case when $a < 0 \Rightarrow \tau = at = -|a|t \Rightarrow d\tau = -|a|dt$

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(t)\delta(at)dt &= -\frac{1}{|a|}\int_{\infty}^{-\infty} \phi(\frac{\tau}{a})\delta(\tau)d\tau \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} \phi(\frac{\tau}{a})\delta(\tau)d\tau \\
&= \frac{1}{|a|}\phi(\frac{\tau}{a})\bigg|_{\tau=0} \\
&= \frac{1}{|a|}\phi(0)
\end{aligned}$$

Thus for any a,

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)dt = \frac{1}{|a|}\phi(0)$$

But we know $\phi(0) = \int_{-\infty}^{\infty} \phi(t)\delta(t)dt$

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(t)\delta(at)dt &= \frac{1}{|a|}\int_{-\infty}^{\infty} \phi(t)\delta(t)dt \\ &= \int_{-\infty}^{\infty} \phi(t)\frac{1}{|a|}\delta(t)dt\end{aligned}$$

Now by the equivalence property mentioned before,

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Solution 3

For a Dirac Delta function $\delta(t)$, the following two properties hold true:

$$\begin{aligned}&x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0) \\ &\int_{-\infty}^{\infty} \delta(\tau)d\tau = 1\end{aligned}$$

(a)

$$\begin{aligned}y(t) &= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau \\ &= x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau \\ &= x(t)\int_{-\infty}^{\infty} \delta(\tau)d\tau \\ &= x(t)\end{aligned}$$

(b)

$$\begin{aligned}y(t) &= \int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau \\ \text{Let } t-\tau &= u \\ y(t) &= \int_{-\infty}^{\infty} x(u)\delta(t-u)du \\ &= x(t)\int_{-\infty}^{\infty} \delta(t-u)du \\ &= x(t)\end{aligned}$$

(c)

$$\begin{aligned}y &= \int_{-\infty}^{\infty} e^{-j\omega t}\delta(t)dt \\ &= e^{-j\omega * 0}\int_{-\infty}^{\infty} \delta(t)dt \\ &= 1\end{aligned}$$

(d)

$$\begin{aligned}y &= \int_{-\infty}^{\infty} sin(\pi t)\delta(2t-3)dt \\ \text{Let} 2t-3 = u &\implies du = 2dt \\ y &= \frac{1}{2}\int_{-\infty}^{\infty} sin\left(\frac{\pi(u+3)}{2}\right)\delta(u)du \\ &= \frac{1}{2}sin\left(\frac{\pi(0+3)}{2}\right)\int_{-\infty}^{\infty} \delta(u)du \\ &= -\frac{1}{2}\end{aligned}$$

(e)

$$\begin{aligned}y &= \int_{-\infty}^{\infty} e^{-t}\delta(t+3)dt \\ \text{Let } t+3 &= u \\ y &= e^3\int_{-\infty}^{\infty} e^{-u}\delta(u)du \\ &= e^3\end{aligned}$$

(f)

$$\begin{aligned}y &= \int_{-\infty}^{\infty} (t^3+4)\delta(1-t)dt \\ &= (1^3+4)\int_{-\infty}^{\infty} \delta(1-t)dt \\ &= 5\int_{-\infty}^{\infty} \delta(t)dt \\ &= 5\end{aligned}$$

(g)

$$\begin{aligned}y &= \int_{-\infty}^{\infty} x(2-t)\delta(3-t)dt \\ 2-t &= u \\ y &= \int_{-\infty}^{\infty} x(u)\delta(1+u)du \\ y &= x(-1)\int_{-\infty}^{\infty} \delta(1+u)du \\ y &= x(-1)\int_{-\infty}^{\infty} \delta(u)du \\ &= x(-1)\end{aligned}$$

(h)

$$y = \int_{-\infty}^{\infty} e^{x-1} cos\left[\frac{\pi}{2}(x-5)\right]\delta(x-3)dx$$

Let $e^{x-1} = e^4 e^{x-5}$

$$y = e^4 \int_{-\infty}^{\infty} e^{x-5} cos\left[\frac{\pi}{2}(x-5)\right]\delta(x-3)dx$$

$x - 5 = u$

$$y = e^4 \int_{-\infty}^{\infty} e^{u} cos\left[\frac{\pi}{2}(u)\right]\delta(u+2)du$$

$$y = e^4 * e^{-2} cos\left[\frac{\pi}{2}(-2)\right]\int_{-\infty}^{\infty} \delta(u+2)du$$

$$y = -e^2 \int_{-\infty}^{\infty} \delta(u)du$$

$$= -e^2$$

Solution 4

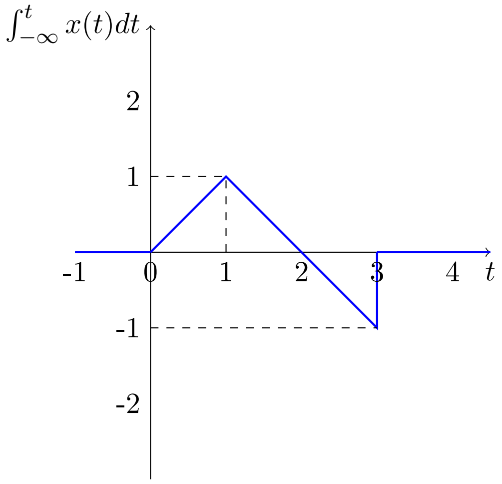
(a) $x(t)$ can be expressed as follows.

$$x(t) = u(t) - 2u(t-1) + u(t-3) + \delta(t-3)$$

$$\begin{aligned}\int_{-\infty}^{t} x(t)dt &= \int_{-\infty}^{t} u(t)dt - 2\int_{-\infty}^{t} u(t-1)dt \\ &+ \int_{-\infty}^{t} u(t-3)dt + \int_{-\infty}^{t} \delta(t-3)dt \\ &= r(t) - 2r(t-1) + r(t-3) + u(t-3)\end{aligned}$$

Thus,

$$\int_{-\infty}^{t} x(t)dt = \begin{cases} 0, & \text{if } t \le 0 \\ t, & \text{if } 0 \le t \le 1 \\ 2-t, & \text{if } 1 \le t < 3 \\ 0, & \text{if } t \ge 3 \end{cases}$$

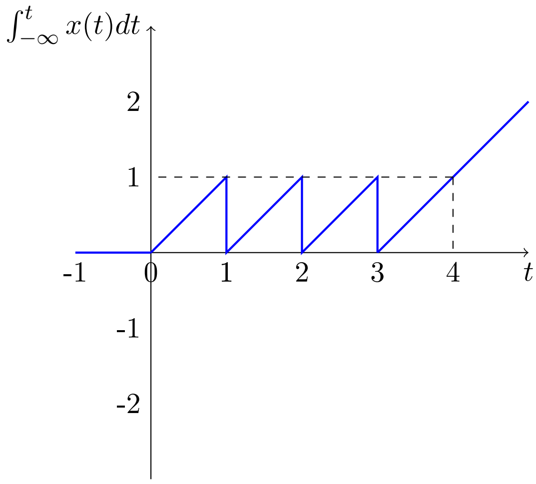


(b) $x(t)$ can be expressed as follows.

$$[x(t) = u(t) - \delta(t-1) - \delta(t-2) - \delta(t-3)$$

$$\begin{aligned}\int_{-\infty}^{t} x(t)dt &= \int_{-\infty}^{t} u(t)dt - \int_{-\infty}^{t} \delta(t-1)dt \\ &- \int_{-\infty}^{t} \delta(t-3)dt - \int_{-\infty}^{t} \delta(t-3)dt \\ &= r(t) - u(t-1) \\ &- u(t-2) - u(t-3)\end{aligned}$$

$$\int_{-\infty}^{t} x(t)dt = \begin{cases} 0, & \text{if } t \le 0 \\ t, & \text{if } 0 \le t < 1 \\ t-1, & \text{if } 1 \le t < 2 \\ t-2, & \text{if } 2 \le t < 3 \\ t-3, & \text{if } t \ge 3 \end{cases}$$



Solution 5

- Suppose $y_1(t)$ is the output of a system to input $x_1(t)$ and $y_2(t)$ is the output to input $x_2(t)$.The system is linear if the response of the system to the input $ax_1(t) + bx_2(t)$ is $ay_1(t) + by_2(t)$ (a and b can be complex in general).
- Suppose $y(t)$ is the output of a system to the input $x(t)$. The system is time invariant if the output of the system to the input $x(t-t_0)$ is $y(t-t_0)$.
- A system is causal if the output of the system at any time depends only on the present and past values of the input and not on the future values of input.

- A system is said to be stable if all bounded inputs to the system result in bounded outputs.
- If a system is invertible then an inverse system exists which when cascaded with the original system gives an output equal to the input to the first system.

Qn	Linear	Time invariant	Causal	Stable	Invertible
(a)$y(t) = \frac{dx}{dt}$	Yes	Yes	Yes	No	No
(b)$y(t) = \int_{-\infty}^{3t} x(\tau)d\tau$	Yes	No	No	No	Yes
(c)$y(t) = x(t/2)$	Yes	No	No	Yes	Yes
(d)$y(t) = \begin{cases} x(t) - x(t-100) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$	Yes	No	Yes	Yes	No
(e)$\frac{dy(t)}{dt} + 3ty(t) = t^2\frac{dx(t)}{dt}$	Yes	No	Yes	No	No
(f)$y(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$	Yes	No	Yes	No	No
(g)$y(t) = x(2t-4)$	Yes	No	No	Yes	Yes

The detailed explanations are given below:

(a)

$$y(t) = \frac{dx(t)}{dt}$$

Suppose $y_1(t)$ is the output of a system to input $x_1(t)$ and $y_2(t)$ is the output to input $x_2(t)$. Let $y(t)$ be the output of the system to the input $ax_1(t) + bx_2(t)$. Then

$$y(t) = \frac{d}{dt}(ax_1(t) + bx_2(t))$$

$$= a\frac{d}{dt}(x_1(t)) + b\frac{d}{dt}(x_2(t))$$
$$= ay_1(t) + by_2(t)$$

Hence the system is **linear**.

Suppose $y(t)$ is the output of a system to input $x(t)$. Let $y_1(t)$ be the output of the system to the input $x(t-t_0)$. Then

$$y_1(t) = \frac{d}{dt}(x(t-t_0))$$
$$= \frac{d}{d(t-t_0)}(x(t-t_0))$$
$$= y(t-t_0) \qquad (\because d(t-t_0) = dt)$$

Hence the system is **time invariant**.

$$y(t) = \frac{dx}{dt} = \lim_{dt \to 0} \frac{x(t) - x(t-dt)}{dt}$$

Since the output of the system depends on the present and past values of the input alone the system is **causal**.

Consider the following input to the system.

$$x(t) = \begin{cases} 1 & -1 \leq t \leq 1 \\ 0 & otherwise \end{cases}$$

The output of the system to the above input will go unbounded at -1 and +1. Hence the system is **unstable**.

The system is **not invertible** since differentiating any constant will give 0 and hence different inputs give the same output.

(b)

$$y(t) = \int_{-\infty}^{3t} x(\tau)d\tau$$

Suppose $y_1(t)$ is the output of a system to input $x_1(t)$ and $y_2(t)$ is the output to input $x_2(t)$. Let $y(t)$ be the output of the system to the input $ax_1(t) + bx_2(t)$.

$$y(t) = \int_{-\infty}^{3t} ax_1(\tau) + bx_2(\tau)d\tau$$
$$= a\int_{-\infty}^{3t} x_1(\tau)d\tau + b\int_{-\infty}^{3t} x_2(\tau)d\tau$$
$$= ay_1(t) + by_2(t)$$

Hence the system is **linear**.

Suppose $y(t)$ is the output of a system to input $x(t)$. Let $y_1(t)$ be the output of the system to the input $x(t-t_0)$.

$$y_1(t) = \int_{-\infty}^{3t} x(\tau - t_0)d\tau$$

$$\tau - t_0 = u$$

$$\text{hence } d\tau = du$$

$$= \int_{-\infty}^{3t-t_0} x(u)du$$

But

$$y(t-t_0) = \int_{-\infty}^{3t-3t_0} x(\tau)d\tau$$

Hence

$$y(t-t_0) \neq y_1(t)$$

Hence the system is **time variant**.

For $t > 0$ the output of the system depends on the future values of the input hence the system is **not causal**.

Consider an input $x(t) = 1$ for all values of t.The output of the system goes unbounded.Hence the system is **unstable**.

If two different inputs give the same output to the above system their values should be the same at every t since the limit of the integral depends on t. Hence the two inputs have to be the same. Hence the system is **invertible**.

(c)

$$y(t) = x(t/2)$$

Suppose $y_1(t)$ is the output of a system to input $x_1(t)$ and $y_2(t)$ is the output to input $x_2(t)$. Let $y(t)$ be the output of the system to the input $ax_1(t) + bx_2(t)$.

$$y(t) = ax_1(t/2) + bx_2(t/2)$$

$$= ay_1(t) + by_2(t)$$

Hence the system is **linear**.

Suppose $y(t)$ is the output of a system to input $x(t)$. Let $y_1(t)$ be the output of the system to the input $x(t-t_0)$.

$$y_1(t) = x(t/2 - t_0)$$

But

$$y(t-t_0) = x((t-t_0)/2)$$

$$= x(t/2 - t_0/2)$$

Hence

$$y(t-t_0) \neq y_1(t)$$

Hence the system is **time variant**.

For negative values of t we find that the output of the system depends on the future values of the input. Hence the system is **not causal**.

For all bounded values of the input we get bounded values at the output. Hence the system is **stable**.

The system is **invertible** since the inverse of the system exists which is $y(t) = x(2t)$.

(d)

$$y(t) = \begin{cases} x(t) - x(t-100) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Suppose $y_1(t)$ is the output of a system to input $x_1(t)$ and $y_2(t)$ is the output to input $x_2(t)$. Let $y(t)$ be the output of the system to the input $ax_1(t) + bx_2(t)$.

$$y(t) = \begin{cases} ax_1(t) + bx_2(t) - ax_1(t-100) - bx_2(t-100) \\ 0 \end{cases}$$

$$y(t) = \begin{cases} a(x_1(t) - x_1(t-100)) + b(x_2(t) - x_2(t-100) \\ 0 \end{cases}$$

$$y(t) = ay_1(t) + by_2(t)$$

Hence the system is **linear**.

Suppose $y(t)$ is the output of a system to input $x(t)$. Let $y_1(t)$ be the output of the system to the input $x(t-t_0)$.

$$y_1(t) = \begin{cases} x(t-t_0) - x(t-t_0-100) & t \geq 0 \\ 0 & otherwise \end{cases}$$

But

$$y(t-t_0) = \begin{cases} x(t-t_0) - x(t-t_0-100) & t \geq t_0 \\ 0 & otherwise \end{cases}$$

Hence the system is **time variant**.

The output of the system depends on the present and past value of the input .Hence the system is **causal**.

The system is **stable** since bounded inputs to the system give bounded outputs.

Consider the constant input function. The constant can be of any value. But they all give the same zero output at all values of t. Hence the system is **not invertible**.

(e)

$$\frac{dy(t)}{dt} + 3ty(t) = t^2\frac{dx(t)}{dt}$$

Suppose $y_1(t)$ is the output of a system to input $x_1(t)$ and $y_2(t)$ is the output to input $x_2(t)$. Let $y(t)$ be the output of the system to the input $ax_1(t) + bx_2(t)$.

$$\frac{dy(t)}{dt} + 3ty(t) = t^2\frac{d}{dt}\left(ax_1(t) + bx_2(t)\right)$$

$$= at^2\frac{dx_1(t)}{dt} + bt^2\frac{dx_2(t)}{dt}$$

$$= a\left(\frac{dy_1(t)}{dt} + 3ty_1(t)\right) + b\left(\frac{dy_2(t)}{dt} + 3ty_2(t)\right)$$

$$= \frac{d}{dt}\left(ay_1(t) + by_2(t)\right) + 3t\left(ay_1(t) + by_2(t)\right)$$

Hence the system is **linear**.

Since the coefficient of $\frac{dx(t)}{dt}$ contains t, the system is **time variant**.

The system is **causal** since the output of the system depends only on the present value of the input.

The system is **unstable**. For a signal having small value of $\frac{dy(t)}{dt}$, $y(t) = \frac{t}{3}\frac{dx(t)}{dt}$. If suppose $x(t)$ is a triangular wave, the output goes unbounded.

The system is **not invertible** due to the presence of $\frac{dx(t)}{dt}$, whose value will be same for $x(t)$ and $x(t) + C$ where C is any constant.

(f)

$$y(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$$

Suppose $y_1(t)$ is the output of a system to input $x_1(t)$ and $y_2(t)$ is the output to input $x_2(t)$. Let $y(t)$ be the output of the system to the input $ax_1(t) + bx_2(t)$.

$$y(t) = \sum_{n=-\infty}^{\infty} (ax_1(t) + bx_2(t))\delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} ax_1(t)\delta(t - nT) + bx_2(t)\delta(t - nT)$$

$$= ay_1(t) + by_2(t)$$

Hence the system is **linear**.

Suppose $y(t)$ is the output of a system to input $x(t)$.Let $y_1(t)$ be the output of the system to the input $x(t - t_0)$.

$$y_1(t) = \sum_{n=-\infty}^{\infty} x(t - t_0)\delta(t - nT)$$

But

$$y(t - t_0) = \sum_{n=-\infty}^{\infty} x(t - t_0)\delta(t - t_0 - nT)$$

Hence $y(t - t_0) \neq y_1(t)$ Hence the system is **time variant**.

The system represents an ideal sampler. The sampler output has non-zero values only at $t = nT$ which is equal to the value of the input $x(t)$ at the same instant. Hence, it is **causal**.

The system is **not stable** since output is a sequence of impulses and impulse function takes an unbounded value.

The system is **not invertible** since the inputs can be different but if they have the same value at nT then they give the same output.

(g)

$$y(t) = x(2t - 4)$$

Suppose $y_1(t)$ is the output of a system to input $x_1(t)$ and $y_2(t)$ is the output to input $x_2(t)$. Let $y(t)$ be the output of the system to the input $ax_1(t) + bx_2(t)$.

$$y(t) = ax_1(2t - 4) + bx_2(2t - 4)$$
$$= ay_1(t) + by_2(t)$$

Hence the system is **linear**.
Suppose $y(t)$ is the output of a system to input $x(t)$.Let $y_1(t)$ be the output of the system to the input $x(t - t_0)$.

$$y_1(t) = x(2t - 4 - t_0)$$

$$y(t - t_0) = x(2(t - t_0) - 4)$$

Hence $y(t-t_0) \neq y_1(t)$ Hence the system is **time variant**.

For all values of $t > 4$ the output of the system depends on the future value of the input. Hence the system is **not causal**.

The system is **stable** since bounded inputs give bounded outputs. The system performs only a time-shifting and time-scaling operation

The system is **invertible** since the inverse of the system exists which is $y(t) = x(\frac{t}{2}+2)$.

Solution 6

We have,

$$y[n] = x[n]x[n-2]$$

(a) The system is **not** memoryless. Since, the output at any time instant n depends on the input at the time instant n-2.

(b) When $x[n] = A\delta[n]$

$$y[n] = A\delta[n]A\delta[n-2] = 0$$

$A\delta[n]$ is non-zero only at $n = 0$ and $A\delta[n-2]$ is non-zero only at $n = 2$. When both are multiplied, we get zero for all n values.

(c) The system is clearly **non-invertible**. To prove non-invertibility, we need one counter-example, which is part (b) itself.

Solution 7

(a) $y(t) = t^2x(t-1)$: System is **linear** and **time variant**

Check for Linearity:Let $y_1(t)$ be the response of $x_1(t)$, $y_2(t)$ be the response of $x_2(t)$ and y(t) be the response of the combined input $x(t) = ax_1(t) + bx_2(t)$.

$$y_1(t) = t^2x_1(t-1)$$
$$y_2(t) = t^2x_2(t-1)$$

$$\begin{aligned}
y(t) &= t^2x(t-1) \\
&= t^2(ax_1(t-1) + bx_2(t-1)) \\
&= at^2x_1(t-1) + bt^2x_2(t-1) \\
&= ay_1(t) + by_2(t)
\end{aligned}$$

The given system is linear.

Check for time-invariance: Delay the input x(t) by t_0. The $y_1(t) = t^2x(t-1-t_0)$. Now delaying the output y(t) by t_0 to get $y(t-t_0) = (t-t_0)^2x(t-1-t_0) \neq y_1(t)$. Hence the system is time variant.

(b) $y[n] = x^2[n-2]$: System is **non-linear** and **time-invariant**.

Check for Linearity:Let $y_1[n]$ be the response of $x_1[n]$, $y_2[n]$ be the response of $x_2[n]$ and $y[n]$ be the response of the combined input $x[n] = ax_1[n] + bx_2[n]$.

$$y_1[n] = x_1^2[n-2]$$

$$y_2[n] = x_2^2[n-2]$$

$$\begin{aligned}
y[n] &= x^2[n-2] \\
&= (ax_1[n-2] + bx_2[n-2])^2 \\
&= a^2x_1^2[n-2] + b^2x_2^2[n-2] \\
&\qquad + 2abx_1[n-2]x_2[n-2] \\
&\neq ay_1[n] + by_2[n]
\end{aligned}$$

The given system is non-linear.

Check for time-invariance: Delay the input $x[n]$ by n_0. Then $y_1[n] = x^2[n-2-n_0]$. Now delaying the output $y[n]$ by n_0 to get $y[n-n_0] = x^2[n-2-n_0] = y_1[n]$. Hence the system is time invariant.

(c) $y[n] = x[n+1] - x[n-1]$: System is **linear** and **time invariant**. (Proof similar to above)

(d) $y[n] = Odd\{x[n]\} = \dfrac{x[n]-x[-n]}{2}$: System is **linear** and **time variant**.

Check for time-invariance: Delay the input $x[n]$ by n_0. Then $y_1[n] = \dfrac{x[n-n_0]-x[-n-n_0]}{2}$. Now delaying the output $y[n]$ by n_0 to get $y[n-n_0] = \dfrac{x[n-n_0]-x[-(n-n_0)]}{2} = \dfrac{x[n-n_0]-x[-n+n_0]}{2}$. Hence the system is time variant.

Solution 8

Let $y(t)$ be the output of the system for input $x(t) = e^{st}$. Then,

$$\mathbf{H}\{e^{st}\} = y(t)$$

Since the system is time-invariant, we have

$$\mathbf{H}\{e^{s(t+t_0)}\} = y(t + t_0)$$

for arbitrary real t_0. Since the system is linear, we have

$$\mathbf{H}\{e^{s(t+t_0)}\} = \mathbf{H}\{e^{st}e^{st_0}\} = e^{st_0}\mathbf{H}\{e^{st}\} = e^{st_0}y(t)$$

Thus,

$$y(t + t_0) = e^{st_0}y(t)$$

Setting $t = 0$, we obtain

$$y(t_0) = y(0)e^{st_0}$$

Since t_0 is arbitrary, by changing t_0 to t, we can rewrite the above equation as

$$y(t) = y(0)e^{st} = \lambda e^{st}$$

$$\implies \mathbf{H}\{e^{st}\} = \lambda e^{st}$$

where $\lambda = y(0)$

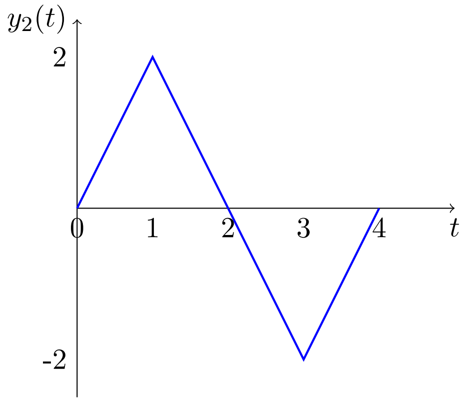
Solution 9

The signal $x_2(t)$ can be represented in terms of the given signal $x_1(t)$ as follows:

$$x_2(t) = x_1(t) - x_1(t - 2)$$

Given that $y_1(t)$ is the response to the input $x_1(t)$, let $y_2(t)$ be the response to the input $x_2(t)$. Since the given system is an LTI system, $y_2(t)$ is given as

$$y_2(t) = y_1(t) - y_1(t - 2)$$



Solution 10

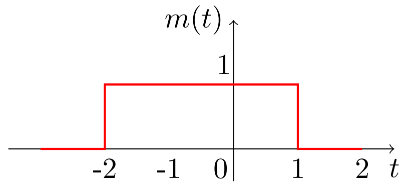
(a)

$$y(t) = A\cos\left(\omega_c t + \omega_\Delta \int_{-\infty}^{t} m(\tau)d\tau\right)$$

Given $\omega_c = 8\pi, \omega_\Delta = 2\pi$ and $m(t) = u(t + 2) - u(t - 1)$.

$$y(t) = A\cos\left(8\pi t + 2\pi \int_{-\infty}^{t} \big(u(\tau + 2) - u(\tau - 1)\big)d\tau\right)$$

m(t) is sketched below:



Case 1: $t < -2$
In this case, the signal $m(t) = 0$.

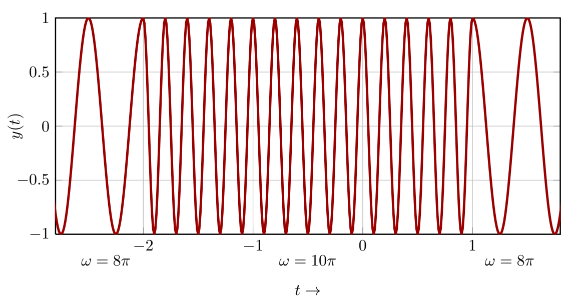
$$y(t) = A\cos(8\pi t)$$

Case 2: $-2 < t < 1$

$$\begin{aligned} y(t) &= A\cos\left(8\pi t + 2\pi \int_{-2}^{t} 1 \cdot d\tau\right) \\ &= A\cos(8\pi t + 2\pi(t + 2)) \\ &= A\cos(10\pi t + 4\pi) \end{aligned}$$

Case 3: $t > 1$

$$\begin{aligned} y(t) &= A\cos\left(8\pi t + 2\pi \int_{-2}^{1} 1 \cdot d\tau\right) \\ &= A\cos(8\pi t + 2\pi(3)) \\ &= A\cos(8\pi t + 6\pi) \end{aligned}$$



(b)

- Let $y_1(t)$ be the response of the system to $m_1(t)$ and $y_2(t)$ be the response of the system to $m_2(t)$.

$$y_1(t) = A\cos\left(\omega_c t + \omega_\Delta \int_{-\infty}^{t} m_1(\tau)d\tau\right)$$

$$y_2(t) = A\cos\left(\omega_c t + \omega_\Delta \int_{-\infty}^{t} m_2(\tau)d\tau\right)$$

Consider $y(t)$ be the response of the system to the input $am_1(t) + bm_2(t)$.

$$\begin{aligned} y(t) = A\cos\Bigg[\omega_c t \\ &+ \omega_\Delta \int_{-\infty}^{t} \Big(am_1(\tau) + bm_2(\tau)\Big)d\tau\Bigg] \\ &\neq ay_1(t) + by_2(t) \end{aligned}$$

Hence the modulation system is **non-linear**.

- The system has **memory** as the output at any instant is dependent on the input at previous time instants.

- The system is **causal** as output at any instant does not depend on the future values of input.

- Let $y_1(t)$ be the response of the system to $m(t - t_0)$.

$$y_1(t) = A\cos\left(\omega_c t + \omega_\Delta \int_{-\infty}^{t} m(\tau - t_0)d\tau\right)$$

However, $y(t)$ delayed by t_0 may be expressed as:

$$y(t-t_0) = A\cos\left(\omega_c(t - t_0) + \omega_\Delta \int_{-\infty}^{t-t_0} m(\tau)d\tau\right)$$

Since $y(t - t_0) \neq y_1(t)$, the system is **time-variant**.