Indian Institute of Technology Madras
PH1020, Tutorial Set-9

Question 1:

Two plane polarized electromagnetic waves propagate in the positive z-direction, with their planes of polarization along the x and y direction, respectively. The electric fields of the two waves have equal amplitudes given by $|E_0|$. The frequency of each wave is ω and the wave vector is $\vec{K}$.

a) Write the expression for the electric and magnetic fields of the two waves.

b) Find the values of $\frac{\partial U}{\partial t}$ and $\vec{\nabla}\cdot\vec{S}$ for the two waves where U is the electromagnetic energy and $\vec{S}$ is the Poynting vector.

Solution:

a) Wave-1: The electric field polarized in the x direction is $\vec{E} = |E_0| \cos(kz-\omega t+\phi_1)\ \hat{e}_x$. Corresponding magnetic filed

$$\begin{aligned}\vec{B} &= \frac{\vec{K}\times\vec{E}}{\omega}\\ &= \frac{k}{\omega}\left|E_0\right| \cos(kz-\omega t+\phi_1)\ \hat{e}_y\end{aligned}$$

Wave-2: The electric field polarized in the y direction is $\vec{E} = |E_0| \cos(kz-\omega t+\phi_2)\ \hat{e}_y$. Corresponding magnetic filed

$$\begin{aligned}\vec{B} &= \frac{\vec{K}\times\vec{E}}{\omega}\\ &= -\frac{k}{\omega}\left|E_0\right| \cos(kz-\omega t+\phi_1)\ \hat{e}_x\end{aligned}$$

b) Electric field and the magnetic field contribute equally to energy. The energy contribution from electric field and the magnetic field are $\frac{1}{2}\epsilon_0\left|\vec{E}\right|^2$ and $\frac{1}{2\mu_0}\left|\vec{B}\right|^2$.
For wave 1,
$U = \epsilon_0\left|\vec{E}\right|^2 = \epsilon_0 E_0^2 \cos^2(kz-\omega t+\phi_1)$
The value of $\frac{\partial U}{\partial t} = 2\omega\epsilon_0\left|E_0\right|^2 \cos(kz-\omega t+\phi_1)\sin(kz-\omega t+\phi_1)$
The Poynting vector

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)\\ &= \frac{k}{\mu_0\omega}\left|E_0\right|^2\cos^2\ (kz-\omega t+\phi_1)\ \hat{e}_z\end{aligned}$$

$$\begin{aligned}\vec{\nabla}\cdot\vec{S} &= -\frac{2k^2}{\mu_0\omega}\left|E_0\right|^2\cos(kz-\omega t+\phi_1)\sin(kz-\omega t+\phi_1)\\ &= -2\omega\epsilon_0\left|E_0\right|^2\cos(kz-\omega t+\phi_1)\sin(kz-\omega t+\phi_1)\\ &= -\frac{\partial U}{\partial t}\end{aligned}$$

For wave 2,
$U = \epsilon_0 \left|\vec{E}\right|^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \phi_2)$
The value of $\frac{\partial U}{\partial t} = 2\omega\epsilon_0 \left|E_0\right|^2 \cos(kz - \omega t + \phi_2)\sin(kz - \omega t + \phi_2)$
The Poynting vector

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) \\ &= \frac{k}{\mu_0 \omega}\left|E_0\right|^2 \cos^2 (kz - \omega t + \phi_2)\ \hat{e}_z\end{aligned}$$

$$\begin{aligned}\vec{\nabla} \cdot \vec{S} &= -\frac{2k^2}{\mu_0 \omega}\left|E_0\right|^2 \cos(kz - \omega t + \phi_2)\sin(kz - \omega t + \phi_2) \\ &= -\frac{\partial U}{\partial t}\end{aligned}$$

Question 2:

The electric field of a plane wave in vacuum is $\vec{E} = E_0 \cos(kx)\cos(\omega t)\ \hat{e}_z$. Write the components of the corresponding magnetic field $\vec{B}$ such that $\vec{B} = 0$ when $t = 0$. Find the mean flux of the energy.

Solution:

$\vec{E} = E_0 \cos(kx)\cos(\omega t)\hat{e}_z$

$$\begin{aligned}-\frac{\partial \vec{B}}{\partial t} &= \vec{\nabla} \times \vec{E} \\ &= \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z \end{vmatrix} \\ &= \frac{\partial E_z}{\partial y}\ \hat{e}_x - \frac{\partial E_z}{\partial x}\ \hat{e}_y \\ &= -\frac{\partial E_z}{\partial x}\ \hat{e}_y \qquad \text{as } E_z \text{ is independent of y.} \\ &= kE_0 \sin(kx)\cos(\omega t)\ \hat{e}_y\end{aligned}$$

$$\begin{aligned}\therefore\ \vec{B} &= -kE_0 \sin(kx)\ \hat{e}_y \int \cos(\omega t)\ dt \\ &= -kE_0 \sin(kx)\frac{\sin(\omega t)}{\omega}\ \hat{e}_y + C\end{aligned}$$

At $t = 0,\ \vec{B} = 0,\ \therefore\ C = 0$

$$\begin{aligned}\therefore\ \vec{B} &= -\frac{kE_0}{\omega}\ \hat{e}_y \sin(kx)\sin(\omega t) \\ \vec{S} &= \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) \\ \vec{S} &= \frac{kE_0^2}{\mu_0 \omega}\sin(kx)\sin(\omega t)\cos(kx)\cos(\omega t)\ \hat{e}_x \\ <\vec{S}> &= 0\end{aligned}$$

Question 3:

Write down the real component of the electric and magnetic fields for a monochromatic plane wave of amplitude E_0, frequency ω, and phase angle zero in vacuum that is

a) travelling in the negative x-direction and polarized in the z-direction,

b) travelling along (111) with polarization parallel to the xz-plane.

Solution:

$$
\begin{aligned}
\tilde{\vec{E}}(\mathbf{r},t) &= \tilde{\vec{E}}_0\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\ \hat{n} \\
\tilde{\vec{B}}(\mathbf{r},t) &= \frac{1}{c}\ \tilde{\vec{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\ (\hat{k}\times\hat{n})
\end{aligned}
\tag{1}
$$

k is the wave propagation vector and $\hat{n}$ is the polarization vector.

a) In the given problem,

$$
\begin{aligned}
\vec{k} &= -\frac{\omega}{c}\ \hat{x} \\
\vec{n} &= \hat{z}
\end{aligned}
\tag{2}
$$

Using equ (2) from equ (1) we get,

$$
\begin{aligned}
\vec{E}(x,t) &= \ E_0\cos(\frac{\omega}{c}x+\omega t)\ \hat{z} \\
\vec{B}(x,t) &= \ \frac{E_0}{c}\cos(\frac{\omega}{c}x+\omega t)\ \hat{y}
\end{aligned}
$$

b) In this case, $\vec{k} = \frac{\omega}{c}\left(\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}\right)$, $\quad \hat{n} = \frac{\hat{x}-\hat{z}}{\sqrt{2}}$

$$
\begin{aligned}
\mathbf{k}\cdot\mathbf{r} &= \frac{\omega}{c\sqrt{3}}\left(\hat{x}+\hat{y}+\hat{z}\right)\cdot\left(x\hat{x}+y\hat{y}+z\hat{z}\right) \\
&= \frac{\omega}{c\sqrt{3}}(x+y+z)
\end{aligned}
$$

$$
\begin{aligned}
\hat{k}\times\hat{n} &= \frac{1}{\sqrt{6}}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} \\
&= \frac{1}{\sqrt{6}}\left[-\hat{x}+2\hat{y}-\hat{z}\right] \\
&= \frac{-\hat{x}+2\hat{y}-\hat{z}}{\sqrt{6}}
\end{aligned}
$$

The electric field is $\vec{E}(\mathbf{r},t) = E_0\cos\left(\frac{\omega}{c\sqrt{3}}(x+y+z)+\omega t\right)\left(\frac{\hat{x}-\hat{z}}{\sqrt{2}}\right)$

The magnetic field is $\vec{B}(\mathbf{r},t) = \frac{E_0}{c}\cos\left(\frac{\omega}{c\sqrt{3}}(x+y+z)+\omega t\right)\left(\frac{-\hat{x}+2\hat{y}-\hat{z}}{\sqrt{6}}\right)$

Question 4:

Calculate the following for a plane sinusoidal electromagnetic wave travelling in free space with an electric field amplitude, $E_0 = 40\ \mu V/m$.

a) Average energy density in the wave,

b) Peak energy density, and

c) Average value of the Poynting vector.

Solution:

a) Average energy density

$$\begin{aligned} < u > &=< u_E > + < u_B > \\ &=< \frac{1}{2}\epsilon_0 E^2 > + < \frac{1}{2\mu_0} B^2 > \\ &=< \epsilon_0 E^2 > \\ &= \frac{1}{2}\epsilon_0 E_0^2 \\ &= \frac{(40\times 10^{-6})^2}{8\pi(9\times 10^9)} \ \ J.m^{-3} \\ &= 7.074\times 10^{-21} \ \ J.m^{-3} \end{aligned}$$

b) Peak energy density $2 < u >= 1.415\times 10^{-20} \ \ J.m^{-3}$

c)

$$\begin{aligned} < \vec{S} > &= \frac{1}{\mu_0} < \vec{E}\times\vec{B} > \\ &= \frac{c}{2}\epsilon_0 E_0^2 \\ &= 2.122\times 10^{-12} \ \ Watt.m^{-2} \end{aligned}$$

Question 5:

An electromagnetic wave propagating in an isotropic medium has its electric vector as $\vec{E}(x,y,z,t) = (70\hat{e}_y)\cos\left[\pi\times 10^7(\frac{x}{3} - 10^8 t)\right] + (50\hat{e}_z)\cos\left[\pi\times 10^7(\frac{x}{3} - 10^8 t)\right]$ in V/m. x is in meter and t is in seconds.

a) Find the refractive index of the material of the medium in which the electromagnetic wave is travelling and

b) The corresponding $\vec{H}(x,y,z,t)$.

c) Hence calculate the time value of the Poynting vector $\vec{S}$.

Solution:

a) $v = \frac{\omega}{k} = 3\times 10^8 \ \ m/s.$
$\therefore \ n = \frac{c}{v} = 1.00$

b)

$$\begin{aligned} \vec{H}_0 &= \frac{k}{\omega\mu_0}\left(\hat{k}\times\vec{E}_0\right) \\ &= \frac{1}{c\mu_0}\left(\hat{k}\times\vec{E}_0\right) \\ &= c\epsilon_0\left(\hat{k}\times\vec{E}_0\right) \end{aligned}$$

$\hat{k} = \hat{e}_x$ and $\vec{E}_0 = \left(70\hat{e}_y + 50\hat{e}_z\right)$

$$\begin{aligned}
\therefore \quad \vec{H} &= 3 \times 10^8 \times (8.854 \times 10^{-12}) \left[\hat{e}_x \times \left(70\hat{e}_y + 50\hat{e}_z\right)\right] \quad A/m \\
&= \frac{1}{120\pi}\left(70\hat{e}_z - 50\hat{e}_y\right) \quad A/m \\
&= \frac{1}{12\pi}\left(7\hat{e}_z - 5\hat{e}_y\right) \quad A/m
\end{aligned}$$

$\therefore \quad \vec{H}(x, y, z, t) = \frac{1}{12\pi}\left(7\hat{e}_z - 5\hat{e}_y\right) \cos\left[\pi \times 10^7 (\frac{x}{3} - 10^8 t)\right]$

c)

$$\begin{aligned}
< \vec{S} > &= < \vec{E} \times \vec{H} > \\
&= < \left(70\ \hat{e}_y + 50\ \hat{e}_z\right) \times \frac{\left(7\ \hat{e}_z - 5\hat{e}_y\right)}{12\pi} \cos^2\left[\pi \times 10^7 \left(\frac{x}{3} - 10^8 t\right)\right] > \\
&= \frac{1}{24\pi}\left[\left(70\hat{e}_y + 50\hat{e}_z\right) \times \left(7\hat{e}_z - 5\hat{e}_y\right)\right] \\
&= \frac{1}{24\pi}(490 + 250)\ \hat{e}_x \\
&= \frac{185}{6\pi}\ \hat{e}_x \quad Watt.m^{-2}
\end{aligned}$$

$< |\vec{S}| > = \frac{185}{6\pi} = 9.815 \ \ Watt.m^{-2}$.