EE1101 Signals and Systems JAN—MAY 2019
Tutorial 6 Solutions

1)

$$H(j\omega) = \begin{cases} 1 & , |\omega| \geq 250 \\ 0 & , \text{otherwise} \end{cases}$$

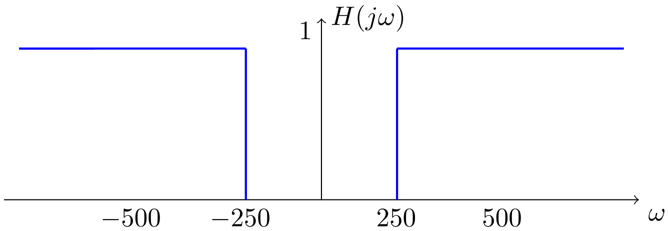


Fig. 1: $H(j\omega)$ of Q1

The system $H(j\omega)$ passes only the frequency components greater than 250 rad/s. The characteristics are shown in Fig. 1.
Since the output is identical to input, this implies that the input contains only frequencies greater than 250.
Hence, for the input $x(t)$, Fourier coefficients, a_k (corresponding to the frequencies: $k\omega_0$) need to be 0 for:

$$\begin{aligned} |k\omega_0| &< 250 \\ |k| &< \frac{250}{14} = 17.85 \end{aligned}$$

Since k is integer, $a_k = 0$ for $|k| \leq 17$.

2)

$$\begin{aligned} x(t) &= 2 + \sum_{k=1}^{3} 3 \sin \frac{k\pi}{2} \cos 100k\pi t \\ &= 2 + 3\big(\sin \frac{\pi}{2} \cos 100\pi t + \sin \pi \cos 200\pi t \\ &\quad + \sin \frac{3\pi}{2} \cos 300\pi t \big) \\ &= 2 + 3 \cos 100\pi t - 3 \cos 300\pi t \\ &= 2 + \frac{3}{2}\big(e^{j100\pi t} + e^{-j100\pi t}\big) \\ &\quad - \frac{3}{2}\big(e^{j300\pi t} + e^{-j300\pi t}\big). \end{aligned}$$

The fundamental frequency is $\omega_0 = 100\pi$, and the non-zero Fourier coefficients of $x(t)$ are

$$a_n = \begin{cases} 2, & \text{for } n = 0 \\ \frac{3}{2}, & \text{for } n = 1, -1 \\ -\frac{3}{2}, & \text{for } n = 3, -3 \end{cases}$$

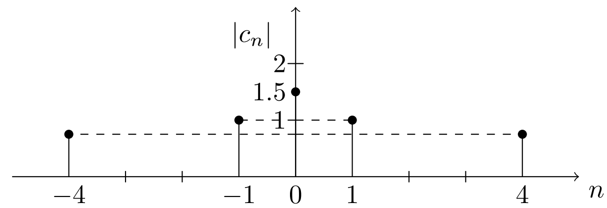
The non-zero Fourier series of coefficients of $\cos(100\pi t)$ are $b_1 = b_{-1} = 1/2$. Using the multiplication property of Fourier series we get

$$y(t) = x(t)\cos(100\pi t) \overset{\text{FS}}{\longleftrightarrow} c_n = \sum_{l=-\infty}^{\infty} a_l b_{n-l}.$$

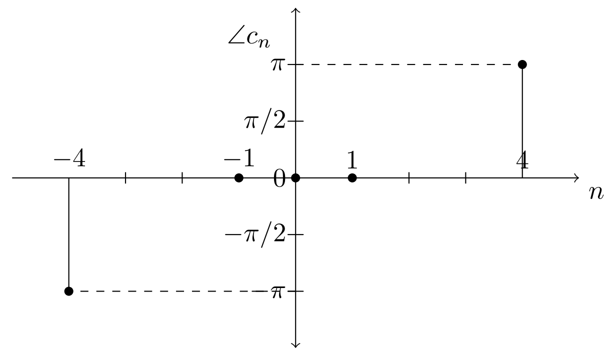
Therefore,

$$\begin{aligned} c_n &= a_n \star b_n \\ &= \big(-\frac{3}{2}\delta[n+3] + \frac{3}{2}\delta[n+1] + 2\delta[n] + \frac{3}{2}\delta[n-1] \\ &\quad - \frac{3}{2}\delta[n-3]\big) \star \big(\frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n-1]\big) \\ &= \frac{3}{2}\delta[n] + \delta[n-1] + \delta[n+1] \\ &\quad - \frac{3}{4}\big(\delta[n-4] + \delta[n+4]\big). \end{aligned}$$

The magnitude and phase spectrum are plotted in Fig. 2a and Fig. 2b respectively



(a) Magnitude spectrum



(b) Phase spectrum

Fig. 2: Q2. Magnitude and phase spectra of the Fourier series coefficients of $y(t)$.

3) The frequency response:

$$H_l(j\omega) = \begin{cases} 1, & |\omega| < 2\pi.500 rad/s \\ 0, & \text{otherwise} \end{cases}$$

The filter characteristics are shown in Fig. 3.

(a) $x(t) = \cos(2\pi.750t) + \sin(2\pi.1500t)$. Fourier

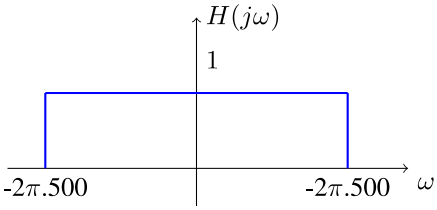


Fig. 3: $H(j\omega)$ of Q3

series expansion of $x(t)$:

$$x(t) = \frac{1}{2}\left(e^{j2\pi.750t} + e^{-j2\pi.750t}\right) + \frac{1}{2j}\left(e^{j2\pi.1500t} - e^{-j2\pi.1500t}\right).$$

The fundamental frequency $\omega_0 = 2\pi.750$. By inspection, the Fourier series coefficients of $x(t)$ are

$$a_{-1} = a_1 = \frac{1}{2}, \qquad a_{-2}^* = a_2 = \frac{1}{2j}.$$

Using the synthesis equation, the output can be written as,

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jn\omega_o)e^{jn\omega_o t}.$$

Since, $H(jn\omega_o)$ is 0 for all n, $y(t) = 0$.

(b) $x(t) = \cos(2\pi.150t) + \sin(2\pi.1500t)$. Fourier series expansion of $x(t)$:

$$x(t) = \frac{1}{2}\left(e^{j2\pi.150t} + e^{-j2\pi.150t}\right) + \frac{1}{2j}\left(e^{j2\pi.1500t} - e^{-j2\pi.1500t}\right).$$

The fundamental frequency $\omega_0 = 2\pi.150$. By inspection, the Fourier series coefficients of $x(t)$ are

$$a_{-1} = a_1 = \frac{1}{2}, \qquad a_{-10}^* = a_{10} = \frac{1}{2j}.$$

Using the synthesis equation, the output can be written as,

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jn\omega_o)e^{jn\omega_o t}.$$

Since, $H(jn\omega_o)$ is 0 for all $n \neq \{1, -1\}$, $y(t) = \cos(2\pi.150t)$.

(c) Periodic square wave with period 4.5 ms, oscillates between +1 V and -1 V with 50% duty cycle and is an even function of time.

$$x(t) = \begin{cases} 1, & 0 < t < T/4 \\ -1, & T/4 < t < 3T/4 \\ 1, & 3T/4 < t < T \end{cases} \tag{1}$$

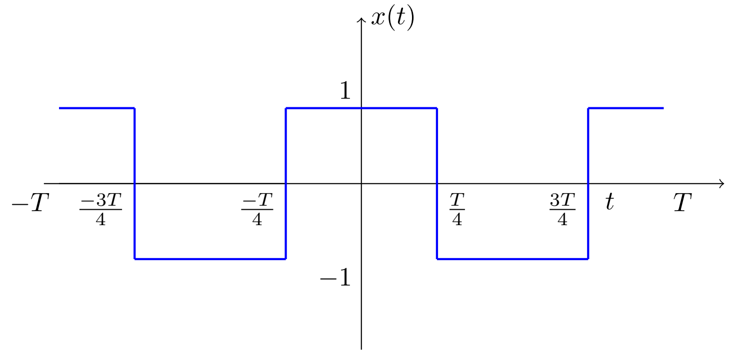


Fig. 4: $x(t)$ of Q3 part c

$$x(t) = x(t + nT), T = 4.5 \text{ ms}$$

Fourier series expansion of $x(t)$:

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_o t}$$

Here $a_o = \frac{1}{T}\int_0^T x(t)dt = 0$

$$\begin{aligned} a_n &= \frac{1}{T}\int_0^T x(t)e^{-jn\omega_o t}dt \\ &= \frac{1}{T}\Big(\int_0^{\frac{T}{4}} e^{-jn\omega_o t}dt \; - \int_{\frac{T}{4}}^{\frac{3T}{4}} e^{-jn\omega_o t}dt \\ &\quad + \int_{\frac{3T}{4}}^{T} x(t)e^{-jn\omega_o t}dt\Big) \end{aligned}$$

simplification yields $a_n = \frac{2}{n\pi}\sin\left(n\frac{\pi}{2}\right)$.
Using the synthesis equation, the output can be written as,

$$\begin{aligned} y(t) &= \sum_{n=-\infty}^{\infty} a_n H(jn\omega_o)e^{jn\omega_o t} \\ &= \sum_{n=-\infty, n \text{ odd}}^{\infty} \frac{2}{n\pi}\sin\left(n\frac{\pi}{2}\right)H(jn\omega_o)e^{jn\omega_o t} \end{aligned}$$

Where, $\omega_o = \dfrac{2\pi}{4.5 \text{ ms}} = 2\pi(222.2)$ rad/s. Thus, $H(j\omega)$ is non-zero only for $n = -1, 1, -2, 2$

$$\begin{aligned} y(t) = &-\frac{2}{\pi}\sin\left(\frac{-\pi}{2}\right)e^{-j\omega_0 t} + \frac{2}{\pi}\sin\left(\frac{\pi}{2}\right)e^{j\omega_0 t} \\ &- \frac{2}{2\pi}\sin\left(\frac{-2\pi}{2}\right)e^{-j2\omega_0 t} + \frac{2}{2\pi}\sin\left(\frac{2\pi}{2}\right)e^{j2\omega_0 t} \\ = &\frac{4}{\pi}\left(\frac{e^{-j\omega_0 t} + e^{j\omega_0 t}}{2}\right) + 0 \\ = &\frac{4}{\pi}\cos(\omega_0 t). \end{aligned}$$

4) The impulse response for the LTI system is

$$h(t) = \delta(t) - e^{-t}u(t),$$
$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt,$$
$$H(j\omega) = 1 - \frac{1}{1+j\omega}$$

(a) $x(t) = \cos(3\pi t) + \dfrac{\pi}{3}$
Fourier series expansion of $x(t)$:

$$x(t) = \frac{\pi}{3} + \frac{1}{2}\left(e^{j3\pi t} + e^{-j3\pi t}\right).$$

The fundamental frequency $\omega_0 = 3\pi$ and period is $T_0 = 2/3$. By inspection, the Fourier series coefficients of $x(t)$ are

$$a_0 = \frac{\pi}{3}, \qquad a_{-1} = \frac{1}{2} \qquad a_1 = \frac{1}{2}.$$

If $e^{j\omega t}$ is the input to a LTI system, then the output is $H(j\omega t)e^{j\omega t}$. Thus, the output of the given LTI system is,

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jnw_o)e^{jnw_o t}$$
$$= \frac{3\pi\big(3\pi\cos 3\pi t - \sin 3\pi t\big)}{1+9\pi^2}$$

(b) $x(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$

Fourier series expansion of $x(t)$:

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jnw_o t}$$

where, $a_o = 1, a_n = 1, \omega_o = 2\pi$

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jnw_o)e^{jnw_o t}$$
$$= \sum_{n=-\infty}^{\infty} \frac{nj2\pi}{1+nj2\pi}e^{jn2\pi t}$$

(c) $x(t) = \sum_{n=-\infty}^{\infty} (-1)^n\delta(t-2n)$

Fourier series expansion of $x(t)$:

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_o t}$$
$$a_n = \frac{-1}{3}\int_0^4 [\delta(t) - \delta(t-2)]e^{jn\omega_0 t}dt$$

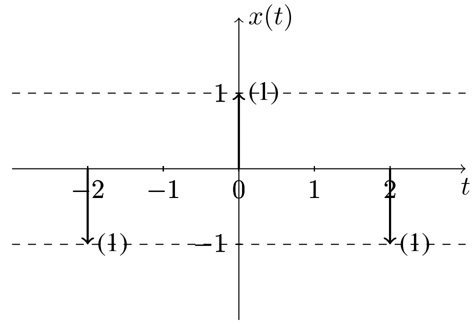


Fig. 5: Q4 (c)

where, $a_o = 0, a_n = \dfrac{1-e^{-jn\pi}}{4}, \omega_o = \pi/2$

$$y(t) = \sum_{n=-\infty}^{\infty} a_n\, H(jn\omega_o)e^{jn\omega_o t}$$
$$= \sum_{n=-\infty, n \text{ odd}}^{\infty} \frac{nj\frac{\pi}{2}}{1+nj\frac{\pi}{2}}\left(\frac{1}{2}e^{jn\frac{\pi}{2}t}\right).$$

5) Given the Fourier Series representation of $x(t)$, its Fourier Series coefficients, a_k are given by

$$a_k = \alpha^{|k|}$$

and its fundamental frequency $\omega_0 = \frac{\pi}{4}$. For inputs of the form $x(t) = e^{j\omega t}$, the output of an LTI system is given by

$$y(t) = H(j\omega)x(t).$$

Therefore we can write

$$y(t) = \sum_{k=-\infty}^{\infty} \alpha^{|k|}H(jk\omega_0)e^{jk\omega_0 t}.$$

Now $H(jk\omega_0)$ is non-zero only for

$$k\omega_0 \le |W|,$$

i.e.,

$$k \le |\frac{4W}{\pi}|.$$

Let

$$\frac{4W}{\pi} = N.$$

Then

$$y(t) = \sum_{k=-N}^{N} \alpha^{|k|}e^{jk\omega_0 t}.$$

The average energy of $x(t)$ over a period is given by

$$\begin{aligned} E_{avg}\{x(t)\} &= \frac{1}{T}\int_T |x(t)|^2 dt \\ &= \sum_{k=-\infty}^{\infty} |a_k|^2 \quad (Parseval's\ relation) \\ &= \sum_{k=-\infty}^{\infty} |\alpha^{|k|}|^2 \\ &= 1 + \sum_{k=1}^{\infty} |\alpha^{|k|}|^2 \\ &= \frac{1+\alpha^2}{1-\alpha^2}. \end{aligned}$$

In last expression Geometric series formula is used. Similarly, the average energy of $y(t)$ is

$$\begin{aligned} E_{avg}\{y(t)\} &= \sum_{k=-N}^{N} |\alpha^{|k|}|^2 \\ &= 1 + \sum_{k=1}^{N} |\alpha^{|k|}|^2 \\ &= \frac{1 - 2\alpha^{2N+2} + \alpha^2}{1-\alpha^2}. \end{aligned}$$

Now, we have to find N for which

$$E_{avg}\{y(t)\} = 0.9 E_{avg}\{x(t)\}.$$

This implies

$$\frac{1 - 2\alpha^{2N+2} + \alpha^2}{1-\alpha^2} = 0.9\frac{1+\alpha^2}{1-\alpha^2}.$$

which gives the value of N as

$$N = \frac{log(0.05) + log(\frac{1+\alpha^2}{\alpha^2})}{2log(\alpha)}.$$

Therefore $W = \frac{\pi}{4}N$ where N is as above.

6) We first evaluate the frequency response of the system. Consider an input $x(t)$ of the form $e^{j\omega t}$. To such an input, the output of the system will be

$$y(t) = H(j\omega)e^{j\omega t},$$

where $H(j\omega)$ is the frequency response of the system. Substituting the above input and output in the given differential equation, we get

$$H(j\omega)j\omega e^{j\omega t} + 4H(j\omega)e^{j\omega t} = e^{j\omega t}.$$

Therefore,

$$H(j\omega) = \frac{1}{j\omega + 4}.$$

Using the synthesis equation for the output, we have

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jn\omega_0)e^{jn\omega_0 t},$$

where a_n are the Fourier series coefficients of input $x(t)$. Therefore, Fourier series coefficients of $y(t)$ are $a_n H(jn\omega_0)$.

We now apply this to the given input $x(t) = \cos(2\pi t)$. Here, the fundamental frequency is $\omega_0 = 2\pi$, and the non-zero Fourier series coefficients of $x(t)$ are $a_1, a_{-1} = 1/2$ (this is left for the reader to derive). Therefore, the non-zero Fourier series coefficients of $y(t)$ are as follows

$$\begin{aligned} b_1 &= a_1 H(j2\pi) = \frac{1}{2(4+j2\pi)}, \\ b_{-1} &= a_{-1}H(-j2\pi) = \frac{1}{2(4-j2\pi)}. \end{aligned}$$

Using the synthesis equation, we get

$$y(t) = \frac{1}{2(4+j2\pi)}e^{j2\pi t} + \frac{1}{2(4-j2\pi)}e^{-j2\pi t},$$

which on simplification yields

$$y(t) = \frac{1}{4+\pi^2}\cos(2\pi t) + \frac{\pi}{8+2\pi^2}\sin(2\pi t).$$

7) (a) The non-zero Fourier series coefficients of $x(t)$ are $a_1 = a_{-1} = \frac{1}{2}$ (left for the reader to derive).

(b) The non-zero Fourier series coefficients of $y(t)$ are $b_1 = b_{-1}^* = \frac{1}{2j}$ (left for the reader to derive).

(c) Using the multiplication property of Fourier series, we know that

$$z(t) = x(t)y(t) \overset{FS}{\longleftrightarrow} c_n = \sum_{l=-\infty}^{\infty} a_l b_{n-l}.$$

Therefore,

$$c_n = a_n \star b_n = \frac{1}{4j}\delta[n-2] - \frac{1}{4j}\delta[n+2].$$

This implies that the nonzero Fourier series coefficients of $z(t)$ are $c_2 = c_{-2}^* = \frac{1}{4j}$.

(d)

$$z(t) = \sin(4\pi t)\cos(4\pi t) = \frac{1}{2}\sin(8\pi t).$$

The derivation of the Fourier series coefficients is left to the reader.