Department of Mathematics, Indian Institute of Technology Madras

MA1102 Series and Matrices

End Semester Examination Solutions

May 02, 2018 Wednesday 9:00-12:00 Maximum Marks: 60

Part-A

Answer the following questions with brief justification.

Each question in this part carries TWO marks.

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1. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 - 8n^2 + 100}{n^5 + 2n^2 + 5}$ converge?

   Sol The given series converges.

   Reason: $\frac{n^3-8n^2+100}{n^5+2n^2+5} \le (1 + 8 + 100)\frac{n^3}{n^5}$. (Or any such estimate)

   And $\sum \frac{1}{n^2}$ converges.

2. Let (a_n) be a sequence of real numbers such that $|a_n| \le 100^{-100}$ for all $n \ge 100$. Does the series $\sum_{n=1}^{\infty} a_n$ converge?

   Sol Not necessarily. For example, $\sum_{n=1}^{\infty} 100^{-100}$ does not converge.

3. Does the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^n}{n^{100}}$ converge?

   Sol No. Reason: $\lim_{n\to\infty} (-1)^{n-1} \frac{e^n}{n^{100}}$ does not exist.

4. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 2 & 4 \\ 6 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 0 & 2 \\ 5 & 0 & 4 \\ 3 & 2 & 1 \end{bmatrix}$. Then

   $(A^T - B)^T + C(B^{-1}C)^{-1}$ is equal to $- - --$?

   Sol $(A^T - B)^T + C(B^{-1}C)^{-1} = A - B^T + CC^{-1}B = A^T - B + B = A.$

5. If $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$ is an orthogonal matrix, then find the value of $|a + b + c|$.

   Sol Columns are orthonormal implies $a^2 + b^2 + c^2 = 1$ and $ab + bc + ca = 0$. Thus $|a + b + c| = 1$.

6. Let A be a nonzero matrix with $A^2 = A$. Does it follow that A is the identity matrix?

   Sol Not necessarily. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = A$.

7. Let B be an invertible matrix such that $I + B + B^2 + B^3 + \cdots + B^n = 0$ for some $n > 1$. Does it follow that $B^{-1} = B^n$?

   Sol $I + B + B^2 + B^3 + \cdots + B^n = 0.$

   Multiplying B^{-1}, we have $B^{-1} + I + B + \cdots + B^{n-1} = 0.$

   Hence $B^{-1} = B^n$.

8. Are the vectors $(2, 3, 1, 1)$, $(-4, 6, 5, 1)$ and $(1, 0, 0, 1)$ linearly independent?
Sol $a(2, 3, 1, 1) + b(-4, 6, 5, 1) + c(1, 0, 0, 1) = 0$. Then solving the equations and getting $a = b = c = 0$. So, they are linearly independent.
Aliter: Take them as rows and convert to RREF. Conclude that they are linearly independent.

9. Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be such that $\sum_{j=1}^{n} a_{ij} = c$ for $i = 1, 2, \ldots, n$.
Is c an eigenvalue of A?
Sol $A[1\ 1\ \cdots\ 1\ 1]^T = [c\ c\ \cdots\ c\ c]^T$. Hence c is an eigenvalue of A.

Part-B

Answer all the questions in detail.

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10. Let (a_n) be a sequence, where any term a_n is either 1 or 2. [4]
If (a_n) is a convergent sequence, then find all possible values of $\sum_{n=1}^{\infty}(a_n - a_{n+1})$.
Sol Let $s_m = \sum_{n=1}^{m}(a_n - a_{n+1}) = a_1 - a_{m+1}$.
As (a_n) is convergent, its limit, say, ℓ is either 1 or 2.
Then $\sum_{n=1}^{\infty}(a_n - a_{n+1}) = \lim_{m\to\infty} s_m = a_1 - \ell$.
This limit is either $1-1,\ 1-2,\ 2-1,\ 2-2$, that is, one of $-1,\ 0$ or 1.

11. For non-negative integers n, let $a_n = \begin{cases} n & \text{for } 0 \le n \le 99 \\ a_{n-100} & \text{for } n \ge 100. \end{cases}$ [3]
For $0 < x < 1$, find the function $f(x)$ to which the power series $\sum_{n=0}^{\infty} a_n x^n$ converges.
Sol For $0 < x < 1$, the power series converges to $f(x)$. Then
$f(x) = x + 2x^2 + \cdots + 99x^{99} + x^{101} + 2x^{102} + \cdots + 99x^{199} + \cdots$.
With $p(x) = x + 2x^2 + \cdots + 99x^{99}$, $\quad f(x) = p(x)(1 + x^{100} + x^{200} + \cdots)$.
Then $f(x) = p(x)\frac{1}{1-x^{100}}$.

12. Let $f(x) = e^{x^3}$. Find the value of $f^{(51)}(0)$. [3]
Sol $f(x) = e^{x^3} = 1 + \frac{x^3}{1!} + \frac{x^6}{2!} + \cdots + \frac{x^{51}}{17!} + \frac{x^{54}}{18!} + \cdots$.
Also, $f(x) = 1 + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. Comparing coefficients of x^{51}, we have
[Aliter: Then $f^{(51)}(x) = \frac{51!}{17!} + x^3 \times g(x)$, where $g(x)$ is a function of x. So,]
$f^{(51)}(0) = \frac{51!}{17!}$.

13. Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} n^2 x^n$.
Also find the function represented by the power series. [4]
Sol The radius of convergence is $\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$.
Suppose it represents the function $f(x)$ in $-1 < x < 1$. For such x,
$\frac{1}{1-x} = \sum x^n$
Differentiating and multiplying with x, $\frac{x}{(1-x)^2} = \sum nx^n$.
Differentiating and multiplying with x, $\frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \sum n^2 x^n$.
Hence $f(x) = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x^2+x}{(1-x)^3}$.

Aliter: The radius of convergence is $\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$.
Suppose it represents the function $f(x)$ in $-1 < x < 1$. For such x, $\frac{1}{1-x} = \sum x^n$.
Differentiating, we have $\frac{1}{(1-x)^2} = \sum nx^n$.
Differentiating once more, we get $\frac{2}{(1-x)^3} = \sum nx^{n-1}$.
Multiplying first one with x and second with x^2 and adding we have
$\frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \sum n^2 x^n$.
Hence $f(x) = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x^2+x}{(1-x)^3}$.

14. Find the Fourier cosine series of the function $f(x) = x(\pi - x)$ for $0 < x < \pi$. [4]
Sol $a_0 = \frac{2}{\pi}\int_0^\pi x(\pi - x)\,dx = \frac{\pi^2}{3}$
$a_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\cos nx\,dx = \frac{2((-1)^n+1)}{n^2}$
Fourier series is $\frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx$.
That is, $\frac{\pi^2}{6} + \sum_{n=1}^\infty \frac{\cos(2nx)}{n^2}$.

15. Using Gram-Schmidt process orthogonalize the set of vectors
$\{(1,0,1,0),\ (0,1,1,0),\ (0,0,1,1)\}$. [3]
Sol $v_1 = (1,0,1,0)$. $v_2 = (0,1,1,0) - \frac{1}{2}(1,0,1,0) = (-\frac{1}{2},1,\frac{1}{2},0)$.
$v_3 = (0,0,1,1) - \frac{1}{2}(1,0,1,0) - \frac{1}{3}(-\frac{1}{2},1,\frac{1}{2},0) = (-\frac{1}{3},-\frac{1}{3},\frac{1}{3},1)$.
$\{v_1, v_2, v_3\}$ is the required orthogonal set.

16. Let $A \in \mathbb{R}^{3\times3}$ satisfy $A(a,b,c)^T = (a-b+2c,\ a-c,\ 2a+b+c)^T$ for all $a,b,c \in \mathbb{R}$.
Determine the matrix A and also its rank. [3]
Sol $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}$.
The RREF of A is I. hence, $\text{rank}(A) = 3$.

17. Find all values of a and b for which the linear system [5]

$$x + 2y + 3z = 6,\ x + 3y + 5z = 9,\ 2x + 5y + az = b$$

has (a) a unique solution, (b) no solutions, (c) infinitely many solutions.

Sol Converting the augmented matrix to RREF, we see, after two pivots, that

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{array}\right]. \text{ Or } \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{array}\right].$$

(a) The system has a unique solution when $a \neq 8$.
(b) The system has no solutions when $a = 8$ and $b \neq 15$.
(c) The system has infinitely many solutions when $a = 8$ and $b = 15$.

18. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$. Determine an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. [6]

Sol Characteristic polynomial of A is $(2+t)(2-t)(t+1)$. Hence eigenvalues are $\lambda = -2, \ -1, 2$.

Write $(a, b, c)^T$ as an eigenvector.

For $\lambda = -2$: $a = 0, b + c = 0$. We choose $(0, 1, -1)^T$.

For $\lambda = -1$: $b = -a, c = -a$. We choose $(-1, 1, 1)^T$.

For $\lambda = 2$: $a = 2c, b = c$. We choose $(2, 1, 1)^T$.

Hence $P = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

19. Let A be a non-invertible 4×4 matrix with real entries. Suppose each diagonal entry of A is equal to 1. If one of the eigenvalues of A is $2 + 3i$, then find the other eigenvalues. [3]

Sol A is non-invertible. So, 0 is an eigenvalue.

A is a real matrix and $2 + 3i$ is an eigenvalue. So, $2 - 3i$ is an eigenvalue.

The trace of A is 4. Hence the other eigenvalue is $4 - (2 + 3i) - (2 - 3i) - 0 = 0$.

20. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Prove that any eigenvalue of A is real. Further, let λ_1 and λ_2 be distinct eigenvalues of A with corresponding eigenvectors v_1 and v_2. Prove that v_1, v_2 are orthogonal. [4]

Sol Let λ be an eigenvalue of A with an eigenvector v. Then $Av = \lambda v$. Then $v^*A^* = \bar{\lambda}v^*$. Then $v^*A^*v = \bar{\lambda}v^*v$. As A is Hermitian, $v^*Av = \bar{\lambda}v^*v$. That is, $v^*\lambda v = \bar{\lambda}v^*v$. Since $v^*v \neq 0$, we have $\lambda = \bar{\lambda}$. That is, λ is real.

For the second part, $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$. Now,

$\langle Av_1, v_2\rangle = \langle \lambda v_1, v_2\rangle = \lambda_1\langle v_1, v_2\rangle$.

Also $\langle Av_1, v_2\rangle = \langle A^*v_1, v_2\rangle = \langle v_1, Av_2\rangle = \langle v_1, \lambda_2 v_2\rangle = \bar{\lambda}_2\langle v_1, v_2\rangle$. Since λ_2 is real, we have $(\lambda_1 - \lambda_2)\langle v_1, v_2\rangle = 0$. Since $\lambda_1 \neq \lambda_2$, we get $\langle v_1, v_2\rangle = 0$.
