EE 1101: SIGNALS AND SYSTEMS JAN-MAY 2019

Tutorial 0 Solutions

Solution 1

In polar form, a complex number $z = x + j\,y$ is represented as $z = re^{j\theta} = r(cos\theta + jsin\theta)$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$. However if the given complex number lies in the second or third quadrant, add π to the θ obtained using the above relation.

In all cases, $r = \sqrt{1+3} = 2$

(a) $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

Hence, $1 + j\sqrt{3} = 2e^{j\frac{\pi}{3}}$

(b) $\theta = \pi + \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \pi + (-\frac{\pi}{3}) = \frac{2\pi}{3}$

Hence, $-1 + j\sqrt{3} = 2e^{j\frac{2\pi}{3}}$

(c) $\theta = \pi + \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

Hence, $-1 - j\sqrt{3} = 2e^{j\frac{4\pi}{3}}$

(d) $\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$

Hence, $1 - j\sqrt{3} = 2e^{-j\frac{\pi}{3}}$

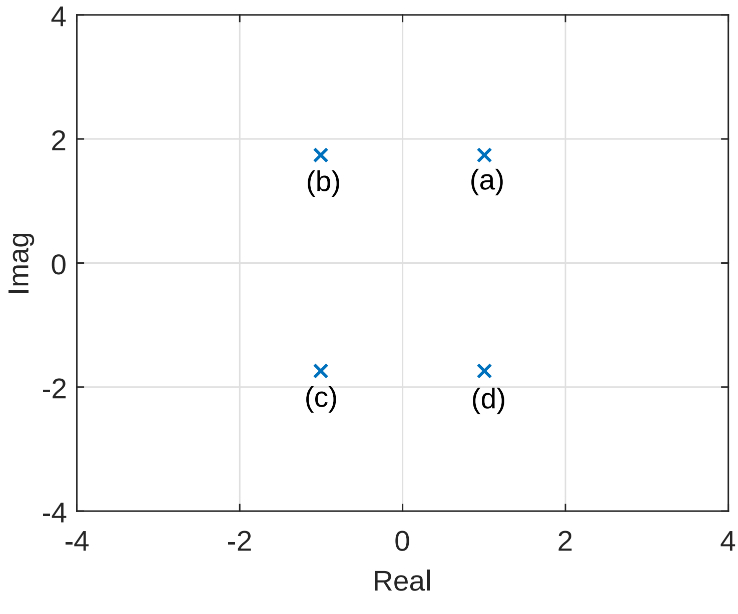


Figure 1: Plot showing the complex numbers in the complex plane

Solution 2

Complex numbers of the given form can be expanded as : $re^{j\theta} \equiv r[\cos(\theta) + j\sin(\theta)]$
Figure 2 shows the numbers plotted in the complex plane.

(a)

$$\begin{aligned} 2e^{j\frac{\pi}{6}} &= 2[\cos(\frac{\pi}{6}) + j\sin(\frac{\pi}{6})] \\ &= 2[\frac{\sqrt{3}}{2} + j\frac{1}{2}] \\ &= 3\sqrt{3} + j \end{aligned}$$

(b)

$$\begin{aligned} -4e^{j\frac{\pi}{3}} &= -4[\cos(\frac{\pi}{3}) + j\sin(\frac{\pi}{3})] \\ &= -2 - j2\sqrt{3} \end{aligned}$$

(c)

$$\begin{aligned} e^{j\frac{\pi}{2}} &= \cos(\frac{\pi}{2}) + j\sin(\frac{\pi}{2}) \\ &= j \end{aligned}$$

(d)

$$\begin{aligned} 3e^{j\frac{-\pi}{3}} &= 3[\cos(\frac{-\pi}{3}) + j\sin(\frac{-\pi}{3})] \\ &= \frac{3}{2} - j\frac{3\sqrt{3}}{2} \end{aligned}$$

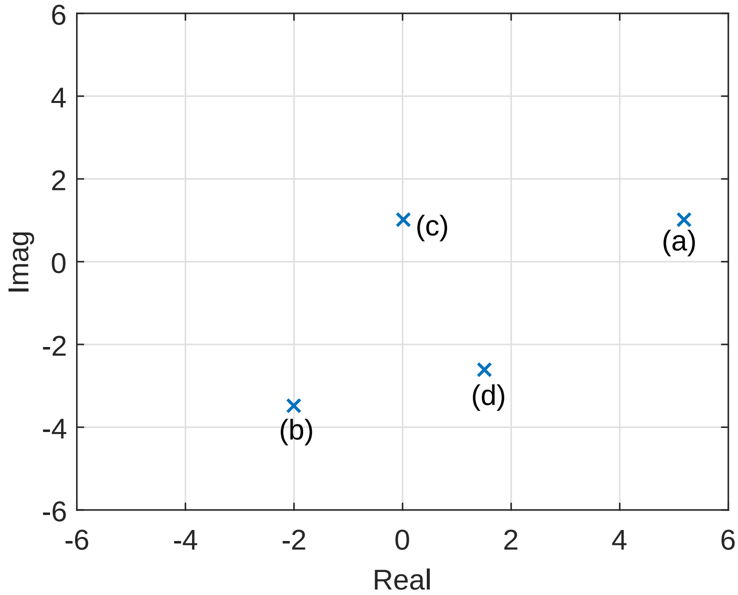


Figure 2: Plot showing the complex numbers in the complex plane

Solution 3

(a) Given $z_1 = -2 + j$ and $z_2 = 3 + j4$

$$\begin{aligned}
z_1 + z_2 &= (-2 + j) + (3 + j4) \\
&= 1 + j5 \\
z_1 z_2 &= (-2 + j)(3 + j4) \\
&= (2.236e^{j153.43^\circ}) \cdot (5e^{j53.13^\circ}) \\
&= 11.18e^{j206.56^\circ} \\
&= -10 - j5 \\
\frac{z_1}{z_2} &= \frac{-2 + j}{3 + j4} \\
&= \frac{2.236e^{j153.43^\circ}}{5e^{j53.13^\circ}} \\
&= 0.447e^{j100.3^\circ} \\
&= -0.08 + j0.44 \\
z_1^{\frac{1}{2}} &= (-2 + j)^{\frac{1}{2}} \\
&= (2.236e^{j153.43^\circ})^{\frac{1}{2}} \\
&= 1.495e^{j76.715^\circ}; 1.495e^{j256.715^\circ} \\
&= 0.34 + j1.45; -0.34 - j1.45 \\
|z_2|^2 &= |(3 + j4)|^2 \\
&= 9 + 16 \\
&= 25
\end{aligned}$$

(b) Given $z_1 = j + e^{\frac{\pi}{4}}$ and $z_2 = \cos j$

$$\begin{aligned}
z_1 &= j + e^{\frac{\pi}{4}} \\
&= 2.193 + j \\
z_2 &= \frac{e^{j \cdot j} + e^{-j \cdot j}}{2} \\
&= 1.543 + j0 \\
z_1 + z_2 &= (2.193 + j) + 1.543 \\
&= 3.736 + j \\
z_1 z_2 &= (2.193 + j) \cdot (1.543) \\
&= 3.384 + j1.543 \\
\frac{z_1}{z_2} &= \frac{2.193 + j}{1.543} \\
&= \frac{2.410e^{j24.51^\circ}}{1.543} \\
&= 1.562e^{j24.51^\circ} \\
&= 1.421 + j0.648 \\
z_1^{\frac{1}{2}} &= (2.193 + j)^{\frac{1}{2}} \\
&= (2.410e^{j24.51^\circ})^{\frac{1}{2}} \\
&= 1.552e^{j12.255^\circ}; 1.552e^{j192.255^\circ} \\
&= 1.517 + j0.329; -1.517 - j0.329 \\
|z_2|^2 &= |1.543|^2 \\
&= 2.381
\end{aligned}$$

Solution 4

Let $z = w - (1+2j)$

$$\begin{aligned}
z^5 &= \frac{32}{\sqrt{2}}(1+j) = 2^5(\frac{1+j}{\sqrt{2}}) = 2^5 e^{j\frac{\pi}{4}} = 2^5 e^{j\frac{\pi}{4}} e^{j2\pi k}, k = 0,1,2,3,4. \\
z^5 &= 2^5 e^{j\frac{\pi}{4}+j2\pi k} \\
z &= 2e^{j\frac{\pi}{20}+\frac{j2\pi k}{5}}; k = 0,1,2,3,4 \\
w &= 1+2j+2[\cos(\frac{\pi}{20}+\frac{2\pi k}{5}) + j\sin(\frac{\pi}{20}+\frac{2\pi k}{5})]
\end{aligned}$$

$$k=0 \Rightarrow w_0 = 1+2j+2[\cos(\frac{\pi}{20}+\frac{2\pi(0)}{5}) + j\sin(\frac{\pi}{20}+\frac{2\pi(0)}{5})] = 1+2\cos(\frac{\pi}{20}) + j(2+2\sin\frac{\pi}{20})$$

$$k=1 \Rightarrow w_1 = 1+2j+2[\cos(\frac{\pi}{20}+\frac{2\pi(1)}{5}) + j\sin(\frac{\pi}{20}+\frac{2\pi(1)}{5})] = 1+2\cos(\frac{9\pi}{20}) + j(2+2\sin\frac{9\pi}{20})$$

$$k=2 \Rightarrow w_2 = 1+2j+2[\cos(\frac{\pi}{20}+\frac{2\pi(2)}{5}) + j\sin(\frac{\pi}{20}+\frac{2\pi(2)}{5})] = 1+2\cos(\frac{17\pi}{20}) + j(2+2\sin\frac{17\pi}{20})$$

$$k=3 \Rightarrow w_3 = 1+2j+2[\cos(\frac{\pi}{20}+\frac{2\pi(3)}{5}) + j\sin(\frac{\pi}{20}+\frac{2\pi(3)}{5})] = 1+2\cos(\frac{25\pi}{20}) + j(2+2\sin\frac{25\pi}{20})$$

$$k=4 \Rightarrow w_4 = 1+2j+2[\cos(\frac{\pi}{20}+\frac{2\pi(4)}{5}) + j\sin(\frac{\pi}{20}+\frac{2\pi(4)}{5})] = 1+2\cos(\frac{33\pi}{20}) + j(2+2\sin\frac{33\pi}{20})$$

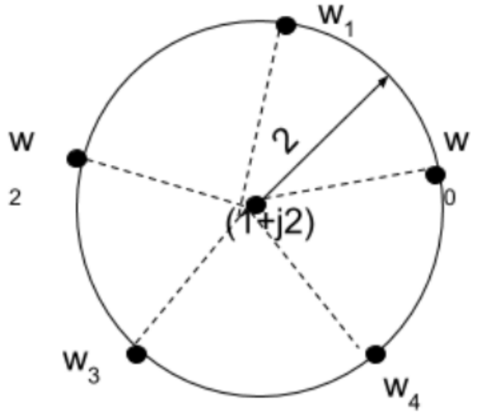


Figure 3: Plot showing the distinct solutions in the complex plane

Solution 5

(a)

$$F(\omega) = \frac{1 + j2\omega}{3 + j4\omega}$$

Multiplying and dividing by $3 - j4\omega$

$$\begin{aligned}
F(\omega) &= \frac{(1 + j2\omega)(3 - j4\omega)}{(3 + j4\omega)(3 - j4\omega)} \\
&= \frac{3 + 8\omega^2 + j2\omega}{9 + 16\omega^2} \\
&= \frac{3 + 8\omega^2}{9 + 16\omega^2} + j\frac{2\omega}{9 + 16\omega^2} \\
\Rightarrow Re(F(\omega)) &= \frac{3 + 8\omega^2}{9 + 16\omega^2} \\
Im(F(\omega)) &= \frac{2\omega}{9 + 16\omega^2}
\end{aligned}$$

At ω=0, $Re(F(\omega)) = \frac{3}{9} = 0.33$
At $\omega = \infty$, $Re(F(\omega)) = \frac{8 + \frac{3}{\omega^2}}{16 + \frac{9}{\omega^2}}|_{\omega=\infty} = 0.5$
The critical points of $Re(F(\omega))$ are given by,

$$\begin{aligned}
\frac{d}{d\omega}Re(F(\omega)) &= 0 \\
\Rightarrow (16\omega)(16\omega^2 + 9) - (8\omega^2 + 3)(32\omega) &= 0 \\
\Rightarrow \omega &= 0
\end{aligned}$$

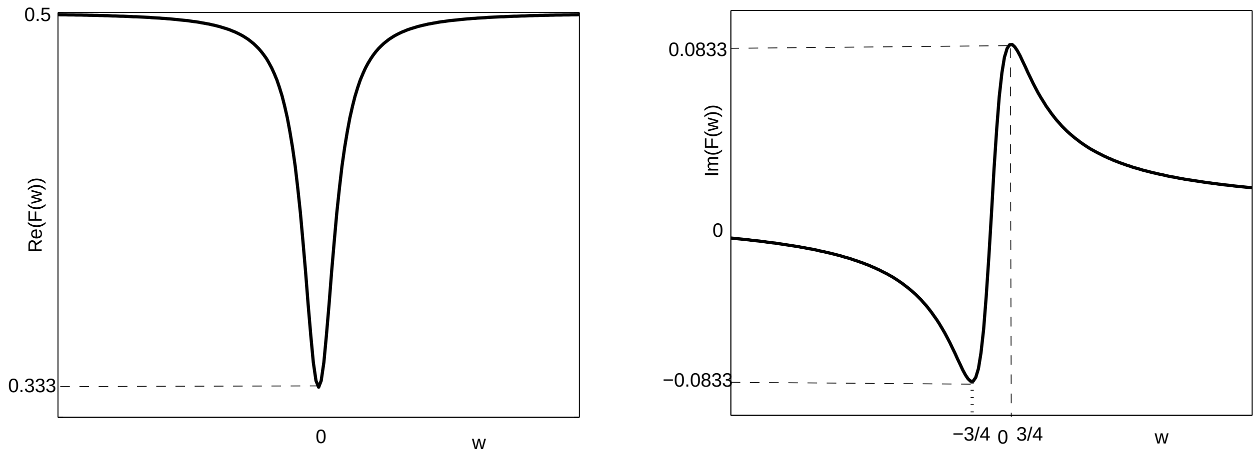
Similarly, at ω=0, $Im(F(\omega)) = 0$
At $\omega = \infty$, $Im(F(\omega)) = \frac{\frac{2}{\omega}}{16 + \frac{9}{\omega^2}}|_{\omega=\infty} = 0$
The critical points of $Im(F(\omega))$ are given by,

$$\begin{aligned}
\frac{d}{d\omega}Im(F(\omega)) &= 0 \\
\Rightarrow 2(16\omega^2 + 9) - (2\omega)(32\omega) &= 0 \\
16\omega^2 &= 9 \\
\Rightarrow \omega &= \pm\sqrt{\frac{9}{16}} \\
&= \pm\frac{3}{4} \\
Im(F(\omega))|_{\omega=\frac{3}{4}} &= \frac{1}{12} \\
Im(F(\omega))|_{\omega=-\frac{3}{4}} &= -\frac{1}{12}
\end{aligned}$$

The maximum and minimum values of $Im(F(\omega))$ are at $\omega = +(3/4)$ and $\omega = -(3/4)$ respectively. The real and imaginary parts of $F(\omega)$ are plotted in Figure 4.

(b) Magnitude of $F(\omega)$

$$\begin{aligned}
|F(\omega)| &= \sqrt{\frac{(8\omega^2 + 3)^2}{(16\omega^2 + 9)^2} + \frac{(2\omega)^2}{(16\omega^2 + 9)^2}} \\
&= \frac{\sqrt{64\omega^4 + 52\omega^2 + 9}}{(16\omega^2 + 9)} \\
|F(\omega = 0)| &= \frac{\sqrt{9}}{9} = \frac{1}{3} \\
|F(\omega = \infty)| &= \frac{\sqrt{64 + \frac{52}{\omega^2} + \frac{9}{\omega^4}}}{16 + \frac{9}{\omega^2}} = \frac{1}{2}
\end{aligned}$$



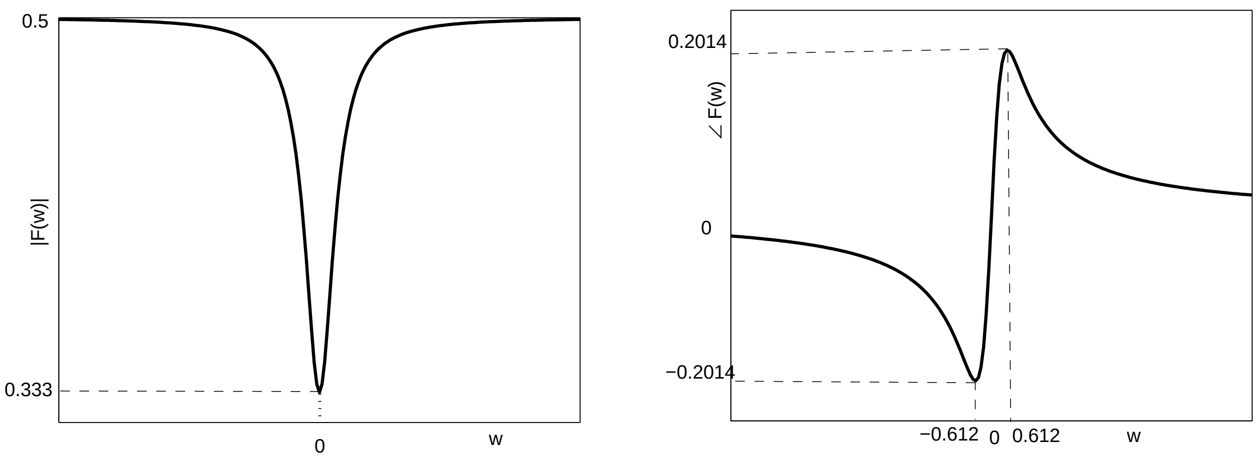
(a) (b)

Figure 4: Real and imaginary parts of $F(\omega)$

Phase of $F(\omega)$

$$\angle F(\omega) = \tan^{-1}\frac{2\omega}{8\omega^2+3}$$
$$\angle F(\omega=0) = \tan^{-1}\frac{0}{3} = 0$$
$$\angle F(\omega=\infty) = \tan^{-1}\frac{\frac{2}{\omega}}{8+\frac{3}{\omega^2}} = 0$$

For finding maximum and minimum points of $\angle F(\omega)$, evaluate the values of ω for which $\frac{d}{d\omega}\angle F(\omega) = 0$
Maximum at $\omega = \frac{\sqrt{6}}{4}$, $\angle F(\omega) = \tan^{-1}\frac{1}{\sqrt{6}} = 0.2014$
Minimum at $\omega = -\frac{\sqrt{6}}{4}$, $\angle F(\omega) = \tan^{-1}\frac{-1}{\sqrt{6}} =$ -0.2014



(a) (b)

Figure 5: Magnitude and Phase of $F(\omega)$

Solution 6

(a)

$$
\begin{aligned}
f(t) &= 2e^{j(2t-\frac{\pi}{3})}, 0 \leq t \leq 3\pi \\
&= 2\cos(2t-\frac{\pi}{3}) + j2\sin(2t-\frac{\pi}{3}) \\
\Rightarrow Re(f(t)) &= 2\cos(2t-\frac{\pi}{3}) \\
Im(f(t)) &= 2\sin(2t-\frac{\pi}{3})
\end{aligned}
$$

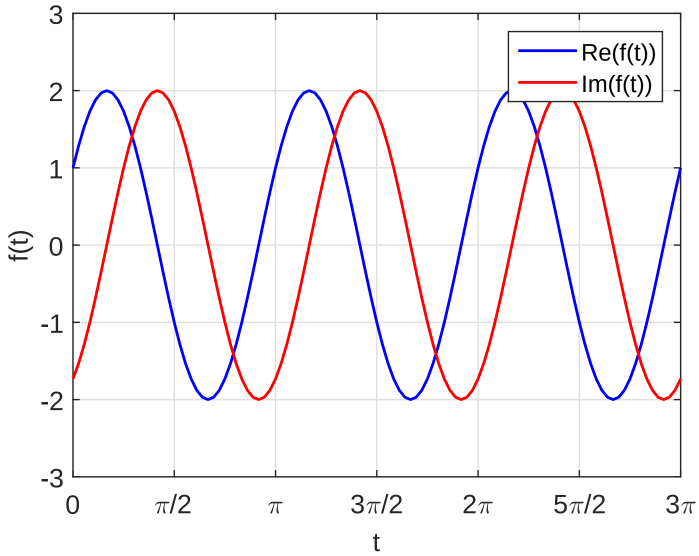


Figure 6: Real and imaginary parts of $f(t) = 2e^{j(2t-\frac{\pi}{3})}$

(b)

$$
\begin{aligned}
f(t) &= 2e^{-2t}e^{j(2t-\frac{\pi}{3})}, t \geq 0 \\
&= 2e^{-2t}\cos(2t-\frac{\pi}{3}) + j2e^{-2t}\sin(2t-\frac{\pi}{3}) \\
\Rightarrow Re(f(t)) &= 2e^{-2t}\cos(2t-\frac{\pi}{3}) \\
Im(f(t)) &= 2e^{-2t}\sin(2t-\frac{\pi}{3})
\end{aligned}
$$

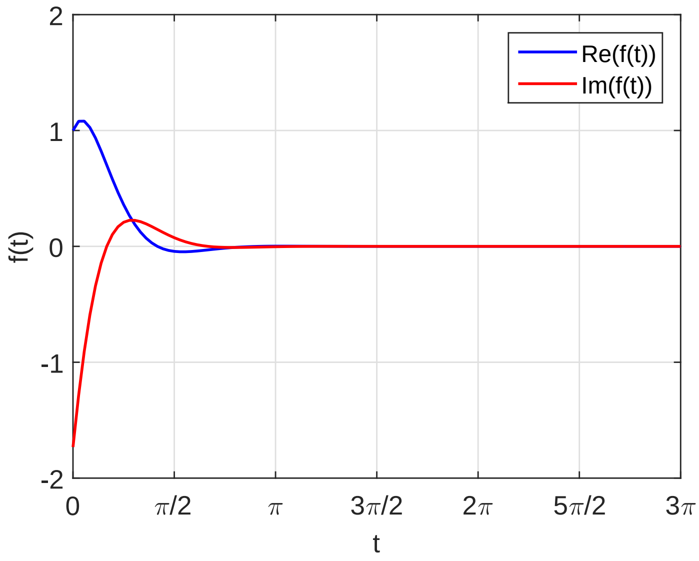


Figure 7: Real and imaginary parts of $f(t) = 2e^{-2t}e^{j(2t-\frac{\pi}{3})}$

(c)

$$\begin{aligned}
f(t) &= 2e^{2t}e^{j(2t-\frac{\pi}{3})}, t \geq 0 \\
&= 2e^{2t}\cos(2t-\frac{\pi}{3}) + j2e^{2t}\sin(2t-\frac{\pi}{3}) \\
\Rightarrow Re(f(t)) &= 2e^{2t}\cos(2t-\frac{\pi}{3}) \\
Im(f(t)) &= 2e^{2t}\sin(2t-\frac{\pi}{3})
\end{aligned}$$

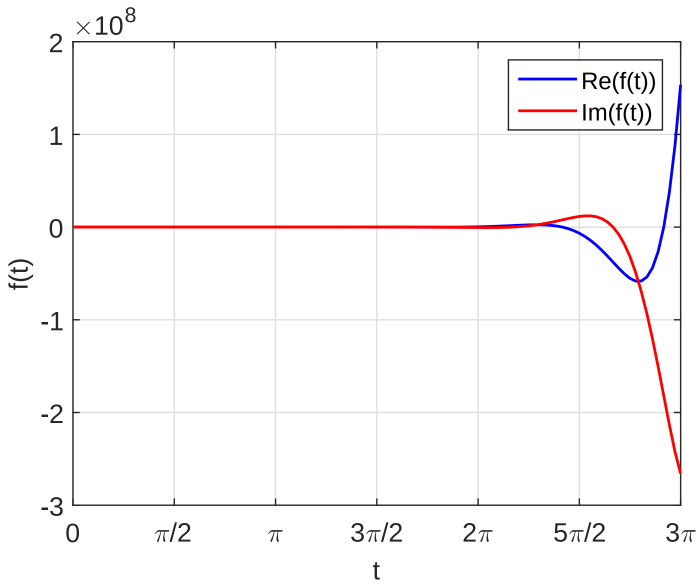


Figure 8: Real and imaginary parts of $f(t) = 2e^{2t}e^{j(2t-\frac{\pi}{3})}$