Department of Mathematics, IIT Madras
MA-1102 Series & Matrices
Assignment-3-Sol **Matrix Operations**

1. Show that given any $n \in \mathbb{N}$ there exist matrices $A, B \in \mathbb{R}^{n\times n}$ such that $AB \neq BA$.

   Let $A = [e_2\ e_1\ e_3\ e_4\ \cdots\ e_n]$ and $B = [v\ u\ u\ \cdots\ u]$, where $e_1, \ldots, e_n$ are standard basis vectors of $\mathbb{R}^{n\times 1}$ and $u = (1,1,1,\ldots,1)^t$, $v = (0,0,0,\ldots,0)^t$.

2. Let $A = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}$. Compute A^n.

   We show that $A^n = \begin{bmatrix}1 & n & n(n-1)\\ 0 & 1 & 2n\\ 0 & 0 & 1\end{bmatrix}$ for $n \in \mathbb{N}$ by induction.
   The basis case $n = 1$ is obvious. Suppose A^n is as given. Now,
   $$A^{n+1} = AA^n = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & n & n(n-1)\\ 0 & 1 & 2n\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & n+1 & (n+1)n\\ 0 & 1 & 2(n+1)\\ 0 & 0 & 1\end{bmatrix}.$$
   Notice that taking $n = 0$ in the matrix A^n, we see that $A^0 = I$.

3. Let $A \in \mathbb{F}^{m\times n}$; $B \in \mathbb{F}^{n\times k}$. Let $A_1, \ldots, A_m$ be the rows of A and let $B_1, \ldots, B_k$ be the columns of B. Show that

   (a) $A_1B, \ldots, A_MB$ are the rows of AB. (b) $AB_1, \ldots, AB_k$ are the columns of AB.

   (a) The jth entry in A_iB is $A_i \cdot B_j$, which is the (i,j)th entry in AB.
   (b) The ith entry in AB_i is $A_i \cdot B_j$, which is the (i,j)th entry in AB.

4. Let $A \in \mathbb{F}^{n\times n}$; I be the identity matrix of order n. Find the inverse of the $2n \times 2n$ matrix $\begin{bmatrix}I & A\\ 0 & I\end{bmatrix}$.

   $\begin{bmatrix}I & A\\ 0 & I\end{bmatrix}^{-1} = \begin{bmatrix}I & -A\\ 0 & I\end{bmatrix}.$

5. If A is a hermitian (symmetric) invertible matrix, then show that A^{-1} is hermitian (symmetric).

   $A^* = A$. Then $(A^{-1})^* = (A^*)^{-1} = A^{-1}$. So, A^{-1} is hermitian.

6. If A is a lower (upper) triangular invertible matrix, then A^{-1} is lower (upper) triangular.

   Suppose A is a lower triangular matrix. Let D be the diagonal matrix whose diagonal entries are exactly the diagonal entries of A in the correct order. Since A is invertible, D is also invertible. Then write $A = D(I+N)$. Here, N is a lower triangular matrix with all diagonal entries as 0. Then verify that $A^{-1} = (I - N)D^{-1}$. Also, verify that this is a lower triangular matrix.

7. Let $x, y \in \mathbb{F}^{1\times n}$ (or in $\mathbb{F}^{n\times 1}$); $\alpha \in \mathbb{F}$. Prove the following:

   (a) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2$. (*Parallelogram Law*)
   (b) $|\langle x, y\rangle| \leq \|x\|\,\|y\|$. (*Cauchy-Schwartz inequality*)
   (c) $\|x+y\| = \|x\| + \|y\|$. (*Triangle inequality*)
   (d) If $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. (*Pythagoras' Law*)

   (a) Expand the norms using inner product.
   (b) A proof of Cauchy-Schwartz inequality goes as follows:

   If $y = 0$, then the inequality clearly holds. Else, $\langle y, y\rangle \neq 0$. Write $\alpha = \frac{\langle x,y\rangle}{\langle y,y\rangle}$. Then $\overline{\alpha} = \frac{\langle y,x\rangle}{\langle y,y\rangle}$ and $\overline{\alpha}\langle x, y\rangle = |\alpha|^2\|y\|^2$. Then
   $$\begin{aligned} 0 &\leq \langle x-\alpha y, x-\alpha y\rangle = \langle x, x\rangle - \overline{\alpha}\langle x, y\rangle + \alpha\big(\overline{\alpha}\langle y, y\rangle - \langle y, x\rangle\big)\\ &= \|x\|^2 - \overline{\alpha}\langle x, y\rangle = \|x\|^2 - |\alpha|^2\|y\|^2 = \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^4}\|y\|^2. \end{aligned}$$

(c) The triangle inequality can be proved using Cauchy-Schwartz, as in the following:

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle \le \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\|.$$

(d) $\langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + \|y\|^2.$

8. Show that each orthogonal 2×2 matrix is either a reflection or a rotation.
If $A = [a_{ij}]$ is an orthogonal matrix of order 2, then $A^t A = I$ implies

$$a_{11}^2 + a_{21}^2 = 1 = a_{12}^2 + a_{22}^2,\ a_{11}a_{12} + a_{21}a_{22} = 0.$$

Thus, there exist α, β such that $a_{11} = \cos\alpha,\ a_{21} = \sin\alpha,\ a_{12} = \cos\beta,\ a_{22} = \sin\beta$ and $\cos(\alpha - \beta) = 0$. It then follows that A is in one of the following forms:

$$O_1 := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},\quad O_2 := \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Let $\overrightarrow{(a,b)}$ be the vector in the plane that starts at the origin and ends at the point (a, b). Writing the point (a, b) as a column vector $[a\ b]^t$, we see that the matrix product $O_1[a\ b]^t$ is the end-point of the vector obtained by rotating the vector $\overrightarrow{(a,b)}$ by an angle θ. Similarly, $O_2[a\ b]^t$ gives a point obtained by reflecting (a, b) along a straight line that makes an angle $\theta/2$ with the x-axis. Thus, O_1 is said to be a *rotation by an angle* θ and O_2 is called a *reflection by an angle* $\theta/2$ along the x-axis.

9. Determine linear independence of $\{(1,2,2,1),\ (1,3,2,1),\ (4,1,2,2),\ (5,2,4,3)\}$ in $\mathbb{C}^{1\times 4}$.

$(5,2,4,3) = 2(1,2,2,1) - 1(1,3,2,1) + 1(4,1,2,2)$. So, the set is lineraly dependent.

10. Let $u, v, w \in \mathbb{F}^{n\times 1}$. Show that $\{u, v, w\}$ is linearly independent iff $\{u+v, v+w, w+u\}$ is linearly independent.

$a(u+v) + b(v+w) + c(w+u) = 0 \Rightarrow (a+c)u + (a+b)v + (b+c)w = 0$
$\Rightarrow a+c = 0,\ a+b = 0,\ b+c = 0 \Rightarrow a = 0,\ b = 0, c = 0.$
Hence $\{u+v, v+w, w+u\}$ is linearly independent.
Conversely, $\alpha u + \beta v + \gamma w = 0 \Rightarrow \frac{\alpha+\beta-\gamma}{2}(u+v) + \frac{\beta+\gamma-\alpha}{2}(v+w) + \frac{\alpha+\gamma-\beta}{2}(w+u) = 0$
$\Rightarrow \alpha+\beta-\gamma = 0,\ \beta+\gamma-\alpha = 0,\ \alpha+\gamma-\beta = 0 \Rightarrow \alpha = \beta = \gamma = 0.$
So, $\{u, v, w\}$ is linearly independent.

11. Find a basis for the subspace $\{(a,b,c) : 2a + 3b - 4c = 0\}$ of $\mathbb{R}^{1\times 4}$.

As a subspace of $\mathbb{R}^{1\times 4}$, $\{(a,b,c) : 2a+3b-4c = 0\} = \{(a, b, \frac{2a+3b}{4}) : a, b \in \mathbb{R}\}$.
The vectors $(1, 0, 1/2)$ and $(0, 1, 3/4)$ are in the subspace.
$(a, b, \frac{2a+3b}{4}) = a(1,0,1/2) + b(0,1,3/4)$. So, these two vectors span the subspace.
Now, $a(1,0,1/2) + b(0,1,3/4) = (0,0,0) \Rightarrow a = 0, b = 0,\ \frac{2a+3b}{4} = 0$. So, the vectors are linearly independent. Hence a basis for the subspace is $\{(1,0,1/2),\ (0,1,3/4)\}$.

12. Let $A \in \mathbb{R}^{3\times 3}$ satisfy $A(a,b,c)^t = (a+b, 2a-b-c, a+b+c)^t$. Determine A and also its rank and nullity.

$Ae_1 = A(1,0,0)^t = (1,2,1)^t,\ Ae_2 = A(0,1,0)^t = (1,-1,1)^t,\ Ae_3 = A(0,0,1)^t = (0,-1,1)^t.$

So, $A = [Ae_1\ \ Ae_2\ \ Ae_3] = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$

Now, $aAe_1 + bAe_2 + cAe_3 = 0 \Rightarrow a+b = 0,\ 2a-b-c = 0,\ a+b+c = 0 \Rightarrow a = b = c = 0.$
Thus, the three columns of A are linearly independent. So, $\operatorname{rank}(A) = 3$ and $\operatorname{null}(A) = 3 - 3 = 0$.

13. Determine a basis of the subspace $U = \{(a,b,c,d,e) : a = c = e, b + d = 0\}$ of $\mathbb{R}^{1\times 5}$.

$U = \{(a, b, a, -b, a) : a, b \in \mathbb{R}\}$. We claim that $\{(1,0,1,0,1), (0,1,0,-1,0)\}$ is a basis of U.
First, these two vectors are in U.
Second, $(a,b,a,-b,a) = a(1,0,1,0,1) + b(0,1,0,-1,0)$. Thus, the set spans U.
Third, if $a(1,0,1,0,1) + b(0,1,0,-1,0) = 0$, then $a = b = 0$. So, the set is linearly independent.

14. Let $A \in \mathbb{F}^{m\times n}$ have rank r. Give reasons for the following:

(a) $\operatorname{rank}(A) \le \min\{m, n\}$.

(b) If $n > m$, then there exist $x, y \in \mathbb{F}^{n\times 1}$ such that $x \neq y$ and $Ax = Ay$.

(c) If $n < m$, then there exists $y \in \mathbb{F}^{m\times 1}$ such that for no $x \in \mathbb{F}^{n\times 1}$, $Ax = y$.

(d) If $n = m$, then as a map, A is one-one iff A is onto.

(a) $\operatorname{rank}(A)$ is the maximum number of linearly independent rows in A. So, $r \leq m$. Also, $\operatorname{rank}(A)$ is the maximum number of linearly independent columns in A. So, $r \leq n$. Therefore, $r \leq \min\{m, n\}$.

(b) Suppose $n > m$. Then $\operatorname{rank}(A) \leq m < n$ and $\operatorname{rank}(A) + \operatorname{null}(A) = n$ implies that $\operatorname{null}(A) > 0$. That is, $N(A)$ is at least one dimensional. So, there exists a nonzero vector $v \in \mathbb{F}^{n\times 1}$ such that $Av = 0$. But $A(0) = 0$. Thus we take $x = 0$ and y as this v so that $Ax = Ay$ but $x \neq y$. In other words, A as a map is not one-one.

(c) Suppose $n < m$. Then $\operatorname{rank}(A) \leq n < m$. So, $R(A)$ is a proper subspace of $\mathbb{F}^{m\times 1}$. That is, as a map, A is not onto. Then the conclusion follows.

(d) Suppose $n = m$. Now, A is one-one iff $N(A) = \{0\}$ iff $\operatorname{null}(A) = 0$ iff $\operatorname{rank}(A) = n = m$ iff $R(A) = \mathbb{F}^{m\times 1}$ iff A is onto.

15. Convert $\begin{bmatrix} 1 & -3 & 2 & -2 & 11 \\ 3 & 8 & -7 & 5 & 8 \\ 5 & 2 & -3 & 1 & 7 \end{bmatrix}$ into its row echelon form and row reduced echelon form using the algorithms. Then determine its rank and nullity.

(a) $\begin{bmatrix} 1 & -3 & 2 & -2 & 11 \\ 3 & 8 & -7 & 5 & 8 \\ 5 & 2 & -3 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & 2 & -2 & 11 \\ 0 & 17 & -13 & 11 & -25 \\ 0 & 0 & 0 & 0 & 23 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -5/17 & -1/17 & 0 \\ 0 & 1 & -13/17 & 11/17 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

rank $= 3$, nullity $= 2$

16. The vectors $u_1 = (1,1,0), u_2 = (0,1,1), u_3 = (1,0,1)$ form a basis for $\mathbb{F}^3$. Apply Gram-Schmidt Orthogonalization.

$v_1 = (1,1,0)$.

$v_2 = u_2 - \dfrac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = (0,1,1) - \dfrac{(0,1,1)\cdot(1,1,0)}{(1,1,0)\cdot(1,1,0)}(1,1,0) = (0,1,1) - \dfrac{1}{2}(1,1,0) = \Big(-\dfrac{1}{2}, \dfrac{1}{2}, 1\Big)$.

$v_3 = u_3 - \dfrac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \dfrac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$

$= (1,0,1) - (1,0,1)\cdot(1,1,0)(1,1,0) - (1,0,1)\cdot\Big(-\dfrac{1}{2}, \dfrac{1}{2}, 1\Big)\Big(-\dfrac{1}{2}, \dfrac{1}{2}, 1\Big)$

$= (1,0,1) - \dfrac{1}{2}(1,1,0) - \dfrac{1}{3}\Big(-\dfrac{1}{2}, \dfrac{1}{2}, 1\Big) = \Big(-\dfrac{2}{3}, \dfrac{2}{3}, -\dfrac{2}{3}\Big)$.

The set $\Big\{(1,1,0),\ \big(-\frac{1}{2}, \frac{1}{2}, 1\big),\ \big(-\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\big)\Big\}$ is orthogonal.

17. Let $A \in \mathbb{R}^{3\times 3}$ have the first two columns as $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3},)^t$ and $(1/\sqrt{2}, 0, -1/\sqrt{2})^t$. Determine the third column of A so that A is an orthogonal matrix.

Notice that the first two columns of A are already orthonormal, and orthogonal to each other. You can start with the third as $(0,0,1)^t$ and use Gram-Schmidt process. Alternatively, let the third column be $(a,b,c)^t$. Then the first two are orthogonal to the third implies $a+b+c=0,\ a-c=0$. This gives $(a,b,c)^t = (a,-2a,a)^t$. Now, the third column has norm 1 implies that $1 = a^2 + 4a^2 + a^2 = 6a^2 \Rightarrow a = \pm 1/\sqrt{6}$. Thus the third column of A is $\pm(1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6})^t$.