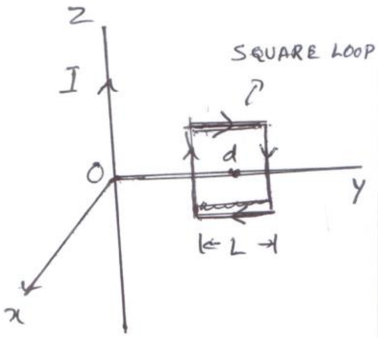
Solutions of Problem sheet-8

1 (a) The field at any point $(0, y, z)$ inside the square loop is

$$\vec{B} = -\frac{\mu_0 I}{2\pi y}\hat{e}_x$$



The flux, Φ_m, through the square loop

$$\Phi_m = \int \vec{B}\cdot d\vec{a}$$

$$= \int_{d-\frac{L}{2}}^{d+\frac{L}{2}} dy \int_{-L/2}^{L/2} dz \left(-\frac{\mu_0 I}{2\pi y}\hat{e}_x\right)\cdot(-\hat{e}_x)$$

Normal to the square loop is $-\hat{e}_x$

$d\vec{a} = -dy\,dz\,\hat{e}_x$

$$= \frac{\mu_0 I L}{2\pi}\ln\left(\frac{d+\frac{L}{2}}{d-\frac{L}{2}}\right)$$

$$\Phi_m \equiv MI \qquad \text{where} \quad M = \frac{\mu_0 L}{2\pi}\ln\left(\frac{d+\frac{L}{2}}{d-\frac{L}{2}}\right)$$

If $d \gg L/2$, $M \approx \frac{\mu_0 L^2}{2\pi d}$ (leading term in 'ln' expansion)

(b) The above result is valid in the quasi-static case; ie when the term $\mu_0\epsilon_0\frac{\partial E}{\partial t}$ term in the Maxwell eq^n $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$ can be neglected compared to the term $\mu_0\vec{J}$.

The ~~em~~ emf induced in the loop is

$$\mathcal{E} = -\frac{d\Phi_m}{dt} = -M\frac{dI}{dt}$$

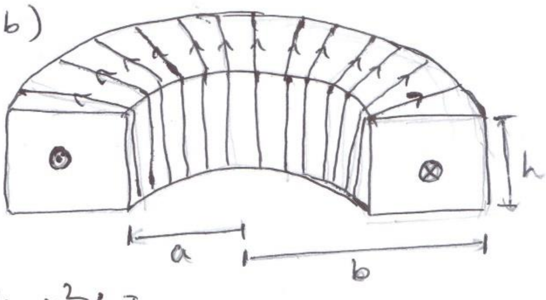
$$\mathcal{E} = M\alpha I_0 e^{-\alpha t} \quad \text{(with M as given above)}$$

Since B decreases ~~at~~ with time, the direction of the induced current in the loop will be such that the field it creates will oppose this decrease in flux (Lenz's Law). Therefore, the current is in the direction shown ~~by~~ in the figure.

Solutions of Problem sheet-8

II

(2) The field $\vec{H}$ at a distance $\rho\,(a \le \rho \le b)$ from the z-axis is $\vec{H} = \frac{nI}{2\pi\rho}\hat{e}_\varphi$

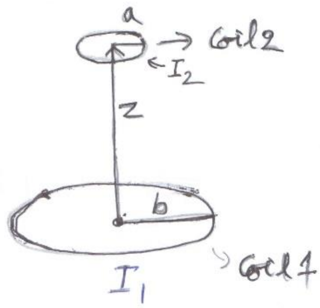


$$\therefore \vec{B} = \frac{\mu_0 n I}{2\pi\rho}\hat{e}_\varphi$$

The flux $\Phi = n\int \vec{B}\cdot d\vec{a} = \frac{\mu_0 n^2 h I}{2\pi}\ln\left(\frac{b}{a}\right) = LI$

$\therefore$ SELF INDUCTANCE IS : $\underline{L = \frac{\mu_0 n^2 h}{2\pi}\ln\left(\frac{b}{a}\right)}$

(3) (a) The magnetic field at coil 2 due to coil 1



$$\vec{B}_1 = \frac{\mu_0 I_1 b^2}{2(b^2+z^2)^{3/2}}\hat{e}_z$$

The flux $(\Phi_{2,1})$ through loop 2 due to I_1 is

$$\Phi_{2,1} = (\pi a^2)\frac{\mu_0 I_1 b^2}{2(b^2+z^2)^{3/2}} = M_{21} I_1$$

where $M_{21} = \frac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}}$

(b) The magnetic dipole moment of the small loop carrying I_2 current is

$$\vec{m} = \pi a^2 I_2\,\hat{e}_z$$

$\therefore$ Magnetic field at a point $(x,y,0)$ in x-y plane due to the dipole located at a point $(0,0,z)$ is

$$\vec{B}_2 = \frac{\mu_0}{4\pi(\rho^2+z^2)^{3/2}}\left[(3\vec{m}\cdot\hat{n})\hat{n} - \vec{m}\right]$$

$\hat{n}$ is unit vector along $(\rho\hat{e}_\rho - z\hat{e}_z)$.

$\therefore$ The flux $\Phi_{1,2}$, due to current I_2 (in loop 2), through loop 1 is

$$\Phi_{1,2} = \int_{loop\,1}(\vec{B}_2\cdot\hat{e}_z)\,\rho\, d\rho\, d\phi$$

Solutions of Problem sheet-8

III

$$\Phi_{1,2} = \int \frac{\mu_0 m}{4\pi(\rho^2+z^2)^{3/2}}\left(\frac{3z^2}{(\rho^2+z^2)} - 1\right) 2\pi \rho\, d\rho$$

$$\Phi_{1,2} = \frac{\mu_0 m b^2}{2(b^2+z^2)^{3/2}}$$

$$\Phi_{1,2} = \frac{\mu_0 a^2 b^2}{2(b^2+z^2)^{3/2}} I_2 = M_{12} I_2$$

where $M_{12} = \dfrac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}}$

(c) Therefore mutual Inductance

$$M = M_{12} = M_{21} = \frac{\mu_0 \pi a^2 b^2}{2(b^2+z^2)^{3/2}}$$

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(4) The displacement current density is

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{I}{A} = \frac{I}{\pi a^2}$$

Drawing an Amperian loop at radius 's'

$$\oint \vec{B}\cdot d\vec{l} = B\cdot 2\pi s = \mu_0 I_{encl} = \mu_0 \frac{I}{\pi a^2}\cdot \pi s^2 = \mu_0 I \frac{s^2}{a^2}$$

$$B = \frac{\mu_0 I s^2}{2\pi s a^2} = \frac{\mu_0 I s}{2\pi a^2}\,\hat{e}_\varphi$$

Solutions of Problem sheet-8

T.V

(5) From Ohm's law, Conduction current is

$$\vec{J} = \sigma \vec{E} = \sigma \vec{E}_0 \cos \omega t \quad \rightarrow (1)$$

The displacement current is

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \frac{\partial (\epsilon \vec{E})}{\partial t} = \frac{\partial}{\partial t} (\epsilon_0 \epsilon_r E_0 \cos \omega t)$$

$$\vec{J}_d = -\omega \epsilon_0 \epsilon_r \vec{E}_0 \sin \omega t$$

$$\vec{J}_d = \omega \epsilon_0 \epsilon_r \vec{E}_0 \cos \left(\omega t + \frac{\pi}{2} \right) \quad \rightarrow (2)$$

From (1) & (2)

$$\frac{J_d}{J} = \frac{\omega \epsilon_0 \epsilon_r E_0 \cos \left(\omega t + \frac{\pi}{2} \right)}{\sigma E_0 \cos \omega t}$$

$$\therefore \left| \frac{\vec{J}_d}{J} \right| = \frac{\omega \epsilon_0 \epsilon_r}{\sigma} = \frac{2\pi f \times 9 \times 10^{-12}}{10^7}$$

$$\simeq f \times 10^{-17}$$

$\therefore$ The displacement current in a good conductor is completely negligible compared to the conduction current at any frequency lower than optical frequency ($\sim 10^{15}$ Hz)

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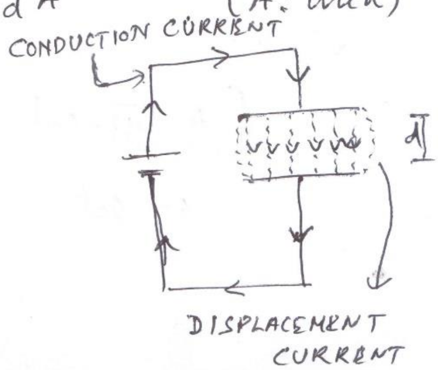
(6) The displacement current $I_d = J_d A$ (A: area)

$$I_d = A \frac{\partial D}{\partial t} \quad \left(\text{Since } \vec{J}_d = \frac{\partial \vec{D}}{\partial t} \right)$$

$$= A \epsilon \frac{\partial E}{\partial t} = \frac{A \epsilon}{d} \frac{\partial V}{\partial t}$$

$$\boxed{\therefore I_d = C \frac{\partial V}{\partial t}} \quad \text{Since } C = \frac{\epsilon A}{d}$$



Solutions of Problem sheet-8

V

Further, $I_d = \frac{\partial}{\partial t}(CV) = \frac{\partial q}{\partial t} = I$

$\left(\text{Since } q = CV \text{ and } I = \frac{dq}{dt}\right)$

Thus displacement current = Conduction current along the discontinuity, such as, a capacitor.

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7(a) We know $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. By taking curl on both sides.
(MAXWELL eqn)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} \simeq -\nabla^2 \vec{E}$$

$$-\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial}{\partial t}\left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right) \simeq -\mu\sigma \frac{\partial \vec{E}}{\partial t}$$

$$\therefore \quad \nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

(b) substituting for $\vec{E} = \vec{E}_0 \exp i(kz - \omega t)$ in the above eqn, we get

$$-k^2 E_0 = (-i\omega\mu\sigma) E_0$$

$$k = \pm \frac{1+i}{\sqrt{2}} \sqrt{\omega\mu\sigma}$$

The electric field takes the ~~form~~ form, for $z > 0$ (conducting region).

$$\therefore \quad \vec{E} = \vec{E}_0 \exp\left[i\left(\sqrt{\frac{\mu\sigma\omega}{2}}\, z - \omega t\right)\right] \exp\left[-\sqrt{\frac{\mu\sigma\omega}{2}}\, z\right].$$

(c) From the above we see the amplitude of electric field decreases exponentially as z increases. The depth at which the field decays to 1/e of its value at $z = 0$ is called "SKIN DEPTH"

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

[A typical metal, with $\sigma \sim 10^7 (\Omega m)^{-1}$, $\mu \sim 10^{-6}\, N/A^2$ we get $\delta = 10^{-8}\, m$ for $\omega = 10^{15} s^{-1}$ (OPTICAL FREQUENCIES)]