4. Evaluate the electrostatic energy W of a charge distribution in the form of a uniform charge density within a sphere of radius $a$ and total charge Q. Express the answer in terms of $\epsilon_0$, $a$, and Q.

Ans:

Three different ways by which the electrostatic energy can be calculated are illustrated

(a) The electrostatic energy is

$$W = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 dV$$

Using Gauss's law replace the surface charge by a uniform volume charge $\rho$. Then,

$$|\vec{E}| = \begin{cases} \rho r / 3\varepsilon_0 & \text{for } r \le a \\ Q / 4\pi\varepsilon_0 r^2 & \text{for } r > a \end{cases}$$

$$W = \frac{\varepsilon_0}{2} \int_0^a \frac{\rho^2 r^2}{9\varepsilon_0^2} r^2 \sin\theta \, dr \, d\theta \, d\varphi + \frac{\varepsilon_0}{2} \int_a^\infty \frac{Q^2}{16\pi^2 \varepsilon_0^2 r^4} r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$= \frac{3Q^2}{40\pi\varepsilon_0 a} = \frac{3Q^2}{20\pi\varepsilon_0 a}$$

Here the total charge in the sphere is $Q = \frac{4}{3}\pi a^3 \rho$

(b) $W = \frac{1}{2}\int \rho V d\tau'$ where $V$ is the potential

Since $\rho = 0$ for $r > a$ we have $W = \frac{1}{2}\int_0^a \rho V d\tau'$

$$= \frac{1}{2}\int_0^a \rho V r'^2 \sin\theta \, dr' d\theta' d\varphi'$$

Now, we need to evaluate V before proceeding further

$$|\vec{E}| = \begin{cases} Q/4\pi\varepsilon_0 r^2 & \text{for} \quad r > a \quad \Rightarrow \quad V = \dfrac{Q}{4\pi\varepsilon_0 r} \\ \rho r/3\varepsilon_0 & \text{for} \quad r \leq a \quad \Rightarrow \quad V = -\dfrac{\rho r^2}{6\varepsilon_0} + C \text{ (constant)} \end{cases}$$

At $r = a$, the electric fields in both the cases must be the same

$$\therefore \quad \frac{-\rho a^2}{6\varepsilon_0} + C = \frac{4\pi a^3 \rho}{12\pi\varepsilon_0 r^2} = \frac{\rho a^2}{3\varepsilon_0} \Rightarrow C = \frac{\rho a^2}{2\varepsilon_0}$$

$$V = \frac{-\rho r^2}{6\varepsilon_0} + \frac{\rho a^2}{2\varepsilon_0}.$$

$$\therefore \quad W = -\frac{1}{2}\int_0^a \frac{\rho^2 r^2}{6\varepsilon_0} r^2 \sin\theta \, dr d\theta d\varphi + \frac{1}{2}\int_0^a \frac{\rho^2 a^2}{2\varepsilon_0} r^2 \sin\theta \, dr d\theta d\varphi$$

$$= \frac{3Q^2}{20\pi\varepsilon_0 a}$$

(C) This is based on the idea that concentric layers of thickness $dr$ can be added to construct the sphere of charges.

The potential $V(r)$ due to a spherical charge distribution enclosed by a surface of radius $r$ is given by

$$V = \frac{Q}{4\pi\varepsilon_0 r} = \frac{\rho r^2}{3\varepsilon_0}$$

The work done in adding a layer of thickness $dr$ to this is

$$dW = V\rho \, d\tau' = \frac{\rho^2 r'^2}{3\varepsilon_0} r'^2 \sin\theta' \, dr' d\theta' d\varphi'$$

$$W = \int_0^a \frac{\rho^2 r^2}{3\varepsilon_0} r^2 \sin\theta \, dr \, d\theta \, d\varphi = \frac{4\pi\rho^2}{3\varepsilon_0}\frac{a^5}{5} = \frac{3Q^2}{20\pi\varepsilon_0 a}$$

5. Verify if $\vec{E} = \frac{a}{\rho^2}[\hat{e_\rho}(1+\cos\phi) + \hat{e_\phi}\sin\phi]$ can be a electric field and if so find the volume charge density that creates it. Note: $(\rho, \phi, z)$ represent the cylindrical polar co-ordinates.

[$\rho_v$ : volume charge density. Here $\rho_v$ is used to avoid confusion with cylindrical coordinates]

Ans:

$$\vec{E} = \frac{a}{\rho^2}\left[\hat{e}_\rho (1+\cos\phi) + \hat{e}_\phi \sin\phi\right]$$

$$\nabla \times \vec{E} = \frac{1}{\rho}\left[\frac{d(\rho E_\phi)}{d\rho} - \frac{dE_\rho}{d\phi}\right]\hat{e}_z$$

$$= \frac{1}{\rho}\left[-\frac{\sin\phi}{\rho^2} + \frac{\sin\phi}{\rho^2}\right] = 0$$

(Can be an electric field)

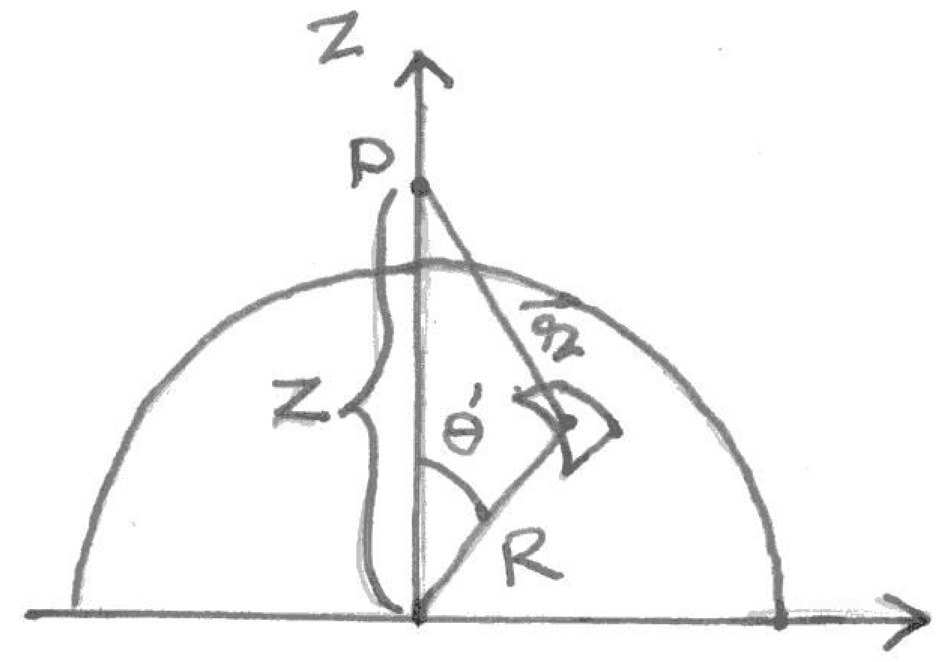
$$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0} = \frac{1}{\rho}\frac{d(\rho E_\rho)}{d\rho} + \frac{1}{\rho}\frac{dE_\phi}{d\phi}$$

$$= \frac{-a}{\rho^3}(1+\cos\phi) + \frac{a}{\rho^3}\cos\phi = \frac{-a}{\rho^3}$$

$$\rho_v = \frac{-\varepsilon_0 a}{\rho^3}$$

6. A hemisphere of radius R has a uniformly distributed surface charge with total charge Q. Find the potential at any position along the z axis due to the entire hemisphere of surface charge.

Ans:




$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma}{\mathscr{r}} da'.$$

We can express $\mathscr{r}$:

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'$$

An element of surface area on the sphere is $R^2 \sin\theta' d\theta' d\varphi'$, so

$$V(z) = \frac{1}{4\pi\epsilon_0} \sigma \int \frac{R^2 \sin\theta' d\theta' d\varphi'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}$$

$$= \frac{1}{4\pi\epsilon_0} \sigma \, 2\pi R^2 \int_0^{\pi/2} \frac{\sin\theta'}{\sqrt{R^2+z^2-2Rz\cos\theta'}} d\theta'$$

$$= \frac{\sigma}{4\pi\epsilon_0} 2\pi R^2 \left( \frac{1}{Rz} \sqrt{R^2+z^2-2Rz\cos\theta'} \right) \Big|_0^{\pi/2}$$

$$= \frac{-Q}{2\pi R^2} \frac{1}{4\pi\epsilon_0} 2\pi R^2 \frac{1}{Rz} \left( |z-R| - \sqrt{z^2+R^2} \right)$$

$$= -\frac{Q}{4\pi\epsilon_0 Rz} \left( |z-R| - \sqrt{z^2+R^2} \right)$$

$$V(z) = \begin{cases} -\dfrac{Q}{4\pi\epsilon_0 Rz}\left(z-R-\sqrt{z^2+R^2}\right) & z > R \\ -\dfrac{Q}{4\pi\epsilon_0 Rz}\left(R-z-\sqrt{z^2+R^2}\right) & z < R \end{cases}$$