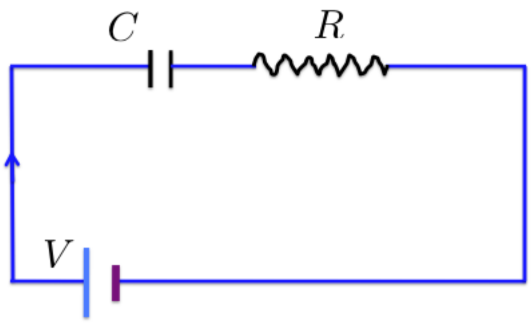
Indian Institute of Technology Madras
PH1020, Tutorial Set-8

Question 1. A 50 pF parallel plate capacitor is getting charged at such a rate that its voltage is increasing at 300 V/s. The plates are circular with a radius of 10 cm. Calculate J_D and the magnetic induction at a distance of 5 cm from the axis of the capacitor in the space between the plates.



Solution:

We have

$$\left|J_D\right| = \frac{\partial D}{\partial t} = \frac{\partial \sigma}{\partial t} = \frac{1}{A}\frac{\partial q}{\partial t} = \frac{C}{A}\frac{\partial V}{\partial t}$$

since $q = CV$. In this case

$$J_D = \frac{50\times 10^{-12}}{\pi \times 10^{-2}} \times 300 = 0.4775 \quad \mu A/m^2$$

In order to calculate the magnetic induction at a distance of 5 cm from the axis of the capacitor plate, this is like calculating B inside a cylindrical conductor having a uniform current density J_D. There is no difference between the magnetic field effect of a displacement current and that of a true current. We get

$$\begin{aligned} B(r = 5cm) &= \frac{\mu_0 J_D}{2} r \\ &= \frac{4\pi \times 10^{-7} \times 0.4775 \times 10^{-6} \times 5 \times 10^{-2}}{2} \\ &= 1.5 \times 10^{-14} \text{ T} \\ &= 1.5 \times 10^{-10} \text{ gauss} \end{aligned}$$

Question 2: Medium 1 comprising the region $z > 0$ of a Cartesian coordinate system is characterized by the permeability $\mu_1 = 4\mu_0$, whereas medium 2 comprising the region $z < 0$ is characterized by $\mu_2 = 2\mu_0$. μ_0 is the permeability of free space. The magnetic induction $\vec{B}_1$ in medium 1 is given by

$$\vec{B}_1 = B_0(2\hat{e}_x + 4\hat{e}_y + 5\hat{e}_z)$$

where B_0 is a constant of suitable dimensions. The boundary $z = 0$ between the two media carries a free surface current density $\vec{K}_f$ given by

$$\vec{K}_f = \left(\frac{B_0}{\mu_0}\right)(\hat{e}_x - 2\hat{e}_y)$$

All fields are independent of time and spatially uniform in both the media. Determine the magnetic induction $\vec{B}_2$ in the medium 2.

Solution:

$\vec{B} = \vec{B}_{2n} + \vec{B}_{2t}$
By the boundary conditions for $\vec{B}$,

$$\begin{aligned}\vec{B}_{2n} &= \vec{B}_{1n} \\ &= 5B_0\hat{e}_z\end{aligned}$$

$$\begin{aligned}\vec{H}_{2t} - \vec{H}_{1t} &= \vec{K}_f \times \hat{n}_2 \\ &= \left(\frac{B_0}{\mu_0}\right)\left(\hat{e}_x - 2\hat{e}_y\right) \times (-\hat{e}_z) \\ &= \left(\frac{B_0}{\mu_0}\right)\left(\hat{e}_y + 2\hat{e}_x\right)\end{aligned}$$

$$\begin{aligned}\therefore \quad \vec{H}_{2t} &= \vec{H}_{1t} + \left(\frac{B_0}{\mu_0}\right)\left(\hat{e}_y + 2\hat{e}_x\right) \\ &= \frac{B_0}{4\mu_0}\left(2\hat{e}_x + 4\hat{e}_y\right) + \left(\frac{B_0}{\mu_0}\right)\left(\hat{e}_y + 2\hat{e}_x\right) \\ &= \left(\frac{B_0}{\mu_0}\right)\left(\frac{5}{2}\,\hat{e}_x + 2\hat{e}_y\right)\end{aligned}$$

$$\vec{B}_{2t} = 2\mu_0\vec{H}_{2t} = B_0\left[5\hat{e}_x + 4\hat{e}_y\right]$$

$\vec{B}_2 = B_0\left[5\hat{e}_x + 4\hat{e}_y + 5\hat{e}_z\right]$

Question 3: A current flowing in a long straight solenoid with the radius R of cross section is varied so that the magnetic field inside the solenoid increases with time according to the law $B = \beta t^2$, where β is a constant. Find the displacement current density as a function of the distance r from the solenoid axis.

Solution:

In order to find the displacement current density, we must find the electric field strength(here it will be a vortex field). Using Maxwell's equation for circulation of vector E, we write

$$2\pi r E = -\pi r^2 \frac{\partial B}{\partial t}, \quad E = -r\beta t \quad (r < R)$$
$$2\pi r E = -\pi R^2 \frac{\partial B}{\partial t}, \quad E = -\frac{R^2\beta t}{r} \quad (r > R)$$

Now, using the formula $j_d = \epsilon_0 \frac{\partial E}{\partial t}$, we can find the displacement current density:

$$j_d = -\epsilon_0\beta r \quad (r < R), \qquad j_d = -\frac{\epsilon_0\beta R^2}{r} \quad (r > R).$$

Question 4: If the electric field in vacuum is $\vec{E} = E_0 cos(\omega t - ky)\hat{x}$, what is the $\vec{H}$ field ? Does the electric field $\vec{E} = E_0 cos(\omega t - kx)\hat{x}$ in vacuum satisfy the Maxwell's equations ? Under what circumstances would this $\vec{E}$ field satisfy the Maxwell's equations ?

Solution:

From Faraday's law,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$
$$\therefore \ \frac{\partial \vec{H}}{\partial t} = \frac{kE_0}{\mu_0} sin(\omega t - ky)\ \hat{z}$$

By integrating this expressing w.r.t time

$$\vec{H} = -\frac{kE_0}{\omega \mu_0} cos\ (\omega t - ky)\ \hat{z}$$

The electric field $\vec{E} = E_0 cos(\omega t - kx)\ \hat{x}$ does not satisfy Gauss's law for vacuum, which requires $\vec{\nabla} \cdot \vec{D} = \rho = 0$.

But it satisfies Gauss's law only for non-zero charge density

$$\rho = \vec{\nabla} \cdot \vec{D} = \epsilon_0 \frac{\partial E_x}{\partial x} = \frac{\partial}{\partial x} (\epsilon_0 E_0 cos(\omega t - kx)) = k\epsilon_0 E_0 sin(\omega t - kx) \neq 0$$