Department of Mathematics, IIT Madras
MA-1102 Series & Matrices
Assignment-2-Sol **Series Representation of Functions**

1. Determine the interval of convergence for each of the following power series:
(a) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^2}$ (c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$
(a) Its radius of convergence is $\displaystyle\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}=\lim_{n\to\infty}\frac{n+1}{n}=1.$
The power series is around $x=0$, i.e., it is in the form $\sum a_n(x-a)^n$, where $a=0$. Thus, the power series converges at every point in the interval $(-1,1)$. To check at the end points:
For $x=-1$, the series $-1+\frac{1}{2}-\frac{1}{3}+\cdots$ converges.
For $x=1$, the series is $1+\frac{1}{2}+\frac{1}{3}+\cdots$ diverges.
Therefore, its interval of convergence is $(-1,1)\cup\{-1\}=[-1,1)$.
(b) Its radius of convergence is $\displaystyle\lim_{n\to\infty}\frac{1/n^2}{1/(n+1)^2}=1$. At $x=\pm 1$, the series $\sum(1/n^2)$ converges. Hence the interval of convergence is $[-1,1]$.
(c) Here, we consider the series in the form $\displaystyle x\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n+1}$.
For the series $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n+1}$, $\quad a_n=(-1)^{n+1}/(n+1)$.
Thus $\lim|a_n/a_{n+1}|=\lim(n+2)/(n+1)=1$. Hence $R=1$. That is, the series is convergent for all $x\in(-1,1)$.
We know that the series converges at $x=-1$ and diverges at $x=1$.
Therefore, the interval of convergence of the original power series is $(-1,1]$.

2. Determine the interval of convergence of the series $\displaystyle\frac{2x}{1}-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}-\cdots.$
Using Ratio test, we see that

$$\lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{n+1}\,\frac{n}{(2x)^n}\right|=\lim_{n\to\infty}\left|\frac{n}{n+1}\right||2x|=|2x|.$$

Thus the series converges for $|2x|<1$, i.e., for $|x|<1/2$. Also, we find that when $x=1/2$, the series converges and when $x=-1/2$, the series diverges. Hence the interval of convergence of the series is $(-1/2,1/2]$.

3. Determine power series expansion of the functions (a) $\ln(1+x)$ (b) $\displaystyle\frac{\ln(1+x)}{1-x}$
(a) For $-1<x<1$, $\displaystyle\frac{1}{1+x}=1-x+x^2-x^3+x^4-\cdots.$
Integrating term by term and evaluating at $x=0$, we obtain

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\quad\text{for } -1<x<1.$$

(b) Using the results in (a) and the geometric series for $1/(1-x)$, we have

$$\frac{\ln(1+x)}{1-x}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,x^n}{n}\cdot\sum_{n=0}^{\infty}x^n\quad\text{for } -1<x<1$$

For obtaining the product of the two power series, we need to write the first in the form $\sum a_nx^n$. (Notice that for the second series, each $b_n=1$.) Here, the first series is

$$\ln(1+x)=\sum_{n=0}^{\infty}a_nx^n,\quad\text{where } a_0=0 \text{ and } a_n=\frac{(-1)^{n-1}}{n}\ \text{ for } n\ge 1.$$

Thus the product above is $\dfrac{\ln(1+x)}{1-x} = \sum_{n=0}^{\infty} c_n x^n$, where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = a_0 + a_1 + \cdots a_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n}.$$

4. The power series for the function $\dfrac{1}{1-x}$ has interval of convergence $(-1, 1)$. However, prove that the function has power series representation around any $c \neq 1$.

$$\frac{1}{1-x} = \frac{1}{1-c} \frac{1}{1 - \frac{x-c}{1-c}} = \frac{1}{1-c} \sum_{n=0}^{\infty} \frac{1}{(1-c)^n} (x-c)^n.$$

This power series converges for all x with $|x-c| < |1-c|$, i.e., for $x \in (c - |1-c|, c + |1-c|)$. We also see that the function $\frac{1}{1-x}$ is well defined for each $x \neq 1$.

5. Find the sum of the alternating harmonic series $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{n}$.

Consider the power series representation of $\dfrac{1}{1+x}$. Differentiate term by term.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{x^n} \Rightarrow \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.$$

Notice that the interval of convergence of the first power series is $(-1, 1)$. But the interval of convergence of the second power series is $(-1, 1]$. Thus, evaluating the second series at $x = 1$, we have

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

6. Give an approximation scheme for $\displaystyle\int_0^a \frac{\sin x}{x}\, dx$ where $a > 0$.

Using the Maclaurin series for $\sin x$, we have

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7} + \cdots$$

Integrating term by term, we get

$$\int_0^a \frac{\sin x}{x}\, dx = a - \frac{a^3}{3! \cdot 3} + \frac{a^5}{5! \cdot 5} - \frac{a^7}{7! \cdot 7} + \cdots$$

Approximations to the integral may be obtained by truncating the series suitably.

7. Give an example of an infinitely diifferentiable function which has a Taylor series expansion at a point but the taylor series does not represent the function around that point.

Consider the function $f(x) = e^{-1/x^2}$ for $x \neq 0$. And $f(0) = 0$. This function is infinitely diferentiable at $x = 0$. The coefficients in the Taylor series are all 0. Thus the Taylor series of the function is the zero series, which clearly does not represent the function $f(x)$ at any nonzero point (near 0).

8. Show that $1 + \dfrac{1}{2}\dfrac{1}{3} + \dfrac{1 \cdot 3}{2 \cdot 4} \cdot \dfrac{1}{5} + \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \dfrac{1}{7} + \cdots = \dfrac{\pi}{2}$.

In the binomial series $(1+t)^m = 1 + mx + \dfrac{m}{m-1} 1 \cdot 2t^2 + \dfrac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} t^3 + \cdots$

for $|t| < 1$, substitute $m = 1/2$ and $m = -1/2$ and then multiply them to obtain

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{1 \cdot 3}{2 \cdot 4} t^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} t^6 + \cdots$$

Integrating this power series from 0 to x for any $x \in (-1, 1)$, we have

$$\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = x + \frac{1}{2} \cdot \frac{1}{3} x^3 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7} x^7 + \cdots$$

This series also converges for $x = 1$. (Show it.) Therefore, for $x = 1$, the series converges to $\sin^{-1} 1 = \frac{\pi}{2}$.

9. Find the Fourier series of $f(x)$ given by: $f(x) = 0$ for $-\pi \le x < 0$; and $f(x) = 1$ for $0 \le x \le \pi$. Say also how the Fourier series represents $f(x)$. Hence give a series expansion of $\pi/4$.

$$a_n = \frac{1}{\pi} \int_0^\pi \cos nx \, dx = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \ne 0 \end{cases}$$

$$b_n = \frac{1}{\pi} \int_0^\pi \sin nx \, dx = \frac{1}{\pi} \left[\frac{1 - \cos n\pi}{n} \right] = \frac{1 - (-1)^n}{n\pi} = \begin{cases} \frac{2}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

Hence the Fourier series for $f(x)$ is $\frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}$.

By the convergence theorem for Fourier series, we know that this Fourier series converges to $f(x)$ for any $x \ne 0$. At $x = 0$, the Fourier series converges to $1/2$.

Taking $x = \pi/2$, we have

$$1 = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\sin(n + 1/2)\pi}{2n+1} = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Therefore, $\frac{\pi}{4} = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

10. Considering the fourier series for $|x|$, deduce that $\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

Cinsider $f(x) = |x|$ in the interval $[-\pi, \pi]$; extended to $\mathbb{R}$ with period 2π. Now, it is an even function. Thus each b_n is 0. Next, $a_0 = (2/\pi) \int_0^\pi x \, dx = \pi$. And for $n > 0$,

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos nx \, dx = \frac{2}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right].$$

That is, $a_{2n} = 0, \; a_{2n+1} = \dfrac{-4}{\pi(2n+1)^2}$ for $n = 1, 2, 3 \ldots$
By the convergence theorem for Fourier series, we have

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2} \quad \text{for } x \in [-\pi, \pi].$$

Taking $x = 0$, we have $\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

11. Considering the fourier series for x, deduce that $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Consider $f(x) = x$ for $x \in [-\pi, \pi]$. It is an odd function. Hence in its Fourier series, each $a_n = 0$. For $n \ge 1$,

$$b_n = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx = \frac{2}{\pi} \left[-x \frac{\cos nx}{n} \right]_0^\pi + \frac{2}{\pi} \int_0^\pi \frac{\cos nx}{n} \, dx = \frac{(-1)^{n+1}}{n}.$$

Thus the Fourier series for $f(x) = x$ in $-\pi, \pi]$ is $2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$.

Taking $x = \pi/2$, we have $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

12. Considering the fourier series for $f(x)$ given by: $f(x) = -1$, for $-\pi \le x < 0$ and $f(x) = 1$ for $0 \le x \le \pi$ deduce that $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.
Here, $f(x)$ is an odd function. Thus in its Fourier series, each a_n is 0. For $n \ge 1$,

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx = \frac{2}{\pi}(1 - \cos n\pi) = \frac{2}{\pi}(1 - (-1)^n).$$

Due to the convergence theorem, we conclude that for $x \neq 0$,

$$f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}.$$

Taking $x = \pi/2$, we obtain the desired expression for $\pi/4$.

13. Considering $f(x) = x^2$, show that for each $x \in [0, \pi]$,

$$\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} = \sum_{n=1}^{\infty} \frac{n\pi^2(-1)^{n+1} + 2(-1)^n - 2}{n^2\pi} \sin nx.$$

We determine sine and cosine series expansions of $f(x) = x^2$ for $0 \le x \le \pi$.
The odd and even expansions of $f(x)$ are

$$f_{odd}(x) = \begin{cases} -x^2 & \text{for } \ -\pi \le x < 0 \\ x^2 & \text{for } \ 0 \le x < \pi, \end{cases} \qquad f_{even}(x) = x^2 \quad \text{for } \ -\pi \le x \le \pi.$$

We see that, as earlier, $f_{even}(x)$ has the Fourier expansion $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$ for $x \in [0, \pi]$.
Due to the convergence theorem of Fourier series, this series sums to x in $[0, \pi]$.

For the sine series expansion, we determine the Fourier series of $f_{odd}(x)$. Here, each a_n is 0. And for $n \ge 1$,

$$b_n = \frac{2}{\pi} \int_0^{\pi} x^2 \sin nx \, dx = 2\pi \frac{(-1)^{n+1}}{n} + \frac{4}{\pi} \left[\frac{(-1)^n - 1}{n^2} \right].$$

Again, due to the convergence theorem of Fourier series, $x = \sum_{n=1}^{\infty} b_n \sin nx$ for $x \in [0, \pi]$.
Equating both the sine and the cosine series for $f(x) = x$ in $[0, \pi]$, we obtain the required result.

14. Represent the function $f(x) = 1 - |x|$ for $-1 \le x \le 1$ as a cosine series.

It is an even function. Thus its Fourier series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$, where $a_0 = 2\int_0^1 (1-x)\, dx = 1$; and for $n \ge 1$,

$$a_n = 2\int_{-1}^{1} (1 - |x|) \cos n\pi x \, dx = 2\int_0^1 (1 - x) \cos n\pi x \, dx = \begin{cases} 0 & \text{for } \ n \ \text{ even} \\ -4/(n^2\pi^2) & \text{for } \ n \ \text{ odd.} \end{cases}$$

Thereofore, $1 - |x| = \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2n+1)\pi x}{(2n+1)^2}$ for $-1 \le x \le 1$.