Department of Physics
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PH1020 Physics II Problem set 4

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1. A cylinder of length $2L$ and radius a is centred at the origin, with the z-axis as its symmetry axis. The cylinder is uniformly polarized with polarization $\mathbf{P} = P_0\hat{e}_z$ where P_0 is a constant. (i) Find the bound charge densities ρ_b and σ_b. (ii) Find the electric field at all points on the positive z-axis, and verify that it satisfied the appropriate boundary condition at $z = L$ (iii) Find the electric field at the origin, and sketch its magnitude as a function of ratio a/L.

2. Consider a uniform spherical free charge distribution of radius a and charge density ρ_0. This region is filled with a medium of dielectric constant K_1, and surrounded by a medium of dielectric constant K_2. Find (i) the bound volume charge density everywhere in space, and (ii) the bound surface charge density on the surface of the sphere.

3. A capacitor is formed of two concentric conducting spheres of radii a and b ($a < b$), and the space between is filled with a substance. The dielectric constant of the substance at a distance r from the centre is $\frac{c+r}{r}$, where c is constant. The outer sphere is earthed and the inner sphere is charged. Calculate the capacitance of the system.

4. An insulated spherical conductor in air carries a charge q. The conductor is now surrounded by a concentric spherical shell of dielectric of radii b and c, ($c > b$), whose dielectric constant is a function $k(r)$ of the radial distance r from the centre. Calculate the electrostatic energy.

5. A dielectric sphere of radius R contains a uniform distribution of free charge with charge density ρ_f. Find the potential at the centre of the sphere.

6. A sphere of radius R and dielectric constant k, centered at the origin of coordinates, is placed in a constant field E_o directed along the z-axis. The corresponding electrostatic potential is given by $\emptyset(r,\theta,\varphi) = (-E_o r + b_1 r^{-2})\cos\theta$ outside the sphere, and $\emptyset(r,\theta,\varphi) = (b_2 r)\cos\theta$ inside the sphere. Find (i) the constants b_1 and b_2, in terms of k, E_o and R, (ii) the electric field at all points in space, (iii) the polarization $\boldsymbol{P}$ of the sphere, and the dipole moment of the sphere about the origin, (iv) the volume and surface densities of the bound charge in the sphere.

Problems & solutions of sheet-4

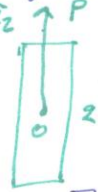
TUTORIAL SHEET – 4

SOLUTIONS

1. (i) Since $\rho_f = 0$, $\rho_b = 0$

$$\sigma_b = \bar{P}\cdot\hat{n} = \begin{cases} P_o & \text{through the top surface} \\ -P_o & \text{through the bottom surface} \\ 0 & \text{through the curved surface} \end{cases}$$

(ii) The electric field is due to the bound surface charges in the top and bottom surfaces alone. $\bar{E}$ at a height z above the center of a circular disc of charge density σ_b in the xy plane with its center at the origin is,



$$\bar{E} = \frac{\sigma_b}{2\epsilon_o}\left[1 - \frac{z}{(a^2+z^2)^{1/2}}\right]\hat{e}_z$$

Here $z > L$

$$\bar{E} = \frac{P_o}{2\epsilon_o}\left[1 - \frac{(z-L)}{[a^2+(z-L)^2]^{1/2}} - 1 + \frac{(z+L)}{[a^2+(z+L)^2]^{1/2}}\right]\hat{e}_z$$

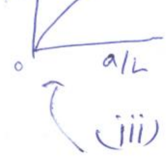
$$= \frac{P_o}{2\epsilon_o}\left[\frac{z+L}{[a^2+(z+L)^2]^{1/2}} - \frac{(z-L)}{[a^2+(z-L)^2]^{1/2}}\right]\hat{e}_z$$

|||ly., for points inside the cylinder, $z < L$

$$\bar{E}(z<L) = -\frac{P_o}{2\epsilon_o}\left[2 - \frac{(L-z)}{[a^2+(L-z)^2]^{1/2}} - \frac{(L+z)}{[a^2+(L+z)^2]^{1/2}}\right]\hat{e}_z$$

At just outside the top surface, $z = L$ and the field is

$$\bar{E}_{out} = \frac{P_o}{\epsilon_o}\left[\frac{L}{(a^2+4L^2)^{1/2}}\right] e_z$$



The field just inside the cylinder is obtained by substituting L for z in the expression for $\bar{E}(z<L)$

$$E_{in} = -\frac{P_o}{\epsilon_o}\left[1 - \frac{L}{(a^2+4L^2)^{1/2}}\right]\hat{e}_z$$

which is consistent with boundary condition: $\bar{E}_{out} - \bar{E}_{in} = \frac{P_o}{\epsilon_o}$

(iii) At the origin the $\underline{E}$ is obtained by putting $z=0$

$$\bar{E} = -\frac{P_o}{\epsilon_o}\left[1 - \frac{L}{(a^2+L^2)^{1/2}}\right]\hat{e}_z = -\frac{P_o}{\epsilon_o}\left[1 - \frac{1}{[1+(a/L)^2]^{1/2}}\right]\hat{e}_z$$

Problems & solutions of sheet-4

T4-2

2) As charge density is uniform, $\bar{E}$, $\bar{D}$ and $\bar{P}$ have only radial components.

(a) From the Gauss's law, in the region $0 < r < a$

$$4\pi r^2 D_r = \frac{4}{3}\pi r^3 \rho_0 \Rightarrow D_{r<a} = \frac{\rho_0 r}{3} \Rightarrow E_{r<a} = \frac{\rho_0 r}{3\epsilon_0 K_1}$$

$$\therefore P_{r<a} = D_{r<a} - \epsilon_0 E_{r<a} = \frac{\rho_0 r}{3}\left[1 - \frac{1}{K_1}\right]$$

$$\Rightarrow \rho_b = -\bar{\nabla}\cdot\bar{P} = -\rho_0\left[1 - \frac{1}{K_1}\right]$$

For $r > a$

$$4\pi r^2 D_r = \frac{4}{3}\pi a^3 \rho_0 \Rightarrow D_{r>a} = \frac{\rho_0 a^3}{3 r^2}$$

$$\Rightarrow E_{r>a} = \frac{\rho_0 a^3}{3\epsilon_0 r^2 K_2}$$

$$\therefore P_{r>a} = \frac{\rho_0 a^3}{3 r^2}\left[1 - \frac{1}{K_2}\right] \text{ and } \rho_b = 0$$

(b) $$\sigma_b = \left|P_{r>a} - P_{r<a}\right|_{r=a} = \frac{\rho_0 a}{3}\left[\frac{1}{K_1} - \frac{1}{K_2}\right]$$

As the free charge density is zero; D_r is continuous at $r = a$ but, P_r is not, leading to the bound charge density.

T4-3

(3) Apply Gauss theorem considering a sphere of radius r:

$$4\pi r^2 D_r = Q \qquad a \le r \le b$$

where Q is the charge on the inner sphere and D_r is the radial component of $\bar{D}$, and the other components of $\bar{D}$ vanish. Therefore.

$$\epsilon E_r = \frac{Q}{4\pi r^2} \; ; \; E_\theta , E_\phi = 0$$

$$\Rightarrow E_r = \frac{Q}{4\pi\epsilon_0 r(c+r)} = -\frac{dV(r)}{dr} = \frac{-Q}{4\pi\epsilon_0 c}\left[\frac{1}{c+r} - \frac{1}{r}\right]$$

$$\therefore , V(r) = \frac{Q}{4\pi\epsilon_0 c} \ln\left(\frac{c+r}{r}\right) + A$$

Given: $V = V_1$ at $r = a$ and $V = 0$ at $r = b$

$$\therefore V_1 = \frac{Q}{4\pi\epsilon_0 c} \ln\left[\frac{b(a+c)}{a(b+c)}\right]$$

$\therefore$ The capacitance C is given by the relation

$$\frac{V_1}{Q} = \frac{1}{C} = \frac{1}{4\pi\epsilon_0 c} \ln\left[\frac{b(a+c)}{a(b+c)}\right]$$

$$\text{or} \quad C = \frac{4\pi\epsilon_0 c}{\ln\left[\frac{b(a+c)}{a(b+c)}\right]}$$

Problems & solutions of sheet-4

T4-4

(4) Let 'a' be the radius of the insulated spherical conductor.

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r}$$ and the electrostatic

energy is $$W_1 = \frac{Q^2}{8\pi\epsilon_0}\int_a^\infty \frac{dr}{r^2}$$

Now, let us place the dielectric shell around spherical conductor

$$E_r = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 r^2}, & a \le r \le b \\ \dfrac{Q}{4\pi\epsilon_0 K(r) r^2} & b \le r \le c \\ \dfrac{Q}{4\pi\epsilon_0 r^2} & r \ge c \end{cases}$$

and $$D_r = \frac{Q}{4\pi r^2}$$

Now the electrostatic energy is given by

$$W_2 = \frac{1}{2}\int \bar{D}\cdot E\, d\tau$$

$$= \frac{Q^2}{32\pi^2}\int_a^b \frac{4\pi r^2 dr}{\epsilon_0 r^4} + \frac{Q^2}{32\pi^2}\int_b^c \frac{4\pi r^2 dr}{\epsilon_0 K(r) r^4} + \frac{Q^2}{32\pi^2}\int_c^\infty \frac{4\pi r^2 dr}{\epsilon_0 r^4}$$

$$= \frac{Q^2}{8\pi\epsilon_0}\left[\int_a^b \frac{dr}{r^2} + \int_b^c \frac{dr}{K(r) r^2} + \int_c^\infty \frac{dr}{r^2}\right]$$

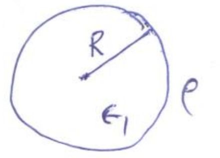
Then $$W_1 - W_2 = \frac{Q^2}{8\pi\epsilon_0}\int_b^c \left[1 - \frac{1}{K(r)}\right]\frac{dr}{r^2}$$

Thus there is a loss of energy. When dielectric intervenes an electrostatic system, the electrostatic energy of the system decreases!!

Problems & solutions of sheet-4

(5)

$$V(0)-V(R) = -\int_R^0 \bar{E}(\bar{r})\cdot d\bar{r}$$



~~ϵ_1: permittivity of sphere.~~

$$V(0) - \frac{Q}{4\pi\epsilon_0 R} = \frac{1}{4\pi\epsilon_1}\int_0^R \frac{Qr}{R^3}\,dr$$

$$= \frac{Q}{4\pi\epsilon_1 R}$$

$$\therefore\ V(0) = \frac{Q}{4\pi R}\left(\frac{1}{\epsilon_0} + \frac{1}{2\epsilon_1}\right)$$

(6) The electrostatic potential

$$\Phi(r,\theta,\phi) = \begin{cases} (-E_0 r + b r^{-2})\cos\theta, & r > R \\ b_2 r\cos\theta; & r < R \end{cases}$$

The corresponding electric field is $E = -\nabla\Phi$

$$\bar{E} = \begin{cases} E_0\,\hat{e}_z + \frac{b_1}{r^3}\left(2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta\right) & r > R \quad \rightarrow (1) \\ -b_2\,\hat{e}_z & r < R \end{cases}$$

The displacement $\bar{D}$ can be computed using. $D = \epsilon_0 k E$ for $r < R$ and $D = \epsilon_0 E$ for $r > R$.

$$\therefore\ \frac{D}{\epsilon_0} = \begin{cases} E_0\,\hat{e}_z + \frac{b_1}{r^3}\left(2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta\right) & \text{for } r > R \\ -b_2 k\,\hat{e}_z & \text{for } r < R \end{cases}$$

(a) To calculate b_1, b_2 constants we use the condition at $r = R$ and arbitrary (θ,ϕ). The normal component of D ie $\bar{D}\cdot\hat{e}_r$ to be continuous as there is no free charge on the interface. The tangential component of E ie $\bar{D} \times \hat{e}_r$ should be continuous. (use $\hat{e}_r \times \hat{e}_z = -\sin\theta\, \hat{e}_\phi$)

$$\left(E_0 + 2\frac{b_1}{R^3}\right)\cos\theta = -b_2 K \cos\theta$$

$$\left(-E_0 + \frac{b_1}{R^3}\right)\sin\theta\, \hat{e}_\theta = b_2 \sin\theta\, \hat{e}_\phi$$

we obtain

$$b_1 = \left(\frac{K-1}{K+2}\right) E_0 R^3, \quad b_2 = -\left(\frac{3}{K+2}\right) E_0$$

(b) Inserting the b_1 & b_2 in eqn ① we get

$$\bar{E} = \begin{cases} E_0 \hat{e}_z + E_0 \left(\frac{K-1}{K+2}\frac{R^3}{r^3}\right)(2\cos\theta\, \hat{e}_r + \sin\theta\, \hat{e}_\theta) & \text{for } r > R \\ \frac{3}{K+2} E_0 \hat{e}_z & \text{for } r < R \end{cases}$$

(c) The polarization is given by $\bar{P} = \bar{D} - \epsilon_0 \bar{E}$ and for $r<R$

$$\bar{P} = \frac{3(K-1)}{(K+2)} \epsilon_0 E_0 \hat{e}_z$$

$\therefore$ Dipole moment of the sphere is

$$\vec{p} = \frac{4}{3}\pi R^3 \vec{P}$$

(d) The volume and bound charge densities are

$$\rho_b(\theta,\phi) = -\bar{\nabla}\bar{P} = 0, \quad \sigma_b(\theta,\phi) = \bar{P}\cdot\hat{n} = \bar{P}\cdot\hat{e}_r$$

$$\boxed{\sigma_b(\theta,\phi) = \frac{3(K-1)}{K+2}\epsilon_0 E_0 \cos\theta}$$