EE1101 : Signals and Systems JAN–MAY 2019

Tutorial 8 Solutions

1. (a) Given $x(t) \longleftrightarrow X(j\omega)$ implies

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Substituting $\omega = 0$, we get,

$$X(0) = \int_{-\infty}^{\infty} x(t)e^{0}dt = \int_{-\infty}^{\infty} x(t)dt$$

(b) Let $x(t) = \dfrac{W}{\pi}\text{sinc}\left(\dfrac{Wt}{\pi}\right)$, then

$$X(j\omega) = \begin{cases} 1, & \text{for } |\omega| < W, \\ 0, & \text{elsewhere.} \end{cases}$$

When $W = \pi$, $x(t) = \text{sinc}(t)$. Now, the Fourier transform of $\text{sinc}(t)$ is a rectangular function of magnitude 1 from $-\pi$ to π.

$$X(j\omega) = \int_{-\infty}^{\infty} \text{sinc}(t)cdt$$

From part (a), $\int_{-\infty}^{\infty} \text{sinc}(t)dt$ is equal to the Fourier transform $X(j\omega)$ at $\omega = 0$ which is equal to 1.

$$\begin{aligned}
\int_{-\infty}^{\infty} \text{sinc}^2(t)dt &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \\
&\text{(From Parseval's Theorem)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 d\omega \\
&= 1
\end{aligned}$$

Alternatively, we can use the multiplication property,

$$\text{sinc}^2(t) = x^2(t) \longleftrightarrow \frac{1}{2\pi}X(j\omega) * X(j\omega) = Y(j\omega)$$

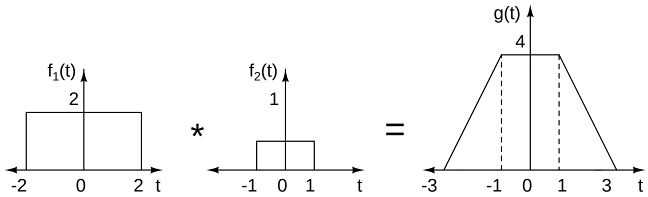
where

$$Y(j\omega) = \begin{cases} 1 - \frac{|\omega|}{2\pi}, & |\omega| \leq 2\pi \\ 0, & |\omega| > 2\pi \end{cases}$$

to get $\int_{-\infty}^{\infty} \text{sinc}^2(t)dt = Y(0) = 1$.

Thus, $\int_{-\infty}^{\infty} \text{sinc}(t)dt = \int_{-\infty}^{\infty} \text{sinc}^2(t)dt = 1$.

2. (a) Given $f_1(t) = 2\,\text{rect}\left(\dfrac{t}{4}\right)$ and $f_2(t) = \text{rect}\left(\dfrac{t}{2}\right)$



$$g(t) = f_1(t) * f_2(t) \quad = \begin{cases} 2(t+3) & , -3 \leq t \leq -1 \\ 4 & , -1 < t \leq 1 \\ 2(3-t) & , 1 < t \leq 3 \\ 0 & , |t| \geq 3 \end{cases}$$

(b) Using the analysis equation,

$$\begin{aligned}
G(j\omega) &= \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt \\
&= \int_{-3}^{-1} 2(t+3)e^{-j\omega t}dt + \int_{-1}^{1} 4e^{-j\omega t}dt \\
&\qquad + \int_{1}^{3} 2(3-t)e^{-j\omega t}dt
\end{aligned}$$

Using the formulae $\int xe^{cx} = \left(\frac{cx-1}{c^2}\right)e^{cx}$ and $\int e^{cx} = \frac{1}{c}e^{cx}$,

$$\begin{aligned}
G(j\omega) &= \left[\frac{(-2j\omega t - 2 - 6j\omega)e^{-j\omega t}}{-\omega^2}\right]_{-3}^{-1} + \left[\frac{4e^{-j\omega t}}{-j\omega}\right]_{-1}^{1} \\
&\qquad + \left[\frac{(2j\omega t + 2 - 6j\omega)e^{-j\omega t}}{-\omega^2}\right]_{1}^{3} \\
&= \frac{(2+4j\omega)e^{j\omega} - 2e^{j3\omega}}{\omega^2} + \frac{8\sin\omega}{\omega} \\
&\qquad + \frac{(2-4j\omega)e^{-j\omega} - 2e^{-j3\omega}}{\omega^2} \\
&= \frac{4\cos(\omega) - 8\omega\sin(\omega) - 4\cos(3\omega)}{\omega^2} + \frac{8\sin\omega}{\omega} \\
&= \frac{4[\cos(\omega) - \cos(3\omega)]}{\omega^2} \\
&= \frac{8\sin(2\omega)\sin(\omega)}{\omega^2} \\
&= 16\,\text{sinc}(2\omega/\pi)\,\text{sinc}(\omega/\pi)
\end{aligned}$$

where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.

(c) We know that

$$\begin{aligned}
\text{rect}\left(\frac{t}{\tau}\right) &\longleftrightarrow \tau\,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right) \\
f_1(t) = 2\,\text{rect}\left(\frac{t}{4}\right) &\longleftrightarrow 2(4)\,\text{sinc}(2\omega/\pi) \\
f_2(t) = \text{rect}\left(\frac{t}{2}\right) &\longleftrightarrow 2\,\text{sinc}(\omega/\pi)
\end{aligned}$$

Using the convolution property of Fourier Transform,

$$\begin{aligned} G(j\omega) &= F_1(j\omega)F_2(j\omega) \\ &= 8\,\mathrm{sinc}(2\omega/\pi)2\,\mathrm{sinc}(\omega/\pi) \\ &= 16\,\mathrm{sinc}(2\omega/\pi)\,\mathrm{sinc}(\omega/\pi) \end{aligned}$$

(d) The magnitude spectrum of $G(j\omega)$ is plotted using matlab in Figure 1. Phase spectrum is plotted in Figure 2.

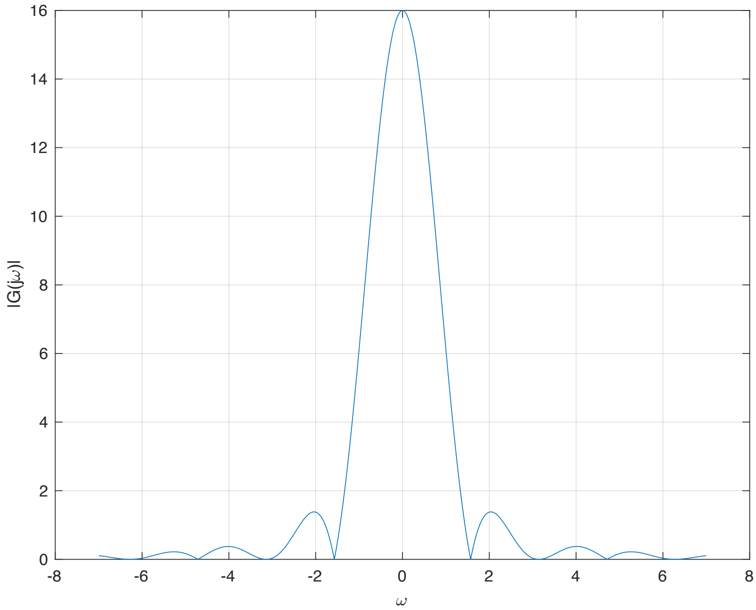


Figure 1: Magnitude spectrum of $G(j\omega)$ in Q2 generated using matlab

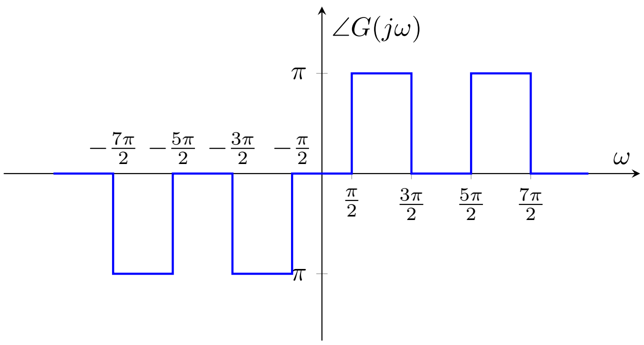


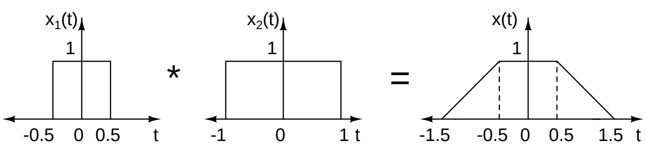
Figure 2: Phase spectrum of $G(j\omega)$ in Q2

Note: $\angle G(j\omega)$ can take either π or $-\pi$ for values of ω for which $G(j\omega)$ is negative. Here we have taken π for when ω is positive and $-\pi$ for when ω is negative to imply the point that phase spectrum is an odd function for real-valued time-domain signals.

3. The given signal $x(t)$ can be written as,

$$x(t) = x_1(t) * x_2(t)$$

$$\text{where, } x_1(t) = \mathrm{rect}(t) \text{ and } x_2(t) = \mathrm{rect}\left(\frac{t}{2}\right)$$



Using the convolution property of Fourier Transform,

$$\begin{aligned} X(j\omega) &= X_1(j\omega)X_2(j\omega) \\ &= \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)2\,\mathrm{sinc}\left(\frac{\omega}{\pi}\right) \\ &= 2\,\mathrm{sinc}\left(\frac{\omega}{\pi}\right)\mathrm{sinc}\left(\frac{\omega}{2\pi}\right) \end{aligned}$$

4. (a) Given $X(j\omega) = \dfrac{2\sin(3(\omega - 2\pi))}{(\omega - 2\pi)}$

$$\mathrm{rect}\left(\frac{t}{\tau}\right) \longleftrightarrow \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right) = \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right) \tag{1}$$

Comparing with standard form in (1),

$$\mathrm{rect}\left(\frac{t}{6}\right) \longleftrightarrow \frac{2}{\omega}\sin(3\omega)$$

Using the frequency shifting property of Fourier Transform,

$$e^{j2\pi t}\,\mathrm{rect}\left(\frac{t}{6}\right) \longleftrightarrow \frac{2\sin(3(\omega - 2\pi))}{(\omega - 2\pi)}$$

(b) Given $X(j\omega) = cos(4\omega + \frac{\pi}{3})$

$$\begin{aligned} X(j\omega) &= cos(4\omega + \frac{\pi}{3}) \\ &= \frac{1}{2}\left[e^{j(4\omega+\frac{\pi}{3})} + e^{-j(4\omega+\frac{\pi}{3})}\right] \\ &= \left(\frac{e^{j\frac{\pi}{3}}}{2}\right)e^{j4\omega} + \left(\frac{e^{-j\frac{\pi}{3}}}{2}\right)e^{-j4\omega} \end{aligned}$$

We know that,

$$\delta(t) \longleftrightarrow 1$$

By using time shifting property,

$$\begin{aligned} \delta(t+4) &\longleftrightarrow e^{j4\omega} \\ \delta(t-4) &\longleftrightarrow e^{-j4\omega} \end{aligned}$$

By using linearity property,

$$x(t) = \left(\frac{e^{j\frac{\pi}{3}}}{2}\right)\delta(t+4) + \left(\frac{e^{-j\frac{\pi}{3}}}{2}\right)\delta(t-4)$$

5. a)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t$$

$$X(0) = \int_{-\infty}^{\infty} x(t)e^{-j(0)t}\mathrm{d}t$$

$$X(0) = \int_{-\infty}^{\infty} x(t)\mathrm{d}t$$

$$X(0) = \int_{-1}^{0} 1\mathrm{d}t + \int_{0}^{1}(-t+1)\mathrm{d}t + \int_{1}^{2}(t-1)\mathrm{d}t + \int_{2}^{3} 1\mathrm{d}t$$

$$\implies X(0) = 3$$

b)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\mathrm{d}\omega$$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(0)}\mathrm{d}\omega$$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\mathrm{d}\omega$$

$$\int_{-\infty}^{\infty} X(j\omega)\mathrm{d}\omega = 2\pi x(0)$$

$$\implies \int_{-\infty}^{\infty} X(j\omega)\mathrm{d}\omega = 2\pi$$

c)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\mathrm{d}\omega$$

$$x(1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(1)}\mathrm{d}\omega$$

$$x(1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega}\mathrm{d}\omega$$

$$\int_{-\infty}^{\infty} X(j\omega)e^{j\omega}\mathrm{d}\omega = 2\pi x(1)$$

$$\implies \int_{-\infty}^{\infty} X(j\omega)e^{j\omega}\mathrm{d}\omega = 2\pi.0$$

$$\implies \int_{-\infty}^{\infty} X(j\omega)e^{j\omega}\mathrm{d}\omega = 0$$

d)

Using Parseval's theorem,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\mathrm{d}\omega = \int_{-\infty}^{\infty} |x(t)|^2\mathrm{d}t$$

$$\implies \int_{-\infty}^{\infty} |X(j\omega)|^2\mathrm{d}\omega = 2\pi\Bigg[\int_{-1}^{0} 1^2\mathrm{d}t + \int_{0}^{1}(1-t)^2\mathrm{d}t + \int_{1}^{2}(t-1)^2\mathrm{d}t + \int_{2}^{3} 1^2\mathrm{d}t\Bigg]$$

$$= 2\pi\left[1 + \frac{1}{3} + \frac{1}{3} + 1\right] = \frac{16\pi}{3}$$

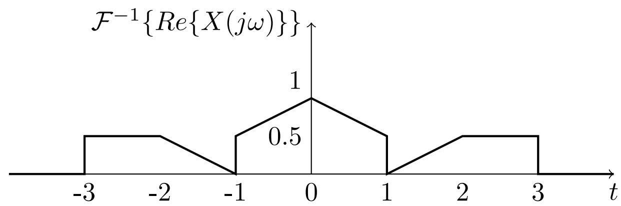
e)

$$Re\{X(j\omega)\} = \frac{X(j\omega) + X^*(j\omega)}{2}$$

$$\mathcal{F}^{-1}\{Re\{X(j\omega)\}\} = \frac{\mathcal{F}^{-1}\{X(j\omega)\} + \mathcal{F}^{-1}\{X^*(j\omega)\}}{2}$$

$$= \frac{x(t) + x^*(-t)}{2}$$

$$= \frac{x(t) + x(-t)}{2} \longrightarrow Ev\{x(t)\}$$



6. For the given signal, $T_0 = 1, \omega_0 = 2\pi$.

(a) Fourier series coefficients P_n of function $p(t)$:

$$P_n = \frac{1}{T_0}\int_{\frac{-1}{4}}^{\frac{1}{4}} p(t)e^{-jn\omega_0 t}dt, \quad n \neq 0$$

$$= \int_{\frac{-1}{4}}^{0} (1+4t)e^{-j2\pi nt}dt + \int_{0}^{\frac{1}{4}}(1-4t)e^{-j2\pi nt}dt$$

$$= \frac{4\sin^2(\pi n/4)}{\pi^2 n^2}, \quad n \neq 0.$$

$$P_0 = \int_{\frac{-1}{4}}^{\frac{1}{4}} p(t)dt = \frac{1}{4}.$$

Note: This can be solved using convolution property as well. Convolution of rect function with itself will give triangular function. In Fourier domain that manifests as $\mathrm{sinc}^2()$ function.

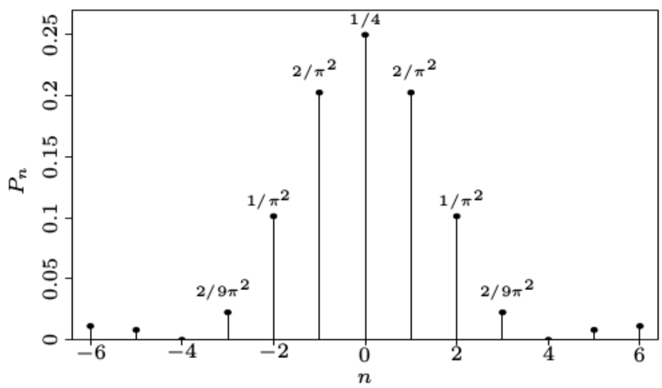


Figure 3: Sketch for P_n vs. n in Q6(a)

(b) Fourier transform of function $p(t)$:

By using Fourier series expansion,

$$p(t) = \sum_{n=-\infty}^{\infty} P_n e^{j2\pi nt}$$

$$P(j\omega) = \sum_{n=-\infty}^{\infty} \mathcal{F}\{P_n e^{j2\pi nt}\}$$

$$= \sum_{n=-\infty}^{\infty} P_n \mathcal{F}\{e^{j2\pi nt}\}$$

$$= \sum_{n=-\infty}^{\infty} 2\pi P_n \delta(\omega - 2\pi n)$$

$$= \sum_{n=-\infty}^{\infty} \frac{8\sin^2(\pi n/4)}{\pi n^2}\delta(\omega - 2\pi n)$$

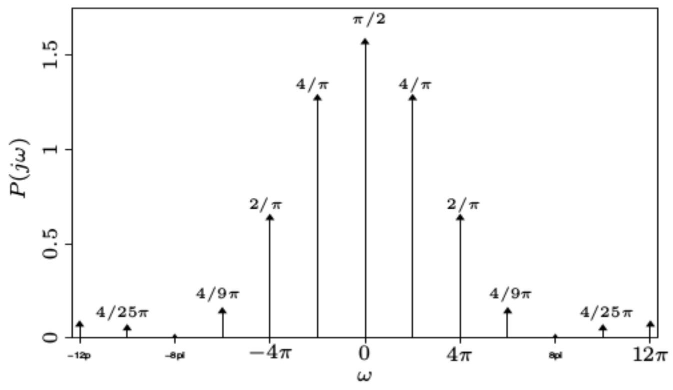


Figure 4: Sketch for $P(j\omega)$ vs. ω in Q6(a)

(c) $y(t) = p(t) \cdot x(t)$. Then the Fourier transform of $y(t)$ is

$$
\begin{aligned}
Y(j\omega) &= \frac{1}{2\pi}\{P(j\omega) * X(j\omega)\} \\
&= \frac{1}{2\pi}\Big\{2\pi \sum_{k=-\infty}^{\infty} P_k \delta(\omega - 2\pi k) * X(j\omega)\Big\} \\
&= \sum_{k=-\infty}^{\infty} P_k X(j(\omega - 2\pi k)) \\
&= \sum_{k=-\infty}^{\infty} \frac{4\sin^2(\pi k/4)}{\pi^2 k^2} X(j(\omega - 2\pi k))
\end{aligned}
$$

(d) $x(t) = \mathrm{sinc}(t)$, has the Fourier transform

$$X(j\omega) = \mathrm{rect}\Big(\frac{\omega}{2\pi}\Big).$$

$$
\begin{aligned}
Y(j\omega) &= \frac{1}{2\pi}\{P(j\omega) * X(j\omega)\} \\
&= \sum_{k=-\infty}^{\infty} \frac{4\sin^2(\pi k/4)}{\pi^2 k^2}\, \mathrm{rect}\Big(\frac{\omega - 2\pi k}{2\pi}\Big).
\end{aligned}
$$

7. Given ,

$$
\begin{aligned}
z_1(t) &= x(t)\cos(\omega_1 t) + y(t)\cos(\omega_2 t) \\
\omega_1 &= 5W \\
\omega_2 &= 7W
\end{aligned}
$$

For $\mathbf{z_1(t)}$:

$$
\begin{aligned}
z_1(t) &= x(t)\cos(\omega_1 t) + y(t)\cos(\omega_2 t) \\
&= x(t)\Big(\frac{e^{j\omega_1 t} + e^{-j\omega_1 t}}{2}\Big) + y(t)\Big(\frac{e^{j\omega_2 t} + e^{-j\omega_2 t}}{2}\Big)
\end{aligned}
$$

The frequency shift property of Fourier Transform states that,

$$f(t)e^{j\omega_0 t} \longleftrightarrow F(j(\omega - \omega_0))$$

Using this property and linearity, $Z_1(j\omega)$ can be written as,

$$
\begin{aligned}
Z_1(j\omega) = \frac{1}{2}\Big(&X(j(\omega - \omega_1)) + X(j(\omega + \omega_1)) + \\
&Y(j(\omega - \omega_2)) + Y(j(\omega + \omega_2))\Big)
\end{aligned}
$$

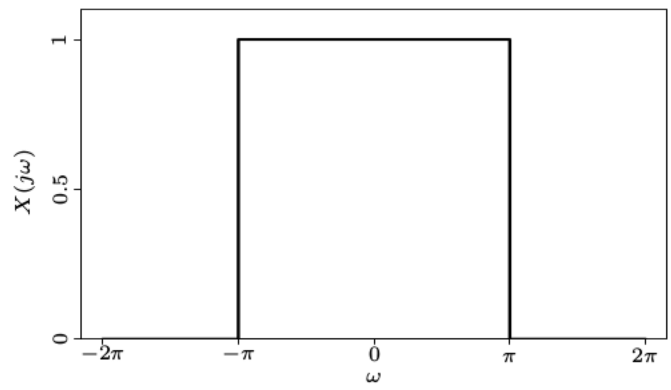


Figure 5: Sketch for $X(j\omega)$ in Q6(c)

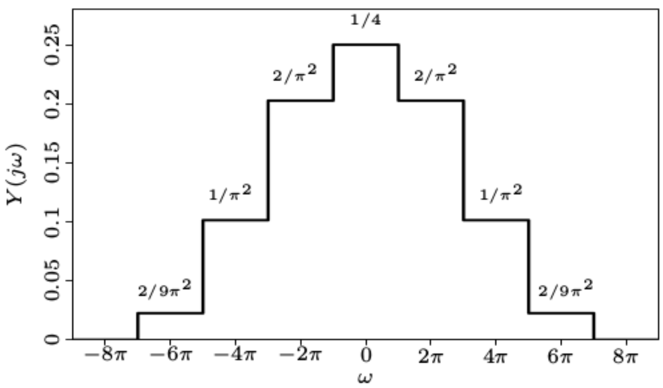


Figure 6: Sketch for $Y(j\omega)$ in Q6(c)

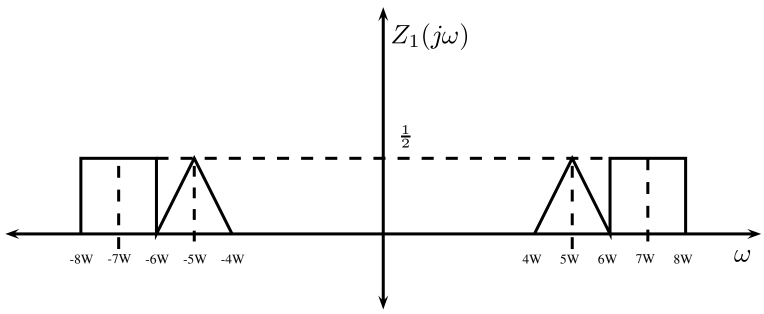


Figure 7: Sketch for $Z_1(j\omega)$

For $\mathbf{z_2(t)}$:

$Z_2(j\omega)$ is $Z_1(j\omega)$ bandlimited to $4W$ and $6W$

$$Z_2(j\omega) = \frac{1}{2}\Big(X(j(\omega - \omega_1)) + X(j(\omega + \omega_1))\Big)$$

For $\mathbf{z_3(t)}$:

$$
\begin{aligned}
z_3(t) &= z_2(t)\cos(\omega t) \\
&= \frac{1}{2}\left(z_2(t)e^{j\omega_1 t} + z_2(t)e^{-j\omega_1 t}\right)
\end{aligned}
$$

Using frequency shifting property of Fourier Transform,

$$
\begin{aligned}
Z_3(j\omega) &= \frac{1}{2}\Big(Z_2(j(\omega - \omega_1)) + Z_2(j(\omega + \omega_1))\Big) \\
&= \frac{1}{4}\Big(X(j(\omega - 2\omega_1)) + 2X(j\omega) + X(j(\omega + 2\omega_1))\Big) \\
&= \frac{1}{2}X(j\omega) + \frac{1}{4}\Big(X(j(\omega - 2\omega_1)) + X(j(\omega + 2\omega_1))\Big)
\end{aligned} \tag{2}
$$

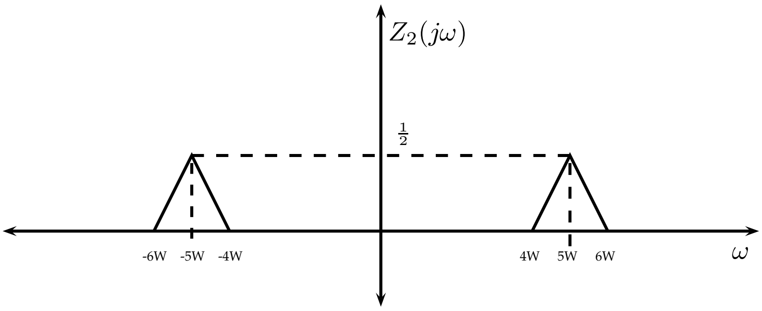


Figure 8: Sketch for $Z_2(j\omega)$

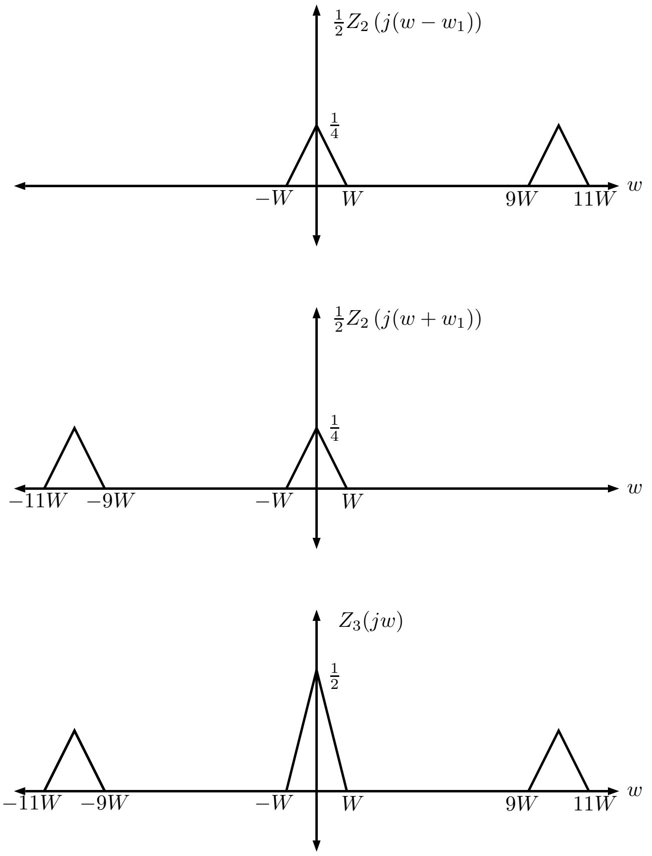


Figure 9: Sketch for $Z_3(j\omega)$

For $\mathbf{z_4(t)}$:

$z_4(t)$ is the LPF output for the input $z_3(t)$ with cutoff freq $= W$. So, only the first summation term from eqn.(2) is available as output in the form of $z_4(t)$, i.e.:

$$Z_4(j\omega) = \frac{1}{2}X(j\omega) \Rightarrow z_4(t) = \frac{1}{2}x(t)$$

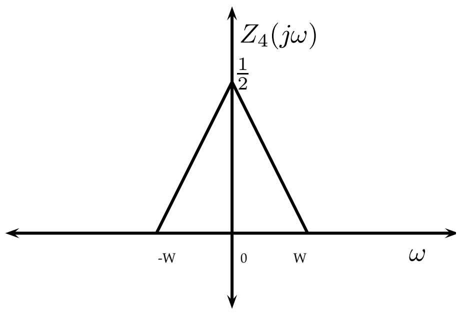


Figure 10: Sketch for $Z_4(j\omega)$

8. Let the impulse train be denoted by $i(t)$.

   (a) So,

$$x_s(t) = x(t)i(t)$$

Which implies,

$$X_s(j\omega) = \frac{1}{2\pi}\left(X(j\omega) * I(j\omega)\right)$$

To find $I(j\omega)$,the Fourier transform of $i(t)$, it can be noted that $i(t)$ is a periodic function and can be written in terms of its Fourier series coefficients as:

$$i(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T}e^{jk\frac{2\pi}{T}t}$$

Now, the Fourier transform can be easily found out by using the modulation property (shifting in frequency).

$$I(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - k\left(\frac{2\pi}{T}\right)\right)$$

Thus,

$$X_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\omega - k\left(\frac{2\pi}{T}\right)\right)\right)$$

   (b) Sketch is as given in Figure 11

   (c) Sketch is as given in Figure 12

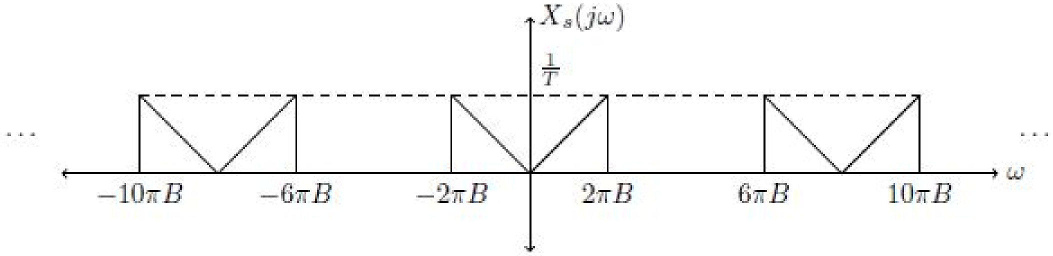
   (d) Largest T such that $X_{sr}(j\omega) = X(j\omega)$ is $\frac{1}{2B}$.

9. (a) Find the Nyquist rates for the signals:
   (i). $x_1(t) = \frac{sin(4000\pi t)}{\pi t}$. This signal is bandlimited to $\omega = 4000\pi$. Thus, the minimum Nyquist rate = 4000 samples per sec.

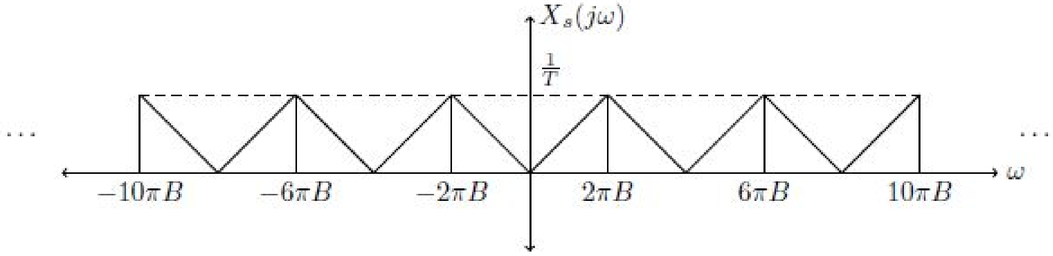
   (ii). $x_2(t) = \left(\frac{sin(4000\pi t)}{\pi t}\right)^2$. This signal is squared of $x_1(t)$. which means its Fourier Transform will be the self convolution of the Fourier Transform of $x_1(t)$. So, this signal is bandlimited to $\omega = 8000\pi$. Thus, the minimum Nyquist rate = 8000 samples per sec.

   (b) $y(t) = x_1(t) * x_2(t)$
   Which implies $Y(j\omega) = X_1(j\omega)X_2(j\omega)$
   Thus, $Y(j\omega) = 0$ for $|\omega| > 1000\pi$.
   Hence, the sampling period range which ensures that $y(t)$ is recoverable from the samples is $(0, 1ms)$, which means the sampling period should be less than 1 ms.

(i). $T = \frac{1}{4B}$



(ii). $T = \frac{1}{2B}$



(iii). $T = \frac{1}{B}$

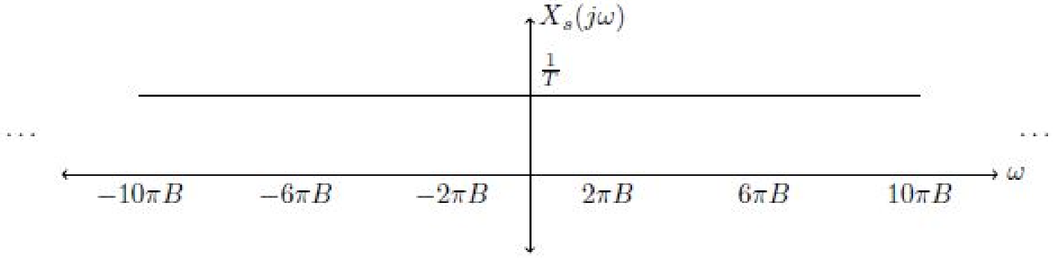
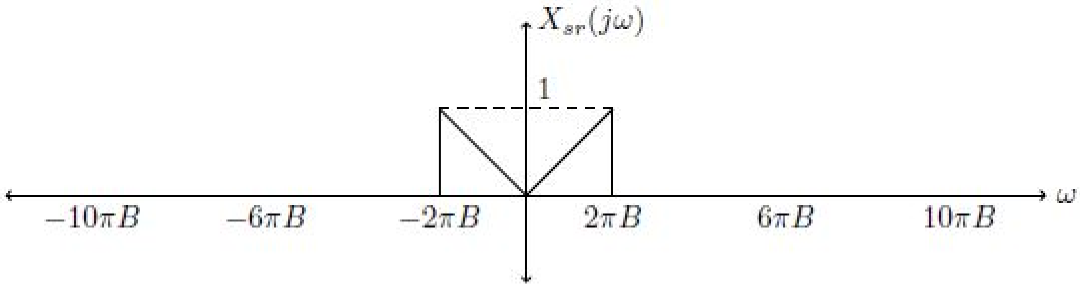
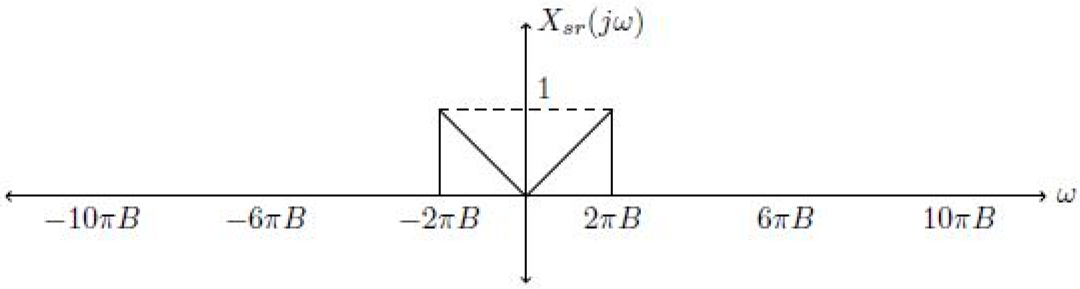


Figure 11: Sketch for $X_s(j\omega)$

(i). $T = \frac{1}{4B}$



(ii). $T = \frac{1}{2B}$



(iii). $T = \frac{1}{B}$

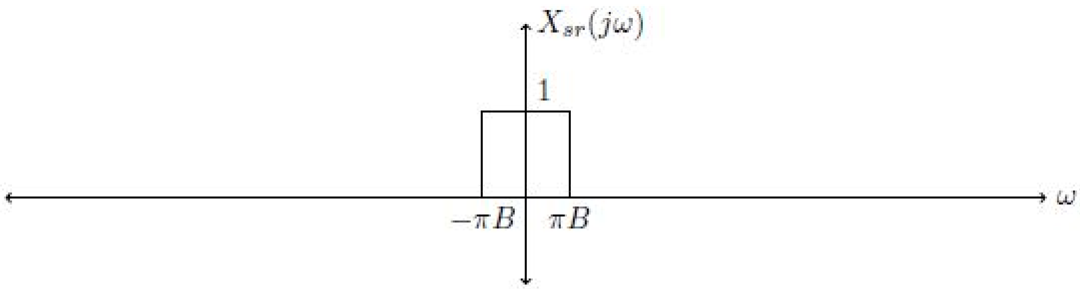


Figure 12: Sketch for $X_{sr}(j\omega)$