EE1101 : Signals and Systems JAN-MAY 2018

Tutorial 9 Solutions

1. (a) $e^{-2t}\cos(3t)u(t)$

$$\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\
&= \int_{-\infty}^{\infty} e^{-2t}\cos(3t)u(t)e^{-st}dt \\
&= \frac{1}{2}\int_{0}^{\infty} e^{-2t}\big(e^{j3t}+e^{-j3t}\big)e^{-st}dt \\
&= \frac{1}{2}\int_{0}^{\infty} \big(e^{-(s+2-j3)t}+e^{-(s+2+j3)t}\big)dt \\
&= \frac{1}{2}\times\Big(\frac{1}{(s+2-j3)}+\frac{1}{(s+2+j3)}\Big) \\
&= \frac{(s+2)}{(s+2)^2+9}
\end{aligned}$$

Since the signal is right sided, ROC is Re{s} > -2.

(b) $f(t)=\sin(t), 0\leq t\leq 1$ and $f(t)=0$, elsewhere

$$\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} f(t)e^{-st}dt \\
&= \int_{0}^{1} \sin(t)e^{-st}dt \\
&= \frac{1}{2j}\int_{0}^{1} \big(e^{jt}-e^{-jt}\big)e^{-st}dt \\
&= \frac{1}{2j}\Big\{\Big(\frac{e^{-(s+j)}}{s+j}-\frac{e^{-(s-j)}}{s-j}\Big)-\Big(\frac{1}{s+j}-\frac{1}{s-j}\Big)\Big\} \\
&= \frac{1}{2j}\Big\{\Big(\frac{e^{-(s+j)}}{s+j}-\frac{e^{-(s-j)}}{s-j}\Big)+\Big(\frac{2j}{s^2+1}\Big)\Big\} \\
&= \frac{1}{2j}\Big\{\Big(\frac{e^{-s}(s(e^{-j}-e^{j})-j(e^{-j}-e^{j}))}{s^2+1}\Big)+\Big(\frac{2j}{s^2+1}\Big)\Big\} \\
&= \frac{1-e^{-s}(s\sin 1+\cos 1)}{s^2+1}
\end{aligned}$$

Since the signal is of finite duration, ROC is the entire s - plane.

(c) $(e^{-4t}+e^{-5t}\sin t)u(t)$

$$\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\
&= \int_{-\infty}^{\infty} (e^{-4t}+e^{-5t}\sin t)u(t)e^{-st}dt \\
&= \int_{0}^{\infty} e^{-(s+4)t}dt+\int_{0}^{\infty} e^{-(s+5)t}\sin t dt \\
&= \int_{0}^{\infty} e^{-(s+4)t}dt+\frac{1}{2j}\int_{0}^{\infty} e^{-(s+5)t}(e^{jt}-e^{-jt})dt \\
&= \frac{1}{s+4}+\frac{1}{2j}\Big(\frac{1}{(s+5-j)}-\frac{1}{(s+5+j)}\Big) \\
&= \frac{1}{s+4}+\frac{1}{(s+5)^2+1}
\end{aligned}$$

The poles are at $s=-4$ and $s=-5\pm$j. Since the signal is right sided, ROC is to the right of the rightmost pole *i.e.* Re{s} > -4

(d) $e^{-2t}u(t-1)$

$$\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\
&= \int_{-\infty}^{\infty} e^{-2t}u(t-1)e^{-st}dt \\
&= \int_{1}^{\infty} e^{-2t}e^{-st}dt \\
&= \int_{1}^{\infty} e^{-(s+2)t}dt \\
&= \frac{e^{-(s+2)}}{(s+2)}
\end{aligned}$$

Since the signal is right sided, ROC is Re{s} > -2.

(e) $e^{-2(t-1)}u(t-1)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$
$$= \int_{-\infty}^{\infty} e^{-2(t-1)}u(t-1)e^{-st}dt$$
$$= \int_{1}^{\infty} e^{-2(t-1)}e^{-st}dt$$
$$= e^2 \int_{1}^{\infty} e^{-(s+2)t}dt$$
$$= \frac{e^{-s}}{(s+2)}$$

Since the signal is right sided, ROC is Re{s} > -2.

(f) $e^{2t}u(-t) + e^{3t}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$
$$= \int_{-\infty}^{\infty} \left(e^{2t}u(-t) + e^{3t}u(-t)\right)e^{-st}dt$$
$$= \int_{-\infty}^{0} e^{-(s-2)}dt + \int_{-\infty}^{0} e^{-(s-3)}dt$$
$$= -\frac{1}{s-2} - \frac{1}{s-3}$$
$$= -\frac{2s-5}{(s-2)(s-3)}$$

The poles are at $s = -2$ and $s = -3$. Since the signal is left sided, ROC is to the left of the left-most pole *i.e.* Re{s} < 2

(g) $te^{-2|t|}$

Let $x(t) = e^{-2|t|} = e^{-2t}u(t) + e^{2t}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$
$$= \int_{-\infty}^{\infty} \left(e^{-2t}u(t) + e^{2t}u(-t)\right)e^{-st}dt$$
$$= \int_{0}^{\infty} e^{-(s+2)}dt + \int_{-\infty}^{0} e^{-(s-2)}dt$$
$$= \frac{1}{s+2} - \frac{1}{s-2}$$
$$= \frac{4}{4-s^2}$$

Since $x(t)$ is two sided, ROC is -2 < Re{s} < 2.
Given signal $f(t) = tx(t)$

$$F(s) = -\frac{dX(s)}{ds}$$
$$= -\frac{8s}{(4-s^2)^2}$$

Since the signal is two sided, ROC is -2 < Re{s} < 2.

2. (a) $\frac{1}{s(s+1)}$, Re(s) > 0

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

Since the ROC is Re(s) > 0, the signal is right sided.

$$\frac{1}{s+a},\ \text{Re(s)} > -a \longleftrightarrow e^{-at}u(t)$$
$$\Longrightarrow \frac{1}{s} - \frac{1}{s+1} \longleftrightarrow (1-e^{-t})u(t)$$

(b) $\frac{1}{s(s+1)}$, Re(s) < -1

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

Since the ROC is Re(s) < -1, the signal is left sided.

$$\frac{1}{s+a},\ \text{Re(s)} < -a \longleftrightarrow -e^{-at}u(-t)$$
$$\Longrightarrow \frac{1}{s} - \frac{1}{s+1} \longleftrightarrow -(1-e^{-t})u(-t)$$

(c) $\frac{1}{s(s+1)}$, -1 < Re(s) < 0

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

Since the ROC is -1 < Re(s) < 0, the signal is two sided.

$$\Longrightarrow \frac{1}{s} - \frac{1}{s+1} \longleftrightarrow -u(-t) - e^{-t}u(t)$$

(d) $\frac{s+1}{(s+1)^2+9}$, Re(s)< -1

$$e^{at}[\cos(bt)]u(t) \longleftrightarrow \frac{s-a}{(s-a)^2+b^2}\text{ROC: Re(s)>a}$$

$$\frac{-(s+1)}{(s+1)^2+9},\text{ROC: Re(s)<-1} \longleftrightarrow e^{-t}[\cos 3t]u(-t)$$
$$\Longrightarrow \frac{(s+1)}{(s+1)^2+9},\text{ROC: Re(s)<-1} \longleftrightarrow -e^{-t}[\cos 3t]u(-t)$$

(e) $\dfrac{s+1}{s^2+5s+6}$, -3< Re(s) < -2

$$\frac{s+1}{s^2+5s+6} = \frac{2}{s+3} - \frac{1}{s+2}$$

Since the ROC is a strip, the signal is two sided.

$$\implies \frac{2}{s+3} - \frac{1}{s+2} \longleftrightarrow 2e^{-3t}u(t) + e^{-2t}u(-t)$$

(f)

$$F(s) = e^{-s}\frac{10s^2}{(s+1)(s+3)}$$

$$\frac{s^2}{(s+1)(s+3)} = 1 - \Big(\frac{4s+3}{(s+1)(s+3)}\Big)$$

Using partial fraction expansion,

$$\frac{10s^2}{(s+1)(s+3)} = 10\Big(1 + \frac{1}{2(s+1)} - \frac{9}{2(s+3)}\Big)$$

$$\begin{aligned}\delta(t) &\longleftrightarrow 1\\ e^{-at}u(t) &\longleftrightarrow \frac{1}{s+a},\ \text{Re(s)} > -a\\ \frac{10s^2}{(s+1)(s+3)} &\longleftrightarrow 10\Big(\delta(t) + \frac{1}{2}e^{-t}u(t) - \frac{9}{2}e^{-3t}u(t)\Big)\end{aligned}$$

Using the time shifting property of Laplace transform,

$$x(t-t_0) \longleftrightarrow e^{-st_0}X(s)$$

Therefore,

$$e^{-s}\frac{10s^2}{(s+1)(s+3)} \longleftrightarrow 10\delta(t-1) + 5e^{-(t-1)}u(t-1) - 45e^{-3(t-1)}u(t-1)$$

(g)

$$F(s) = \frac{d}{ds}\Big(e^{-2s}\frac{1}{(s+2)^2}\Big)$$

$$\frac{1}{(s+2)^2} = -\frac{d}{ds}\Big(\frac{1}{s+2}\Big)$$

Using the differentiation in s domain property,

$$tx(t) \longleftrightarrow -\frac{d}{ds}X(s)$$

$$\implies \frac{1}{(s+2)^2} \longleftrightarrow te^{-2t}u(t)$$

Using the time shifting property of Laplace transform,

$$\begin{aligned}x(t-t_0) &\longleftrightarrow e^{-st_0}X(s)\\ \frac{e^{-2s}}{(s+2)^2} &\longleftrightarrow (t-2)e^{-2(t-2)}u(t-2)\\ \frac{d}{ds}\Big(\frac{e^{-2s}}{(s+2)^2}\Big) &\longleftrightarrow -t(t-2)e^{-2(t-2)}u(t-2)\end{aligned}$$

3. (a)

$$E(s) = \frac{s+1}{(s+1)^2+4}$$

Laplace transform of e(t) is given by:

$$E(s) = \int_{t=-\infty}^{\infty} e(t)e^{-st}\mathrm{d}t$$

Since $s=0$ is included in the ROC, let us find the Laplace Transform at $s=0$

$$E(0) = \int_{t=-\infty}^{\infty} e(t)\mathrm{d}t$$

$$\begin{aligned}\int_{t=-\infty}^{\infty} e(t)\mathrm{d}t &= E(0)\\ &= \frac{0+1}{(0+1)^2+4}\\ &= \frac{1}{5}\end{aligned}$$

(b) Using the differentiation in s domain property,

$$te(t) \longleftrightarrow -\frac{d}{ds}E(s)$$

$$\begin{aligned}\int_{t=-\infty}^{\infty} te(t)\mathrm{d}t &= -\frac{d}{ds}E(s)|_{s=0}\\ &= -\frac{4-(s+1)^2}{((s+1)^2+4)^2}|_{s=0}\\ &= -\frac{4-1}{(1+4)^2}\\ &= -\frac{3}{25}\end{aligned}$$

4. (a) $x(t) = u(t-2)$

1.

$$\begin{aligned}X(s) &= \int_0^\infty u(t-2)e^{-st}dt\\ &= \int_2^\infty e^{-st}dt\\ &= \frac{e^{-2s}}{s}\end{aligned}$$

ROC is Re(s) > 0.

2. No. Since Re(s) does not include σ=0(jω axis) on the s-plane, Fourier transform does not exist.

(b) $x(t) = u(t) - u(t-3)$

1.

$$\begin{aligned} X(s) &= \int_0^{\infty} (u(t) - u(t-3))e^{-st} dt \\ &= \int_0^{3} e^{-st} dt \\ &= \frac{1-e^{-3s}}{s} \end{aligned}$$

ROC is the entire s-plane.

2. Yes. Since Re(s) includes σ=0(jω axis) on the s-plane, $X(j\omega) = \dfrac{1-e^{-3j\omega}}{j\omega}$

$$\begin{aligned} \frac{2}{j\omega} &\longleftrightarrow sgn(t) = 2u(t) - 1 \\ \frac{1}{j\omega} &\longleftrightarrow u(t) - \frac{1}{2} \\ \frac{e^{-3j\omega}}{j\omega} &\longleftrightarrow u(t-3) - \frac{1}{2} \end{aligned}$$

(Time shifting property of Fourier transform)

$X(j\omega) \longleftrightarrow u(t) - u(t-3)$

(c) $x(t) = e^{3t}u(t)$

1.

$$\begin{aligned} X(s) &= \int_0^{\infty} e^{3t} e^{-st} dt \\ &= \int_0^{\infty} e^{(-s+3)t} dt \\ &= \frac{1}{s-3} \end{aligned}$$

ROC is Re(s)>3.

2. No. Since Re(s) does not include σ=0(jω axis) on the s-plane, Fourier transform does not exist.

(d) $x(t) = te^{-t}u(t)$

1.

$$\begin{aligned} X(s) &= \int_0^{\infty} te^{-t} e^{-st} dt \\ &= \frac{1}{(s+1)^2} \end{aligned}$$

ROC is Re(s)>-1.

2. Yes. Since Re(s) does include jω axis on the s-plane, $X(j\omega) = \dfrac{1}{(j\omega+1)^2}$

$$\begin{aligned} \frac{1}{(j\omega+1)^2} &= j\frac{d}{d\omega}\Big(\frac{1}{j\omega+1}\Big) \\ \frac{1}{j\omega+1} &\longleftrightarrow e^{-t}u(t) \\ j\frac{d}{d\omega}\Big(\frac{1}{j\omega+1}\Big) &\longleftrightarrow te^{-t}u(t) \\ \frac{1}{(j\omega+1)^2} &\longleftrightarrow te^{-t}u(t) \end{aligned}$$

(e) $x(t) = \sin t u(t)$

1.

$$\begin{aligned} X(s) &= \int_0^{\infty} sin(t) e^{-st} dt \\ &= \frac{1}{2j}\int_0^{\infty} [e^{jt} - e^{-jt}]e^{-st} dt \\ &= \frac{1}{s^2+1} \end{aligned}$$

ROC is Re(s)>0.

2. No. Since Re(s) does not include jω axis on the s-plane, Fourier transform does not exist.

5. $x_1(t) \quad \leftrightarrow \dfrac{1}{s+2}$, $ROC: Re\{s\} > -2$

Using the time shifting property,

$x_1(t-2) \quad \leftrightarrow \dfrac{e^{-2s}}{s+2}$, $Re\{s\} > -2$

$x_2(t) \quad \leftrightarrow \dfrac{1}{s+3}$, $ROC: Re\{s\} > -3$

Using the time scaling property,

$x_2(-t) \quad \leftrightarrow \dfrac{1}{-s+3}$, $ROC: Re\{s\} < 3$

Using the time shifting property,

$x_2(-(t-3)) \quad \leftrightarrow \dfrac{e^{-3s}}{-s+3}$, $ROC: Re\{s\} < 3$

Using the convolution property,

$x_1(t-2) * x_2(-t+3) \quad \leftrightarrow \dfrac{e^{-2s}}{s+2}\,\dfrac{e^{-3s}}{-s+3} = \dfrac{e^{-5s}}{(2+s)(3-s)}$

$ROC: -2 < Re\{s\} < 3$

6. (a)

$$\begin{aligned} X(s) &= \int_{-\infty}^{+\infty} x(t)e^{-st} dt \\ &= \int_{-\infty}^{+\infty} x(-t)e^{-st} dt \quad (x \ is \ even, put - t = p) \\ &= \int_{-\infty}^{+\infty} x(p)e^{sp} dp \\ &= X(-s) \end{aligned}$$

Therefore a even function of time has a even Laplace transform

(b) The Laplace transform $X(s)$ with zeros at $s = z_1, z_2, .., z_m$ and poles at $s = p_1, p_2, .., p_n$ can be expressed as,

$$X(s) = \frac{(s - z_1)(s - z_2)...(s - z_m)}{(s - p_1)(s - p_2)...(s - p_n)}$$

Using the above, Laplace transform for the given pole-zero plots can be written as:
1. $X_1(s) = \dfrac{s}{(s+1)(s-1)}$
2. $X_2(s) = \dfrac{(s+1)(s-1)}{s}$
3. $X_3(s) = \dfrac{(s+j)(s-j)}{(s+1)(s-1)}$
4. $X_4(s) = \dfrac{s-1}{s+1}$

Using the result from part (a), only $X_3(s)$ satisfies the property that $X_3(s) = X_3(-s)$. Hence $x_3(t)$ is an even function of time. The ROC is $-1 < Re\{s\} < 1$ since the signal is even and has to be two sided.

7.

$$\delta(t) \longleftrightarrow 1$$

Using the time shifting property,

$$\delta(t - nT) \longleftrightarrow e^{-nTs}$$

$$\begin{aligned} \text{Given } x(t) &= \sum_{n=0}^{\infty} e^{-nT} \delta(t - nT) \\ X(s) &= \sum_{n=0}^{\infty} e^{-nT} e^{-snT} \\ &= \sum_{n=0}^{\infty} e^{-nT(s+1)} \\ &= \frac{1}{1 - e^{-T(s+1)}} \end{aligned}$$

To find the poles :
$1 - e^{-T(s+1)} = 0$
$\implies -T(s+1) = j2\pi k$
$\implies s_k = -1 - j\dfrac{2\pi k}{T}, k = 0, \pm 1, \pm 2, ..$
Since the signal is right sided, ROC:Re{s} > -1.