Department of Mathematics, IIT Madras
MA-1102 Series & Matrices
Assignment-4-Sol **Linear Systems & Eigenvalue Problem**

1. Solve the following system by (a) Gauss elimination, and (b) Gauss-Jordan elimination:

$$\begin{array}{rrrrrl} x_1 & +x_2 & +x_3 & +x_4 & -3x_5 & = 6 \\ 2x_1 & +3x_2 & +x_3 & +4x_4 & -9x_5 & = 17 \\ x_1 & +x_2 & +x_3 & +2x_4 & -5x_5 & = 8 \\ 2x_1 & +2x_2 & +2x_3 & +3x_4 & -8x_5 & = 14 \end{array}$$

(b) We reduce the augmented matrix to row reduced echelon form.

$$\left[\begin{array}{rrrrr|r} \boxed{1} & 1 & 1 & 1 & -3 & 6 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right] \rightarrow \left[\begin{array}{rrrrr|r} \boxed{1} & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrrrr|r} \boxed{1} & 0 & 2 & -1 & 0 & 1 \\ 0 & \boxed{1} & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{rrrrr|r} \boxed{1} & 0 & 2 & 0 & -2 & 3 \\ 0 & \boxed{1} & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & \boxed{1} & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Find out the row operations used in each step. Since no pivot is on the b portion, the system is consistent. To solve this system, we consider only the pivot rows, ignoring the bottom zero rows. The basis variables are x_1, x_2, x_4 and the free variables are x_3, x_5. Write $x_3 = \alpha$ and $x_5 = \beta$. Then

$$x_1 = 3 - 2\alpha + 2\beta, \quad x_2 = 1 + \alpha - \beta, \quad x_4 = 2 + 2\beta.$$

We can write the solution set as in the following:

$$Sol(A,b) = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} : \alpha, \beta \in \mathbb{F} \right\}.$$

2. Let $A \in \mathbb{F}^{m\times n}$ have columns $A_1, \ldots, A_n$. Let $b \in \mathbb{F}^m$. Show the following:
   (a) The equation $Ax = 0$ has a non-zero solution iff $A_1, \ldots, A_n$ are linearly dependent.
   (b) The equation $Ax = b$ has at least one solution iff $b \in \operatorname{span}\{A_1, \ldots, A_n\}$.
   (c) The equation $Ax = b$ has at most one solution iff $A_1, \ldots, A_n$ are linearly independent.
   (d) The equation $Ax = b$ has a unique solution iff $\operatorname{rank} A = \operatorname{rank}[A|b] =$ number of unknowns.

(a) We have scalars $\alpha_1, \ldots, \alpha_n$ not all 0 such that $\sum \alpha_i A_i = 0$. But each $A_i = Ae_i$. So, $A(\sum \alpha_i e_i) = 0$. Here, take $x = \sum \alpha_i e_i$. See that $x \neq 0$.

(b) $R(A) = \operatorname{span}\{A_1, \ldots, A_n\}$. So, $b \in R(A)$. That is, we have a $x \in \mathbb{F}^{n\times 1}$ such that $b = Ax$.

(c) If $Au = b$ and $Av = b$, then $A(u - v) = 0$. Let $u - v = (\alpha_1, \ldots, \alpha_n)^t$. Then $A(u - v) = 0$ can be rewritten as $\alpha_1 A_1 + \cdots \alpha_n A_n = 0$. Since $A_1, \ldots, A_n$ are linearly independent, each α_i is 0. That is, $u - v = 0$.

(d) If the system $Ax = b$ has a unique solution, then it is a consistent system and $\operatorname{null}(A) = \{0\}$. Then $\operatorname{rank}(A) = \operatorname{rank}[A|b]$ and $\operatorname{rank}(A) = n - \operatorname{null}(A) = n =$ number of unknowns.

3. Check if the system is consistent. If so, determine the solution set.

(a) $x_1 - x_2 + 2x_3 - 3x_4 = 7,\ 4x_1 + 3x_3 + x_4 = 9,\ 2x_1 - 5x_2 + x_3 = -2,\ 3x_1 - x_2 - x_3 + 2x_4 = -2.$

(b) $x_1 - x_2 + 2x_3 - 3x_4 = 7, 4x_1 + 3x_3 + x_4 = 9, 2x_1 - 5x_2 + x_3 = -2, 3x_1 - x_2 - x_3 + 2x_4 = -2.$

4. Using Gaussian elimination determine the values of $k \in \mathbb{R}$ so that the system of linear equations

$$x + y - z = 1,\ 2x + 3y + kz = 3,\ x + ky + 3z = 2$$

has (a) no solution, (b) infinitely many solutions, (c) exactly one solution.

Gaussian elimination on $[A|b]$ yields the matrix $\begin{bmatrix} \boxed{1} & 1 & -1 & 1 \\ 0 & \boxed{1} & k+2 & 1 \\ 0 & 0 & (k+3)(2-k) & 2-k \end{bmatrix}$.

(a) The system has no solution when $(k+3)(2-k) = 0$ but $2 - k \neq 0$, that is, when $k = -3$.
(b) It has infinitely many solutions when $(k+3)(2-k) = 0 = 2 - k$, that is, when $k = 2$.
(c) It has exactly one solution when $(k+3)(2-k) \neq 0$, that is, when $k \neq -3, k \neq 2$.

5. Find the eigenvalues and the associated eigenvectors for the matrices given below.

(a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 0 & 3 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(d) Call the matrix A. Its characteristic polynomial is $-(2+t)(3-t)(5-t)$.
So, the eigenvalues are $\lambda = -2,\ 3,\ 5$.
For $\lambda = -2$, $A(a,b,c)^t = -2(a,b,c)^t \Rightarrow -2a + 3c = -2a,\ -2a + 3b = -2b,\ 5c = -2c$.
One of the solutions for $(a,b,c)^t$ is $(5,2,0)^t$. It is an eigenvector for $\lambda = -2$.
For $\lambda = 3$, $A(a,b,c)^t = 3(a,b,c)^t \Rightarrow -2a + 3c = 3a,\ -2a + 3b = 3b,\ 5c = 3c$.
One of the solutions for $(a,b,c)^t$ is $(0,0,1)^t$. It is an eigenvector for $\lambda = 3$.
For $\lambda = 5$, $A(a,b,c)^t = 5(a,b,c)^t \Rightarrow -2a + 3c = 5a,\ -2a + 3b = 5b,\ 5c = 5c$.
One of the solutions for $(a,b,c)^t$ is $(3,-3,7)^t$. It is an eigenvector for $\lambda = 5$.

6. Let $A \in \mathbb{C}^{n\times n}$ be invertible. Show that $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $1/\lambda$ is an eigenvalue of A^{-1}.

Since A is invertible, its determinant is nonzero. As $\det(A)$ is the product of eigenvalues of A, no eigenvalue of A is 0. Thus for each eigenvalue λ, $1/\lambda$ makes sense. Now,
$A - \lambda I = -\lambda A(A^{-1} - \lambda^{-1} I)$.
Since A is invertible, $\lambda \neq 0$, we see that $(-\lambda A)$ is invertible. Therefore, as linear transformations, $A - \lambda I$ is one-one iff $A^{-1} - \lambda^{-1} I$ is one-one.
Or that $A - \lambda I$ is not one-one iff $A^{-1} - \lambda^{-1} I$ is not one-one.
This shows that λ is an eigenvalue of A iff λ^{-1} is an eigenvlaue of A^{-1}.

7. Let A be an $n \times n$ matrix and α be a scalar such that each row (or each column) sums to α. Show that α is an eigenvalue of A.

If each row sums to α, then $A(1,1,\ldots,1)^t = \alpha(1,1,\ldots,1)^t$. Thus α is an eignevlaue with an eigenvector as $(1,1,\ldots,1)^t$.
If each column sums to α, then each row sums to α in A^t. Thus A^t has an eigenvalue as α. However, A^t and A have the same eigenvalues. Thus α is also an eigenvalue of A.

8. Give examples of matrices which cannot be diagonalized.

One such matrix is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Reason?
A has the single eigenvalue as 1. Its algebraic multiplicity is 2.
To find its geometric multiplicity, consider $A(a,b)^t = 1(a,b)^t$. It gives $b = 0$ and a arbitrary. That is, $N(A - 1\,I) = \{(a,0)^t : a \in \mathbb{F}\}$. It has dimension 1. That is, the geometric multiplicity of the eigenvalue 1 is 1. This is not equal to the algebraic multiplicity.

Construct $A \in \mathbb{F}^{n\times n}$ whose first row is $e_1 + e_2$, and ith row is e_i for $i > 1$. Prove similarly that the eigenvalue 1 has geometric multiplicity $n-1$ and algebraic multiplicity is n.

9. Which of the following matrices is/are diagonalizable? If it is diagonalizable, diagonalize it.

   (a) $A \in \mathbb{R}^{3\times 3}$ is such that $A(a,b,c)^t = (a+b+c,\ a+b-c,\ a-b+c)^t$.

   (b) $A \in \mathbb{R}^{3\times 3}$ is such that $Ae_1 = 0, \quad Ae_2 = e_1, \quad Ae_3 = e_2.$

   (c) $A \in \mathbb{R}^{3\times 3}$ is such that $Ae_1 = e_2, \quad Ae_2 = e_3, \quad Ae_3 = 0.$

   (d) $A \in \mathbb{R}^{3\times 3}$ is such that $Ae_1 = e_3, \quad Ae_2 = e_2, \quad Ae_3 = e_1.$

(a) $A = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}$. Its charachteristic polynomial is $-(t+1)(t-2)^2$.
So, eigenvalues are 1 and 2. Solving $A(a,b,c)^t = \lambda(a,b,c)^t$ for $\lambda = 1, 2$, we have
$\lambda = 1\ :\ a+b+c = -a,\ a+b-c = -b,\ a-b+c = -c \Rightarrow a = -c, b = -c.$
Thus a corresponding eigenvector is $(-1,1,1)^t$.
$\lambda = 2\ :\ a+b+c = 2a,\ a+b-c = 2b,\ a-b+c = 2c \Rightarrow a = b+c.$
Thus two linearly independent corresponding eigenvectors are $(1,1,0)^t$ and $(1,0,1)^t$.
Therefore, we have a basis of eigenvectors for $\mathbb{F}^{3\times 1}$; thus A is diagonalizable.
Take the matrix $P = \begin{bmatrix}-1 & 1 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$. Then verify that $P^{-1}AP = diag(-1,2,2)$.

(b) The eigenvalue 0 has algebraic multiplicity 3 but geometric multiplicity 1. So, A is not diagonalizable.

(c) Similar to (b).

(d) Proceed as in (a) to get $P = \begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\\ -1 & 1 & 0\end{bmatrix}$ and verify $P^{-1}AP = diag(-1,1,1)$.

10. Check whether each of the following matrix is diagonalizable. If diagonalizable, find a basis of eigenvectors for the space $\mathbb{R}^{3\times 1}$:

(a) $\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$ (b) $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$ (c) $\begin{bmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$ (d) $\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$.

(a) It is a real symmetric matrix; so diagonalizable. Its eigenvalues are $-2, 1, 2$. Also since the 3×3 matrix has three distinct eigenvalues, it is diagonalizable. Proceed like 9(a).

(b) 1 is a triple eigenvalue. It leads to solve $a+b+c = a, b+c = b, c = c$. Solution is a is arbitrary, $b = c = 0$. The geometric multiplicity is 1. So, it is not diagonalizable.

(c) Its eigenvalues are $2,\ (-1 \pm \sqrt{3}\,i)/2$. Since three distinct eigenvalues; it is diagonalizable. Here, P will be a complex matrix. Proceed as in 9(a).

(d) $(1,0,-1)^t$ and $(1,-1,0)^t$ are two linearly independent eigenvectors for the eigenvalue -1.
$(0,1,-1)^t$ is an eigenvector for the eigenvalue 2.
Hence taking $P = \begin{bmatrix}1 & 1 & 0\\ 0 & -1 & 0\\ -1 & 0 & -1\end{bmatrix}$, we have $P^{-1}AP = diag(-1,-1,2)$.