Department of Mathematics, IIT Madras
MA1020 Series & Matrices
Assignment-1-Sol **Series**

1. Show the following:

(a) $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0.$ (b) $\lim\limits_{n\to\infty} n^{1/n} = 1.$ (c) $\lim\limits_{n\to\infty} x^n = 0$ for $|x| < 1.$

(d) $\lim\limits_{n\to\infty} \dfrac{n^p}{x^n} = 0$ for $x > 1.$ (e) $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$ (f) $\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$

(a) $\ln x$ is defined on $[1, \infty)$. Using L' Hospital's rule, $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} = \lim\limits_{x\to\infty} \dfrac{1}{x} = 0.$
Therefore, $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} = 0.$
(b) $\lim n^{1/n} = \lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} e^{\frac{1}{x}\ln x} = e^{\lim_{x\to\infty} \frac{\ln x}{x}} = e^0 = 1.$
Here, we have used continuity of e^x.
(c) Write $|x| = \dfrac{1}{1+r}$ for some $r > 0$. By the Binomial theorem, $(1+r)^n \geq 1 + nr > nr$. So,

$$0 < |x|^n = (1+r)^n < \frac{1}{nr}.$$

By Sandwich theorem, $\lim |x|^n = 0$. Now, $-|x|^n \leq x^n \leq |x|^n$. Again, by Sandwich theorem, $\lim x^n = 0$.
(d) Let $x > 1$. We know that $\lim\limits_{t\to\infty} \dfrac{t^p}{x^T} = 0$ for $p \in \mathbb{N}$. Therefore, $\lim \dfrac{n^p}{x^n} = 0.$
If $m < p < m + 1$ for an $m \in \mathbb{N}$, then $n^p < n^{m+1}$. Use Sandwich theorem to get the limit. If $p < 1$, then similarly, use $n^p < n$.

Analogously, show that the limits in (e) and (f) hold.

2. Prove the following:

   (a) It is not possible that a series converges to a real number ℓ and also diverges to $-\infty$.
   (b) It is not possible that a series diverges to ∞ and also to $-\infty$.

(a) Suupose $\sum a_j$ converges to ℓ and also diverges to $-\infty$. Then we have natural numbers k, m such that for every $n \geq k$, $\ell - 1 < \sum_{j=1}^n a_j < \ell| + 1$. And also for all $n \geq m$, $\sum_{j=1}^n a_j < \ell - 2$. Choose $M = \max\{k, m\}$. Then both inequalities hold for $n = M$. But this is not possible.

(b) Suppose $\sum a_j$ diverges to both ∞ and to $-\infty$. Then we have natural number s k, m such that for each $n \geq k$, $\sum_{j=1}^n a_j > 1$ and for each $n \geq m$, $\sum_{j=1}^n a_j < -1$. Choose $M = \max\{k, m\}$. Then both the inequalirties hold for $n = M$. But this is impossible.

3. Prove the following:

   (a) If both the series $\sum a_n$ and $\sum b_n$ converge, then the series $\sum(a_n + b_n)$, $\sum(a_n - b_n)$ and $\sum ka_n$ converge; where k is any real number.
   (b) If $\sum a_n$ converges and $\sum b_n$ diverges to $\pm\infty$, then $\sum(a_n+b_n)$ diverges to $\pm\infty$, and $\sum(a_n-b_n)$ diverges to $\mp\infty$.
   (c) If $\sum a_n$ diverges to $\pm\infty$, and $k > 0$, then $\sum ka_n$ diverges to $\pm\infty$.
   (d) If $\sum a_n$ diverges to $\pm\infty$, and $k < 0$, then $\sum ka_n$ diverges to $\mp\infty$.

(a) Suppose $\sum a_n$ converges to ℓ and $\sum b_n$ converges to s. Let $\epsilon > 0$. Then we have natural numbers k, m such that for all $n \ge k$, $|\sum_{j=1}^n a_j - \ell| < \epsilon/2$; and for all $n \ge m$, $|\sum_{j=1}^n b_j - s| < \epsilon/2$. Choose $M = \max\{k, m\}$. Then for all $n \ge M$, both the inequalities hold. So, we obtain

$$|\sum_{j=1}^n (a_j + b_j) - (\ell + s)| \le |\sum_{j=1}^n a_j - \ell| + |\sum_{j=1}^n b_j - s| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Similarly, the other two are proved.

(b) Suppose $\sum a_n$ converges to ℓ and $\sum b_n$ diverges to ∞. Let $r > 0$. Then, we have natural numbers k, m such that for all $n \ge k$, $\ell - 1 < \sum_{j=1}^n a_j < \ell| + 1$; and for all $n \ge m$, $\sum_{j=1}^n b_j > r + |\ell| + 1$. Choose $M = \max\{k, m\}$. Then all the three inequalities hold for $n \ge M$. But then for all $n \ge M$,

$$\ell - 1 < \sum_{j=1}^n a_j, \quad r + |\ell| + 1 < \sum_{j=1}^n b_j.$$

That is, for all $n \ge M$, $r \le \ell - 1 + r + |\ell| + 1 < \sum_{j=1}^n (a_j + b_j)$.
Similarly, other cases are proved.

(c) Suppose $\sum a_n$ diverges to $\pm\infty$, and $k > 0$. Let $r \in \mathbb{R}$. We have $m \in \mathbb{N}$ such that for all $n \ge m$, $\sum_{j=1}^n a_j > r/k$. Then for all such n, $\sum_{j=1}^n (ka_j) > r$.
Similarly other cases are proved.

(d) Suppose $\sum a_n$ diverges to ∞, and $k < 0$. Let $r \in \mathbb{R}$. We have $m \in \mathbb{N}$ such that for all $n \ge m$, $\sum_{j=1}^n a_j > -r/k$. Then for all such n, $\sum_{j=1}^n (ka_j) < r$, since $k < 0$.
Similarly other cases are proved.

4. Give examples for the following:

   (a) $\sum a_n$ and $\sum b_n$ both diverge, but $\sum (a_n + b_n)$ converges to a nonzero number.

   (b) $\sum a_n$ and $\sum b_n$ both diverge, and $\sum (a_n + b_n)$ diverges to ∞.

   (c) $\sum a_n$ and $\sum b_n$ both diverge, and $\sum (a_n + b_n)$ diverges to $-\infty$.

   (a) $1 + 1 + 1 + \cdots$ diverges; $(-1) + (-1) + (-1) + \cdots$ also diverges.
   But $(1 + (-1)) + (1 + (-1)) + \cdots = 0 + 0 + \cdots$ converges to 0.

   (b) $1 + 2 + 3 + 4 + \cdots$ diverges; $-1 - 1 - 1 - 1 - \cdots$ also diverges.
   And $(1 - 1) + (2 - 1) + (3 - 1) + \cdots = 0 + 1 + 2 + 3 + \cdots$ diverges to ∞.

   (c) $-1 - 2 - 3 - 4 - \cdots$ diverges; $1 + 1 + 1 + 1 + \cdots$ also diverges.
   And $(-1 + 1) + (-2 + 1) + (-3 + 1) + \cdots = 0 - 1 - 2 - 3 - \cdots$ diverges to $-\infty$.

5. Show that the sequence $1,\ 1.1,\ 1.1011,\ 1.10110111, \ldots$ converges.

6. Compute the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n - 4}{6^n}$.

$$\sum_{n=1}^{\infty} \frac{3^n - 4}{6^n} = \sum_{n=1}^{\infty} \frac{1/2}{2^n} - \frac{4}{6} \sum_{n=1}^{\infty} \frac{1}{6^n} = \frac{1/2}{1 - 1/2} - \frac{4/6}{1 - 1/6} = 1 - \frac{4}{5} = \frac{1}{5}.$$

7. Determine whether the following series converge:

   (a) $\displaystyle\sum \frac{1}{n(n+1)}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{-n}{3n+1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1 + n \ln n}{1 + n^2}$

(a) Since $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$, we have

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} = 1 + \sum_{k=2}^{n} \frac{1}{k} - \sum_{k=1}^{n-1} \frac{1}{k+1} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

Since $1/(n+1) \to 0$ as $n \to \infty$, the series converges to 1. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

(b) It diverges because $\lim\limits_{n\to\infty} \dfrac{-n}{3n+1} = -\dfrac{1}{3} \neq 0.$

(c) Take $a_n = \dfrac{\ln n}{n^{3/2}}$ and $b_n = \dfrac{1}{n^{5/4}}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} = \lim_{n\to\infty} \frac{1/n}{(1/4)n^{-3/4}} = \lim_{n\to\infty} \frac{4}{n^{1/4}} = 0.$$

Since $\sum b_n$ converges, by the Limit comparison test, $\sum a_n$ converges.

(d) Take $a_n = \dfrac{1 + n\ln n}{1+n^2}$ and $b_n = \dfrac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n + n^2 \ln n}{1+n^2} = \infty.$$

As $\sum b_n$ diverges to ∞, by the Limit comparison test, $\sum a_n$ diverges to ∞.

8. Test for convergence the series $\dfrac{1}{3} + \left(\dfrac{2}{3}\right)^2 + \left(\dfrac{3}{7}\right)^3 + \cdots + \left(\dfrac{n}{2n+1}\right)^n + \cdots$.

Using Cauchy root test, $\lim\limits_{n\to\infty} (a_n)^{1/n} = \lim\limits_{n\to\infty} \dfrac{n}{2n+1} = \dfrac{1}{2} < 1$. Therefore, the series converges.

9. Is the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ convergent?

$$\int_a^b \frac{1}{1+x^2}\,dx = \tan^{-1} b - \tan^{-1} a.$$

So,

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} (-\tan^{-1} a) = -\Big(-\frac{\pi}{2}\Big) = \frac{\pi}{2}.$$

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} (\tan^{-1} b) = \frac{\pi}{2}.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx$$

is convergent and its value is $\pi/2 + \pi/2 = \pi$.

10. Is the area under the curve $y = (\ln x)/x^2$ for $1 \le x < \infty$ finite?

The question is whether $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ converges?

Let $b > 1$. Integrating by parts,

$$\int_1^b \frac{\ln x}{x^2}\,dx = \left[\ln x\Big(-\frac{1}{x}\Big)\right]_1^b - \int_1^b \Big(\frac{-1}{x}\Big)\frac{1}{x}\,dx = -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

$$\lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] = 1.$$

Therefore, the improper integral $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$ converges to 1. That is, the required area is finite and it is equal to 1.

11. Evaluate (a) $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$ (b) $\displaystyle\int_0^3 \frac{dx}{x-1}$

(a) The integrand is not defined at $x = 1$. We consider it as an improper integral.

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to1-}\int_0^b \frac{dx}{(x-1)^{2/3}} + \lim_{a\to1+}\int_a^3 \frac{dx}{(x-1)^{2/3}}.$$

$$\lim_{b\to1-}\int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to1-} 3(x-1)^{1/3}\Big|_0^b = \lim_{b\to1-}(3(b-1)^{1/3} - 3(-1)^{1/3}) = 3.$$

$$\lim_{a\to1+}\int_a^3 \frac{dx}{(x-1)^{2/3}} = \lim_{a\to1+} 3(x-1)^{1/3}\Big|_a^3 = \lim_{a\to1+}(3(3-1)^{1/3} - 3(a-1)^{1/3}) = 3(2)^{1/3}.$$

Hence $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3(1+2^{1/3})$.

Had we not noticed that the integrand has discontinuity in the interior, we would have ended up at a wrong computation such as

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3(x-1)^{1/3}\Big|_0^3 = 3(2^{1/3} - (-1)^{1/3}),$$

even though the answer happens to be correct here. See the next problem.

(b) Overlooking the point $x = 1$, where the integrand is not defined, we may compute

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big]_0^3 = \ln 2 - \ln 1 = \ln 2.$$

However, it is an improper integral and its value, if exists, must be computed as follows:

$$\int_0^3 \frac{dx}{x-1} = \lim_{b\to1-}\int_0^b \frac{dx}{x-1} + \lim_{a\to1+}\int_a^3 \frac{dx}{x-1}.$$

The integral converges provided both the limits are finite. However,

$$\lim_{b\to1-}\int_0^b \frac{dx}{x-1} = \lim_{b\to1-}\big(\ln|b-1| - \ln|-1|\big) = \lim_{b\to1-}\ln(1-b) = -\infty.$$

Therefore, $\displaystyle\int_0^3 \frac{dx}{x-1}$ does not converge.

12. Show that $\displaystyle\int_1^\infty \frac{\sin x}{x^p}\,dx$ converges for all $p > 0$.

For $p > 1$ and $x \ge 1$, $\left|\dfrac{\sin x}{x^p}\right| \le \dfrac{1}{x^p}$. Since $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converges, $\displaystyle\int_1^\infty \left|\frac{\sin x}{x^p}\right| dx$ converges. Therefore,

$\displaystyle\int_1^\infty \frac{\sin x}{x^p}\,dx$ converges.

For $0 < p \le 1$, use integration by parts: $\displaystyle\int_1^b \frac{\sin x}{x^p}\,dx = -\frac{\cos b}{b^p} + \frac{\cos 1}{1^p} + p\int_1^b \frac{\cos x}{x^{p+1}}\,dx.$
Taking the limit as $b \to \infty$, we see that the first term goes to 0; the second term is already a real number, the third term, an improper integral converges as in the case for $p > 1$ above. Therefore, the given improper integral also converges in this case.

13. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x^p}\,dx$ converges for $0 < p \le 1$.

    For $p = 1$, the integral $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ is not an improper integral. Since $\dfrac{\sin x}{x}$ with its value at 0 as 1 is continuous on $[0, 1]$, this integral exists.

    For $0 < p < 1$ and $0 < x \le 1$, since $\dfrac{\sin x}{x^p} \le \dfrac{1}{x^p}$ and $\displaystyle\int_0^1 \frac{dx}{x^p}$ converges due to last problem; the improper integral $\displaystyle\int_0^1 \frac{\sin x}{x^p}\,dx$ converges.

    Next, the improper integral $\displaystyle\int_1^\infty \frac{\sin x}{x}\,dx$ converges due to last problem.

    Hence $\displaystyle\int_0^\infty \frac{\sin x}{x^p}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$ converges.

14. Show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^\alpha}$ converges for $\alpha > 1$ and diverges to ∞ for $\alpha \le 1$.
    The function $f(x) = \frac{1}{x(\ln x)^\alpha}$ is continuous, positive, and decreasing on $[2, \infty)$. By the integral test, it converges when $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^\alpha}\,dx$ converges. Evaluating the integral, we have

    $$\int_2^\infty \frac{1}{x(\ln x)^\alpha}\,dx = \int_{\ln 2}^\infty \frac{1}{t^\alpha}\,dt.$$

    We conclude that the series converges for $\alpha > 1$ and diverges to ∞ for $\alpha \le 1$.

15. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{4^n (n!)^2}{(2n)!}$ converge?
    The tests either give no information or are difficult to apply. However,

    $$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}((n+1)!)^2}{(2(n+1))!}\,\frac{(2n)!}{4^n (n!)^2} = \frac{2(n+1)}{2n+1} > 1.$$

    Since $a_1 = 2$, we see that each $a_n > 2$. That is, $\lim a_n \ge 2 \ne 0$. Therefore, the series diverges. Since it is a series of positive terms, it diverges to ∞.

16. Does the series $1 - \dfrac{1}{4} - \dfrac{1}{16} + \dfrac{1}{9} + \dfrac{1}{25} + \dfrac{1}{49} - \cdots$ converge?
    Here, the series has been made up from the terms $1/n^2$ by taking first one term, next two negative terms of squares of next even numbers, then three positive terms which are squares of next three odd numbers, and so on. This is a rearrangement of the series

    $$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots$$

    which is absolutely convergent (since $\sum(1/n^2)$ is convergent). Therefore, the given series is convergent and its sum is the same as that of the alternating series $\sum(-1)^{n+1}(1/n^2)$.