Problems & solutions of sheet-2

Department of Physics
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PH1020 Physics II   Problem set 2

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1. Two equal and opposite charges ($+q$ and $-q$) separated by a distance d constitute a dipole. Assume the charges to be situated on the z-axis so that its coordinates are $(0, 0, +d/2)$ and $(0, 0, -d/2)$.
   (i) Calculate the electrostatic potential at an arbitrary point $P(x, y, z)$ and show that at large distance from the origin the potential is given by $V = (p\cos\theta)/4\pi\varepsilon_0 r^2$, where $p = qd$ dipole moment and θ is the polar angle.
   (ii) Calculate the corresponding electric field in Cartesian coordinates. Plot the electric field and equipotential surfaces.

2. A sphere of radius R_1 has uniform charge density ρ within its volume, except for a small spherical hollow region of radius R_2 located at a distance a from the center (center to center).
   (i) Find the electric field at the center of the hollow region. (ii) Find the potential at the same point.

3. The total charge within a sphere of radius r is given by $q\frac{r^2}{a^2}e^{-r/a}$, where a is real constant of appropriate dimensions. Determine: (i) the electric potential (ii) the charge density.

4. Find the electric potential on the axis of a charged ring of radius a lying in the xoy plane when the charge density on the ring varies as $\rho = \alpha(1 + \sin\varphi)\delta(r - a)\delta(z)$, where α is real constant of suitable dimensions.

5. Consider a situation with cylindrical symmetry, where the potential is independent of the coordinates φ and z i.e, $\Phi = \Phi(\rho)$. Show that the most general solution to Laplace's equation in this case takes the form $\Phi = A\log\rho + B$ for suitable constants A and B. Determine A and B using the following boundary conditions: $\Phi(\rho)\big|_{\rho=a} = V_0$ and $\Phi(\rho)\big|_{\rho=b} = 0$ for $a > b > 0$. Take the region of interest to be $b \leq \rho \leq a$. Compute the electric field for this electrostatic potential. Plot lines of equipotential as well as electrical lines of force in the plane $z = 0$.

6. Consider the situation where two (infinite) metal plates are parallel to each other and separated by a distance d. One plate is grounded and the other is kept at V_0. Choose the direction of the separation to be the y-axis. Solve Laplace's equation in the region between the two plates subject to the given boundary conditions. Compute the electric field for this electrostatic potential. Plot lines of equipotential as well as electrical lines of force.

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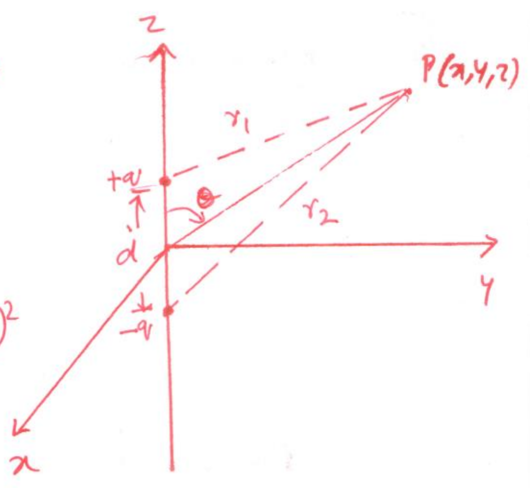
SOLUTIONS OF PROBLEM SHEET 2



Sol.1 The electrostatic potential is given by

(i) $$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r_1} - \frac{q}{r_2}\right]$$

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{\sqrt{x^2+y^2+(z-d/2)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+\frac{d}{2})^2}}$$

$$\simeq \frac{1}{r}\left[\left(1 - \frac{zd}{r^2}\right)^{-1/2} - \left(1+\frac{zd}{r^2}\right)^{-1/2}\right]$$

$$\therefore\ V(x,y,z) \simeq \frac{q}{4\pi\epsilon_0}\frac{zd}{r^3} = \frac{pz}{4\pi\epsilon_0 r^3}$$ $p = qd$ dipole moment.

$$V(r,\theta,\phi) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$ [In spherical polar coordinates]

(ii) $$V(x,y,z) = \frac{p}{4\pi\epsilon_0}\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

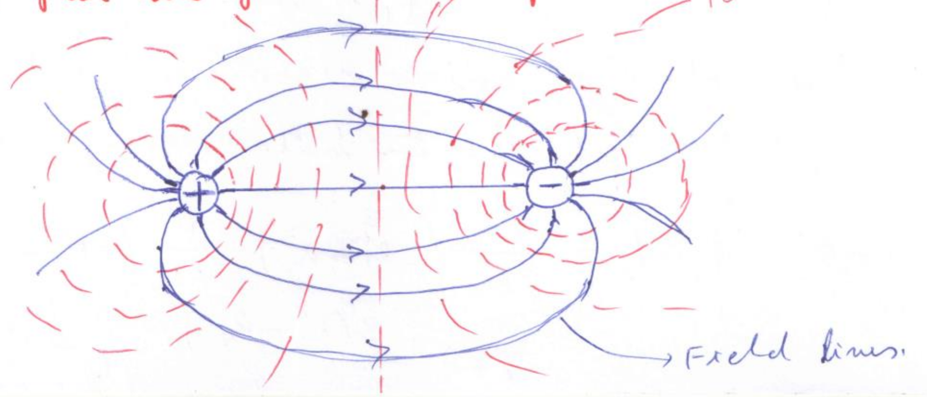
$$E_x = -\frac{\partial V}{\partial x} = \frac{p}{4\pi\epsilon_0}\frac{3zx}{r^5}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{p}{4\pi\epsilon_0}\frac{3zy}{r^5}$$

$$E_z = -\frac{\partial V}{\partial z} = \frac{p}{4\pi\epsilon_0}\left[\frac{3z^2}{r^5} - \frac{1}{r^3}\right]$$

or $$E_z = \frac{p}{4\pi\epsilon_0}\frac{1}{r^5}\left(3z^2 - r^2\right) = \frac{p}{4\pi\epsilon_0}\frac{(3\cos^2\theta - 1)}{r^3}$$

We plot the field in y-z plane



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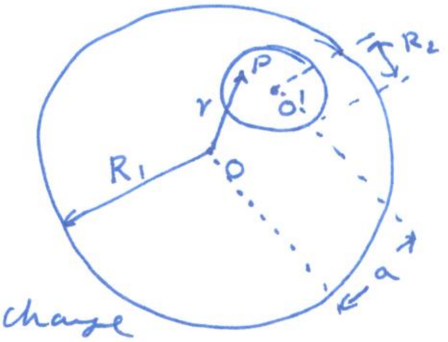
Q2 soln

Consider an arbitrary point P of the hollow region

$OP = r$, $O'P = r'$, $OO' = a$, $r' = r - a$



If there were no hollow region inside, the E at P would be.

$$E_1 = \frac{\rho}{3\epsilon_0}\vec{r}$$

If only spherical hollow region has charge density ρ the electric field at P would be

$$E_2 = \frac{\rho}{3\epsilon_0} r'$$

superposition theorem gives us $E = E_1 - E_2 = \frac{\rho}{3\epsilon_0}\vec{a}$

The field inside the hollow region is uniform which includes center of the hollow..

The electric field inside and outside sphere is

$$E(r) = \begin{cases} \frac{\rho \bar{r}}{3\epsilon_0} & r < R \\ \frac{\rho R^3 \hat{r}}{3\epsilon_0 r^2} & r > R \end{cases}$$

This will given potential at an arbitrary point inside sphere

$$\phi = \left(\int_r^R + \int_R^\infty\right)\bar{E}\cdot d\bar{r} = \frac{\rho}{6\epsilon_0}\left(3R^2 - r^2\right). \longrightarrow \circledast$$

r is the distance between point in question and center of sphere.

Let ϕ_1 be the potential at the center O' of the hollow region. If charge distribution is replaced by a small sphere of uniform charge density ρ of radius R_2 in the hollow region, Assume potential at O' be ϕ_2. Using eqn $\circledast$ and superposition theorem, we get.

$$\phi_{O'} = \phi_1 - \phi_2 = \frac{\rho}{6\epsilon_0}\left(3R_1^2 - a^2\right) - \frac{\rho}{6\epsilon_0}\left(3R_2^2 - 0\right)$$

$$= \frac{\rho}{6\epsilon_0}\left[3\left(R_1^2 - R_2^2\right) - a^2\right]$$

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③ Gauss theorem gives us.

(i) $$\vec{E}\, 4\pi r^2 = \frac{\alpha}{\epsilon_0}\, \frac{r^2}{a^2}\, e^{-r/a}$$

$$\therefore \vec{E}(r) = \frac{\alpha}{4\pi\epsilon_0 a^2}\, e^{-r/a}$$

$\therefore$ potential

$$-\int_{\infty}^{r} \vec{E}(\vec{r}')\, dr' = -\frac{\alpha}{4\pi\epsilon_0 a^2} \int_{\infty}^{r} e^{-r'/a}\, dr'$$

$$= \frac{\alpha}{4\pi\epsilon_0 a}\, e^{-r/a}$$

(ii) $$\rho = \frac{\epsilon_0}{r^2}\frac{d}{dr}(r^2 E_r) = \frac{\alpha}{4\pi a^2}\, e^{-r/a}\left(\frac{2}{r} - \frac{1}{a}\right)$$

④ The potential at a point z on the Z-axis is

$$\phi(z) = \frac{1}{4\pi\epsilon_0} \int \frac{\alpha(1+\sin\phi')\,\delta(r'-a)\,\delta(z')\, r'dr'dz'd\phi'}{\sqrt{z^2+a^2}}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{a\alpha(1+\sin\phi')\,d\phi'}{\sqrt{z^2+a^2}}$$

$$= \frac{\alpha a}{4\pi\epsilon_0}\, \frac{2\pi}{\sqrt{z^2+a^2}} = \frac{\alpha a}{2\epsilon_0\sqrt{z^2+a^2}}$$

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(5) The Laplace's eqn in cylindrical coordinates

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\phi}{d\rho}\right) = 0$$

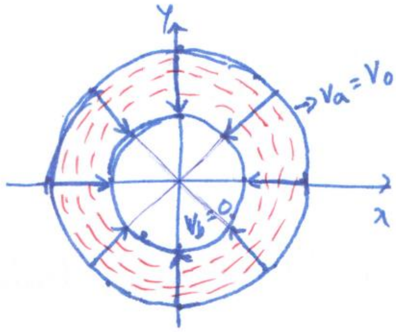
$$\Rightarrow \quad \phi(\rho) = A\log\rho + B$$

Applying given boundary conditions we get

$$A = \frac{V_0}{(\log a - \log b)}, \quad B = \frac{V_0 \log b}{(\log b - \log a)}$$

$$\Rightarrow \quad \phi(\rho) = V_0 \frac{\log(\rho/b)}{\log(a/b)}$$

$$\therefore \quad \vec{E} = -\frac{V_0}{\rho\log(a/b)}\,\hat{e}_\rho$$



(6) The Laplace's eqn

$$\frac{d^2\phi}{dy^2} = 0 \quad \Rightarrow \quad \phi(y) = ay + b$$

Applying given boundary conditions.

$$\phi(y) = V_0 \frac{y}{d}$$

The ~~corresponding~~ corresponding electric field.

$$\vec{E}(y) = -\nabla\phi = -\frac{V_0}{d}\,\hat{e}_y$$