Department of Mathematics, IIT Madras
MA1020 Series & Matrices
Assignment-4 Linear Systems & Eigenvalue Problem

1. Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{cccccl} x_1 & +x_2 & +x_3 & +x_4 & -3x_5 & =6 \\ 2x_1 & +3x_2 & +x_3 & +4x_4 & -9x_5 & =17 \\ x_1 & +x_2 & +x_3 & +2x_4 & -5x_5 & =8 \\ 2x_1 & +2x_2 & +2x_3 & +3x_4 & -8x_5 & =14 \end{array}$$

2. Let $A \in \mathbb{F}^{m\times n}$ have columns $A_1, \ldots, A_n$. Let $b \in \mathbb{F}^m$. Show the following:
   (a) The equation $Ax = 0$ has a non-zero solution iff $A_1, \ldots, A_n$ are linearly dependent.
   (b) The equation $Ax = b$ has at least one solution iff $b \in \operatorname{span}\{A_1, \ldots, A_n\}$.
   (c) The equation $Ax = b$ has at most one solution iff $A_1, \ldots, A_n$ are linearly independent.
   (d) The equation $Ax = b$ has a unique solution iff $\operatorname{rank} A = \operatorname{rank}[A|b] =$ number of unknowns.

3. Let $x, y \in \mathbb{F}^{1\times n}$ (or in $\mathbb{F}^{n\times 1}$); $\alpha \in \mathbb{F}$. Prove the following:
   (a) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2$. (*Parallelogram Law*)
   (b) $|\langle x, y\rangle| \le \|x\|\,\|y\|$. (*Cauchy-Schwartz inequality*)
   (c) $\|x+y\| = \|x\| + \|y\|$. (*Triangle inequality*)
   (d) If $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. (*Pythagoras' Law*)

4. Find the eigenvalues and the associated eigenvectors for the matrices given below.

   (a) $\begin{bmatrix} 3 & 10 \\ 8 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 13 & 2 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & -1 \\ 15 & 12 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 0 & 3 \\ -2 & 3 & 0 \\ 10 & 0 & 5 \end{bmatrix}$

5. Let $A \in \mathbb{C}^{n\times n}$ be invertible. Show that $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $1/\lambda$ is an eigenvalue of A^{-1}.

6. Let A be an $n \times n$ matrix and α be a scalar such that each row (or each column) sums to α. Show that α is an eigenvalue of A.

7. Give examples of matrices which cannot be diagonalized.

8. Which of the following matrices is/are diagonalizable? If it is diagonalizable, diagonalize it.
   (a) $A \in \mathbb{R}^{3\times 3}$ is such that $A(a, b, c)^t = (a+b+c,\ a+b-c,\ a-b+c)^t$.
   (b) $A \in \mathbb{R}^{3\times 3}$ is such that $Ae_1 = 0, \quad Ae_2 = e_1, \quad Ae_3 = e_2$.
   (c) $A \in \mathbb{R}^{3\times 3}$ is such that $Ae_1 = e_2, \quad Ae_2 = e_3, \quad Ae_3 = 0$.
   (d) $A \in \mathbb{R}^{3\times 3}$ is such that $Ae_1 = e_3, \quad Ae_2 = e_2, \quad Ae_3 = e_1$.

9. Check whether each of the following matrix is diagonalizable. If diagonalizable, find a basis of eigenvectors for the space $\mathbb{R}^{3\times 1}$:

   (a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

10. Show that each orthogonal 2×2 matrix is either a reflection or a rotation.