Department of Mathematics, IIT Madras
MA1020 Series & Matrices
Assignment-4 Linear Systems & Eigenvalue Problem

1. Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{cccccl} x_1 & +x_2 & +x_3 & +x_4 & -3x_5 & = 6 \\ 2x_1 & +3x_2 & +x_3 & +4x_4 & -9x_5 & = 17 \\ x_1 & +x_2 & +x_3 & +2x_4 & -5x_5 & = 8 \\ 2x_1 & +2x_2 & +2x_3 & +3x_4 & -8x_5 & = 14 \end{array}$$

2. Let $A \in \mathbb{F}^{m\times n}$ have columns $A_1, \ldots, A_n$. Let $b \in \mathbb{F}^m$. Show the following:
   (a) The equation $Ax = 0$ has a non-zero solution iff $A_1, \ldots, A_n$ are linearly dependent.
   (b) The equation $Ax = b$ has at least one solution iff $b \in \text{span}\{A_1, \ldots, A_n\}$.
   (c) The equation $Ax = b$ has at most one solution iff $A_1, \ldots, A_n$ are linearly independent.
   (d) The equation $Ax = b$ has a unique solution iff $\text{rank}\, A = \text{rank}[A|b] =$ number of unknowns.

3. Check if the system is consistent. If so, determine the solution set.
   (a) $x_1 - x_2 + 2x_3 - 3x_4 = 7,\ 4x_1 + 3x_3 + x_4 = 9,\ 2x_1 - 5x_2 + x_3 = -2,\ 3x_1 - 2x_2 - 2x_3 + 10x_4 = -12.$
   (b) $x_1 - x_2 + 2x_3 - 3x_4 = 7, 4x_1 + 3x_3 + x_4 = 9, 2x_1 - 5x_2 + x_3 = -2, 3x_1 - 2x_2 - 2x_3 + 10x_4 = -14.$

4. Using Gauss-Jordan elimination determine the values of $k \in \mathbb{R}$ so that the system of linear equations

$$x + y - z = 1,\ 2x + 3y + kz = 3,\ x + ky + 3z = 2$$

   has (a) no solution, (b) infinitely many solutions, (c) exactly one solution.

5. Find the eigenvalues and the associated eigenvectors for the matrices given below.
   (a) $\begin{bmatrix} 3 & 10 \\ 8 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 13 & 2 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & -1 \\ 15 & 12 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 0 & 3 \\ -2 & 3 & 0 \\ 10 & 0 & 5 \end{bmatrix}$

6. Let $A \in \mathbb{C}^{n\times n}$ be invertible. Show that $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $1/\lambda$ is an eigenvalue of A^{-1}.

7. Let A be an $n \times n$ matrix and α be a scalar such that each row (or each column) sums to α. Show that α is an eigenvalue of A.

8. Give an example of an $n \times n$ matrix that cannot be diagonalized.

9. Find the matrix $A \in \mathbb{R}^{3\times 3}$ that satisfies the given condition. Diagonalize it if possible.
   (a) $A(a, b, c)^T = (a + b + c,\ a + b - c,\ a - b + c)^T$ for all $a, b, c \in \mathbb{R}$.
   (b) $Ae_1 = 0, \quad Ae_2 = e_1, \quad Ae_3 = e_2.$
   (c) $Ae_1 = e_2, \quad Ae_2 = e_3, \quad Ae_3 = 0.$
   (d) $Ae_1 = e_3, \quad Ae_2 = e_2, \quad Ae_3 = e_1.$

10. Which of the following matrices is/are diagonalizable? If one is diagonalizable, then diagonalize it.
   (a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.