Lecture Notes on

CONTROL SYSTEM THEORY AND DESIGN

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Preface

This is a collection of the lecture notes of the three authors for a firstyear graduate course on control system theory and design (ECE 515, formerly ECE 415) at the ECE Department of the University of Illinois at Urbana-Champaign. This is a fundamental course on the modern theory of dynamical systems and their control, and builds on a first-level course in control that emphasizes frequency-domain methods (such as the course ECE 486, formerly ECE 386, at UIUC). The emphasis in this graduate course is on state space techniques, and it encompasses modeling, analysis (of structural properties of systems, such as stability, controllability, and observability), synthesis (of observers/compensators and controllers) subject to design specifications, and optimization. Accordingly, this set of lecture notes is organized in four parts, with each part dealing with one of the issues identified above. Concentration is on *linear systems*, with nonlinear systems covered only in some specific contexts, such as stability and dynamic optimization. Both continuous-time and discrete-time systems are covered, with the former, however, in much greater depth than the latter.

The notions of "control" and "feedback", in precisely the sense they will be treated and interpreted in this text, pervade our everyday operations, oftentimes without us being aware of it. Leaving aside the facts that the human body is a large (and very complex) feedback mechanism, and an economy without its build-in (and periodically fine-tuned) feedback loops would instantaneously turn to chaos, the most common examples of control systems in the average person's everyday life is the thermostat in one's living room, or the cruise control in one's their automobile. Control systems of a similar nature can be found in almost any of today's industries. The most obvious examples are the aerospace industry, where control is required in fly by wire positioning of ailerons, or in the optimal choice of trajectories in space flight. The chemical industry requires good control designs to ensure safe and accurate production of specific products, and the paper industry requires accurate control to produce high quality paper. Even in applications

where control has not found any use, this may be a result of inertia within industry rather than lack of need! For instance, in the manufacture of semiconductors currently many of the fabrication steps are done without the use of feedback, and only now are engineers seeing the difficulties that such open-loop processing causes.

There are only a few fundamental ideas that are required to take this course, other than a good background in linear algebra and differential equations. All of these ideas revolve around the concept of feedback, which is simply the act of using measurements as they become available to control the system of interest. The main objective of this course is to teach the student some fundamental principles within a solid conceptual framework, that will enable her/him to design feedback loops compatible with the information available on the "states" of the system to be controlled, and by taking into account considerations such as stability, performance, energy conservation, and even robustness. A second objective is to familiarize her/him with the available modern computational, simulation, and general software tools that facilitate the design of effective feedback loops.

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Part I System Modeling and Analysis

Chapter 1

State Space Models

After a first course in control system design one learns that intuition is a starting point in control design, but that intuition may fail for complex systems such as those with multiple inputs and outputs, or systems with nonlinearities. In order to augment our intuition to deal with problems as complex as high speed flight control, or flow control for high speed communication networks, one must start with a useful mathematical model of the system to be controlled. It is not necessary to "take the system apart" to model every screw, valve, and axle. In fact, to use the methods to be developed in this book it is frequently more useful to find a simple model which gives a reasonably accurate description of system behavior. It may be difficult to verify a complex model, in which case it will be difficult to trust that the model accurately describes the system to be controlled. More crucial in the context of this course is that the control design methodology developed in this text is model based. Consequently, a complex model of the system will result in a complex control solution, which is usually highly undesirable in practice.

Although this text treats nonlinear models in some detail, the most far reaching approximation made is *linearity*. It is likely that no physical system is truly linear. The reasons that control theory is effective in practice even when this ubiquitous assumption fails are that (i) physical systems can frequently be approximated by linear models; and (ii) control systems are designed to be robust with respect to inaccuracies in the system model. It is not simply luck that physical systems can be modeled with reasonable accuracy using linear models. Newton's laws, Kirchoff's voltage and current laws, and many other laws of physics give rise to linear models. Moreover, in this chapter we show that a generalized Taylor series expansion allows

the approximation of even a grossly nonlinear model by a linear one.

The linear models we consider are primarily of the state space form

$$\dot{x} = Ax + Bu
y = Cx + Du$$
(1.1)

where x is a vector signal in \mathbb{R}^n , y and u are the output and input of the system evolving in \mathbb{R}^p and \mathbb{R}^m , respectively, and A, B, C, D are matrices of appropriate dimensions. If these matrices do not depend on the time variable, t, the corresponding linear system will be referred to as *linear time invariant* (LTI), whereas if any one of these matrices has time-varying entries, then the underlying linear system will be called *linear time varying* (LTV). If both the input and the output are scalar, then we refer to the system as *single input-single output* (SISO); if either a control or output are of dimension higher than one, then the system is *multi input-multi output* (MIMO).

Our first example now is a simple nonlinear system which we approximate by a linear state space model of the form (1.1) through linearization.

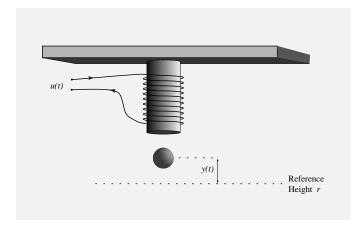


Figure 1.1: Magnetically Suspended Ball

1.1 An electromechanical system

The magnetically suspended metallic ball illustrated in Figure 1.1 is a simple example which illustrates some of the important modeling issues addressed

in this chapter. The input u is the current applied to the electro-magnet, and the output y is the distance between the center of the ball and some reference height. Since positive and negative inputs are indistinguishable at the output of this system, it follows that this cannot be a linear system. The upward force due to the current input is approximately proportional to u^2/y^2 , and hence from Newton's law for translational motion we have

$$ma = m\ddot{y} = mg - c\frac{u^2}{y^2},$$

where g is the gravitational constant and c is some constant depending on the physical properties of the magnet and ball. This input-output model can be converted to (nonlinear) state space form using $x_1 = y$ and $x_2 = \dot{y}$:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = g - \frac{c}{m} \frac{u^2}{x_1^2}$$

where the latter equation follows from the formula $\dot{x}_2 = \ddot{y}$. This pair of equations forms a two-dimensional state space model

$$\dot{x}_1 = x_2 = f_1(x_1, x_2, u) \tag{1.2}$$

$$\dot{x}_2 = g - \frac{c}{m} \frac{u^2}{x_1^2} = f_2(x_1, x_2, u)$$
 (1.3)

It is nonlinear, since f_2 is a nonlinear function of $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Letting $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, the state equations may be written succinctly as

$$\dot{x} = f(x, u).$$

The motion of a typical solution to a nonlinear state space model in \mathbb{R}^2 is illustrated in Figure 1.2.

1.2 Linearization about an equilibrium state

Suppose that a fixed value u_e , say positive, is applied, and that the state $x_e = \begin{pmatrix} x_{e1} \\ x_{e2} \end{pmatrix}$ has the property that

$$f(x_{e1}, x_{e2}, u_e) = \begin{pmatrix} f_1(x_{e1}, x_{e2}, u_e) \\ f_2(x_{e1}, x_{e2}, u_e) \end{pmatrix} = \vartheta.$$

From the definition of f we must have $x_{e2} = 0$, and

$$x_{e1} = \sqrt{\frac{c}{mg}} u_e$$

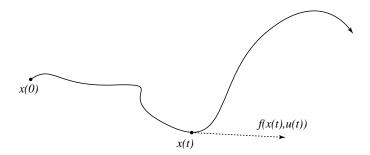


Figure 1.2: Trajectory of a nonlinear state space model in two dimensions: $\dot{x} = f(x, u)$

which is unique when restricted to be positive. The state x_e is called an equilibrium state since the velocity vector f(x, u) vanishes when $x = x_e$, and $u = u_e$. If the signals $x_1(t)$, $x_2(t)$ and u(t) remain close to the fixed point (x_{e1}, x_{e2}, u_e) , then we may write

$$x_1(t) = x_{e1} + \delta x_1(t)$$

$$x_2(t) = x_{e2} + \delta x_2(t)$$

$$u(t) = u_e + \delta u(t),$$

where $\delta x_1(t)$, $\delta x_2(t)$, and $\delta u(t)$ are small-amplitude signals. From the state equations (1.2) and (1.3) we then have

$$\delta \dot{x}_1 = x_{e2} + \delta x_2(t) = \delta x_2(t)$$

 $\delta \dot{x}_2 = f_2(x_{e1} + \delta x_1, x_{e2} + \delta x_2, u_e + \delta u)$

Applying a Taylor series expansion to the right hand side (RHS) of the second equation above gives

$$\delta \dot{x}_{2} = f_{2}(x_{e1}, x_{e2}, u_{e}) + \frac{\partial f_{2}}{\partial x_{1}} \Big|_{(x_{e1}, x_{e2}, u_{e})} \delta x_{1} + \frac{\partial f_{2}}{\partial x_{2}} \Big|_{(x_{e1}, x_{e2}, u_{e})} \delta x_{2} + \frac{\partial f_{2}}{\partial u} \Big|_{(x_{e1}, x_{e2}, u_{e})} \delta u + H.O.T.$$

After computing partial derivatives we obtain the formulae

$$\begin{aligned}
\delta \dot{x}_1 &= \delta x_2. \\
\delta \dot{x}_2 &= 2 \frac{c}{m} \frac{u_e^2}{x_{e1}^3} \delta x_1 - \frac{2c}{m} \frac{u_e}{x_{e1}^2} \delta u + H.O.T.
\end{aligned}$$

Letting x denote the bivariate signal $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ we may write the linearized system in state space form:

$$\delta \dot{x} = \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ \beta \end{bmatrix} \delta u$$
$$\delta y = \delta x_1,$$

where

$$\alpha = 2\frac{c}{m}\frac{u_e^2}{x_{e1}^3}, \qquad \beta = -2\frac{c}{m}\frac{u_e}{x_{e1}^2}.$$

This linearized system is only an approximation, but one that is reasonable and useful in control design as long as the state δx and the control δu remain small. For example, we will see in Chapter 4 that "local" stability of the nonlinear system is guaranteed if the simpler linear system is stable.

1.3 Linearization about a trajectory

Consider now a general nonlinear model

$$\dot{x}(t) = f(x(t), u(t), t)$$

where f is a continuously differentiable (C^1) function of its arguments. Suppose that for an initial condition x_0 given at time t_0 (i.e., $x(t_0) = x_0$), and a given input $u^n(t)$, the solution of the nonlinear state equation above exists and is denoted by $x^n(t)$, which we will refer to as the nominal trajectory corresponding to the nominal input $u^n(t)$, for the given initial state. For example, in the control of a satellite orbiting the earth, the nominal trajectory $x^n(t)$ might represent a desired orbit (see Exercise 7 below). We assume that the input and state approximately follow these nominal trajectories, and we again write

$$x(t) = x^{n}(t) + \delta x(t)$$

$$u(t) = u^{n}(t) + \delta u(t)$$

From a Taylor series expansion we then have

$$\dot{x} = \dot{x}^n + \delta \dot{x} = f(x^n, u^n, t) + \left(\underbrace{\frac{\partial f}{\partial x}\Big|_{(x^n, u^n, t)}}_{A(t)}\right) \delta x + \left(\underbrace{\frac{\partial f}{\partial u}\Big|_{(x^n, u^n, t)}}_{B(t)}\right) \delta u + H.O.T.$$

Since we must have $\dot{x}^n = f(x^n, u^n, t)$, this gives the state space description

$$\delta \dot{x} = A(t)\delta x + B(t)\delta u,$$

where the higher order terms have been ignored.

1.4 A two link inverted pendulum

Below is a photograph of the *pendubot* found at the robotics laboratory at the University of Illinois, and a sketch indicating its component parts. The pendubot consists of two rigid aluminum links: link 1 is directly coupled to the shaft of a DC motor mounted to the end of a table. Link 1 also includes the bearing housing for the second joint. Two optical encoders provide position measurements: one is attached at the elbow joint and the other is attached to the motor. Note that no motor is directly connected to link 2 - this makes vertical control of the system, as shown in the photograph, extremely difficult!

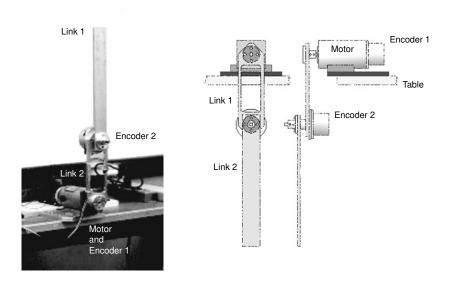


Figure 1.3: The Pendubot

Since the pendubot is a two link robot with a single actuator, its dynamic equations can be derived using the so-called Euler-Lagrange equations found in numerous robotics textbooks [11]. Referring to the figure, the equations

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of motion are

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + h_1 + \phi_1 = \tau \tag{1.4}$$

$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + h_2 + \phi_2 = 0 (1.5)$$

where q_1 , q_2 are the joint angles and

$$d_{11} = m_1 \ell_{c1}^2 + m_2 (\ell_1^2 + \ell_{c2}^2 + 2\ell_1 \ell_{c2} \cos(q_2)) + I_1 + I_2$$

$$d_{22} = m_2 \ell_{c2}^2 + I_2$$

$$d_{12} = d_{21} = m_2 (\ell_{c2}^2 + \ell_1 \ell_{c2} \cos(q_2)) + I_2$$

$$h_1 = -m_2 \ell_1 \ell_{c2} \sin(q_2) \dot{q}_2^2 - 2m_2 \ell_1 \ell_{c2} \sin(q_2) \dot{q}_2 \dot{q}_1$$

$$h_2 = m_2 \ell_1 \ell_{c2} \sin(q_2) \dot{q}_1^2$$

$$\phi_1 = (m_1 \ell_{c1} + m_2 \ell_1) g \cos(q_1) + m_2 \ell_{c2} g \cos(q_1 + q_2)$$

$$\phi_2 = m_2 \ell_{c2} g \cos(q_1 + q_2)$$

The definitions of the variables q_i, ℓ_1, ℓ_{ci} can be deduced from Figure 1.4.

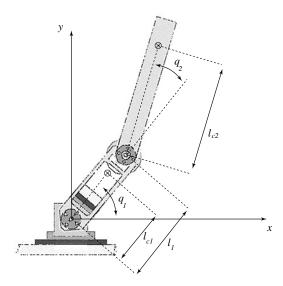


Figure 1.4: Coordinate description of the pendubot: ℓ_1 is the length of the first link, and ℓ_{c1} , ℓ_{c2} are the distances to the center of mass of the respective links. The variables q_1, q_2 are joint angles of the respective links.

This model may be written in state space form as a nonlinear vector differential equation, $\dot{x} = f(x, u)$, where $x = (q_1, q_2, \dot{q}_1, \dot{q}_2)'$, and f is defined

from the above equations. For a range of different constant torque inputs τ , this model admits various equilibria. For example, when $\tau = 0$, the vertical downward position $x_e = (-\pi/2, 0, 0, 0)$ is an equilibrium, as illustrated in the right hand side of Figure 1.3. When $\tau = 0$ it follows from the equations of motion that the upright vertical position $x_e = (+\pi/2, 0, 0, 0)$ is also an equilibrium. It is clear from the photograph given in the left hand side of Figure 1.3 that the upright equilibrium is strongly unstable in the sense that with $\tau = 0$, it is unlikely that the physical system will remain at rest. Nevertheless, the velocity vector vanishes, $f(x_e, 0) = 0$, so by definition the upright position is an equilibrium when $\tau = 0$. Although complex, we may again linearize these equations about the vertical equilibrium. The control design techniques introduced later in the book will provide the tools for stabilization of the pendubot in this unstable vertical configuration via an appropriate controller.

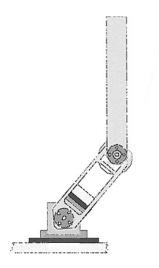


Figure 1.5: There is a continuum of different equilibrium positions for the Pendubot corresponding to different constant torque inputs τ .

1.5 An electrical circuit

A simple electrical circuit is illustrated in Figure 1.6. Using Kirchoff's voltage and current laws we may obtain a state space model in which the current through the inductor and the capacitor voltage become state variables.

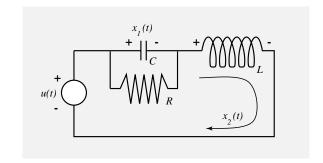


Figure 1.6: A simple RLC circuit

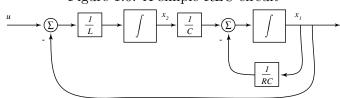


Figure 1.7: A simulation diagram for the corresponding state space model with x_1 = the voltage across the capacitor, and x_2 = the current through the inductor.

Kirchoff's Current Law (KCL) gives

$$x_2 = C\dot{x}_1 + \frac{1}{R}x_1,$$

which may be written as

$$\dot{x}_1 = -\frac{1}{RC}x_1 + \frac{1}{C}x_2 \tag{1.6}$$

From Kirchoff's Voltage Law (KVL) we have

$$x_1 + L\dot{x}_2 = +u,$$

or,

$$\dot{x}_2 = -\frac{1}{L}x_1 + \frac{1}{L}u. \tag{1.7}$$

Equations (1.7) and (1.6) thus give the system of state space equations

$$\dot{x} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u$$

1.6 Transfer Functions & State Space Models

In the previous examples we began with a physical description of the system to be controlled, and through physical laws obtained a state space model of the system. In the first and second examples, a linear model could be obtained through linearization, while in the last example, a linear model was directly obtained through the KVL and KCL circuit laws. For a given LTI system, a state space model which is constructed from a given transfer function G(s) is called a realization of G. Realization of a given G is not unique, and in fact there are infinitely many such realizations, all of which are equivalent however from an input-output point of view. In this section we show how to obtain some of the most common realizations starting from a transfer function model of a system.

Consider the third-order model

$$\ddot{y} + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_2 \ddot{u} + b_1 \dot{u} + b_0 u \tag{1.8}$$

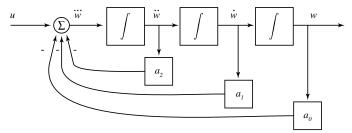
where the coefficients $\{a_i, b_i\}$ are arbitrary real numbers. The corresponding transfer function description is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{B(s)}{A(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

where Y and U are the Laplace transforms of the signals y and u, respectively. The numerator term $B(s) = b_2 s^2 + b_1 s + b_0$ can create complexity in determining a state space model. So, as a starting point, we consider the "zero-free system" where $B(s) \equiv 1$, which results in the model

$$\ddot{w} + a_2 \ddot{w} + a_1 \dot{w} + a_0 w = u$$

An equivalent simulation diagram description for this system is:



With zero initial conditions one has the relation Y(s) = B(s)W(s), so that the signal y can be obtained by adding several interconnections to the above simulation diagram, yielding the simulation diagram depicted in

Figure 1.8. Letting the outputs of the integrators form states for the system we obtain the state space model

$$x_1 = w$$
 $\dot{x}_1 = x_2$
 $x_2 = \dot{w}$ $\dot{x}_2 = x_3$
 $x_3 = \ddot{w}$ $\dot{x}_3 = -a_2x_3 - a_1x_2 - a_0x_1 + u$

and

$$y = b_0 x_1 + b_1 x_2 + b_2 x_3$$

This may be written in matrix form as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u
y = [b_0 \ b_1 \ b_2] x + [0] u.$$

This final state space model is called the *controllable canonical form* (CCF) - one of the most important system descriptions from the point of view of analysis.

Several alternative descriptions can be obtained by manipulating the transfer function G(s) before converting to the time domain, or by defining states in different ways. One possibility is to take the description

$$(s^3 + a_2s^2 + a_1s + a_0)Y(s) = (b_0 + b_1s + b_2s^2)U(s)$$
(1.9)

and divide throughout by s^3 to obtain

$$\left(1 + \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_0}{s^3}\right)Y(s) = \left(\frac{b_0}{s^3} + \frac{b_1}{s^2} + \frac{b_2}{s}\right)U(s)$$

Rearranging terms then gives

$$Y = \frac{1}{s^3}(b_0U - a_0Y) + \frac{1}{s^2}(b_1U - a_1Y) + \frac{1}{s}(b_2U - a_2Y)$$

We may again describe this equation using a simulation diagram, as given in Figure 1.9. As before, by letting x_1, x_2, x_3 denote the outputs of the integrators we obtain a state space model which now takes the form

$$\dot{x}_1 = x_2 - a_2 x_1 + b_2 u
\dot{x}_2 = x_3 - a_1 x_1 + b_1 u
\dot{x}_3 = -a_0 x_1 + b_0 u
y = x_1$$

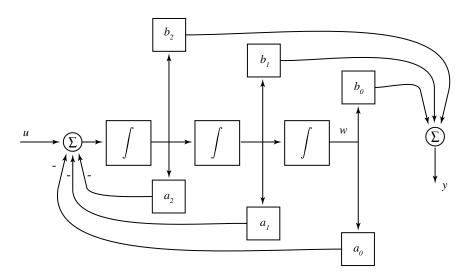


Figure 1.8: Controllable Canonical Form

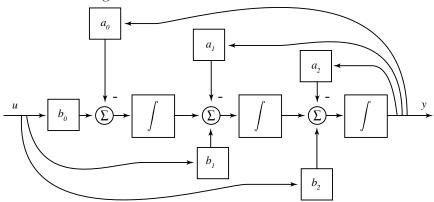


Figure 1.9: Observable Canonical Form

or in matrix form

$$\dot{x} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u
y = [1 & 0 & 0]x + [0]u.$$

This final form is called the *observable canonical form* (OCF).

In the example above, the degree n_0 of the denominator of G(s) is 3, and the degree m_0 of the numerator is 2, so that $n_0 > m_0$. In this case the model is called *strictly proper*. In the case where $n_0 = m_0$, the "D" matrix in (1.1) will be non-zero in any state space realization. To see this, try adding the term b_3s^3 to the right hand side of (1.9), or solve Exercise 10 of this chapter.

Both controllable and observable canonical forms admit natural generalizations to the n-dimensional case. For a SISO LTI system, let the input-output transfer function be given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{B(s)}{A(s)} = \frac{b_{n-1}s^{n-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_2s^2 + a_1s + a_0}$$

Then, the n-dimensional state space realization in *controllable canonical* form is identified by the following A, B, and C matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ -a_0 & -a_1 & -a_2 & a_3 & \cdots & a_{n-1} \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}; \quad C = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}^T.$$

The observable canonical form, on the other hand, will have the following A, B, and C matrices:

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}; \quad B = \begin{bmatrix} b_{n-1} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T.$$

Another alternative is obtained by applying a partial fraction expansion to the transfer function G:

$$G(s) = \frac{b(s)}{a(s)} = d + \sum_{i=1}^{n} \frac{k_i}{s - p_i},$$

where $\{p_i : 1 \leq i \leq n\}$ are the poles of G, which are simply the roots of a. A partial expansion of this form is always possible if all of the poles are distinct. In general, a more complex partial fraction expansion must be employed. When a simple partial fraction expansion is possible, as above, the system may be viewed as a parallel network of simple first order simulation diagrams; see Figure 1.10.

The significance of this form is that it yields a strikingly simple system description:

$$\begin{vmatrix} \dot{x}_1 = p_1 x_1 + k_1 u \\ \vdots \\ \dot{x}_n = p_n x_n + k_n u \end{vmatrix}$$
 decoupled dynamic equations.

This gives the state space model

$$\dot{x} = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & p_n \end{bmatrix} x + \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} u$$

$$y = [1, \dots, 1]x + [d]u.$$

This is often called the *modal form*, and the states $x_i(t)$ are then called *modes*. It is important to note that this form is not always possible if the roots of the denominator polynomial are not distinct. Exercises 12 and 13 below address some generalizations of the modal form.

Matlab Commands

Matlab is not well suited to nonlinear problems. However, the Matlab program Simulink can be used for simulation of both linear and nonlinear models. Some useful Matlab commands for system analysis of linear systems are

RLOCUS calculates the root locus. e.g. rlocus(num,den), or rlocus(A,B,C,D).

STEP computes the step response of a linear system.

BODE computes the Bode plot.

NYQUIST produces the Nyquist plot.

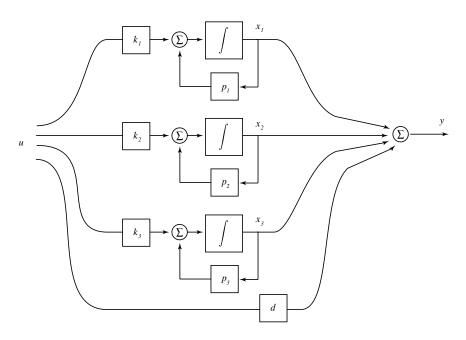


Figure 1.10: Partial fraction expansion of a transfer function

 $\mathbf{TF2SS}$ gives the CCF state space representation of a transfer function model, but in a different form than given here.

SS2TF computes the transfer function of a model, given any state space representation.

RESIDUE may be used to obtain the partial fraction expansion of a transfer function.

Summary and References

State space models of the form

$$\dot{x} = f(x, u)
y = g(x, u)$$

occur naturally in the mathematical description of many physical systems, where u is the input to the system, y is the output, and x is the state variable. In such a state space model, each of these signals evolves in Euclidean space. The functions f and g are often linear, so that the model takes the form

$$\dot{x} = Ax + Bu
y = Cx + Du.$$

If not, the functions f, g may be approximated using a Taylor series expansion which again yields a linear state space model. The reader is referred to [7] for more details on modeling from a control perspective.

A given linear model will have many different state space representations. Three methods for constructing a state space model from its transfer function have been illustrated in this chapter:

- (a) Construct a simulation diagram description, and define the outputs of the integrators as states. This approach was used to obtain the controllable canonical form, but it is also more generally applicable.
- (b) Manipulate the transfer function to obtain a description which is more easily described in state space form. For instance, a simple division by s^n led to the observable canonical form.
- (c) Express the transfer function as a sum of simpler transfer functions This approach was used to obtain a modal canonical form.

These three system descriptions, the modal, controllable, and observable canonical forms, will be applied in control analysis in later chapters.

Much more detail on the synthesis of state space models for linear systems may be found in Chapter 6 of [6], and Chapter 3 of [5].

1.7. EXERCISES 17

1.7 Exercises

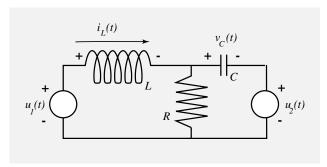
1.7.1 You are given a nonlinear input-output system which satisfies the non-linear differential equation:

$$\ddot{y}(t) = 2y - (y^2 + 1)(\dot{y} + 1) + u.$$

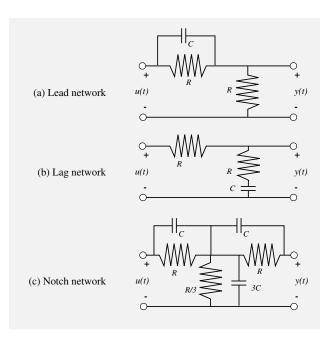
- (a) Obtain a nonlinear state-space representation.
- (b) Linearize this system of equations around its equilibrium output trajectory when $u(\cdot) \equiv 0$, and write it in state space form.
- 1.7.2 Repeat Exercise 1 with the new system

$$\ddot{y}(t) = 2y - (y^2 + 1)(\dot{y} + 1) + u + 2\dot{u}.$$

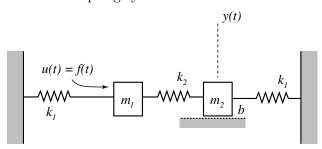
1.7.3 Obtain state equations for the following circuit. For the states, use the voltage across the capacitor, and the current through the inductor.



- 1.7.4 In each circuit below,
 - (a) Obtain a transfer function and a state space realization.
 - (b) Sketch a frequency response.
 - (c) Use the step command in *Matlab* to obtain a step response.



1.7.5 Consider the mass-spring system shown below



Assume that a force is acting on m_1 , and let the horizontal position of m_2 represent the output of this system.

- (a) Derive a set of differential equations which describes this inputoutput system. To solve this problem you will require Newton's law of translational motion, and the following facts: (i) The force exerted by a spring is proportional to its displacement, and (ii) the force exerted by a frictional source is proportional to the relative speed of the source and mass.
- (b) Find the transfer function for the system.
- (c) Obtain a state space description of the system.
- **1.7.6** The *n*-dimensional nonlinear vector differential equation $\dot{x} = f(x)$ has

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a unique solution from any $x \in \mathbb{R}^n$ if the function f has continuous partial derivatives. To see that just continuity of f is not sufficient for uniqueness, and that some additional conditions are needed, consider the scalar differential equation

$$\dot{x} = \sqrt{1 - x^2}, \quad x(0) = 1.$$

Show that this differential equation with the given initial condition has at least two solutions: One is $x(t) \equiv 1$, and another one is $x(t) = \cos(t)$.

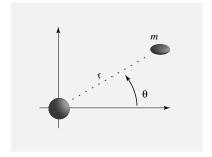
1.7.7 Consider a satellite in *planar* orbit about the earth. The situation is modeled as a point mass m in an inverse square law force field, as sketched below. The satellite is capable of thrusting (using gas jets, for example) with a radial thrust u_1 and a tangential (θ direction) thrust u_2 . Recalling that acceleration in polar coordinates has a radial component $(\ddot{r}-r\dot{\theta}^2)$, and a tangential component $(r\ddot{\theta}+2\dot{r}\dot{\theta})$, Newton's Law gives

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2} + u_1$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = u_2,$$

where k is a gravitational constant.

- (a) Convert these equations to (nonlinear) state space form using $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \theta$, $x_4 = \dot{\theta}$.
- (b) Consider a nominal circular trajectory $r(t) = r_0$; $\theta(t) = \omega_0 t$, where r_0 and ω_0 are constants. Using $u_1(t) = u_2(t) = 0$, obtain expressions for the nominal state variables corresponding to the circular trajectory. How are k, r_0 , m, and ω_0 related?
- (c) Linearize the state equation in (a) about the state trajectory in (b). Express the equations in matrix form in terms of r_0 , ω_0 and m (eliminate k).



- **1.7.8** Using *Matlab* or *Simulink*, simulate the nonlinear model for the magnetically suspended ball.
 - (a) Using proportional feedback $u = -k_1y + k_2r$, can you stabilize the ball to a given reference height r? Interpret your results by examining a root locus diagram for the linearized system.
 - (b) Can you think of a better control law? Experiment with other designs.

Transfer Functions & State Space Models

1.7.9 A SISO LTI system is described by the transfer function

$$G(s) = \frac{s+4}{(s+1)(s+2)(s+3)}$$

- (a) Obtain a state space representation in the controllable canonical form;
- (b) Now obtain one in the observable canonical form;
- (c) Use partial fractions to obtain a representation with a diagonal state matrix A (modal form).

In each of (a)–(c) draw an all-integrator simulation diagram.

1.7.10 A SISO LTI system is described by the transfer function

$$G(s) = \frac{s^3 + 2}{(s+1)(s+3)(s+4)}$$

- (a) Obtain a state space representation in the controllable canonical form;
- (b) Now obtain one in the observable canonical form;
- (c) Use partial fractions to obtain a representation of this model with a diagonal state matrix A.

In each of (a)–(c) draw an all-integrator simulation diagram.

1.7.11 For the multiple input-multiple output (MIMO) system described by the pair of differential equations

$$\ddot{y}_1 + 2\dot{y}_1 + 3y_2 = u_1 + \ddot{u}_1 + \dot{u}_2$$

$$\ddot{y}_2 - 3\dot{y}_2 + \dot{y}_1 + y_2 + y_1 = u_2 + \dot{u}_3 - u_3$$

obtain a state space realization by choosing y_1 and y_2 as state variables. Draw the corresponding simulation diagram. 1.7. EXERCISES

1.7.12 This exercise generalizes modal form to the case where some eigenvalues are repeated. For each of the following transfer functions, obtain a state-space realization for the corresponding LTI system by breaking H into simple additive terms, and drawing the corresponding simulation diagrams for the sum. Choose the outputs of the integrators as state variables.

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(a)
$$H(s) = \frac{2s^2}{s^3 - s^2 + s - 1}$$

(b) $H(s) = \frac{s^2 + s + 1}{s^3 + 4s^2 + 5s + 2}$

(b)
$$H(s) = \frac{s^2 + s + 1}{s^3 + 4s^2 + 5s + 2}$$

- 1.7.13 This exercise indicates that a useful generalization of the modal form may be constructed when some eigenvalues are complex.
 - (a) Use partial fractions to obtain a diagonal state space representation for a SISO LTI system with transfer function

$$G(s) = \frac{s+6}{s^2+2s+2}.$$

Note that complex gains appear in the corresponding all-integrator diagram.

(b) Given a transfer function in the form

$$G(s) = \frac{s+\beta}{(s-\lambda_1)(s-\lambda_2)}$$

and a corresponding state space realization with A matrix Compute the eigenvalues λ_1, λ_2 of the matrix

$$A = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

where $\sigma \geq 0$, $\omega > 0$, find the relationships between $\lambda_1, \lambda_2, \beta$ and σ, ω . In view of this, complete the state space realization by obtaining the B and C matrices, and draw the corresponding simulation diagram.

- (c) For the transfer function $H(s) = \frac{s^2 + s + 1}{s^3 + 4s^2 + 5s + 2}$, obtain a statespace realization for the corresponding LTI system by breaking H into simple additive terms, and drawing the corresponding simulation diagrams for the sum.
- (d) Apply your answer in (b) to obtain a state space realization of G(s) in (a) with only real coefficients.

Chapter 2

Vector Spaces

Vectors and matrices, and the spaces where they belong, are fundamental to the analysis and synthesis of multivariable control systems. The importance of the theory of vector spaces in fact goes well beyond the subject of vectors in finite-dimensional spaces, such as \mathbb{R}^n . Input and output signals may be viewed as vectors lying in infinite dimensional function spaces. A system is then a mapping from one such vector to another, much like a matrix maps one vector in a Euclidean space to another. Although abstract, this point of view greatly simplifies the analysis of state space models and the synthesis of control laws, and is the basis of much of current optimal control theory. In this chapter we review the theory of vector spaces and matrices, and extend this theory to the infinite-dimensional setting.

2.1 Fields

A field is any set of elements for which the operations of addition, subtraction, multiplication, and division are defined. It is also assumed that the following axioms hold for any $\alpha, \beta, \gamma \in \mathcal{F}$

- (a) $\alpha + \beta \in \mathcal{F}$ and $\alpha \cdot \beta \in \mathcal{F}$.
- (b) Addition and multiplication are commutative:

$$\alpha + \beta = \beta + \alpha, \qquad \alpha \cdot \beta = \beta \cdot \alpha.$$

(c) Addition and multiplication are associative:

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma), \qquad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

(d) Multiplication is distributive with respect to addition:

$$(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$$

- (e) There exists a unique null element 0 such that $0 \cdot \alpha = 0$ and $0 + \alpha = \alpha$.
- (f) There exists a unique identity element 1 such that $1 \cdot \alpha = \alpha$.
- (g) For every $\alpha \in \mathcal{F}$ there exists a unique element $\beta \in \mathcal{F}$ such that $\alpha + \beta = 0$; this unique element is sometimes referred to as the *additive inverse* or *negative* of α , and is denoted as $-\alpha$.
- (h) To every $\alpha \in \mathcal{F}$ which is not the element 0 (i.e., $\alpha \neq 0$), there corresponds an element γ such that $\alpha \cdot \gamma = 1$; this element is referred to as the *multiplicative inverse* of α , and is sometimes written as α^{-1} .

Fields are a generalization of \mathbb{R} , the set of all real numbers. The next example is the set of all complex numbers, denoted \mathbb{C} . These are the only examples of fields that will be used in the text, although we will identify others in the exercises at the end of the chapter.

2.2 Vector Space

A set of vectors \mathcal{X} is a set on which vector addition, and scalar multiplication are defined, generalizing the concept of the vector space \mathbb{R}^n . In this abstract setting a *vector space* over a field \mathcal{F} , denoted $(\mathcal{X}, \mathcal{F})$, is defined as follows:

- (a) For every $x_1, x_2 \in \mathcal{X}$, the vector sum $x_1 + x_2 \in \mathcal{X}$.
- (b) Addition is commutative: For every $x_1, x_2 \in \mathcal{X}$, the sum $x_1 + x_2 = x_2 + x_1$.
- (c) Addition is associative: For every $x_1, x_2, x_3 \in \mathcal{X}$,

$$(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$$

- (d) The set \mathcal{X} contains a vector ϑ such that $\vartheta + x = x$ for all $x \in \mathcal{X}$.
- (e) For every $x \in \mathcal{X}$, there is a vector $y \in \mathcal{X}$ such that $x + y = \vartheta$.
- (f) For every $x \in \mathcal{X}$, $\alpha \in \mathcal{F}$, the scalar product $\alpha \cdot x \in \mathcal{X}$.
- (g) Scalar multiplication is associative: for every $\alpha, \beta \in \mathcal{F}$, and $x \in \mathcal{X}$,

$$\alpha(\beta x) = (\alpha \beta) x.$$

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Below is a list of some of the vector spaces which are most important in applications

- $(\mathbb{R}^n, \mathbb{R})$ the real vector space of n dimensional real-valued vectors.
- $(\mathbb{C}^n,\mathbb{C})$ the complex vector space of n dimensional complex-valued vectors.
- $(C^n[a,b],\mathbb{R})$ the vector space of real-valued continuous functions on the interval [a,b], taking values in \mathbb{R}^n .
- $(D^n[a,b],\mathbb{R})$ the vector space of real-valued piecewise-continuous functions on the interval [a,b], taking values in \mathbb{R}^n .
- $(L_p^n[a,b],\mathbb{R})$ the vector space of functions on the interval [a,b], taking values in \mathbb{R}^n , which satisfy the bound

$$\int_a^b |f(t)|^p dt < \infty, \qquad f \in L_p[a, b].$$

 $(\mathcal{R}(\mathbb{C}), \mathbb{R})$ – the vector space of rational functions $\frac{b(s)}{a(s)}$ of a complex variable s, with real coefficients.

A subspace \mathcal{Y} of a vector space \mathcal{X} is a subset of \mathcal{X} which is itself a vector space with respect to the operations of vector addition and scalar multiplication. For example, the set of complex n-dimensional vectors whose first component is zero is a subspace of $(\mathbb{C}^n, \mathbb{C})$, but \mathbb{R}^n is not a subspace of $(\mathbb{C}^n, \mathbb{C})$.

2.3 Bases

A set of vectors $S = (x^1, ..., x^n)$ in $(\mathcal{X}, \mathcal{F})$ is said to be *linearly independent* if the following equality

$$\alpha_1 x^1 + \alpha_2 x^2 + \dots + \alpha_n x^n = 0$$

holds for a set of n elements $\{\alpha_i : 1 \leq i \leq n\} \subset \mathcal{F}$, then $\alpha_1 = \alpha_2 = \cdots \alpha_n = 0$. If the set S contains an infinite number of vectors, then we call S linearly independent if every finite subset of S is linearly independent, as defined above.

In the case where $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^n, \mathbb{R})$, we will have linear independence of $\{x^1, \ldots, x^n\}$ if and only if

$$\det[x^1 x^2 \cdots x^n] = \det \begin{bmatrix} x_1^1 & x_1^n \\ \vdots & \dots & \vdots \\ x_n^1 & x_n^n \end{bmatrix} \neq 0.$$

The maximum number of linearly independent vectors in $(\mathcal{X}, \mathcal{F})$ is called the *dimension* of $(\mathcal{X}, \mathcal{F})$. For example, $(\mathbb{R}^n, \mathbb{R})$ and $(\mathbb{C}^n, \mathbb{C})$ both have dimension n. What is the dimension of $(\mathbb{C}^n, \mathbb{R})$? (see Exercise 6).

More interesting examples can be found in function spaces. If for example $(\mathcal{X}, \mathcal{F}) = (C[0, 1], \mathbb{R})$, where C[0, 1] is the set of real-valued continuous functions on [0, 1], then we can easily find a set S of infinite size which is linearly independent. One such set is the collection of simple polynomials $S = \{t, t^2, t^3, \ldots\}$. To see that S is linearly independent, note that for any n.

$$\sum_{i=1}^{n} \alpha_i t^i = \vartheta \quad \text{only if} \quad \alpha_i = 0, \ 1 \le i \le n,$$

where $\vartheta \in C[0,1]$ is the function which is identically zero on [0,1]. We have thus shown that the dimension of $(C[0,1],\mathbb{R})$ is infinite.

A set of linearly independent vectors $S = \{e^1, \dots, e^n\}$ in $(\mathcal{X}, \mathcal{F})$ is said to be a *basis* of \mathcal{X} if every vector in \mathcal{X} can be expressed as a unique linear combination of these vectors. That is, for any $x \in \mathcal{X}$, one can find $\{\beta_i, 1 \leq i \leq n\}$ such that

$$x = \beta_1 e^1 + \beta_2 e^2 + \dots + \beta_n e^n.$$

Because the set S is linearly independent, one can show that for any vector x, the scalars $\{\beta_i\}$ are uniquely specified in \mathcal{F} . The n-tuple $\{\beta_1, \ldots, \beta_n\}$ is often called the *representation* of x with respect to the basis $\{e^1, \ldots, e^n\}$.

We typically denote a vector $x \in \mathbb{R}^n$ by

$$x = \left[\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right].$$

There are two interpretations of this equation:

(a) x is a vector (in \mathbb{R}^n), independent of basis.

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(b) x is a representation of a vector with respect to the natural basis:

$$x = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The following theorem is easily proven in \mathbb{R}^n using matrix manipulations, and the general proof is similar:

Theorem 2.1. In any n-dimensional vector space, any set of n linearly independent vectors qualifies as a basis.

In the case of Euclidean space $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^n, \mathbb{R})$, with $\{e^1, \dots, e^n\}$ a given basis, any vector $x \in \mathbb{R}^n$ may be expressed as

$$x = \beta_1 e^1 + \dots + \beta_n e^n,$$

where $\{\beta_i\}$ are all real scalars. This expression may be equivalently written as $x = E\beta$, where

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad E = \begin{bmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \cdots & e_{nn} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix},$$

which through inversion shows that β is uniquely given by

$$\beta = E^{-1}x\tag{2.1}$$

Here E^{-1} stands for the matrix inverse of E (i.e., $E^{-1}E = EE^{-1} = I$, where I is the identity matrix), which exists since e^{i} 's are linearly independent.

Consider the numerical example with $e^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Then we have $x = \beta_1 e^1 + \beta_2 e^2$, with

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

2.4 Change of basis

Suppose now we are given two sets of basis vectors:

$$\{e^1,\cdots,e^n\}$$
 $\{\bar{e}^1,\cdots,\bar{e}^n\}$

A vector x in \mathcal{X} can be represented in two possible ways, depending on which basis is chosen:

$$x = \beta_1 e^1 + \dots + \beta_n e^n = \sum_{k=1}^n \beta_k e^k$$
 (2.2)

or,

$$x = \bar{\beta}_1 \bar{e}^1 + \cdots \bar{\beta}_n \bar{e}^n = \sum_{k=1}^n \bar{\beta}_k \bar{e}^k.$$
 (2.3)

Since $\{e^i\} \subset \mathcal{X}$, there exist scalars $\{p_{ki} : 1 \leq k, i \leq n\}$ such that for any i,

$$e^{i} = p_{1i}\bar{e}^{1} + \dots + p_{ni}\bar{e}^{n} = \sum_{k=1}^{n} p_{ki}\bar{e}^{k}.$$

From (2.2) it then follows that a vector x may be represented as

$$x = \sum_{i=1}^{n} \beta_i \left(\sum_{k=1}^{n} p_{ki} \bar{e}^k \right) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} p_{ki} \beta_i \right) \bar{e}^k.$$
 (2.4)

In view of (2.3) and (2.4) we have by subtraction

$$\sum_{k=1}^{n} \left[\sum_{i=1}^{n} p_{ki} \beta_i - \bar{\beta}_k \right] \bar{e}^k = \vartheta.$$

By linear independence of $\{\bar{e}^k\}$, this implies that each coefficient in brackets is 0. This gives a matrix relation between the coefficients $\{\beta_i\}$ and $\{\bar{\beta}_i\}$:

$$\bar{\beta}_k = \sum_{i=1}^n p_{ki}\beta_i, \quad k = 1, \dots, n$$

or using compact matrix notation,

$$\bar{\beta} = \begin{bmatrix} \bar{\beta}_1 \\ \vdots \\ \bar{\beta}_n \end{bmatrix} = \begin{bmatrix} p_{11} & \dots & p_{1n} \\ \vdots & & \\ p_{n1} & \dots & p_{nn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = P\beta.$$

The transformation P maps $\mathcal{F}^n \to \mathcal{F}^n$, and is one to one, and onto. It therefore has an inverse P^{-1} , so that β can also be computed through $\bar{\beta}$:

$$\beta = P^{-1}\bar{\beta}.$$

For the special case where $(\mathcal{X}, \mathcal{F}) = (\mathbb{R}^n, \mathbb{R})$, the vectors $\{e_i\}$ can be stacked to form a matrix to obtain as in (2.1),

$$x = E\beta = \bar{E}\bar{\beta}.$$

Hence the transformation P can be computed explicitly: $\bar{\beta} = \bar{E}^{-1}E\beta$, so that $P = \bar{E}^{-1}E$. The inverse \bar{E}^{-1} again exists by linear independence.

2.5 Linear Operators

A linear operator \mathcal{A} is simply a function from one vector space $(\mathcal{X}, \mathcal{F})$ to another $(\mathcal{Y}, \mathcal{F})$, which is linear. This means that for any scalars α_1, α_2 , and any vectors x_1, x_2 ,

$$\mathcal{A}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \mathcal{A}(x_1) + \alpha_2 \mathcal{A}(x_2).$$

For a linear operator \mathcal{A} or a general function from a set \mathcal{X} into a set \mathcal{Y} we adopt the terminology

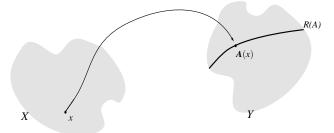
 \mathcal{X} : Domain of the mapping \mathcal{A}

 \mathcal{Y} : Co-Domain of the mapping \mathcal{A}

When \mathcal{A} is applied to every $x \in \mathcal{X}$, the resulting set of vectors in \mathcal{Y} is called the *range* (or *image*) of \mathcal{A} , and is denoted by $\mathcal{R}(\mathcal{A})$:

$$\mathcal{R}(\mathcal{A}) := \bigcup_{x \in \mathcal{X}} \mathcal{A}(x)$$

Pictorially, these notions are illustrated as follows:



The rank of A is defined to be the dimension of $\mathcal{R}(A)$.

In the special case of a linear operator $\mathcal{A}: \mathbb{R}^n \to \mathbb{R}^m$ defined as $\mathcal{A}(x) = Ax$ for an $m \times n$ matrix A,

$$y = Ax = [a^{1} \dots a^{n}] \begin{bmatrix} x_{1} \\ \vdots \\ x_{n} \end{bmatrix}$$
$$= x_{1}a^{1} + \dots + x_{n}a^{n},$$

the range of A is the set of all possible linear combinations of the columns $\{a_i\}$ of A. That is, the space spanned by the columns of A. The dimension of $\mathcal{R}(A)$ is then the maximum number of linearly independent columns of A.

Theorem 2.2. For a linear operator A, the set $\mathcal{R}(A)$ is a subspace of \mathcal{Y} .

Proof To prove the theorem it is enough to check closure under addition and scalar multiplication. Suppose that $y^1, y^2 \in \mathcal{R}(\mathcal{A})$, and that $\alpha_1, \alpha_2 \in \mathcal{F}$. Then by definition of $\mathcal{R}(\mathcal{A})$ there are vectors x_1, x_2 such that

$$\mathcal{A}(x^1) = y^1, \qquad \mathcal{A}(x^2) = y^2,$$

and then by linearity of the mapping \mathcal{A} ,

$$\mathcal{A}(\alpha_1 x^1 + \alpha_2 x^2) = \alpha_1 \mathcal{A}(x^1) + \alpha_2 \mathcal{A}(x^2) = \alpha_1 y^1 + \alpha_2 y^2.$$

Hence, $\alpha_1 y^1 + \alpha_2 y^2 \in \mathcal{R}(\mathcal{A})$, which establishes the desired closure property.

By assumption, the set \mathcal{Y} contains a zero vector, which could be mapped from numerous $x \in \mathcal{X}$. The set of all such $x \in \mathcal{X}$ is called the *nullspace* of \mathcal{A} , denoted $\mathcal{N}(\mathcal{A})$:

$$\mathcal{N}(\mathcal{A}) := \{x \in \mathcal{X} \text{ such that } \mathcal{A}(x) = 0\}.$$

Theorem 2.3. For any linear operator $A: \mathcal{X} \to \mathcal{Y}$, the nullspace $\mathcal{N}(A)$ is a subspace of \mathcal{X} .

Proof Again we check that $\mathcal{N}(\mathcal{A})$ is closed under addition and scalar multiplication. Suppose that $x^1, x^2 \in \mathcal{N}(\mathcal{A})$, and that $\alpha_1, \alpha_2 \in \mathcal{F}$. Then by definition,

$$\mathcal{A}(x^1) = \mathcal{A}(x^2) = \vartheta.$$

Again by linearity of \mathcal{A} it is clear that $\mathcal{A}(\alpha_1 x^1 + \alpha_2 x^2) = \vartheta$, which proves the theorem.

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2.6 Linear operators and matrices

Suppose that \mathcal{V} and \mathcal{W} are two finite-dimensional vector spaces over the field \mathcal{F} , with bases $\{v^1, \ldots, v^n\}$ and $\{w^1, \ldots, w^m\}$ respectively. If $\mathcal{A}: \mathcal{V} \to \mathcal{W}$, then \mathcal{A} may be represented by a matrix. To see this, take any $v \in \mathcal{V}$, and write

$$v = \sum_{i=1}^{n} \alpha_i v^i$$

for scalars $\{\alpha_i\} \subset \mathcal{F}$. By linearity we then have,

$$\mathcal{A}(v) = \mathcal{A}\left(\sum_{i=1}^{n} \alpha_i v^i\right) = \sum_{i=1}^{n} \alpha_i \mathcal{A}(v_i)$$

But for any i we have that $\mathcal{A}(v^i) \in \mathcal{W}$, which implies that for some scalars $\{a_{ji}\}$,

$$\mathcal{A}(v^i) = \sum_{j=1}^m a_{ji} w^j, \qquad 1 \le i \le n.$$

From the form of v we must therefore have

$$\mathcal{A}(v) = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m} a_{ji} w^j$$
$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ji} \alpha_i \right) w^j$$
(2.5)

Recall that the vector $w = \mathcal{A}(v)$ in \mathcal{W} has a unique representation

$$\mathcal{A}(v) = \sum_{j=1}^{m} \beta_j w^j$$

Consequently, the terms in parentheses in (2.5) are identical to the $\{\beta_j\}$:

$$\beta_j = \sum_{i=1}^n a_{ji} \alpha_i \quad j = 1, \dots, m.$$

From this we see how the representations of v and w are transformed through the linear operator A:

$$\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{mi} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = A\alpha$$

so that A is the representation of A with respect to $\{v^i\}$ and $\{w^i\}$.

The special case where \mathcal{A} is a mapping of $(\mathcal{X}, \mathcal{F})$ into itself is of particular interest. A question that frequently arises is "if the linear operator is represented by a matrix A of elements in \mathcal{F} , how is A affected by a change of basis?". Let $x^b = \mathcal{A}(x^a)$, and write

$$x^{a} = \sum_{i=1}^{n} \alpha_{i} e^{i} = \sum_{i=1}^{n} \bar{\alpha}_{i} \bar{e}^{i}$$
$$x^{b} = \sum_{i=1}^{m} \beta_{i} e^{i} = \sum_{i=1}^{m} \bar{\beta}_{i} \bar{e}^{i}$$

where the α and β are related by

$$\beta = A\alpha \quad \bar{\beta} = \bar{A}\bar{\alpha}.$$

To see how A and A are related, recall that there is a matrix P such that

$$\bar{\alpha} = P\alpha$$
 $\bar{\beta} = P\beta$

Combining these four equations gives

$$PA\alpha = P\beta = \bar{\beta} = \bar{A}\bar{\alpha} = \bar{A}P\alpha.$$

Since $\alpha \in \mathcal{X}$ is arbitrary, we conclude that $PA = \bar{A}P$, and hence we can also conclude that

$$\bar{A} = PAP^{-1}$$

$$A = P^{-1}\bar{A}P$$

When these relationships hold, we say that the matrices A and A are similar.

2.7 Eigenvalues and Eigenvectors

For a linear operator $\mathcal{A} \colon \mathcal{X} \to \mathcal{X}$ on an arbitrary vector space $(\mathcal{X}, \mathcal{F})$, a scalar λ is called an *eigenvalue* of \mathcal{A} if there exists a non-zero vector x for which

$$\mathcal{A}(x) = \lambda x. \tag{2.6}$$

The vector x in (2.6) is then called an *eigenvector*.

Let $\mathcal{X} = \mathbb{C}^n$, $\mathcal{F} = \mathbb{C}$, and A be a matrix representation for \mathcal{A} . If an eigenvalue λ of \mathcal{A} does exist, then one may infer from the equation

$$(A - \lambda I)x = 0,$$

that the matrix $A - \lambda I$ is singular. For nontrivial solutions, we must then have

$$\Delta(\lambda) := \det(\lambda I - A) = 0. \tag{2.7}$$

The function $\Delta(\cdot)$ is called the *characteristic polynomial* of the matrix A, and (2.7) is known as the *characteristic equation*. The characteristic polynomial is a polynomial of degree n, which must therefore have n roots. Any root of Δ is an eigenvalue, so at least in the case of operators on $(\mathbb{C}^n, \mathbb{C})$, eigenvalues always exist. Note that if \bar{A} is some other matrix representation for A, since A and \bar{A} are necessarily *similar*, \bar{A} has the same characteristic polynomial as A. Hence, the eigenvalues do not depend on the specific representation picked.

If the roots of the characteristic polynomial are distinct, then the vector space \mathbb{C}^n admits a basis consisting entirely of eigenvectors:

Theorem 2.4. Suppose that $\lambda_1, \ldots, \lambda_n$ are the distinct eigenvalues of the $n \times n$ matrix A, and let v^1, \ldots, v^n be the associated eigenvectors. Then the set $\{v^i, i = 1 \ldots n\}$ is linearly independent over \mathbb{C} .

When the eigenvalues of A are distinct, the modal matrix defined as

$$M := [v^1 \dots v^n]$$

is nonsingular. It satisfies the equation $AM = M\Lambda$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

and therefore A is similar to the diagonal matrix Λ :

$$\Lambda = M^{-1}AM. \tag{2.8}$$

Unfortunately, not every matrix can be diagonalized, and hence the eigenvectors do not span \mathbb{C}^n in general. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

The characteristic polynomial for A is

$$\Delta(\lambda) = \det(\lambda I - A) = (\lambda - 1)(\lambda - 1)(\lambda - 2)$$

So, eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 2$.

Considering the eigenvector equation

$$(A - \lambda_1 I)x = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} x = \vartheta,$$

we see that $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and its constant multiples are the only eigenvec-

tors associated with λ_1 . It then follows that there does not exist a state transformation P for which

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = PAP^{-1}.$$

To obtain a nearly diagonal representation of A we use *generalized eigenvectors*, defined as follows. Search for a solution to

$$(A - \lambda_1 I)^k x = 0$$

such that $(A - \lambda_1 I)^{k-1} x \neq 0$. Then x is called a generalized eigenvector of grade k.

Note that it then follows that the vector $(A - \lambda_1 I)x$ is a generalized eigenvector of grade k - 1. In the example,

$$(A - \lambda_1 I)^2 x = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} x$$

$$= \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} x.$$

So $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a generalized eigenvector of grade k.

Letting $y = (A - \lambda_1 I)x$, we have

$$(A - \lambda_1 I)y = (A - \lambda_1 I)^2 x = 0.$$

So y is an eigenvector to A, and x, y are linearly independent. In fact, y = (1,0,0)' is the eigenvector computed earlier. To obtain an approximately

diagonal form, let $x_1 = y$, $x_2 = x$, x_3 any eigenvector corresponding to $\lambda_3 = 2$. We then have

$$Ax_1 = \lambda_1 x_1$$

$$Ax_2 = Ax = y + \lambda_1 x = x_1 + \lambda_1 x_2$$

$$Ax_3 = \lambda_3 x_3$$

Letting $M = [x_1|x_2|x_3]$ it follows that

$$AM = M \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} M \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix} \middle| M \begin{pmatrix} 1 \\ \lambda_1 \\ 0 \end{pmatrix} \middle| M \begin{pmatrix} 0 \\ 0 \\ \lambda_3 \end{pmatrix} \end{bmatrix}$$
$$= MJ$$

where

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = M^{-1}AM$$

This representation of A with respect to a basis of generalized eigenvectors is known as the *Jordan form*.

2.8 Inner Products

Inner products are frequently applied in the solution of optimization problems because they give a natural measure of distance between vectors. This abstract notion of distance can often be interpreted as a cost in applications to finance, or as *energy* in mechanical systems. Inner products also provide a way of defining angles between vectors, and thereby introduce geometry to even infinite dimensional models where any geometric structure is far from obvious at first glance.

To define an inner product we restrict our attention to a vector space \mathcal{X} over the complex field \mathbb{C} . An inner product is then a complex-valued function of two vectors, denoted $\langle \cdot, \cdot \rangle$, such that the following three properties hold:

- (a) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (complex conjugate).
- (b) $\langle x, \alpha_1 y^1 + \alpha_2 y^2 \rangle = \alpha_1 \langle x, y^1 \rangle + \alpha_2 \langle x, y^2 \rangle$, for all $x, y^1, y^2 \in \mathcal{X}$, $\alpha_1, \alpha_2 \in \mathbb{C}$
- (c) $\langle x, x \rangle \geq 0$ for all $x \in \mathcal{X}$, and $\langle x, x \rangle = 0$ if and only if x = 0.

In the special case where $\mathcal{X} = \mathbb{C}^n$ we typically define

$$\langle x, y \rangle = x^* y$$

where x^* denotes the complex conjugate transpose of the vector x. Another important example is the function space $L_p[a, b]$ with p = 2. It can be shown that the formula

$$\langle f, g \rangle := \int_a^b f(t)g(t) dt, \qquad f, g \in L_2[a, b],$$

defines on inner product on $L_2[a, b]$.

The most obvious application of the inner product is in the formulation of a *norm*. In general, the norm of a vector x in a vector space $(\mathcal{X}, \mathbb{C})$, denoted ||x||, is a real-valued function of a vector x such that

- **1.** $||x|| \ge 0$, and ||x|| = 0 if and only if x = 0.
- **2.** $\|\alpha x\| = |\alpha| \|x\|$, for any $\alpha \in \mathbb{C}$.
- 3. $||x+y|| \le ||x|| + ||y||$.

The third defining property is known as the triangle inequality.

In \mathbb{R}^n we usually define the norm as the usual Euclidean norm,

$$||x|| = \sqrt{x^T x} = \sqrt{\sum_{i=1}^n x_i^2},$$

which we will henceforth write as |x|, and reserve the notation $\|\cdot\|$ for norm of an infinite-dimensional vector. This Euclidean norm can also be defined using the inner product:

$$|x| = \sqrt{\langle x, x \rangle} \tag{2.9}$$

In fact, one can show that the expression (2.9) defines a norm in an arbitrary (finite- or infinite-dimensional) inner product space.

We define the norm of a vector $f \in L_p[a, b]$ as

$$||f||_{L_p} := \left(\int_a^b |f(t)|^p dt\right)^{1/p}.$$

In the case p = 2, this norm is derived from the inner product on $L_2[a, b]$, but for general p this norm is not consistent with any inner product.

2.9 Orthogonal vectors and reciprocal basis vectors

Two vectors x, y in an inner product space $(\mathcal{X}, \mathbb{C})$ are said to be *orthogonal* if $\langle x, y \rangle = 0$. This concept has many applications in optimization, and orthogonality is also valuable in computing representations of vectors. To see the latter point, write

$$x = \sum_{i=1}^{n} \alpha_i v^i,$$

where $\{v^i, i=1\dots n\}$ is a basis for $(\mathcal{X}, \mathbb{C})$. By orthogonality we then have

$$\langle v^j, x \rangle = \langle v^j, \sum_{i=1}^n \alpha_i v^i \rangle = \sum_{i=1}^n \alpha_i \langle v^j, v^i \rangle \quad j = 1 \dots n$$

This may be written explicitly as

$$\langle v^{1}, x \rangle = \alpha_{1} \langle v^{1}, v^{1} \rangle + \alpha_{2} \langle v^{1}, v^{2} \rangle + \dots + \alpha_{n} \langle v^{1}, v^{n} \rangle$$

$$\vdots$$

$$\langle v^{n}, x \rangle = \alpha_{1} \langle v^{n}, v^{1} \rangle + \alpha_{2} \langle v^{1}, v^{2} \rangle + \dots + \alpha_{n} \langle v^{n}, v^{n} \rangle$$

or in matrix form

$$\begin{bmatrix} \langle v^1, x \rangle \\ \vdots \\ \langle v^n, x \rangle \end{bmatrix} = \underbrace{\begin{bmatrix} \langle v^1, v^1 \rangle & \cdots & \langle v^1, v^n \rangle \\ \vdots \\ \langle v^n, v^1 \rangle & \cdots & \langle v^n, v^n \rangle \end{bmatrix}}_{G} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

The $n \times n$ matrix G is called the *Grammian*. Its inverse gives a formula for the representation α :

$$\alpha = G^{-1} \left[\begin{array}{c} \langle v^1, x \rangle \\ \vdots \\ \langle v^n, x \rangle \end{array} \right]$$

If the basis is orthogonal, then G is diagonal, in which case the computation of the inverse G^{-1} is straightforward.

A basis is said to be orthonormal if

$$\langle v^j, v^i \rangle = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

In this case G = I (the identity matrix), so that $G^{-1} = G$.

A basis $\{r^i\}$ is said to be reciprocal to (or dual to) the basis $\{v^i\}$ if

$$\langle r^i, v^j \rangle = \delta_{ij}, \qquad i = 1, \dots, n, \quad j = 1, \dots, n.$$
 (2.10)

If a dual basis is available, then again the representation of x with respect to the basis $\{v^i\}$ is easily computed. For suppose that x is represented by α :

$$x = \sum_{i=1}^{n} \alpha_i v^i$$

Then, by the dual property and linearity we have

$$\langle r^j, x \rangle = \langle r^j, \sum_{i=1}^n \alpha_i v^i \rangle = \sum_{i=1}^n \alpha_i \langle r^j, v^i \rangle.$$

Since $\langle r^j, v^i \rangle = \delta_{ij}$, this shows that $x = \sum_{i=1}^n \langle r^i, x \rangle v^i$. Of course, to to apply this formula we must have the reciprocal basis $\{r^i\}$, which may be as difficult to find as the inverse Grammian.

In the vector space $(\mathbb{C}^n, \mathbb{C})$ define the matrices

$$M = [v^1 \cdots v^n] \qquad R = \begin{bmatrix} r^{1*} \\ \vdots \\ r^{n*} \end{bmatrix}$$

From the defining property (2.10) of the dual basis, we must have RM = I, so that $R = M^{-1}$.

2.10 Adjoint transformations

Suppose that \mathcal{A} is a linear operator from the vector space $(\mathcal{X}, \mathbb{C})$ to another vector space $(\mathcal{Y}, \mathbb{C})$, that is, $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$. Then the *adjoint* is a linear operator working in the reverse direction:

$$\mathcal{A}^*: \mathcal{V} \to \mathcal{X}$$
.

Its definition is subtle since it is not directly defined through \mathcal{A} . We say that \mathcal{A}^* is the adjoint of \mathcal{A} , if for any $x \in \mathcal{X}$ and any $y \in \mathcal{Y}$,

$$\langle \mathcal{A}(x), y \rangle = \langle x, \mathcal{A}^*(y) \rangle.$$

To illustrate this concept, let us begin with the finite dimensional case

$$\mathcal{X} = \mathbb{C}^n, \qquad \mathcal{Y} = \mathbb{C}^m$$

and suppose that \mathcal{A} is defined through an $m \times n$ matrix A, so that $\mathcal{A}(x) = Ax$, $x \in \mathcal{X}$. We may then compute the adjoint using the definition of the inner products on \mathcal{X} and \mathcal{Y} as follows

$$\langle \mathcal{A}(x), y \rangle = (Ax)^* y = \bar{x}^T \bar{A}^T y = \langle x, \bar{A}^T y \rangle$$

Thus, the adjoint of A is defined through the complex conjugate transpose of A:

$$\mathcal{A}^*(y) = \bar{A}^T y = A^* y$$

Matlab Commands

Matlab is virtually designed to deal with the numerical aspects of the vector space concepts described in this chapter. Some useful commands are

INV to compute the inverse of a matrix.

DET to compute the determinant.

EIG finds eigenvalues and eigenvectors.

RANK computes the rank of a matrix.

Summary and References

In this chapter we have provided a brief background on several different topics in linear algebra, including

- (a) Fields and vector spaces.
- (b) Linear independence and bases.
- (c) Representations of vectors, and how these representations change under a change of basis.
- (d) Linear operators and matrices.
- (e) Inner products and norms.
- (f) Adjoint operators.
- (g) Eigenvectors.

Good surveys on linear algebra and matrix theory may be found in Chapter 2 of [6], or Chapters 4 and 5 of [5].

2.11. EXERCISES

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2.11 Exercises

2.11.1 Determine conclusively which of the following are fields:

- (a) The set of integers.
- (b) The set of rational numbers.
- (c) The set of polynomials of degree less than 3 with real coefficients.
- (d) The set of all $n \times n$ nonsingular matrices.
- (e) The set {0,1} with addition being binary "exclusive-or" and multiplication being binary "and".
- **2.11.2** Define rules of addition and multiplication such that the set consisting of three elements $\{a, b, c\}$ forms a field. Be sure to define the zero and unit elements.
- **2.11.3** Let (X, \mathcal{F}) be a vector space, and $Y \subset X$ a subset of X. If Y satisfies the closure property $y_1, y_2 \in Y$, $\alpha_1, \alpha_2 \in \mathcal{F} \Longrightarrow \alpha_1 y_1 + \alpha_2 y_2 \in Y$, show carefully using the definitions that (Y, \mathcal{F}) is a subspace of (X, \mathcal{F}) .

Linear independence and bases

2.11.4 Which of the following sets are linearly independent?

(a)
$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$
, $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in $(\mathbb{R}^3, \mathbb{R})$.

(b)
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
, $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$, $\begin{bmatrix} 7912 \\ -314 \\ 0.098 \end{bmatrix}$ in $(\mathbb{R}^3, \mathbb{R})$.

(c)
$$\begin{bmatrix} 1\\4j\\2 \end{bmatrix}$$
, $\begin{bmatrix} 0\\-2\\j \end{bmatrix}$, $\begin{bmatrix} j\\0\\0 \end{bmatrix}$ in $(\mathbb{C}^3,\mathbb{C})$.

- (d) $\sin(t), \cos(t), t$ in $(C[0,1], \mathbb{R})$ the set of real-valued continuous functions on [0,1] over the real field.
- (e) e^{jt} , $\sin(t)$, $\cos(t)$, t in $(C[0,1],\mathbb{C})$ the set of complex-valued continuous functions on [0,1] over the complex field.
- **2.11.5** Determine which of the following sets of vectors are linearly independent in \mathbb{R}^3 by computing the determinant of an appropriate matrix.

(a)
$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}.$$

(b)
$$\begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

- **2.11.6** For the vector space $(\mathbb{C}^n, \mathbb{R})$,
 - (a) Verify that this is indeed a vector space.
 - (b) What is its dimension?
 - (c) Find a basis for $(\mathbb{C}^n, \mathbb{R})$.
- **2.11.7** Let $\mathbb{R}^{2\times 2}$ be the set of all 2×2 real matrices.
 - (a) Briefly verify that $\mathbb{R}^{2\times 2}$ is a vector space under the usual matrix addition and scalar multiplication.
 - (b) What is the dimension of $\mathbb{R}^{2\times 2}$?
 - (c) Find a basis for $\mathbb{R}^{2\times 2}$.
- **2.11.8** Is the set $\{I, A, A^2\}$ linearly dependent or independent in $(\mathbb{R}^{2\times 2}, \mathbb{R})$, with $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$?

$Representations\ of\ vectors$

2.11.9 Given the basis
$$\left\{ v^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ v^2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \ v^3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$
 and the vector

$$x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \alpha_1 v^1 + \alpha_2 v^2 + \alpha_3 v^3$$

- (a) Compute the reciprocal basis.
- (b) Compute the Grammian.
- (c) Compute the representation of x with respect to $\{v^i\}$ using your answer to (a).
- (d) Compute the representation of x with respect to $\{v^i\}$ using your answer to (b).

Linear operators and matrices

2.11.10 Compute the null space, range space, and rank of the following matrices.

2.11. EXERCISES

(a)
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \end{bmatrix}$$
.
(b) $A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 0 & 1 & -1 \\ -1 & 1 & 0 & 1 \end{bmatrix}$.

2.11.11 Let $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times m}$. Give necessary and sufficient conditions on A and b in order that the linear system of equations Ax = b has a solution $x \in \mathbb{R}^m$.

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2.11.12 Let $\mathcal{A}: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator. Consider the two sets $B = \{b_1, b_2, b_3\}$ and $C = \{c_1, c_2, c_3\}$ below

$$B = \left\{ \begin{bmatrix} 1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1 \end{bmatrix} \right\}, C = \left\{ \begin{bmatrix} 1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1 \end{bmatrix}, \begin{bmatrix} 1\\0\\1 \end{bmatrix} \right\}$$

It should be clear that these are bases for \mathbb{R}^3 .

- (a) Find the transformation P relating the two bases.
- (b) Suppose the linear operator \mathcal{A} maps

$$\mathcal{A}(b_1) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \ \mathcal{A}(b_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{A}(b_1) = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

Write down the matrix representation of \mathcal{A} with respect to the basis B and also with respect to the basis C.

2.11.13 Find the inverse of the matrix A, where B is a matrix.

$$A = \begin{bmatrix} I & B \\ 0 & I \end{bmatrix}$$

- **2.11.14** Consider the set P_n of all polynomials of degree *strictly less than* n, with real coefficients, where $x \in P_n$ may be written $x = a_0 + a_1t + \cdots + a_{n-1}t^{n-1}$.
 - (a) Verify that P_n is a vector space, with the usual definitions of polynomial addition, and scalar multiplication.
 - (b) Explain why $\{1, t, \dots, t^{n-1}\}$ is a basis, and thus why P_n is n-dimensional.

- (c) Suppose $x = 10 2t + 2t^2 3t^3$. Find the representation α of x with respect to the basis in (b) for n = 4.
- (d) Consider differentiation, $\frac{d}{dt}$, as an operator $\mathcal{A}: P_n \to P_{n-1}$. That is, $\mathcal{A}(x) = \frac{d}{dt}x$. Show that \mathcal{A} is a linear operator, and compute its null space and range space.
- (e) Find A, the matrix representation of \mathcal{A} for n=4, using the basis in (b). Use your A and α to compute the derivative of x in (c).

Inner products and norms

- **2.11.15** Let $(\mathcal{V}, \mathbb{C})$ be an inner product space.
 - (a) Let $x, y \in \mathcal{V}$ with x orthogonal to y. Prove the Pythagorean theorem:

$$||x + y||^2 = ||x||^2 + ||y||^2$$

(b) Prove that in an inner product space,

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$$

where $\|\cdot\|$ is the norm induced by the inner product. This is called the *Parallelogram law*. Can you give a geometric interpretation of this law in \mathbb{R}^2 ?

Adjoint operators

- **2.11.16** If $\mathcal{A}: \mathcal{X} \to \mathcal{Y}$ where \mathcal{X} and \mathcal{Y} are inner product spaces, the adjoint \mathcal{A}^* is a mapping $\mathcal{A}^*: \mathcal{Y} \to \mathcal{X}$. Hence, the composition $Z = \mathcal{A}^* \circ \mathcal{A}$ is a mapping from \mathcal{X} to itself. Prove that $\mathcal{N}(Z) = \mathcal{N}(\mathcal{A})$.
- **2.11.17** For $1 \leq p < \infty$, let L_p denote functions $f: (-\infty, \infty) \to \mathbb{C}$ such that $\int_{-\infty}^{\infty} |f(s)|^p ds < \infty$. For $p = \infty$, L_p denotes bounded functions $f: (-\infty, \infty) \to \mathbb{C}$. The set L_p is a vector space over the complex field. Define the function $\mathcal{A}: L_p \to L_p$ as $\mathcal{A}(f) = a * f$, where "*" denotes convolution. We assume that for some constants $C < \infty$, c > 0, we have the bound $|a(t)| \leq Ce^{-c|t|}$ for all $t \in \mathbb{R}$. This is sufficient to ensure that $\mathcal{A}: L_p \to L_p$ for any p.
 - (a) First consider the case where $p = \infty$, and let $f_{\omega}(t) = e^{j\omega t}$, where $\omega \in \mathbb{R}$. Verify that $f_{\omega} \in L_{\infty}$ is an eigenvector of \mathcal{A} . What is the corresponding eigenvalue?

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(b) In the special case $p=2,\,L_p$ is an inner product space with

$$< f, g>_{L_2} = \int f^*(s)g(s) ds, \qquad f, g \in L_2.$$

Compute the adjoint \mathcal{A}^* : $L_2 \to L_2$, and find conditions on a under which \mathcal{A} is self adjoint.

2.11.18 Let $X = \mathbb{R}^n$ with the usual inner product. Let $Y = L_2^n[0,\infty)$, the set of functions $f : [0,\infty) \to \mathbb{R}^n$ with $\int_0^\infty |f(s)|^2 ds < \infty$. We define the inner product as before:

$$\langle f, g \rangle_Y = \int f^\top(s)g(s) ds$$
 $f, g \in Y$

For an $n \times n$ Hurwitz matrix A, consider the differential equation $\dot{x} = Ax$. By stability, for each initial condition $x_0 \in X$, there exists a unique solution $x \in Y$. Define $A: X \to Y$ to be the map which takes the initial condition $x_0 \in X$ to the solution $x \in Y$.

- (a) Explain why A is a linear operator.
- (b) What is the null space N(A)? What is the rank of A?
- (c) Compute the adjoint A^* .

Eigenvectors

2.11.19 Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 5 & 1 & 5 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

2.11.20 An $n \times n$ matrix A is called *positive definite* if it is symmetric,

$$A = A^* = \bar{A}^T$$
.

and if for any $x \neq \vartheta$, $x \in \mathbb{R}^n$,

$$x^*Ax > 0$$
.

The matrix A is positive semi-definite if the strict inequality in the above equation is replaced by " \geq ". Show that for a positive definite matrix,

- (a) Every eigenvalue is real and strictly positive.
- (b) If v^1 and v^2 are eigenvectors corresponding to different eigenvalues λ_1 and λ_2 , then v^1 and v^2 are orthogonal.
- **2.11.21** For a square matrix X suppose that (i) all of the eigenvalues of X are strictly positive, and (ii) the domain of X possesses an orthogonal basis consisting entirely of eigenvectors of X. Show that X is a positive definite matrix (and hence that these two properties completely characterize positive definite matrices).

Hint: Make a change of basis using the modal matrix $M = [v^1 \cdots v^n]$, where $\{v_i\}$ is an orthonormal basis of eigenvectors.

- **2.11.22** Left eigenvectors $\{\omega^i\}$ of an $n \times n$ matrix A are defined by $\omega^i A = \lambda_i \omega^i$, where $\{\omega^i\}$ are row vectors $(1 \times n)$ matrices and the $\{\lambda_i\}$ are the eigenvalues of A. Assume that the eigenvalues are distinct.
 - (a) How are the $\{\omega^i\}$ related to the ordinary (right) eigenvectors of A^* ?
 - (b) How are the $\{\omega^i\}$ related to the reciprocal (dual) eigenvectors of A?

Chapter 3

Solutions of State Equations

In this chapter we investigate solutions to the linear state space model (1.1) in the general setting with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$. We begin with the linear, time-invariant (LTI) case where the matrices (A, B, C, D) do not depend upon time.

3.1 LTI state space models

Assume that an initial condition x_0 is given at time t = 0. The LTI case is then particularly simple to treat since we can take Laplace transforms in (1.1) to obtain the transformed equations

$$sX(s) - x_0 = AX(s) + BU(s)$$

 $Y(s) = CX(s) + DU(s)$

Solving for X(s) gives $[sI - A]X(s) = x_0 + BU(s)$, and therefore

$$X(s) = [sI - A]^{-1}x_0 + [sI - A]^{-1}BU(s)$$
(3.1)

Letting

$$\Phi(s) := [sI - A]^{-1}$$

we finally obtain

$$X(s) = \Phi(s)x_0 + \Phi(s)BU(s). \tag{3.2}$$

To convert this into the time domain, we will take inverse Laplace transforms. First define

$$\phi(t) = \mathcal{L}^{-1}(\Phi(s)) = \mathcal{L}^{-1}([sI - A]^{-1})$$
(3.3)

where the inverse Laplace transform of the matrix-valued function $\Phi(s)$ is taken term by term. Since products in the transform domain are mapped to convolutions in the time domain, the formula (3.2) may be inverted to give

$$x(t) = \phi(t)x_0 + \int_0^t \phi(t - \tau)Bu(\tau) d\tau.$$

The function ϕ defined in (3.3) is evidently important in understanding the solutions of (1.1). Since in the absence of inputs the state evolves according to the equation

$$x(t) = \phi(t)x_0, \tag{3.4}$$

the function ϕ is known as the state transition matrix.

As for the output, in the transform domain we have

$$Y(s) = CX(s) + DU(s)$$

Assuming for simplicity that x(0) = 0, this gives

$$Y(s) = CX(s) + DU(s) = C\Phi(s)BU(s) + DU(s)$$

Therefore the transfer function is a $p \times m$ matrix-valued function of s which takes the form

$$G(s) = C\Phi(s)B + D,$$

and the impulse response is the inverse Laplace transform of the transfer function:

$$g(t) := \mathcal{L}^{-1}\{G(s)\} = C\phi(t)B + D\delta(t).$$

In the time domain, we conclude that for zero initial conditions,

$$y(t) = g * u(t) := \int_0^t g(t - \tau)u(\tau)d\tau = \int_0^t C\phi(t - \tau)Bu(\tau)d\tau + Du(t)$$

3.2 Other descriptions of the state transition matrix

We have seen that the solution of the differential equation

$$\dot{x} = Ax, \qquad x(0) = x_0 \tag{3.5}$$

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is given by the formula (3.4). We also know that in the scalar case where n = 1, the solution of (3.5) is given by the exponential $x(t) = e^{At}x_0$, so that $\phi(t) = e^{At}$ in this special case. In the multivariable case, define the matrix exponential through the power series

$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = I + tA + \frac{t^2}{2} A^2 \cdots,$$

so that e^{At} is a matrix-valued function of the matrix A. We now show that we do indeed have $\phi(t) = e^{At}$ with this definition of the matrix exponential. To see that $x(t) = e^{At}x_0$ is a solution to (3.5), first note that the specified initial conditions are satisfied:

$$x(0) = e^{A \cdot 0} x_0 = I x_0 = x_0.$$

To see that the differential equation is satisfied, take derivatives term by term to obtain the formula

$$\frac{d}{dt}e^{At} = A + tA^2 + \dots = Ae^{At}$$

This implies that $\frac{d}{dt}(e^{At}x_0) = A(e^{At}x_0)$, so that the equation (3.5) is satisfied, and hence it is correct to assert that $\phi(t) = e^{At}$.

The matrix exponential satisfies the following three properties, for any t,s,

- 1. $e^{A0} = I$.
- 2. $(e^{At})^{-1} = e^{-At}$.
- 3. $e^{At}e^{As} = e^{A(t+s)}$.

The third property, which is known as the *semigroup property*, is analogous to the equation $e^A e^B = e^{A+B}$ when A and B are scalars. However, this equation *does not* hold in the matrix case unless the matrices A and B commute.

To summarize, we now have three descriptions of the state transition matrix $\phi(t)$:

(a)
$$\phi(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

(b)
$$\phi(t) = e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

(c)
$$\dot{\phi}(t) = A\phi(t), \ \phi_0 = I$$

3.3 Change of state variables

We now consider the effect of a change of basis on the state transition matrix ϕ . Suppose that the $n \times n$ matrix P is used to make a change of basis, and define the variable \bar{x} as

$$\bar{x} = Px$$
 $x = P^{-1}\bar{x}$

After pre-multiplying by P, the state equation

$$\dot{x} = Ax + Bu$$

becomes a state equation for \bar{x} :

$$\dot{\bar{x}} = \underbrace{PAP^{-1}}_{\bar{A}} \bar{x} + \underbrace{PB}_{\bar{B}} u$$

The output equation y = Cx + Du can be similarly transformed:

$$y = \underbrace{CP^{-1}}_{\bar{C}} \bar{x} + \underbrace{D}_{\bar{D}} u$$

Hence for any invertible matrix P, we can form a new state space model

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u
y = \bar{C}\bar{x} + \bar{D}u,$$
(3.6)

with $\bar{A} = PAP^{-1}$, $\bar{B} = PB$, $\bar{C} = CP^{-1}$, and $\bar{D} = D$. Moreover, if $\bar{x}(0) = Px(0)$ and identical inputs are applied to both systems, then the outputs of (1.1) and (3.6) will be identical.

The state transition matrix of the system (3.6) can be expressed in terms of the state transition matrix of (1.1) as follows. Starting with the defining formula $\bar{\Phi}(s) = (sI - \bar{A})^{-1} = (sI - PAP^{-1})^{-1}$ and noting that $I = PP^{-1}$, we have

$$\bar{\Phi}(s) = (P[sI - A]P^{-1})^{-1} = P[sI - A]^{-1}P^{-1}.$$

It follows that $\bar{\Phi}(s) = P\Phi(s)P^{-1}$, and hence that

$$\bar{\phi}(t) = P\phi(t)P^{-1} \tag{3.7}$$

To show how (3.7) may be applied, suppose that the eigenvalues of A are distinct, so that the modal matrix $M = [v^1 \cdots v^n]$ exists and is non-singular. Defining $P = M^{-1}$, we have seen in equation (2.8) that

$$\bar{A} = PAP^{-1} = \Lambda$$

The point is, when \bar{A} is diagonal, the state transition matrix is given by the direct formula

$$\bar{\phi}(t) = e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

and from (3.7) we have

$$\phi(t) = M\bar{\phi}(t)M^{-1},$$

so that $\phi(t)$ is quickly computed via the modal matrix.

3.4 Numerical examples

Take the two-dimensional case, with A given by

$$A = \left[\begin{array}{cc} 0 & 1 \\ 8 & -2 \end{array} \right]$$

We will compute $\phi(t)$ for the system (3.5) using the various methods described in the previous sections. The simple-minded approach is to write down the formula

$$\phi(t) = e^{At} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 8 & -2 \\ -16 & 12 \end{bmatrix} + \cdots,$$

but this does not give a closed form expression unless the infinite series can be computed.

In the transform domain, we have

$$\Phi(s) := (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ -8 & s + 2 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s + 2 & 1 \\ 8 & s \end{bmatrix}}{s^2 + 2s - 8}$$

Taking inverse Laplace transforms then gives

$$\phi(t) = \begin{bmatrix} \frac{1}{3}e^{-4t} + \frac{2}{3}e^{2t}, & -\frac{1}{6}e^{-4t} + \frac{1}{6}e^{2t} \\ -\frac{4}{3}e^{-4t} + \frac{4}{3}e^{2t}, & \frac{2}{3}e^{-4t} + \frac{1}{3}e^{2t} \end{bmatrix}$$
(3.8)

Observe that $\phi(0) = I$, as it should be.

A third approach is to convert the state space model to modal form. To begin, we must compute the eigenvalues of the matrix A. The characteristic polynomial is given by

$$\Delta(s) := \det \left[\begin{array}{cc} s & -1 \\ -8 & s+2 \end{array} \right] = (s+4)(s-2)$$

The roots of Δ , which are also the eigenvalues of A, are $\lambda = -4, 2$. Hence the eigenvalues are distinct, and a modal form does exist. Solving the equation $Ax = \lambda x$ for the eigenvector x in this example gives for $\lambda = -4$,

$$\begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which is solved by $x_2 = -4x_1$. It follows that the vector

$$v^1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

is an eigenvector for the eigenvalue $\lambda_1 = -4$. Similarly, the vector $v^2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ can be shown to be an eigenvector for the eigenvalue $\lambda_2 = 2$.

The modal matrix becomes $M = \begin{bmatrix} v^1 \mid v^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix}$, and the state transition matrix may be expressed as

$$\phi(t) = M\bar{\phi}(t)M^{-1} = \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix} \begin{pmatrix} e^{-4t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{bmatrix} 1 & 1 \\ -4 & 2 \end{bmatrix}^{-1}.$$

After computing the matrix inverse and products, one again obtains the expression (3.8).

3.5 Cayley-Hamilton Theorem

Another useful tool in systems theory which also gives a method for computing ϕ is the following theorem attributed to Cayley and Hamilton. For an $n \times n$ matrix A with real coefficients, we recall that the characteristic polynomial defined as $\Delta(x) = \det(xI - A)$ is a polynomial of degree n, which therefore may also be expressed as

$$\Delta(x) = x^n + \alpha_1 x^{n-1} + \dots + \alpha_n$$

for real coefficients $\{\alpha_i\}$.

Theorem 3.1 (Cayley-Hamilton Theorem). The matrix A satisfies the matrix characteristic equation

$$\Delta(A) = A^n + \alpha_1 A^{n-1} + \dots + \alpha_n I = 0_{n \times n}$$
(3.9)

Proof We provide a proof for the case of distinct eigenvalues only - for the general case, see [5], pages 286, 294-5.

When the eigenvalues are distinct, we can work with the transformed matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ & \ddots \\ 0 & \lambda_m \end{bmatrix} = M^{-1}AM$$

where M is the modal matrix, $M = [v^1 \cdots v^n]$. It follows that

$$A = M\Lambda M^{-1}$$
.

and that for any k,

$$A^k = M\Lambda^k M^{-1}$$

Applying this to (3.9) gives

$$\Delta(A) = M(\Lambda^{n} + \alpha_{1}\Lambda^{n-1} \cdots + \alpha_{n}I)M^{-1}$$

$$= M\left(\begin{bmatrix} \lambda_{1}^{n} & 0 \\ & \ddots & \\ 0 & \lambda_{n}^{n} \end{bmatrix} + \alpha_{1}\begin{bmatrix} \lambda_{1}^{n-1} & 0 \\ & \ddots & \\ 0 & \lambda_{n}^{n-1} \end{bmatrix} + \cdots + \alpha_{n}\begin{bmatrix} 1 & 0 \\ & \ddots & \\ 0 & 1 \end{bmatrix}\right)M^{-1}$$

$$= M\begin{bmatrix} \lambda_{1}^{n} + \alpha_{1}\lambda_{1}^{n-1} \cdots + \alpha_{n} & 0 \\ & \lambda_{2}^{n} + \alpha_{1}\lambda_{2}^{n-1} + \cdots + \alpha_{n} \\ & & \ddots \\ & & & \lambda_{n}^{n} + \alpha_{1}\lambda_{n}^{n-1} + \cdots + \alpha_{n} \end{bmatrix}$$

$$M^{-1}$$

For each i, the eigenvalue λ_i satisfies the characteristic equation $\Delta(\lambda_i) = 0$. So, $\Delta(A) = 0$, as claimed.

Example 3.5.1. Consider the 2×2 matrix

$$A = \left[\begin{array}{cc} 0 & 1 \\ -2 & -3 \end{array} \right]$$

The characteristic polynomial in this example is

$$\Delta(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{bmatrix} = \lambda^2 + 3\lambda + 2$$

We therefore have

$$\Delta(A) = A^2 + 3A + 2I = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is evidently equal to 0 (the 2×2 null matrix).

In view of Theorem 3.1, the infinite series used to define the transition matrix may be written as a finite sum

$$\phi(t) = e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{n-1} \beta_k(t) A^k$$

To see this, apply the Cayley-Hamilton theorem to replace any A^k for $k \ge n$ by a linear combination of I, A, \ldots, A^{n-1} .

To compute $\phi(t)$, we must then find $\beta_k(t)$. We illustrate this with a numerical example, using the matrix A defined above.

Example 3.5.2. Our goal is to compute

$$\phi(t) = e^{At} = \beta_0(t)I + \beta_1(t)A \tag{3.10}$$

by computing the scalar functions β_0, β_1 . Solving the characteristic equation $\lambda^2 + 3\lambda + 2 = 0$ then gives the distinct eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = -2$$

From the definition of the matrix exponential we have for i = 1, 2,

$$e^{At}v^{i} = Iv^{i} + tAv^{i} + \frac{t^{2}}{2}A^{2}v^{i} + \dots = v^{i} + t\lambda_{i}v^{i} + \frac{t^{2}}{2}\lambda_{i}^{2}v^{i} + \dots = e^{\lambda_{i}t}v^{i}.$$

That is, the eigenvectors $\{v^i\}$ for the matrix A are also eigenvectors for the matrix e^{At} . We also have

$$[\beta_0(t)I + \beta_1(t)A]v^i = [\beta_0(t) + \beta_1(t)\lambda_i]v^i$$

and since $e^{At} = [\beta_0(t)I + \beta_1(t)A]$, this gives the system of equations

$$e^{\lambda_1 t} = e^{-t} = \beta_0(t) + \beta_1(t)(-1)$$

$$e^{\lambda_2 t} = e^{-2t} = \beta_0(t) + \beta_1(t)(-2)$$

Rewriting this as

$$e^{-t} = \beta_0(t) - \beta_1(t)$$

 $e^{-2t} = \beta_0(t) - 2\beta_1(t)$

we may solve to obtain

$$\beta_0(t) = 2e^{-t} - e^{-2t}$$

 $\beta_1(t) = e^{-t} - e^{-2t}$

Applying the formula (3.10) then gives

$$\phi(t) = e^{At} = \begin{bmatrix} \beta_0 & 0 \\ 0 & \beta_0 \end{bmatrix} + \begin{bmatrix} 0 & \beta_1 \\ -2\beta_1 & -3\beta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \beta_0 & \beta_1 \\ -2\beta_1 & \beta_0 - 3\beta_1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

3.6 Linear Time-Varying Systems

We conclude this chapter with a development of the linear time-varying (LTV) case

$$\dot{x} = A(t)x, \qquad x(t_0) = x_0$$
 (3.11)

Given our experience with the LTI case, one might presume that the response to (3.11) may be expressed as $x(t) = \exp(\int_{t_0}^t A(s) \, ds) x_0$. While this expression is valid in the scalar case, it is unfortunately *not* correct in general. This is a more complicated situation since we can no longer take Laplace transforms, and in general we do not even know if (3.11) has a solution. However, we do have the following result.

Theorem 3.2. If A(t) is a piecewise-continuous function of t, then:

- (a) For any $x_0 \in \mathbb{R}^n$, the differential equation (3.11) possesses a unique solution x(t) defined on $(-\infty, \infty)$;
- (b) The set of all possible solutions to $\dot{x} = A(t)x$ forms an n-dimensional vector space.

Proof The proof of (a) is outside of the scope of this book. To see (b), let $\{x_0^i\}$ be a basis for \mathbb{R}^n , and define for each $1 \leq i \leq n$ the function ψ^i to be the solution to (3.11) with $x_0 = x_0^i$. For any $x_0 = \sum \alpha_i x_0^i \in \mathbb{R}^n$ one may easily verify that $x(t) = \sum \alpha_i \psi^i(t)$ is the unique solution to (3.11). Hence the span of $\{\psi^i\}$ contains all solutions to (3.11). We now show that the $\{\psi^i\}$ are linearly independent, and hence provide a basis for the solution set.

To show that the $\{\psi^i(t)\}$ are linearly independent time functions, assume that there exist $\alpha_1, \ldots, \alpha_n$ such that

$$\alpha_1 \psi^1(t) + \dots + \alpha_n \psi^n(t) = \vartheta, \qquad t \in \mathbb{R}.$$

Setting $t = t_0$ gives $\sum \alpha_i x_0^i = \vartheta$, and since $\{x_0^i\}$ was assumed to be a basis, we must have $\alpha_i = 0$ for all i. By definition, the set $\{\psi^i(t), 1 \le i \le n\}$ is linearly independent, and hence forms a basis for the solution set of (3.11).

3.7 Fundamental matrices

Using Theorem 3.2 we may define an analog of the state transition matrix $\phi(t) = e^{At}$ constructed earlier in the LTI case. Let $\{\psi^i(t), 1 \leq i \leq n\}$ be any set of solutions to (3.11) which are linearly independent as vectors in $C^n(-\infty,\infty)$, and define the $n \times n$ fundamental matrix U as

$$U(t) = [\psi^1(t) \cdots \psi^n(t)] \tag{3.12}$$

Since each ψ_i is a solution to (3.11), the fundamental matrix satisfies the matrix differential equation

$$\dot{U}(t) = A(t)U(t), \tag{3.13}$$

just as the state transition matrix does in the LTI case. To proceed, we need to invert the fundamental matrix.

Lemma 3.3. The inverse $U^{-1}(t)$ exists for all $-\infty < t < \infty$.

Proof Fix t_0 , and suppose that for some vector $x \in \mathbb{R}^n$,

$$U(t_0)x = \vartheta \in \mathbb{R}^n$$
.

We will show that x is necessarily equal to ϑ , and from this it will follow that $U(t_0)$ is invertible as claimed.

The function $x(t) = U(t)x = \sum x_i \psi^i(t)$ is evidently a solution to (3.11) with $x(t_0) = \vartheta$. By uniqueness of solutions, we must then have $x(t) = \vartheta \in$

 \mathbb{R}^n for all t, and hence $\sum x_i \psi^i = \vartheta$ as a vector in $C^n(-\infty, \infty)$. Since the $\{\psi^i(t), 1 \leq i \leq n\}$ are linearly independent, it follows that $x = \vartheta$. \square Given Lemma 3.3, we may define

$$\phi(t,\tau) := U(t)U^{-1}(\tau), \qquad t,\tau \in (-\infty,\infty)$$

The matrix valued function ϕ is again called the *state transition matrix* of the differential equation $\dot{x} = A(t)x$. The state transition matrix shares many of the properties of the matrix e^{At} defined earlier, although we do not have $\phi(t) = e^{A(t)t}$ in the LTV case! We do have, however, the following properties, which hold even in the LTV case: For any t_0, t_1, t_2 ,

- 1. $\phi(t_0, t_0) = I$.
- 2. $\phi^{-1}(t_0, t_1) = \phi(t_1, t_0)$.
- 3. $\phi(t_2, t_0) = \phi(t_2, t_1)\phi(t_1, t_0)$.

The third property is again known as the *semigroup property*.

The solution to (3.11) can now be conveniently expressed in terms of the state transition matrix

$$x(t) = \phi(t, t_0)x(t_0).$$

To see this, use property (1) above, and the matrix differential equation (3.13). The state transition matrix also gives the solution to the controlled system (1.1) in the time varying case. We have that for any control u, initial time t_0 , and any initial condition $x(t_0) = x_0$,

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^t \phi(t, \tau)B(\tau)u(\tau)d\tau$$

To see this, check the equation $\dot{x} = Ax + Bu$ by differentiating both sides of the equation above.

3.8 Peano-Baker Series

There are few tools available for computing the state transition matrix in the LTV case. One approach is by iteration, which is related to the Taylor series expansion used in the LTI case. From the differential equation

$$\frac{d}{dt}\phi(t,t_0) = A(t)\phi(t,t_0), \qquad \phi(t_0,t_0) = I.$$

and the fundamental theorem of calculus, we have the integral equation

$$\phi(t, t_0) = I + \int_{t_0}^t A(\sigma)\phi(\sigma, t_0) d\sigma$$

This suggests the following iterative scheme for finding ϕ : Set $\phi_0(t, t_0) \equiv I$, and

$$\phi_{k+1}(t,t_0) = I + \int_{t_0}^t A(\sigma)\phi_k(\sigma,t_0) d\sigma, \qquad k \ge 0.$$

It can be shown that if A is piecewise continuous, then this Peano-Baker series does converge to the state transition matrix.

We can show this explicitly in the LTI case where $A(t) \equiv A$. The Peano-Baker series then becomes

$$\phi_0(t, t_0) = I$$

$$\phi_1(t, t_0) = I + A \int_{t_0}^t I \, d\sigma = I + A(t - t_0)$$

$$\phi_2(t, t_0) = I + A \int_{t_0}^t (I + A(\sigma - t_0)) \, d\sigma$$

$$= I + (t - t_0)A + \frac{(t - t_0)^2}{2}A^2$$

Thus, we see again that in the LTI case,

$$\phi(t, t_0) = \sum_{i=0}^{k} \frac{(t - t_0)^k}{k!} A^k.$$

Matlab Commands

EXPM computes the matrix exponential.

Summary and References

We have provided in this chapter a detailed description of solutions to (1.1) through the construction of the state transition matrix. Chapter 9 of [5] contains a similar exposition on the solution of state equations; for a more detailed exposition, the reader is referred to [12].

3.9 Exercises

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3.9.1 Use the (matrix) exponential series to evaluate e^{At} for

(a)
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
 (b) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}$.

3.9.2 If A is an $n \times n$ matrix of full rank, show that

$$\int_0^t e^{A\sigma} d\sigma = A^{-1} [e^{At} - I] .$$

Hint: This is easy once you realize that both functions of t solve the same differential equation.

Using this result, obtain the solution to the linear time-invariant equation

$$\dot{x} = Ax + B\bar{u} \; , \quad x(0) = x_0 \; ,$$

where \bar{u} is a constant m-dimensional vector and B is an $(n \times m)$ -dimensional matrix.

3.9.3 Given the single-input single-output system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; ; \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; , \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \; ,$$

let $\phi(t,s)$ denote the corresponding state transition matrix.

- (a) Evaluate $\phi(t,s)$ using the transfer function approach (i.e. $(sI-A)^{-1}$).
- (b) Evaluate $\phi(t, s)$ by transforming the system into diagonal form (i.e. using modal decomposition).
- (c) With $u(t) \equiv 1$, compute y(1).

3.9.4 Given the second-order system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} x \qquad ; \quad x(0) = x_0 \; ,$$

find the conditions on the initial state vector x_0 such that only the mode corresponding to the smaller (in absolute value) eigenvalue is excited.

3.9. EXERCISES

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3.9.5 Given the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix} ,$$

- (a) Compute its inverse using the Cayley-Hamilton Theorem.
- (b) Compute A^6 .
- **3.9.6** Revisit Exercise 8 in Chapter 2, and using the Cayley-Hamilton Theorem, explain why these three matrices must be linearly dependent.
- **3.9.7** Suppose that A and B are constant square matrices. Show that the state transition matrix for the time-varying system described by

$$\dot{x}(t) = e^{-At} B e^{At} x(t)$$

is

$$\Phi(t,s) = e^{-At}e^{(A+B)(t-s)}e^{As} .$$

3.9.8 Suppose that A and Q are $n \times n$ matrices, and consider the differential equation

$$\dot{Z} = AZ + ZA^*, \qquad Z(0) = Q.$$
 (3.14)

(a) Show using the product rule that the unique solution to (3.14) is

$$Z(t) = e^{At} Q e^{A^* t}$$

(b) Integrating both sides of (3.14) from 0 to t_f , and applying the fundamental theorem of calculus, conclude that if $e^{At} \to 0$, then

$$P = \lim_{t_f \to \infty} \int_0^{t_f} Z(t) \, dt$$

is a solution to the Lyapunov equation

$$AP + PA^* + Q = 0.$$

This fact will be very useful when we study stability, and it will come up again in our study of optimization.

3.9.9 If ϕ denotes the state transition matrix for the system $\dot{x}(t) = A(t)x(t)$, show that $V(t) = \phi(t, t_0)Q\phi^*(t, t_0)$ is a solution to the equation

$$\frac{d}{dt}V(t) = A(t)V(t) + V(t)A^*(t), \qquad V(t_0) = Q.$$

3.9.10 Associated with the control free differential equation (3.11) is the so-called *adjoint equation*

$$\dot{z} = -A^*z,$$

where $A^* = \bar{A}^T$. Show that the state transition matrix ϕ_a for the adjoint equation satisfies

$$\phi_a(t_1, t_0) = \phi^*(t_0, t_1) \,,$$

where $\phi(t, t_0)$ is the state transition matrix for (3.11).

3.9.11 The LTV differential equation (1.1), with D=0, has the adjoint equations

$$\dot{z} = -A^*z + C^*v$$

$$w = B^*z.$$

(i) Show that the impulse response matrix g_a for the adjoint model satisfies

$$g_a(t_1, t_0) = g^*(t_0, t_1).$$

(ii) In the linear case, this relation becomes

$$q_a(s) = q^*(-s).$$

Comment on this result, in view of Exercise 17 of the previous chapter.

- **3.9.12** Derive the three properties of the state transition matrix given on page 57,
 - (a) In the LTI case, using properties of the matrix exponential.
 - (b) In the LTV case, using the definition of ϕ in terms of the fundamental matrix.

Part II System Structural Properties

Chapter 4

Stability

Depicted in Figure 4.1 are results from several simulations conducted using Matlab. Three different linear state space models of the form (1.1) have been simulated using three different initial conditions.

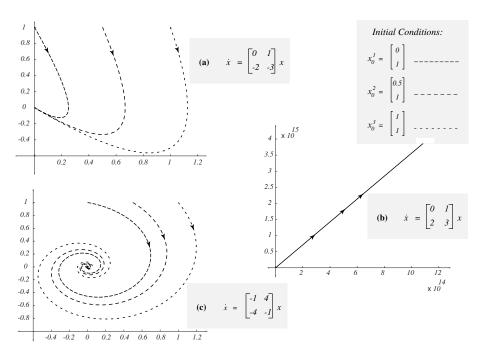


Figure 4.1: Typical solutions of the linear state space model

One can see in these plots three substantially different modes of behavior. In (a) the solutions are attracted to the origin along the line $x_2 = -x_1$, while

in (c) solutions are also attracted to the origin, but they move there in a cyclical fashion. In (b) solutions explode to infinity approximately along a straight line.

These simulations suggest that some solidarity among different states exists, but this is not true in general. Consider for example the interconnection shown in Figure 4.2. From the state equations

$$\dot{x}_1 = x_2
\dot{x}_2 = 2x_1 + x_2 + u$$

we have, in the Laplace transform domain,

$$X_1(s) = \frac{1}{s^2 - s - 2}U(s)$$
 $X_2(s) = \frac{s}{s^2 - s - 2}U(s)$

Thus the output may be expressed in terms of the input as

$$Y(s) = X_2(s) - 2X_1(s) = \frac{s-2}{s^2 - s - 2}U(s) = \frac{1}{s+1}U(s)$$

Since the single pole p=-1 is in the left hand plane, this is a BIBO stable model, as defined in any undergraduate signals and systems course. However, the state equations also give

$$A = \left[\begin{array}{cc} 0 & 1 \\ 2 & 1 \end{array} \right]$$

Solving for the eigenvalues of A yields

$$det(sI - A) = 0 \Longrightarrow \lambda_1 = -1, \lambda_2 = 2$$

Thus, we have an unstable root $\lambda_2 = 2$, but it is cancelled by the numerator of the transfer function due to the feed forward of x_2 . For certain initial conditions one would expect the state to diverge to infinity, even if the input and output remain bounded.

In this chapter we present an introduction to stability theory for nonlinear models, and in the process we give a complete characterization of the types of behavior possible for a linear state space model.

4.1 Stability in the sense of Lyapunov

Once a feedback control law has been specified, it is frequently possible to express the closed-loop system in the form of a nonlinear state space model

$$\dot{x} = f(x); \qquad x(0) = x_0,$$
 (4.1)

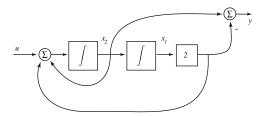


Figure 4.2: Is this model stable, or unstable?

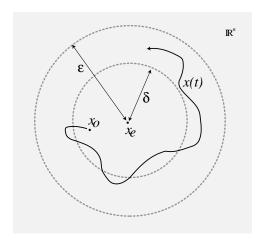


Figure 4.3: If $x_0 \in B_{\delta}(x_e)$, then $x(t) = \varphi(t; x_0) \in B_{\epsilon}(x_0)$ for all $t \ge 0$.

where x(t) evolves in \mathbb{R}^n . We assume throughout this chapter that the function $f \colon \mathbb{R}^n \to \mathbb{R}^n$ has continuous first and second partial derivatives (denoted $f \in C^2(\mathbb{R}^n)$). The solution is denoted by x(t), or $\varphi(t; x_0)$ when we wish to emphasize the dependence of the state on its initial condition. Recall that x_e is an equilibrium of (4.1) if $f(x_e) = \vartheta$, in which case $\varphi(t; x_e) = x_e$ for all t. Our most basic, and also the weakest form of stability considered in this book is the following: The equilibrium x_e is stable in the sense of Lyapunov if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $||x_0 - x_e|| < \delta$, then

$$\|\varphi(t;x_0) - \varphi(t;x_e)\| < \epsilon$$
 for all $t \ge 0$.

In words, if an initial condition is close to the equilibrium, then it will stay close forever. An illustration is provided in Figure 4.3.

Example 4.1.1. Consider the frictionless pendulum illustrated in Figure 4.4.

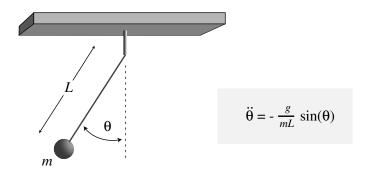


Figure 4.4: Frictionless pendulum

Letting $x_1 = \theta$, $x_2 = \dot{\theta}$ we obtain the state space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\frac{g}{mL}\sin(x_1) \end{pmatrix}.$$

In Figure 4.5 some sample trajectories of x(t) are illustrated. As Figure 4.5 shows, the equilibrium $x_e = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$ is not stable in any sense, which agrees with physical intuition. Trajectories which begin near the equilibrium $x_e = \vartheta$ will remain near this equilibrium thereafter. Consequently, the origin is stable in the sense of Lyapunov.

Although the equilibrium $x_e = \vartheta$ is stable, there is no reason to expect that trajectories x(t) will converge to x_e . Since it is frequently a goal in control design to make the model fall into a specific regime starting from any initial condition, we often require the following stronger forms of stability.

(a) An equilibrium x_e is said to be asymptotically stable if (i) x_e is stable in the sense of Lyapunov; and (ii) for some $\delta_0 > 0$, whenever $||x_0 - x_e|| < \delta_0$,

$$\lim_{t \to \infty} \varphi(t; x_0) = x_e.$$

(b) An equilibrium x_e is globally asymptotically stable if it is asymptotically stable with $\delta_0 = \infty$. Hence $x(t) \to x_e$ from any initial condition.

Consider now the case where f is linear, so that $\dot{x} = Ax$. When is this LTI model asymptotically stable? First note that by the *homogeneity* property of linear systems, asymptotic stability and global asymptotic stability are equivalent concepts. In the linear case, x_e is an equilibrium if and only

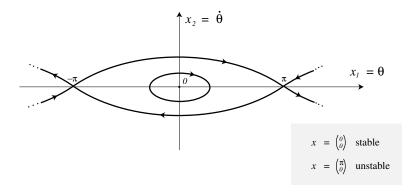


Figure 4.5: Stable and unstable equilibria.

if $x_e \in \mathcal{N}(A)$. Hence the origin is always an equilibrium. For asymptotic stability we must in fact have $\mathcal{N}(A) = \{0\}$ since, by global asymptotic stability, we cannot have more than one equilibrium. Unstable modes are also ruled out. The following result can be demonstrated using these facts:

Theorem 4.1. An LTI model is asymptotically stable if and only if Re (λ) < 0 for every eigenvalue λ of A.

A matrix A satisfying the conditions of Theorem 4.1 is said to be Hurwitz.

Stability in the sense of Lyapunov is more subtle. For example, consider the two matrices

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The eigenvalues of both matrices are the same: $\lambda_1 = \lambda_2 = 0$. In the first case,

$$(sI - A_1)^{-1} = \begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix},$$

so that the state transition matrix becomes $\phi_1(t) = I$. Since state trajectories are constant, it follows that every state is an equilibrium, and that every equilibrium is stable in the sense of Lyapunov.

However for A_2 ,

$$(sI - A_2)^{-1} = \begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}$$

Therefore,

$$\phi_2(t) = \left[\begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right]$$

which is growing with time. This implies that $x(t) \to \infty$ for certain initial conditions, and so this system is not stable in any sense.

Thus, the eigenvalues alone do not foretell stability in the sense of Lyapunov. Looking at the eigenvectors, we see in the first instance from the equation

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = 0 \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$$

that we may take as eigenvectors

$$x^1 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right] \qquad x^2 = \left[\begin{array}{c} 0 \\ 1 \end{array} \right]$$

However, in the second case, the eigenvector equation becomes

$$\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = 0 \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$$

This has a solution

$$x^1 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right],$$

but in this example it is not possible to choose another linearly independent eigenvector. This is a special case of the more general result:

Theorem 4.2. The LTI model is stable in the sense of Lyapunov if and only if

- (a) Re $(\lambda) \leq 0$ for every eigenvalue λ .
- (b) If $\operatorname{Re}(\lambda) = 0$, where λ is an eigenvalue of multiplicity m, then there exist m linearly independent corresponding eigenvectors.

Proof This follows from Theorem 8-14 of [6].

4.2 Lyapunov's direct method

Lyapunov's direct method is a general approach to verifying stability for linear or nonlinear models. This method allows us to determine stability without solving an ordinary differential equation. The idea is to search for a "bowl-shaped" function V on the state space \mathbb{R}^n . If V(x(t)) is decreasing in time, then we can qualitatively describe the behavior of the trajectory x(t), and determine if the model is stable in the sense of Lyapunov.

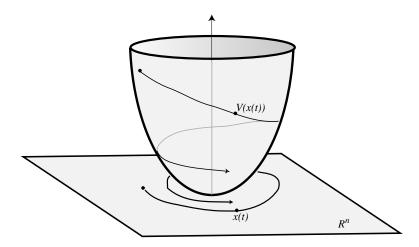


Figure 4.6: V(x(t)) represents the height of the 'lifted' trajectory from the state space \mathbb{R}^n

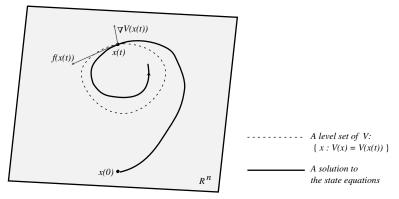


Figure 4.7: V(x(t)) is decreasing with time if $\nabla V(x) f(x) \leq 0$

To capture what we mean by "bowl-shaped", we give the following definition. A scalar-valued function $V: \mathbb{R}^n \to \mathbb{R}$ with continuous partial deriva-

tives is called *positive definite* on an (open) region Ω about the equilibrium x_e if $V(x_e) = 0$, and V(x) > 0 for all $x \in \Omega$, $x \neq x_e$.

Theorem 4.3. Suppose $x_e \in \mathbb{R}^n$ is an equilibrium, and that $\Omega \subset \mathbb{R}^n$ is an open set containing x_e .

(a) x_e is stable in the sense of Lyapunov if there exists a positive definite function V such that

$$\frac{d}{dt}V(x(t)) \le 0 \qquad \text{whenever } x(t) \in \Omega$$

(b) x_e is asymptotically stable if there exists a positive definite function V such that

$$\frac{d}{dt}V(x(t)) < 0$$
 whenever $x(t) \in \Omega$, $x(t) \neq x_e$

(c) x_e is globally asymptotically stable if there exists a positive definite function V on \mathbb{R}^n such that $V(x) \to \infty$ as $x \to \infty$, and

$$\frac{d}{dt}V(x(t)) < 0 \qquad \text{for all } x(t), \ x(t) \neq x_e.$$

Proof See for example [9].

Using the formula

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t)) f(x(t)),$$

the condition for asymptotic stability becomes

$$\nabla V(x) f(x) < 0$$
 for $x \in \Omega, x \neq x_e$.

This is illustrated geometrically in Figure 4.7.

As an application of Theorem 4.3 consider the Van der Pol oscillator described by the pair of equations

$$\dot{x}_1 = x_2
\dot{x}_2 = -(1 - x_1^2)x_2 - x_1.$$
(4.2)

We first search for equilibria: If $f(x) = \vartheta$, it follows that $x_2 = 0$ and $-(1-x_1^2)x_2 - x_1 = 0$. The unique equilibrium is therefore $x_e = \vartheta$.

To see if x_e is stable in any sense, try the Lyapunov function $V(x) = \frac{1}{2}|x|^2$. We then have

$$\nabla V(x) = x^T, \qquad f(x) = \begin{pmatrix} x_2 \\ -(1 - x_1^2)x_2 - x_1 \end{pmatrix}$$

and so

$$\nabla V(x) f(x) = x_1 x_2 - (1 - x_1^2) x_2^2 - x_1 x_2 = -(1 - x_1^2) x_2^2 \le 0$$
 If $|x_1| < 1$.

The set $\Omega = \{x \in \mathbb{R}^2 : |x_1| < 1\}$ contains x_e and is open. Hence, x_e is stable in the sense of Lyapunov.

Is the origin asymptotically stable? Considering Figure 4.8, it is apparent that we need to be more careful in choosing V(x). To obtain elliptical level sets of the form shown in the second illustration of Figure 4.8, set

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) + \epsilon x_1 x_2$$

where ϵ is small. If $|\epsilon| < 1$, the function V is then positive definite. The derivative is now

$$\nabla V(x) = \begin{pmatrix} x_1 + \epsilon x_2 \\ x_2 + \epsilon x_1 \end{pmatrix}^T$$

and the equation to be analyzed is

$$\nabla V(x) f(x) = x_1 x_2 + \epsilon x_2^2 - \left[(1 - x_1^2) x_2^2 + x_1 x_2 + \epsilon (1 - x_1^2) x_1 x_2 + \epsilon x_1^2 \right]$$

= $-\left[(1 - x_1^2) - \epsilon \right] x_2^2 - \epsilon (1 - x_1^2) x_1 x_2 - \epsilon x_1^2.$

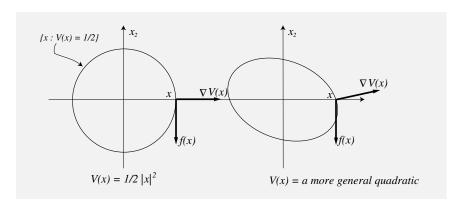


Figure 4.8: The vectors $\nabla V(x)$ and f(x) meet at ninety degrees when $x = (\gamma, 0)^T$, and $V(x) = |x|^2$. By distorting V, and hence also its level set $\{x : V(x) = 1/2\}$, one can increase this angle beyond ninety degrees.

If we take $x_1^2 < \frac{1}{2}$ this then gives

$$\nabla V(x) f(x) \le -\left(\frac{1}{2} - \epsilon\right) x_2^2 + \epsilon |x_1 x_2| - \epsilon x_1^2$$

Now use the bound $|x_1x_2| \leq \frac{1}{2}(x_1^2 + x_2^2)$:

$$\nabla V(x) f(x) \leq -\left(\frac{1}{2} - \epsilon\right) x_2^2 + \frac{1}{2} \epsilon (x_1^2 + x_2^2) - \epsilon x_1^2$$
$$= -\left(\frac{1}{2} - \epsilon - \frac{1}{2} \epsilon\right) x_2^2 - \frac{\epsilon}{2} x_1^2.$$

Finally, take $\epsilon = \frac{1}{6}$. Then for any $x \neq \vartheta$,

$$\nabla V(x) f(x) \leq -\left(\frac{1}{2} - \frac{1}{6} - \frac{1}{12}\right) x_2^2 - \frac{1}{12} x_1^2$$
$$= -\frac{1}{4} x_2^2 - \frac{1}{12} x_1^2 < 0.$$

With $\Omega = \{x : x_1^2 < \frac{1}{2}\}$, we see that

$$\nabla V(x) f(x) < 0 \text{ for } x \in \Omega, x \neq x_e = 0.$$

It follows that the equilibrium x_e is asymptotically stable.

4.3 Region of asymptotic stability

Let V(x) be a positive-valued function on \mathbb{R}^n with continuous first partial derivatives, and let Ω_ℓ designate the open region within which V(x) is less

than ℓ :

$$\Omega_{\ell} := \{ x : V(x) < \ell \}.$$

Suppose that, within Ω_{ℓ} , V(x(t)) is always decreasing. This will be the case if and only if

$$\nabla V(x) f(x) \le 0$$
 for all $x \in \Omega_{\ell}$. (4.3)

Since under (4.3) we have $V(x(t)) < \ell$ for all t if $V(x(0)) < \ell$, it follows that the set Ω_{ℓ} is invariant: if $x_0 \in \Omega_{\ell}$, then $x(t) \in \Omega_{\ell}$ for all t > 0. Under a stronger set of conditions, the set Ω_{ℓ} becomes a region of asymptotic stability.

Theorem 4.4. Suppose that $V: \mathbb{R}^n \to \mathbb{R}$ is C^1 , and that the nonlinear model (4.1) satisfies for some $\ell > 0$,

The set
$$\Omega_{\ell}$$
 is bounded. (4.4)

$$V(x) > 0$$
 for all $x \neq \vartheta, x \in \Omega_{\ell}$. (4.5)

$$V(x) > 0 for all x \neq \vartheta, x \in \Omega_{\ell}. (4.5)$$

$$\nabla V(x) f(x) < 0 for all x \neq \vartheta, x \in \Omega_{\ell}. (4.6)$$

Then $x_e = \vartheta$ is an asymptotically stable equilibrium, and every solution beginning in Ω_{ℓ} tends to ϑ as $t \to \infty$. That is, Ω_{ℓ} is a region of asymptotic stability.

Since V(x(t)) is decreasing in t and is bounded from below, it must have a limit $v_0 \geq 0$. If $v_0 = 0$, then by (4.5) it follows that $x(t) \to \vartheta$, as claimed. To prove the theorem we will show that it is impossible to have $v_0 > 0$.

We proceed by contradiction: If $v_0 > 0$ then for all t, x(t) lies in the closed and bounded set

$$S = \{x : v_0 \le V(x) \le \ell\}.$$

Since $\nabla V(x) f(x)$ is strictly negative on S, by continuity it follows that for some $\varepsilon > 0$,

$$\nabla V(x) f(x) < -\varepsilon, \qquad x \in S.$$

Hence, from this bound, the bound $V(x(0)) \leq \ell$, and the fundamental theorem of calculus, for any T > 0,

$$-\ell \le V(x(t_f)) - V(x(0)) = \int_0^{t_f} \nabla V(x(t)) f(x(t)) dt \le -\varepsilon t_f.$$

Letting $t_f \to \infty$, we obtain a contradiction, and this proves the theorem. \square

Example 4.3.1. Consider the nonlinear state space model

$$\dot{x}_1 = -x_1 - x_2
\dot{x}_2 = x_1 - x_2 + x_2^3$$

With the Lyapunov function $V(x) = x_1^2 + x_2^2$, we have for all $x \neq \vartheta$

$$\nabla V(x) f(x) = 2(x_1, x_2) \cdot (-x_1 - x_2, x_1 - x_2 + x_2^3)$$

= $-2[x_1^2 + x_2^2(1 - x_2^2)]$

which will be negative for $x \neq \vartheta$ so long as $x_2^2 < 1$. Considering Figure 4.9, it follows from Theorem 4.4 that $\Omega_1 = \{x : x_1^2 + x_2^2 < 1\}$ is a region of asymptotic stability. \square

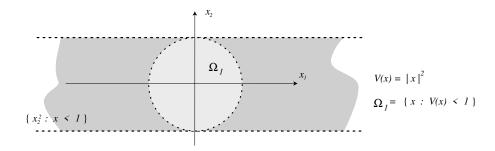


Figure 4.9: The sub-level set of V shown as a grey disk in the figure is a region of asymptotic stability

4.4 Stability of linear state space models

We have already seen in Theorem 4.1 that to determine asymptotic stability of a linear state space model, it is sufficient to compute the eigenvalues of an $n \times n$ matrix. By specializing Theorem 4.3 to the LTI case, we derive here an algebraic characterization of stability which is simpler to apply than the eigenvalue test. Before we proceed, we need a better understanding of positive definite quadratic functions. An $n \times n$ matrix M is called Hermitian if $M = M^*$ (complex-conjugate transpose). A matrix M is said to be positive definite if M is Hermitian, and

$$x^*Mx > 0$$
 for all $x \neq \vartheta, x \in \mathbb{C}$.

If the strict inequality above is replaced by $x^*Mx \geq 0$, then the matrix M is called *positive semi-definite*. We adopt the notation M > 0, $M \geq 0$, respectively. If $V(x) = x^T Mx$ for a real matrix M > 0, then the function V is positive definite on \mathbb{R}^n .

There are several tests for positive definiteness which avoid computation of x^*Mx for every $x \in \mathbb{R}^n$. Suppose for instance that M is Hermitian, and is partitioned as shown below:

In this 4×4 example, we have taken four submatrices found on the diagonal, and indicated their respective determinants, $\Delta_1, \Delta_2, \Delta_3, \Delta_4$. These are called the (leading) *principle minors* of M. One of the most readily verified tests for positivity is given in the following theorem.

Theorem 4.5 (Sylvester-Principal Minors Test). A Hermitian matrix M is positive definite if and only if each of its leading principle minors is strictly positive.

Proof See
$$[3]$$
.

Consider now the LTI model

$$\dot{x} = Ax$$

Choose $V(x) = x^T P x$, where we can assume without loss of generality that P is real and Hermitian $(P = P^T = P^*)$. If P is a positive definite matrix, then V is also positive definite.

The gradient $\nabla V(x)$, which is a row vector, may be computed using the product rule,

$$\nabla V(x)^T = Px + P^T x = 2Px.$$

Also, f(x) = Ax, so the condition for asymptotic stability is

$$\nabla V(x) f(x) = 2x^T P^T A x < 0, \quad x \neq 0.$$

Since $x^T P^T A x = x^T A^T P x$, we have that $2x^T P^T A x = x^T [PA + A^T P] x$. Thus, we need to find P > 0 such that the Lyapunov equation

$$A^T P + PA = -Q (4.7)$$

is satisfied for some positive definite matrix Q. This approach characterizes stability for LTI models:

Theorem 4.6. The LTI model $\dot{x} = Ax$ is asymptotically stable if and only if for some Q > 0, there exists a solution P > 0 to the Lyapunov equation.

Proof If for one Q > 0 we can find a P > 0 which solves the Lyapunov equation (4.7), then we have seen that that $V(x) = x^T P x$ is a valid Lyapunov function. Hence the model is globally asymptotically stable by Theorem 4.3.

Conversely, if the model is asymptotically stable, then $\operatorname{Re}(\lambda) < 0$ for any λ , so $e^{At} \to 0$ as $t \to \infty$ at an exponential rate. Let

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

where Q > 0 is given. Given the form of the state transition matrix ϕ , it follows that the matrix P has the following interpretation:

$$x^T P x = \int_0^\infty x(t)^T Q x(t) dt$$
 when $x(0) = x$.

To see that P solves the Lyapunov equation, let $F(t) = e^{A^T t} Q e^{At}$, and apply the product rule for differentiation:

$$\frac{d}{dt}F(t) = A^{T}e^{A^{T}t}Qe^{At} + e^{A^{T}t}Qe^{At}A = A^{T}F(t) + F(t)A.$$

So, by the Fundamental Theorem of Calculus,

$$F(s) - F(0) = \int_0^s \frac{d}{dt} F(t) \, dt = A^T \left(\int_0^s F(t) \, dt \right) + \left(\int_0^s F(t) \, dt \right) A.$$

We have F(0) = Q, and letting $s \to \infty$ we have $F(s) \to 0$, and $\int_0^s F(t)dt \to P$. In the limit we thus see that the identity $-Q = A^T P + PA$ holds, so the Lyapunov equation is satisfied.

Example 4.4.1. Consider the two-dimensional model

$$\dot{x} = Ax = \left[\begin{array}{cc} 0 & 1 \\ -2 & -2 \end{array} \right] x.$$

To see if this model is stable we will attempt to solve the Lyapunov equation with $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

$$\begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving the resulting three equations gives:

$$-2p_{11} - 2p_{12} = -1$$

$$-2p_{22} + p_{11} - 2p_{12} = 0$$

$$2p_{12} - 4p_{22} = -1,$$

which has the solution $P = \begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 3/8 \end{bmatrix}$.

To see if the system is stable, we must now check to see if the matrix P is positive definite. The principle minors can be computed as

$$5/4 > 0$$
, $(5/4)(3/8) - (1/4)(1/4) > 0$

Hence the Sylvester-Principal Minors test is positive. From this we can conclude that P > 0, and that the model is globally asymptotically stable.

4.5 Stability subspaces

The stable subspace Σ_s of $\dot{x} = Ax$ is the set of all $x \in \mathbb{R}^n$ such that

$$e^{At}x \to 0$$
 as $t \to \infty$.

In the special case where the matrix A has distinct eigenvalues, the stable subspace is easily characterized using the modal form. Letting $\{\lambda_i\}$ denote the distinct eigenvalues, and $\{v^i\}$ the corresponding eigenvectors, we may write any solution x as

$$x(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v^i$$

where $\{\alpha_i\}$ is the representation of x(0) with respect to the basis $\{v^i\}$. From this equation it is evident that

$$\Sigma_s = \operatorname{Span} \{ v^i : \operatorname{Re}(\lambda_i) < 0 \}.$$

In this case we may also define an unstable subspace as

$$\Sigma_u = \operatorname{Span} \{ v^i : \operatorname{Re}(\lambda_i) \ge 0 \}.$$

Strictly speaking, if the stable subspace is to be real, then these formulae will not be valid if there are complex eigenvalues. Consider for example the three-dimensional model with

$$A = \begin{bmatrix} -2 & 1 & -1 \\ -2 & -5 & 6 \\ -1 & -3 & 4 \end{bmatrix}$$

The eigenvalues of A are distinct, $\lambda = 1, -2 \pm j$, and hence a modal form of the state equations is possible with a complex matrix Λ .

The unstable subspace is the span of the eigenvector v^1 corresponding to $\lambda_1 = 1$:

$$\Sigma_u : \{x : (\lambda_1 I - A)x = 0\},\$$

from which we compute $x_1 = 0$, $x_2 = x_3$. Therefore,

$$\Sigma_u = \operatorname{Span} \left\{ \begin{bmatrix} 0\\1\\1 \end{bmatrix} \right\}$$

The following vectors are eigenvectors corresponding to λ_2 , λ_3 :

$$v^2 = \begin{bmatrix} 1 \\ 2j \\ j \end{bmatrix}, \qquad v^3 = \begin{bmatrix} 1 \\ -2j \\ -j \end{bmatrix}.$$

Since the stable subspace is real, it follows that

$$\Sigma_s: x = k \begin{bmatrix} 1 \\ j2 \\ j1 \end{bmatrix} + \bar{k} \begin{bmatrix} 1 \\ -j2 \\ -j1 \end{bmatrix}, \qquad k \in \mathbb{C}.$$

where \bar{k} denotes the complex conjugate of k. The formula for the stable subspace may be equivalently expressed as

$$\Sigma_s = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

4.6 Linearization of nonlinear models and stability

We now return to the nonlinear case, and show that the simply stated characterization of stability given in Theorem 4.6 for linear models has remarkably

strong implications to general nonlinear state equations. Consider again the nonlinear model (4.1) with $x_e = \vartheta$. With $A = \frac{d}{dx} f(\vartheta)$, the linearized model may be expressed

$$\dot{\delta x} = A \, \delta x. \tag{4.8}$$

Theorem 4.7. For the nonlinear state space model with $x_e = \vartheta$,

- (a) If x_e is asymptotically stable for (4.8), then x_e is also asymptotically stable for (4.1).
- (b) If A has an eigenvalue with $\operatorname{Re}(\lambda) > 0$, then x_e is not stable in the sense of Lyapunov for (4.1).
- (c) If $\operatorname{Re}(\lambda) \leq 0$ for all λ , but $\operatorname{Re}(\lambda) = 0$ for some λ , nothing can be said about the stability of the nonlinear model.

Proof (sketch) For (a), the proof proceeds by constructing a Lyapunov function for the linear model, and then applying this to the nonlinear model. The result then follows from Theorem 4.3.

The proof of (b) is similar: If the linear model is strictly unstable in this sense, then there exists an initial condition x_0 for which the corresponding solution to the linearization tends to infinity. One may then construct $\varepsilon > 0, \delta > 0$ such that the solution to the nonlinear equations with initial condition $x(0) = cx_0$ will eventually leave the ball $B_{\varepsilon}(\vartheta)$ centered at the origin with radius ε , whenever $c < \delta$. This follows from the Taylor series expansion. Hence, the origin cannot be stable in the sense of Lyapunov.

For (c), there is nothing to prove, but we may give an example. Consider the two state space models on \mathbb{R}^1 :

(a)
$$\dot{x} = -x^3$$
 (b) $\dot{x} = +x^3$

In the first instance, using the Lyapunov function $V(x) = x^2$ we see that $\nabla V(x) f_1(x) = -2x^4$, and we conclude that the model is globally asymptotically stable. In the second case, we have $\nabla V(x) f_2(x) = +2x^4$, which shows that trajectories explode to infinity from any initial condition. Although the behavior of these two models is very different, both have the same linearization with A = 0.

We can now revisit the Van der Pol equations defined in (4.2). The corresponding matrix A can be computed as

$$A = \frac{d}{dx}f(x)\Big|_{x=\vartheta} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Taking determinants, $|Is - A| = \det \begin{pmatrix} s & -1 \\ 1 & s+1 \end{pmatrix} = s^2 + s + 1$, which gives $\lambda_i = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}j$. So, the origin is asymptotically stable. Previously we found that the Lyapunov function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \epsilon x_1 x_2$$

is effective for the nonlinear model. The function V may be written

$$V(x) = x^T P x$$
 where $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\epsilon \\ \frac{1}{2}\epsilon & \frac{1}{2} \end{bmatrix}$

This also works for the linearized model, as can be seen from the following Lyapunov equation:

$$A^{T}P + PA = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$
$$= \frac{1}{2} \begin{pmatrix} -\epsilon & -1 \\ 1 - \epsilon & \epsilon - 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\epsilon & 1 - \epsilon \\ -1 & \epsilon - 1 \end{pmatrix}$$
$$= -\begin{pmatrix} \epsilon & \frac{1}{2}\epsilon \\ \frac{1}{2}\epsilon & 1 - \epsilon \end{pmatrix} = -Q.$$

Since the matrix Q is positive definite for $\epsilon < 4/5$, we again see that the origin is an asymptotically stable for the linearized model.

Although this is obviously a powerful method, it does not in general tell the complete story about the stability of a nonlinear system. In some applications we are in the situation of part (c) where the linearized model is not asymptotically stable, and in this case the nonlinear equations must be tackled directly. Another shortcoming of this method is that when using a linearized model it is not possible to estimate a region of asymptotic stability for the nonlinear model.

4.7 Input-output stability

A function $f:[0,\infty)\to\mathbb{R}^n$ is called bounded if there exists a constant M such that $|f(t)|\leq M$ for all t. A typical bounded function is illustrated in Figure 4.10. A state space model with input u and output y is bounded input/bounded output stable (BIBO stable) if for any t_0 , and any bounded input u(t), $t\geq t_0$, the output is bounded on $t_0\leq t<\infty$ if $x(t_0)=\vartheta$.

Consider for example the LTI model whose state space and transfer function descriptions are expressed as

$$\dot{x} = Ax + Bu$$

 $y = Cx + Du$, $G(s) = C[Is - A]^{-1}B + D$.

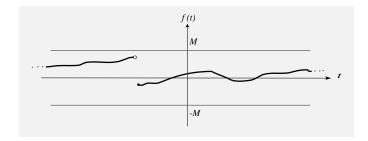


Figure 4.10: A typical bounded function: The graph of f remains in a tube of radius M.

This model is BIBO stable if and only if each pole of every entry of G(s) has a *strictly* negative real part.

For an LTI model, we see that asymptotic stability of $\dot{x} = Ax$ implies BIBO stability. The converse is *false* since pole-zero cancellations may occur. The connections between BIBO stability and asymptotic stability may be strengthened, but first we must develop controllability and observability for LTI models.

Matlab Commands

EIG finds eigenvalues and eigenvectors.

ROOTS finds the roots of a polynomial.

LYAP solves the Lyapunov matrix equation (4.7).

Summary and References

In this chapter we have introduced a powerful approach to understanding the qualitative behavior of dynamical systems. The main tool is the *Lyapunov function*, which is interpreted geometrically as "lifting" the state trajectory onto the surface defined by the Lyapunov function, and checking to see if the trajectory travels "down hill". Starting with these ideas, we then developed

- (a) Criteria for stability and instability of nonlinear state space models.
- (b) Strong connections between stability of a nonlinear state space model, and stability of its linearization.
- (c) An algebraic characterization of stability for linear systems, based on the matrix *Lyapunov equation*.

There are many excellent texts on stability and the Lyapunov function approach. See in particular [9]. A historical development of stability and the Lyapunov equation is contained in [8].

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4.8 Exercises

4.8.1 Explain the behavior exhibited in Figure 4.1.

- (a) In this first simulation, the trajectories appear to converge to a line within the two dimensional state space. Show that this is the case for the given initial conditions. Will this occur for *every* nonzero initial condition? Is the model stable in any sense?
- (b) Explain the behavior in simulation (b). Will the trajectories approximately follow a line from each initial condition?
- (c) Explain the behavior in the last simulation. Will the trajectories cycle to zero from each nonzero initial condition?
- **4.8.2** This exercise shows how to use state space methods to solve numerical equations and optimization problems.
 - (a) Suppose that one wishes to compute numerically a solution x^* to f(x) = g(x), where $f, g: \mathbb{R}^n \to \mathbb{R}^n$ are C^1 functions. Find conditions on f and g which ensure that solutions to the differential equation

$$\dot{x} = A(f(x) - g(x))$$

will converge to x^* for initial conditions near x^* . In this equation A is an $n \times n$ matrix which can be chosen by the user. Experiment with the algorithm using Simulink with $f(x) = e^x$, and $g(x) = \sin(x)$, for a suitable scalar A.

(b) Suppose that one wants to minimize a C^2 function $V: \mathbb{R}^n \to \mathbb{R}_+$. A necessary condition for a point $x^* \in \mathbb{R}^n$ to be a minimum is that it be a *stationary point*; that is, $\nabla V(x^*) = \vartheta$.

Consider the differential equation

$$\dot{x} = -P\nabla V(x)^T,$$

where P is a positive definite matrix. Find conditions on the function V to ensure that a given stationary point x^* will be asymptotically stable for this equation. *Hint*: Find conditions under which the function V is a Lyapunov function. Try this algorithm out using Simulink with a function of your choice with n=2. For example, try $V(x)=x_1^6-2x_1^2+x_1x_2+6+|x_2|^3$.

4.8.3 Consider the pendulum described in Figure 4.4, with the addition of an applied torque u. This is a nonlinear system which is described by the differential equation $\ddot{\theta} = -\sin(\theta) + u$.

Find a PD controller (state feedback) which stabilizes the system about $\theta = \pi$. That is, for controlled state space model with $x = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$, the equilibrium $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$ is asymptotically stable.

Is $\begin{pmatrix} \pi \\ 0 \end{pmatrix}$ globally asymptotically stable with your control law?

- **4.8.4** Suppose that $V(x) = x^T A x$ for a real matrix A, and suppose that the function V is positive definite. Prove that the matrix $M = \frac{1}{2}(A + A^T)$ is positive definite. Can you find an example in which the matrix A is not symmetric?
- **4.8.5** Study the stability of the equilibrium state $x^e = 0$ $(x := (y \ \dot{y})^T)$ for the nonlinear system

$$\ddot{y} + \dot{y} + y^3 = 0$$

using the function

$$V(y, \dot{y}) = y^4 + y^2 + 2y\dot{y} + 2\dot{y}^2$$

as a candidate Lyapunov function.

4.8.6 For the LTI system described by

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -2 & 3 \end{bmatrix} x ,$$

investigate asymptotic stability using the Lyapunov equation

$$A^T P + P A = -Q$$
 with $Q = I$.

Can you arrive at any definitive conclusion?

4.8.7 Consider the linear model

$$\dot{x} = \begin{pmatrix} 1 & 1 \\ -1 - 2\alpha + \alpha^2 & -1 - \alpha + \alpha^2 \end{pmatrix} x,$$

where α is a scalar parameter. For what values of α is the origin (i) asymptotically stable? (ii) stable in the sense of Lyapunov?

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4.8.8 Consider the simple scalar model

$$\dot{y} = \alpha_0 u$$
.

We would like to use feedback of the form u = -Ky + Nr so that the closed-loop pole is at -1, where K and N are scalar constants. However, the parameter α_0 is not known!

To solve this problem, we estimate α_0 using the "observer":

$$\frac{d}{dt}\hat{\alpha} = \ell(\dot{y} - \hat{\alpha}u)u.$$

Here ℓ is a positive constant.

(a) Verify that $\frac{d}{dt}(\hat{\alpha} - \alpha_0)^2 \leq 0$ for all time. Hence the absolute identification error $|\hat{\alpha} - \alpha_0|$ can never increase, regardless of the control. Find a control signal u(t) which forces the error to go to zero.

Assuming that $\hat{\alpha}(t)$ does converge to α_0 , a natural control law is

$$u(t) = \frac{1}{\hat{\alpha}(t)}(r(t) - y(t))$$

We now consider the regulation problem, where $r(t) = r_0 \in \mathbb{R}$ for all t.

- (b) Obtain a nonlinear state space model for the *closed-loop* system, with states y and $\hat{\alpha}$.
- (c) Show that $y = r_0$; $\hat{\alpha} = \beta$ is an equilibrium for any $\beta \neq 0$. Deduce that the equilibrium $\begin{pmatrix} r_0 \\ \alpha_0 \end{pmatrix}$ cannot be asymptotically stable.
- (d) Linearize the nonlinear model about an equilibrium. What can you conclude about the stability of an equilibrium $\binom{r_0}{\beta}$?
- (e) Show that $V(y, \hat{\alpha}) = \frac{1}{2}(y-r_0)^2 + \frac{1}{2}(\hat{\alpha}-\alpha_0)^2$ is a Lyapunov function for this model. Can you conclude that the equilibrium (r_0, α_0) is stable in the sense of Lyapunov?

These ideas are the starting point of *adaptive control*; an approach based on simultaneous system identification and control.

4.8.9 We have seen that stability of an LTI model may be characterized in terms of the existence of a Lyapunov function, and we found that the Lyapunov function in the stable case can be taken as

$$V(x) = x^T \int_0^\infty \phi(\tau)^T Q \phi(\tau) d\tau x = x^T P x \quad \text{with } Q > 0.$$

A similar result is possible in the nonlinear case under suitable stability conditions, although the function V will not necessarily be a quadratic if the model is not linear.

Suppose that for the nonlinear model $\dot{x}=f(x)$, the origin $x_e=0$ is globally asymptotically stable, and suppose moreover that the stability is such that the solutions $\phi(t;x)$ are square integrable over time. Hence for any Q>0 the function $V(x)=\int_0^\infty \phi(\tau;x)^TQ\phi(\tau;x)\,d\tau$ is finite valued.

(a) Compute V(x(t)) for t > 0, and simplify using properties of the state transition function, e.g.,

$$\phi(0;x) = x,$$
 $\phi(t;\phi(s;x)) = \phi(t+s;x)$

(b) Using (a), show that $\frac{d}{dt}V(x(t))|_{t=0} = -x(0)^TQx(0)$. Conclude that V is an effective Lyapunov function for this model.

Chapter 5

Controllability

In this chapter we consider exclusively the LTV state space model

$$\dot{x} = A(t)x + B(t)u, \qquad x(t_0) = x_0,$$
 (5.1)

and as a special case its LTI version. Our interest lies in the effectiveness of the input u to control the state x to arbitrary points in \mathbb{R}^n . Consider for example the two-dimensional LTI model illustrated in Figure 5.1.

From the state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u.$$

we see that when $b_1 = b_2 = 0$, this is the example illustrated in Figure 4.1 (a) at the beginning of the previous chapter. Obviously, in this case the control cannot affect the state, so when $b = \vartheta$ the model is uncontrollable in the strongest sense. Consider now $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and suppose that it is desired to choose a control which takes the initial state $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to a final state $x_f = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ at some time t_f . One approach is illustrated in Figure 5.1: Simply follow the control-free trajectory until a time t_s at which the state is nearly "below" the desired final state x_f . Then, apply a large positive value so that the vector bu dominates the state derivative Ax + bu. This procedure will place $x(t_f)$ at x_f for $some \ t_f$, and a similar strategy can be used to drive any state to any other state.

A substantially different situation occurs with $b = \binom{-1}{1}$. In this case the vector b is parallel to the subspace indicated by the dotted line Span $\{v^1\}$ = Span $\{b\}$, where v^1 is an eigenvector for A. It can be shown in this case that no control will take a state from one side of this line to the other, and that if

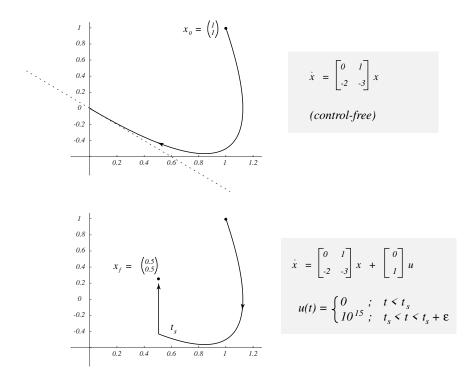


Figure 5.1: The controlled model can be steered to any state in \mathbb{R}^n using impulsive control.

the initial state x_0 lies on this line, then it will stay there forever, regardless of the control.

The general LTV model (5.1) is called *controllable* at time t_0 if there exists a finite time, t_f , such that for any initial state x_0 , and any final state x_f , there is a control u defined on $[t_0, t_f]$ such that $x(t_f) = x_f$. This concept is not obviously directly useful in control design, except possibly in a ballistics problem where x_f denotes a target! However, we will see that this is a central idea in the theory of state space models, and that controllability will be an underlying assumption in many of the control design methods to be developed. The purpose of this chapter is to characterize controllability to better understand examples such as these simple models. We will see in Theorem 5.6 that the pathology illustrated in Figure 5.2 is the only way that a lack of controllability can occur for an LTI model.

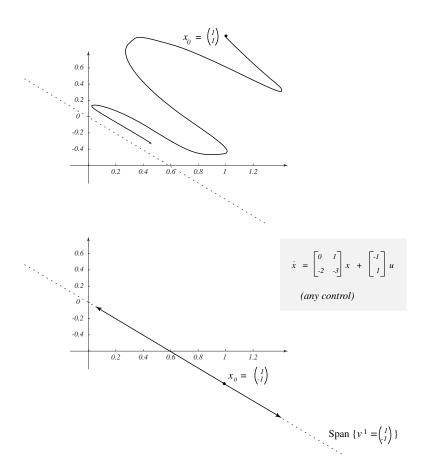


Figure 5.2: If the initial condition satisfies $x_1 = -x_2$, so that the state lies on the indicated diagonal line, then the state can never leave this line, regardless of the control. If the initial condition does not start on the diagonal, then it may never reach it, though it will come arbitrarily close to it for large t.

5.1 A preview: The LTI discrete-time case

Controllability is most easily understood in the LTI discrete-time case

$$x(k+1) = Ax(k) + Bu(k), k = 0, 1, 2, 3, \dots,$$

where again x is of dimension n and u is of dimension m. Assume that we start at $x(0) = \vartheta$, and that it is desired to move to another state x_f in finite time. We have x(1) = Bu(0), and in general

$$\begin{array}{lll} k=2 & x(2) & = & Ax(1)+Bu(1)=ABu(0)+Bu(1) \\ k=3 & x(3) & = & Ax(2)+Bu(2) \\ & = & A^2Bu(0)+ABu(1)+Bu(2) \\ & & \vdots \\ k=r & x(r) & = & A^{r-1}Bu(0)+A^{r-2}Bu(1)+\dots Bu(r-1). \end{array}$$

This can be written compactly as

$$x(r) = [B \ AB \cdots A^{r-1}B] \begin{bmatrix} u(r-1) \\ u(r-2) \\ \vdots \\ u(0) \end{bmatrix}$$

which shows that we can reach arbitrary $x_f \in \mathbb{R}^n$ at time $t_f = r$ if and only if $\mathcal{R}([B \ AB \cdots A^{r-1}B]) = \mathbb{R}^n$, which holds if and only if the rank of $[B \ AB \cdots A^{r-1}B]$ is equal to n.

How do we know when to stop iterating? The Cayley-Hamilton Theorem 3.1 tells us that A^k can be written as a linear combination of A^i , i < n. Hence the rank of the matrix $[B \mid AB \mid \cdots \mid A^{r-1}B]$ cannot increase for $r \geq n$. The controllability matrix is the $n \times (nm)$ matrix

$$\mathcal{C} := [B \mid AB \mid \dots \mid A^{n-1}B].$$

From this discussion it follows that in the discrete time model it is possible to steer from the origin to any other state in \mathbb{R}^n if and only if rank $(\mathcal{C}) = n$.

5.2 The general LTV continuous-time case

We saw in Section 3.7 that the solution to the state space model (5.1) may be expressed

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^t \phi(t, \tau)B(\tau)u(\tau) d\tau,$$

Controllability is thus determined by the range of the linear operator $\mathcal{A}: D[t_0, t_f] \to \mathbb{R}^n$ defined as

$$\mathcal{A}(u) = \int_{t_0}^{t_f} \phi(t_f, \tau) B(\tau) u(\tau) d\tau. \tag{5.2}$$

The input space $D[t_0, t_f]$ is the set of all piecewise continuous time functions on $[t_0, t_f]$. We leave the proof of the following as an exercise.

Theorem 5.1. The following are equivalent for the LTV model (5.1)

- (a) The model is controllable at time t_0 ;
- (b) $\mathcal{R}(\mathcal{A}) = \mathbb{R}^n$ for some $t_f > t_0$.
- (c) There exists a time $t_f > t_0$ such that for any $x_0 \in \mathbb{R}^n$, there is a control u defined on $[t_0, t_f]$ such that $x(t_f) = \vartheta$ when $x(t_0) = x_0$.

The controllability grammian $W(t_0, t_f)$ for the LTV model (5.1) is the positive definite matrix

$$W(t_0, t_f) := \int_{t_0}^{t_f} \phi(t_0, \tau) B(\tau) B^*(\tau) \phi^*(t_0, \tau) d\tau.$$
 (5.3)

The following deeper characterization of controllability gives a finite dimensional test in terms of the $n \times n$ matrix W. The essence of the proof is this: the grammian is full rank if and only if the n rows of the function $\phi(t_0, \cdot)B(\cdot)$ are linearly independent as functions in $D[t_0, t_f]$. This then is equivalent to controllability of the model.

Theorem 5.2. The LTV model (5.1) is controllable at time t_0 if and only if there exists a finite time $t_f > t_0$ such that $W(t_0, t_f)$ is nonsingular.

Proof

(i) Assume $W(t_0, t_f)$ is nonsingular, so that $W(t_0, t_f)^{-1}$ exists. We will show that the model is controllable by showing that it is possible to control any x_0 to θ at time t_f . Consider the control

$$u(t) = -B^*(t)\phi^*(t_0, t)W^{-1}(t_0, t_f)x_0.$$

Given the formula

$$x(t_f) = \phi(t_f, t_0)x_0 + \int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau)u(\tau) d\tau$$

for the final value of the state, we may substitute the formula for the control to give

$$x(t_f) = \phi(t_f, t_0)x_0 + \int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau) \left[-B^*(\tau)\phi^*(t_0, \tau)W^{-1}(t_0, t_f)x_0 \right] d\tau$$
$$= \phi(t_f, t_0)x_0 - \left(\int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau)B^*(\tau)\phi^*(t_0, \tau) d\tau \right)W^{-1}(t_0, t_f)x_0.$$

The term $W^{-1}(t_0, t_f)x_0$ is taken outside of the integral since it does not depend on τ . By the semigroup property

$$\phi(t_f, t_0)\phi(t_0, \tau) = \phi(t_f, \tau),$$

the integral becomes $\phi(t_f, t_0)W(t_0, t_f)$. Therefore

$$x(t_f) = \phi(t_f, t_0)x_0 - \left(\phi(t_f, t_0)W(t_0, t_f)\right)W^{-1}(t_0, t_f)x_0$$

which gives $x(t_f) = \vartheta$. Hence by Theorem 5.1 the model is controllable at time t_0 .

(ii) We now show that controllability of the LTV model implies that $W(t_0, t_f)^{-1}$ exists. We prove the contrapositive: Assuming $W(t_0, t_f)$ is singular, we show that the model is not controllable.

If W is singular, there exists an $\alpha \neq \vartheta$ such that

$$W(t_0, t_f)\alpha = \vartheta$$
 and hence also $\alpha^* W(t_0, t_f)\alpha = 0.$ (5.4)

From the definition of W it then follows that

$$\int_{t_0}^{t_f} \alpha^* \phi(t_0, \tau) B(\tau) B^*(\tau) \phi^*(t_0, \tau) \alpha \, d\tau = 0$$

This can be equivalently expressed as

$$\int_{t_0}^{t_f} \langle B^*(\tau)\phi^*(t_0,\tau)\alpha, B^*(\tau)\phi^*(t_0,\tau)\alpha\rangle \ d\tau = 0,$$

or

$$\int_{t_0}^{t_f} |B^*(\tau)\phi^*(t_0,\tau)\alpha|^2 d\tau = 0.$$

Since the integral of the square norm is zero, and all of these functions are assumed to be piecewise continuous, we must have $B^*(\tau)\phi^*(t_0,\tau)\alpha = \vartheta$ and hence

$$\alpha^* \phi(t_0, \tau) B(\tau) = \vartheta^* \tag{5.5}$$

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for all $\tau \in [t_0, t_f]$.

If the model is controllable at time t_0 , then starting at the particular state $x_0 = \alpha$, there is a control u which makes $x(t_f) = \vartheta$:

$$x(t_f) = \vartheta = \phi(t_f, t_0)\alpha + \int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau)u(\tau) d\tau.$$

From the property $\phi^{-1}(t_f, t_0) = \phi(t_0, t_f)$, we can multiply this equation through by ϕ^{-1} to obtain

$$\vartheta = \alpha + \int_{t_0}^{t_f} \phi(t_0, t_f) \phi(t_f, \tau) B(\tau) u(\tau) d\tau.$$

From the semigroup property we must therefore have

$$\alpha = -\int_{t_0}^{t_f} \phi(t_0, \tau) B(\tau) u(\tau) d\tau.$$

Multiplying both sides on the left by α^* gives

$$|\alpha|^2 = -\int_{t_0}^{t_f} \alpha^* \phi(t_0, \tau) B(\tau) u(\tau) d\tau$$

which by (5.5) must be zero. However, $|\alpha|^2 \neq 0$ by assumption. Hence the model cannot be controllable if W is singular, and the proof is complete. \Box

5.3 Controllability using the controllability matrix

In the LTI discrete-time case we have shown that the controllability matrix is key to understanding controllability. To show that this is still true for the continuous-time LTI model

$$\dot{x} = Ax + Bu
y = Cx + Du,$$
(5.6)

we consider how the controllability grammian W is related to the controllability matrix

$$\mathcal{C} = [B \mid AB \mid A^2B \mid \dots \mid A^{n-1}B].$$

Using this approach we will prove the following theorem.

Theorem 5.3. For the LTI model (5.6), the model is controllable if and only if the $n \times (nm)$ matrix C has rank n.

Proof To begin, note that in the LTI case we have for $t_0 = 0$,

$$W(0,t_f) = \int_0^{t_f} \phi(-\tau)BB^*\phi^*(-\tau) d\tau$$
$$= \int_0^{t_f} e^{-A\tau}BB^*e^{-A^*\tau} d\tau.$$
 (5.7)

To proceed, we show that

$$\operatorname{rank}(\mathcal{C}) < n \iff W \text{ is singular.}$$

(i) We first show that assuming rank (C) < n, we must also have that W is singular.

If rank $(\mathcal{C}) < n$, there exists $\alpha \neq \vartheta$ such that

$$\alpha^*[B \ AB \cdots A^{n-1}B] = \vartheta^*$$

which can be equivalently expressed as

$$\alpha^* A^k B = \vartheta^*, \quad k = 0, \dots, n-1.$$

From the Cayley-Hamilton Theorem 3.1, the state transition matrix can be expressed as

$$\phi(-\tau) = e^{-A\tau} = \sum_{k=0}^{n-1} \beta_k(-\tau)A^k$$

from which it follows that $\alpha^*\phi(-\tau)B = \vartheta^*$ for all τ . From (5.7) we see that $\alpha^*W(0,t_f) = \vartheta^*$, which shows that W is singular.

(ii) Assume now that W is singular. We will show that rank (C) < n. We have seen in equation (5.5) that if $W(0, t_f)$ is singular, then there exists an $\alpha \in \mathbb{R}^n$ such that

$$\alpha^* \phi(0, \tau) B(\tau) = \vartheta^*, \qquad 0 \le \tau \le t_f.$$

In the LTI case, it follows that $\alpha^* e^{-A\tau} B = \vartheta^*$ for all τ . By setting t = 0 this shows that $\alpha^* B = \vartheta^*$. Moreover, if a function is zero over an interval, so is its derivative:

$$\frac{d}{dt}(\alpha^* e^{-At}B) = \alpha^T(-Ae^{-At}B), \Big|_{t=0} = 0$$

which shows that also $-\alpha^*AB = \vartheta^*$. Continuing this procedure, we see that

$$\left.\frac{d^2}{dt^2}(\alpha^*e^{-At}B)\right|_{t=0} = \alpha^*A^2B = \vartheta^*,$$

and for any k

$$(-1)^{k-1}\alpha^*A^{k-1}B = \vartheta^*.$$

It then follows that

$$\alpha^*[B|AB|A^2B \mid \dots \mid A^{n-1}B] = \vartheta^*,$$

which establishes the desired result that rank $(\mathcal{C}) < n$.

A careful examination of the proof shows that we in fact have established the stronger result

$$\mathcal{R}(\mathcal{C}) = \mathcal{R}(\mathcal{A}).$$

Hence, the subspace of \mathbb{R}^n

$$\Sigma_c := \mathcal{R}(\mathcal{C}) = \{\mathcal{C}z : z \in \mathbb{R}^{mn}\}$$

is equal to the set of all states which can be reached from the origin. The set Σ_c is called the *controllable subspace*.

5.4 Other tests for controllability

The controllability of an LTI state space model is much more apparent when the model is in modal form. To see this, we must construct the modal form using a similarity transformation, so we begin with the following:

Theorem 5.4. Controllability of the LTI model

$$\dot{x} = Ax + Bu$$

is invariant under any equivalence transformation $\bar{x} = Px$

Proof For the transformed model $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$ we have

$$\bar{C} = [\bar{B} \mid \bar{A}\bar{B} \mid \dots \mid \bar{A}^{n-1}\bar{B}]
= [PB \mid (PAP^{-1})PB \mid (PAP^{-1})(PAP^{-1})PB \mid \dots \mid (PAP^{-1}) \dots (PAP^{-1})PB].$$

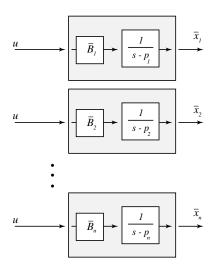


Figure 5.3: The modes of the model are decoupled, and hence the model cannot be controllable if some \bar{B}_i is zero.

Therefore, the controllability matrix for the transformed model is

$$\bar{\mathcal{C}} = [PB \mid PAB \mid PA^2B \mid \dots \mid PA^{n-1}B],$$

or $\bar{\mathcal{C}}=P\mathcal{C}$. Since the matrix P is nonsingular, it follows that rank $\bar{\mathcal{C}}=\mathrm{rank}\,\mathcal{C}$.

Suppose that the eigenvalues of A are distinct, so that A is similar to a diagonal matrix Λ :

$$\bar{A} = \Lambda, \qquad P = M^{-1},$$

where $M = [v^1 \cdots v^n]$. This gives the modal form

$$\dot{\bar{x}} = \Lambda \bar{x} + \bar{B}u,$$

where the modes of the model are decoupled. Figure 5.3 illustrates the structure of the transformed model. If some \bar{B}_i is zero, which means that a row of the matrix \bar{B} is zero, it is apparent that the model cannot be controllable. It turns out that the converse is also true:

Theorem 5.5. Suppose that the matrix A has distinct eigenvalues. Then, (A, B) is controllable if and only if $\overline{B} = M^{-1}B$ has no zero rows.

Proof We prove the theorem in two steps.

- (i) If row i of of $\bar{B} = M^{-1}B$ is zero, it follows that if the initial state satisfies $x_i(0) = 0$, then $x_i(t) = 0$ for all t. Obviously then the model is not controllable.
- (ii) Conversely, if the model is not controllable, there exists an $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$, such that

$$0 = \alpha^* \bar{W} \alpha = \int_0^{t_f} \alpha^* e^{-\Lambda \tau} \bar{B} \, \bar{B}^* e^{-\Lambda^* \tau} \alpha \, d\tau.$$

Equivalently, we must have

$$\sum_{i=1}^{n} \alpha_i \bar{B}_i^* e^{-\lambda_i \tau} \equiv \vartheta^*.$$

If none of the $\{\bar{B}_i\}$ are zero, choose k such that $\alpha_k \neq 0$. We then have

$$e^{-\lambda_k \tau} = \frac{1}{\alpha_k \bar{B}_k \bar{B}_k^*} \sum_{\substack{i=1\\i \neq k}}^n \alpha_i \bar{B}_k \bar{B}_i^* e^{-\lambda_i \tau}.$$

This is impossible, since the functions $\{e^{\lambda_i t}\}$ are linearly independent in $C[t_0, t_f]$. Hence, some row B_k must be zero, which thus proves the theorem.

Theorem 5.5 may also be understood in terms of reciprocal eigenvectors. Recall that the matrix \bar{B} may be expressed as

$$\bar{B} = M^{-1}B = \begin{bmatrix} r^{1*} \\ \vdots \\ r^{n*} \end{bmatrix} B$$

The jth row of \bar{B} is $r^{j*}B$. From Theorem 5.5 we see that (A, B) is controllable if and only if there is no r^j within the null space of B^* .

Since r^{j*} is also a left eigenvector (see Exercise 22 of Chapter 2), we always have

$$r^{j*}(\lambda_j I - A) = \vartheta^*.$$

Then, (A, B) is uncontrollable if and only if there is a j such that

$$r^{j*}(\lambda_j I - A) = \vartheta^*$$
 and $r^{j*}B = \vartheta^*$,

which can be written compactly as

$$r^{j*} \left[\lambda_j I - A | B \right] = \vartheta^*.$$

To summarize, we see that the pair (A, B) in uncontrollable if and only if for some j,

$$rank [\lambda_j I - A \mid B] < n.$$

This final test was derived using a modal form, but it turns out to be true in general:

Theorem 5.6 (Hautus-Rosenbrock test). The pair (A, B) is controllable if and only if for every $s \in \mathbb{C}$,

$$rank [sI - A \mid B] = n. (5.8)$$

Proof One direction is easy: If $r^*[\lambda I - A \mid B] = \vartheta^*$, then we must have for any input u,

$$\frac{d}{dt}(r^*x(t)) = r^*Ax(t) + r^*Bu(t) = \lambda r^*x(t),$$

so that $r^*x(t) = e^{\lambda t}r^*x(0)$, regardless of the control! Hence, if (5.8) is violated for $s = \lambda$, then the model is not controllable.

We have seen that the converse is a direct consequence of Theorem 5.5 in the case of distinct eigenvalues, but the proof is more subtle in full generality. The reader is referred to page 184 of [6] for details.

Note that when s is not an eigenvalue of A the matrix sI - A already has rank n, so that we also have rank $([sI - A \mid B]) = n$. Hence the rank of the matrix (5.8) need only be checked for s equal to the eigenvalues of A.

Example 5.4.1. We illustrate some of the controllability tests developed so far using the simple model

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$$

(i) The controllability matrix for this model is

$$C = [B \mid AB] = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}.$$

Since the rank of C is 1, the model is not controllable. The controllable subspace Σ_c is equal to the range space of C, which in this case is the diagonal in \mathbb{R}^2 :

$$\Sigma_c = \{ \alpha \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) : \alpha \in \mathbb{R} \}.$$

(ii) The eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 - 2$, and the modal matrix can be taken as $M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $M^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. We can then compute

$$\Lambda = \left[\begin{array}{cc} -1 & 0 \\ 0 & -2 \end{array} \right] \qquad \bar{B} = M^{-1}B = \left[\begin{array}{c} 0 \\ 1 \end{array} \right],$$

and again we conclude that the model is not controllable. Since the first row is zero, it follows that the first mode $(\lambda_1 = -1)$ is not connected to the input in modal form.

(iii) We have

$$[sI - A \mid B] = \left[\begin{array}{ccc} s+2 & 0 & 1 \\ 1 & s+1 & 1 \end{array} \right].$$

Evaluating this matrix with s = -1 gives

$$\left[\begin{array}{ccc} 1 & 0 & -2 \\ 1 & 0 & -2 \end{array}\right].$$

Since the rank of this matrix is one, we again see from the Hautus-Rosenbrock test that the model is not controllable. \Box

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5.5 Exercises

5.5.1 Investigate the controllability properties of the LTI model $\dot{x} = Ax + Bu$, for the three pairs of (A, B) matrices given below.

(a)
$$A = \begin{bmatrix} -5 & 1 \\ 0 & 4 \end{bmatrix}$$
, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b)
$$A = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$
 , $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(c)
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- **5.5.2** For the system model corresponding to $A = \begin{bmatrix} -5 & 1 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, obtain a control that drives the system from $x = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$ at t = 0, to $x = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ at t = 1. Simulate your control using *Simulink*.
- **5.5.** Given the linear time-invariant model

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u = Ax + Bu$$

$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x = Cx,$$

check controllability using

- (a) the controllability matrix
- (b) the rows of $\bar{B} = M^{-1}B$, where M is chosen such that $M^{-1}AM$ is diagonal
- (c) The Hautus-Rosenbrock condition.
- **5.5.4** Transform the state space model below into CCF, and from the resulting equations compute its transfer function.

$$\dot{x} = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u \qquad y = (0 \ 0 \ 1)x$$

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5.5.5 Consider the LTI model

$$\dot{x} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u
y = [1 \ 0 \ 1] x.$$

For what initial states $x(0) = x_0$ is it possible to choose a control u so that $y(t) = te^{-3t}$ for t > 1?

- **5.5.6** Let A be an $n \times n$ matrix and B be an $n \times r$ matrix, both with real entries. Assume that the pair (A, B) is controllable. Prove or disprove the following statements. (If the statement is false, then producing a counterexample will suffice.)
 - (a) The pair (A^2, B) is controllable.
 - (b) Let $k(\cdot)$ be an r-dimensional (known) function, piecewise continuous in t. Then, the model described by

$$\dot{x} = Ax + Bu + k(t)$$

is completely controllable, in the sense that any state can be reached from any other state using some control.

- (c) Given that the model $\dot{x} = Ax + Bu$ has the initial condition $x(0) = x_0 \neq \vartheta$, it is possible to find a piecewise continuous control, defined on the interval $[0, \infty)$, so that the model is brought to rest at t = 1 (that is, $x(t) = \vartheta$ for all $t \geq 1$).
- (d) Assume now that the model above is initially (at t = 0) at rest, and we wish to find a piecewise continuous control which will bring the state to $\bar{x} \in \mathbb{R}^n$ by time t = 1 and maintain that value for all $t \geq 1$. Such a control can always be found.
- **5.5.7** Consider the LTI model $\dot{x} = Ax + Bu$. Suppose that A has n distinct eigenvalues, and corresponding eigenvectors $\{v^1, \ldots, v^n\}$.

Show that if $\mathcal{R}(B) \subset \operatorname{Span}(v^1, \dots, v^{j-1}, v^{j+1}, \dots, v^n)$ for some j then the model is not controllable.

Hint: Look at the reciprocal basis vectors: How are r^j and $\mathcal{R}(B)$ related?

5.5.8 Compute the adjoint \mathcal{A}^* of the operator \mathcal{A} defined in (5.2), and then compute the composition

$$V = A \circ A^* \cdot \mathbb{R}^n \to \mathbb{R}^n$$

as an $n \times n$ matrix. How are the controllability grammian W and the operator V related? Comment on this result in view of Theorem 5.2 and Exercise 16 of Chapter 2.

5.5.9 Consider the *m*-input, *p*-output LTI model described by the convolution

$$y(t) = \int_{-\infty}^{\infty} g(t - \tau)u(\tau) d\tau,$$

where g is a $p \times m$ matrix-valued function.

Call the model output controllable if for any $y_0 \in \mathbb{R}^p$, $t_0 \in \mathbb{R}$, there exists a control u_0 such that $y(t_0) = y_0$.

- (a) Formulate output controllability as a range space dimension problem for a specific linear operator.
- (b) Compute the adjoint of the linear operator determined in (a).
- (c) Derive a finite dimensional test for output controllability, based on the rank of some $p \times p$ matrix (see the previous problem).
- **5.5.10** Prove Theorem **5.1**.

Chapter 6

Observability, Duality and Minimality

In many control problems we obtain measurements y(t), but we may not be able to measure directly all of the system states. For a complex system with a large number of states, it may be expensive to position a sensor to measure every state, and it may even be impossible to do so. In the light of this, the question we address in this chapter is the following: Is it possible to determine the state using only the input-output information $\{u(t), y(t)\}$? Informally, we say that some states are observable if one can determine them unambiguously based on these input-output measurements. This is obviously a very important property for a physical system.

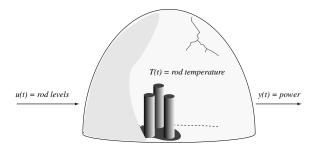


Figure 6.1: If the input to this plant does not affect the internal temperature, another actuator is needed. If the temperature cannot be estimated based on input-output measurements, another sensor is needed to directly measure the temperature.

To give a simple, if artificial example, consider the academic model of

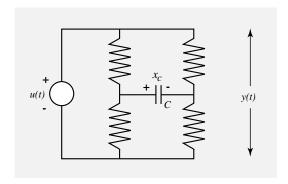


Figure 6.2: If the input to this circuit is zero, then the output is zero. Hence the charge across the capacitor is *not* observable.

a nuclear reactor given in Figure 6.1. Suppose that there is a state space model for this plant with the input equal to the rod levels, the output equal to the power generated, and one of the states equal to the rod temperature. Suppose first that using this control it is not possible to influence the temperature T. Then the system is not controllable, and for safe operation it will be necessary to add another actuator to raise or lower the internal temperature. This situation seems highly unlikely for a real reactor. Suppose now that given the input-output measurements, it is not possible to estimate the internal temperature T. We then say that the system is not observable. This again is a highly undesirable situation, though it can be resolved by adding another measurement - a sensor to directly measure T. Hence, an uncontrollable or unobservable plant can be "repaired" by expanding the number of inputs or outputs for the plant.

To take a more realistic example, consider the circuit given in Figure 6.2. It is obvious here that if the input voltage is zero, then the output voltage is zero, regardless of the voltage across the capacitor. Hence, the voltage x_c is not observable.

To give a formal definition of observability, consider the LTV model with an output equation y of the form

$$\dot{x}(t) = A(t)x + B(t)u;
y(t) = C(t)x + D(t)u.
x(t_0) = x_0.$$
(6.1)

Since we are assuming that the system is defined by a linear state space model, to determine x(t) for all t it is sufficient to compute the initial state $x_0 = x(t_0)$.

The model (6.1) is said to be observable at time t_0 if there exists a finite time $t_1 > t_0$ such that for any initial x_0 at t_0 , knowledge of the input u(t) and the output y(t) for $t_0 \le t \le t_1$ suffices to determine x_0 . Given the expression for the state

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^t \phi(t, \tau)B(\tau)u(\tau) d\tau$$

the output may be expressed as

$$y(t) = \underbrace{C(t)\phi(t, t_0)x_0}_{\text{unknown}} + \underbrace{C(t)\int_{t_0}^t \phi(t, \tau)B(\tau)u(\tau)\,d\tau + D(t)u(t)}_{\bar{u}(t) \text{ known}}.$$

Since \bar{u} is known, we can simplify matters by considering the known quantity

$$\bar{y}(t) = y(t) - \bar{u}(t) = C(t)\phi(t, t_0)x_0.$$
 (6.2)

Equivalently, when analyzing observability we may assume without loss of generality that the input is zero. Defining the linear operator $\mathcal{B} \colon \mathbb{R}^n \to D[t_0, t_1]$ as

$$\mathcal{B}(x) = f$$
, where $f(t) = C(t)\phi(t, t_0)x$,

we see that the model is observable if, for some t_1 , the operator \mathcal{B} is one to one.

6.1 The observability matrix

In the LTI case we have

$$\bar{y}(t) = Ce^{At}x_0 \qquad t \in [0, t_1],$$

where in this case we have taken $t_0 = 0$. We have some information about x_0 through the equation $y(0) = Cx_0$. However, typically C is not square, so inversion cannot be used directly to solve this equation for x_0 . Taking derivatives yields

$$\dot{\bar{y}}(t) = CAe^{At}x_0$$

$$\dot{\bar{y}}(0) = CAx_0,$$

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and this process can be continued until sufficient information has been generated to find x_0 :

$$\begin{bmatrix} \bar{y}(0) \\ \dot{\bar{y}}(0) \\ \ddot{\bar{y}}(0) \\ \vdots \\ \bar{y}^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0.$$

Defining the observability matrix as the $(np) \times n$ -matrix

$$\mathcal{O} := \left[\begin{array}{c} C \\ CA \\ CA^2 \\ \cdots \\ CA^{n-1} \end{array} \right],$$

where p is the dimension of y, we can now prove the following theorem.

Theorem 6.1. The LTI model (5.6) is observable if and only if

$$\operatorname{rank} \mathcal{O} = n$$
.

Proof If \mathcal{O} is full rank, then we have seen that x_0 can be computed using y and its derivatives.

Conversely, suppose that there is an $x_0 \in \mathbb{R}^n$, $x \neq \vartheta$, such that $\mathcal{O}x = \vartheta$. We then have $CA^kx_0 = \vartheta$ for any k, and hence also $Ce^{At}x_0 = \vartheta$ for all t. Since x_0 is indistinguishable from the state ϑ , it follows that the model is not observable.

The unobservable subspace $\Sigma_{\bar{o}}$ is defined to be the null space of the observability matrix. From the proof above, we see that for an initial condition $x(0) \in \Sigma_{\bar{o}}$, the output y is identically zero if the input is equal to zero.

6.2 LTV models and the observability grammian

In the LTV case we define the observability grammian as

$$H(t_1, t_0) := \int_{t_0}^{t_1} \phi^*(\tau, t_0) C^*(\tau) C(\tau) \phi(\tau, t_0) d\tau.$$
 (6.3)

Theorem 6.2. The model

$$\dot{x} = A(t)x, \quad y = C(t)x$$

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is observable at time t_0 if and only if there exists a finite time t_1 such that the observability grammian $H(t_1, t_0)$ has rank n.

Proof We first prove that non-singularity of the observability grammian is *sufficient* for observability:

(i) From (6.2) we have

$$\int_{t_0}^t \phi^*(\tau, t_0) C^*(\tau) \bar{y}(\tau) d\tau = \int_{t_0}^t \phi^*(\tau, t_0) C^*(\tau) C(\tau) \phi(\tau, t_0) d\tau x_0.$$

Thus

$$\underbrace{\int_{t_0}^{t_1} \phi^*(\tau, t_0) C^*(\tau) \bar{y}(\tau) d\tau}_{\text{known } n \times 1 \text{ vector}} = \underbrace{H(t_1, t_0)}_{\text{known } n \times n} x_0.$$
(6.4)

If H is nonsingular, x_0 is uniquely determined by (6.4), and we conclude that the model is observable. We now prove the converse:

(ii) Assuming the model is observable, we prove that H is nonsingular by contradiction. If H is singular, then there exists $\alpha \neq 0$ such that

$$H(t_1, t_0)\alpha = \vartheta$$
 and hence $\alpha^* H(t_1, t_0)\alpha = \vartheta$.

This may also be written as

$$\alpha^* \left(\int_{t_0}^{t_1} \phi^*(\tau) C^*(\tau) C(\tau) \phi(\tau) d\tau \right) \alpha = 0$$

or equivalently

$$\langle C\phi\alpha, C\phi\alpha\rangle_{L_2} = ||C\phi\alpha||_{L_2}^2 = 0.$$

This implies that $C(t)\phi(t)\alpha = 0$ for all $t_0 \le t \le t_1$. Since the two initial conditions $x(t_0) = x_0 + \alpha$ and $x(t_0) = x_0$ yield the same output $\bar{y}(t)$, we conclude that the model is not observable.

6.3 Duality

By exploiting the similarities between controllability and observability, we can generate further tests for observability. This approach is known as duality: observability and controllability are dual concepts. This can be made precise by defining a model which is "dual" to the model under study.

Given two models

(I)
$$\dot{x} = A(t)x + B(t)u$$

 $y = C(t)x + D(t)u, \quad x(t_0) \in \mathbb{R}^n.$

(II)
$$\dot{z} = -A^*(t)z + C^*(t)v$$

$$\gamma = B^*(t)z + D^*(t)v, \qquad z(t_0) \in \mathbb{R}^n.$$

The second model is called the dual of the first

In the LTI case we know that rank $(C_{II}) = n$ if and only if (II) is controllable, where

$$C_{II} = [C^* \mid A^*C^* \mid A^{*2}C^* \mid \cdots \mid A^{*n-1}C^*].$$

Since we have

$$\mathcal{C}_{II}^* = \left[\begin{array}{c} C \\ CA \\ CA^2 \\ \vdots \end{array} \right] = \text{observability matrix for model I,}$$

it is obvious that the controllability test for model II is equivalent to the observability test for I in the LTI case. This is a special case of the following result.

Theorem 6.3. For the general LTV model,

- (a) The LTV model (I) is controllable at t₀ if and only if (II) is observable at t₀.
- (b) The model (I) is observable at t_0 if and only if (II) is controllable at t_0 .

Proof We just prove (a) since the proof of (b) is identical.

Exercise 10 of Chapter 3 gives the relation

$$\phi_{II}(t_1, t_0) = \phi_I(t_0, t_1)^*.$$

Hence the controllability grammian for I and the observability grammian for II are identical:

$$W_{I} := \int_{t_{0}}^{t_{f}} \phi_{I}(t_{0}, \tau) B_{I}(\tau) B_{I}^{*}(\tau) \phi_{I}^{*}(t_{0}, \tau) d\tau$$

$$= \int_{t_{0}}^{t_{f}} \phi_{II}^{*}(\tau, t_{0}) C_{II}^{*}(\tau) C_{II}(\tau) \phi_{II}(\tau, t_{0}) d\tau$$

$$= H_{II}.$$

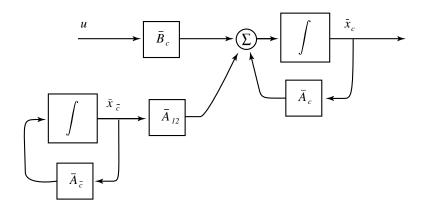


Figure 6.3: The Kalman Controllability Canonical Form

In view of Theorem 5.2 and Theorem 6.2 the proof of (a) is complete. \Box As a direct application of Theorem 6.3 and Theorem 5.6 we have the following dual of the Hautus-Rosenbrock test.

Theorem 6.4. The LTI model is observable if and only if

$$\operatorname{rank}\left[sI-A^*\mid C^*\right]=n\ \text{for any }s\in\mathbb{C}.$$

We then call a mode λ unobservable if rank $[\lambda I - A^* \mid C^*] < n$.

6.4 Kalman canonical forms

If the model is not controllable, then we can construct a state transformation P in which the state $\bar{x} = Px$ has the form $\bar{x} = \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix}$, as shown in Figure 6.3. After this transformation, the state space model takes the form

$$\dot{\bar{x}} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}^c \\ \bar{x}^{\bar{c}} \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}^c \\ \bar{x}^{\bar{c}} \end{bmatrix}.$$

This is called the Kalman Controllability Canonical Form (KCCF). In the case of distinct eigenvalues, the eigenvalues of (A_c) are precisely those for

which rank $[sI - A \mid B] = n$. The matrix P can be defined as

$$P^{-1} = \begin{bmatrix} \leftarrow n_1 \rightarrow & \leftarrow (n - n_1) \rightarrow \\ \text{linear independent} & \text{set of vectors} \\ \text{columns} & \text{linearly independent of} \\ \text{of } \mathcal{C} & \text{columns of } \mathcal{C} \end{bmatrix}$$

where rank $(C) = n_1 < n$ (see Brogan, §11.7).

Example 6.4.1. Consider the LTI model with

$$A = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The controllability matrix is $C = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$, which has rank one. Hence the model is not controllable. Define the matrix P by

$$P^{-1} = \left[\begin{array}{c|c} 1 & 1 \\ 1 & 0 \end{array} \right]$$

where the column $\begin{pmatrix} 1\\1 \end{pmatrix}$ was taken as the first column of the controllability matrix, and the column $\begin{pmatrix} 1\\0 \end{pmatrix}$ was chosen to make P^{-1} full rank. Thus,

$$P = \left[\begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right]$$

The Kalman controllability canonical form can be written as

$$\bar{A} = PAP^{-1} = \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix} \qquad \bar{B} = PB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For general LTI models, the observability canonical form can be obtained using duality: Using a state transformation, the transformed state may be written $\bar{x} = \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix}$ with

$$\dot{x} = \begin{bmatrix} A_O & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}^{\bar{o}} \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u$$

$$y = [C_o \ 0] \begin{bmatrix} \bar{x}^o \\ \bar{x}^{\bar{o}} \end{bmatrix}$$

This Kalman Observability Canonical Form (KOCF) is illustrated pictorially in Figure 6.4.

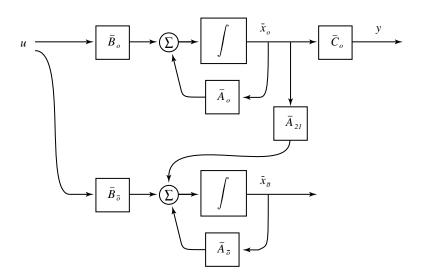


Figure 6.4: The Kalman Observability Canonical Form

6.5 State space models and their transfer functions

It should now be clear that given a transfer function, there are many possible state space descriptions. Moreover, the dimension of the realization is not unique since we can always append uncontrollable or unobservable dynamics. A state space realization of a transfer function P(s) is called *minimal* if it has minimum dimension over all state space models for which

$$C(sI - A)^{-1}B + D = P(s).$$

Theorem 6.5. A realization (A, B, C, D) of a transfer function P(s) is minimal if and only if it is both controllable and observable.

Proof For a general proof, see [5], #12.17, page. 439. We prove the theorem here in the special case where the eigenvalues of A are distinct. When the eigenvalues are distinct, we have the partial fraction expansion

$$P(s) = D + \sum_{i=1}^{u} \frac{k_i}{s - \lambda_i}$$

$$(6.5)$$

where in the MIMO case, the $\{k_i\}$ are $p \times m$ matrices. By considering the modal form, the transfer function can also be written as

$$P(s) = \bar{D} + \bar{C}(sI - \bar{A})^{-1}\bar{B}$$
(6.6)

where

This latter form will allow us to compute the gains $\{k_i\}$ in (6.5). First break up the matrices \bar{C} , \bar{B} into their component columns and rows, respectively:

$$\bar{C} = [\gamma^1 \cdots \gamma^n]$$

$$\bar{B} = [\beta^1 \cdots \beta^n]^T = \begin{bmatrix} \beta^{1T} \\ \vdots \\ \beta^{nT} \end{bmatrix}$$

Then the expansion (6.6) can be written explicitly as

$$P(s) = \bar{D} + \sum_{i=1}^{n} \frac{\gamma^{i} \beta^{iT}}{s - \lambda_{i}}.$$

The model is controllable if and only if each β_i is non-zero, and it is observable if and only if each γ_i is non-zero. From the equation above, it follows that the modal form is controllable and observable if and only if none of the poles λ_i are cancelled in this sum. That is, there are no pole/zero cancellations in the expression $C(sI-A)^{-1}B+D$. Thus, the transfer function has n poles, and hence any realization must have dimension no less than n.

Consider for example

$$P(s) = \frac{s+1}{s^2 + 6s + 5} = \frac{s+1}{(s+5)(s+1)}.$$

The second order modal realization is not minimal, but the first order realization is.

6.6 Realization of MIMO transfer functions

For a SISO model with transfer function P it is straightforward to obtain a minimal realization. First, cancel any common pole-zero pairs, count the number of poles remaining, say n, and construct any n-dimensional state space model. One can for example use the CCF. For a MIMO transfer function the situation is more complex. We refer the reader to [6] for details - here we provide an example to illustrate some of the issues involved.

Consider the transfer function P which describes a 2-input, 2-output model (m = p = 2):

$$P(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{-1}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix}.$$

By examining each entry of the matrix P we see that there are no direct pole-zero cancellations. Since there are two poles, one might expect that a realization can be constructed of degree two. On the other hand, since P consists of three first-order, and one second-order transfer functions, one might suspect that a degree four state space model will be required. It turns out that for this model both predictions are incorrect. The degree of any minimal realization is three, as we now show.

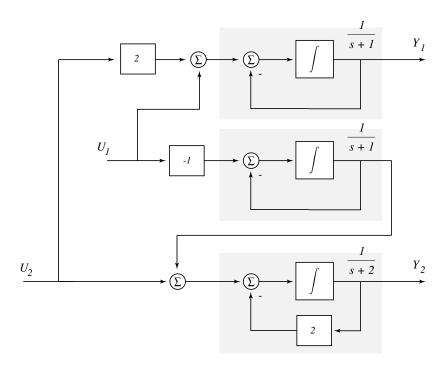


Figure 6.5: A block diagram realization of P.

The input-output description Y(s) = P(s)U(s) may be written as

$$Y_1(s) = \frac{1}{s+1}[U_1(s) + 2U_2(s)]$$

$$Y_2(s) = \frac{1}{s+2}[-\frac{1}{s+1}U_1(s) + U_2(s)]$$

Hence, using just three first order filters we may simulate this model using the block diagram description shown in Figure 6.5. Taking the outputs of the integrators as states, we obtain the following state space realization:

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} u
y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x.$$

By computing the controllability and observability matrices one can show that this model is both controllable and observable. Hence, this is a minimal realization.

A general approach to obtaining a minimal realization for a MIMO transfer function is to mimic the partial fractions expansion used in the SISO case to form the modal form. This will again yield a diagonal, minimal realization. For the transfer function P given above we may write

$$P(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 2(s+2) \\ -1 & s+1 \end{bmatrix}.$$

Performing a partial fractions expansion on each term we obtain

$$P(s) = \frac{1}{(s+1)} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + \frac{1}{(s+2)} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} := \frac{1}{(s+1)} R_1 + \frac{1}{(s+2)} R_2.$$

The rank of R_1 is two, and the rank of R_2 is one. The sum of these ranks gives the degree 3 of the minimal realization.

To compute the diagonal realization let

$$R_1 = \underbrace{C_1}_{2\times 2} \underbrace{B_1}_{2\times 2}$$
 $R_2 = \underbrace{C_2}_{2\times 1} \underbrace{B_2}_{1\times 2}$

For instance, we can take

$$R_{1} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_{2} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

With this factorization, the transfer function P may be expressed

$$P(s) = \frac{1}{(s+1)}C_1B_1 + \frac{1}{(s+2)}C_2B_2$$

$$= C_1 \begin{bmatrix} \frac{1}{(s+1)} & 0 \\ 0 & \frac{1}{(s+1)} \end{bmatrix} B_1 + C_2 \begin{bmatrix} \frac{1}{(s+2)} \end{bmatrix} B_2$$

$$= [C_1 \mid C_2] \begin{bmatrix} \frac{1}{(s+1)} & 0 & 0 \\ 0 & \frac{1}{(s+1)} & 0 \\ 0 & 0 & \frac{1}{(s+2)} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+1)} & 0 & 0 \\ 0 & \frac{1}{(s+1)} & 0 \\ 0 & 0 & \frac{1}{(s+2)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= C(Is - A)^{-1}B.$$

These values of A, B and C define a minimal realization of the transfer function.

In summary, when a simple partial fractions expansion is possible, then a minimal realization is obtained by a factorization of the matrix gains found in this expansion. In general, this approach might fail if one of the denominator polynomials in P possesses a repeated root. However, a minimal realization may still be found by a more general factorization. Write the transfer function P as

$$P(s) = N_R(s)D_R(s)^{-1}$$

where N_R , D_R are matrices whose entries are polynomial functions of s. This is known as a right factorization. A left factorization can be defined analogously: $P(s) = D_L(s)^{-1}N_L(s)$. Using the right factorization one may construct a generalization of the controllable canonical form, and using the left factorization an observable canonical form may be constructed. To ensure minimality, one must impose a co-primeness condition on the matrices N and D - for details, the reader is referred to [6].

6.7 Exercises

6.7.1 Consider the transfer function

$$P(s) = \frac{s+1}{s^2 + 3s + 2}$$

- (a) Obtain a second-order state space realization in controllable canonical form. Is your realization controllable? Observable?
- (b) Obtain a second-order state space realization in observable canonical form. Is your realization controllable? Observable?
- **6.7.2** Given the linear time-invariant model

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u = Ax + Bu$$

$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x = Cx$$

check observability using

- (a) The observability matrix
- (b) columns of $\bar{C}=CM$, where M is chosen such that $M^{-1}AM$ is diagonal
- (c) the Hautus-Rosenbrock test.
- **6.7.3** Prove the following assertion: If the state space model

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

is a minimal realization of the transfer function G(s), and if the model is BIBO stable, then the state space model $\dot{x} = Ax$ is asymptotically stable.

For example: Take the BIBO stable model Y(s) = (s-1)/[(s-1)(s+1)]U(s). Any second order realization will have characteristic polynomial $s^2 - 1$, and hence the second order model $\dot{x} = Ax$ cannot be asymptotically stable. A minimal realization will be first order with A = -1, so that $\dot{x} = Ax$ is asymptotically stable.

6.7.4 Show that the observability grammian given by (6.3) is equal to $\mathcal{B}^* \circ \mathcal{B}$, where $\mathcal{B}(x) = C(t)\phi(t, t_0)x$.

Part III Feedback

Chapter 7

Pole Placement

In this chapter, we discuss a number of feedback control issues for the LTI state space model

$$\dot{x} = Ax + Bu
y = Cx + Du.$$
(7.1)

We consider initially the regulation problem: By translating the state space, we may assume that we are regulating to the value $x = \vartheta$. To perform this translation we must assume that the DC gain of the plant is known exactly, and that there are no disturbances. We will develop more "robust" designs which work in spite of disturbances later in Section 8.1.

The following example serves to illustrate most of the concepts introduced in this chapter. Consider the special case of (7.1),

$$\dot{x} = Ax + Bu = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u
y = Cx = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$$
(7.2)

The system has three modes:

 $\lambda_1 = 1$ controllable and observable. $\lambda_2 = 2$ controllable but not observable. $\lambda_3 = -1$ observable but not controllable.

Define the control as u = -Kx + r. This is *linear state feedback*, with an auxiliary input r. With this controller, the *closed-loop system* becomes

$$\dot{x} = (A - BK)x + Br = A_{cl}x + Br
 y = Cx$$
(7.3)

where A_{cl} , the closed-loop system matrix, can be explicitly written as

$$A_{cl} = \begin{bmatrix} 1 - k_1 & -k_2 & -k_3 \\ -k_1 & 2 - k_2 & -k_3 \\ 0 & 0 & -1 \end{bmatrix}$$

with $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$. We now study properties of the closed-loop system (7.3), such as controllability, observability, stability, as a function of K.

Fact 1. The controllability matrix

$$\mathcal{C} = [B \mid A_{cl}B \mid A_{cl}^2B]$$

has rank 2 for all k_1, k_2, k_3 . Hence,

The rank of C is invariant under state feedback. The controllable subspace is invariant under state feedback.

Fact 2. Using the Hautus-Rosenbrock test, one may show that $\lambda = -1$ is the eigenvalue corresponding to the uncontrollable mode of A_{cl} , independent of K. The other two modes of A_{cl} are controllable for all K, but their values depend explicitly on K. Hence,

Uncontrollable modes are invariant under state feedback.

Fact 3. The observability matrix for the controlled system is

$$\mathcal{O} = \begin{bmatrix} C \\ CA_{cl} \\ CA_{cl}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 - k_1 & -k_2 & -k_3 - 1 \\ (1 - k_1)^2 + k_1 k_2 & -k_2 (3 - k_1 - k_2) & 1 + k_3 (k_1 + k_2) \end{bmatrix}.$$

This matrix does not have a fixed rank. It could be

one (corresponding to $k_1 = 1, k_2 = 0, k_3 = -1$),

two (corresponding to $k_1 = k_2 = k_3 = 0$), or

three (corresponding to $k_1 = k_3 = 0, k_2 = -1$)

Hence,

Observability is not invariant under state feedback.

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Fact 4. The eigenvalues of A_{cl} can be found by solving the characteristic equation

$$\det(\lambda I - A_{cl}) = 0 \Rightarrow (\lambda + 1)(\lambda^2 - (3 - k_1 - k_2)\lambda + 2 - 2k_1 - k_2) = 0.$$

One eigenvalue is $\lambda = -1$, the other two depend on k_1 and k_2 . Since the two terms $k_1 + k_2$ and $2k_1 + k_2$ are linearly independent, these two eigenvalues can be chosen arbitrarily (subject to the restriction that complex eigenvalues can appear only as conjugate pairs) by proper choices of real scalars k_1 and k_2 . Hence,

Arbitrary pole placement is possible by linear state feedback, in the controllable subspace.

The LTI model (7.2) is *stabilizable* by linear state feedback, since the uncontrollable mode is asymptotically stable.

After gaining an understanding of the state feedback control problem we will turn to output feedback of the form u = -Gy + r. Even if an LTI system is both controllable and observable, it is in general not possible to assign the closed-loop poles arbitrarily by static output feedback. We will see shortly that all poles can be assigned arbitrarily (subject to the complex conjugate pair restriction) if we use dynamic output feedback, where G is a transfer function rather than a static matricial gain.

7.1 State feedback

Returning to the regulation problem, assuming that we can measure all of the states, the simplest controller is given by the state feedback control law

$$u = -Kx$$
,

where K is a $n \times m$ matrix. This gives rise to the closed-loop system

$$\dot{x} = (A - BK)x = A_{cl}x.$$

To determine whether or not $x(t) \to \vartheta$ from any initial condition, we must consider the eigenvalues of the closed-loop matrix A_{cl} .

Consider the SISO case, where we initially assume that the pair (A, B)

is in CCF (controllable canonical form):

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & \cdots & -\alpha_1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Recall that

(a) The characteristic polynomial of this model is

$$\Delta(s) = \det(sI - A) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$$

(b) (A, B) is a controllable pair.

Writing the control as $u = -k_1x_1 - k_2x_2 \cdots - k_nx_n$, it is immediate that the closed-loop system has the form

$$A - BK = A_{cl} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -(\alpha_n + k_1) & -(\alpha_{n-1} + k_2) & \cdots & \cdots & -(\alpha_1 + k_n) \end{bmatrix},$$

so that the closed-loop characteristic polynomial is

$$\Delta(s) = \det(sI - [A - BK]) = s^n + (\alpha_1 + k_n)s^{n-1} + \dots + (\alpha_n + k_1).$$
 (7.4)

We can assign the coefficients of this polynomial, and hence the closed-loop poles can be assigned arbitrarily.

This analysis can be applied to any controllable SISO system. It is performed on a system in CCF, but we can transform to CCF for any time invariant SISO state space model, provided the model is controllable. The question is, what is the state transformation P that is required to ensure that $\bar{x} = Px$ is described by a model in CCF? The controllability matrices \mathcal{C} and $\bar{\mathcal{C}}$ are related by the formula

$$\bar{C} = PC$$
.

Hence the matrix P is given by $P = \bar{\mathcal{C}}\mathcal{C}^{-1}$. The inverse exists if the original state space model is controllable. System eigenvalues do not change with

a state transformation, so by converting to CCF and then applying pole placement to the transformed model, one can place the poles as desired. This is summarized in the following theorem. While this result was derived for SISO systems, it is also true in the MIMO case.

Theorem 7.1. The eigenvalues of (A - BK) can be placed arbitrarily, respecting complex conjugate constraints, if and only if (A, B) is a controllable pair.

All of this is applicable if (A, B) is controllable. What do we do if (A, B) is not controllable? Consider the Kalman controllability canonical form introduced previously in Figure 6.3, and given by the equations below

$$\dot{x} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} x + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u$$

$$x = \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix}.$$

We can still apply a control law of the form

$$u = -[K_1 \ K_2] \left[\begin{array}{c} x_c \\ x_{\bar{c}} \end{array} \right]$$

to obtain the closed-loop system

$$\dot{x} = \begin{bmatrix} A_c - B_c K_1 & A_{12} - B_c K_2 \\ 0 & A_{\bar{c}} \end{bmatrix} x \tag{7.5}$$

From the form of the closed-loop system, we find that the eigenvalues are dependent only upon $(A_c - B_c K_1)$ and not on $A_{\bar{c}}$. In fact, the characteristic polynomial for the closed-loop system becomes

$$\Delta(s) = \det(sI - A_{cl}) = \underbrace{\det(sI - (A_c - B_cK_1))}_{\text{arbitrary Based on } K} \underbrace{\det(sI - A_{\bar{c}})}_{\text{independent of } K}$$

In other words, one can place the controllable modes arbitrarily, but the uncontrollable modes remain unchanged.

The pair (A, B) is said to be *stabilizable* if there exists a K such that (A - BK) is a Hurwitz matrix. From these arguments we see that

Theorem 7.2. If the eigenvalues are distinct, then (A, B) is stabilizable if and only if the eigenvalues of the uncontrollable modes are in the strict LHP.

For a general LTI model, (A, B) is stabilizable if and only if the eigenvalues of $A_{\bar{c}}$ lie in the strict LHP, where $A_{\bar{c}}$ is the matrix defined in the KCCF.

Geometrically, Theorem 7.2 implies that the model is stabilizable if and only if the unstable subspace is contained in the controllable subspace.

7.2 Observers

The state feedback approach can be generalized to the situation where only partial measurements of the state are available. Consider again the LTI model

$$\dot{x} = Ax + Bu,
 y = Cx.$$

We would like to define an observer, of the form illustrated in Figure 7.1, in which input-output measurements are collected on-line to give estimates $\hat{x}(t)$ of the state x(t). To mimic the behavior of the system one can try

$$\dot{\widehat{x}}(t) = A\widehat{x}(t) + Bu(t).$$

Defining the error as $\widetilde{x}(t) = x(t) - \widehat{x}(t)$, this gives the error equation $\dot{\widetilde{x}}(t) = A\widetilde{x}(t)$, from which we deduce that

$$\widetilde{x}(t) = e^{At}\widetilde{x}(0).$$

This is a poor approach since there is no flexibility in design. If for example the open-loop system is unstable, then for some initial conditions the error will not converge to zero, and may diverge to infinity.

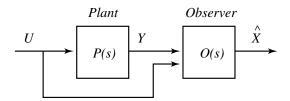


Figure 7.1: Can the observer \mathcal{O} be designed so that $\hat{x}(t) - x(t) \to 0$, as $t \to \infty$, at a predesignated rate?

The observers we consider are of a similar form, but we adjoin an output error term:

$$\dot{\widehat{x}} = A\widehat{x} + Bu + L(y - \widehat{y}), \qquad \widehat{x}(0) \in \mathbb{R}^n,$$

where $\hat{y} = C\hat{x}$. For any fixed $n \times p$ matrix L one obtains

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu - L(Cx - C\hat{x}) = (A - LC)e.$$

To ensure that $\widetilde{x}(t) \to 0$ as $t \to \infty$, we must compute the eigenvalues of the matrix (A - LC) as a function of L. But note:

$$eig(A - LC) = eig(A^* - C^*L^*)$$

We are thus exactly in the same position as when we considered pole placement. It follows that the eigenvalues of (A - LC) can be placed arbitrarily, provided that the matrix pair (A^*, C^*) is controllable. Based on duality, this is simply observability of the pair (A, C). Thus, we can place the observer poles arbitrarily if and only if (A, C) is observable.

This again raises the question, what do we do if this basic assumption fails? The pair (A, C) is said to be *detectable* if there exists some L such that (A - LC) is a Hurwitz matrix. This property is the dual of stabilizability, in the sense that (A, C) is detectable if and only if (A^*, C^*) is stabilizable. We then obtain the following dual statement to Theorem 7.2:

Theorem 7.3. If the eigenvalues are distinct, then (A, C) is detectable if and only if the eigenvalues corresponding to the unobservable modes lie in the strict LHP.

For a general LTI model, (A, C) is detectable if and only if the eigenvalues of $A_{\bar{o}}$ lie in the strict LHP, where $A_{\bar{o}}$ is the defined in the KOCF.

A state space model is thus detectable if and only if the unobservable subspace is contained in the stable subspace.

7.3 Observer feedback

The objective in the previous section was to construct a useful state estimator for a state space model, but this of course was rooted in the desired to control the system. Suppose that we ignore that \hat{x} is an estimate, and we apply the observer feedback control law

$$u(t) = -K\widehat{x}(t).$$

This control law is designed based on the faith that $\tilde{x}(t)$ will converge to zero fast enough so that this control law is essentially equivalent to full state feedback. To see if this faith is in vain, we must first see if the overall system

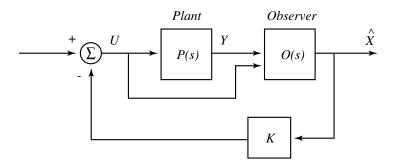


Figure 7.2: The separation principle justifies the use of state estimates in place of the true state values in the feedback control law. However, full state feedback is not completely equivalent to feedback based on state estimation.

is asymptotically stable. The overall system is linear, so we can check this by computing the closed-loop eigenvalues.

First apply the following state transformation for the overall state

$$\begin{bmatrix} x \\ e \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}}_{P} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \qquad e = x - \hat{x}$$
$$\hat{x} = x - e.$$

The transformed state $\begin{pmatrix} x \\ e \end{pmatrix}$ is described by the closed-loop equations

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}}_{A \cdot i} \begin{bmatrix} x \\ e \end{bmatrix}$$

Thus, the closed-loop poles of the overall system are defined by the characteristic polynomial

$$\det(sI - A_{cl}) = \det(sI - [A - BK]) \cdot \det(sI - [A - LC]) \tag{7.6}$$

where

- (a) $det(sI [A BK]) \equiv state$ feedback eigenvalues arbitrary placement if and only if (A, B) is controllable.
- (b) $det(sI [A LC]) \equiv observer$ eigenvalues arbitrary placement if and only if (A, C) is observable.

Equation (7.6) is known as the *separation principle*. Its obvious consequence to control design is that the design of the feedback gain can be conducted independently of the design of an observer. One must be cautious when interpreting this result however, since we are only considering stability and the placement of closed-loop poles. In particular, we are ignoring such issues as sensitivity to plant uncertainty.

The separation principle allows us to separately place the state feedback eigenvalues for good dynamic response, and the observer-eigenvalues so they are faster than the desired closed-loop response. This still leaves open the question, where do we place the observer poles? A typical rule of thumb is that the observer poles should be faster than the "slowest" state feedback pole by a factor of 2 to 5. This however is very problem specific. In some applications, the location of the observer poles is not very important, so long as the poles are faster than those of the state feedback poles. In other examples, the observer must be designed with care.

7.4 Reduced-order (Luenberger) observers

Considering the state space equations (7.1) we see that at least part of the state is directly observed through y. Assume for simplicity that D=0, and construct a state transformation $\bar{x}=Px$ so that y is explicitly a part of the state. This can be accomplished if C has rank p, since then we can define the matrix P so that

$$\bar{x} = \begin{bmatrix} y \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{C}{\text{anything for}} \end{bmatrix} x.$$

Then we do have a transformed model in the desired form:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{PAP^{-1}} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{PB} u.$$

Since by definition we have $\bar{x}_1 = y$, the output equation is evidently

$$y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}.$$

The state estimation problem has been reduced in complexity through this transformation: rather than construct an estimator of dimension n, we only have to estimate the (n-p) dimensional vector \bar{x}_2 . The next step then is to construct an observer for \bar{x}_2 alone. To see how this can be accomplished, write the transformed state equations as follows. The equation for \bar{x}_2 may be written in the suggestive form,

$$\dot{\bar{x}}_2 = A_{22}\bar{x}_2 + \underbrace{A_{21}y + B_2u}_{\text{known from}}.$$
(7.7)

The equation for \bar{x}_1 can be written

$$\underbrace{\dot{y} - A_{11}y - B_1 u}_{\text{known from}} = A_{12}\bar{x}_2 \tag{7.8}$$

Defining $\bar{u} = A_{21}y + B_2u$ and $\bar{y} = \dot{y} - A_{11}y - B_1u$, we obtain the state space model

$$\dot{\bar{x}}_2 = A_{22}\bar{x}_2 + \bar{u}$$

$$\bar{y} = A_{12}\bar{x}_2$$

We can now write down the corresponding observer to estimate \bar{x}_2 . The only technicality is that \bar{y} contains the derivative \dot{y} , which strictly speaking is not known. We will assume for now that we know \dot{y} , but through some manipulations of the observer equations we will relax this assumption below. An observer for \bar{x}_2 takes the form

$$\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + \bar{u} + L(\bar{y} - \hat{y}) \tag{7.9}$$

where $\hat{y} = A_{12}\hat{x}_2$. Since $\bar{y} = A_{12}\bar{x}_2$, the error equation for $e := \bar{x}_2 - \hat{x}_2$ takes the form

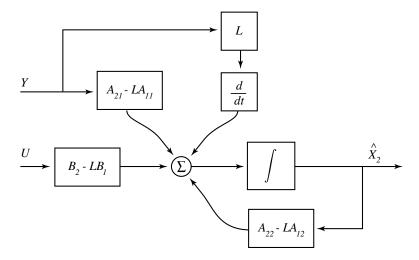
$$\dot{e} = \dot{\bar{x}}_2 - \dot{\hat{x}}_2 = (A_{22} - LA_{12})e$$

So we can place the observer poles arbitrarily if and only if the pair (A_{22}, A_{12}) is observable.

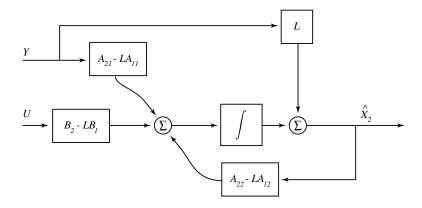
We now show that it is unnecessary to differentiate the output measurements. First write the observer equations in the expanded form

$$\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + A_{21}y + B_2u + L(\dot{y} - A_{11}y - B_1u - A_{12}\hat{x}_2).$$

These equations can be represented through the block diagram



It is evident in this diagram that the term $L\dot{y}$ is immediately integrated. To eliminate the derivative, we simply cancel the derivative and integral to obtain the equivalent block diagram



We then obtain an estimate \hat{x} of the original state through

$$\begin{array}{cccc}
Y & & & & \\
& \hat{X}_2 & & & & \\
\end{array}$$

$$\begin{array}{cccc}
& & \hat{X} & & \\
& & \hat{X} & & \\
\end{array}$$

In conclusion, provided (A_{22}, A_{12}) is an observable pair, the state can be estimated based upon an (n-p)-dimensional observer using the measurements u, y, and it is not necessary to differentiate these measurements. This is important, since if the measurement y is noisy, its derivative may be much worse.

If we define

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} I & 0 \end{bmatrix}$$

then by the Hautus-Rosenbrock test we know that (\bar{A}, \bar{C}) is observable if and only if

$$\operatorname{rank} \left[\begin{array}{c} sI - \bar{A} \\ \bar{C} \end{array} \right] = n \qquad \text{for any } s \in \mathbb{C}.$$

This matrix may be written as

$$\begin{bmatrix} sI - A_{11} & -A_{12} \\ -A_{21} & sI - A_{22} \\ I & 0 \end{bmatrix}.$$

The first p columns are automatically independent because of the identity matrix I. Because of the 0, the matrix is full rank if the last n-p columns are independent of each other. That is, we must have

$$\operatorname{rank} \left[\begin{array}{c} -A_{12} \\ sI - A_{22} \end{array} \right] = (n - p).$$

This condition holds if (A_{22}, A_{12}) is observable, since this is the Hautus test on the reduced complexity observer problem. We conclude that the condition for observability of the reduced order observer is equivalent to observability of (\bar{A}, \bar{C}) , which is equivalent to observability of the original pair (A, C).

Example 7.4.1. Consider the magnetically suspended ball, whose linearized and normalized state space model is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Full state feedback is simply proportional-derivative (PD) feedback:

$$u = -Kx = -K_1y - K_2\dot{y}.$$

If the derivative cannot be measured directly, then there are several possibilities:

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- (a) The second state $x_2 = \dot{y}$ could be obtained by differentiating y, but this will greatly amplify any high-frequency noise.
- (b) An approximate derivative could be obtained through a filter:

$$\dot{\hat{x}}_2 + \ell \hat{x}_2 = \ell \dot{y},$$

or in the frequency domain

$$\dot{\hat{X}}_2(s) = \frac{\ell s}{s + \ell} Y(s),$$

where ℓ is a large constant. The trouble with this approach is that it does not make use of the system model, and for this reason one cannot expect that $\hat{x}_2(t) - \dot{y}(t) \to 0$ as $t \to 0$.

(c) A reduced order observer can be constructed to estimate \dot{y} . If there is noise, the observer can be designed to obtain a good trade off between tracking of \dot{y} , and rejection of this noise. In the noise free case we have that $\hat{x}_2(t) - \dot{y}(t) \to 0$, as long as the observer gain is chosen to give a stable observer.

The reduced order observer is defined by the equations

$$\dot{\hat{x}}_2 = x_1 + u + \ell(\dot{x}_1 - \hat{x}_2).$$

Taking transforms gives a formula which is similar to the estimator (b) given above:

$$\dot{\hat{X}}_2(s) = \frac{\ell s + 1}{\ell + s} X_1(s) + \frac{1}{\ell + s} U(s).$$

However, this observer does make use of system information, and hence does exhibit desirable convergence properties.

The controller can be written in the frequency domain as

$$U(s) = -K_1 X_1(s) - K_2 \frac{\ell s + 1}{\ell + s} X_1(s) - K_2 \frac{1}{\ell + s} U(s).$$

Solving for U, it is readily seen that for large ℓ this is a lead compensator. That is, the above expression may be written as

$$U(s) = -g\frac{s-z}{s-p}Y(s)$$

where p < z < 0. In conclusion, we see that in simple models, observer based feedback generalizes classical control design.

Matlab Commands

- In Matlab there are two methods for assigning eigenvalues. These can be used to obtain the state feedback gain K, or in observer design by placing the eigenvalues of $(A^* C^*L^*)$ to compute the observer gain L.
- **ACKER** This command is named after Ackerman, who derived a formula for explicit solution of the feedback gain K using the controller canonical form. This is only suitable for SISO plants.
- **PLACE** A more numerically robust algorithm which can be used for MIMO systems. This command cannot be used with repeated roots.

7.5. EXERCISES

7.5 Exercises

7.5.1 For the LTI model

$$\dot{x} = \begin{bmatrix} -1 & 2\\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1\\ 2 \end{bmatrix} u$$

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- (a) Find the controllable subspace.
- (b) Show that the system is unstable, and compute the unstable modes and corresponding eigenvector.
- (c) Is the system stabilizable? Explain carefully using (a) and (b). If the system is stabilizable, find a stabilizing state feedback control law.

7.5.2 Consider the SISO, LTI system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u; \qquad y = \begin{bmatrix} 7 & 8 \end{bmatrix} x$$

- (a) There is a matrix P such that $(\bar{A}, \bar{B}) = (PAP^{-1}, PB)$ is in controllable canonical form. Compute P and P^{-1} using the formula $P = \bar{\mathcal{C}}\mathcal{C}^{-1}$.
- (b) Compute by hand a feedback control law $\bar{K} = [\bar{k}_1, \bar{k}_2]$ which places the eigenvalues of $\bar{A} \bar{B}\bar{K}$ at -2 (twice).
- (c) Let $K = \bar{K}P$. Verify that the eigenvalues of A BK are at -2.

7.5. Given the system

$$\dot{x} = \begin{bmatrix} -1 & 4\\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 2\\ 1 \end{bmatrix} u,$$

- (a) Show that when $u \equiv \vartheta$ the origin is unstable.
- (b) Determine the stable subspace Σ_s , the span of the eigenvectors corresponding to unstable modes, Σ_u , and the controllable subspace Σ_c . Plot all on the same graph.
- (c) Now choose the feedback control

$$u = (1 \beta)x + r$$

where β is a scalar, and r is a reference input. Show that the closed-loop system will be asymptotically stable for some value(s) of β .

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- (d) What are the closed-loop poles? Do different choices of β affect the stable mode?
- **7.5.4** Consider the single input/single output second-order linear time-invariant system

$$\dot{x} = Ax - \begin{bmatrix} 0 \\ a \end{bmatrix} u$$

$$y = (1, b)x$$

where a and b are scalar real parameters.

It is known that for some constant c > 0,

$$A^{T}\begin{bmatrix} 1 \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ -2c \end{bmatrix}, \quad \text{and} \quad A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix},$$

where A^T denotes the transpose of A. Also, it is known that λ , one of the eigenvalues of A, is strictly positive.

- (a) Compute the left eigenvector and the right eigenvector of A corresponding to the positive eigenvalue λ . Hint: left eigenvectors are reciprocal basis vectors for the basis consisting of right eigenvectors.
- (b) For what values of (a, b) is the system observable?
- (c) For what values of (a, b), if any, is this system BIBO (bounded-input bounded-output) stable?

Explain your answers using appropriate system structural concepts. It may help to draw pictures of left and right eigenvectors: Think geometrically!

7.5.5 This problem is a follow-up to Exercise 5 of Chapter 5. Consider again the LTI model

$$\dot{x} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u
y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x.$$

Suppose now that the initial condition of the model is not known, and that x(t) is not observed directly. Is it possible to choose a control so that $y(t) = te^{-3t}$ for t > 1?

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7.5.6 The system $\dot{x} = Ax + Bu$; y = Cx is controlled using static output feedback u = -Hy + v. Show that the resulting closed-loop system

$$\dot{x} = (A - BHC)x + Bv; \qquad y = Cx$$

has the same controllability/observability properties as the original system. *Hint: use the Hautus-Rosenbrock test.*

7.5.7 In this problem you will see that the feedback gain K may not be unique for multivariate models. Consider the state space model defined by the pair of matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

(a) Find vectors $w^a, w^b \in \mathbb{R}^2$ such that each pair $(A, B_a), (A, B_b)$ is controllable, where

$$B_a = Bw^a \qquad B_b = Bw^b.$$

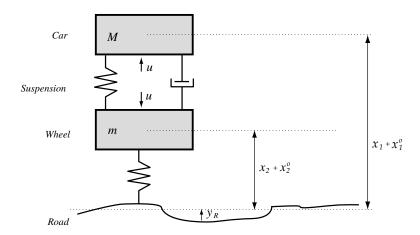
Try to pick the vectors so that the two 3×1 matrices B_a, B_b are significantly different.

(b) Now, find two controllers of the form

$$u_a = -w^a K_a x \qquad u_b = -w^b K_b x$$

so that in each case, the closed-loop poles are placed at -1 and $-1 \pm j$. These designs should be performed by placing the poles of $A - B_a K_a$, $A - B_b K_b$, respectively.

- (c) Provide simultations of the step response $u(t) = -Kx(t) + (1,1)^T$ with zero initial conditions for the two designs. Include in your plots both $x_1(t)$ and the two dimensional input u(t), for $0 \le t \le 1$.
- **7.5.8** A schematic of an active suspension system is illustrated below.



A state space model in scientific units is given below, where $v_i = x_i'$. The states x_i have been normalized so that their nominal values are 0.

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 10 & -2 & 2 \\ 60 & -660 & 12 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3.34 \\ -20 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 600 \end{bmatrix} y_R$$

- (a) Obtain an open-loop simulation of x_1 and x_2 for $0 \le t \le 5$ s., with $x_1(0) = 0.5 m$; $x_2(0) = 0 m$; $v_1(0) = v_1(0) = 0$; $u(t) = y_R(t) \equiv 0$.
- (b) Repeat(a), but let y_R be a square wave of amplitude 0.2 m, and fundamental frequency of 0.2 Hz. You may take zero initial conditions.
- (c) Obtain a state feedback control law for a comfortable ride, but make sure that the car does not bottom-out Given the normalized state variables, this means that $x_1(t) x_2(t) > -0.5m$. You should obtain a critically damped response for $X_1(s)/Y_R(s)$, and you should also speed up the settling time to approximately half the open loop response.

Show simulations under the conditions of (a) and (b) above.

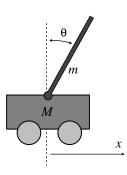
- (d) Full state feedback is not feasible, since it is not practical to measure the distance from the car to the road. Letting $y = x_1 x_2$ denote the actual measurement available to the controller, is the resulting system observable?
- (e) Repeat (c) with a full order observer, where the measurement is taken to be y.

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7.5.9 The linearized (and normalized) magnetically suspended ball is described by

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

- (a) Show that the system is unstable with u = 0.
- (b) Explain why it is possible to place the poles of the system at arbitrary locations (with the restriction of conjugate pairs) by linear state feedback.
- (c) Find a state feedback which would place the poles of the closed-loop system at $-1 \pm j1$.
- (d) Simulate your controller for the nonlinear model $\ddot{y} = 1 y^2/u^2$, with the nominal values $u_0 = y_0 = 1$, and experiment with initial conditions to find a region of asymptotic stability for the controlled system.
- **7.5.10** In the previous problem, the ball position x_1 can be measured using a photocell, but the velocity x_2 is more difficult to obtain. Suppose, therefore, that the output is $y = x_1$.
 - (a) Design a full-order observer having eigenvalues at -5, -6, and use the observer feedback to produce closed-loop eigenvalues at $-1 \pm j1, -5, -6$.
 - (b) Simulate your controller for the nonlinear model $\ddot{y} = 1 y^2/u^2$, as in the previous problem, and again experiment with initial conditions to find a region of asymptotic stability.
- **7.5.11** You will now construct a reduced-order observer for the previous model.
 - (a) Repeat Exercise 10 using a reduced-order observer to yield closed-loop eigenvalues at $-1 \pm j1$ and -5.
 - (b) Letting $z = \dot{y}$ in (a), compute the transform $\ddot{Z}(s)$, and show that it is approximately equal to sY(s) when the observer gain is large.
- **7.5.12** Below is a drawing of a cart of mass M with a uniform stick of mass m pivoted on top:



In appropriate dimensionless units, the equations of motion may be written as

$$\ddot{\theta} = \theta + u$$
, $\ddot{x} = -\beta\theta - u$,

where $\beta := \frac{3}{4}[m/(M+m)]$ is a parameter of the system, and u is the torque applied to the wheels of the cart by an electric motor. We wish to find a linear feedback control that will balance the stick (i.e., keep $\theta \approx 0$) and keep the cart near x=0. To do this, find the gains k_1, k_2, k_3 and k_4 in the state-variable feedback

$$u = k_1\theta + k_2\dot{\theta} + k_3x + k_4\dot{x}$$

such that the closed-loop system has a double pole at s=-1 and a pair of complex poles at $s=-1\pm j1$

7.5.13 On the line connecting the center of the earth to the center of the moon, there is a so-called *libration point* where the pull of the earth on a satellite (in an orbit about the earth with the same period as the moon's orbit) exactly equals the pull of the moon plus the centrifugal force. The dynamic equations for small deviations in position away from the libration point can be shown to be:

$$\ddot{x} - 2\omega \dot{y} - 9\omega^2 x = 0$$
$$\ddot{y} + 2\omega \dot{x} + 4\omega^2 y = u$$

where x:= radial position perturbation, y:= azimuthal position perturbation, $u = F/m\omega^2$ (control exerted by a small reaction engine), F:= engine thrust in the y direction, m:= satellite mass, and $\omega := 2\pi/29$ rad/day.

(a) With u = 0, show that the equilibrium point $x \equiv y \equiv 0$ is unstable.

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(b) To stabilize the position, one can use state feedback

$$u = k_1 x + k_2 \dot{x} + k_3 y + k_4 \dot{y}.$$

Show that it is possible to stabilize the system with a control law of this form. Determine the constants $k_1, ..., k_4$ such that the closed-loop system has poles at $s = -3\omega, s = -4\omega$, and $s = (-3 \pm i3)\omega$.

7.5.14 Given the LTI system: $\dot{x} = Ax + Bu$, y = Cx, where

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \beta \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

- (a) Determine the range of values of the scalar real parameter β for which we can place 2 poles arbitrarily, using static output feedback (as usual, we assume that if the two poles are complex then they should be in conjugate pairs).
- (b) for $\beta = 1$, determine static output feedback so that two of the eigenvalues of the closed-loop system will be $\lambda_{1,2} = -1 \pm j1$.

7.5.15 Consider the linear system with transfer function

$$P(s) = \frac{s + 28}{(s + 27)(s + 29)}$$

We would like to find a control law which places the poles at (-29, -28), so that the zero is canceled.

- (a) Place the system in modal canonical form, and apply the feedback control law $u = -K_m x + r$. Compute K_m for the desired closed-loop pole locations.
- (b) Place the system in observable canonical form, and apply the feedback control law $u = -K_o x + r$. Compute K_o for the desired closed-loop pole locations.
- (c) Why do you think the gains are larger in the second case.

7.5.16 Design a first-order controller of the form

$$G(s) = K \frac{(s+a)}{(s+b)}$$

(K, a, b free parameters) for the plant with transfer function

$$P(s) = \frac{s + 28}{(s + 27)(s + 29)}$$

to place poles at (-30.5, -31, -31.5). Repeat this for poles at (-31, -32, -33). Note that the two controllers are quite different, even though the closed-loop poles differ only slightly.

7.5.17 Design a first-order controller of the form

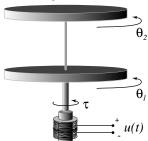
$$G(s) = K \frac{(s+a)}{(s+b)}$$

(K, a, b free parameters) for the plant with transfer function

$$P(s) = \frac{s - \alpha}{(s+4)}$$

to place poles at $-10 \pm 3j$, where $\alpha > 0$ is a scalar parameter. Plot the magnitude of the sensitivity function $S = \frac{1}{PG}$ for several values of α ranging from 1 to 100. For what values of α is the controlled system most sensitive to plant uncertainty?

7.5.18 Below is a diagram of a flexible structure found in the undergraduate controls lab at the University of Illinois



An accurate state space model can be found, which is defined by the following matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -500.3513 & -2.626 & 500.3513 & 0.3972 \\ 0 & 0 & 0 & 1 \\ 558.964 & 0 & -558.964 & -2.1267 \end{bmatrix}; \qquad B = \begin{bmatrix} 0 \\ 299.535 \\ 0 \\ 0 \end{bmatrix}$$
(7.10)

The input is the voltage to a DC motor, and the states of this model are

$$x_1 = \theta_1, x_2 = \dot{\theta}_1, x_3 = \theta_2, x_4 = \dot{\theta}_2.$$

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(a) Make a Simulink model for this system and plot the open-loop response $\theta_2(t)$ to a non-zero initial condition with $\theta_1(0) \neq \theta_2(0)$.

- (b) Design a full state feedback controller u=-Kx+gr to place the poles in the region $-10 \le \sigma \le -5$, and choose the gain g so that the DC gain of the closed-loop system from r to θ_2 is one. Plot step responses for your controlled system, with $r=\pi/2$ radians. In one step response, use zero initial conditions, in the other, choose the same initial conditions as in (a). Design K so that the overshoot is less than 20%.
- (c) Make a Simulink model for an observer, assuming only θ_1 is available through direct measurements. Combine the observer, your full state feedback controller, and plant as on page 129 of the lecture notes. After designing appropriate observer gains, obtain two step responses of the controlled system, as in (b).
- (d) Repeat (c) with a reduced order observer, assuming only θ_1 is directly measured.

Chapter 8

Tracking and Disturbance Rejection

8.1 Internal model principle

Tracking and disturbance rejection are two of the basic goals in control design. This chapter addresses each of these issues by applying the state space theory developed so far.

Assume that one wishes to make the output y track a constant reference input r. One approach is to modify the state feedback control law

$$u = -Kx + Nr$$

where N is chosen based upon the DC gain of the system. Using the final value theorem

$$y(\infty) = \lim_{s \to 0} sY(s)$$

a formula for N is easily obtained. This is a reasonable approach if the DC gain of the plant is known. However, the scaling matrix N does not account for disturbances and parameter variations that will hamper tracking.

To obtain a solution which is robust to DC disturbances we explicitly model the unknown constant disturbance as follows:

$$\dot{x} = Ax + Bu + Ew
y = Cx + Fw,$$

where $w \in \mathbb{R}^n$ is a constant. Our goal then is to make $e(t) = y(t) - r \to 0$, as $t \to \infty$, regardless of the value of the disturbance w, or the reference

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input r. The approach taken allows us to achieve these three objectives simultaneously:

- (a) Asymptotic tracking: $y(t) \to r$ as $t \to \infty$.
- (b) Complete insensitivity to w.
- (c) Tuned transient response through pole placement.

Conceptually, the approach taken is to note that the exogenous signal $\begin{pmatrix} r \\ w \end{pmatrix}$ is generated by a state space model

$$\dot{z} = A_m z \qquad z(0) \in \mathbb{R}^q;
\begin{pmatrix} r \\ w \end{pmatrix} = C_m z$$

where in this special case of constant disturbances and a constant reference input, the matrix A_m is equal to zero. This reference/disturbance model is simply an integrator, and based upon this we incorporate an integrator in the control law. Thus, the control law we adopt is of the general form

$$u = -K_1 x - K_2 \eta, \qquad \dot{\eta} = e = y - r.$$

This procedure is known as the *internal model principle*.

The internal model principle is most easily understood when viewed in the frequency domain. The controller transfer function (from x to u) possesses a pole at the origin, which in the frequency domain is equivalent to demanding infinite gain at DC. A look at the sensitivity function then shows that sensitivity with respect to plant uncertainty or disturbances at DC will be zero, provided the plant itself does not possess a zero at DC. However in this course we remain in the time domain, and so our approach is to show that a desired equilibrium is asymptotically stable.

First note that since the external inputs r and w are constant, assuming stability one can expect that all signals will converge to some constant values, which will form an equilibrium for the controlled system. If the integrated error η converges in this sense, then the error itself e will necessarily converge to ϑ . To make this precise, consider the closed-loop system equations, given by

$$\left[\begin{array}{c} \dot{x} \\ \dot{\eta} \end{array}\right] = \left[\begin{array}{cc} A - BK_1 & -BK_2 \\ C & 0 \end{array}\right] \left[\begin{array}{c} x \\ \eta \end{array}\right] + \left[\begin{array}{c} 0 \\ -I \end{array}\right] r + \left[\begin{array}{c} E \\ F \end{array}\right] w.$$

At an equilibrium we have $\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} = \vartheta$, from which we conclude that

$$\vartheta = \dot{\eta} = Cx - r + Fw = y - r.$$

Thus, we will have asymptotic disturbance rejection and asymptotic tracking if the controlled system is stable. To determine if this is the case, we must consider the eigenvalues of the matrix

$$A_{cl} = \left[\begin{array}{cc} A - BK_1 & -BK_2 \\ C & 0 \end{array} \right].$$

Writing the closed-loop matrix as

$$\begin{bmatrix} A - BK_1 & -BK_2 \\ C & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}}_{\widetilde{A}} - \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\widetilde{K}} \underbrace{[K_1 \ K_2]}_{\widetilde{K}}$$

we conclude that we can achieve arbitrary pole placement if and only if the pair $(\widetilde{A}, \widetilde{B})$ is controllable. If this is the case then the two gains $(K_1, K_2) = \widetilde{K}$ can be designed simultaneously using a single Matlab command.

We use the Hautus-Rosenbrock test to better understand the controllability of $(\widetilde{A}, \widetilde{B})$. We have for any complex s,

$$\operatorname{rank}\left(sI-\widetilde{A}\mid\widetilde{B}\right)=\operatorname{rank}\left[\begin{array}{cc|c}sI-A & 0 & B\\ -C & sI & 0\end{array}\right].$$

For controllability, we must have for all s,

$$\operatorname{rank} \left[\begin{array}{cc|c} sI - A & 0 & B \\ -C & sI & 0 \end{array} \right] = n + p. \tag{8.1}$$

That is, all (n+p) rows must be linearly independent.

First suppose that $s \neq 0$. Then the bottom p rows (given by $[-C \ sI \ | \ 0]$) are linearly independent because of the presence of the rank p matrix sI. The remaining upper n rows will be linearly independent if (A, B) is controllable (the 0 term does not affect the rank of these rows). Thus if (A, B) is controllable, the rank condition is met for $s \neq 0$. For s = 0, the rank condition (8.1) becomes

$$\operatorname{rank} \left[\begin{array}{cc} -A & 0 & B \\ -C & 0 & 0 \end{array} \right] = n + p.$$

To summarize, the following two conditions are equivalent to controllability of the pair $(\widetilde{A}, \widetilde{B})$:

(a) (A, B) is controllable

(b) rank
$$\begin{bmatrix} -A & B \\ -C & 0 \end{bmatrix} = n + p$$
.

For $m \ge p$, condition (b) will be true if and only if no transmission zero exists at $s = s_0 = 0$ (see page 162).

Note that if the rank condition (b) fails then there exists a non-zero pair $(x_0, u_0) \in \mathbb{R}^{n+p}$ such that

$$\left[\begin{array}{cc} A & B \\ C & 0 \end{array}\right] \left(\begin{array}{c} x_0 \\ u_0 \end{array}\right) = \left(\begin{array}{c} \vartheta \\ \vartheta \end{array}\right).$$

If the constant control $u \equiv u_0$ is applied when the initial condition is x_0 then we have

$$\dot{x} = Ax_0 + Bu_0 = \vartheta.$$

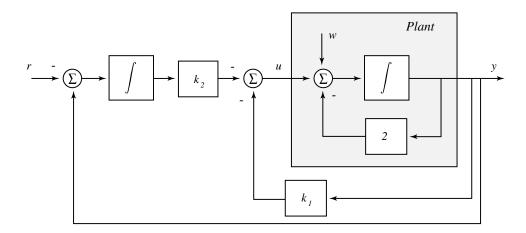
That is, the state x_0 is an equilibrium when this constant input is applied. The output equation is $y = Cx_0 = \vartheta$, so that the initial state/input signal is invisible to the output. It is this complete lack of response to a DC input that must be ruled out if integral control, or any other approach to DC tracking is to be successful.

If the states are not available, we can still use an observer. If there are disturbances then the estimates may not converge, but one can still show that the unique equilibrium will satisfy e=0. In Exercise 3 you will see how the estimator behaves in the presence of a disturbance.

Example 8.1.1. Consider the first order model

$$\begin{array}{rcl} \dot{x} & = & -2x + u + w \\ y & = & x. \end{array}$$

To regulate this system to track a constant reference r, the control law will be of the form illustrated below.



To place the closed-loop poles we consider the closed-loop system matrix

$$A_{cl} = \widetilde{A} - \widetilde{B}\widetilde{K} = \begin{bmatrix} -2 - k_1 & -k_2 \\ 1 & 0 \end{bmatrix}.$$

For example, to place the closed-loop poles at $-2 \pm i2$ we set

$$\det[sI - (\widetilde{A} - \widetilde{B}\widetilde{K})] = s^2 + (2 + k_1)s + k_2 = s^2 + 4s + 8.$$

This gives $k_1 = 2, k_2 = 8$, so that we obtain the "PI" controller

$$u = -2y - 8 \int_0^t e \, dt.$$

The internal model principle can be extended to the more general situation where the disturbance and reference signal are the sum of periodic signals of known frequency. If this is the case, then we may again find a matrix A_m such that

$$\dot{z} = A_m z \qquad z(0) \in \mathbb{R}^q;
\begin{pmatrix} r \\ w \end{pmatrix} = C_m z,$$

where the eigenvalues of the matrix A_m lie on the $j\omega$ -axis. To design a controller one replicates this model to define the signal η :

$$\dot{w} = A_m w + B_1 e \qquad w(0) \in \mathbb{R}^q;$$

 $\eta = C_1 w.$

As before, the control can then be defined as $u = -K_1x - K_2\eta$.

8.2 Transfer function approach

When we use an observer together with state feedback to control a plant, we are actually designing a form of dynamic feedback compensation, similar to what is obtained through loop shaping in a frequency domain design. We show here that equivalent dynamic compensation may be conducted entirely in the Laplace transform domain by manipulating transfer functions. We concern ourselves only with the SISO case, and for simplicity we treat only the regulator problem. Integral control or reduced order observer based feedback can be analyzed in the transform domain in the same manner.

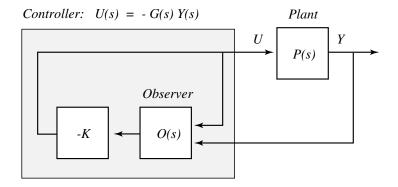


Figure 8.1: The observer/state feedback combination can be regarded as dynamic compensation

The regulator problem using a full order observer is illustrated in Figure 8.1. To compute the controller transfer function G(s), we first construct the transfer function from Y to \hat{X} . The state estimates are described by the equation

$$\dot{\widehat{x}} = A\widehat{x} + Bu + L(y - C\widehat{x}) = (A - BK - LC)\widehat{x} + Ly$$

where the second equation follows from the control law

$$u = -K\widehat{x}$$
.

Taking transforms, we see that

$$U(s) = -K\hat{X}(s) = -K(sI - [A - BK - LC])^{-1}LY(s).$$

This results in the formula

$$G(s) = K(sI - [A - BK - LC])^{-1}L.$$
(8.2)

Since there is no direct feedthrough term (no "D" matrix) in the formula (8.2), the transfer function G is strictly proper.

This transfer function can be directly designed, without consideration of a state space model. To see this, first write the plant and controller transfer functions as a ratio of polynomials:

$$Plant \ transfer \ function = P(s) = \frac{b(s)}{a(s)}$$

$$Controller \ transfer \ function = G(s) = \frac{n(s)}{d(s)}.$$

If the control U(s) = -G(s)Y(s) + R(s) is applied, then the closed loop transfer function becomes

$$P_{cl}(s) = \frac{Y(s)}{R(s)} = \frac{\frac{b(s)}{a(s)}}{1 + \frac{n(s)}{d(s)} \frac{b(s)}{a(s)}} = \frac{b(s)d(s)}{a(s)d(s) + b(s)n(s)}.$$

The poles of the closed-loop system are determined by the corresponding characteristic equation

$$\Delta_{cl}(s) = a(s)d(s) + b(s)n(s) = 0.$$

Thus, if we let α_c , α_o denote the polynomials which define the state feedback poles, and observer poles, respectively:

$$\alpha_c(s) = \det(sI - (A - BK))$$

 $\alpha_o(s) = \det(sI - (A - LC))$ $= \text{known},$

then the polynomials n(s), d(s) must satisfy the following *Diophantine equation*:

$$a(s)d(s) + b(s)n(s) = \alpha_c(s)\alpha_o(s). \tag{8.3}$$

We assume that both a(s) and d(s) are monic, so that each of these polynomials has degree n. We have already noted that G is strictly proper, which means that the degree of n(s) is strictly less than n. The product $\alpha_c(s)\alpha_o(s)$ should also be monic. With this information, the unknown polynomials d(s) and n(s) can be computed. However, some conditions must be imposed. When we treated this problem using state space arguments we found that the system must be controllable and observable to compute the gains K and L. This is equivalent to minimality, which means that the transfer function $P(s) = C(Is - A)^{-1}B$ cannot have any common pole-zero pairs, or that the polynomials a and b do not have common roots.

Example 8.2.1. Consider the transfer function description of the magnetically suspended ball

$$P(s) = \frac{1}{s^2 - 1}.$$

There can be no pole zero cancellations since b(s) = 1, and hence any two dimensional state space model is minimal. We conclude that the degree of n(s) is less than or equal to 1, and the degree of d(s) is equal to 2. So, the controller can be written

$$U(s) = \frac{n_1 s + n_0}{s^2 + d_1 s + d_0} Y(s)$$

From equation (8.3) we must solve the following equation to place the two state feedback poles, and the two observer poles:

$$(s^{2}-1)(s^{2}+d_{1}s+d_{0})+(1)(n_{1}s+n_{0})=(s-p_{1})(s-p_{2})(s-\ell_{1})(s-\ell_{2}),$$

which gives four equations and four unknowns through equating of the coefficients. These equations are called the Sylvester equations.

Based on our criteria for solving the Diophantine equation, it follows that for the Sylvester equations to be independent and solvable for any set of observer poles and state feedback poles, the plant transfer function cannot have common pole-zero pairs. For example, consider the plant transfer function

$$P(s) = \frac{s+1}{(s+1)(s+2)}.$$

The corresponding Diophantine equation is

$$(s+1)(s+2)d(s) + (s+1)n(s) = \alpha_c(s)\alpha_o(s)$$

Since the left hand side of this equation vanishes when s = -1, obviously this equation cannot be solved if neither α_c nor α_o have a root at -1. Pole placement is not possible in this example because the transfer function P does contain a common pole-zero pair.

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8.3 Exercises

8.3.1 Consider the servo motor with transfer function

$$Y(s) = \frac{1}{s(1+s)}U(s),$$

where u is the voltage input to the motor, and y is its position. In this exercise you will compare closed-loop sensitivity with and without integral control.

(a) Design a first order dynamic compensator

$$U(s) = k \frac{s-z}{s-p} (gR(s) - Y(s))$$

so that the closed-loop poles are at -3, $-2 \pm j$. Choose g so that the closed-loop transfer function Y/R has unity gain at DC.

(b) Plot the step response, and a Bode plot of the sensitivity transfer function S_1 .

8.3.2 Consider the LTI system

$$\dot{x}_1 = x_2
\dot{x}_2 = u + w$$
 $y = x_1,$
(8.4)

where w is a constant but unknown disturbance. Rather than introduce integral control, if we could obtain estimates of w, then we could design a control law to cancel the effect of this disturbance.

One way to attempt to do just this is to consider $x_3 = w$ as an additional state variable:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & u + x_3 & \quad y = x_1. \\ \dot{x}_3 & = & 0 \end{array}$$

- (a) Carefully show that this third-order "augmented" system is observable.
- (b) Design a full-order observer for this system. Draw the corresponding all-integrator diagram of the observer in full detail, including all integrators.
- (c) Based upon this observer, design a controller for this system to place the closed-loop poles at -1, -2, -3, -4, -5. Explain carefully why the poles will be placed, and the state estimates will converge.

- (d) Is it possible to design the observer-based feedback control law in (c) so that the closed-loop system matrix is Hurwitz?
- **8.3.3** Consider again the LTI system given in (8.4).
 - (a) Design a feedback compensator of the form $u = -K_1 \hat{x} K_2 \eta$ to regulate the output to zero. Take $\dot{\eta} = y$, and \hat{x}_2 as an estimate of x_2 based on a reduced order observer, assuming $w \equiv 0$.
 - (b) With this controller, compute $\widehat{x}_2^{\infty} = \lim_{t \to \infty} \widehat{x}_2(t)$, as a function of w. Note: you do not need to compute the entire transfer function.
 - (c) Provide simulations of your controller using Simulink for various initial conditions, and values of w.
- **8.3.4** An electromagnetic suspension system for a magnetic ball is modeled by

$$m\ddot{y} = mg - ci^2/y^2$$

where m is the mass of the ball (= 10^{-3} kg.), g = 9.8m./sec², $c = 9.8 \times 10^{-9}$ newton-meter²/amp², y is the distance between the magnet and the ball, and i is the current through the electromagnet. Suppose that transducers are available for measuring $x_1 = y, x_2 = \dot{y}$. It is desired that in the steady state, $x_1 \to v$, where v is a constant reference input.

To accomplish this, it is proposed to feed back the integral of the error $x_1 - v$ in addition to the states x_1 and x_2 . That is, the control i is to be implemented as

$$i = k_0 x_0 + k_1 x_1 + k_2 x_2$$

where x_0 is the integral of $x_1 - v$, and k_0, k_1, k_2 are constant gains, chosen by the designer as $k_0 = 100$, $k_1 = 1200$, $k_2 = 1$. Find the equilibria of the closed-loop system when v = 0.025 m, and determine whether or not they are asymptotically stable.

Chapter 9

Control Design Goals

9.1 Performance

Control can be broadly defined as the use of feedback to obtain satisfactory performance in the presence of uncertainty. *Feedback* is simply the act of incorporating measurements as they become available to define the inputs to the plant. The definition of *performance* depends of course on the nature of the particular control application, but there are some universal performance issues. Among these are:

- (i) Stability of the controlled system is of course the most basic goal. Anyone who has held a microphone too close to its speaker system knows that feedback can result in instability, but when properly designed, feedback can prevent unstable behavior.
- (ii) Frequently one desires reasonably accurate *tracking* of some desired trajectory. If the trajectory is a constant reference value, this is known as the *regulation* problem. In the production of chemicals, particularly in the pharmacuetical industry, yields must be controlled within stringent specifications. In the automatic control of an aircraft in flight, again one objective is to keep the flight at a level altitude and a constant speed.
- (iii) Another time domain specification concerns *speed of response*. For example, if the pilot of an aircraft desires a change in altitude, this should occur quickly, but not at the expense of the comfort of the passengers.

(iv) Rarely do we have available a perfect model of the plant to be controlled. Even if an accurate model is available, it may be convenient to linearize the nonlinear model, which can introduce significant inaccuracies. Such inaccuracies combined with unforeseen influences on the plant are collectively known as disturbances, and we must always consider *robustness* to such disturbances when designing a control system.

All of these issues are typically addressed in the design of a control system in the face of physical and economic constraints, and in most current industries, time constraints as well!

One cannot expect good performance if the physical system is poorly designed. In particular, it is important from the start that the measurement devices be reasonably precise, and that actuation - the final synthesis of the control action - be performed accurately. However, in this course we assume that these issues were considered in the construction of the plant, so that what is left for the control engineer can be largely completed with a pen, paper, and a computer.

9.2 Measurements

This course concerns state space models, which in many of our examples are naturally constructed based upon signals which have physical significance, and which can be measured if sensors are adjoined to the plant. For instance, in the pendubot illustrated in Figure 1.3, the states are the link angles, and their velocities. The link angles can be measured with high precision using an optical encoder. The velocities are more difficult to measure, but they can be estimated using finite differences. If, as in this example, an accurate state space model is available together with accurate state measurements, then control based upon pole placement or optimal control is frequently simple and very effective.

However, there are times when full state information is not feasible. For instance, in many applications the state space model is constructed by first finding a frequency domain model (i.e. a Bode plot) based upon input-output measurements. A state space model can then be constructed so that its frequency response accurately matches this empirical Bode plot. In this case, the state has no physical significance, and the only information available for controlling the system may be the input and output. In Section 7.2 we circumvent this difficulty by constructing *observers* to estimate the state as input-output measurements are collected. In principle at least, one can then

treat the estimates as if they are perfect measurements of the state, and then apply one of the state feedback control techniques. This gives a mathematically elegant approach to control design which is also easily analyzed. In particular, one can easily verify that the goals (i)–(iii) outlined above will be met if the observer and state feedback controller are appropriately designed.

The fourth goal, robustness, can be easily forgotten in a state space design, and this can lead to absurd control designs! To see this, consider once again the pendubot shown in Figure 1.3. With the input u taken as the torque applied to the lower link, and with the output y taken to be the lower link angle, the linearized plant in the vertical position is both controllable and observable. Theoretically then, it is easy to design a controller which will keep the pendubot balanced in the vertical position based upon measurements of the lower angle alone. Physically, this seems very unlikely, and in fact such a controller has never been successfully implemented on this plant. The reason for this divergence between theory and practice is that when evaluating a controller one must also consider robustness to plant uncertainty. Using a frequency domain analysis, one can more easily see that any controller for this plant based upon the measurements u, y alone will be highly sensitive to model inaccuracy. An insensitive design is possible if both link angles are used in the feedback compensator.

9.3 Robustness and sensitivity

The Bode sensitivity function is a measure of the sensitivity of the closed-loop transfer function to a deviation in the plant model. For the control system illustrated in Figure 9.1, the closed-loop transfer function may be expressed

$$T = \frac{Y}{R} = \frac{GP}{1 + GP}$$

If the transfer function P changes to form a new transfer function P', the controller will not function precisely as it was intended, since the transfer function T will undergo a corresponding change. Rather than consider absolute changes, it is more appropriate to analyze the affect of the relative change in the plant

$$\frac{\Delta P}{P}$$
,

where $\Delta P = P' - P$ denotes the difference between the true transfer function and the nominal one. For small changes, this can be interpreted as

a differential. The *sensitivity* is then the ratio of this differential, and the corresponding differential in T:

$$S := \frac{dT/T}{dP/P} = \frac{P}{T}\frac{dT}{dP}.$$
(9.1)

For the error feedback configuration given in Figure 9.1, the derivative can be computed to give

$$S = \frac{1}{1 + PG} = \frac{1}{1 + L}$$

where L(s) is the loop gain P(s)G(s). The quantity 1 + L(s) is known as the return difference.

In the MIMO case we can again write for the error feedback configuration

$$Y = \left[(I + PG)^{-1} PG \right] R,$$

and the sensitivity function is then defined to be the matrix-valued function of ω ,

$$S(j\omega) = (I + P(j\omega)G(j\omega))^{-1}.$$

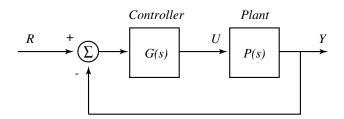


Figure 9.1: An error feedback configuration

We can conclude from this formula that if we want a system which is insensitive to changes in the plant, then we should make the loop gain L(s) as large as possible. For SISO systems the size of the sensitivity function is measured by means of the magnitude of its frequency response. We might require for example that

$$|S(j\omega)| < 1, \qquad \omega \ge 0,$$

which does correspond to a "large" loop gain. In making the sensitivity small however, we must keep all of the previous design goals in mind, and there are also other constraints which are absolute. For instance, for the error feedback configuration we will always have

$$S + T = 1$$
.

Moreover, if the transfer function L is relative degree at least two, and all poles lie in the left half plane in \mathbb{C} , we have the constraint

$$\int_0^\infty \log |S(j\omega)| d\omega = 0.$$

In this case it is impossible to make $|S(j\omega)| < 1$ for all ω , since then $\log |S(j\omega)|$ is always negative.

9.4 Zeros and sensitivity

It is clear that the structure of the open-loop poles will play an important role in the design of an effective controller. What is less obvious at first glance is the importance of the open-loop zeros.

To illustrate the importance of zeros, we again take the pendubot introduced in Chapter 1. The pendubot can be linearized in the upright position illustrated in Figure 1.3. With the input u equal to the applied torque, and the output y equal to the lower link angle, the resulting state space model is defined by the following set of matrices:

$$A = \begin{pmatrix} 0 & 1.0000 & 0 & 0 \\ 51.9243 & 0 & -13.9700 & 0 \\ 0 & 0 & 0 & 1.0000 \\ -52.8376 & 068.4187 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} 0 \\ 15.9549 \\ 0 \\ -29.3596 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} D = 0. \tag{9.2}$$

Using Matlab, the corresponding transfer function model is found to be

$$P(s) = \frac{Y(s)}{U(s)} = 15.9549 \frac{(s - 6.5354)(s + 6.5354)}{(s - 9.4109)(s + 9.4109)(s - 5.6372)(s + 5.6372)}$$

This is of the general form

$$P(s) = k \frac{(s - \gamma)(s + \gamma)}{(s - \alpha)(s + \alpha)(s - \beta)(s + \beta)},$$

where $0 < \alpha < \gamma < \beta$.

Because P contains no common pole-zero pairs, the state space model must be both controllable and observable, i.e., minimal. In this chapter we will show that it is therefore possible to design a fourth order dynamic compensator U(s) = -G(s)Y(s) which stabilizes this model at $x = \vartheta$. We will find in Chapter 10 that a set of closed-loop poles which will perform well using full state feedback are

$$p_1 = -36$$
 $p_2 = -2.9$ $p_3 = -5.3 + 0.84j$ $p_4 = -5.3 - 0.84j$. (9.3)

The compensation will introduce four poles, which we will place at

$$p_5 = -3.9932$$
 $p_6 = -7.4167$ $p_7 = -8.4862$ $p_8 = -11.3012$.

All of these poles are based upon an "optimal" design. By examining a root locus plot it may be shown that any compensator which stabilizes this system will have poles in the right half plane in \mathbb{C} . In this example, we find that the compensator which places these poles is given by

$$G(s) = 24,424 \frac{(s+9.4)(s+5.64)(s-7.41)}{(s+60.24+63.5j)(s+60.24-63.5j)(s-46.37)(s+6.54)}.$$

As expected, this compensator has an unstable pole at s = 46.37.

A Nyquist plot of the loop transfer function L(s) = G(s)P(s) is given in Figure 9.2. If the plant is changed slightly, then the number of encirclements of -1 will change, resulting in an unstable closed-loop system. A Bode plot of S = 1/(1+GP) is shown in Figure 9.3. Here it is seen that the controlled system is highly sensitive to plant uncertainty in the low frequency range. By applying higher feedback gain, it is possible to reduce sensitivity in this range, but then one will find high sensitivity at higher frequencies.

One way to explain these difficulties is the presence of zeros in the right half plane, which impose strong limits on achievable sensitivity. For the zero free model with transfer function

$$P(s) = \frac{1}{(s-\alpha)(s+\alpha)(s-\beta)(s+\beta)},$$

it is easy to obtain a low sensitivity design. Another explanation is again common sense - it is difficult to imagine that this plant can be controlled using only information at the lower link. By adding another sensor, an insensitive design may be obtained using state space methods.

For a SISO transfer function P(s) = b(s)/a(s), the poles are the roots of a, and the zeros are the roots of b, so that we have $P(s_0) = 0$ for a

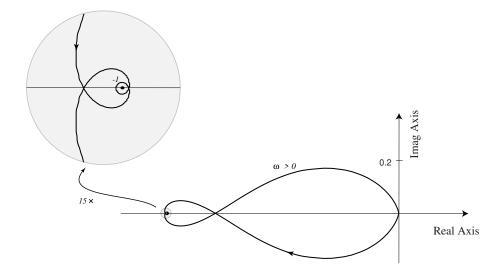


Figure 9.2: The Nyquist plot of $G(j\omega)P(j\omega)$. The compensated open-loop system with transfer function GP possesses two poles in the right half plane. The three encirclements of -1 ensure that the controlled system is asymptotically stable, from the Nyquist Stability Criterion, but this stability is very fragile.

zero s_0 . In the MIMO case, we define the characteristic polynomial as $\Delta(s) = \det(Is - A)$, and the roots of Δ are then the poles of P. We define zeros similarly, but it would be far too strong to suppose that $P(s_0) = \vartheta$ at a zero s_0 . For the state space model

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x_0 = \vartheta$$

we can take Laplace transforms to obtain

$$(sI - A)X(s) = BU(s),$$
 $Y(s) = Cx(s).$

Combining these equations in one matrix expression gives

$$\underbrace{\left[\begin{array}{cc} sI-A & -B \\ C & \vartheta \end{array}\right]}_{(n+p)\times(n+m)} \left[\begin{array}{c} X(s) \\ U(s) \end{array}\right] = \left[\begin{array}{c} \vartheta \\ Y(s) \end{array}\right].$$

Assume that there are as many plant inputs as outputs (p = m), so that the plant is *square*. A natural definition of a zero can be formulated as follows:

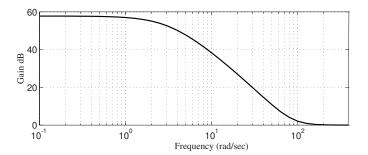


Figure 9.3: The magnitude plot of the frequency response for the sensitivity function S = 1/(1+GP). The controlled system is highly sensitive to plant uncertainty at low frequencies.

At s_0 , can I find a fixed $U(s_0)$ and $X(s_0)$ so that the resulting $Y(s_0)$ is zero? That is,

$$\begin{bmatrix} s_0 I - A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} X(s_0) \\ U(s_0) \end{bmatrix} = \begin{bmatrix} \vartheta \\ \vartheta \end{bmatrix}.$$

This is possible if and only if the matrix above is rank deficient:

$$\rho\left(\left[\begin{array}{cc} s_0I - A & -B \\ C & 0 \end{array}\right]\right) < n + m.$$

Provided that s_0 is not also a pole of P, this is equivalent to rank deficiency of the transfer function P at s_0 :

$$\rho(P(s_0)) = \rho(C(Is_0 - A)^{-1}B) < m,$$

which does generalize the definition given in the SISO case. The complex number s_0 is called a transmission zero. If all of the transmission zeros lie within the open left half plane in \mathbb{C} then the model is called minimum phase. In the multivariate case, the location of the transmission zeros can play an important role in the achievable sensitivity of the controlled plant, just as in the SISO case.

9.5. EXERCISES

9.5 Exercises

9.5.1 For the Pendubot described by the state space model (9.2), you will design two controllers. The design can be done using the PLACE commands within *Matlab*, and the formula (8.2).

(a) Design a fifth order dynamic compensator U = -GY of the form

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$$G(s) = \frac{n(s)}{sd(s)},$$

where d is a fourth order, monic polynomial. The dominant closed loop poles should lie to the left of the line Re(s) = -10 in the complex plane. Obtain a Bode plot of the sensitivity function for your design.

(b) You will now design a compensator based on the two angle measurements. We then have

$$U(s) = \frac{n_1(s)}{sd_1(s)}Y_1(s) + \frac{n_2(s)}{sd_2(s)}Y_2(s)$$

where both d_1 and d_2 are fourth order polynomials. See if you can find an observer based dynamic feedback controller of this form which has significantly improved sensitivity. Obtain a Bode plot of the sensitivity function for your design.

Part IV Optimal Control

Chapter 10

Dynamic Programming and the HJB Equation

10.1 Problem formulation

This chapter concerns optimal control of dynamical systems. Most of this development concerns linear models with a particularly simple notion of optimality. However, to understand the most basic concepts in optimal control, and not become lost in complex notation, it is most convenient to consider first the general model given in nonlinear state space form

$$\dot{x} = f(x, u, t), \qquad x(t_0) = x_0.$$
 (10.1)

The cost V of a particular control input u is defined by the following integral

$$V(u) = \int_{t_0}^{t_1} \ell(x, u, \tau) d\tau + m(x(t_1))$$
 (10.2)

where

- (a) t_1 is the *final time* of the control problem.
- (b) ℓ is a scalar-valued function of x, u, and t
- (c) m is a function of x. It is called the terminal penalty function.

We assume that x_0 , t_0 , and t_1 are known, fixed values, and $x(t_1)$ is free. Our goal is to choose the control $u_{[t_0,t_1]}$ to minimize V.

A case which is typical in applications is where m and ℓ are quadratic functions of their arguments. For an LTV model, the system description

and cost are then given by

$$\dot{x} = A(t)x + B(t)u, \qquad x(t_0) = x_0$$

$$V(u) = \int_{t_0}^{t_1} (x^T Q(t)x + u^T R(t)u) dt + x^T(t_1) M x(t_1)$$
(10.3)

where M, Q and R are positive semidefinite matrix-valued functions of time. These matrices can be chosen by the designer to obtain desirable closed-loop response. The minimization of the quadratic cost V for a linear system is known as the *linear quadratic regulator* (LQR) problem.

We will study the LQR problem in detail, but first we develop some general results for the nonlinear state space model (10.1).

10.2 Hamilton-Jacobi-Bellman equations

The value function $V^{\circ} = V^{\circ}(x_0, t_0)$ is defined to be the minimum value of V over all controls. This is a function of the two variables x and t which can be written explicitly as

$$V^{\circ}(x,t) = \min_{u_{[t,t_1]}} \left[\int_t^{t_1} \ell(x(\tau), u(\tau), \tau) \, d\tau + m(x(t_1)) \right].$$

Under very general conditions, the value function satisfies a partial differential equation (PDE) known as the *Hamilton-Jacobi-Bellman* (*HJB*) equation. To derive this result, let x and t be an arbitrary initial time and initial state, and let t_m be an intermediate time, $t < t_m < t_1$. Assuming that $x(\tau)$, $t \le \tau \le t_1$, is a solution to the state equations with x(t) = x, we must have

$$V^{\circ}(x,t) = \min_{u_{[t,t_{1}]}} \left[\int_{t}^{t_{m}} \ell(x(\tau), u(\tau), \tau) d\tau + \int_{t_{m}}^{t_{1}} \ell(x(\tau), u(\tau), \tau) d\tau + m(x(t_{1})) \right]$$

$$= \min_{u_{[t,t_{m}]}} \left[\int_{t}^{t_{m}} \ell(x(\tau), u(\tau), \tau) d\tau + \min_{u_{[t,t_{1}]}} \left(\int_{t_{m}}^{t_{1}} \ell(x(\tau), u(\tau), \tau) d\tau + m(x(t_{1})) \right) \right].$$

$$V^{\circ}(x(t_{m}), t_{m})$$

This gives the functional equation

$$V^{\circ}(x,t) = \min_{u_{[t,t_m]}} \left[\int_t^{t_m} \ell(x(\tau), u(\tau), \tau) \, d\tau + V^{\circ}(x(t_m), t_m) \right]. \tag{10.4}$$

As a consequence, the optimal control over the whole interval has the property illustrated in Figure 10.1: If the optimal trajectory passes through the

state x_m at time $x(t_m)$ using the control u° , then the control $u^{\circ}_{[t_m,t_1]}$ must be optimal for the system starting at x_m at time t_m . If a better u^* existed on $[t_m,t_1]$, we would have chosen it. This concept is called the *principle of optimality*.

Optimal trajectory starting from x at time t

---- Optimal trajectory starting from x_m at time t_m



Figure 10.1: If a better control existed on $[t_m, t_1]$, we would have chosen it.

By letting t_m approach t, we can derive a partial differential equation for the value function V° . Let Δt denote a small positive number, and define

$$t_m = t + \Delta t$$

 $x_m = x(t_m) = x(t + \Delta t) = x(t) + \Delta x.$

Assuming that the value function is sufficiently smooth, we may perform a Taylor series expansion on V° using the optimality equation (10.4) to obtain

$$V^{\circ}(x,t) = \min_{u_{[t,t_m]}} \left\{ \ell(x(t), u(t), t) \Delta t + V^{\circ}(x,t) + \frac{\partial V^{\circ}}{\partial x} (x(t), t) \Delta x + \frac{\partial V^{\circ}}{\partial t} (x(t), t) \Delta t \right\} + p(\Delta t).$$

Dividing through by Δt and recalling that x(t) = x then gives

$$0 = \min_{u_{[t,t_m]}} \left\{ \ell(x,u(t),t) \frac{\Delta t}{\Delta t} + \frac{\partial V^\circ}{\partial x}(x,t) \frac{\Delta x}{\Delta t} + \frac{\partial V^\circ}{\partial t}(x,t) \frac{\Delta t}{\Delta t} \right\}.$$

Letting $\Delta t \to 0$, the ratio $\Delta x/\Delta t$ can be replaced by a derivative to give

$$0 = \min_{u} \left[\ell(x, u, t) + \frac{\partial V^{\circ}}{\partial x}(x, t) \dot{x}(t) \right] + \frac{\partial V^{\circ}}{\partial t}$$

where

$$\frac{\partial V^{\circ}}{\partial x} = \left[\frac{\partial V^{\circ}}{\partial x_1} \dots \frac{\partial V^{\circ}}{\partial x_n} \right] = (\nabla_x V^{\circ})^T.$$

Thus, we have obtained the following partial differential equation which the value function must satisfy if it is smooth. The resulting equation (10.5) is the *Hamilton-Jacobi-Bellman (HJB) equation*,

$$-\frac{\partial V^{\circ}}{\partial t}(x,t) = \min_{u} \left[\ell(x,u,t) + \frac{\partial V^{\circ}}{\partial x}(x,t)f(x,u,t) \right]. \tag{10.5}$$

The terminal penalty term gives a boundary condition for the HJB equation

$$V^{\circ}(x(t_1), t_1) = m(x(t_1)). \tag{10.6}$$

The term in brackets in (10.5) is called the *Hamiltonian*, and is denoted H:

$$H(x, p, u, t) := \ell(x, u, t) + p^{T} f(x, u, t), \tag{10.7}$$

where $p = \nabla_x V^{\circ}$. We thus arrive at the following

Theorem 10.1. If the value function V° has continuous partial derivatives, then it satisfies the following partial differential equation

$$-\frac{\partial V^{\circ}}{\partial t}(x,t) = \min_{u} H(x, \nabla_{x} V^{\circ}(x,t), u, t),$$

and the optimal control $u^{\circ}(t)$ and corresponding state trajectory $x^{\circ}(t)$ must satisfy

$$\min_{u} H(x^{\circ}(t), \nabla_{x}V^{\circ}(x^{\circ}(t), t), u, t) = H(x^{\circ}(t), \nabla_{x}V^{\circ}(x^{\circ}(t), t), u^{\circ}(t), t).$$

$$(10.8)$$

Conversely, if there exists a value function $V^{\circ}(x,t)$ solving (10.5) subject to (10.6), and a control u° obtained from (10.8), then u° is the optimal controller minimizing (10.2) for the system (10.1) and $V^{\circ}(x_0,t_0)$ is the minimum value of (10.2).

An important consequence of Theorem 10.1 is that the optimal control can be written in state feedback form $u^{\circ}(t) = \bar{u}(x^{\circ}(t), t)$, where the function \bar{u} is defined through the minimization in (10.8).

Proof Proof of Theorem 10.2.1 The first part of the statement of the theorem has already been proved through the derivation that led to (10.5). For the second part of the statement (the sufficiently part), let us first note that for an arbitrary control u, from (10.5):

$$-\frac{\partial V^{\circ}}{\partial t}(x,t) = \ell(x,u^{\circ},t) + \frac{\partial V^{\circ}}{\partial x}(x,t)f(x,u^{\circ},t)$$

$$\leq \ell(x,u,t) + \frac{\partial V^{\circ}}{\partial x}(x,t)f(x,u,t)$$

which is equivalent to

$$0 = \ell(x, u^{\circ}, t) + \frac{d}{dt}V^{\circ}(x, y) \le \ell(x, u, t) + \frac{d}{dt}V^{\circ}(x, t)$$
(10.9)

where the total derivative (with respect to t) is evaluated on the trajectory generated by (10.1), with $u = u^{\circ}$ on the left hand-side and arbitrary u on the right hand-side. Now integrate both side of the inequality in (10.9) from t_0 to t_1 , the left hand-side over the trajectory of (10.1) with $u = u^{\circ}$, and the right hand-side also over the trajectory of (10.1) but for an arbitrary u. Further use (10.6) to arrive at

$$V^{\circ}(x_0, t_0) = m(x^{\circ}(t_1)) + \int_{t_0}^{t_1} \ell(x^{\circ}, u^{\circ}, t) dt$$

$$\leq m(x(t_1)) + \int_{t_0}^{t_1} \ell(x, u^{\circ}, t) dt$$

where x° is the trajectory from (10.1) corresponding to u° . The inequality above shows that u° is indeed optimal, and $V^{\circ}(x_0, t_0)$ is the optimal value of the cost.

Example 10.2.1. Consider the simple integrator model, with the polynomial cost criterion

$$\dot{x} = u$$
 $V(u) = \int_0^{t_1} [u^2 + x^4] dt$

Here we have f(x,u,t)=u, $\ell(x,u,t)=u^2+x^4$, and $m(x,t_1)\equiv 0$. The Hamiltonian is thus

$$H(x, p, u, t) = pu + u^2 + x^4,$$

and the HJB equation becomes

$$-\frac{\partial V^{\circ}}{\partial t} = \min_{u} \left\{ \frac{\partial V^{\circ}}{\partial x} u + u^{2} + x^{4} \right\}.$$

Minimizing with respect to u gives

$$u^{\circ} = -\frac{1}{2} \frac{\partial V^{\circ}}{\partial x} (x, t),$$

which is a form of state feedback. The closed-loop system has the appealing form

$$\dot{x}^{\circ}(t) = -\frac{1}{2} \frac{\partial V^{\circ}}{\partial x}(x^{\circ}(t), t)$$

This equation shows that the control forces the state to move in the direction in which the "cost to go" V° decreases.

Substituting the formula for u° back into the HJB equation gives the PDE

$$-\frac{\partial V^{\circ}}{\partial t}(x,t) = -\frac{1}{4} \left(\frac{\partial V^{\circ}}{\partial x}(x,t) \right)^{2} + x^{4},$$

with the boundary condition $V^{\circ}(x, t_1) = 0$. This is as far as we can go, since we do not have available methods to solve a PDE of this form. If a solution is required, it may be found numerically. However, a simpler set of equations is obtained in the limit as $t_1 \to \infty$. This simpler problem is treated in Exercise 1 below.

We now leave the general nonlinear model and concentrate on linear systems with quadratic cost. We will return to the more general problem in Chapter 11.

10.3 A solution to the LQR problem

For the remainder of this chapter we consider the LQR problem whose system description and cost are given in (10.3). For this control problem, x_0 , t_0 and t_1 are given, and $x(t_1)$ is free. To ensure that this problem has a solution we assume that R is strictly positive definite (R > 0).

To begin, we now compute the optimal control u° by solving the HJB partial differential equation. The Hamiltonian for this problem is given by

$$H(x, p, u, t) = \ell + p^T f = x^T Q x + u^T R u + p^T (A x + B u).$$

To minimize H with respect to u we compute the derivative

$$\nabla_u H = 0 + 2Ru + B^T p.$$

Setting this equal to zero, we see that the optimal control is given by

$$u^{\circ} = -\frac{1}{2}R^{-1}B^{T}p \tag{10.10}$$

where we recall that $p = \nabla_x V^{\circ}$. Hence the controlled state evolves according to the equation

$$\dot{x}(t) = A(t)x(t) - \frac{1}{2}BR^{-1}B^{T}\nabla_{x}V^{\circ}(x,t).$$

Since the matrix $BR^{-1}B^T$ is positive definite, we see once again that the control tends to force x in a direction in which V° is decreasing.

Equation (10.10) shows that the optimal control may be written in state feedback form. However, we cannot compute the optimal control law unless we can compute the value function. Substituting the formula for the optimal control into the HJB equation gives

$$-\frac{\partial V^{\circ}}{\partial t} = H(x, \nabla_x V^{\circ}, u^{\circ}, t)$$

$$= x^T Q x + \frac{1}{4} (\nabla_x V^{\circ})^T B R^{-1} R R^{-1} B^T \nabla_x V^{\circ}$$

$$+ (\nabla_x V^{\circ})^T A x - \frac{1}{2} (\nabla_x V^{\circ})^T B R^{-1} B^T (\nabla_x V^{\circ}).$$

This yields the partial differential equation

$$-\frac{\partial V^{\circ}}{\partial t} = x^{T}Qx + (\nabla_{x}V^{\circ})^{T}Ax^{\circ} - \frac{1}{4}(\nabla_{x}V^{\circ})^{T}BR^{-1}B^{T}(\nabla_{x}V^{\circ}). \quad (10.11)$$

To solve this PDE, start at time t_1 when the value function is known:

$$V^{\circ}(x, t_1) = x^T M x.$$

The terminal V° is quadratic, which suggests that we try a quadratic solution for V° over the entire interval

$$V^{\circ}(x,t) = x^T P(t) x$$

where P is a positive semidefinite, $n \times n$ matrix. If V° is of this form then we have

$$\frac{\partial V^{\circ}}{\partial t} = x^{T} \dot{P} x \qquad \nabla_{x} V^{\circ} = 2P(t) x.$$

Thus, the PDE (10.11) becomes

$$-x^T \dot{P}x = x^T Q x + 2x^T P A x - x^T P B R^{-1} B^T P x.$$

This can be equivalently expressed

$$-x^T \dot{P}x = x^T Qx + x^T (PA + A^T P)x - x^T PBR^{-1}B^T Px.$$

Since this equation must hold for any x, the matrix P must satisfy the following matrix differential equation:

$$-\dot{P} = Q + PA + A^{T}P - PBR^{-1}B^{T}P.$$
 (10.12)

This is an *ordinary* differential equation for P, but it is time-varying, and nonlinear due to the term which is quadratic in P. Adding the boundary condition

$$V(x, t_1) = x^T M x = x^T(t_1) P x(t_1)$$

gives

$$P(t_1) = M. (10.13)$$

Equation (10.12) is called the *Riccati Differential Equation* (RDE), and its boundary condition is given by (10.13). Since the boundary condition is given at the final time, the solution to this differential equation may be viewed as *starting* at time t_1 , and then traveling backwards in time until time t_0 . The Riccati equation possesses a unique solution, provided that the matrices A, B, Q and R are piecewise continuous in t. Further note that this unique solution is symmetric, because P^T satisfies the same differential equation (to see this simply take transpose of both side of (10.12)).

The solution to the LQR problem is now summarized in the following

Theorem 10.2. For the LQR optimal control problem (10.3), assuming that A, B, Q and R are piecewise continuous in t, and that R(t) > 0 for all t.

(a) The optimal control u° is given in linear state feedback form $u^{\circ}(t) = -K(t)x^{\circ}(t)$, where K(t) is a matrix of time-varying gains:

$$u^{\circ} = -\frac{1}{2}R^{-1}B^{T}\nabla_{x}V^{\circ} = -R^{-1}(t)B^{T}(t)P(t)x^{\circ}(t) = -K(t)x^{\circ}(t).$$

- (b) The matrix-valued function P(t) is positive semidefinite for each t, and is defined by the RDE (10.12) with boundary condition (10.13).
- (c) The value function V° is quadratic:

$$V^{\circ}(x_0, t_0) = x_0^T P(t_0) x_0.$$

10.4 The Hamiltonian matrix

We now show how the Riccati equation can be solved by computing the solution to a *linear* ODE. In addition to providing a complete solution to the LQR problem, this will provide a solution to the infinite time horizon

optimal control problem where $t_1 = \infty$. Consider the 2n dimensional LTV model, where the state (X(t), Y(t))' is a $2n \times n$ matrix-valued function of t.

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{2n \times 2n} \underbrace{\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}}_{2n \times n}, \quad \begin{bmatrix} X(t_1) \\ Y(t_1) \end{bmatrix} = \begin{bmatrix} I \\ M \end{bmatrix} 10.14)$$

The $2n \times 2n$ system-matrix of (10.14) is called the *Hamiltonian matrix*, denoted by \mathcal{H} .

Theorem 10.3. The solution to the RDE (10.12) with boundary condition (10.13) is given by the formula

$$P(t) = Y(t)X^{-1}(t), t_0 \le t \le t_1.$$

Proof To prove the theorem, observe that the desired initial conditions are satisfied:

$$\underbrace{Y(t_1)}_{M}\underbrace{X^{-1}(t_1)}_{I} = M = P(t_1).$$

We next verify that the Riccati differential equation is also satisfied. Using the product rule and the matricial quotient rule $\frac{d}{dt}(X^{-1}) = -X^{-1}\dot{X}X^{-1}$, we have

$$\frac{d}{dt}(YX^{-1}) = Y\frac{d}{dt}(X^{-1}) + \dot{Y}X^{-1} = -YX^{-1}\dot{X}X^{-1} + \dot{Y}X^{-1}.$$

Substituting the definitions of \dot{X} and \dot{Y} then gives

$$\begin{split} \frac{d}{dt}(YX^{-1}) &= -YX^{-1}(AX - BR^{-1}B^TY)X^{-1} + (-QX - A^TY)X^{-1} \\ &= -(YX^{-1})A + (YX^{-1})BR^{-1}B^T(YX^{-1}) - Q - A^T(YX^{-1}) \end{split}$$

which is precisely Riccati equation for $P = YX^{-1}$. Note that this immediately tells us that YX^{-1} is symmetric and postive semidefinite.

In the LTI case we may extend this analysis further to explicitly solve the RDE. We first require the following result, which shows that the eigenvalues of the Hamiltonian matrix possess symmetry about the complex axis in \mathbb{C} .

Lemma 10.4. For an LTI model, if λ is an eigenvalue of \mathcal{H} , so is $-\lambda$.

Proof Define the $2n \times 2n$ matrix J as

$$J := \left[\begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right],$$

where each element is $n \times n$. The inverse of J is equal to

$$J^{-1} = \left[\begin{array}{cc} 0 & -I \\ I & 0 \end{array} \right] = J^T.$$

Using the definition of \mathcal{H} gives

$$J\mathcal{H}J^T = -\mathcal{H}^T$$
.

By the definition of similarity transformations, we have for any eigenvalue λ of \mathcal{H} ,

$$0 = \det(\lambda I - \mathcal{H}) = \det(\lambda I - J\mathcal{H}J^{-1}).$$

Moreover, for this special case

$$\det(\lambda I - J\mathcal{H}J^{-1}) = \det(\lambda I + \mathcal{H}^T) = (-1)^n \det((-\lambda)I - J\mathcal{H}J^{-1}).$$

These two equations combined show that when λ is equal to an eigenvalue of \mathcal{H} then so is $-\lambda$, as claimed.

We will see below that the matrix \mathcal{H} cannot have eigenvalues on the $j\omega$ axis, provided the model is stabilizable and detectable. Consider the special case where the eigenvalues of \mathcal{H} are distinct. In this case we can solve the RDE by solving a linear differential equation. We first diagonalize the Hamiltonian matrix:

$$U^{-1}\mathcal{H}U = \left[\begin{array}{cc} \Lambda_s & 0\\ 0 & -\Lambda_s \end{array} \right]$$

where the diagonal elements of Λ_s lie in the strict left hand plane in \mathbb{C} . Note that if the eigenvalues of \mathcal{H} are distinct then, by Lemma 10.4, none can lie on the $j\omega$ axis. The matrix of eigenvectors U can be written

$$U = \left[\begin{array}{c|c} U_{11} & U_{12} \\ U_{21} & U_{22} \end{array} \right]$$

where U_{11}, U_{21} correspond to the stable eigenvalues. The solution to the differential equation (10.14) can be computed in this case:

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = U \begin{bmatrix} e^{\Lambda_s(t-t_1)} & 0 \\ 0 & e^{-\Lambda_s(t-t_1)} \end{bmatrix} U^{-1} \begin{bmatrix} I \\ M \end{bmatrix}$$
(10.15)

This is a special case of the solution to an LTI model

$$x(t) = \phi(t - t_1)x(t_1)$$

using the state transition matrix ϕ .

With considerable algebra, equation (10.15) can be solved to give closed-form expressions for X(t) and Y(t), which can then be combined to form $P(t, t_1)$:

$$P(t,t_1) = [U_{21} + U_{22}e^{-\Lambda_s(t-t_1)}Ge^{-\Lambda_s(t-t_1)}][U_{11} + U_{12}e^{-\Lambda_s(t-t_1)}Ge^{-\Lambda_s(t-t_1)}]^{-1}$$
(10.16)

where

$$G = -[U_{22} - MU_{12}]^{-1}[U_{21} - MU_{11}].$$

Hence the RDE can be solved explicitly for an LTI model.

10.5 Infinite horizon regulator

We now consider the infinite-horizon or steady-state control problem. We assume that the model is LTI, and since the time horizon is infinite, the cost only depends upon the initial condition of the model:

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$

$$V(x) = \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t)) dt$$
(10.17)

There is no need to include a terminal penalty function because the assumptions we impose will imply that $x(t) \to \vartheta$ as $t \to \infty$.

Our approach is to consider an optimal control problem on the finite interval $[0,t_1]$, and then let $t_1 \to \infty$. Let $P(t,t_1)$ denote the solution to the RDE with time horizon t_1 , so that the optimal control may be written $u(t) = -K(t,t_1)x(t) = -R^{-1}B^TP(t,t_1)x(t)$. Suppose that $P(t,t_1) \to \bar{P}$ as $t_1 \to \infty$, where \bar{P} is an $n \times n$ matrix independent of time. Then the limiting control law $u(t) = -Kx(t) = -R^{-1}B^T\bar{P}x(t)$ is optimal for the infinite horizon control problem. To see this, observe that for any control u on $[0,\infty)$, since $x^TP(0,t_1)x$ is equal to the optimal cost for the finite horizon problem,

$$x^{T}P(0,t_{1})x \leq \int_{0}^{t_{1}} (x^{T}(t)Qx(t) + u^{T}(t)Ru(t)) dt$$

 $\leq \int_{0}^{\infty} (x^{T}(t)Qx(t) + u^{T}(t)Ru(t)) dt.$

Letting V° denote the optimal cost for the infinite horizon problem, it follows that $x^T P(0, t_1) x \leq V^{\circ}(x)$ for all x and t_1 , and hence

$$\lim_{t_1 \to \infty} x^T P(0, t_1) x \le V^{\circ}(x).$$

This gives a lower bound on the cost $V^{\circ}(x)$, and this lower bound is attained with the limiting controller.

Under certain conditions we can show that the limit does exist, and thereby solve the infinite horizon control problem.

Example 10.5.1. Consider the simple model

$$\dot{x} = x + u$$

$$V(u) = \int_0^{t_1} (x^2 + u^2) dt + 5x(t_1)^2.$$

The RDE becomes

$$-\frac{d}{dt}P(t,t_1) = 2P(t,t_1) + 1 - P(t,t_1)^2, \qquad P(t_1,t_1) = 5.$$

A solution to this differential equation is shown in Figure 10.2. These plots suggest that for any fixed t, $P(t, t_1)$ converges to a constant \bar{P} as $t_1 \to \infty$. If this is the case, then the limit \bar{P} must be an equilibrium for the ARE. That is,

$$0 = 2\bar{P} + 1 - \bar{P}^2$$
.

Solving this equation gives $\bar{P}=1+\sqrt{2}$, which is consistent with the limiting value shown in the plots. We will see that the limiting control law $u=-(1+\sqrt{2})x$ is optimal for the infinite horizon cost with $t_1=\infty$.

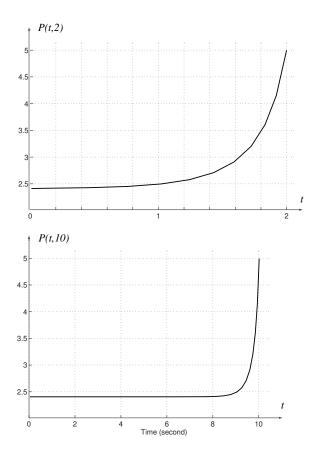


Figure 10.2: Solutions to the ARE for a scalar model with $t_1=2$ and $t_1=10$. It appears that for any fixed t, $P(t,t_1)$ converges to a constant \bar{P} as $t_1\to\infty$.

Using the formula (10.16) it is easy to show that the limiting matrix \bar{P} does exist in the special case where the eigenvalues of the Hamiltonian are distinct. Because the matrix Λ_s is Hurwitz, in this case it follows from (10.16) that

$$\lim_{t_1 \to \infty} P(t, t_1) = U_{21} U_{11}^{-1} := \bar{P}$$
 (10.18)

where \bar{P} is an $n \times n$ symmetric matrix which does not depend on t. The optimal control for (10.17) is given by the limiting formula

$$u^{\circ}(t) = -R^{-1}B^T \bar{P}x^{\circ}(t).$$

Note that u° is given by a state feedback control law of the form $u^{\circ} = -Kx$, which is a form of pole placement. We will see in Section 10.6 that the location of the closed-loop poles possess desirable properties.

We now use the Hamiltonian matrix to characterize the closed-loop poles under optimal control. From the eigenvalue equation $\mathcal{H}U = U\Lambda$ we have

$$\mathcal{H} \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} \Lambda_s. \tag{10.19}$$

The upper set of equations defined by (10.19) may be written $U_{11}\Lambda_s = AU_{11} - BR^{-1}B^TU_{21}$, and this can be transformed to obtain

$$U_{11}\Lambda_s = AU_{11} - BR^{-1}B^TU_{21}(U_{11}^{-1}U_{11})$$

= $(A - BR^{-1}B^TU_{21}U_{11}^{-1})U_{11}$.

Since $K = R^{-1}B^T\bar{P}$ and $\bar{P} = U_{21}U_{11}^{-1}$, this gives the formula

$$(A - BK)U_{11} = U_{11}\Lambda_s.$$

The matrix U_{11} is thus a modal matrix for the optimal closed-loop system matrix $A_{cl} = (A - BK)$. That is,

Theorem 10.5. Assume that the controlled system is asymptotically stable, so that the closed-loop poles determined by

$$\Delta(s) = \det[Is - (A - BR^{-1}B^T\bar{P})]$$

lie in the strict left half plane. Then,

(a) The stable eigenvalues Λ_s for \mathcal{H} are also the optimal closed-loop poles.

(b) The columns of
$$U_{11}$$
 are the eigenvectors for $A_{cl} = A - BR^{-1}B^T\bar{P}$.

Using the Hamiltonian we now give a quadratic algebraic equation which the matrix \bar{P} must satisfy. Considering both the first and second rows of (10.19) gives

$$AU_{11} - BR^{-1}B^TU_{21} = U_{11}\Lambda_s (10.20)$$

$$-QU_{11} - A^T U_{21} = U_{21} \Lambda_s (10.21)$$

Multiplying the first equation by $(U_{21}U_{11}^{-1})$ on the left hand side and by U_{11}^{-1} on the right gives

$$(U_{21}U_{11}^{-1})AU_{11}U_{11}^{-1} - (U_{21}U_{11}^{-1})BR^{-1}B^{T}U_{21}U_{11}^{-1} = (U_{21}U_{11}^{-1})U_{11}\Lambda_{s}U_{11}^{-1}$$

or

$$\bar{P}A - \bar{P}BR^{-1}B^T\bar{P} = U_{21}\Lambda_s U_{11}^{-1}$$
 (10.22)

Multiplying (10.21) by U_{11}^{-1} on the right we obtain

$$-QU_{11}U_{11}^{-1} - A^TU_{21}U_{11}^{-1} = U_{21}\Lambda_sU_{11}^{-1}$$

or

$$-Q - A^T \bar{P} = U_{21} \Lambda_s U_{11}^{-1}. \tag{10.23}$$

Combining (10.22) and (10.23) gives $\bar{P}A - \bar{P}BR^{-1}B^T\bar{P} = -Q - A^T\bar{P}$, or

$$A^{T}\bar{P} + \bar{P}A + Q - \bar{P}BR^{-1}B^{T}\bar{P} = 0$$
 (10.24)

This final formula is known as the algebraic Riccati equation (ARE). It contains $\frac{n(n+1)}{2}$ equations to solve for the components of \bar{P} , since \bar{P} is symmetric.

Example 10.5.2. Consider the system/cost combination

$$\dot{x} = x + u$$

$$V = \int_0^\infty u^2 dt.$$

We have R=1, which is strictly positive, and $Q=0\geq 0$, so our conditions on the cost function are satisfied. However, it is obvious that the optimal control is to set u equal to zero so that V is also zero. The resulting closed-loop system is unstable, so this is obviously a poor design! Clearly, some additional conditions on the cost function must be imposed.

Example 10.5.3. The difficulty in the previous example is that the cost is not a good measure of the internal stability of the model. This is reminiscent of detectability, a connection that we further explore in two more examples.

Consider first the LTI model with quadratic cost

$$\dot{x} = \begin{bmatrix} -3 & -2\\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0\\ 1 \end{bmatrix} u;$$

$$V(u) = \int_0^\infty \left[(x_1 + x_2)^2 + u^2 \right] dt$$
(10.25)

In this example

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \ge 0, \qquad R = 1.$$

The matrix A is Hurwitz, with eigenvalues at -2, -1.

Letting $y = Cx = x_1 + x_2$, the cost becomes $V(u) = \int (y^2 + u^2) dt$. Consider a very special case where the initial condition x(0) belongs to the unobservable subspace $\Sigma_{\bar{o}}$. What then is the optimal control? Recall that if $u \equiv 0$ then the output becomes

$$y(t) = Ce^{At}x(0), \qquad t \ge 0.$$

Since $x(0) \in \Sigma_{\bar{o}}$ we then have $y \equiv 0$. Obviously then, this is the optimal control since it yields V(u) = 0! It follows from the definition of the optimal cost that

$$V^{\circ}(x) = 0 \text{ for } x \in \Sigma_{\bar{o}}.$$

This can be seen explicitly for this example by solving the ARE, and computing $\Sigma_{\bar{o}}$. The observability matrix for this example is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}.$$

Hence the unobservable subspace becomes

$$\Sigma_{\bar{o}} = \mathcal{N}(\mathcal{O}) = \operatorname{Span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Using the lqr command in Matlab, the solution to the ARE is approximately

$$P = 0.24 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Hence we do indeed find that $V^{\circ}(x) = x^T P x = 0$ for $x \in \Sigma_{\bar{o}}$.

Setting u = 0 may be reasonable for a stable system such as (10.25). Consider however the reversed situation where the matrix A is replaced by -A:

$$\dot{x} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

The new state matrix has eigenvalues at +2, +1. Hence for any non-zero initial condition the state explodes as $t \to \infty$ when the input is set to zero. For this system can we again conclude that a zero input is optimal? For this cost criterion, the answer is yes, for precisely the same reason as before. The observability matrix for this model is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix},$$

so that the unobservable subspace is again equal to $\Sigma_{\bar{o}}$ is again equal to the span of the single vector $(1,-1)^T$. If $x(0) \in \Sigma_{\bar{o}}$, and $u \equiv 0$, then $y = Cx = x_1 + x_2$ is identically zero.

We conclude that V(u) = 0, even though the state $x(t) \to \infty$ if $x(0) \neq 0$ lies in the unobservable subspace. This unstable behavior of the optimized system is a consequence of a lack of *detectability*.

For a general LQR problem, suppose that the matrix Q is factored as $Q = C^T C$ for a $p \times n$ matrix C. As in the previous example, we then define y = Cx so that the cost function becomes

$$V = \int_0^\infty (x^T Q x + u^T R u) \, dt = \int_0^\infty (y^T y + u^T R u) \, dt.$$

If the pair (A, C) is observable, then for any control u, and any initial state $x(0) \neq \vartheta$, the cost V is strictly positive. To see this, note that if u is identically zero, then by observability we must have $\int x^T Qx \, dt = \int |y|^2 \, dt > 0$. If u is not identically zero, then V > 0 since R > 0. Since for an initial condition x_0 the value function may be expressed $V^{\circ}(x_0) = x_0^T \bar{P} x_0$, it follows that the matrix \bar{P} is positive definite $(\bar{P} > 0)$. From the previous example it is clear that P will be singular if the system is not observable. If the system is not detectable, then the last example shows that we might expect disaster. These results can be refined to give the following

Theorem 10.6. For the LTI system (10.17) with R > 0 and $Q = C^T C \ge 0$,

- (i) If (A, B) is stabilizable, and (A, C) is detectable then
 - (a) There exists a positive semi-definite solution \bar{P} to the ARE (10.24), which is unique in the class of positive semi-definite matrices.
 - (b) The closed-loop system matrix $A_{cl} = A BR^{-1}B^T\bar{P}$ is Hurwitz.
 - (c) The infinite horizon optimal control is $u^{\circ} = -R^{-1}B^T\bar{P}x$, and the infinite horizon optimal cost is $V^{\circ}(x_0) = x_0^T\bar{P}x_0$.
 - (d) If $P(t, t_1)$ is the solution to the RDE (10.12) subject to $P(t_1, t_1) = 0$, then $\lim_{t_1 \to \infty} P(t, t_1) = \bar{P}$.

(ii) If (A, B) is stabilizable, and (A, C) is observable, then (a)–(d) hold, and in addition the matrix \bar{P} is positive definite.

Example 10.5.4. Consider the multivariate model

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$V = \int_0^\infty (4x_1^2 + 4x_1x_2 + x_2^2 + u_1^2 + u_2^2) dt.$$

From these equations we may deduce that the weighting matrices are

$$Q = \left[\begin{array}{cc} 4 & 2 \\ 2 & 1 \end{array} \right], \qquad R = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right].$$

The matrix Q has rank one, and $Q \ge 0$. It follows from the spectral representation of a positive definite matrix that there is a 1×2 matrix C such that $Q = C^T C$. In fact C = (2, 1), so we define $y = Cx = 2x_1 + x_2$. Thus, the controllability and observability matrices become

$$\mathcal{C} = \left[\begin{array}{ccc} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{array} \right] \qquad \mathcal{O} = \left[\begin{array}{ccc} 2 & 1 \\ 0 & -2 \end{array} \right].$$

Since both of these matrices have rank 2, the conditions of Theorem 10.6 (ii) are satisfied.

We now check directly to see if the controlled system is stable by computing the closed-loop system poles. The Hamiltonian matrix for this model is

$$\mathcal{H} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline -4 & -2 & 0 & 0 \\ -2 & -1 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of \mathcal{H} are found by solving the characteristic equation

$$det[sI - \mathcal{H}] = s^4 - 5s^2 + 4$$
$$= (s - 2)(s + 2)(s - 1)(s + 1).$$

Note that the roots of this equation possess the symmetry required by Lemma 10.4. The eigenvectors $\{w^i\}$ of the stable modes $\{\lambda^i\}$ are

$$\lambda_1 = -1, \quad w^1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \qquad \lambda_2 = -2, \quad w^2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}.$$

Thus, the matrices $\{U_{ij}\}$ may be computed:

$$\left[\frac{U_{11}}{U_{21}} \right] = \left[\begin{array}{ccc} 1 & 1 \\ 0 & -1 \\ \hline 2 & 2 \\ 0 & -1 \end{array} \right].$$

Applying (10.18) we can compute the solution to the ARE:

$$\bar{P} = \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

and the optimal feedback gain matrix is

$$K = R^{-1}B^T \bar{P} = \left[\begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right].$$

The resulting optimal control

$$u^{\circ} = \begin{pmatrix} -2x_1 \\ -x_2 \end{pmatrix}.$$

places the closed-loop eigenvalues at (-2,-1), which are the stable eigenvalues of \mathcal{H} . The closed-loop eigenvectors are contained in the columns of U_{11} .

Example 10.5.5. To show how the ARE can be solved directly without constructing the Hamiltonian, consider the model

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u; \qquad y = x_1
V = \int_0^\infty (x_1^2 + u^2) dt$$
(10.26)

In this example we have R = 1, and $Q = C^T C$, where $C = (1 \ 0)$. The controllability and observability matrices become

$$C = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$
 $O = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Again we see that the conditions of Theorem 10.6 (ii) are satisfied, and it follows that the solution \bar{P} to the ARE will be positive definite. We now verify this directly.

The ARE for this model is

$$A^{T}\bar{P} + \bar{P}A - \bar{P}BR^{-1}B^{T}\bar{P} + Q = 0.$$

Writing \bar{P} as $\bar{P}=\left[\begin{array}{cc} \ell & m \\ m & n \end{array}\right]$, this gives the equation

$$\left[\begin{array}{cc} 0 & 0 \\ \ell-m & m-n \end{array}\right] + \left[\begin{array}{cc} 0 & \ell-m \\ 0 & m-n \end{array}\right] - \left[\begin{array}{cc} m^2 & mn \\ mn & n^2 \end{array}\right] + \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] = 0.$$

Examining the (1,1) entry of each matrix gives

$$-m^2 = -1 \Rightarrow m = +1$$
 (since $\bar{P} \ge 0$).

Examining the (1,2) entry of the ARE we see that $\ell - 1 - n = 0$, and from the (2,2) entry $n^2 + 2n - 2 = 0$. These equations result in the unique positive definite solution

$$\bar{P} = \left[\begin{array}{cc} \sqrt{3} & 1\\ 1 & -1 + \sqrt{3} \end{array} \right]$$

The optimal feedback control law is $u^{\circ} = -R^{-1}B^{T}Px = -[1\sqrt{3}-1]x$. To find the closed-loop poles, we must compute the eigenvalues of

$$A_{cl} = A - BK = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & \sqrt{3} - 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{3} \end{bmatrix}.$$

The characteristic equation is $\Delta(s) = s^2 + 2\frac{\sqrt{3}}{2}s + 1 = 0$, which gives

$$p_i = -\frac{1}{2}(\sqrt{3} \pm j).$$

The closed-loop response will be somewhat under-damped, but of course it is stable. \Box

10.6 Return difference equation

The LQR solution has some striking and highly desirable properties when viewed in the frequency domain. These results have many applications to design and analysis of linear control systems. In particular,

- (a) For a SISO system and certain MIMO systems, a generalization of the root locus method can be used to find the optimal state feedback controller.
- (b) For an LQR design based upon full state feedback, the closed-loop system is insensitive to plant uncertainty, regardless of the choice of weighting matrices Q and R. In particular, the gain margin is always infinite, and the phase margin at least 60%.

The conclusion (b) is surprising, given the purely time domain approach that we have taken. In the end this serves as perhaps the greatest motivation for using LQR methods in control design.

For the LTI model with infinite horizon cost criterion expressed in Equation (10.17), we have the following identity for any $s \in \mathbb{C}$, known as the return difference equation:

$$R + B^{T}(-sI - A)^{-T}Q(sI - A)^{-1}B$$

= $[I + K(-sI - A)^{-1}B]^{T}R[I + K(sI - A)^{-1}B].$ (10.27)

The vector K is the optimal feedback gain

$$K = R^{-1}B^T \bar{P}.$$

The left hand side of (10.27) involves only the open-loop system, while the right hand side depends on the feedback gain K. The return difference equation is derived by simply expanding both sides of the equation, and substituting the definition of K and the algebraic Riccati equation.

The return difference equation appears complex, but after the matrices are given proper interpretation it has a relatively simple form. First, factor the state weighting matrix as $Q = C^T C$. By defining y = Cx we obtain the controlled model illustrated in Figure 10.3. With $P(s) = C(Is - A)^{-1}B$, the left hand side of (10.27) becomes

$$R + P^T(-s)P(s).$$

From the feedback from u = -Kx we define the *loop transfer function* described in Chapter 9 as $L(s) = K(Is - A)^{-1}B$, so that the right hand side

of (10.27) becomes

$$[I + L(-s)]^T R[I + L(s)].$$

Hence the return difference equation can be expressed succinctly as

$$R + P^{T}(-s)P(s) = [I + L(-s)]^{T}R[I + L(s)].$$
 (10.28)

On the complex axis we have $H^T(-j\omega) = H^*(j\omega)$ for any transfer function H, so that (10.28) may be written

$$R + P^*(j\omega)P(j\omega) = [I + L(j\omega)]^*R[I + L(j\omega)]. \tag{10.29}$$

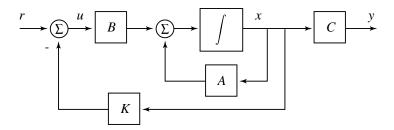


Figure 10.3: The loop transfer function of the LTI system under full state feedback u = -Kx + r is $L(s) = K(Is - A)^{-1}B$.

In the SISO case with R=r>0 a scalar, the return difference equation becomes

$$r + P(-s)P(s) = r[1 + L(-s)][1 + L(s)].$$
(10.30)

The poles of the closed-loop system are found by solving the equation

$$1 + L(s) = 0.$$

In view of (10.30), the return difference equation gives the closed-loop poles as a function of the weighting parameter r and the open-loop transfer function P(s) through the formula

$$1 + \frac{1}{r}P(-s)P(s) = 0. (10.31)$$

The set of all solutions to (10.31), with r ranging from 0 to ∞ , is called the *symmetric root locus*. For fixed r, the roots of this equation are the eigenvalues of the Hamiltonian matrix \mathcal{H} with R = r. We have seen in

Section 10.5 that the stable (left hand plane) eigenvalues of the Hamiltonian matrix are the poles of the closed-loop system.

As an illustration consider the general second order system with transfer function $P(s) = (s-z)/(s^2+2\zeta s+1)$, where z<0, and $\zeta>0$. The symmetric root locus is illustrated in Figure 10.5. Note the symmetry of the poles and zeros with respect to the $j\omega$ axis. It is seen here that when $r\approx\infty$, the closed-loop poles are approximately equal to the open loop poles of the system. For small r, one closed-loop pole converges to ∞ along the negative real axis, and the other converges to the zero z.

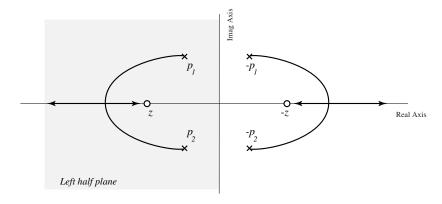


Figure 10.4: The symmetric root locus for the system with transfer function $P(s) = (s-z)/(s^2 + 2\zeta s + 1)$. The roots of $1 + \frac{1}{r}P(-s)P(s) = 0$ give the optimal closed-loop poles, where r is the control weighting in the LQR cost. As r varies from 0 to ∞ , this gives the symmetric root locus.

Example 10.6.1. We now revisit (10.26) to show how a control design may be accomplished using the symmetric root locus. The point is that everything on the left hand side of the return difference equation is known. The desired closed-loop feedback poles are unknown, but are defined through the right hand side of this equation. In the SISO case, this can be exploited to solve for the feedback gain K directly, without solving a Riccati equation.

We must first compute the transfer function P(s). From the state equations we have

$$sX_1(s) = X_2(s)$$

 $sX_2(s) = -X_2(s) + U(s)$
 $Y(s) = X_1(s)$.

Combining these equations gives $s^2Y(s) = -sY(s) + U(s)$, or

$$P(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+1)}.$$

To compute the symmetric root locus we solve for the roots of the equation

$$0 = 1 + r^{-1} \left[\frac{1}{s(s+1)} \right] \left[\frac{1}{-s(-s+1)} \right] = 1 + r^{-1} \left[\frac{1}{s^4 - s^2} \right].$$

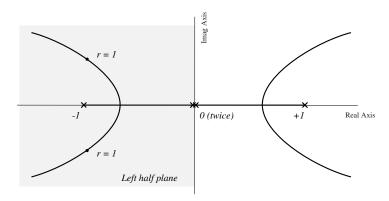


Figure 10.5: The symmetric root locus for the model $\ddot{y} = -\dot{y} + u$, with cost criterion $V = \int y^2 + ru^2 dt$. A more highly damped closed-loop response can be obtained by increasing r below the nominal value of 1, while decreasing r will result in greater closed-loop bandwidth.

A sketch of the symmetric root locus is provided in Figure 10.5. When r=1 we have already seen that the optimal control is $u^{\circ}=-[1, \sqrt{3}-1]x$, and that the closed-loop poles are $-\frac{1}{2}(\sqrt{3}\pm j)$. From Figure 10.5 it is evident that a more highly damped closed loop response can be obtained by increasing r, while reducing r will result in greater closed-loop bandwidth.

The return difference equation leads to an elegant approach to pole placement. The resulting symmetric root locus gives the designer a 'good' set of poles to choose from, which can then be narrowed down based upon other constraints, such as closed-loop bandwidth or time domain specifications.

The most striking implication of the return difference equation is perhaps its application to sensitivity analysis. In the SISO case, we have seen that by letting $s = j\omega$ we obtain

$$|1 + L(j\omega)|^2 = 1 + \frac{1}{r}|P(j\omega)|^2.$$

This implies Kalman's inequality,

$$|1 + L(j\omega)|^2 \ge 1, \qquad \omega \in \mathbb{R}$$

If we define the sensitivity function as S = 1/(1 + L), then we also have

$$|S(j\omega)| \le 1, \qquad \omega \in \mathbb{R}.$$
 (10.32)

It follows from (9.1) that the controlled system is insensitive to plant uncertainty at all frequencies. A typical plot of the loop transfer function L is shown in Figure 10.6. We see from this plot that the controlled system has a gain margin of infinity and a phase margin of ± 60 degrees. These results hold for *any* full state LQR design.

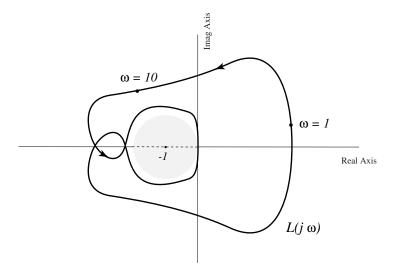


Figure 10.6: The loop transfer function $L(j\omega) = K(Ij\omega - A)^{-1}B$ for the LTI system under optimal control cannot enter the shaded disk centered at -1.

Example 10.6.2. As an illustration of the frequency domain properties of a state space model under optimal control, we now return to the Pendubot model linearized in the upright position. To control the system we define the cost as

$$V = \int_0^\infty [|x(t)|^2 + ru(t)^2] dt.$$

where $x = (q_1, \dot{q}_1, q_2, \dot{q}_2)^T$. We therefore have $Q = C^T C$, where Q = C = I. Since the matrix Q has rank n, so that C is square, the transfer function

defined as $P(s) = C(Is - A)^{-1}B$ is an $n \times 1$ vector-valued function of s. The return difference equation for the Pendubot with this cost criterion becomes

$$r + \sum_{i=1}^{4} P_i(-s)P_i(s) = r[1 + L(-s)][1 + L(s)],$$

so that a generalization of the symmetric root locus is still possible, since the closed-loop poles are still equal to the roots of 1 + L(s). For any i, the poles of P_i are equal to the eigenvalues of A. These are simply the open-loop poles, which are roughly located at $\{-9.5, -5.5, 5.5, 9.5\}$. The numerator of P_i depends upon i, but the sum on the left hand side is readily computable using Matlab. The resulting symmetric root locus is shown in Figure 10.7.

For r=1 the closed-loop poles are placed at (9.3), as described in the beginning of Chapter 9. We saw then that when this controller was combined with an observer based on measurements at the lower link alone, the controlled system is highly sensitive to modeling errors. For instance, the gain margin is found to be less than 10^{-2} . With full state feedback we know that the controlled system is far more robust. A Nyquist plot of the loop transfer function $L(j\omega)$ is shown in Figure 10.8. We see in this figure that, as expected, the gain margin with this design is infinite, and that the gain can also be reduced by one half, and the system will remain stable.

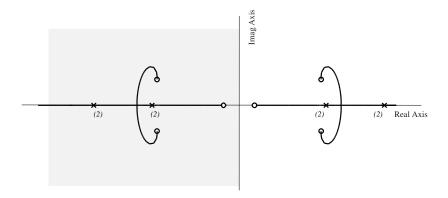


Figure 10.7: The symmetric root locus for the Pendubot with Q=I. Each of the poles and zeros in this diagram are indicated with crosses and zeros, respectively. Each pole appears twice because of the symmetry of the open-loop transfer function.

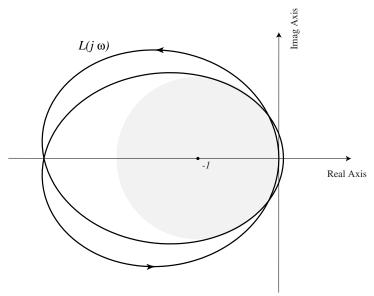


Figure 10.8: The Nyquist plot for the Pendubot controlled with full state feedback, based on an LQR design. If all of the states can be measured, a far more robust design is possible compared with what is found using a single output measurement.

Summary and References

This chapter has provided only a brief survey on the LQR problem. For a more detailed discussion, see the book [1]. Some historic papers on the topic are:

- R. E. Kalman, "Contributions to the Theory of Optimal Control," *Bol. Soc. Matem. Mex.*, (1960), pp. 102–119.
- R. E. Kalman, "When is a Linear Control System Optimal?", *Trans. ASME Ser. D: J. Basic Eng.*, Vol. 86 (March 1964), pp. 1–10.

Another good source for survey/tutorial papers on the linear quadratic theory is the *December 1971* special issue of the *IEEE Transactions on Automatic Control*.

Matlab Commands

LQG Solves the LQR problem using the data A, B, Q, and R.

LQG2 A different numerical algorithm for solving the LQR problem.

ARE Gives solution to the algebraic Riccati equation.

RLOCUS Used together with the convolution command **CONV**, this can be used to graph the symmetric root locus.

10.7. EXERCISES

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10.7 Exercises

10.7.1 The purpose of this exercise is to review the derivation of the HJB equation, and to provide an introduction to a general theory of infinite horizon optimal control. Consider the time invariant nonlinear system

$$\dot{x} = f(x, u), \qquad x(0) = x_0 \in \mathbb{R}^n.$$

We assume that there is an equilibrium at $\vartheta \in \mathbb{R}^n$, in the sense that $f(\vartheta,\vartheta) = \vartheta$. Assuming that the goal is to force the state to zero, consider the cost criterion

$$J(u) = \int_0^\infty \ell(x, u) \, dt,$$

where $\ell \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ is smooth, with $\ell(\vartheta,\vartheta) = 0$. We typically also assume that ℓ is positive definite, as defined in Section 4.2. We let $J^{\circ}(x)$ denote the minimum of J over all controls, when the initial condition of the model is x.

- (a) Derive a "principle of optimality" for this problem by considering an optimal trajectory (x°, u°) , as done in the derivation of the HJB equations.
- (b) Using (a), derive a differential equation that J° must satisfy.
- (c) Apply your solution in (b) to solve the optimal control problem defined by the simple integrator model with polynomial cost defined below:

$$\dot{x} = u$$
 $J(u) = \int_0^\infty [u^2 + x^4] dt$

10.7.2 Solve by hand the ARE with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ R = 1.$$

10.7.3 For the double integrator $\ddot{y} = u$, what is the minimal value of the cost

$$V(u) = \int_0^\infty y(t)^2 + u(t)^2 dt,$$

over all controls $u(\cdot)$, when y(0) = 1, $\dot{y}(0) = 0$? Explain why the minimum exists.

10.7.4 For the scalar time-invariant plant $\dot{x} = ax(t) + u(t)$, and under the quadratic performance index

$$V = \int_0^\infty [x^2(t) + ru^2(t)]dt \; ,$$

- (a) Obtain the optimal feedback control as a function of the scalar parameters a and r;
- (b) Study the eigenvalues of the optimal closed-loop system as
 - (i) $r \to \infty$ (expensive control)
 - (ii) $r \to 0$ (cheap control)
- **10.7.5** The linearized (and normalized) magnetically suspended ball is described by $\dot{x}_1 = x_2$, $\dot{x}_2 = x_1 + u$, where the initial state x(0) is specified. Answer the following three questions by constructing the Hamiltonian matrix. You may check your work using Matlab.
 - (a) Obtain the feedback control u(x) that minimizes the infinite-horizon quadratic performance index

$$V = \int_0^\infty (x_1^2 + 2x_1x_2 + x_2^2 + 4u^2) dt.$$

- (b) Determine the eigenvalues of the closed-loop system matrix A_{cl} .
- (c) Find the minimum value of V as a function of x(0). Explain why in this example we can have $V(x_0) = 0$ even though $x_0 \neq \vartheta$.
- **10.7.6** The following is the so-called *generalized regulator problem*: The system equation is

$$\dot{x}(t) = A(t)x(t) + b(t)u(t) + c(t),$$

where $c(\cdot)$ is an *n*-dimensional vector function defined on the finite interval $[t_0, t_1]$, with continuous entries; and the performance index is

$$V = x^{T}(t_{1})Q_{f}x(t_{1}) + \int_{t_{0}}^{t_{1}} [x^{T}(t)R_{1}(t)x(t) + 2x^{T}(t)R_{12}(t)u(t) + u^{T}(t)R_{2}(t)u(t)] dt$$

where all matrices have continuous entries, and

$$R_2(\cdot) > 0,$$
 $Q_f \ge 0,$
$$\begin{bmatrix} R_1(\cdot) & R_{12}(\cdot) \\ R_{12}^T(\cdot) & R_2(\cdot) \end{bmatrix} \ge 0.$$

(a) Using the *Hamilton-Jacobi-Bellman equation*, show that the optimal control for this problem can be written as

$$u^{\circ}(t) = \gamma^{\circ}(x, t) = -R_2^{-1}(t) \left[[B^T(t)P(t) + R_{12}^T(t)]x + B^T(t)k(t) \right],$$

where $P(\cdot)$ is a matrix function (≥ 0) and $k(\cdot)$ is a vector function. Obtain the expressions (differential equations) satisfied by P and k.

- (b) Obtain an expression for the minimum value of V.
- 10.7.7 In the previous problem, assume that all matrices and the vector $c(\cdot)$ are time-invariant, and $t_1 \to \infty$. Obtain an expression for the steady-state optimal feedback control under appropriate controllability and observability assumptions. Is this a stabilizing control?
- **10.7.8** The Pendubot arranged in the vertical position may be approximately described by the linear system $\dot{x} = Ax + Bu$; $y = x_1$ where

$$A = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 42.0933 & 0 & -4.3893 & 0 \\ 0 & 0 & 0 & 1.0000 \\ -29.4557 & 0 & 43.3215 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 6.2427 \\ 0 \\ -9.7741 \end{bmatrix}.$$

There exists a feedback control law u = -K(r)x which minimizes the quadratic performance index

$$V(u) = \int_0^\infty \left(y^2 + \alpha \dot{y}^2 + ru^2 \right) d\tau$$

Plot the resulting closed-loop poles as r is varied from 0 to ∞ , and discuss your results. How does the introduction of \dot{y} in the cost functional affect the root locus? Note that this will require a *generalization* of the symmetric root locus derived in these notes. Include the derivation of this result with your plots.

10.7.9 LQR theory can also be extended to problems where also the derivative of the control variable appears in the performance index (*i.e.* we have a soft constraint not only on the control variable but also on its derivative). Specifically, the LQR problem with cost

$$V(u) = \int_0^\infty [x^T(t)Qx(t) + u^T(t)R_1u(t) + \dot{u}^T(t)R_2\dot{u}(t)] dt$$

subject to $\dot{x} = Ax + Bu$ can be solved by introducing the additional state variable, z = u, and treating $v = \dot{u}$ as the control variable.

- (a) By following this procedure, assuming that $Q \ge 0$, $R_1 \ge 0$, $R_2 > 0$, and $x(0) = x_0$, $u(0) = u_0$ are specified, obtain the general solution of the above by simply applying the LQR theory developed in these notes.
- (b) Can you formulate a generalization of the symmetric root locus for this more general problem?
- (c) After obtaining the general solution, apply it to the specific problem:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u; \quad x_1(0) = x_2(0) = 1, u(0) = 1$$

$$V(u) = \int_0^\infty [x_1^2(t) + \rho \dot{u}^2(t)] dt.$$

Provide a root locus plot for $0 < \rho < \infty$, and compute the optimal control for $\rho = 2$.

Note: For this numerical example, your solution will be in the form $v \equiv \dot{u} = k_1 x_1 + k_2 x_2 + k_0 u$, where k_1, k_2, k_0 are scalar quantities, whose values will have to be determined as part of the solution process.

10.7.10 This problem deals with optimal control problems in which the closed-loop system is required to have a prescribed degree of stability. Optimal control is an expression of the tradeoff between the desire to place closed-loop poles deep into the left-half-plane and the desire not to use excessive control (because of possible actuator saturation) or have large excursions of the state trajectories (because of modeling uncertainty). Sometimes we want to guarantee that the closed-loop poles are stable by a "margin" of at least α . That is, the poles lie in the region $\Omega_{\alpha} = \{s : \text{Re}(s) \leq -\alpha\}$.

To this end, consider the plant $\dot{x} = Ax + Bu$, and the cost functional

$$V(u) = \int_0^\infty e^{2\alpha t} \left(x^T Q x + u^T R u \right) dt, \qquad \alpha > 0$$

where $Q \geq 0$ and R > 0. Observe that if $V(u^{\circ}) < \infty$ then $\sqrt{Q}x^{\circ}(t)$ must decay faster than $e^{-\alpha t}$ which, under appropriate conditions, means the closed-loop poles lie in the left-half plane as desired. In this problem you will rigorously establish this assertion.

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(a) Set $z(t) = e^{\alpha t}x(t)$ and $v(t) = e^{\alpha t}u(t)$, and denote the resulting as a function v by $\tilde{V}(v)$. Compute $\tilde{V}(v)$ and the corresponding state equations.

- (b) Carefully write down a complete solution to this optimal control problem including necessary and sufficient conditions under which solutions exist and conditions under which these solutions are stabilizing. In the case where the eigenvalues are distinct, interpret your conditions using the Hautus-Rosenbrock criteria.
- (c) Show that the closed-loop system has all its poles inside the left hand plane as prescribed under appropriate conditions on the open-loop system.
- (d) With $A = B = Q = R = x_0 = 1$ find the optimal control law and the optimal cost, and observe that the poles are indeed placed within Ω_{α} .
- 10.7.11 Consider again the flexible structure defined in (7.10). The input is the voltage to a DC motor, and the states of this model are

$$x_1 = \theta_1, x_2 = \dot{\theta}_1, x_3 = \theta_2, x_4 = \dot{\theta}_2.$$

(a) Plot the loop transfer function with your previous design. You should use the Matlab command NYQUIST, as follows

$$nvquist(A-.00001*eve(4),B,K,0)$$

where K is the feedback gain. This will give a plot of $L(j\omega) = K(Ij\omega - A + \epsilon I)^{-1}B$. The term ϵ is needed since the open-loop system has a pole at the origin. Does your plot enter the region $\operatorname{dist}(s,-1) < 1$?

(b) Plot the optimal closed-loop poles for the cost function

$$V(u) = \int_0^\infty \theta_2^2 + \beta(\dot{\theta}_2)^2 + \rho u^2 d\tau$$

where ρ varies from 0 to ∞ . Try $\beta = 0$, and then some $\beta > 1$.

- (c) Obtain an optimal design for which the dominant poles satisfy $-10 \le \sigma \le -5$, and plot the loop transfer function with this design.
- 10.7.12 Consider the input-output system described in the frequency domain by

$$Y(s) = \frac{s-1}{s^2 + s + 1}U(s) + W(s)$$

where y is the measurement, u is the input, and w is a disturbance.

Two control designs will be obtained below by setting $w \equiv 0$. In practice however, $w \neq 0$, and hence its affect on the output must be understood.

- (a) Obtain a state space model in observable canonical form: Let $x_1 = y w$, and $\dot{x}_2 = -x_1 u$, so that $y = x_1 + w$.
- (b) Derive a state feedback control law u = -Kx for an approximately critically damped response which minimizes $V(u) = \int y^2 + ru^2 d\tau$ for some r. Your closed-loop poles should be constrained in magnitude between 1 and 5.

Sketch the symmetric root locus by hand - use Matlab to verify your plot, and to determine an appropriate feedback gain.

What is the frequency response $\frac{Y}{W}(j\omega)$ for the closed-loop system when the state x is perfectly observed?

(c) Design a reduced order observer, and combine this with the control law obtained above so that now $u = -K_1y - K_2\hat{x}_2$. Note that the observer will have to use the approximation $y \approx x_1$ since neither x_1 nor x_2 are directly measurable.

Your observer pole should be at least three times as large in magnitude than the magnitude of the largest closed-loop pole obtained in (b).

Plot the frequency response $\frac{Y}{W}(j\omega)$ using the BODE command, and discuss your findings.

10.7.13 In this problem you will design a controller for a model of a Boeing 747 aircraft.

A linearized model for the longitudinal motion of this plant when operating shortly after take-off, at sea level, with a speed of 190 miles per hour is given by

$$\frac{d}{dt} \begin{bmatrix} \delta V \\ \delta \alpha \\ q \\ \delta \theta \end{bmatrix} = \begin{bmatrix} -0.0188 & 11.5959 & 0 & -32.2 \\ -0.0007 & -0.5357 & 1 & 0 \\ 0.000048 & -0.4944 & -0.4935 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta V \\ \delta \alpha \\ q \\ \delta \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -0.5632 \\ 0 \end{bmatrix} \delta e$$

where δV is the deviation from the nominal speed of 190 mph, $\delta \alpha$ is the angle of attack, which is the longitudinal angle of the plane with respect to the longitudinal velocity vector, $\delta \theta$ is the pitch angle, which is the longitudinal angle of the plane with respect to the ground, and $q = \dot{\delta} \theta$.

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The input δe is the deviation of the elevators with respect to some nominal value. Of course, the throttle position is also a significant control, but we will hold the throttle value constant in this design.

The altitude of the plane h is the variable that we would like to control. Linearizing, we find that a reasonable approximation to the rate of change of h is given by

$$\dot{\delta}h = 290(\delta\theta - \delta\alpha)$$

(a) Adjoin the variable h to the state space model above to obtain a fifth order system with state x. Obtain an LQR design to obtain a feedback control u = -Kx + r for this system using symmetric root locus. Plot the step response of both the altitude h and the pitch angle θ of the controlled system. Try to obtain a rise time of 250 seconds with reasonable undershoot.

Note that the response is always slow due to one closed-loop pole close to the origin.

- (b) Can you obtain a better controller with a more general choice of Q and R? For your best design, where are the resulting closed-loop poles?
- (c) Using your best choice of K in (b), improve the response by modifying the controller as follows: Pass the reference signal r through a first order filter G(s). Hence, in the frequency domain, your controller is of the form

$$U(s) = -KX(s) + k\frac{s-z}{s-p}R(s).$$

You should place the zero z at the location of the slow pole (approximately), and place the pole p further from the origin. This should cancel the effect of the slow closed-loop pole in the closed-loop response. The gain k should be chosen to achieve good steady state tracking for d.c. reference inputs.

Again plot the step response of both the altitude h and the pitch angle θ of the controlled system. Can you choose k, z and p to improve your response over that obtained in (b)?

Chapter 11

An Introduction to the Minimum Principle

We now return to the general nonlinear state space model, with general cost criterion of the form

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0 \in \mathbb{R}^n$$

$$V(u) = \int_{t_0}^{t_1} \ell(x(t), u(t), t) dt + m(x(t_1))$$
(11.1)

We have already shown under certain conditions that if the input u minimizes V(u), then a partial differential equation known as the HJB equation must be satisfied. One consequence of this result is that the optimal control can be written in state feedback form through the derivative of the value function V° . The biggest drawback to this result is that one must solve a PDE which in general can be very complex. Even for the relatively simple LQR problem, the solution of the HJB equations required significant ingenuity.

The minimum principle is again a method for obtaining necessary conditions under which a control u is optimal. This result is based upon the solution to an ordinary differential equation in 2n dimensions. Because it is an ordinary differential equation, in many instances it may be solved even though the HJB equation is intractable. Unfortunately, this simplicity comes with a price. The solution u° to this ordinary differential equation is in open-loop form, rather than state space form. The two view points, the Minimum Principle and the HJB equations, each have value in nonlinear optimization, and neither approach can tell the whole story.

11.1 Minimum Principle and the HJB equations

In a course at this level, it is not possible to give a complete proof of the Minimum Principle. We can however give some heuristic arguments to make the result seem plausible, and to gain some insight. We initially consider the optimization problem (11.1) where x_0 , t_0 , t_1 are fixed, and $x(t_1)$ is free. Our first approach is through the HJB equation

$$\begin{split} -\frac{\partial}{\partial t} V^{\circ}(x,t) &= & \min_{u} (\ell(x,u,t) + \frac{\partial V^{\circ}}{\partial x} f(x,u,t)) \\ u^{\circ}(t) &= & \arg\min_{u} (\ell(x,u,t) + \frac{\partial V^{\circ}}{\partial x} f(x,u,t)). \end{split}$$

For x, t fixed, let $\bar{u}(x, t)$ denote the value of u which attains the minimum above, so that $u^{\circ} = \bar{u}(x^{\circ}(t), t)$. In the derivation below we assume that \bar{u} is a smooth function of x. This assumption is false in many models, which indicates that another derivation of the Minimum Principle which does not rely on the HJB equation is required to create a general theory. With \bar{u} so defined, the HJB equation becomes

$$-\frac{\partial}{\partial t}V^{\circ}(x,t) = \ell(x,\bar{u}(x,t),t) + \frac{\partial V^{\circ}}{\partial x}(x,t)f(x,\bar{u}(x,t),t).$$

Taking partial derivatives of both sides with respect to x gives the term $\frac{\partial V^{\circ}}{\partial x}$ on both sides of the resulting equation:

$$-\frac{\partial^{2}}{\partial x \partial t} V^{\circ}(x,t) = \frac{\partial \ell}{\partial x}(x, \bar{u}(x,t),t)$$

$$+ \frac{\partial^{2} V^{\circ}}{\partial^{2} x}(x,t) f(x, \bar{u}(x,t),t)$$

$$+ \frac{\partial V^{\circ}}{\partial x}(x,t) \frac{\partial f}{\partial x}(x, \bar{u}(x,t),t)$$

$$+ \underbrace{\frac{\partial}{\partial u} \left(\ell + \frac{\partial V^{\circ}}{\partial x} f\right) (x, \bar{u}(x,t),t)}_{derivative\ vanishes\ at\ \bar{u}} \underbrace{\frac{\partial \bar{u}(x,t)}{\partial x}}.$$

As indicated above, since \bar{u} is the unconstrained minimum of $H = \ell + \frac{\partial V^{\circ}}{\partial x} f$, the partial derivative with respect to u must vanish.

This PDE holds for any state x and time t - they are treated as independent variables. Consider now the optimal trajectory $x^{\circ}(t)$ with optimal

input $u^{\circ}(t) = \bar{u}(x^{\circ}(t), t)$. By a simple substitution we obtain

$$0 = \frac{\partial^{2}}{\partial x \partial t} V^{\circ}(x^{\circ}(t), t) + \frac{\partial \ell}{\partial x} (x^{\circ}(t), u^{\circ}(t), t)$$

$$+ \frac{\partial^{2} V^{\circ}}{\partial x^{2}} (x^{\circ}(t), t) f(x^{\circ}(t), u^{\circ}(t), t)$$

$$+ \frac{\partial V^{\circ}}{\partial x} (x^{\circ}(t), t) \frac{\partial f}{\partial x} (x^{\circ}(t), u^{\circ}(t), t). \quad (11.2)$$

We can now convert this PDE into an ODE. The main trick is to define

$$p(t) := \frac{\partial V^{\circ}}{\partial x}(x^{\circ}(t), t).$$

This function is interpreted as the *sensitivity of the cost with respect to* current state, and takes values in \mathbb{R}^n . The derivative of p with respect to t can be computed as follows:

$$\frac{d}{dt}p(t) = \frac{\partial^2 V^{\circ}}{\partial x^2}(x^{\circ}(t), t)\dot{x}^{\circ}(t) + \frac{\partial^2 V^{\circ}}{\partial x \partial t}(x^{\circ}(t), t)
= \frac{\partial^2 V^{\circ}}{\partial x^2}(x^{\circ}(t), t)f(x^{\circ}(t), u^{\circ}(t), t) + \frac{\partial^2 V^{\circ}}{\partial x \partial t}(x^{\circ}(t), t).$$

The two mixed partial terms in this equation are also included in (11.2). Combining these two equations, we thereby eliminate these terms to obtain

$$0 = \frac{d}{dt}p(t) + \frac{\partial \ell}{\partial x}(x^{\circ}(t), u^{\circ}(t), t) + \underbrace{\frac{\partial V^{\circ}}{\partial x}(x^{\circ}(t), t)}_{p^{T}(t)} \underbrace{\frac{\partial f}{\partial x}(x^{\circ}(t), u^{\circ}(t), t).$$
(11.3)

From the form of the Hamiltonian

$$H(x, p, u, t) = p^{T} f(x, u, t) + \ell(x, u, t),$$

the differential equation (11.3) may be written

$$\dot{p}(t) = -\nabla_x H(x^{\circ}(t), p(t), u^{\circ}(t), t)$$

with the boundary condition

$$p(t_1) = \nabla_x V^{\circ}(x^{\circ}(t_1), t_1) = \nabla_x m(x^{\circ}(t_1), t_1).$$

This is not a proof, since for example we do not know if the value function V° will be smooth. However, it does make the following result seem plausible. For a proof see [10].

Theorem 11.1 (Minimum Principle). Suppose that t_1 is fixed, $x(t_1)$ is free, and suppose that u° is a solution to the optimal control problem (11.1). Then

- (a) There exists a costate vector p(t) such that for $t_0 \le t \le t_1$,
 - $u^{\circ}(t) = \operatorname*{arg\,min}_{u} H(x^{\circ}(t), p(t), u, t).$
- (b) The pair (p, x°) satisfy the 2-point boundary value problem:

$$\dot{x}^{\circ}(t) = \nabla_{p} H(x^{\circ}(t), p(t), u^{\circ}(t), t) \qquad \left(= f(x^{\circ}(t), u^{\circ}(t), t)\right)$$
$$\dot{p}(t) = -\nabla_{x} H(x^{\circ}(t), p(t), u^{\circ}(t), t) \qquad (11.4)$$

with the two boundary conditions

$$x(t_0) = x_0;$$
 $p(t_1) = \nabla_x m(x(t_1), t_1).$

11.2 Minimum Principle and Lagrange multipliers*

Optimal control involves a functional minimization which is similar in form to ordinary optimization in \mathbb{R}^m as described in a second year calculus course. For *unconstrained* optimization problems in \mathbb{R}^m , the main idea is to look at the derivative of the function V to be minimized, and find points in \mathbb{R}^m at which the derivative is zero, so that $\nabla V(x) = \vartheta$. Such an x is called a stationary point of the optimization problem. By examining all stationary points of the function to be minimized one can frequently find among these the optimal solution. The calculus of variations is the infinite dimensional generalization of ordinary optimization in \mathbb{R}^m . Conceptually, it is no more complex than its finite dimensional counterpart.

In more advanced calculus courses, solutions to constrained optimization problems in \mathbb{R}^m are addressed using Lagrange multipliers. Suppose for example that one desires to minimize the function V(x) subject to the constraint $g(x) = \vartheta$, $x \in \mathbb{R}^m$, where $g \colon \mathbb{R}^m \to \mathbb{R}^d$. Consider the new cost function $\widehat{V}(x,p) = V(x) + p^T g(x)$, where $p \in \mathbb{R}^d$. The vector p is known as a Lagrange multiplier. The point of extending the state space in this way is that we can now solve an unconstrained optimization problem, and very often this gives a solution to the constrained problem. Indeed, if x^0, p^0 is a stationary point for \widehat{V} then we must have

$$\vartheta = \nabla_x \widehat{V}(x^0, p^0) = \nabla V(x^0) + \nabla g(x^0) p^0$$
(11.5)

$$\vartheta = \nabla_p \hat{V}(x^0, p^0) = g(x^0).$$
 (11.6)

Equation (11.5) implies that the gradient of V is a linear combination of the gradients of the components of g at x_0 , which is a necessary condition for optimality under very general conditions. Figure 11.1 illustrates this with m=2 and d=1. Equation (11.6) is simply a restatement of the constraint $g=\vartheta$. This Lagrange multiplier approach can also be generalized to infinite dimensional problems of the form (11.1), and this then gives a direct derivation of the Minimum Principle which does not rely on the HJB equations.

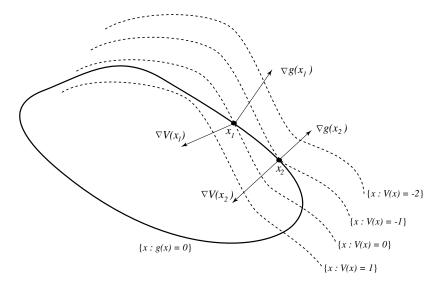


Figure 11.1: An optimization problem on \mathbb{R}^2 with a single constraint g = 0. The point x_1 is not optimal: By moving to the right along the constraint set, the function V(x) will decrease. The point x_2 is a minimum of V subject to the constraint g(x) = 0 since at this point we have $V(x_2) = -1$. V can get no lower - for example, V(x) is never equal to -2 since the level set $\{x: V(x) = -2\}$ does not intersect the constraint set $\{x: g(x) = 0\}$. At x_2 , the gradient of V and the gradient of q are parallel.

To generalize the Lagrange multiplier approach we must first generalize the concept of a stationary point. Suppose that F is a functional on $D^r[t_0, t_1]$. That is, for any function $z \in D^r[t_0, t_1]$, F(z) is a real number. For any $\eta \in D^r[t_0, t_1]$ we can define a directional derivative as follows:

$$D_{\eta}F\left(z\right)=\lim_{\epsilon\rightarrow0}\frac{F(z+\epsilon\eta)-F(z)}{\epsilon},$$

whenever the limit exists. The function $z(t, \epsilon) = z(t) + \epsilon \eta(t)$ may be viewed

as a perturbation of z, as shown in Figure 11.2

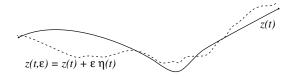


Figure 11.2: A perturbation of the function $z \in D[t_0, t_1]$.

We call z_0 a stationary point of F if $D_{\eta}F(z_0) = 0$ for any $\eta \in D^r[t_0, t_1]$. If z_0 is a minimum of F, then a perturbation cannot decrease the cost, and hence we must have for any η ,

$$F(z_0 + \epsilon \eta) \ge F(z_0), \quad \epsilon \in \mathbb{R}.$$

From the definition of the derivative it then follows that an optimal solution must be a stationary point, just as in the finite dimensional case!

The problem at hand can be cast as a constrained functional minimization problem,

Minimize
$$V(x,u) = \int \ell d\tau + m$$

Subject to $\dot{x} - f = \vartheta, \quad x \in D^n[t_0, t_1], \ u \in D^m[t_0, t_1].$

To obtain necessary conditions for optimality for constrained problems of this form we can extend the Lagrange multiplier method. We outline this approach in five steps:

Step 1. Append the state equations to obtain the new cost functional

$$\widehat{V}(x,u) = \int_{t_0}^{t_1} \ell \, dt + m(x(t_1)) + \int_{t_0}^{t_1} p^T(f - \dot{x}) \, dt.$$

The Lagrange multiplier vector p lies in $D^n[t_0, t_1]$. The purpose of this is to gain an expression for the cost in which x and u can be varied independently.

Step 2. Use integration by parts to eliminate the derivative of x in the definition of \hat{V} :

$$\int_{t_0}^{t_1} p^T \dot{x} \, dt = \int_{t_0}^{t_1} p^T dx = (p^T x) \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{p}^T x \, dt.$$

Step 3. Recall the definition of the Hamiltonian,

$$H(x, p, u, t) := \ell(x, u, t) + p^T f(x, u, t).$$

Combining this with the formulas given in the previous steps gives

$$\widehat{V}(x,u) = \int_{t_0}^{t_1} H(x,p,u,t) dt + \int_{t_0}^{t_1} \dot{p}^T x dt + p^T(t_0) x(t_0) - p^T(t_1) x(t_1) + m(x(t_1)).$$
(11.7)

Step 4. Suppose that u° is an optimal control, and that x° is the corresponding optimal state trajectory. The Lagrange multiplier theorem asserts that the pair (x°, u°) is a stationary point of \widehat{V} for some p° . Perform perturbations of the optimal control and state trajectories to form

$$u(t,\delta) = u^{\circ}(t) + \delta\psi(t)$$
 $x(t,\epsilon) = x^{\circ}(t) + \epsilon\eta(t).$ (11.8)

Since we are insisting that $x(t_0) = x_0$, we may assume that $\eta(t_0) = \vartheta$. Consider first variations in ϵ , with δ set equal to zero. Letting $\hat{V}(\epsilon) = \hat{V}(x^{\circ} + \epsilon \eta, u^{\circ})$, we must have

$$\frac{d}{d\epsilon}\widehat{V}(\epsilon) = 0.$$

Using (11.7) to compute the derivative gives

$$0 = \int_{t_0}^{t_1} \frac{\partial}{\partial x} H(x^{\circ}, p, u^{\circ}, t) \eta(t) dt + \int_{t_0}^{t_1} \dot{p}^T \eta dt$$
$$-p^T(t_1) \eta(t_1) + p^T(t_0) \eta(t_0) + \frac{\partial}{\partial x} m(x^{\circ}(t_1)) \eta(t_1). \tag{11.9}$$

Similarly, by considering perturbations in u° we obtain for any $\psi \in D^m[t_0, t_1]$,

$$\vartheta = \int_{t_0}^{t_1} \frac{\partial}{\partial u} H(x^{\circ}, p, u^{\circ}, t) \psi(t) dt.$$
 (11.10)

This simpler expression is obtained because only the first term in (11.7) depends upon u.

Step 5. We can choose $\eta(t)$ freely in (11.9). From this it follows that

$$\frac{\partial H}{\partial x} + \dot{p}^T = \vartheta^T \Rightarrow \dot{p} = -\nabla_x H$$

and since $\eta(t_0) = \vartheta$, and $\eta(t_1)$ is free,

$$-p^{T}(t_1) + \frac{\partial m}{\partial x}(t_1) = \vartheta^{T} \Rightarrow p(t_1) = \nabla_x m(x(t_1)).$$

Similarly, by (11.10) we have

$$\frac{\partial H}{\partial u} = \vartheta^T \Rightarrow \nabla_u H = \vartheta.$$

In fact, if u is to be a minimum of \hat{V} , then in fact it must minimize H pointwise. These final equations then give the Minimum Principle Theorem 11.1.

From this proof it is clear that many generalizations of the Minimum Principle are possible. Suppose for instance that the final state $x(t_1) = x_1$ is specified. Then the perturbation η will satisfy $\eta(t_1) = \vartheta$, and hence using (11.9), it is impossible to find a boundary condition for p. None is needed in this case, since to solve the 2n-dimensional coupled state and costate equations, it is enough to know the initial and final conditions of x° .

11.3 The penalty approach^{*}

A third heuristic approach to the Minimum Principle involves relaxing the hard constraint $\dot{x} - f = \vartheta$, but instead impose a large, yet "soft" constraint by defining the cost

$$\widehat{V}(x,u) = \int_{t_0}^{t_1} \ell(x(t), u(t), t) dt + \frac{k}{2} \int_{t_0}^{t_1} |\dot{x}(t) - f(x(t), u(t), t)|^2 dt + m(x(t_1)).$$

The constant k in this equation is assumed large, so that $\dot{x}(t) - f(x(t), u(t), t) \approx \vartheta$.

We assume that a pair (x_k, u_k) exists which minimizes \hat{V}_k . Letting (x°, u°) denote a solution to the original optimization problem, we have by optimality,

$$\widehat{V}_k(x_k, u_k) \le \widehat{V}_k(x^\circ, u^\circ) = V^\circ.$$

Assuming ℓ and m are positive, this gives the following uniform bound

$$\int_{t_0}^{t_1} |\dot{x}(t) - f(x(t), u(t), t)|^2 dt \le \frac{2}{k} V^{\circ}.$$

Hence, for large k, the pair (x_k, u_k) will indeed approximately satisfy the differential equation $\dot{x} = f$.

If we perturb x_k to form $x_k + \epsilon \eta$ and define $\hat{V}(\epsilon) = \hat{V}(x_k + \epsilon \eta, u_k)$ then we must have $d/d\epsilon \hat{V}(\epsilon) = 0$ when $\epsilon = 0$. Using the definition of \hat{V} gives

$$\widehat{V}(\epsilon) = \int_{t_0}^{t_1} \ell(x_k(t) + \epsilon \eta(t), u_k(t), t) dt + m(x_k(t_1 + \epsilon \eta(t_1))) + \frac{k}{2} \int_{t_0}^{t_1} |\dot{x}_k(t) + \epsilon \dot{\eta}(t) - f(x_k(t) + \epsilon \eta(t), u_k(t), t)|^2 dt.$$

The derivative of this expression with respect to ϵ can be computed as follows:

$$0 = \frac{d}{d\epsilon} \widehat{V}(0) = \int_{t_0}^{t_1} \frac{d}{dx} \ell(x(t), u(t), t) \eta(t) dt$$
$$+ k \int_{t_0}^{t_1} (\dot{x}(t) - f(x(t), u(t), t))^T [\dot{\eta} - \frac{\partial}{\partial x} f(x(t), u(t), t) \eta(t)] dt$$
$$+ \frac{\partial}{\partial x} m(x(t_1)) \eta(t_1).$$

To eliminate the derivative term $\dot{\eta}$ we integrate by parts, and this gives the expression

$$0 = \int_{t_0}^{t_1} \left\{ \frac{d}{dx} \ell(x_k(t), u_k(t), t) dt + p_k(t)^T \frac{\partial}{\partial x} f(x_k(t), u_k(t), t) + \frac{d}{dt} \left(p_k(t)^T \right) \right\} \eta(t) dt$$
$$- p_k(t_1) + \frac{\partial}{\partial x} m(x_k(t_1) \eta(t_1),$$
(11.11)

where we have set $p_k(t) = -k(\dot{x}_k(t) - f(x_k(t), u_k(t), t))$. Since η is arbitrary, we see that

$$\vartheta = \frac{d}{dt} p_k(t)^T + \frac{\partial \ell}{\partial x} (x_k(t), u_k(t), t) + p_k(t)^T \frac{\partial}{\partial x} f(x_k(t), u_k(t), t),$$

and we again obtain the boundary condition

$$p_k(t_1) = \frac{\partial}{\partial x} m(x(t_1).$$

Considering perturbations in u gives the equation $\nabla_u H(x_k(t), p_k(t), u_k(t), t) = 0$, which is a weak form of the Minimum Principle for the perturbed problem.

By letting $k \to \infty$, this penalty approach can be used to prove both the complete Minimum Principle and the Lagrange Multiplier Theorem.

11.4 Application to LQR

The LQR problem is a good test case to see if the Minimum Principle is a useful tool for the construction of optimal policies. Consider again the general LTI model with quadratic cost:

$$\dot{x} = Ax + Bu$$

$$V = \int_{t_0}^{t_1} (x^T Q x + u^T R u) dt + x^T (t_1) M x(t_1).$$

The Hamiltonian is written

$$H = x^T Q x + u^T R u + p^T (A x + B u)$$

and hence the control can be computed through

$$\nabla_u H = 0 = 2Ru + B^T p \Longrightarrow u = -\frac{1}{2}R^{-1}B^T p.$$

This then gives the first set of differential equations:

$$\dot{x} = Ax + Bu = Ax - \frac{1}{2}BR^{-1}B^{T}p. \tag{11.12}$$

Through the expression $\nabla_x H = 2Qx + A^T p$, we find that the derivative of p is

$$\dot{p} = -\nabla_x H = -2Qx - A^T p. \tag{11.13}$$

Equations (11.12) and (11.13) form the coupled set of differential equations

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -\frac{1}{2}BR^{-1}B^T \\ -2Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$$

with boundary conditions

$$x(t_0) = x_0$$

$$p(t_1) = 2Mx(t_1).$$

If we scale p we again arrive at the Hamiltonian defined in Section 10.4: With $\lambda = \frac{1}{2}p$,

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \mathcal{H} \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix}, \tag{11.14}$$

with $u^{\circ}(t) = -R^{-1}B^T\lambda(t)$.

This ODE can be solved using the "sweep method". Suppose that $\lambda(t) = P(t)x(t)$. Then

$$\dot{\lambda} = \dot{P}x + P\dot{x} = \dot{P}x + P(Ax + Bu),$$

and substituting $u^{\circ} = -R^{-1}B^{T}\lambda = -R^{-1}B^{T}Px$ gives

$$\dot{\lambda} = \dot{P}x + P(Ax - BR^{-1}B^TPx).$$

From (11.14) we also have

$$\dot{\lambda} = -Qx - A^T \lambda = -Qx - A^T Px.$$

Equating the two expressions for $\dot{\lambda}$ gives

$$-Qx - A^T Px = \dot{P}x + PAx - PBR^{-1}B^T x$$

which yields the RDE given in (10.12). The boundary condition for λ is $\lambda = \frac{1}{2}p = \frac{1}{2}m'(x_1)$, which from the definitions of p and m implies that

$$P(t_1) = M.$$

Solving for P(t) gives $\lambda(t)$, which in turn gives p(t).

11.5 Nonlinear examples

We now solve some nonlinear problems where it is possible to obtain an explicit solution to the coupled state-costate equations given in the Minimum Principle. At the same time, we also give several extensions of this result.

Our first extension involves constraints on the input. Suppose that \mathcal{U} is a subset of \mathbb{R}^m , and that we require that $u(t) \in \mathcal{U}$ for all t. Since this is a hard constraint, Theorem 11.1 is not directly applicable, but we have the following simple extension:

Theorem 11.2 (Minimum Principle with constraints). Suppose that $x(t_1)$ is free, t_1 is fixed, and suppose that u° is a solution to the optimal control problem (11.1) under the constraint $u \in \mathcal{U}$. That is,

$$V^{\circ} = \min\{V(u) : u(t) \in \mathcal{U} \text{ for all } t\}.$$

We then have

(a) There exists a costate vector p(t) such that

$$u^{\circ}(t) = \operatorname*{arg\,min}_{u \in \mathcal{U}} H(x^{\circ}(t), p(t), u, t).$$

(b) The pair (p, x°) satisfy the 2-point boundary value problem (11.4), with the two boundary conditions

$$x(t_0) = x_0;$$
 $p(t_1) = \frac{\partial}{\partial x} m(x(t_1), t).$

Example 11.5.1. To illustrate Theorem 11.2 consider the control of a bilinear system, defined by the differential equation

$$\dot{x} = ux$$
, $x(0) = x_0 > 0$, $0 \le u(t) \le 1$.

Suppose that the goal is to make x large, while keeping the derivative of x small on average. Then a reasonable cost criterion is

$$V(u) = \int_0^{t_1} \dot{x}(\tau) - x(\tau) d\tau = \int_0^{t_1} [u(\tau) - 1] x(\tau) d\tau.$$

We assume that $t_1 \geq 1$ is fixed, and that $x(t_1)$ is free.

The Hamiltonian becomes

$$H(x, p, u, t) = pf + \ell = p(ux) + [u - 1]x$$

= $x\{u(p + 1) - 1\}$

and by the Minimum Principle, the optimal control takes the form

$$u^{\circ}(t) = \underset{u \in \mathcal{U}}{\operatorname{arg \, min}} H(x^{\circ}(t), p(t), u, t).$$

Since $x^{\circ}(t) > 0$ for all t, the minimization leads to

$$u^{\circ}(t) = \begin{cases} 1 & \text{if } p(t) + 1 < 0 \\ 0 & \text{if } p(t) + 1 > 0 \\ \text{unknown} & \text{if } p(t) = -1 \end{cases}$$

The costate variable and state variable satisfy the differential equations

$$\dot{p} = -\frac{\partial H}{\partial x} = -(p+1)u^{\circ} + 1$$
 $\dot{x^{\circ}} = \frac{\partial H}{\partial p} = u^{\circ}x^{\circ}$

with the boundary conditions

$$x^{\circ}(0) = x_0, \qquad p(t_1) = \frac{\partial m}{\partial x} = 0.$$

How do we solve these coupled differential equations to compute u° ? First note that if $t = t_1$ then p(t) + 1 = 1 > 0 so $u^{\circ}(t) = 0$. By continuity, if $t \approx t_1$, then p(t) > 0, so we still must have $u^{\circ}(t) = 0$. Thus $\dot{p}(t) = +1$ for $t \approx t_1$, which implies

$$p(t) = t - t_1, \qquad t \approx t_1.$$

As we move backwards in time from time t_1 , the costate variable will continue to have this form up until the first time that p(t) + 1 changes sign. This will occur when $0 = t - t_1 + 1$, or $t = t_1 - 1$.

For $t < t_1 - 1$ we then have p(t) + 1 < 0 so that $u^{\circ}(t) = 1$. For these time values, the costate vector is described by the equation

$$\dot{p} = -(p+1) + 1 = -p.$$

Since $p(t_1 - 1) = -1$, we obtain

$$p(t) = -e^{-t+t_1-1}, t < t_1 - 1.$$

The costate trajectory is sketched in Figure 11.3.

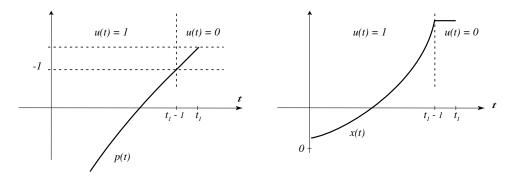


Figure 11.3: The costate trajectory, optimal control, and optimal state trajectory for the bilinear model.

We conclude that the optimal control is *bang-bang*. That is, it takes on only two values, which are the upper and lower limits:

$$u(t) = \begin{cases} 1 & t < t_1 - 1 \\ 0 & t > t_1 - 1. \end{cases}$$

The time $t_1 - 1$ is called the *switching point*. The optimal controlled state trajectory is defined by the differential equation

$$\dot{x}^{\circ}(t) = \begin{cases} x^{\circ}(t) & t < t_1 - 1 \\ 0 & t > t_1 - 1 \end{cases}$$

so the resulting state trajectory is expressed

$$x^{\circ}(t) = \begin{cases} x_0 e^t & t < t_1 - 1 \\ x_0 e^{t_1 - 1} & t > t_1 - 1. \end{cases}$$

This is also illustrated in Figure 11.3.

The next example we treat has a hard constraint on the *final state value*, so that $t_0, t_1, x(t_0) = x_0$, and $x(t_1) = x_1$ are all fixed and prespecified. In this situation the Minimum Principle is the same, but the boundary conditions for differential equations must be modified.

Theorem 11.3 (Minimum Principle with final value constraints). Suppose that t_0 , t_1 , $x(t_0) = x_0$, and $x(t_1) = x_1$ are prespecified, and suppose that u° is a solution to the optimal control problem (11.1), subject to these constraints. Then

(a) There exists a costate vector p(t) such that

$$u^{\circ}(t) = \underset{u}{\operatorname{arg\,min}} H(x^{\circ}(t), p(t), u, t).$$

(b) The pair (p, x°) satisfy the 2-point boundary value problem (11.4), with the two boundary conditions

$$x(t_0) = x_0;$$
 $x(t_1) = x_1.$

Example 11.5.2. To illustrate the application of Theorem 11.3 we consider the LQR problem where the terminal state has been specified. Note that in general we cannot make the state stay at the terminal state x_1 . The cost criterion is

$$\begin{split} V &= \frac{1}{2} \int_0^{t_1} (x^T Q x + u^T R u) dt \qquad R > 0, Q \geq 0 \\ H &= p(A x + B u) + \frac{1}{2} (x^T Q x + u^T R u). \end{split}$$

The Minimum Principle again implies that the optimal control has the form

$$u^{\circ}(t) = -R^{-1}B^T p(t),$$

and that the costate vector satisfies the differential equation

$$\dot{p}(t) = -\frac{\partial H}{\partial x} = -p^T A - x^T Q$$

Thus we again arrive at the linear differential equation based on the Hamiltonian:

Let $\psi(t,\tau)$ denote the state transition matrix, so that

$$\frac{d}{dt}\psi(t,\tau) = \mathcal{H}(t)\psi(t,\tau); \qquad \psi(t,t) = I.$$

This is a $2n \times 2n$ matrix which we decompose as

$$\psi(t,\tau) = \begin{bmatrix} \psi_{11}(t,\tau) & \psi_{12}(t,\tau) \\ \psi_{21}(t,\tau) & \psi_{22}(t,\tau) \end{bmatrix}.$$

To solve for the optimal control, we will compute the initial condition $p(t_0) = p_0$. This will allow us to compute p(t) for all t, and hence also the optimal control expressed in terms of p above as $u^{\circ} = -R^{-1}B^Tp$. The unknown term p_0 can be expressed in terms of the other known quantities as follows:

$$x_1 = x(t_1) = \psi_{11}(t_1, t_0)x_0 + \psi_{12}(t_1, t_0)p_0$$

Assuming that $\psi_{12}(t_1,t_0)$ is invertible, this gives the formula

$$p_0 = \psi_{12}(t_1, t_0)^{-1} \{ x_1 - \psi_{11}(t_1, t_0) x_0 \}.$$

For all t we then have

$$p(t) = \psi_{21}(t, t_0)x_0 + \psi_{22}(t, t_0)p_0.$$

The optimal control is

$$u^{\circ}(t) = -R^{-1}B^T p(t).$$

and the optimal state trajectory becomes

$$x^{\circ}(t) = \psi_{11}(t, t_0)x_0 + \psi_{12}(t, t_0)p_0, \qquad t \ge t_0.$$

Our last extension concerns the case where the terminal time is not fixed. At the same time, suppose that *some* components of $x(t_1)$ are specified. We assume that t_0 and $x(t_0) = x_0$ are fixed.

Theorem 11.4 (Minimum Principle with free terminal time). Suppose that t_0 and $x(t_0) = x_0$ are fixed, and that for some index set $I \subset \{1, ..., n\}$,

$$x_i(t_1) = x_{1i}, \quad \text{if } i \in I.$$

Suppose that u° is a solution to the optimal control problem (11.1), subject to these constraints. Then

(a) There exists a costate vector p(t) such that

$$u^{\circ}(t) = \underset{u}{\operatorname{arg\,min}} H(x^{\circ}(t), p(t), u, t).$$

(b) The pair (p, x°) satisfy the 2-point boundary value problem (11.4), with the boundary conditions

$$x_i(t_i) = x_{1i}, i \in I$$

 $p_j(t_1) = \frac{\partial m}{\partial x_j}(x^{\circ}(t_1), t_1), i \in I^c.$

The unspecified terminal time t_1 satisfies

$$\frac{\partial m}{\partial t}(x^{\circ}(t_1), t) + H(x^{\circ}(t_1), p(t_1), u^{\circ}(t_1), t_1) = 0.$$
(11.15)

The most common example in which the terminal time is free is a minimum time problem. As an example consider a single input linear state space model

$$\dot{x} = Ax + bu$$
.

We wish to find the input u° which drives x from $x(0) = x_0$ to $x(t_1) = x_1$ in *minimum time*, under the constraint $|u(t)| \leq 1$ for all t. Hence the cost criterion we consider is

$$V(u) = t_1 = \int_0^{t_1} 1 \, dt,$$

so that $\ell \equiv 1$, and $m \equiv 0$.

The Hamiltonian for this problem is

$$H = \ell + p^T f = 1 + p^T A x + p^T b u.$$

Minimizing over all H again gives a bang-bang control law

$$u^{\circ}(t) = \begin{cases} 1 & p(t)^T b < 0 \\ -1 & p(t)^T b > 0 \end{cases}$$

If A, b are time invariant and the eigenvalues $\{\lambda_i\}$ of A are real, distinct, and negative, then since the modes are all decreasing, the quantity $p(t)^T b$ changes sign at most n-1 times. Hence, the number of switching times for the optimal control is also bounded above by n-1.

The costate equation is

$$\dot{p} = -\nabla_x H = -A^T p.$$

Using the final time boundary condition (11.15), or $H|_{t=t_1} = 0$, we obtain

$$1 + p^{T}(t_1)Ax^{\circ}(t_1) + p^{T}(t_1)bu^{\circ}(t_1) = 1 + p^{T}(t_1)Ax^{\circ}(t_1) - |p^{T}(t_1)b| = 0.$$

We now show how these equations may be solved using a second order example.

Example 11.5.3. Consider the special case where the model is the double integrator $\ddot{y} = u$, which is described by the state space model with

$$A = \left(\begin{array}{cc} 0 & 1\\ 0 & 0 \end{array}\right) \qquad b = \left(\begin{matrix} 0\\ 1 \end{matrix}\right)$$

so that $\dot{x}_1 = x_2$, $\dot{x}_2 = u$. The optimal control has the form $u^{\circ} = -\operatorname{sgn}(p^T b) = -\operatorname{sgn}(p_2(t))$. From the costate equations

$$\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = -A^T \begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \end{pmatrix}$$

we find that for constants c_1 and c_2 ,

$$p_1(t) = c_1$$
 $p_2(t) = -c_1t + c_2$.

It follows that $p_2(t)$ changes sign on $(0, t_1)$ at most once.

If $p_2(t) < 0$ then $u^{\circ}(t) = +1$, so that by solving the differential equations

$$\dot{x}_1^{\circ} = x_2^{\circ}, \qquad \dot{x}_2^{\circ} = 1,$$

we find that $x_1^{\circ}(t) = \frac{1}{2}(x_2^{\circ}(t))^2 + K_1$ for some constant K_1 . If $p_2(t) > 0$ so that $u^{\circ}(t) = -1$, then for another constant, $x_1^{\circ}(t) = -\frac{1}{2}(x_2^{\circ}(t))^2 + K_2$. Hence the optimal state trajectories follow the quadratic paths illustrated in Figure 11.4.

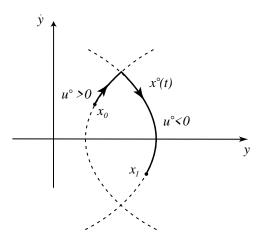


Figure 11.4: The optimal trajectories for the minimum time problem in this two dimensional example follow quadratic paths, and may switch paths at most once. A typical optimal path is illustrated as a union of two dark line segments in this drawing.

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11.6 Exercises

11.6.1 Consider the system/performance index

$$\dot{x} = u$$
 $V(u) = \frac{q}{2}x^2(T) + \frac{1}{2}\int_0^T u^2 dt,$

with $x(0) = x_0$, T given, and $q \ge 0$. Find u° using the Minimum Principle. First consider the case where $\infty > q > 0$. Then, obtain the control for " $q = \infty$ ". That is, impose the constraint that x(T) = 0. Do the controllers converge as $q \to \infty$?

11.6.2 A simplified model for a reactor is given by the bilinear system $\dot{x} = xu$, where x is the concentration, and u is the rate, taken as the control variable. Suppose that x(0) = 0.5, and the terminal time is $t_1 = 1$. Using the Minimum Principle, find the control function that minimizes the performance index

$$V = [x(1) - 1]^2 + \int_0^1 u^2(t) dt.$$

11.6.3 The temperature of a room is described by $\dot{x} = ax + u$, where x is the room temperature, a is a negative constant, and u is the heat input rate. Suppose that $x(0) = x_0$ is given and it is desired to find a control function $u(\cdot)$ so that $x(t_1) = q$, where $q > x_0$, and the performance index

$$V = \int_0^{t_1} u^2(t) dt$$

is minimized. Note that, in this problem, the state trajectory $x(\cdot)$ is specified at two end points, while the terminal time t_1 is free. Using the Minimum Principle, determine the optimal open-loop control function and the corresponding value of t_1 , in terms of a, x_0 and q.

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