EC1101 : Signals and Systems

Tutorial -8 Solutions

1. (a) Given $x(t) \longleftrightarrow X(j\omega)$ implies

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Substituting $\omega = 0$, we get,

$$X(0) = \int_{-\infty}^{\infty} x(t)e^{0}dt = \int_{-\infty}^{\infty} x(t)dt$$

(b) Let $x(t) = \frac{W}{\pi}\text{sinc}\left(\frac{Wt}{\pi}\right)$, then

$$X(j\omega) = \begin{cases} 1, & \text{for } |\omega| < W, \\ 0, & \text{elsewhere.} \end{cases}$$

When $W = \pi$, $x(t) = \text{sinc}(t)$. Now, the Fourier transform of $\text{sinc}(t)$ is a rectangular function of magnitude 1 from $-\pi$ to π.

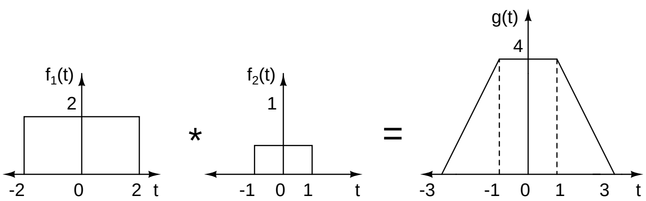
$$X(j\omega) = \int_{-\infty}^{\infty} \text{sinc}(t)e^{-j\omega t}dt$$

From part (a), $\int_{-\infty}^{\infty} \text{sinc}(t)dt$ is equal to the Fourier transform $X(j\omega)$ at $\omega = 0$ which is equal to 1.

$$\begin{aligned}\int_{-\infty}^{\infty} \text{sinc}^2(t)dt &= \int_{-\infty}^{\infty} |x(t)|^2dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \\ &\text{(From Parseval's Theorem)} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 d\omega \\ &= 1\end{aligned}$$

Thus, $\int_{-\infty}^{\infty} \text{sinc}(t)dt = \int_{-\infty}^{\infty} \text{sinc}^2(t)dt = 1$.

2. (a) Given $f_1(t) = 2\,\text{rect}\left(\frac{t}{4}\right)$ and $f_2(t) = \text{rect}\left(\frac{t}{2}\right)$



$$g(t) = f_1(t) * f_2(t) \quad = \begin{cases} 2(t+3) & , -3 \le t \le -1 \\ 4 & , -1 < t \le 1 \\ 2(3-t) & , 1 < t \le 3 \\ 0 & , |t| \ge 3 \end{cases}$$

$$\begin{aligned}\text{rect}\left(\frac{t}{\tau}\right) &\longleftrightarrow \tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right) \\ f_1(t) = 2\,\text{rect}\left(\frac{t}{4}\right) &\longleftrightarrow 2(4)\,\text{sinc}(2\omega) \\ f_2(t) = \text{rect}\left(\frac{t}{2}\right) &\longleftrightarrow 2\,\text{sinc}(\omega)\end{aligned}$$

(b) Using the convolution property of Fourier Transform,

$$\begin{aligned}G(j\omega) &= F_1(j\omega)F_2(j\omega) \\ &= 8\,\text{sinc}(2\omega)2\,\text{sinc}(\omega) \\ &= 16\,\text{sinc}(2\omega)\,\text{sinc}(\omega)\end{aligned}$$

(c) The magnitude and phase spectrum of $G(j\omega)$ are plotted using matlab in Figure 1. Phase spectrum manually plotted is given in Figure 2.

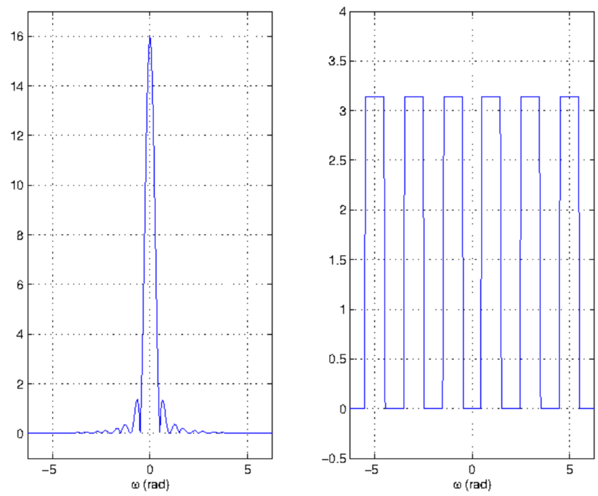


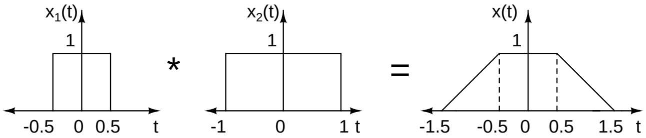
Figure 1: Magnitude and phase spectrum of $G(j\omega)$ in Q2 generated using matlab

Figure 2: Magnitude spectrum of $G(j\omega)$ in Q2

3. The given signal $x(t)$ can be written as,

$$x(t) = x_1(t) * x_2(t)$$

where, $x_1(t) = \text{rect}(t)$ and $x_2(t) = \text{rect}\left(\frac{t}{2}\right)$



Using the convolution property of Fourier Transform,

$$\begin{aligned} X(j\omega) &= X_1(j\omega)X_2(j\omega) \\ &= \text{sinc}\left(\frac{\omega}{2\pi}\right)2\,\text{sinc}\left(\frac{\omega}{\pi}\right) \\ &= 2\,\text{sinc}\left(\frac{\omega}{\pi}\right)\text{sinc}\left(\frac{\omega}{2\pi}\right) \end{aligned}$$

4. (a) Given $X(j\omega) = \dfrac{2\sin(3(\omega - 2\pi))}{(\omega - 2\pi)}$

$$\text{rect}\left(\frac{t}{\tau}\right) \longleftrightarrow \tau\,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right) = \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right) \qquad (1)$$

Comparing with standard form in (1),

$$\text{rect}\left(\frac{t}{6}\right) \longleftrightarrow \frac{2}{\omega}\sin(3\omega)$$

Using the frequency shifting property of Fourier Transform,

$$e^{j2\pi t}\,\text{rect}\left(\frac{t}{6}\right) \longleftrightarrow \frac{2\sin(3(\omega - 2\pi))}{(\omega - 2\pi)}$$

(b) Given $X(j\omega) = cos(4\omega + \frac{\pi}{3})$

$$\begin{aligned} X(j\omega) &= cos(4\omega + \frac{\pi}{3}) \\ &= \frac{1}{2}\left[e^{j(4\omega + \frac{\pi}{3})} + e^{-j(4\omega + \frac{\pi}{3})}\right] \\ &= \left(\frac{e^{j\frac{\pi}{3}}}{2}\right)e^{j4\omega} + \left(\frac{e^{-j\frac{\pi}{3}}}{2}\right)e^{-j4\omega} \end{aligned}$$

We know that,

$$\delta(t) \longleftrightarrow 1$$

By using time shifting property,

$$\begin{aligned} \delta(t+4) &\longleftrightarrow e^{j4\omega} \\ \delta(t-4) &\longleftrightarrow e^{-j4\omega} \end{aligned}$$

By using linearity property,

$$x(t) = \left(\frac{e^{j\frac{\pi}{3}}}{2}\right)\delta(t+4) + \left(\frac{e^{-j\frac{\pi}{3}}}{2}\right)\delta(t-4)$$

5. a)

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\text{d}t \\ X(0) &= \int_{-\infty}^{\infty} x(t)e^{-j(0)t}\text{d}t \\ X(0) &= \int_{-\infty}^{\infty} x(t)\text{d}t \end{aligned}$$

$$\begin{aligned} X(0) &= \int_{-1}^{0} 1\text{d}t + \int_{0}^{1}(-t+1)\text{d}t \\ &\quad + \int_{1}^{2}(t-1)\text{d}t + \int_{2}^{3} 1\text{d}t \end{aligned}$$

$$\implies X(0) = 3$$

b)

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\text{d}\omega \\ x(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(0)}\text{d}\omega \\ x(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\text{d}\omega \end{aligned}$$

$$\int_{-\infty}^{\infty} X(j\omega)\text{d}\omega = 2\pi x(0)$$

$$\implies \int_{-\infty}^{\infty} X(j\omega)\text{d}\omega = 2\pi$$

c)

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\text{d}\omega \\ x(1) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(1)}\text{d}\omega \\ x(1) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega}\text{d}\omega \end{aligned}$$

$$\int_{-\infty}^{\infty} X(j\omega)e^{j\omega}\text{d}\omega = 2\pi x(1)$$

$$\implies \int_{-\infty}^{\infty} X(j\omega)e^{j\omega}\text{d}\omega = 2\pi.0$$

$$\implies \int_{-\infty}^{\infty} X(j\omega)e^{j\omega}\text{d}\omega = 0$$

d)

Using Parseval's theorem,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2\mathrm{d}\omega = \int_{-\infty}^{\infty}|x(t)|^2\mathrm{d}t$$

$$\implies \int_{-\infty}^{\infty}|X(j\omega)|^2\mathrm{d}\omega = 2\pi\left[\int_{-1}^{0}1^2\mathrm{d}t + \int_{0}^{1}(1-t)^2\mathrm{d}t + \int_{1}^{2}(t-1)^2\mathrm{d}t + \int_{2}^{3}1^2\mathrm{d}t\right]$$

$$= 2\pi\left[1+\frac{1}{3}+\frac{1}{3}+1\right] = \frac{16\pi}{3}$$

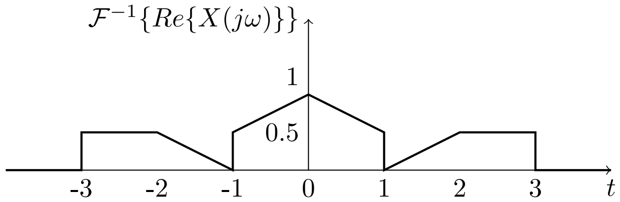
e)

$$Re\{X(j\omega)\} = \frac{X(j\omega)+X^*(j\omega)}{2}$$

$$\mathcal{F}^{-1}\{Re\{X(j\omega)\}\} = \frac{\mathcal{F}^{-1}\{X(j\omega)\}+\mathcal{F}^{-1}\{X^*(j\omega)\}}{2}$$

$$= \frac{x(t)+x^*(-t)}{2}$$

$$= \frac{x(t)+x(-t)}{2} \longrightarrow Ev\{x(t)\}$$



6. For the given signal, $T_0 = 1, \omega_0 = 2\pi$.

(a) Fourier series coefficients P_n of function $p(t)$:

$$P_n = \frac{1}{T_0}\int_{\frac{-1}{4}}^{\frac{1}{4}} p(t)e^{-jn\omega_0 t}dt, \quad n \neq 0$$

$$= \int_{\frac{-1}{4}}^{0}(1+4t)e^{-j2\pi nt}dt + \int_{0}^{\frac{1}{4}}(1-4t)e^{-j2\pi nt}dt$$

$$= \frac{4\sin^2(\pi n/4)}{\pi^2 n^2}, \quad n \neq 0.$$

$$P_0 = \int_{\frac{-1}{4}}^{\frac{1}{4}} p(t)dt = \frac{1}{4}.$$

Fourier transform of function $p(t)$:

Method 1 :

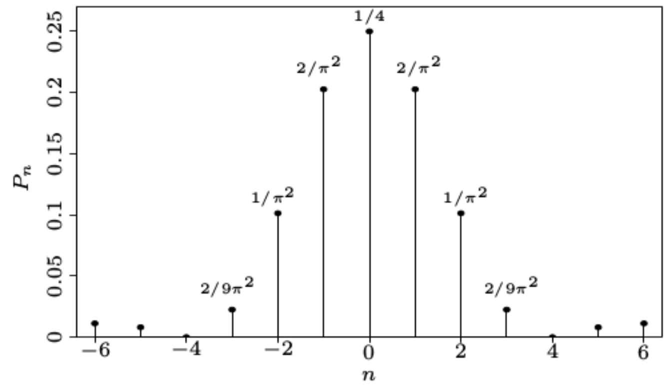


Figure 3: Sketch for P_n vs. n in Q6(a)

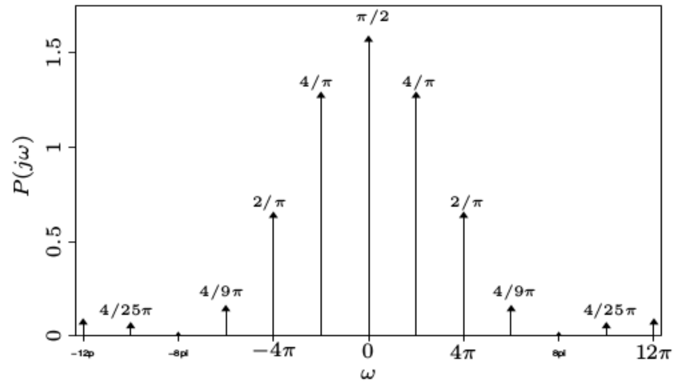


Figure 4: Sketch for $P(j\omega)$ vs. ω in Q6(a)

By using Fourier series expansion,

$$p(t) = \sum_{n=-\infty}^{\infty} P_n e^{j2\pi nt}$$

$$P(j\omega) = \sum_{n=-\infty}^{\infty}\mathcal{F}\{P_n e^{j2\pi nt}\}$$

$$= \sum_{n=-\infty}^{\infty} P_n\mathcal{F}\{e^{j2\pi nt}\}$$

$$= \sum_{n=-\infty}^{\infty} 2\pi P_n\delta(\omega-2\pi n)$$

$$= \sum_{n=-\infty}^{\infty}\frac{8\sin^2(\pi n/4)}{\pi n^2}\delta(\omega-2\pi n)$$

(b) $y(t) = p(t)\cdot x(t)$. Then the Fourier transform of $y(t)$ is

$$Y(j\omega) = \frac{1}{2\pi}\{P(j\omega) * X(j\omega)\}$$

$$= \frac{1}{2\pi}\left\{2\pi\sum_{k=-\infty}^{\infty}P_k\delta(\omega-2\pi k) * X(j\omega)\right\}$$

$$= \sum_{k=-\infty}^{\infty}P_k X(j(\omega-2\pi k))$$

$$= \sum_{k=-\infty}^{\infty}\frac{4\sin^2(\pi k/4)}{\pi^2 k^2}X(j(\omega-2\pi k))$$

(c) $x(t) = \mathrm{sinc}(t)$, has the Fourier transform

$$X(j\omega) = \mathrm{rect}\left(\frac{\omega}{2\pi}\right).$$

$$
\begin{aligned}
Y(j\omega) &= \frac{1}{2\pi}\{P(j\omega) * X(j\omega)\} \\
&= \sum_{k=-\infty}^{\infty} \frac{4\sin^2(\pi k/4)}{\pi^2 k^2}\,\mathrm{rect}\Big(\frac{\omega - 2\pi k}{2\pi}\Big).
\end{aligned}
$$

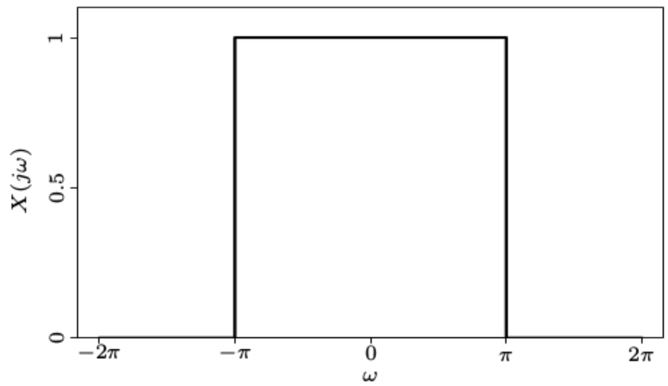


Figure 5: Sketch for $X(j\omega)$ in Q6(c)

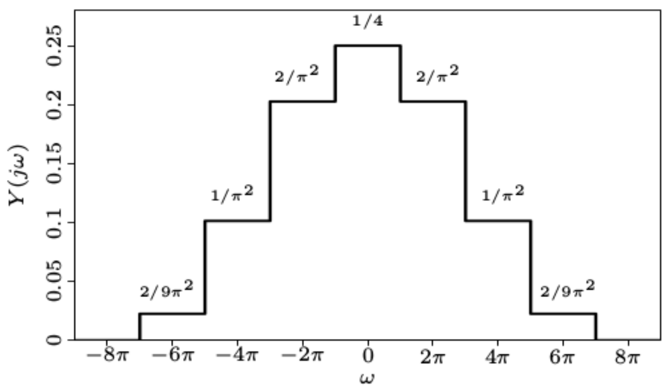


Figure 6: Sketch for $Y(j\omega)$ in Q6(c)

7. Given ,

$$
\begin{aligned}
z_1(t) &= x(t)\cos(\omega_1 t) + y(t)\cos(\omega_2 t) \\
\omega_1 &= 5W \\
\omega_2 &= 7W
\end{aligned}
$$

For $\mathbf{z_1(t)}$:

$$
\begin{aligned}
z_1(t) &= x(t)\cos(\omega_1 t) + y(t)\cos(\omega_2 t) \\
&= x(t)\Big(\frac{e^{j\omega_1 t} + e^{-j\omega_1 t}}{2}\Big) + y(t)\Big(\frac{e^{j\omega_2 t} + e^{-j\omega_2 t}}{2}\Big)
\end{aligned}
$$

The frequency shift property of Fourier Transform states that,

$$
f(t)e^{j\omega_0 t} \;\longleftrightarrow\; F(j(\omega - \omega_0))
$$

Using this property and linearity, $Z_1(j\omega)$ can be written as,

$$
\begin{aligned}
Z_1(j\omega) = \frac{1}{2}\Big(&X(j(\omega-\omega_1)) + X(j(\omega+\omega_1)) + \\
&Y(j(\omega-\omega_2)) + Y(j(\omega+\omega_2))\Big)
\end{aligned}
$$

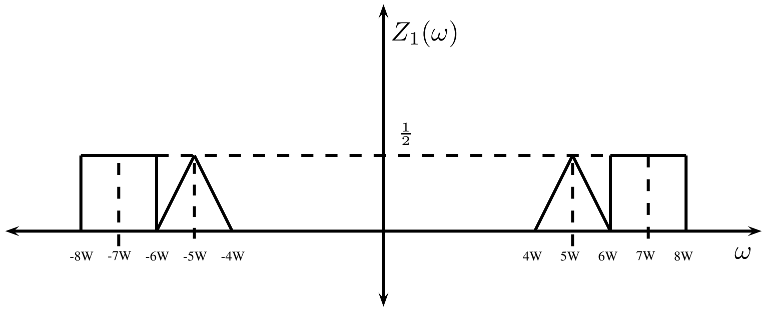


Figure 7: Sketch for $Z_1(j\omega)$

For $\mathbf{z_2(t)}$:

$Z_2(j\omega)$ is $Z_1(j\omega)$ bandlimited to $4W$ and $6W$

$$
Z_2(j\omega) = \frac{1}{2}\Big(X(j(\omega-\omega_1)) + X(j(\omega+\omega_1))\Big)
$$

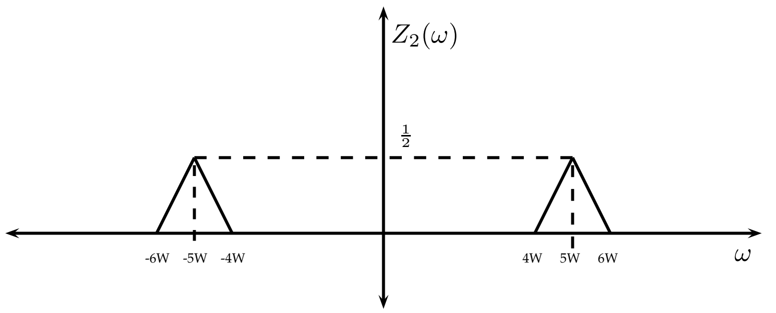


Figure 8: Sketch for $Z_2(j\omega)$

For $\mathbf{z_3(t)}$:

$$
\begin{aligned}
z_3(t) &= z_2(t)\cos(\omega t) \\
&= \frac{1}{2}\left(z_2(t)e^{j\omega_1 t} + z_2(t)e^{-j\omega_1 t}\right)
\end{aligned}
$$

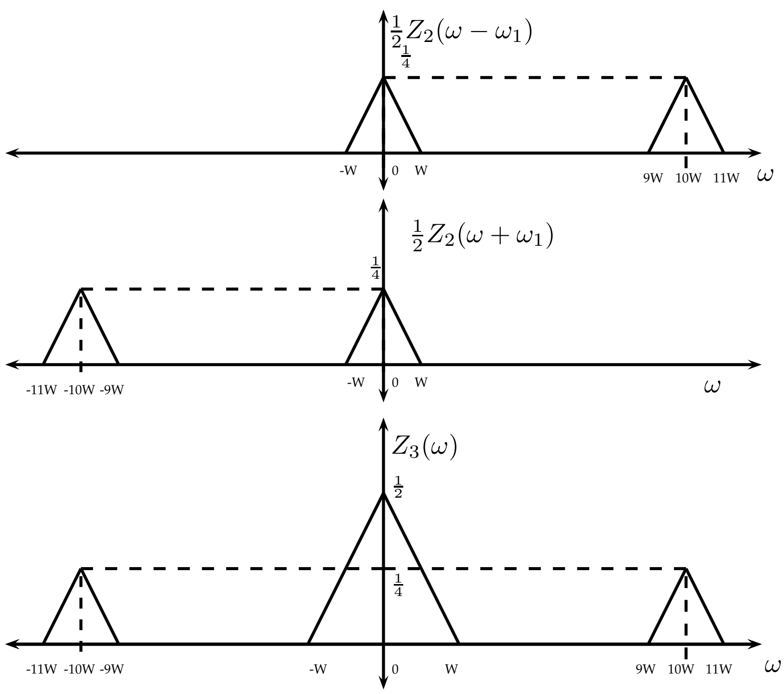


Figure 9: Sketch for $Z_3(j\omega)$

Using frequnecy shifting property of Fourier Transform,

$$\begin{aligned} Z_3(j\omega) &= \frac{1}{2}\Big(Z_2(j(\omega-\omega_1)) + Z_2(j(\omega+\omega_1))\Big) \\ &= \frac{1}{4}\Big(X(j(\omega-2\omega_1)) + 2X(j\omega) + X(j(\omega+2\omega_1))\Big) \\ &= \frac{1}{2}X(j\omega) + \frac{1}{4}\Big(X(j(\omega-2\omega_1)) + X(j(\omega+2\omega_1))\Big) \end{aligned} \tag{2}$$

For $\mathbf{z_4(t)}$:

$z_4(t)$ is the LPF output for the input $z_3(t)$ with cutoff freq $= W$. So, only the first summation term from eqn.(2) is available as output in the form of $z_4(t)$, i.e.:

$$Z_4(j\omega) = \frac{1}{2}X(j\omega) \Rightarrow z_4(t) = \frac{1}{2}x(t)$$

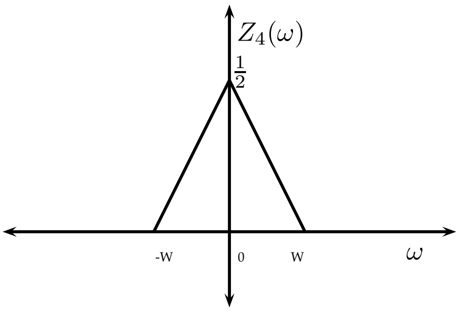


Figure 10: Sketch for $Z_4(j\omega)$

8. Let the impulse train be denoted by $i(t)$.

   (a) So,

   $$x_s(t) = x(t)i(t)$$

   Which implies,

   $$X_s(j\omega) = \frac{1}{2\pi}\left(X(j\omega) * I(j\omega)\right)$$

   To find $I(j\omega)$,the fourier transform of $i(t)$, it can be noted that $i(t)$ is a periodic function and can be written in terms of its fourier series coefficients as:

   $$i(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\frac{2\pi}{T}t}$$

   Now, the fourier transform can be easily found out by using the modulation property (shifting in frequency).

   $$I(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - k\left(\frac{2\pi}{T}\right)\right)$$

   Thus,

   $$X_s(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\omega - k\left(\frac{2\pi}{T}\right)\right)\right)$$

   (b) Sketch is as given in Figure 10

   (c) Sketch is as given in Figure 11

   (d). Largest T such that $X_{sr}(j\omega) = X(j\omega)$ is $\frac{1}{2B}$.

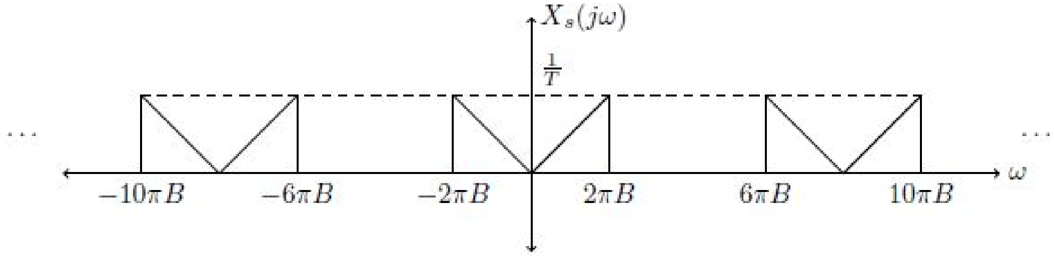
9. (a) Find the Nyquist rates for the signals:

   (i). $x_1(t) = \frac{sin(4000\pi t)}{\pi t}$. This signal is bandlimited to $\omega = 4000\pi$. Thus, the minimum Nyquist rate $= 4000$ samples per sec.

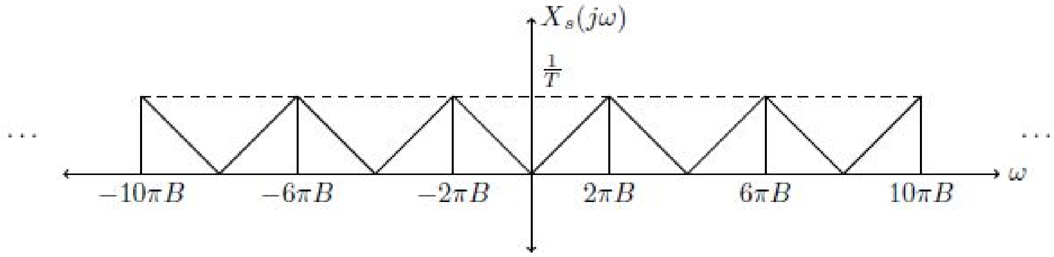
   (ii). $x_2(t) = \left(\frac{sin(4000\pi t)}{\pi t}\right)^2$. This signal is squared of $x_1(t)$. which means its Fourier Transform will be the self convolution of the Fourier Transform of $x_1(t)$. So, this signal is bandlimited to $\omega = 8000\pi$. Thus, the minimum nyquist rate $= 8000$ samples per sec.

   (b) $y(t) = x_1(t) * x_2(t)$
   Which implies $Y(j\omega) = X_1(j\omega)X_2(j\omega)$
   Thus, $Y(j\omega) = 0$ for $|\omega| > 1000\pi$.
   Hence, the sampling period range which ensures that $y(t)$ is recoverable from the samples is $(0, 1ms)$, which means the sampling period should be less than 1 ms.

(i). $T = \frac{1}{4B}$



(ii). $T = \frac{1}{2B}$



(iii). $T = \frac{1}{B}$

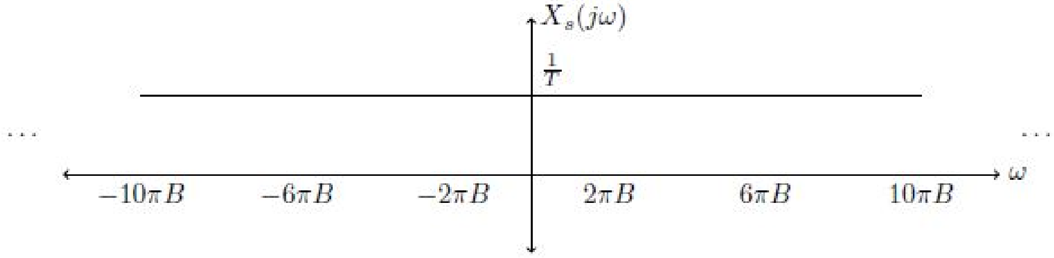
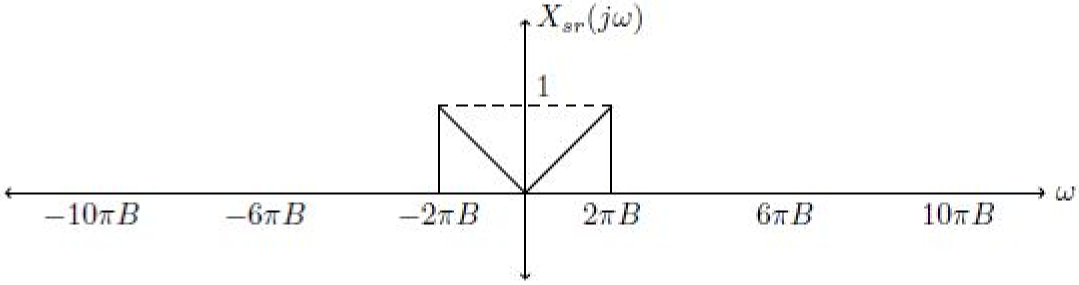
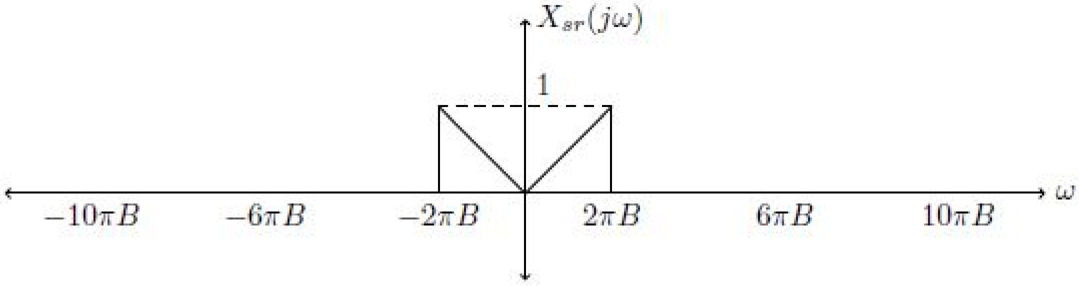


Figure 11:

(i). $T = \frac{1}{4B}$



(ii). $T = \frac{1}{2B}$



(iii). $T = \frac{1}{B}$

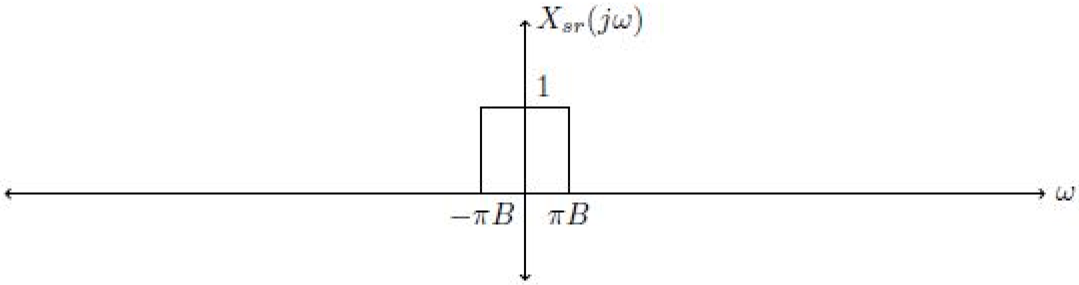


Figure 12: