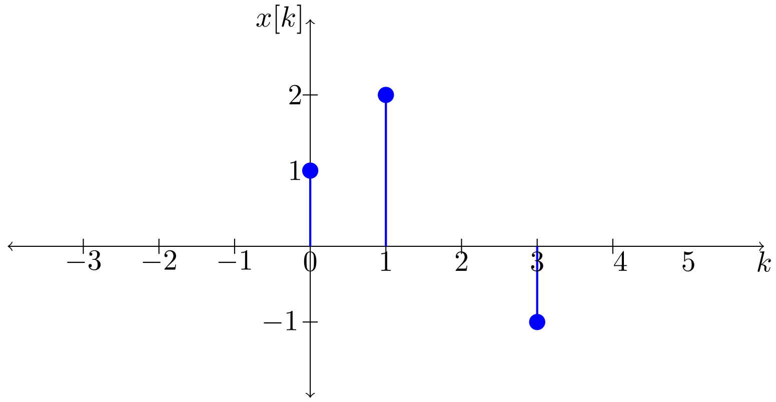
EE1101 : Signals and Systems

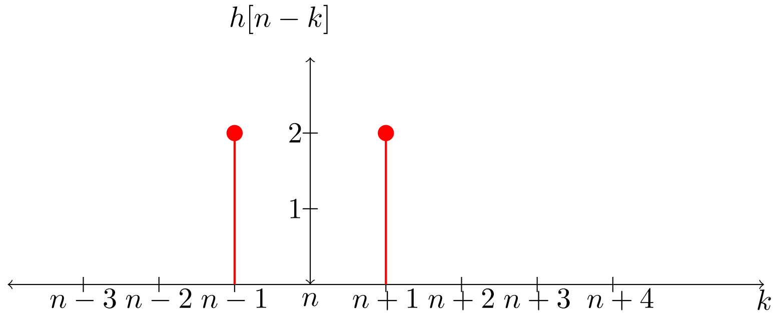
Tutorial 3 Solutions

1. Convolution can be calculated by using,

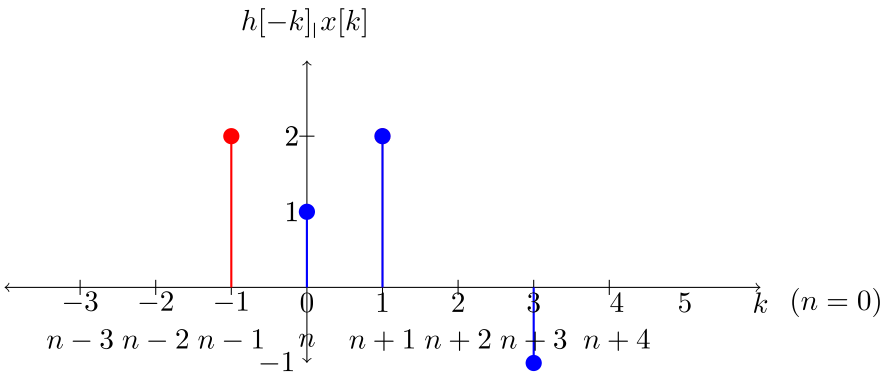
$$x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n]$$

So we plot $x[k]$,$h[-k]$ and then $h[n-k]$ for finding convolution using graphical method.

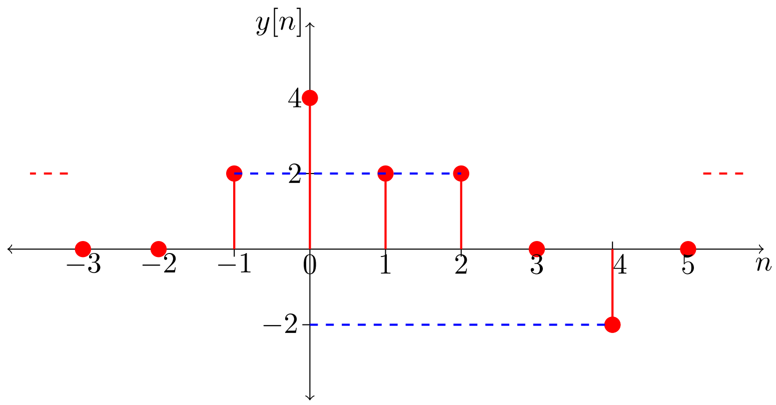




$h[-k]$ is constructed first and then $h[n-k]$ is drawn as shown in the figure. Note here that the $x - axis$ is k.



$h[n-k]$ and $x[k]$ are to be multiplied for different values of n and then sum of each product is the convolution. The figure above dipicts when n=0. Similarly by drawing at different values of n and summing them we get $y[n]$



2. (a)

$$x[n] = \alpha^n u[n]$$
$$h[n] = \beta^n u[n], \alpha \neq \beta$$

$$\text{Now,} \quad y[n] = x[n] * h[n]$$
$$y[n] = \sum_{k=-\infty}^{\infty} \alpha^k u[k] \beta^{(n-k)} u[n-k]$$
$$y[n] = \sum_{k=-\infty}^{n} \alpha^k \beta^{(n-k)} u[k]$$

if $n \geq 0$,

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{n} \alpha^k \beta^{(n-k)} = \beta^n + \alpha\beta^{n-1} + \alpha^2\beta^{n-2} + \cdots \\
&= \beta^n [1 + \frac{\alpha}{\beta} + (\frac{\alpha}{\beta})^2 + \cdots] \\
&= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.
\end{aligned}
$$

For $n < 0$ the convolution result is 0.

(b) Let

$$
\begin{aligned}
x[n] &= u[n] \\
h[n] &= a^n u[-n-1], |a| > 1
\end{aligned}
$$

$$
\begin{aligned}
\text{Now,} \quad y[n] &= x[n] * h[n] \\
y[n] &= \sum_{k=-\infty}^{\infty} a^k u[-k-1] u[n-k] \\
y[n] &= \sum_{k=-\infty}^{n} a^k u[-k-1]
\end{aligned}
$$

if $n > -1$,

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{-1} a^k = a^{-1} + a^{-2} + \cdots \\
&= a^{-1} [1 + \frac{1}{a} + \frac{1}{a^2} + \cdots] \\
&= \frac{1}{a-1}.
\end{aligned}
$$

if $n \leq -1$,

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{n} a^k = a^n + a^{n-1} + a^{n-2} + \cdots \\
&= a^n(1 + \frac{1}{a} + \frac{1}{a^2} + \cdots) = a^n\Big(\frac{1}{1-\frac{1}{a}}\Big) \\
&= \frac{a^{n+1}}{a-1} \\
\therefore\ y[n] &= \begin{cases} \frac{a^{n+1}}{a-1}, & n \leq -1 \\ \frac{1}{a-1}, & n > -1 \end{cases}
\end{aligned}$$

3. (a) We have to prove the commutative property of convolution operator.

$$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$

Let $l = n - k \implies k = n - l$

$$\begin{aligned}
x[n] * y[n] &= \sum_{l=-\infty}^{\infty} x[n-l]y[l] \\
&= \sum_{l=-\infty}^{\infty} y[l]x[n-l] \\
x[n] * y[n] &= y[n] * x[n]
\end{aligned}$$

(b) We have to prove the distributive property of convolution operator.

$$\begin{aligned}
x[n] * (y[n] + z[n]) &= \sum_{k=-\infty}^{\infty} x[k](y[n-k] + z[n-k]) \\
&= \sum_{k=-\infty}^{\infty} x[k]y[n-k] + \sum_{k=-\infty}^{\infty} x[k]z[n-k] \\
x[n] * (y[n] + z[n]) &= x[n] * y[n] + x[n] * z[n]
\end{aligned}$$

(c)

$$
\begin{aligned}
x[n] * \delta[n-a] &= \sum_{k=-\infty}^{\infty} x[k]\delta[(n-a)-k] \\
&= \sum_{k=-\infty}^{\infty} x[n-a]\delta[(n-a)-k] \qquad (\text{as } x[n]\delta[n-a] = x[a]\delta[n-a]) \\
&= x[n-a] \sum_{k=-\infty}^{\infty} \delta[(n-a)-k] \\
x[n] * \delta[n-a] &= x[n-a] \qquad (\text{as } \sum_{k=-\infty}^{\infty} \delta[k] = 1)
\end{aligned}
$$

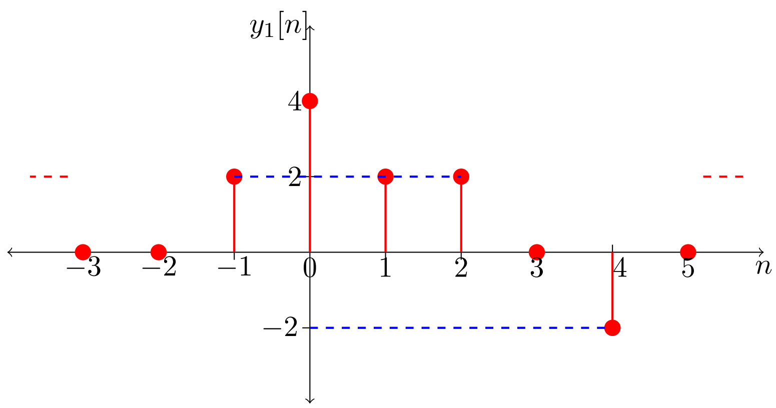
4. To compute the given convolutions, we first compute $x[n] * \delta[n-a]$ (a is an integer) and then use linearity and time invariance of convolution operation.

$$
x[n] * \delta[n-a] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k-a] = x[n-a]
$$

.

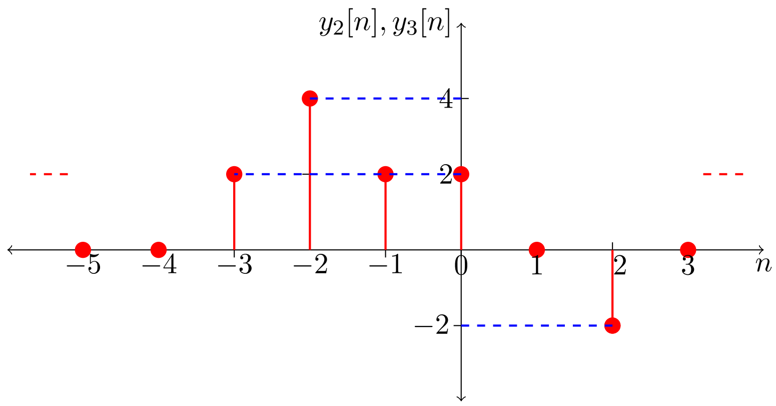
(a)

$$
\begin{aligned}
y_1[n] &= x[n] * (2\delta[n+1] + 2\delta[n-1]) = 2(x[n+1] + x[n-1]) \\
&= 2\delta[n+1] + 4\delta[n] + 2\delta[n-2] + 2\delta[n-1] - 2\delta[n-4].
\end{aligned}
$$



(b) Here, we use commutative and associative properties of convolution operator and get,

$$
\begin{aligned}
y_2[n] &= x[n+2] * h[n] = (x[n] * \delta[n+2]) * h[n] = x[n] * h[n] * \delta[n+2] \\
&= y_1[n] * \delta[n+2] = y_1[n+2].
\end{aligned}
$$



5. The signal y[n] is

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

In this case, the summation reduces to

$$y[n] = \sum_{k=0}^{9} x[k]h[n-k] = \sum_{k=0}^{9} h[n-k]$$

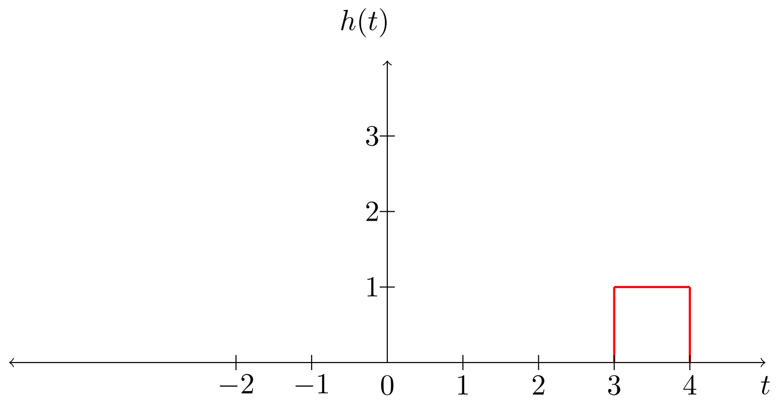
$$y[4] = \sum_{k=0}^{9} h[4-k]$$
$$\Rightarrow 5 = h[4] + h[3] + h[2] + h[1] + h[0] + h[-1] + h[-2] + h[-3] + h[-4] + h[-5]$$
$$\Rightarrow 5 = h[4] + h[3] + h[2] + h[1] + h[0] \;(\because h[n] = 0 \;\forall n < 0)$$

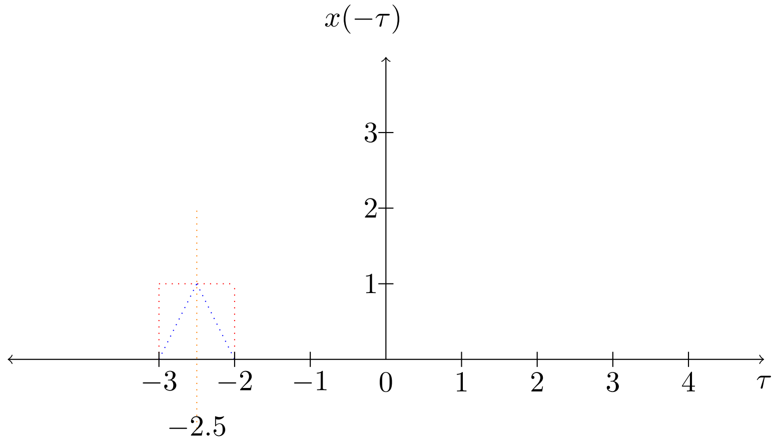
$\therefore N \geq 4$

$$y[14] = \sum_{k=0}^{9} h[14-k]$$
$$\Rightarrow 0 = h[14] + h[13] + h[12] + h[11] + h[10] + h[9] + h[8] + h[7] + h[6] + h[5]$$

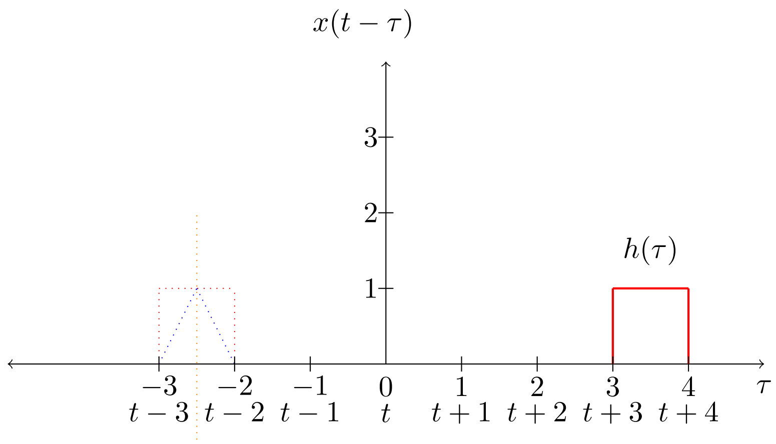
As value of h[n] is either 0 or 1, in order to satisfy the above condition we need $h[14] = h[13] = h[12] = h[11] = h[10] = h[9] = h[8] = h[7] = h[6] = h[5] = 0$. Therefore $N = 4$

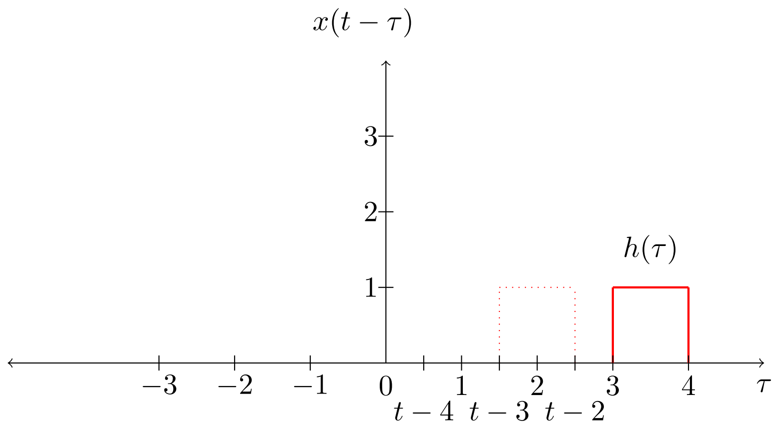
6. (a) We know that $y(t) = \int\limits_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$. Here, $h(\tau)$ is non-zero only in $(3, 4)$, then the above integral becomes $y(t) = \int\limits_{3}^{4} x(t-\tau)d\tau$. Further, $x(t)$ is non-negative only in $(2, 3)$, and zero elsewhere. Eventually, $x(t-\tau)$ will be non-zero only between $t-3$ and $t-2$. $y(t)$ will be zero $t - 3 > 4 \Rightarrow t > 7$ and $t - 2 < 3 \Rightarrow t < 5$. This implies that the above integral is non-zero for $5 \leq t \leq 7$. Hence, y(t) is non-zero for $t \in (5, 7)$. This can be seen in the figures shown below.





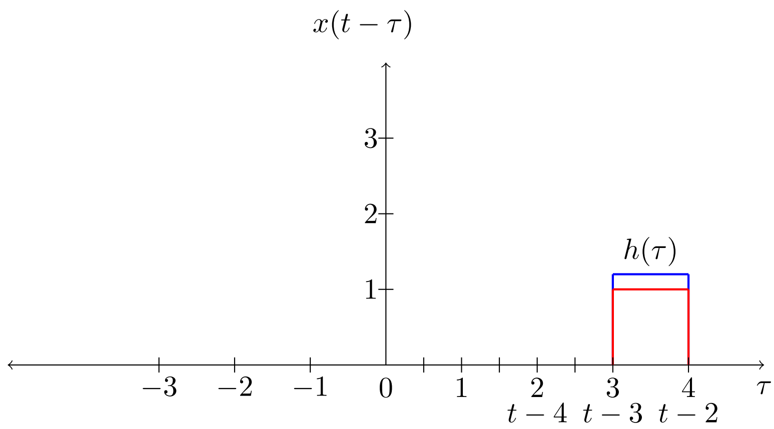
$x(\tau)$ can be any signal which is non negative and symmetric about 2.5. Here $x(-\tau)$ is plotted.





$x(t-\tau)$ is moved towards $h(\tau)$ and we check conditions for $y(t)$ to be non-zero by seeing where the multiplication between $x(t-\tau)$ and $h(\tau)$ yields a non-zero value.

(b) $y(t) = \int_{3}^{4} x(t-\tau)d\tau = \int_{t-4}^{t-3} x(\tau)d\tau$. Again, $x(\tau)$ is **non-negative** only in $\tau \in (2,3)$, with symmetry around $\tau = \frac{5}{2}$. The integral computes the complete area occupied by $x(\tau)$ only when $t = 6$, as only at $t = 6$ the limits of the integral is 2 to 3. For other values of t, the integration will either be equal to area of a part of $x(\tau)$ or zero. Therefore, $y(t)$ will have maximum value at $t = 6$. It can be seen from the figure that the area will be maximum when the $x(t-\tau)$ is superimposed on the $h(\tau)$. Note that signal $x(t-\tau)$ can be any signal, here it is been assumed to be square pulse.



7. Given: $y(t) = \int_{-\infty}^{t+1} \sin(t-\tau)x(\tau)d\tau$.

(a) The response to a delayed input will be,

$$y_1(t) = \int_{-\infty}^{t+1} \sin(t-\tau)x(\tau - t_o)d\tau = \int_{-\infty}^{t-t_o+1} \sin(t-\tau^{'}-t_o)x(\tau^{'})d\tau^{'}$$

However, the delayed response of the system is given by,

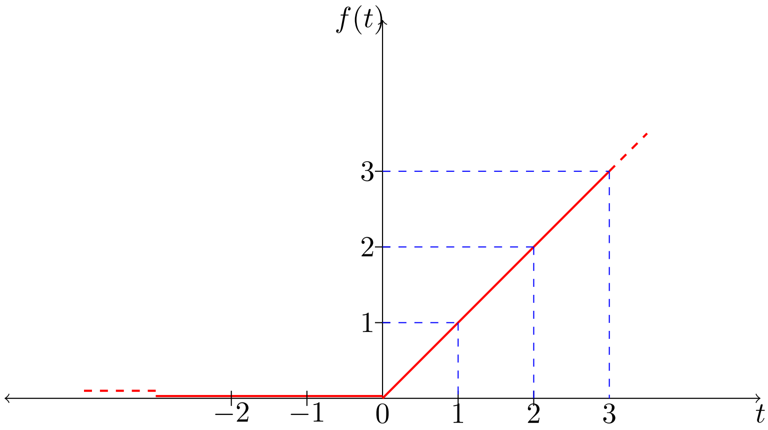
$$y_2(t) = \int_{-\infty}^{t-t_o+1} \sin(t - t_o - \tau)x(\tau)d\tau.$$

Since, $y_1(t) = y_2(t)$, the given system is time-invariant.

(b) Now, $y(t) = \int_{-\infty}^{t+1} \sin(t-\tau)x(\tau)d\tau = \int_{-\infty}^{\infty} \sin(t-\tau)u(t+1-\tau)x(\tau)d\tau$. Hence, the impulse response of the system is given by, $h(t) = \sin(t)u(t+1)$.

(c) The system given is non-causal since the output depends on future values of the input.

8. (a) $f(t) = u(t) * u(t) = \int_{-\infty}^{\infty} u(t-\tau)u(\tau)d\tau = \int_0^t 1\, d\tau = t$, (for $t \geq 0$). Thus $f(t) = tu(t)$.



(b)

$$f(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} (-e^{-\tau} + 2e^{-2\tau})u(\tau)10e^{-3(t-\tau)}u(t-\tau)d\tau$$

$$= \int_0^t 10(-e^{-\tau} + 2e^{-2\tau})e^{-3(t-\tau)}d\tau = 10\int_0^t (-e^{2\tau-3t} + 2e^{\tau-3t})d\tau = -5e^{-t} + 20e^{-2t} - 15e^{-3t}.$$

Hence, $f(t) = -5e^{-t} + 20e^{-2t} - 15e^{-3t}$ for $t \geq 0$ and zero elsewhere.

(c) (i) Given: $h(t) = 2e^{-2t}u(t)$ and $x(t) = 1, \forall 2 \leq t \leq 4$ and zero otherwise. Let $y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$. Now, $x(t-\tau)$, as a function of τ, will be 1 from $\tau = t-4$ to $\tau = t-2$, for any given t, and zero outside this range. Consider the following cases:

Case 1: When $t - 2 < 0 \Rightarrow t < 2$.

The product $h(\tau)x(t-\tau) = 0$, as there is no common overlap between the non-zero regions of these two signals. Hence, $y(t) = 0$, $\forall t < 2$.

Case 2: Suppose $t-2 \geq 0$ and $t-4 < 0$, i.e., $2 \leq t < 4$.

Then, $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{t-2} 2e^{-2\tau}d\tau = 1 - e^{-2t+4}$, $\forall 2 \leq t < 4$.

Case 3: Finally, $t-4 \geq 0$, i.e., $t \geq 4$.

Now, $y(t) = \int_{t-4}^{t-2} 2e^{-2\tau}d\tau = -(e^{-2t+4} - e^{-2t+8})$. This value of $y(t)$ is for the range $t \geq 4$.

Hence, the signal $y(t)$ is given by,

$$y(t) = \begin{cases} 0, & t < 2 \\ 1 - e^{-2t+4}, & 2 \leq t < 4 \\ e^{-2t+8} - e^{-2t+4}, & t \geq 4 \end{cases}$$

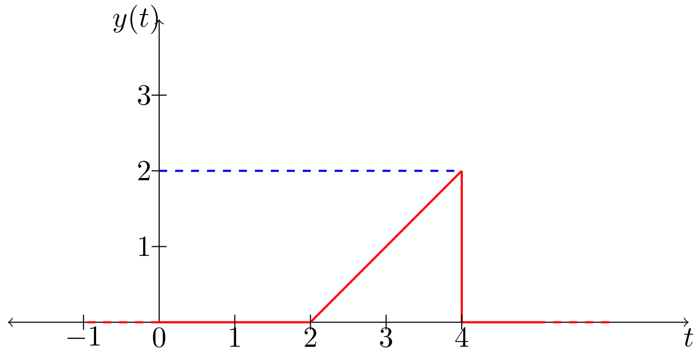
(ii) Given: $h(t) = 2e^{-2t}u(t)$ and $x(t) = \cos(4\pi t) = \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})$. Then, we obtain,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{\infty} e^{j4\pi t}e^{-j4\pi\tau-2\tau} + e^{-j4\pi t}e^{j4\pi\tau-2\tau}\mathrm{d}\tau$$

$$= \left(e^{j4\pi t}\int_{0}^{\infty} e^{-(j4\pi+2)\tau}d\tau + e^{-j4\pi t}\int_{0}^{\infty} e^{-(2-j4\pi)\tau}\mathrm{d}\tau\right)$$

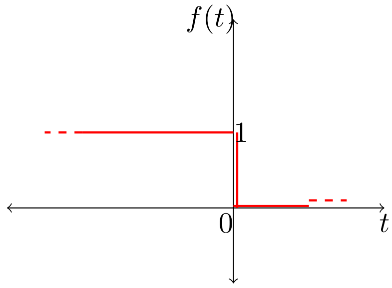
$$= \left(\frac{e^{j4\pi t}}{2+j4\pi} + \frac{e^{-j4\pi t}}{2-j4\pi}\right) = \frac{4\cos(4\pi t) + 8\pi\sin(4\pi t)}{4+16\pi^2}.$$

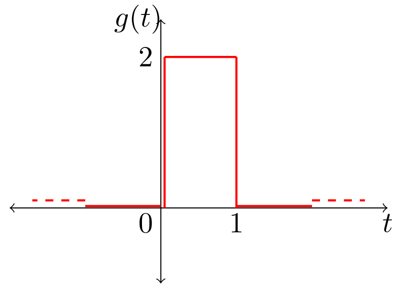
(d) $y(t) = [u(t) * u(t-2)]u(4-t) = r(t-2)u(4-t)$.



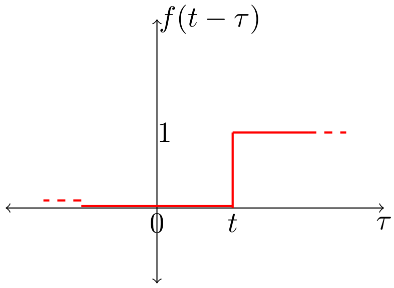
(e) **i)**

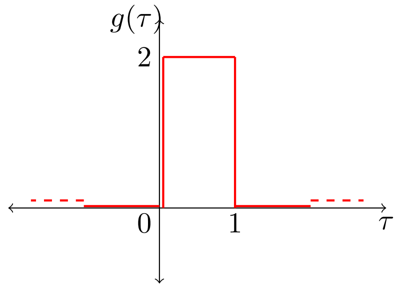
Given: $f(t) = u(-t)$ and $g(t) = 2\Big(u(t) - u(t-1)\Big)$. The signals look like,





Now, $h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau$.

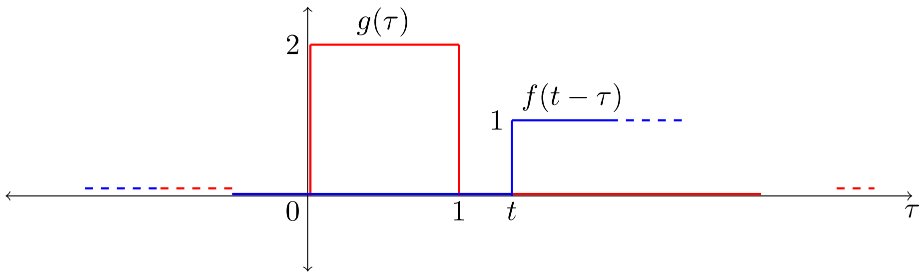




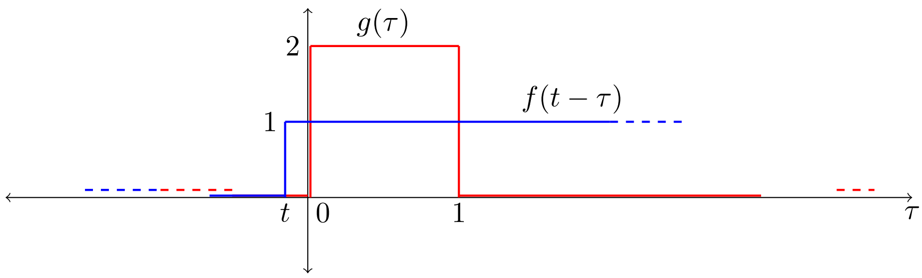
Consider the following cases:

Case 1: $t > 1$.

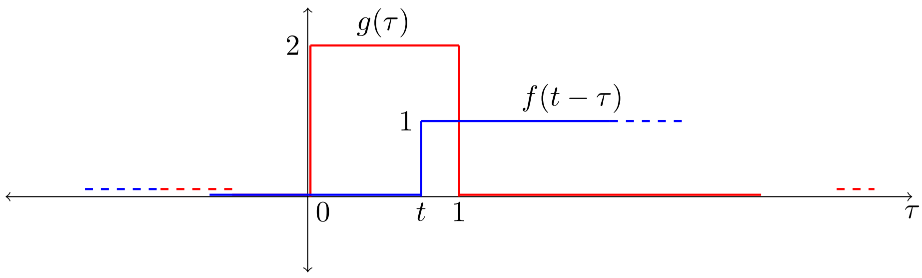
In this case, there is no overlap between $g(\tau)$ and $f(t-\tau)$. Thus, $h(t) = 0, \forall t > 1$.



Case 2: $t \leq 0$. Here, we get, $h(t) = \int_{0}^{1} 2\,d\tau = 2$.

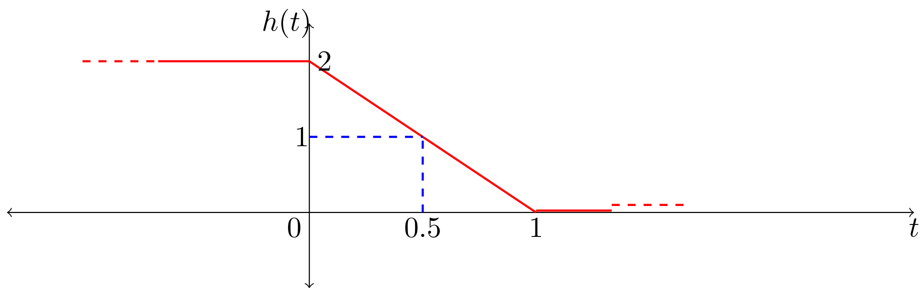


Case 3: $0 < t \leq 1$. Then, $h(t) = \int_{t}^{1} 2\,d\tau = 2(1-t)$.



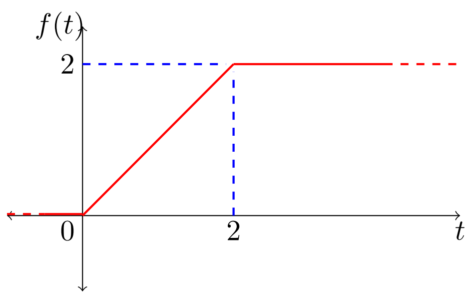
The final expression for the signal $h(t)$ is given by,

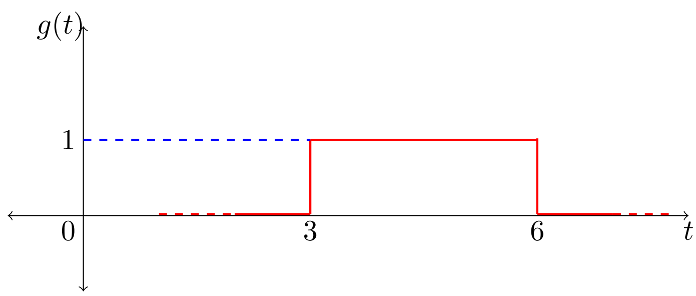
$$h(t) = \begin{cases} 2 & t < 0 \\ 2(1-t) & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}$$



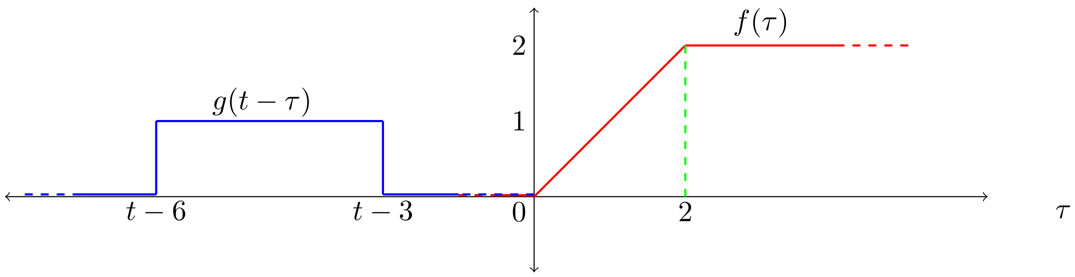
ii)

The signals $f(t)$ and $g(t)$ are as given below,



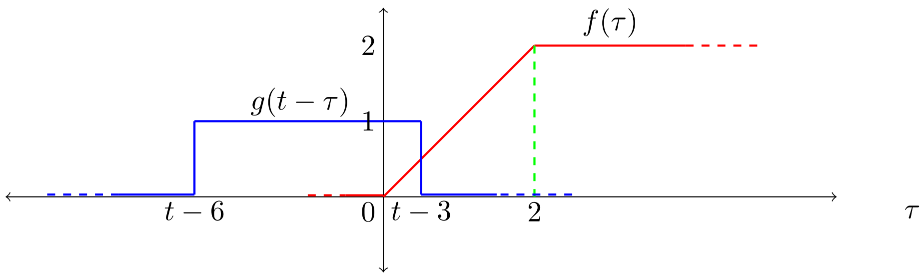


Case 1: For $t < 3$, we have the following:



Hence, $h(t)$ will be zero for $t < 3$, as there is no overlap between $g(t-\tau)$ and $f(\tau)$.

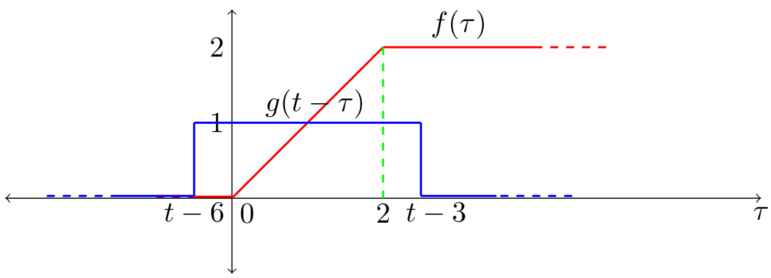
Case 2: For $3 \leq t \leq 5$,



Here, $h(t)$ will be,

$$h(t) = \int_0^{t-3} \tau d\tau = \frac{(t-3)^2}{2}.$$

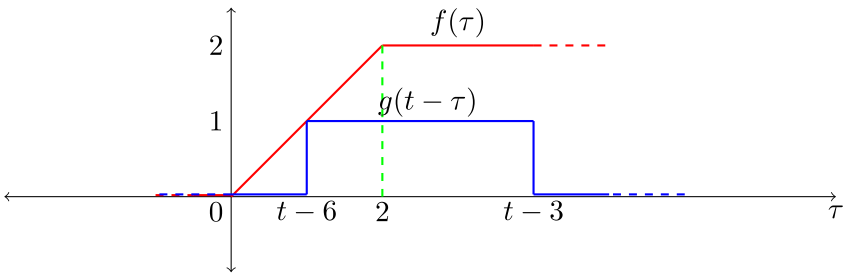
Case 3: For $5 \leq t \leq 6$,



In this case, we get,

$$h(t) = \int_0^{2} \tau d\tau + \int_2^{t-3} 2 d\tau = 2(t-4).$$

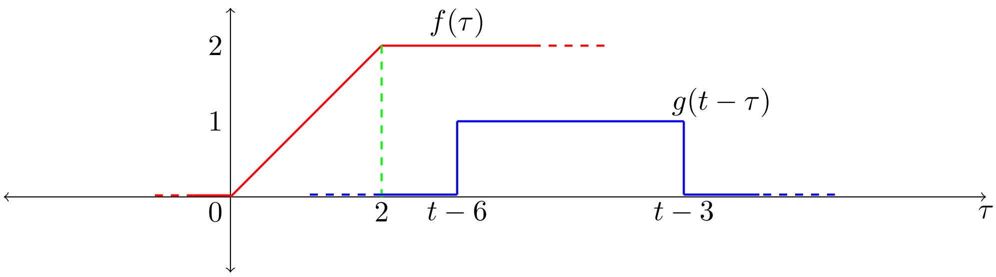
Case 4: For $6 \leq t \leq 8$,



Due to the above overlapping fashion, $h(t)$ for $6 \le t \le 8$ will be,

$$h(t) = \int_{t-6}^{2} \tau d\tau + \int_{2}^{t-3} 2 d\tau = -\frac{t^2}{2} + 8t - 26.$$

Case 5: For $t > 8$, we obtain,



$$h(t) = \int_{t-6}^{t-3} 2 d\tau = 6.$$

Finally, the signal $h(t)$ is given by, $h(t) = \begin{cases} 0 & t \le 3 \\ \frac{(t-3)^2}{2}, & 3 < t < 5, \\ 2(t-4), & 5 \le t < 6 \\ \frac{-t^2}{2} + 8t - 26, & 6 \le t < 8, \\ 6, & t \ge 8 \end{cases}$

9. Given, $x[n] = 0$, outside $0 \leq n \leq N-1$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=0}^{N-1} x[k]h[n-k] \end{aligned}$$

Now, substitute diiferent values for 'n' and expand the sumation,

$$\begin{aligned} y[0] &= x[0]h[0] + x[1]h[-1] + \cdots + x[N-1]h[-(N-1)] \\ y[1] &= x[0]h[1] + x[1]h[0] + \cdots + x[N-1]h[-N+2] \\ &\vdots \\ y[N-1] &= x[0]h[N-1] + x[1]h[N-2] + \cdots + x[N-1]h[0] \end{aligned}$$

We can write the above equations in matrix form ($\mathbf{y{=}Hx}$) as follows ,

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[-1] & \cdots & h[-N+1] \\ h[1] & h[0] & \cdots & h[-N+2] \\ \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

10. (a) Given that $f(t) * g(t) = y(t)$. Hence,

$$y(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau.$$

Let us consider $f(t-T_1) * g(t-T_2)$, and by using the definition of convolution, we have

$$f(t-T_1) * g(t-T_2) = \int_{-\infty}^{\infty} f(\tau - T_1)g(t-\tau-T_2)d\tau = \int_{-\infty}^{\infty} f(\tau-T_1)g(t-T_2-\tau)d\tau.$$

Denote $\tau' = \tau - T_1$, note that the limits and derivative does not change.

$$f(t-T_1) * g(t-T_2) = \int_{-\infty}^{\infty} f(\tau')g\big(t - T_2 - (\tau' + T_1)\big)d\tau' = \int_{-\infty}^{\infty} f(\tau')g\big(t - (T_1+T_2) - \tau'\big)d\tau'$$

$$= y\big(t - (T_1 + T_2)\big) \text{ [On comparing with the first equation].}$$

(b) If $u(t) * u(t) = r(t)$, then

$$\begin{aligned}&\big(u(t+1) - u(t-2)\big) * \big(u(t-3) - u(t-4)\big) \\ &\quad = u(t+1) * u(t-3) - u(t+1) * u(t-4) + u(t-2) * u(t-4) - u(t-2) * u(t-3) \\ &\quad = r(t-2) - r(t-3) + r(t-6) - r(t-5).\end{aligned}$$

The last equality is a consequence of the result obtained in (a). We now sketch $r(t-2) - r(t-3) + r(t-6) - r(t-5)$ in Fig. [1].

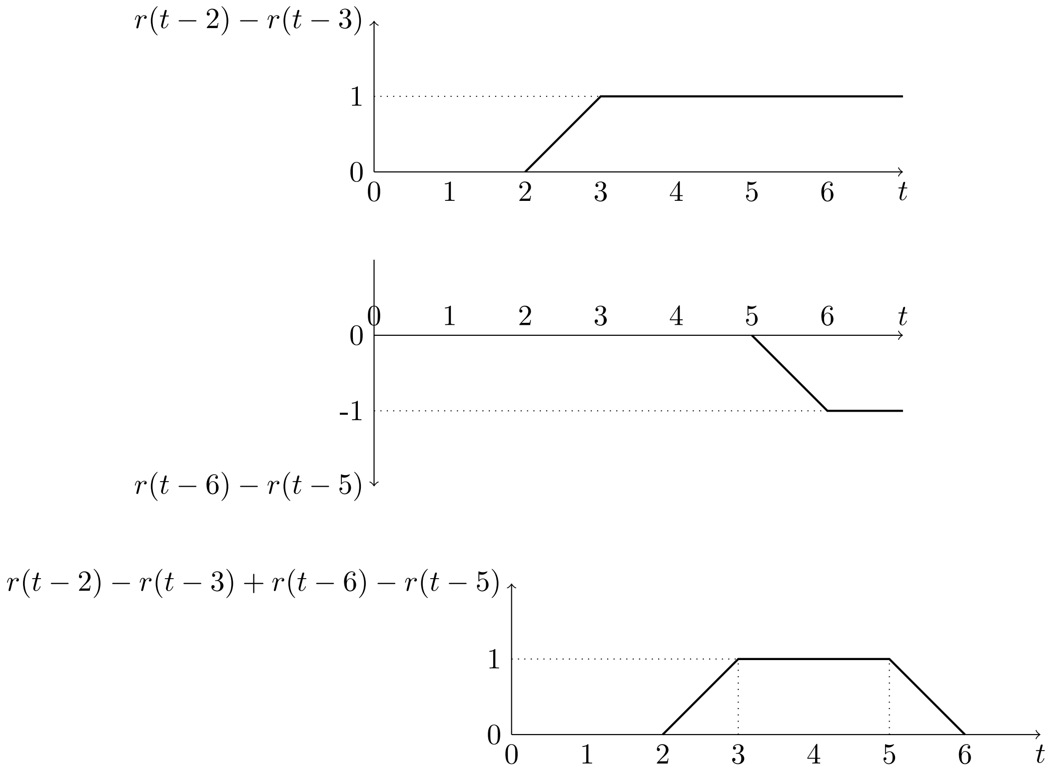


Figure 1: Sketch of the signal $\big(u(t+1) - u(t-2)\big) * \big(u(t-3) - u(t-4)\big)$ using the distributive and shift property of convolution.

One can verify the result by performing convolution of pulses $\big(u(t+1) - u(t-2)\big)$ and $\big(u(t-3) - u(t-4)\big)$, shown in Fig. [2].

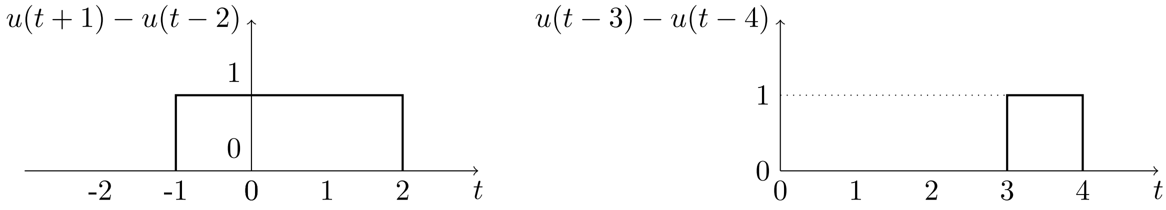


Figure 2: Signals $x(t) = \big(u(t+1) - u(t-2)\big)$ and $y(t) = \big(u(t-3) - u(t-4)\big)$.

Hence, if $x(t) = \Big(u(t+1) - u(t-2)\Big)$ and $y(t) = u(t-3) - u(t-4)$, then $x(t) * y(t)$ is given by,

$$x(t) * y(t) = \begin{cases} 0 & t < 2 \\ t-2 & 2 \leq t < 3 \\ 1 & 3 \leq t \leq 5 \\ 6-t & 5 \leq t \leq 6 \\ 0 & t > 6 \end{cases}$$

11. Given: $y(t) = f(t) * g(t)$. Now, consider the following:

$$y(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau \Longrightarrow y(ct) = \int_{-\infty}^{\infty} f(\tau) g(ct-\tau) d\tau$$

At the same time,

$$f(ct) * g(ct) = \int_{-\infty}^{\infty} f(c\tau) g(ct - c\tau) d\tau$$

Case 1: Let $c > 0$. Then, $c = |c|$. Let $\tau' = c\tau = |c|\tau \Longrightarrow d\tau = \frac{d\tau'}{|c|}$. Hence,

$$f(ct) * g(ct) = \int_{-\infty}^{\infty} f(\tau') g(ct - \tau') \frac{d\tau'}{|c|} = \frac{1}{|c|} \int_{-\infty}^{\infty} f(\tau') g(ct-\tau') d\tau' = \frac{1}{|c|} y(ct).$$

Case 2: Suppose $c < 0$, then $c = -|c|$. In which case, let $\tau' = c\tau = -|c|\tau \Longrightarrow d\tau' = -\frac{d\tau}{|c|}$.

$$f(ct) * g(ct) = \int_{\infty}^{-\infty} f(\tau') g(ct-\tau') \Big(-\frac{d\tau'}{|c|} \Big) = \frac{1}{|c|} \int_{-\infty}^{\infty} f(\tau') g(ct-\tau') d\tau' = \frac{1}{|c|} y(ct).$$

Therefore, if $y(t) = f(t) * g(t)$, then $f(ct) * g(ct) = \frac{1}{|c|} y(ct)$, for all $c \neq 0$.

Fig. [3] shows $f(t)$ and $g(t)$.

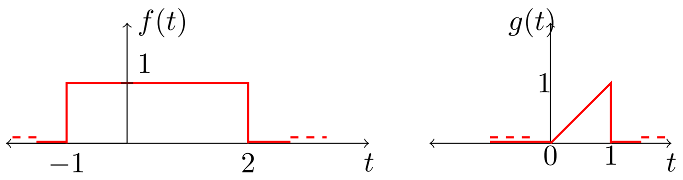


Figure 3: $f(t)$

Then, $y(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$ and $g(t-\tau)$, as a function of τ, will be non-zero from $\tau = t-1$ to $\tau = t$, as shown in figure [4].

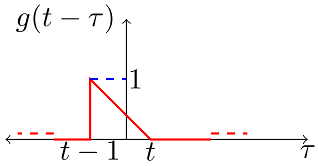


Figure 4: $g(t-\tau)$

Case 1: If $t < -1$. Then, $f(\tau)g(t-\tau) = 0$ in this range. Hence, $y(t) = 0, \forall t < -1$.

Case 2: If $t \geq -1$ but $t-1 < -1$, i.e., $-1 \leq t < 0$.

Here, $y(t) = \int_{-1}^{t} (-\tau + t)\, d\tau = \frac{t^2}{2} + t + \frac{1}{2}$.

Case 3: If $t < 2$ but $t-1 \geq -1$, i.e., $0 \leq t < 2$.

Then, $y(t) = \int_{t-1}^{t} (-\tau + t)\, d\tau = 0.5$.

Case 4: If $t \geq 2$ but $t-1 < 2$, i.e., $2 \leq t < 3$.

Now, $y(t) = \int_{t-1}^{2} (-\tau + t)\, d\tau = -\frac{t^2}{2} + 2t - \frac{3}{2}$.

Case 5: If $t-1 \geq 2$, $f(\tau)g(t-\tau) = 0 \implies y(t) = 0$, $\forall t \geq 3$.

$$y(t) = f(t) * g(t) = \begin{cases} 0, & t < -1 \\ \frac{t^2}{2} + t + \frac{1}{2}, & -1 \leq t < 0 \\ 0.5, & 0 \leq t < 2 \\ -\frac{t^2}{2} + 2t - \frac{3}{2}, & 2 \leq t < 3 \\ 0, & t \geq 3 \end{cases}$$

Using the result derived initially, we get,

$$f(2t) * g(2t) = \frac{1}{2}y(2t) = \begin{cases} 0, & t < -0.5 \\ t^2 + t + 0.25, & -0.5 \leq t < 0 \\ 0.25, & 0 \leq t < 1 \\ -t^2 + 2t - 0.75, & 1 \leq t < 1.5 \\ 0, & t \geq 1.5 \end{cases}$$