DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II

Tutorial 3 (12.2.2018)

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1. Consider a spherical medium of radius a and dielectric constant $\epsilon_r^{(1)}$, carrying uniform free-charge distribution ρ. It is surrounded by a medium of dielectric constant $\epsilon_r^{(2)}$. If the two mediums are linear dielectrics, then find (i) the bound volume-charge density everywhere in space, and (ii) the bound surface-charge density on the surface of the sphere.

   Solution: To find the bound-charge densities, we need to find the polarization of the medium. Once we have $\mathbf{P}$, the bound charge density ρ_b can be found using $\rho_b = -\nabla.\mathbf{P}$

   Using Gauss' Law in a dielectric medium ($\nabla.\mathbf{D} = \rho_f$),

$$\mathbf{D} = \begin{cases} \frac{\rho_0 a^3}{3r^2}\hat{e_r}, & \text{for } r \geq a. \\ \frac{\rho_0 r}{3}\hat{e_r}, & \text{for } r < a. \end{cases}$$

   Given that the dielectric constants are $\epsilon_r^{(1)}$ inside the spherical region and $\epsilon_r^{(2)}$ outside it, we can find the polarization using $\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}$ and $\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \epsilon_0(\epsilon_r - 1)\mathbf{E}$.

$$\mathbf{P} = \frac{(\epsilon_r - 1)}{\epsilon_r}\mathbf{D}$$

$$\Rightarrow \mathbf{P} = \begin{cases} \left(1 - \frac{1}{\epsilon_r^{(2)}}\right)\frac{\rho_0 a^3}{3r^2}\hat{e_r}, & \text{for } r > a. \\ \\ \left(1 - \frac{1}{\epsilon_r^{(1)}}\right)\frac{\rho_0 r}{3}\hat{e_r}, & \text{for } r < a. \end{cases}$$

   Therefore, the bound volume charge density is given by

$$\rho_b = -\nabla.\mathbf{P} = \begin{cases} 0, & \text{for } r > a. \\ -\left(1 - \frac{1}{\epsilon_r^{(1)}}\right)\rho_0, & \text{for } r < a. \end{cases}$$

(ii) The bound surface charge density is given by

$$\sigma_b = -(\vec{P}_2 - \vec{P}_1).\hat{e_r}$$

So,

$$\mathbf{P}.\hat{e_r} = \begin{cases} \left(1 - \frac{1}{\epsilon_r^{(2)}}\right)\frac{\rho_0 a}{3}, & \text{on the outer surface.} \\ \\ \left(1 - \frac{1}{\epsilon_r^{(1)}}\right)\frac{\rho_0 a}{3}, & \text{on the inner surface.} \end{cases}$$

Therefore, the total charge density on the surface of the sphere is

$$\sigma_b = \left(\frac{1}{\epsilon_r^{(2)}} - \frac{1}{\epsilon_r^{(1)}}\right)\frac{\rho_0 a}{3}.$$

2. A cylindrical coaxial cable has conducting surfaces at $s = a$ and $s = 4a$, which carry uniform surface charge densities σ_0 and $-\sigma_0/4$, respectively. Two linear dielectric media with dielectric constants $\epsilon_r^{(1)}$ and $\epsilon_r^{(2)}$ fill the regions $a < s \leq 2a$ and $2a < s < 4a$, respectively. (a) Find the energy density between $a < s \leq 2a$.(b) Determine the ratio of the magnitude of the polarization just inside and just outside the boundary at $s = 2a$. (c) Sketch $|\mathbf{E}|$ as a function of s in the interval $0 < s \leq 5a$. Given $\epsilon_r^{(1)} = 1.5$ and $\epsilon_r^{(2)} = 2$.

Solution: We use the Gauss's law for dielectric to evaluate $\mathbf{D}$ in the various regions of interest: For $0 < s < a$, cylindrical symmetry and $Q_f = 0$ implies $\mathbf{D} = 0$ and hence $\mathbf{E} = 0$. Similarly, for $s > 4a$, the enclosed $Q_f = 0$ thus once again $\mathbf{D} = 0$, and $\mathbf{E} = 0$ in this region. For $a < s < 4a$, $Q_f^{enc} = 2\pi\sigma_0 aL$. Thus by using Gauss's law for dielectrics we get

$$\int_S \mathbf{D}.d\mathbf{S} = Q_f^{enc}$$

Using a cylindrical Gaussian surface, one sees that

$$D_s = \frac{\sigma_0 a}{s}$$

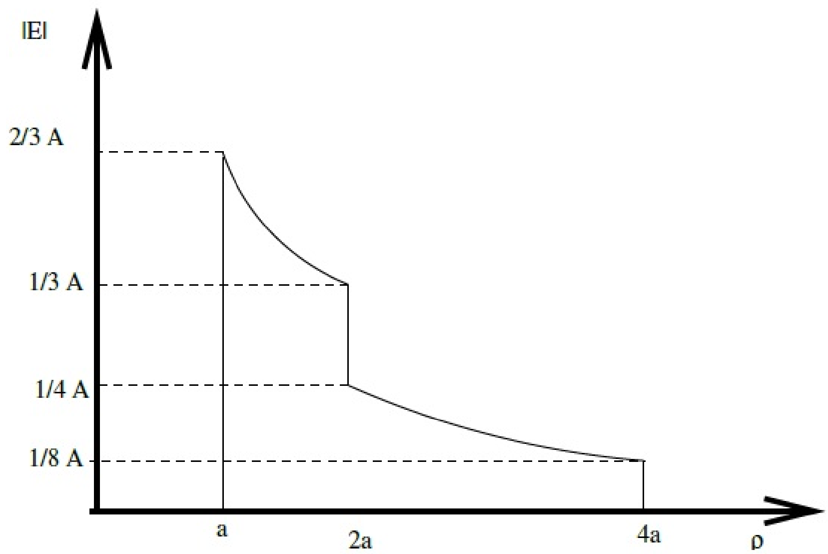


Figure 1: A plot of E(s) vs s : The discontinuities of the Electric field across the various boundaries are shown. The constant A = $\frac{\sigma_0}{\epsilon_0}$.

- To evaluate the energy density in the region $a < s < 2a$, we use Energy density $= \frac{1}{2}\mathbf{D}.\mathbf{E}$, with $\mathbf{E}(a < s < 2a) = \frac{\sigma_0 a}{\epsilon_0 \epsilon_r^{(1)} s}$. Thus the energy density

$$W = \frac{1}{2}\frac{\sigma_0^2 a^2}{\epsilon_0 \epsilon_r^{(1)} s^2}$$

- We have

$$P_s = D_s - \epsilon_0 E_s$$

Thus, for $0 < s < 2a$, we have

$$P_s = \left(1 - \frac{1}{\epsilon_r^{(1)}}\right)\frac{\sigma_0 a}{s},$$

and for $s > 2a$ we get

$$P_s = \left(1 - \frac{1}{\epsilon_r^{(2)}}\right)\frac{\sigma_0 a}{s}$$

The ratio of polarization at the boundary is thus

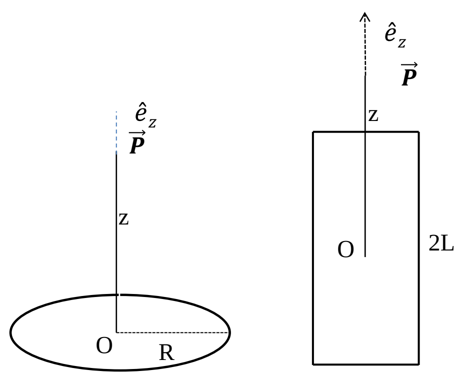
$$= \left(\frac{\epsilon_r^{(1)} - 1}{\epsilon_r^{(1)}}\right)\left(\frac{\epsilon_r^{(2)}}{\epsilon_r^{(2)} - 1}\right)$$

- From above we see that $|\mathbf{E}(a < s < 2a)| = \frac{\sigma_0 a}{\epsilon_0 \epsilon_r^{(1)} s}$, and

  $|\mathbf{E}(2a < s < 4a)| = \frac{\sigma_0 a}{\epsilon_0 \epsilon_r^{(2)} s}$

3. Consider a wire of length $2l$ and radius a, centered at the origin and its symmetry axis being the $z-$axis. The wire carries a uniform polarization $\mathbf{P} = P_0 \hat{e_z}$, with P_0 constant. (a) Find the surface and volume bound-charge densities. (b) Electric field on the positive $z-$axis. Check that it satisfies appropriate boundary condition at $z = L$. (c) Sketch the magnitude of the electric field at the origin as function of $\frac{a}{L}$

   Solution:



$\mathbf{E}$ at any point is the vector sum of the fields due to the bound charges on the two plane surfaces of the dielectric cylinder. $\mathbf{E}$ at a distance z along the axis of a circular disc of radius R and uniform surface charge density σ_0 is given by $\frac{\sigma_0}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{R^2+z^2}}\right]\hat{e_z}$. For the flat surfaces $\sigma_b(z = L) = P_0$ and $\sigma_b(z = -L) = -P_0$

$$\begin{aligned}
\mathbf{E}(z > L) &= \frac{P_0}{2\epsilon_0}\left[1 - \frac{z-L}{\sqrt{a^2 + (z-L)^2}}\right]\hat{e_z} - \frac{P_0}{2\epsilon_0}\left[1 - \frac{z+L}{\sqrt{a^2 + (z+L)^2}}\right]\hat{e_z} \\
&= \frac{P_0}{2\epsilon_0}\left[-\frac{z-L}{\sqrt{a^2 + (z-L)^2}} + \frac{z+L}{\sqrt{a^2 + (z+L)^2}}\right]\hat{e_z} \qquad (1)
\end{aligned}$$

$$\begin{aligned}\mathbf{E}(0<z<L) &= -\frac{P_0}{2\epsilon_0}\left[1-\frac{L-z}{\sqrt{a^2+(L-z)^2}}\right]\hat{e_z} - \frac{P_0}{2\epsilon_0}\left[1-\frac{L+z}{\sqrt{a^2+(L+z)^2}}\right]\hat{e_z} \\ &= -\frac{P_0}{2\epsilon_0}\left[2-\frac{L-z}{\sqrt{a^2+(L-z)^2}}-\frac{L+z}{\sqrt{a^2+(L+z)^2}}\right]\hat{e_z} \qquad (2)\end{aligned}$$

The electric field at a point just outside the dielectric,

$$\mathbf{E}_{no} = \mathbf{E}(z>L)|_{z=L} = \frac{P_0}{2\epsilon_0}\left[\frac{2L}{\sqrt{a^2+4L^2}}\right]\hat{e_z}$$

The electric field at a point just inside the dielectric,

$$\mathbf{E}_{ni} = \mathbf{E}(0<z<L)|_{z=L} = -\frac{P_0}{2\epsilon_0}\left[2-\frac{2L}{\sqrt{a^2+4L^2}}\right]\hat{e_z}$$

$$\mathbf{E}_{no} - \mathbf{E}_{ni} = \frac{P_0}{2\epsilon_0}\hat{e_z} = \frac{P_0}{\epsilon_0}\hat{e_z} = \frac{\vec{P_0}}{\epsilon_0}$$

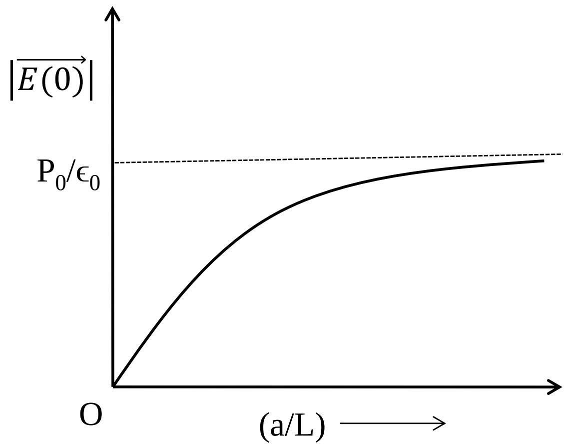
or

$$\left[\mathbf{E}_{no} - \mathbf{E}_{ni}\right].\hat{n} = \frac{P_0}{\epsilon_0}$$

This is to be expected from the boundary conditions for $\mathbf{E}$.
In the absence of free surface charge density at the boundary;
$\mathbf{D}_{no} = \mathbf{D}_{ni}$ or $\epsilon_0\mathbf{E}_{no} = \epsilon_r\epsilon_0\mathbf{E}_{ni}$. Thus $\mathbf{E}_{no} = \epsilon_r\mathbf{E}_{ni}$.

$$\mathbf{E}_{no} - \mathbf{E}_{ni} = (\epsilon_r - 1)\mathbf{E}_{ni} = \frac{\mathbf{P}}{\epsilon_0}$$

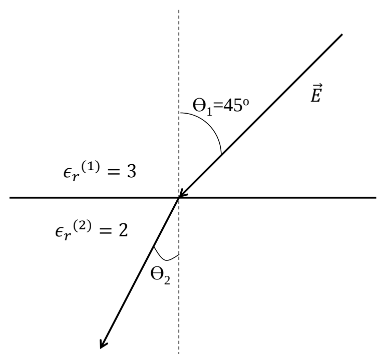
$$\begin{aligned}\mathbf{E}(0) &= -\frac{P_0}{2\epsilon_0}\left[2-\frac{2L}{\sqrt{a^2+4L^2}}\right]\hat{e_z} \\ &= -\frac{P_0}{\epsilon_0}\left[1-\frac{1}{\sqrt{1+(a/L)^2}}\right]\hat{e_z}\end{aligned}$$



For $a = 0$, $|\mathbf{E}(0)|_{min} = 0$. $|\mathbf{E}(0)| \to \frac{P_0}{\epsilon_0}$ as $\frac{a}{L} \to \infty$
For $\frac{a}{L} \to \infty$, the two end surfaces become infinite parallel surfaces and so the field at the origin becomes

$$\frac{P_0}{2\epsilon_0} + \frac{P_0}{2\epsilon_0} = \frac{P_0}{\epsilon_0}$$

4. At the planar boundary between two dielectrics with dielectric constants, = 3 and $\epsilon_r^{(2)} = 2$, electric field $E_1 = 1200V/m$ in medium 1 makes an angle $\theta = 45^0$ with the normal to the boundary. Find the electric field in medium 2 and also the polarization charge density on the interface.
Solution:



Since $\sigma_f = 0$ at the interface, the normal component of $\mathbf{D}$ is continuous at the interface. Since $\mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E}$,

$$\epsilon_r^{(1)} E_1 \cos\theta_1 = \epsilon_r^{(2)} E_2 \cos\theta_2$$

Continuity of the tangential component of $\mathbf{E}$ gives

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

Combining the above two equations, we have

$$\epsilon_r^{(1)} \cot\theta_1 = \epsilon_r^{(2)} \cot\theta_2$$

which may be considered as the law of refraction of the lines of $\mathbf{E}$ at the interface between two dielectrics. Using the given values of $\epsilon_r^{(1)}$, $\epsilon_r^{(2)}$ and θ_1, we can get $\theta_2 = \cot^{-1}(1.5) = 33.69°$. Therefore $E_2 = E_1(\sin 45°)/(\sin 33.7°) = 1529.7\ V/m$ The polarization surface charge density is given by

$$\mathbf{P} \cdot \hat{n} = \epsilon_0(\epsilon_r - 1)\mathbf{E} \cdot \hat{n}$$

For the common interface, $\mathbf{E}_1 \cdot \hat{n}_1$ is positive while $\mathbf{E}_2 \cdot \hat{n}_2$ is negative. The total surface charge density on the interface is thus

$$\begin{aligned}\sigma_P &= \epsilon_0(\epsilon_r^{(1)} - 1)E_1 \cos\theta_1 - \epsilon_0(\epsilon_r^{(2)} - 1)E_2 \cos\theta_2 \\ &= (424.23)\epsilon_0 = 3756\ \times 10^{-12} C/m^2\end{aligned}$$