Department of Mathematics, IIT Madras
MA1020 Series & Matrices
Assignment-3 Matrix Operations & Linear Independence

1. Show that given any $n \in \mathbb{N}$ there exist matrices $A, B \in \mathbb{R}^{n\times n}$ such that $AB \neq BA$.

2. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Compute A^n.

3. Let $A \in \mathbb{F}^{m\times n}$; $B \in \mathbb{F}^{n\times k}$; $A_1, \ldots, A_m$ be the rows of A; $B_1, \ldots, B_k$ be the columns of B. Show that

   (a) $A_1B, \ldots, A_mB$ are the rows of AB. (b) $AB_1, \ldots, AB_k$ are the columns of AB.

4. Let $A \in \mathbb{F}^{n\times n}$; I be the identity matrix of order n. Find the inverse of the $2n \times 2n$ matrix $\begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$.

5. If A is a hermitian (symmetric) invertible matrix, then show that A^{-1} is hermitian (symmetric).

6. If A is a lower (upper) triangular invertible matrix, then A^{-1} is lower (upper) triangular.

7. Let $x, y \in \mathbb{F}^{1\times n}$ (or in $\mathbb{F}^{n\times 1}$); $\alpha \in \mathbb{F}$. Prove the following:

   (a) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2$. (*Parallelogram Law*)
   (b) $|\langle x, y\rangle| \leq \|x\|\,\|y\|$. (*Cauchy-Schwartz inequality*)
   (c) $\|x+y\| \leq \|x\| + \|y\|$. (*Triangle inequality*)
   (d) If $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. (*Pythagoras' Law*)

8. Show that each orthogonal 2×2 matrix is either a reflection or a rotation.

9. Let $u, v, w \in \mathbb{F}^{n\times 1}$. Show that $\{u, v, w\}$ is linearly independent iff $\{u+v, v+w, w+u\}$ is linearly independent.

10. Find linearly independent vectors from $U = \{(a, b, c) : 2a + 3b - 4c = 0\}$ whose span is U.

11. The vectors $u_1 = (1, 1, 0), u_2 = (0, 1, 1), u_3 = (1, 0, 1)$ are linearly independent in $\mathbb{F}^3$. Apply Gram-Schmidt Orthogonalization.

12. Let $A \in \mathbb{R}^{3\times 3}$ have the first two columns as $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3},)^T$ and $(1/\sqrt{2}, 0, -1/\sqrt{2})^T$. Determine the third column of A so that A is an orthogonal matrix.

13. Convert the following matrices into RREF and determine their ranks.

    (a) $\begin{bmatrix} 5 & 2 & -3 & 1 & 7 \\ 1 & -3 & 2 & -2 & 11 \\ 3 & 8 & -7 & 5 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 2 & -3 & 1 & 30 \\ 1 & -3 & 2 & -2 & 11 \\ 3 & 8 & -7 & 5 & 8 \end{bmatrix}$

14. Determine linear independence of $\{(1, 2, 2, 1),\ (1, 3, 2, 1),\ (4, 1, 2, 2),\ (5, 2, 4, 3)\}$ in $\mathbb{C}^{1\times 4}$.

15. Let $A \in \mathbb{R}^{3\times 3}$ satisfy $A(a, b, c)^T = (a + b, 2a - b - c, a + b + c)^T$. Determine A and also its rank.

16. Determine linearly independent vectors whose span is $U = \{(a, b, c, d, e) \in \mathbb{R}^5 : a = c = e, b + d = 0\}$.

17. Let $A \in \mathbb{F}^{m\times n}$ have rank r. Give reasons for the following:

    (a) $\operatorname{rank}(A) \leq \min\{m, n\}$.
    (b) If $n > m$, then there exist $x, y \in \mathbb{F}^{n\times 1}$ such that $x \neq y$ and $Ax = Ay$.
    (c) If $n < m$, then there exists $y \in \mathbb{F}^{m\times 1}$ such that for no $x \in \mathbb{F}^{n\times 1}$, $Ax = y$.
    (d) If $n = m$, then as a map, A is one-one iff A is onto.