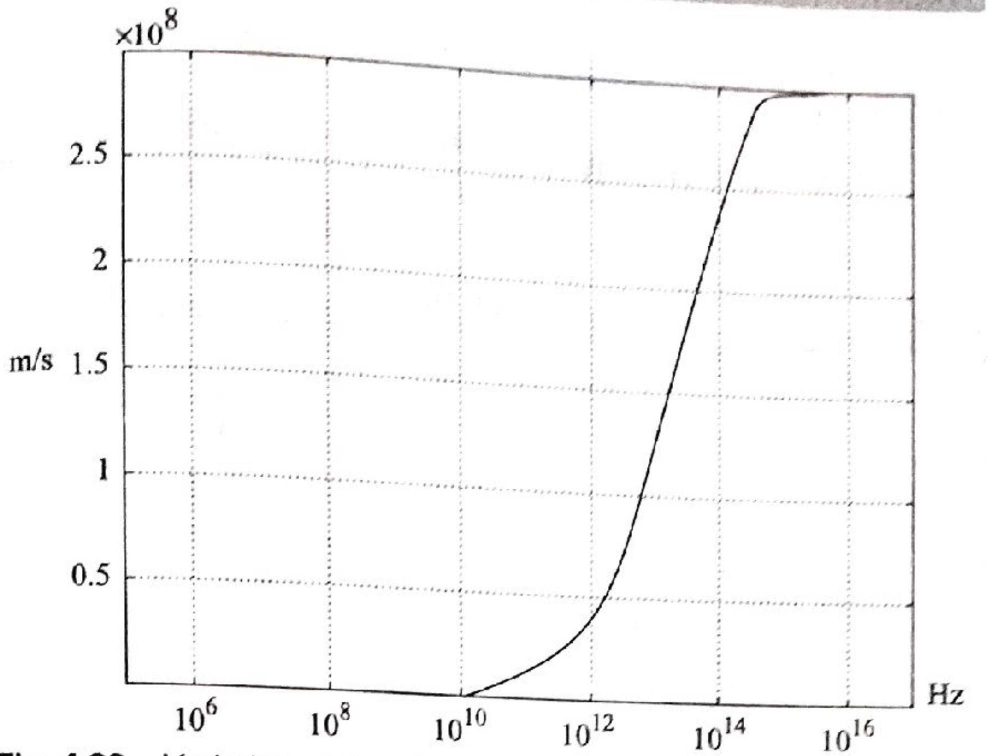

Fig. 4.20 *Variation of the phase velocity as a function of frequency.*

4.8 POWER FLOW AND POYNTING VECTOR

As seen above, the time varying electric and magnetic fields have to form an electromagnetic wave which propagates in the space. Naturally the wave carries some energy with it. It is then worthwhile to investigate, the quantity energy or power (i.e. energy per unit time) is carried by an electromagnetic wave? Note that in general, the electromagnetic wave need neither be a plane wave nor be travelling in an unbound medium. We, therefore, would like to develop a general frame work for power flow from arbitrary time varying fields. Since the electromagnetic phenomenon is completely governed by the Maxwell's equations, we again fall back upon the Maxwell's equations to find the power flow due to time varying fields.

Let us take the two Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{4.146}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{4.147}$$

We assume here that μ, ϵ are not varying as a function of time. From the vector identity we have

$$\nabla \cdot (\mathbf{A} \times \mathbf{C}) = \mathbf{C} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot \nabla \times \mathbf{C} \tag{4.148}$$

where $\mathbf{A}$ and $\mathbf{C}$ are any two arbitrary vectors.

Taking $\mathbf{A} = \mathbf{E}$ and $\mathbf{C} = \mathbf{H}$ we have the vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \tag{4.149}$$

Substituting for $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}$ from Eqns (4.146) and (4.147) in Eqn (4.149) we get

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \left(-\mu \frac{\partial \mathbf{H}}{\partial t}\right) - \mathbf{E} \cdot \left\{\mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t}\right\} \tag{4.150}$$

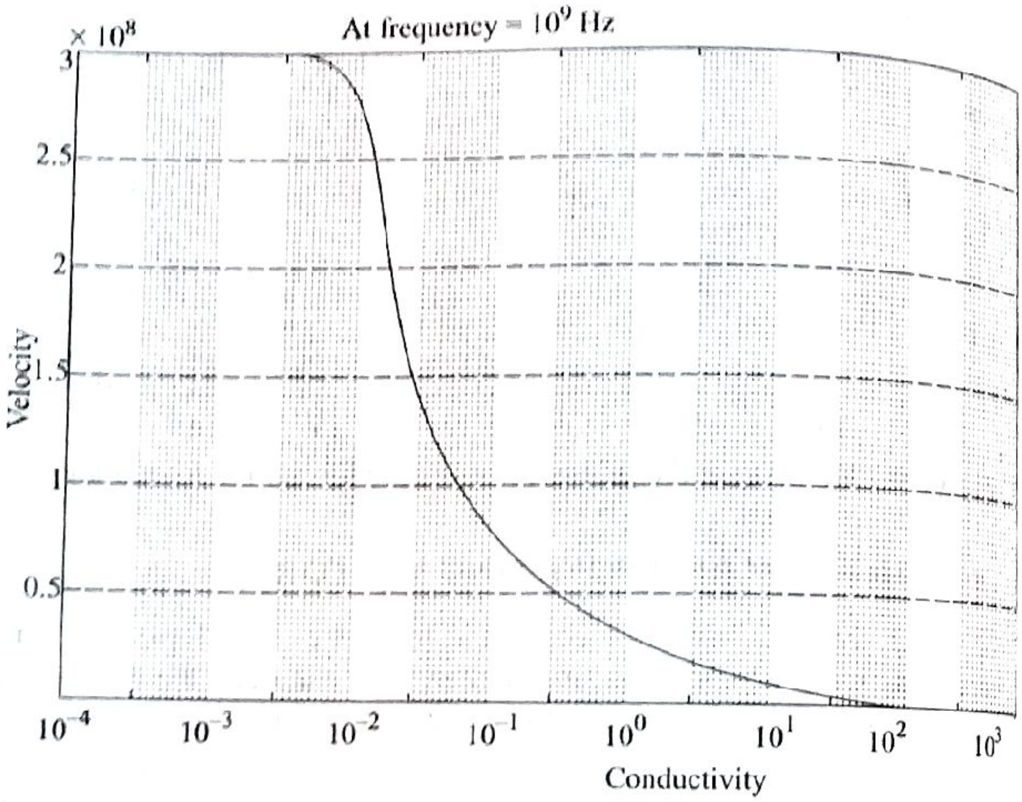

Fig. 4.21 *Variation of the phase velocity as a function of conductivity.*

Here we have assumed that the medium is isotropic and hence μ, ϵ and σ are scalar quantities. Also note that, for any two vectors **A** and **C** we have

$$\frac{\partial(\mathbf{A} \cdot \mathbf{C})}{\partial t} = \mathbf{A} \cdot \frac{\partial \mathbf{C}}{\partial t} + \mathbf{C} \cdot \frac{\partial \mathbf{A}}{\partial t} \tag{4.151}$$

$$\Rightarrow \quad \frac{\partial(\mathbf{A} \cdot \mathbf{A})}{\partial t} = 2\mathbf{A} \cdot \frac{\partial \mathbf{A}}{\partial t} \tag{4.152}$$

$$\Rightarrow \quad \mathbf{A} \cdot \frac{\partial \mathbf{A}}{\partial t} = \frac{1}{2} \frac{\partial(\mathbf{A} \cdot \mathbf{A})}{\partial t} = \frac{1}{2} \frac{\partial |\mathbf{A}|^2}{\partial t} \tag{4.153}$$

Noting that $\mathbf{J} = \sigma \mathbf{E}$, and taking σ not a function of time, we can write from Eqn (4.150)

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\mu}{2} \frac{\partial |\mathbf{H}|^2}{\partial t} - \frac{\epsilon}{2} \frac{\partial |\mathbf{E}|^2}{\partial t} - \sigma |\mathbf{E}|^2 \tag{4.154}$$

The Eqn (4.154) is essentially a point relation. That is, it should be valid at every point in the space at every instant of time.

If we now integrate Eqn (4.154) over a volume we get

$$\oint_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) dv = \oint_V (-\frac{\mu}{2}\frac{\partial |\mathbf{H}|^2}{\partial t} - \frac{\epsilon}{2}\frac{\partial |\mathbf{E}|^2}{\partial t} - \sigma |\mathbf{E}|^2) dv \tag{4.155}$$

Application of the divergence theorem (see Eqn (3.16)) on the LHS yields

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{da} = \oint_V -\frac{\mu}{2}\frac{\partial |\mathbf{H}|^2}{\partial t} dv - \oint_V \frac{\epsilon}{2}\frac{\partial |\mathbf{E}|^2}{\partial t} dv - \oint_V \sigma |\mathbf{E}|^2 dv \tag{4.156}$$

Taking the volume V constant as a function of time, we can interchange the integral and the $\frac{\partial}{\partial t}$ giving

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{da} = -\frac{\partial}{\partial t}\oint_V \frac{\mu}{2}|H|^2 dv - \frac{\partial}{\partial t}\oint_V \frac{\epsilon}{2}|\mathbf{E}|^2 dv - \oint_V \sigma |E|^2 dv \tag{4.157}$$

$$= -\frac{\partial}{\partial t}\left(\oint_V \frac{\mu}{2}|H|^2 dv + \oint_V \frac{\epsilon}{2}|E|^2 dv\right) - \oint_V \sigma |E|^2 dv \tag{4.158}$$

In Eqn (4.158), the first term within the brackets gives the magnetic energy stored in the volume, whereas the second term represents the electric energy stored in the volume V. The quantity within the brackets therefore represents the total energy stored in the volume, and the first term on the RHS of Eqn (4.158) represents the rate of decrease of total energy stored in the volume V, i.e. decrease in power in the volume V.

The second term on the RHS gives the ohmic power loss in the volume V. This term then represents the electromagnetic energy converted to heat per unit time. The two terms on the RHS consequently represent the total decrease in the electromagnetic energy per unit time, i.e. the power loss from the volume. Since there are no other losses in the medium, the law of conservation of energy dictates that this power loss should be due to energy leaving the volume. The LHS of Eqn (4.158) hence gives the power coming out of the volume, i.e.

$$\text{Net Outward Power } W = \oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{da} \tag{4.159}$$

Since the surface integral of $(\mathbf{E} \times \mathbf{H})$ gives the total power flow from the surface, the quantity $(\mathbf{E} \times \mathbf{H})$, therefore, represents the power density on the surface of the volume. The vector defined as

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} \tag{4.160}$$

is called the 'Poynting vector' and it gives the power flow density (power flow per unit area) at any point on the surface.

The statement that 'the surface integral of **P** over a closed surface (the LHS in Eqn (4.158)) is equal to the total power leaving the closed surface' is called the *Poynting theorem*. It should be emphasized however, that the Poynting theorem is strictly valid only for a closed surface. That is to say that the Eqn (4.160) is not a point relationship according to the Poynting theorem. The use of $(\mathbf{E} \times \mathbf{H})$ as power density at every point on the closed surface is arbitrary. There are certain classical cases which contradict the use of point relationship for the Poynting vector. For example, let us place a charge at one of the poles of a bar magnet as shown in Fig. 4.22. Now, let us take some point, say P, as shown in Fig. 4.22. At this point since $\mathbf{E}$ and $\mathbf{H}$ are not parallel, we have finite $\mathbf{E} \times \mathbf{H}$ which is pointing out of the plane of the paper. In other words, we have a power coming out of the paper at point P. Obviously, since in this case there is no sustained power flow the answer obtained from the Poynting vector is absurd. If we consider however a symmetrically located point P', the $\mathbf{E} \times \mathbf{H}$ vector here is of same magnitude as that at P but going inside the plane of the paper. The sum of the power flow at P and P' hence is zero. Extending the result to any closed surface around the magnet we find that the net power flow from the surface is zero even though the poynting vector $\mathbf{P}$ is non-zero on the surface. It might, therefore, appear that the use of $\mathbf{E} \times \mathbf{H}$ for power density at any location might give erroneous results at times. However, except for few pathological cases (like the one discussed above) the Poynting vector correctly gives the power density at a point in the space. The Eqn (4.160) hence is normally used as a point relation although it is not strictly a point relation.

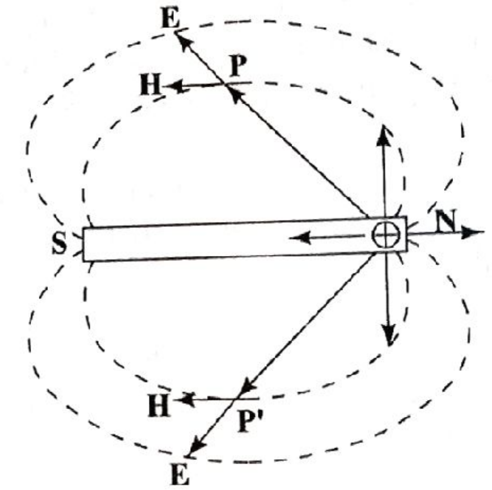

Fig. 4.22 *Poynting Vector for a system consisting of a charge placed on a bar magnet.*

The Poynting vector is a rather useful concept as it tells that the power flow is perpendicular to both the electric and the magnetic fields. The Poynting vector has the unit of Watts/m^2.

For a uniform plane wave travelling in $+z$ direction, the electric and the magnetic fields are oriented in $+x$ and $+y$ directions respectively. The $(\mathbf{E} \times \mathbf{H}) = (\hat{\mathbf{x}} \times \hat{\mathbf{y}})$ vector then is along $+z$ direction, i.e. the direction of the wave propagation. The Poynting vector hence correctly gives the direction of the power flow as the direction of the wave propagation.

EXAMPLE 4.12 A uniform plane wave has a power density of 20 W/m^2 and is travelling along $-y$ direction in the free space. If the electric field makes an angle of 30° with the $+x$ axis, find electric and magnetic field strengths and the direction of the magnetic field.

Solution:
For a uniform plane wave E and H are perpendicular to each other. Therefore, the poynting vector is

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = |\mathbf{E}||\mathbf{H}|$$

$$= \frac{|\mathbf{E}|^2}{\eta_0} \qquad \text{since} \quad |\mathbf{H}| = \frac{|\mathbf{E}|}{\eta_0}$$

$$\Rightarrow \qquad |E| = \sqrt{P\eta_0} = \sqrt{20 \times 120\pi} = 86.83 \text{ V/m}$$

$$\text{and} \qquad |H| = \frac{E}{120\pi} = 0.23 \text{ A/m}$$

Since, **E** and **H** are perpendicular to the direction of the wave, $-y$ axis, they must lie in xz plane. Also **E** × **H** should be in the direction of the wave. The **H** vector therefore will make an angle of 30° with the z-axis (see Fig. 4.23).

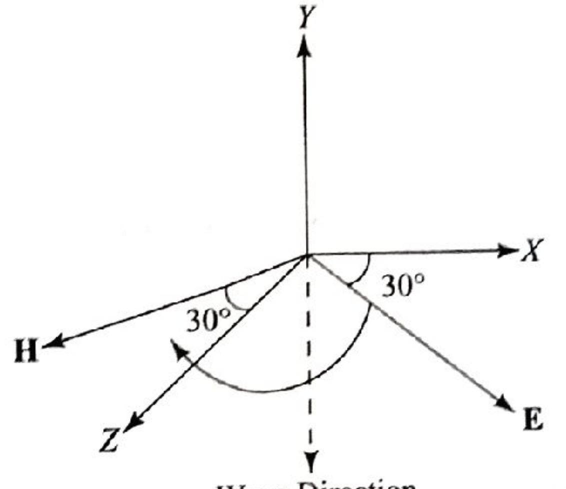

Fig. 4.23

4.8.1 Instantaneous and Average Poynting Vector

For time harmonic fields, it is rather useful to have the average power density (average over time) or average Poynting vector at any point in space. In the following sections, starting with the instantaneous **E**, **H** and **P** we will derive the average power density or the average Poynting vector.

Writing **E** and **H** explicitly for the time harmonic function we have

$$\mathbf{E}(x, y, z, t) = \mathbf{E_0}(x, y, z)e^{j\omega t} \tag{4.161}$$

and

$$\mathbf{H}(x, y, z, t) = \mathbf{H_0}(x, y, z)e^{j\omega t} \quad (4.16$$

The instantaneous values of the electric and the magnetic fields are

$$\mathbf{E}(x, y, z, t) = Re\{\mathbf{E_0}(x, y, z)e^{j\omega t}\} \quad (4.16$$

$$= Re\{E_0(x, y, z)e^{j\phi_e}e^{j\omega t}\}\hat{\mathbf{e}} \quad (4.16$$

$$= E_0(x, y, z)\cos(\phi_e + \omega t)\hat{\mathbf{e}} \quad (4.16$$

and

$$\mathbf{H}(x, y, z, t) = Re\{\mathbf{H_0}(x, y, z)e^{j\omega t}\} \quad (4.16$$

$$= Re\{H_0(x, y, z)e^{j\phi_h}e^{j\omega t}\}\hat{\mathbf{h}} \quad (4.16$$

$$= H_0(x, y, z)\cos(\phi_h + \omega t)\hat{\mathbf{h}} \quad (4.168$$

where $\hat{\mathbf{e}}$ and $\hat{\mathbf{h}}$ are the unit vectors in the directions of **E** and **H** respectively and ϕ_e and ϕ_h are the time phases of the electric and the magnetic fields respectively at a point (x, y, z) in the space (see Fig. 4.24). The instantaneous Poynting vector is

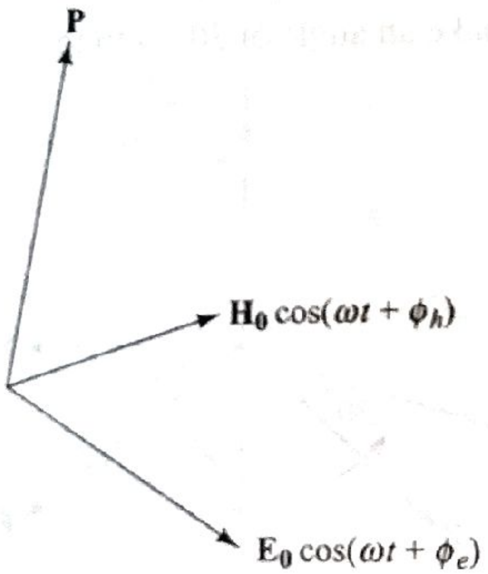

Fig. 4.24 *Electric and magnetic fields, and the Poynting vector.*

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = E_0 H_0 \cos(\omega t + \phi_e)\cos(\omega t + \phi_h)(\hat{\mathbf{e}} \times \hat{\mathbf{h}}) \quad (4.169)$$

Equation (4.169) can be re-written as

$$\mathbf{P} = \frac{E_0 H_0}{2}\{\cos(\phi_e - \phi_h)\hat{\mathbf{e}} \times \hat{\mathbf{h}} + \cos(2\omega t + \phi_e + \phi_h)\hat{\mathbf{e}} \times \hat{\mathbf{h}}\} \quad (4.170)$$

The instantaneous power density harmonically varies at a frequency 2ω, double of that of **E** or **H**. The average power density

$$\mathbf{P_{av}} = \frac{1}{T}\int_0^T \mathbf{P}dt \quad (4.171)$$

where T is the time period of the time harmonic fields, $T = 2\pi/\omega$. Substituting Eqn (4.170) in Eqn (4.171) and noting that the average of the second term is zero, we get

$$\mathbf{P}_{av} = \frac{1}{2} E_0 H_0 \cos(\phi_e - \phi_h) \hat{\mathbf{e}} \times \hat{\mathbf{h}} \tag{4.172}$$

$$= \frac{1}{2} Re\left([E_0 e^{j\phi_e} e^{j\omega t} \hat{\mathbf{e}}] \times [H_0 e^{-j\phi_h} e^{-j\omega t} \hat{\mathbf{h}}] \right) \tag{4.173}$$

$$= \frac{1}{2} Re\left([E_0 e^{j\phi_e + j\omega t} \hat{\mathbf{e}}] \times [H_0 e^{j\phi_h + j\omega t} \hat{\mathbf{h}}]^* \right) \tag{4.174}$$

$$\Rightarrow \quad \mathbf{P}_{av} = \frac{1}{2} Re\,(\mathbf{E} \times \mathbf{H}^*) \tag{4.175}$$

The average power density is a much meaningful quantity as it gives the actual power flow at that location. The instantaneous power on the other hand, does not correctly represent the power flow as it can be negative or positive. It can probably then give the amount of power oscillating back and forth around a point plus the actual power flow at that point.

From Eqns (4.172) and (4.175) it is clear that for a real power flow, two conditions should be satisfied.

1. **E** and **H** fields should cross each other
2. **E** and **H** should not be in time quardrature, i.e. 90° out of phase with each other.

E and **H** fields which are parallel and/or in time quardrature, do not constitute any power flow. In the later section, we will observe that in complex situations, this argument comes very handy in checking whether the fields carry any power along with them.

EXAMPLE 4.13 In a region the **E** and **H** fields are gives as

$$\mathbf{E} = 100\,(j\hat{\mathbf{x}} + 2\hat{\mathbf{y}} - j\hat{\mathbf{z}})\,e^{j\omega t}$$

$$\mathbf{H} = (-\hat{\mathbf{x}} + j\hat{\mathbf{y}} + \hat{\mathbf{z}})\,e^{j\omega t}$$

Find the average power flow density and direction of power flow in the region.

Solution:

The average poynting vector

$$\mathbf{P}_{av} = \frac{1}{2} Re\,\{\mathbf{E} \times \mathbf{H}^*\}$$

$$= \frac{100}{2} Re \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ j & 2 & -j \\ -1 & -j & 1 \end{vmatrix}$$

$$= 150\sqrt{2}\frac{(\hat{\mathbf{x}} + \hat{\mathbf{z}})}{\sqrt{2}}$$

where T is the time period of the time harmonic fields, $T = 2\pi/\omega$. Substituting Eqn (4.170) in Eqn (4.171) and noting that the average of the second term is zero, we get

$$\mathbf{P}_{\mathbf{av}} = \frac{1}{2} E_0 H_0 \cos(\phi_e - \phi_h)\hat{\mathbf{e}} \times \hat{\mathbf{h}} \tag{4.172}$$

$$= \frac{1}{2} Re\left([E_0 e^{j\phi_e} e^{j\omega t}\hat{\mathbf{e}}] \times [H_0 e^{-j\phi_h} e^{-j\omega t}\hat{\mathbf{h}}]\right) \tag{4.173}$$

$$= \frac{1}{2} Re\left([E_0 e^{j\phi_e + j\omega t}\hat{\mathbf{e}}] \times [H_0 e^{j\phi_h + j\omega t}\hat{\mathbf{h}}]^*\right) \tag{4.174}$$

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$$\mathbf{H} = (-\hat{\mathbf{x}} + j\hat{\mathbf{y}} + \hat{\mathbf{z}})\,e^{j\omega t}$$

Find the average power flow density and direction of power flow in the region.

Solution:

The average poynting vector

$$\mathbf{P}_{\mathbf{av}} = \frac{1}{2} Re\left\{\mathbf{E} \times \mathbf{H}^*\right\}$$

$$= \frac{100}{2} Re \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ j & 2 & -j \\ -1 & -j & 1 \end{vmatrix}$$

$$= 150\sqrt{2}\frac{(\hat{\mathbf{x}} + \hat{\mathbf{z}})}{\sqrt{2}}$$

The average power density is $150\sqrt{2}\ \mathrm{W/m^2} = 212.132\ \mathrm{W/m^2}$ and the power flows in the direction $\frac{(\hat{\mathbf{x}}+\hat{\mathbf{z}})}{\sqrt{2}}$, i.e. in the xz plane at an angle of 45° with respect to x and z axes.

4.8.2 Power Density of a Uniform Plane Wave

For a uniform plane wave, **E** and **H** are perpendicular to each other and the ratio of their magnitudes is equal to the intrinsic impedance of the medium, η. Without losing generality let us take the **E**-field oriented along the $+x$ direction and the **H**-field oriented along the $+y$ direction as

$$\mathbf{E} = E_0 e^{j\omega t}\hat{\mathbf{x}} \tag{4.176}$$

$$\mathbf{H} = H_0 e^{j\omega t}\hat{\mathbf{y}} = \frac{\mathbf{E_0}}{\eta}\mathbf{e}^{j\omega t}\hat{\mathbf{y}} \tag{4.177}$$

The average power density of the wave is

$$\mathbf{P_{av}} = \frac{1}{2}Re\{\mathbf{E}\times\mathbf{H}^*\} \tag{4.178}$$

$$= \frac{1}{2}Re\{E_0 e^{j\omega t}[\frac{E_0}{\eta}e^{j\omega t}]^*\}\hat{\mathbf{x}}\times\hat{\mathbf{y}} \tag{4.179}$$

$$= \frac{1}{2}Re\{\frac{|E_0|^2}{\eta^*}\}\hat{\mathbf{z}} = \frac{1}{2}Re\{\eta|H_0|^2\}\hat{\mathbf{z}} \tag{4.180}$$

$$\Rightarrow \qquad P_{av} = \frac{|E_0|^2}{2}Re\{\frac{1}{\eta^*}\} \equiv \frac{|H_0|^2}{2}Re\{\eta\} \tag{4.181}$$

(a) For a **loss-less dielectric medium** $\eta = \sqrt{\frac{\mu}{\epsilon}}$ = Real number
The average power density of the wave is

$$P_{av} = \frac{1}{2}\frac{|E_0|^2}{\eta} = \frac{1}{2}|E_0|^2\sqrt{\frac{\epsilon}{\mu}} \tag{4.182}$$

(b) For a **lossy medium** however, η is complex, and **E** and **H** are not in time phase. Consequently, P_{av} has to be calculated using exact Eqn (4.181).

(c) For a **good conductor** $\sigma >> \omega\epsilon$, and $\eta \approx \sqrt{\frac{\omega\mu}{2\sigma}} + j\sqrt{\frac{\omega\mu}{2\sigma}}$. The phase angle between **E** and **H** is approximately 45°, and the average power density is

$$P_{av} = \frac{1}{2}|E_0|^2 Re\{\frac{1}{\eta^*}\} = \frac{1}{2}\frac{|E_0|^2}{|\eta|^2}Re\{\eta\} \tag{4.183}$$

$$= \frac{1}{2}|E_0|^2\sqrt{\frac{\sigma}{2\omega\mu}} \tag{4.184}$$

It is interesting to note from Eqn (4.184) that as the frequency increases, the power density of the wave reduces and at very high frequency very little power penitrates the conducting medium.

EXAMPLE 4.14 At some location inside a lossy dieletric material the measured peak electric field of a wave is 10 V/m. The material has relative permittivity of 8 and conductivity of 100 ℧/m. Find the average power density of the wave at that location. Also find the power density at a distance of 1 cm in the direction of the wave propagation. The frequency of the wave is 300 MHz.

Solution:

$$\omega = 2\pi \times 300 \times 10^6 = 1.8849 \times 10^9 \text{ rad/sec}$$

The intrinsic impedance of the medium is

$$\eta = \sqrt{\frac{j\omega\mu_0}{\sigma + j\omega\epsilon_0\epsilon_r}} = 3.44 + j3.44\ \Omega$$

From Eqn (4.181), the average power density of the wave is

$$P_{av} = \frac{|E_0|^2}{2} Re\{\frac{1}{\eta^*}\} = 7.27\ \text{W/m}^2$$

Now, since the conductivity of the medium is non-zero, there is attenuation in the medium. The propagation constant of the medium is

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu_0(\sigma + j\omega\epsilon_0\epsilon_r)}$$
$$= 343.9 + j344.36 \text{ per meter}$$

$\Rightarrow$ Attenuation constant α =343.9 nepers/m

Power density at 1 cm distance is

$$P_{av}e^{-2\times 0.01\times\alpha} = P_{av}e^{-6.878} = 7.49\ \text{mW/m}^2$$

4.9 SURFACE CURRENT AND POWER LOSS IN A CONDUCTOR

4.9.1 Surface Current

We have seen in the previous chapter that the tangential component of the electric field is zero at the surface of an ideal conductor. The tangential magnetic field is balanced by the surface current J_s giving,

$$\mathbf{J_s} = \hat{\mathbf{n}} \times \mathbf{H} = \mathbf{H_t} \tag{4.185}$$

where $\hat{\mathbf{n}}$ is the outward normal unit vector to the conductor surface and $\mathbf{H}$ is the total magnetic field at the surface. $\mathbf{H_t}$ denotes the tangential component of the magnetic field.

Inside an ideal conductor no time varying fields exist and hence the current is truely the surface current. One may wonder at this point, as to the force that drives this current, which needs electric field to move charges. Since the electric field along the conducting surface is zero, there is no force for driving the current. This is however, a steady state picture. To excite the current, some electric field must have been present momentarily. The electric field would have put the charges into motion and must have disappeared. Since the conductivity is infinite for an ideal conductor, once the charges are placed in motion, the current keeps flowing for infinite time without the electric force. It is, therefore, important that one clearly distinguishes between the two cases (i) no field and no current and (ii) no field but current. In steady state both cases have no electric field along the conductor surface.

The surface current is a concept of an ideal conductor. As we do not have ideal conductors in real life, one would wonder regarding the purpose of the concept of surface current. In the following sections we will observe that although there is nothing like surface current for a non-ideal conductor ($\sigma \neq \infty$), the concept can still be useful in analysing good conductors.

Let us consider an electric field $\mathbf{E_0}$ tangential to the conducting surface (along x-direction) and just inside it. It should be made clear that $\mathbf{E_0}$ is not the incident field but the field which would exist at the surface (why the field is not the same as the incident field will become clear later after we have discussed the reflection from the conducting surface). Since we are considering good conductors, the magnetic field $\mathbf{H_0}$ is almost tangential to the surface (along y-axis) and the ratio $|\mathbf{E_0}|/|\mathbf{H_0}|$ is the intrinsic impedance of the conductor η_c. The $\mathbf{E_0}$ and $\mathbf{H_0}$ constitute a wave going inside the conductor as shown in Fig. 4.25.

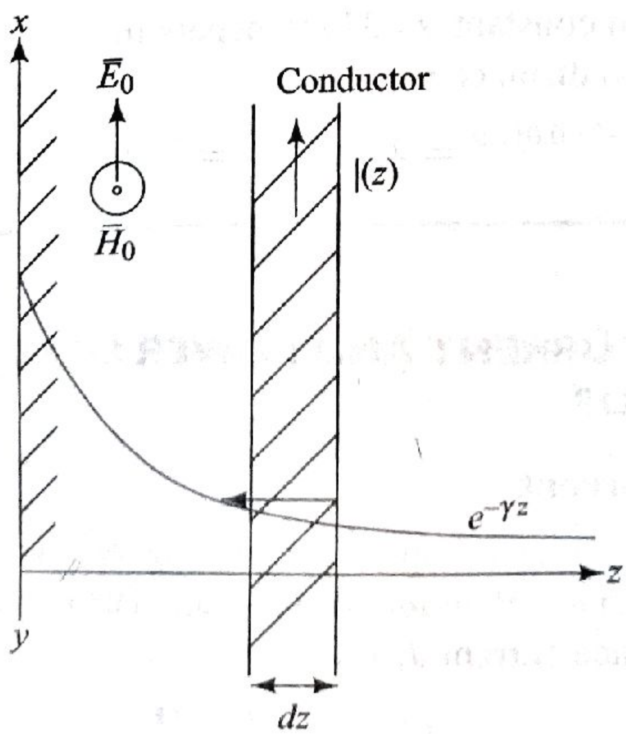

Fig. 4.25 *Electrical field variation inside a conductor.*

The propagation constant of the wave inside the conductor is

$$\gamma = \alpha + j\beta = \sqrt{\frac{\omega\mu_0\sigma}{2}} + j\sqrt{\frac{\omega\mu_0\sigma}{2}} \tag{4.186}$$

The field amplitude decreases exponentially inside the conductor. The field at a depth z from the surface then is

$$\mathbf{E}(z) = \mathbf{E}_0 e^{-\gamma z} = \mathbf{E}_0 e^{-\alpha z}.e^{-j\beta z} \tag{4.187}$$

Now, due to conductivity σ of the conductor, a conduction current flows in the same direction as the electric field **E**. The density of the conduction current at a depth z is

$$\mathbf{J}(z) = \sigma\mathbf{E} = \sigma E_0 e^{-\alpha z} e^{-j\beta z}\hat{\mathbf{x}} \tag{4.188}$$

Let us now consider a slab parallel to the surface having unit width along y-direction, and thickness dz. The sheet is located at a distance of z from the surface. The current in the sheet can be written as

$$\mathbf{I}(z) = \mathbf{J}(z)dz = \sigma E_0 e^{-\gamma z} dz\hat{\mathbf{x}} \tag{4.189}$$

The current flows along x-direction (same as that of the electric field). If we now integrate $\mathbf{I}(z)$ along z, we get total current flow per unit width of the conductor surface (the width is perpendicular to the direction of the current and is along y-direction) as

$$\mathbf{J}_s = \int_0^\infty \mathbf{E}_0\sigma e^{-\gamma z} dz = \mathbf{E}_0\sigma\left[-\frac{e^{-\gamma z}}{\gamma}\right]_0^\infty \tag{4.190}$$

Since γ has +ve real part, $e^{-\gamma\infty} \to 0$ giving

$$\mathbf{J}_s = \frac{E_0\sigma}{\gamma}\hat{\mathbf{x}} \tag{4.191}$$

For a good conductor, since the field decays very rapidly inside the conductor, the current $\mathbf{J}_s$ effectively flows close to the surface, and we may treat this current as the surface current. Note that dimensionally $\mathbf{J}_s$ has dimensions Ampere/meter, which is same as the dimension of the surface current. To completely justify that $\mathbf{J}_s$ is equivalent to the surface current, it must be related to the tangential magnetic field $\mathbf{H}_0$ through Eqn (4.185). For the conductor, the intrinsic impedance is

$$\eta_c = \sqrt{\frac{j\omega\mu}{\sigma}} = \frac{\gamma}{\sigma} \tag{4.192}$$

since for a good conductor the propagation constant $\gamma = \sqrt{j\omega\mu\sigma}$.

The magnetic field at the surface of the conductor is

$$|\mathbf{H}_0| = \frac{|\mathbf{E}_0|}{|\eta_c|} = \frac{|\mathbf{E}_0|}{|\gamma|}\sigma = |\mathbf{J}_s| \tag{4.193}$$

From Fig. 4.25, we see that the normal vector $\hat{\mathbf{n}}$ to the conductor surface is in $-z$ direction, and the magnetic field is in y-direction. We, therefore, get

$$\hat{\mathbf{n}} \times \mathbf{H_0} = -\hat{\mathbf{z}} \times \hat{\mathbf{y}}|H_0| = \mathbf{J_s} \tag{4.194}$$

Hence, the Eqn (4.185) is indeed satisfied by $\mathbf{J_s}$ given by Eqn (4.191).

It is important to emphasize that for a non-ideal conductor, there is only a volume current density $\mathbf{J}$ and truly there is no surface current. However, the total integrated current $\mathbf{J_s}$ can be treated like the surface current. One can observe that for an ideal conductor, as $\sigma \to \infty$, the skin depth tends to be zero, and the current $\mathbf{J_s}$ truly becomes the surface current.

From Eqn (4.193), we can define a parameter called the surface impedance Z_s which is the ratio of the tangential electric field E_0 and the surface current J_s as

$$Z_s = \frac{|\mathbf{E}_{\tan}|}{|\mathbf{J_s}|} = \frac{E_0}{J_s} = \frac{\gamma}{\sigma} = \eta_c \tag{4.195}$$

Seperation of real and imaginary parts yields

$$Z_s = R_s + jX_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} + j\sqrt{\frac{\omega\mu_0}{2\sigma}} \tag{4.196}$$

As will be seen later the surface impedance is a useful parameter in computation of the conductor losses.

4.9.2 Power Loss in a Conductor

Let us again consider the thin sheet in Fig. 4.25. Take a piece of this sheet which has unit length and unit width as shown in Fig. 4.26. If we treat this slab as a resistor of resistivity $\rho = 1/\sigma$, the area of cross-section of the resistor is $A = 1 \times dz = dz$ and the length of the resistor is $l = 1$.

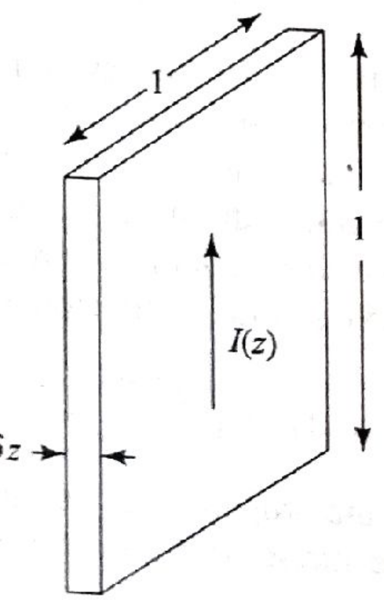

Fig. 4.26 *Power Loss in a thin sheet.*

The resistance of the slab is

$$dR = \frac{\rho l}{A} = \frac{1}{\sigma\, dz} \tag{4.197}$$

The ohmic loss in the slab is

$$dW = \frac{1}{2}|I(z)|^2 dR \tag{4.198}$$

Substituting for $I(z)$ from Eqn (4.189) we get

$$dW = \frac{1}{2}|\sigma E_0 e^{-\gamma z} dz|^2 \frac{1}{\sigma\, dz} \tag{4.199}$$

$$= \frac{1}{2}\sigma |E_0|^2 e^{-2\alpha z} dz \tag{4.200}$$

The total loss per unit area of the conductor surface can be obtained by integrating Eqn (4.200) from $z = 0$ to ∞ as

$$W = \frac{1}{2}\int_0^{\infty} \sigma |E_0|^2 e^{-2\alpha z} dz \tag{4.201}$$

$$= \frac{1}{2}\sigma |E_0|^2 \left[\frac{e^{-2\alpha z}}{-2\alpha}\right]_0^{\infty} \tag{4.202}$$

$$\Rightarrow \quad W = \frac{1}{2}\frac{\sigma |E_0|^2}{2\alpha} = \frac{1}{2}\frac{\sigma}{2\alpha}\frac{|\gamma|^2}{\sigma^2}|\mathbf{J}_s|^2 \tag{4.203}$$

Substituting for γ and α from Eqn (4.186) we get

$$W = \frac{1}{2}\frac{|\mathbf{J}_s|^2 \omega\mu_0\sigma}{2\sigma\sqrt{\omega\mu_0\sigma/2}} \tag{4.204}$$

$$= \frac{1}{2}|\mathbf{J}_s|^2 \sqrt{\frac{\omega\mu_0}{2\sigma}} \tag{4.205}$$

$$= \frac{1}{2}R_s|\mathbf{J}_s|^2 \tag{4.206}$$

The power loss, therefore, is proportional to the surface resistance ($R_s = \sqrt{\omega\mu_0/2\sigma}$) which increases with frequency and decreases with the conductivity. It is then interesting to see that as the conductivity increases, the wave attenuates rapidly inside the conductor but this attenuation is not due to the ohmic loss. This case, therefore, is not similar to a lossy transmission line where the power is lost in the heating of the line due to ohmic loss. This means that, as the conductivity increases, the energy finds it difficult to enter the conducting surface. For ideal conductor, i.e. for $\sigma = \infty$, there is no penitration of the wave. The current flows only on the conductor surface and there is no power loss.

EXAMPLE 4.15 The magnetic field at the surface of a good conductor is 2 A/m. The frequency of the field is 600 MHz. If the conductivity of the