INDIAN INSTITUTE OF TECHNOLOGY MADRAS
PH1020
Tutorial set-5

Question 1:-

The vector potential in a region is given by $\vec{A}(\rho,\phi,z) = -\frac{\mu_0}{2}k\rho^2\ \hat{e}_z$ for $0 < \rho \le a$ and $-\frac{\mu_0}{2}ka(2\rho - a)\hat{e}_z$ for $\rho > a$. Find the corresponding $\vec{J}$ and sketch its magnitude as a function of ρ.

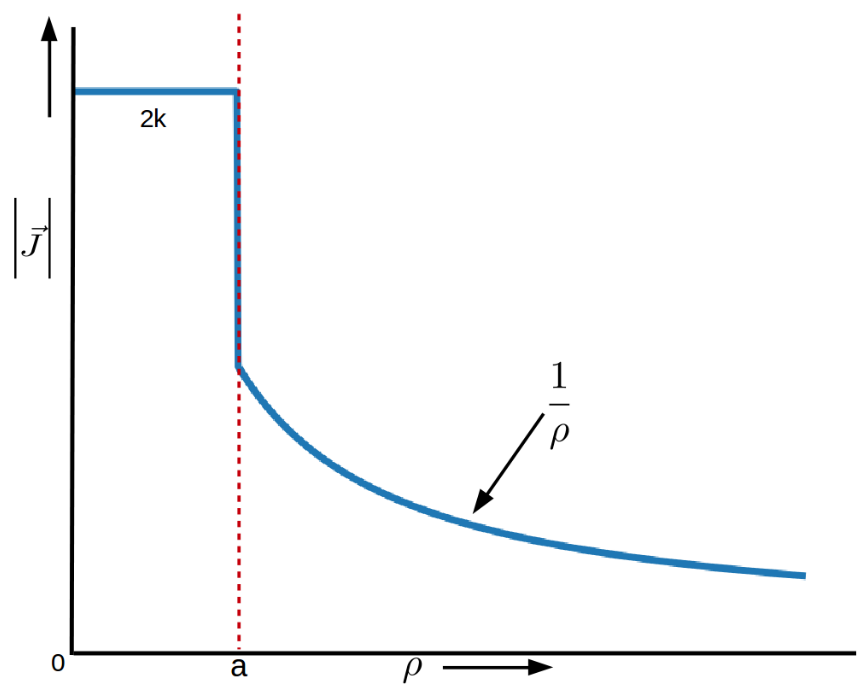
Solution:-

$\vec{A} = -\frac{\mu_0}{2}k\rho^2\ \hat{e}_z$ (inside) $0 < \rho \le a$

$$\therefore\quad \vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{B} = -\frac{\partial A_z}{\partial \rho}\ \hat{e}_\phi$$
$$\vec{B} = \mu_0 k\rho\hat{e}_\phi$$
$$\vec{J} = \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)$$
$$= \frac{1}{\rho\mu_0}\frac{\partial}{\partial\rho}\left(\rho B_\phi\right)\hat{e}_z$$
$$\vec{J} = 2k\ \hat{e}_z$$

For $\rho > a$

$$\vec{A} = -\frac{\mu_0}{2}\ ka\,(2\rho - a)\ \hat{e}_z \qquad \therefore\quad \vec{J} = \frac{ka}{\rho}$$



Question 2:-

(a) Find the vector potential due to an infinite wire carrying a steady current 'I' along z- axis. Using the following relations:

$$(i)\quad \vec{A} = \frac{\mu_0}{4\pi}\int \frac{I}{r}\,\overrightarrow{dl'} \qquad (ii)\quad \vec{B} = \vec{\nabla}\times\vec{A}.$$

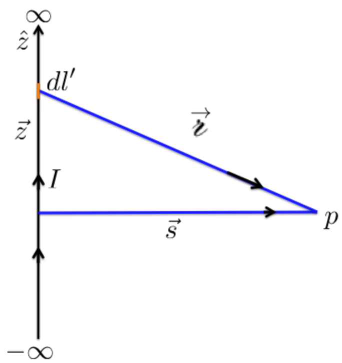
Analyse the results obtained from both the relations.

(b) Now bring a second infinite straight wire with steady current I and aligned it parallel to the first wire at a distance d. Calculate the vector potential when both the wires have (i) parallel currents (ii) anti-parallel currents.

Solution:-

(a)

$I = I\ \hat{z}$



$$\vec{A} = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} \frac{I}{\mathscr{r}}\,dl'$$

$$= \frac{\mu_0}{4\pi} I \int_{-\infty}^{\infty} \frac{dz}{\sqrt{s^2+z^2}}\ \hat{z}$$

$$= \frac{\mu_0 I}{4\pi} \ln\left| z + \sqrt{z^2+s^2}\right|_{-\infty}^{\infty}\ \hat{z}$$

(ii) $\vec{B} = \vec{\nabla}\times\vec{A}$, The magnetic field for a infinite wire

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\ \hat{\phi}$$

$$\therefore \qquad \frac{\mu_0 I}{2\pi s}\ \hat{\phi} = -\frac{\partial A_z}{\partial s}\ \hat{\phi}$$

$$\Rightarrow \qquad \frac{\mu_0 I}{2\pi s} = -\frac{\partial A_z}{\partial s}$$

$$\Rightarrow \qquad A_z = -\frac{\mu_0 I}{2\pi}\int \frac{1}{s}\,ds$$

$$\Rightarrow \qquad A_z = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{s}{a}\right) \qquad \text{where 'a' is a constant}$$

Remarks:

1. In expression (1), $\vec{A}$ diverges. This is not serious, since $\vec{A}$ is arbitrary up to a constant which in this case is ∞. For instance, if instead of integrating from $-\infty$ to ∞, we can integrate it from 0 to ∞ and then double the result. In that case the integral diverges only in the upper limit.

2. In expression (2) the constant 'a' is arbitrary. One can make it unity. However, in this case the units are questionable. To validate (1) and (2), one should ensure that $\vec{\nabla}.\vec{A} = 0$ and $\vec{\nabla} \times \vec{A} = \vec{B}$

(b) Using expression (2), we get

$$\vec{A} = \frac{\mu_0 I}{2\pi}\left[\ln\left(\frac{r_2}{r_1}\right)\right] \qquad \text{for parallel current}$$
$$\vec{A} = \frac{\mu_0 I}{2\pi}\left[\ln\left(r_2 r_1\right)\right] \qquad \text{for anti-parallel current}$$

Hence r_1 and r_2 are the distance from the axis of wire (1) and (2) respectively.

Question 3:-

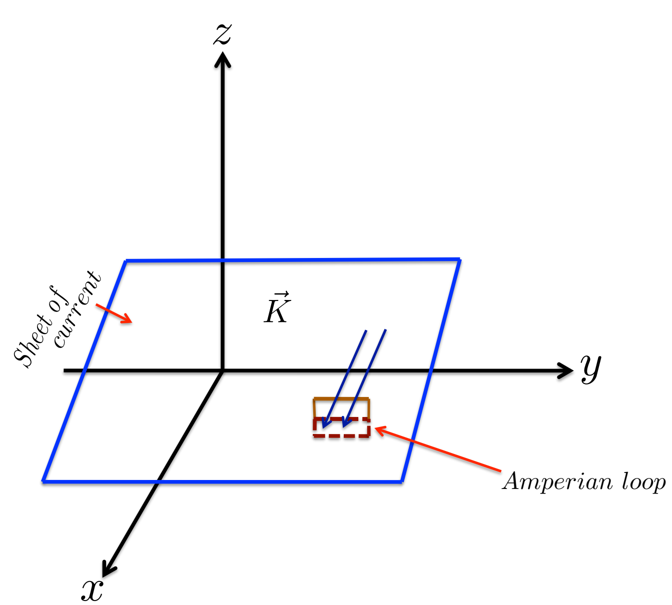
Find the vector potential above and below the current sheet, lies in the XY-plane, with uniform current density $\vec{K} = k\hat{x}$. Also verify the magneto-static boundary condition for the vector potential.

Solution:-

The magnetic field

$$\vec{B} = \begin{cases} +\frac{\mu_0}{2}k\hat{y}, & \text{for } z < 0. \\ -\frac{\mu_0}{2}k\hat{y}, & \text{for } z > 0. \end{cases}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$



$$\frac{\partial A_x}{\partial z} = \mp \frac{\mu_0 k}{2}$$

$$A_x = \mp \int \frac{\mu_0 k}{2} dz \quad + \quad c$$

$$= -\frac{\mu_0 k}{2} \; |z| \quad + \qquad c$$

$$\therefore \qquad \vec{A} = -\frac{\mu_0 k}{2}|z|\hat{x} \quad + c \qquad \text{is the desired solution.}$$

$$\tfrac{\partial}{\partial z}\vec{A}_{\text{above}} - \tfrac{\partial}{\partial z}\vec{A}_{\text{below}} = -\mu_0 k\hat{x} = -\mu_0\vec{K}$$

The magneto-static boundary conditions for the vector potential is verified.

Question 4:-

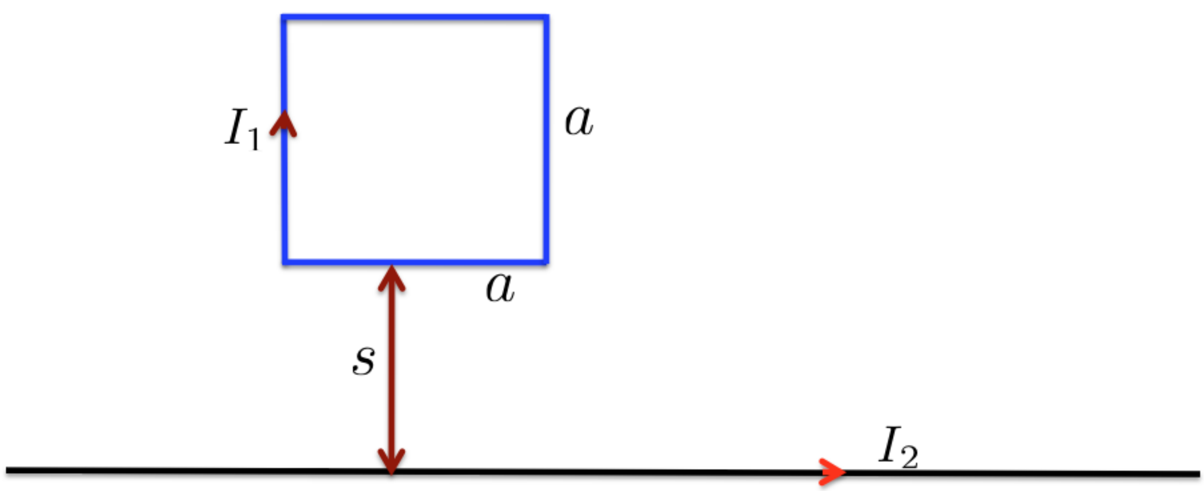
A loop of dimensions $a \times a$, carrying a steady current I_1, is placed at a distance 's' from an infinite wire carrying a steady current I_2, as shown in the figure.

(i) Calculate the total force exerted on the loop by the wire.

(ii) Assuming $a << s$, where the loop can be approximated as a point magnetic dipole. Using this approximation calculate:

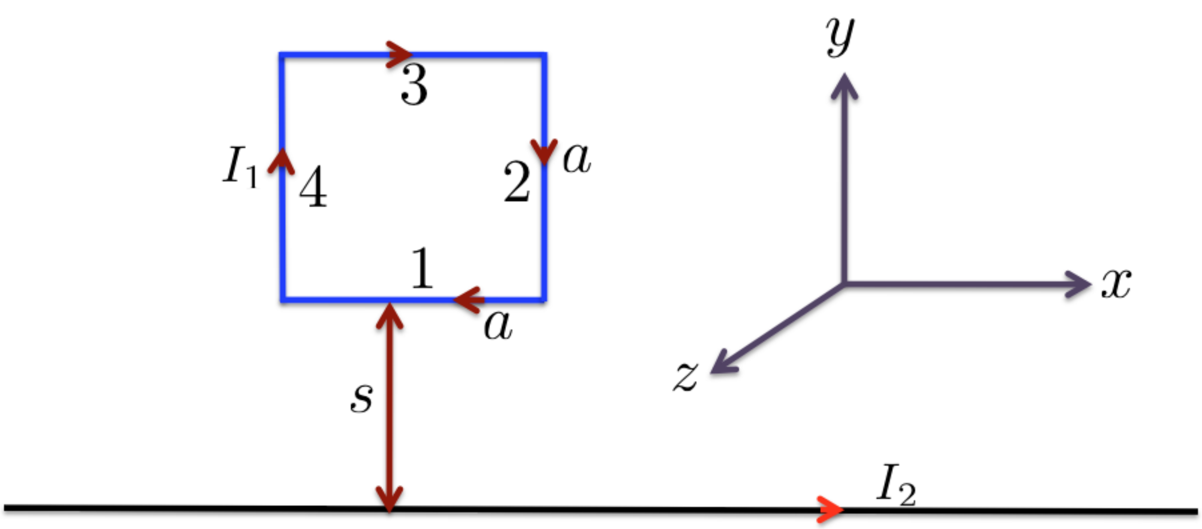
a) Force on the loop using the expression $\vec{\nabla}(\vec{m}.\vec{B})$.

b) Force on the wire due to the loop.



Solution:-

Force exerted on component (2) and (4) cancels each other.



$$\vec{F_1} = I_1 \int \ \vec{dl} \times \vec{B} \ \text{ and } \vec{B} = \tfrac{\mu_0 I_2}{2\pi s} \ \hat{z}$$

$$\vec{F_1} = \frac{\mu_0 I_1 I_2}{2\pi s} a \ \hat{y}$$

Similarly $$\vec{F_3} = -\frac{\mu_0 I_1 I_2 a}{2\pi(s+a)} \ \ \hat{y}$$

$\therefore \quad \vec{F}_{total} = \vec{F_1} + \vec{F_3} = \frac{\mu_0 I_1 I_2 ab}{2\pi s(s+a)} \ \ \hat{y}$

(ii)-(a)

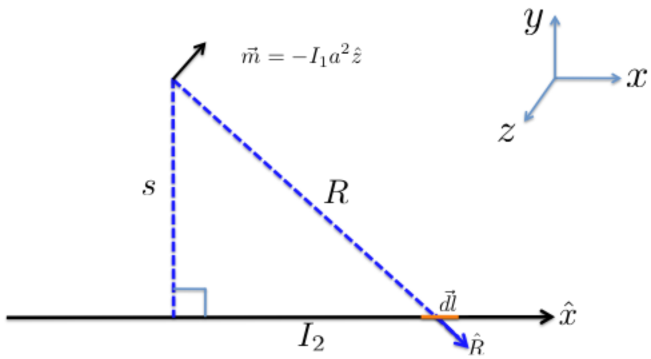
$$\begin{aligned} \vec{m} &= I \int \vec{da} \\ &= -Ia^2 \ \ \hat{z} \end{aligned}$$

assuming the point dipole is at a distance of ‘s’ from the wire

$$\begin{aligned} \vec{B} &= \frac{\mu_0 I_2}{2\pi s} \quad \hat{z} \\ \vec{m} \cdot \vec{B} &= -\frac{\mu_0 I_1 I_2 a^2}{2\pi s} \end{aligned}$$

$$\vec{F}_{\text{dip}} = \vec{\nabla}(\vec{m} \cdot \vec{B}) = \tfrac{\mu_0 I_1 I_2 a^2}{2\pi s^2} \ \hat{y} \qquad (2)$$

(ii)-(b)



$$\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{R^3}\left(3(\vec{m}\cdot\hat{R})\hat{R} - \vec{m}\right)$$

$\vec{m}\cdot\hat{R} = 0$ since $\vec{m}\perp\vec{R}$.

$\therefore \quad \vec{B}_{dip} = -\frac{\mu_0}{4\pi}\frac{1}{R^3}\vec{m}$ and $\overrightarrow{dl} = dx\ \hat{x}$

$$\begin{aligned}
\vec{F}_{wire} &= I_2 \int \overrightarrow{dl} \times \vec{B}_{dip} \\
&= I_2 \left(-\frac{\mu_0}{4\pi}\right)\left(-I_1 a^2\right) \int\limits_{-\infty}^{\infty} dx\,(\hat{x}\times\hat{z})\,\frac{1}{R^3} \\
&= \left(\frac{\mu_0 I_1 I_2 a^2}{4\pi}\right)(-\hat{y})\left[2\int\limits_0^{\infty} dx \frac{1}{(s^2+x^2)^{\frac{3}{2}}}\right] \\
&= \left(\frac{\mu_0 I_1 I_2 a^2}{4\pi}\right)(-\hat{y})\left[\frac{2}{s^3}\int\limits_0^{\infty} \frac{dx}{\left(1+\left(\frac{x}{s}\right)^2\right)^{\frac{3}{2}}}\right]
\end{aligned}$$

put $\frac{x}{s} = t \quad \Rightarrow \quad dx = s dt.$

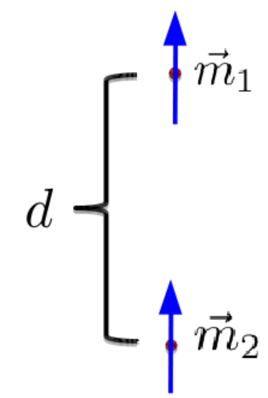
$\therefore \quad \vec{F}_{\text{wire}} = \left(\frac{\mu_0 I_1 I_2 a^2}{4\pi}\right)(-\hat{y})\left[\frac{2}{s^3}\int\limits_0^{\infty}\frac{s\ dt}{(1+t^2)^{\frac{3}{2}}}\right]$

As $\int\limits_0^{\infty}\frac{dt}{(1+t^2)^{\frac{3}{2}}} = 1, \quad \therefore \quad \vec{F}_{\text{wire}} = -\left(\frac{\mu_0 I_1 I_2 a^2}{2\pi s^2}\right)(\hat{y})$ (3)

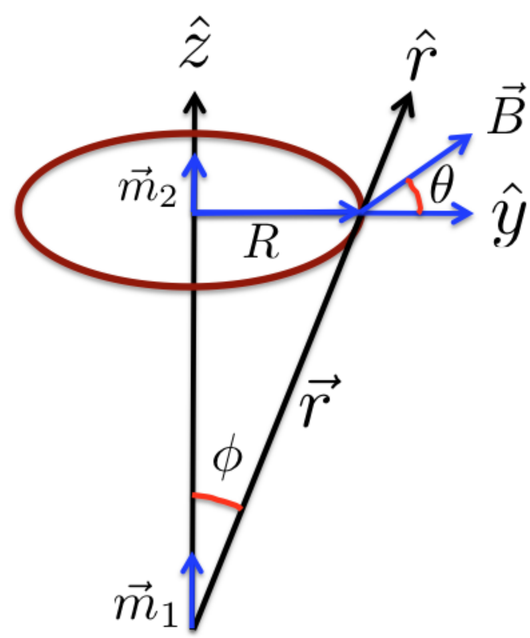
From equation (2) and (3) it can be noted that $\vec{F}_{\text{dip}} = -\vec{F}_{\text{wire}}$ as expected from Newton's 3rd law.

Question 5:-

Find the force of attraction between two magnetic dipoles, m_1 and m_2, oriented as shown in figure, a distance 'd' apart.



Solution:-



The force on $\vec{m}_2$ is $F = 2\pi IRB\ cos\theta$. But $\vec{B} = \frac{\mu_0}{4\pi}\frac{\left[3\left(\vec{m}_1\cdot\hat{r}\right)\hat{r}-\vec{m}_1\right]}{r^3}$ and $B\ cos\theta = \vec{B}\cdot\hat{y}$.
So,

$$B\ \cos\theta = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\vec{m}_1\cdot\hat{r})(\hat{r}\cdot\hat{y}) - (\vec{m}_1.\hat{y})\right]$$

but $\vec{m}_1.\hat{y} = 0$ and $\hat{r}\cdot\hat{y} = sin\phi$, while $\vec{m}_1.\hat{r} = m_1 cos\phi$
$\therefore \quad B\ cos\theta = \frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\ sin\phi\ cos\phi$ and $\therefore \quad F = 2\pi IR\frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\ sin\phi\ cos\phi$
Now $sin\phi = \frac{R}{r}$, $cos\phi = \frac{\sqrt{r^2-R^2}}{r}$ $\quad\therefore\quad F = \frac{3\mu_0}{2}m_1 IR^2\frac{\sqrt{r^2-R^2}}{r^5}$
But $IR^2\pi = m_2$, So, $F = \frac{3\mu_0}{2\pi}m_1 m_2\frac{\sqrt{r^2-R^2}}{r^5}$
For a dipole, $R << r$, and moreover in this case $r \approx d$

$$\therefore \qquad \vec{F} = -\frac{3\mu_0}{2\pi}\frac{m_1 m_2}{d^4}\ \hat{y}$$