Tutorial 5

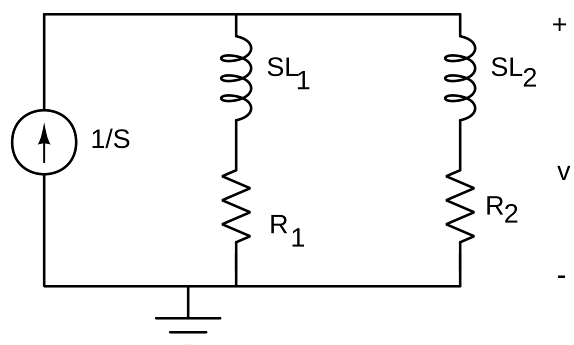
September 13, 2019

1. Let the initial voltages across C_1 and C_2 be denoted as v_{C_1} and v_{C_2}. From the circuit, $v_{C_1} = v_o$ and $v_{C_2} = 0$.

   After the switch is closed ($t > 0$), both the capacitors arrive at a common voltage. Let this common voltage be called as v_C. Now, the total charge in the circuit remains constant. Therefore, equating charges before and after closing the switch, we get:

$$\begin{aligned} C_1 v_{C_1} + C_2 v_{C_2} &= v_C(C_1 + C_2) \\ C_1 v_o &= v_C(C_1 + C_2) \\ \implies v_C &= \frac{C_1 v_o}{(C_1 + C_2)} \end{aligned}$$

2. In Laplace domain, the circuit is given as:

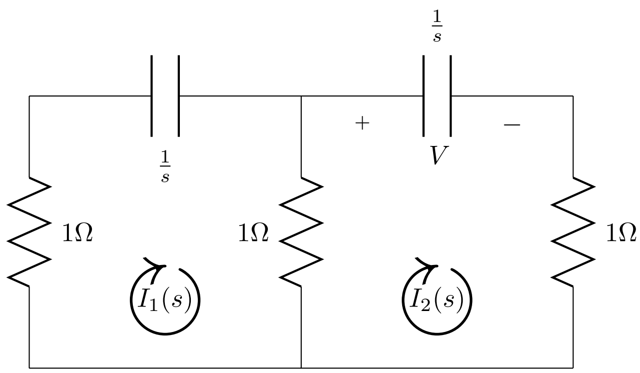


Q. 2

Now, the step response is given by:

$$\begin{aligned} V(s) &= I(s)Z(s) \\ I(s) &= \frac{1}{s}, \quad Z(s) = \frac{(sL_1 + R_1)(sL_2 + R_2)}{(sL_1 + R_1) + (sL_2 + R_2)} \\ \implies V(s) &= \frac{(sL_1 + R_1)(sL_2 + R_2)}{s((sL_1 + R_1) + (sL_2 + R_2))} \\ \implies v(t) &= \frac{R_1 R_2}{R_1 + R_2} + \Big(\frac{R_1 {L_2}^2 + R_2 {L_1}^2}{(L_1 + L_2)^2} - \frac{R_1 R_2}{R_1 + R_2}\Big) e^{-\left(\frac{R_1 + R_2}{L_1 + L_2}\right)t} + \frac{L_1 L_2}{L_1 + L_2}\delta(t); \quad t \geq 0 \end{aligned}$$

3. (a) . Laplace form of the circuit:



Using mesh analysis, we get the matrix as:

$$\begin{bmatrix} 2+\frac{1}{S} & -1 \\ -1 & 2+\frac{1}{S} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Therefore

$$\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} 2+\frac{1}{S} & 1 \\ 1 & 2+\frac{1}{S} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

where Δ is the determinant of mesh basis matrix. Equating the determinant to zeros:

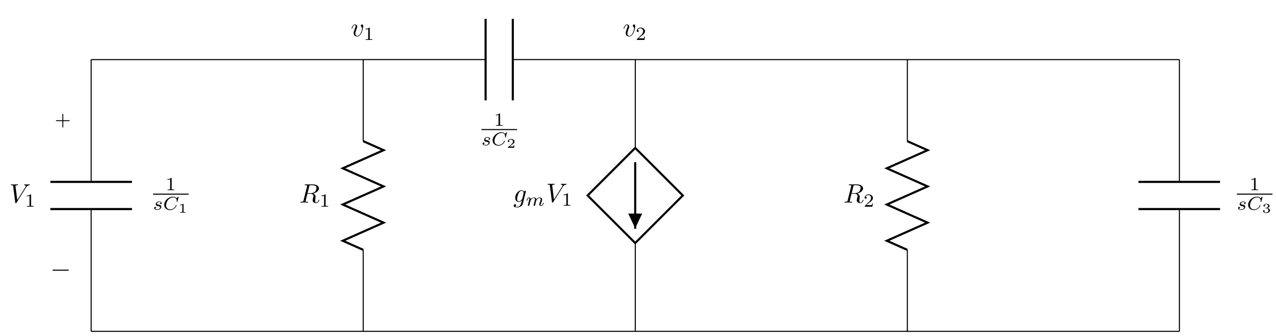
$$\begin{vmatrix} 2+\frac{1}{S} & -1 \\ -1 & 2+\frac{1}{S} \end{vmatrix} = 0$$

Solving this we get

$$(2+\frac{1}{s})^2 - 1 = 0$$
$$3s^2 + 4s + 1 = 0$$
$$(3s+1)(s+1) = 0$$

s=-1,$\frac{-1}{3}$ are the natural frequencies.

(b)



Using nodal analysis, we get the matrix as:

$$\begin{bmatrix} s(C_1+C_2)+\frac{1}{R_1} & -sC_2 \\ g_m - sC_2 & s(C_2+C_3)+\frac{1}{R_2} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Equating the determinant to zero:

$$\begin{vmatrix} s(C_1+C_2)+\frac{1}{R_1} & -sC_2 \\ g_m - sC_2 & s(C_2+C_3)+\frac{1}{R_2} \end{vmatrix} = 0$$

$$\begin{aligned} s^2(C_1+C_2)(C_2+C_3) + s(\frac{C_1+C_2}{R_2} + \frac{C_2+C_3}{R_1}) + \frac{1}{R_1R_2} + sC_2(g_m - sC_2) = 0 \\ s^2(C_1C_2 + C_2C_3 + C_1C_3) + s(\frac{C_1+C_2}{R_2} + \frac{C_2+C_3}{R_1} + g_mC_2) + \frac{1}{R_1R_2} = 0 \end{aligned}$$

Let

$$\begin{aligned} a &= C_1C_2 + C_2C_3 + C_1C_3 \\ b &= \frac{C_1+C_2}{R_2} + \frac{C_2+C_3}{R_1} + g_mC_2 \\ c &= \frac{1}{R_1R_2} \end{aligned}$$

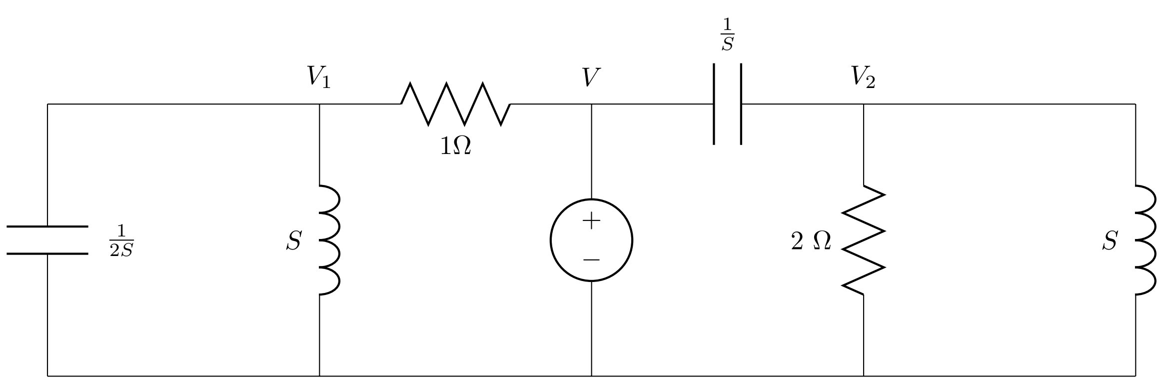
Then

$$\begin{aligned} s_1 &= \frac{-b+\sqrt{b^2-4.a.c}}{2.a} \\ s_2 &= \frac{-b-\sqrt{b^2-4.a.c}}{2.a} \end{aligned}$$

Here we are getting only two natural frequencies though we have three capacitors.This is because as the three capacitors are connected in a loop, only two independent initial conditions are required. For example if we know initial conditions of capacitors $C_1 and\ C_3$ then initial condition of capacitor C_2 is

$$V_{C2}(0) = V_{C1}(0) - V_{C3}(0)$$

(c)



Using nodal analysis, we get the following matrix:

$$\begin{bmatrix} 1+\frac{1}{s}+2s & 0 \\ 0 & s+\frac{1}{2}+\frac{1}{s} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} V \\ sV \end{bmatrix}$$

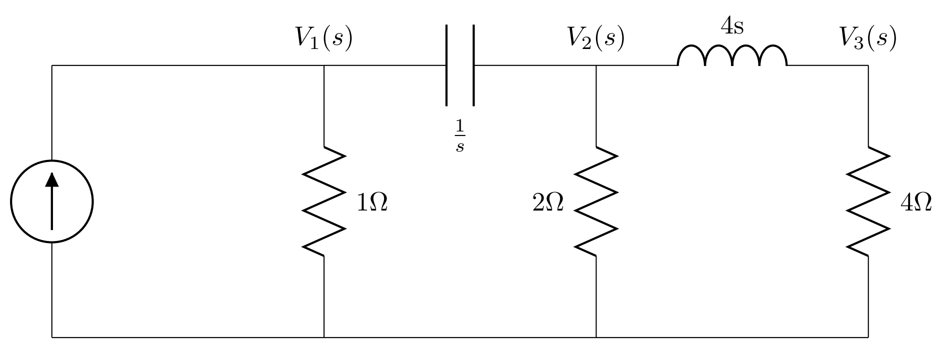
Equating the determinant to zero:

$$\begin{vmatrix} 1+\frac{1}{s}+2s & 0 \\ 0 & s+\frac{1}{2}+\frac{1}{s} \end{vmatrix} = 0$$

$$(2s^2+s+1)(2s^2+s+2)=0$$

By solving this we get natural frequencies=$\frac{-1}{4}-\frac{i\sqrt{15}}{4}, \frac{-1}{4}+\frac{i\sqrt{15}}{4}, \frac{-1}{4}-\frac{i\sqrt{7}}{4}and\frac{-1}{4}+\frac{i\sqrt{7}}{4}$

(d) . Laplace form of the circuit:



Using nodal analysis, we get the following matrix:

$$\begin{bmatrix} 1+s & -s & 0 \\ -s & s+\frac{1}{2}+\frac{1}{4S} & -\frac{1}{4s} \\ 0 & -\frac{1}{4s} & \frac{1}{4s}+\frac{1}{4} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{bmatrix} = \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}$$

Equating the determinant to zero:

$$\begin{vmatrix} 1+s & -s & 0 \\ -s & s+\frac{1}{2}+\frac{1}{4S} & -\frac{1}{4s} \\ 0 & -\frac{1}{4s} & \frac{1}{4s}+\frac{1}{4} \end{vmatrix} = 0$$

$$\frac{1}{s}+2s+3=0$$

$$2s^2+3s+1=0$$

By solving this we get s=$-\frac{1}{2}$ and s=-1
$\therefore$ natural frequencies of the circuit are $-\frac{1}{2}$ and -1

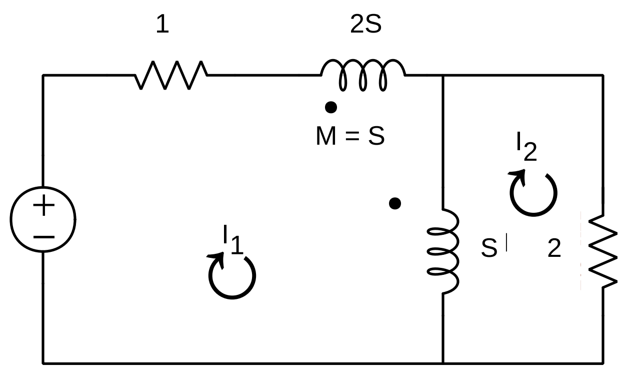
(e)

The circuit in s domain is given below:

Upon applying Loop equations, we get the following matrix:

$$\begin{bmatrix} 1+(2s+s)+(s+s) & -s-s \\ -s-s & s+2 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V \\ 0 \end{bmatrix}$$

Upon equating the determinant of Z matrix to zero, we get $s=\frac{-11+\sqrt{113}}{2}$, $s=\frac{-11-\sqrt{113}}{2}$.



Q. 3(e)

4. (At the end)
5. Given, The quality factor of a series RLC circuit as Q = 500. α is the damping factor

$$Q = \frac{\omega_o}{2\alpha}$$
$$Q = 500$$
$$\implies \frac{\omega_o}{\alpha} = 1000$$
$$\implies \alpha < \omega_o$$

Since $\alpha < \omega_o$, the RLC circuit is under damped.
Zero input response of an under damped RLC circuit is given as

$$v(t) = \exp^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

It is a sinusoidal which decays with period $\frac{2\pi}{\omega_d}$
Then half period is $\frac{\pi}{\omega_d}$
$\omega_o = \frac{1}{\sqrt{LC}}$ is the natural frequency
$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$ is the damping frequency.
Consider magnitude of v(t)

$$|v(t)| = \exp^{-\alpha t}$$

$\zeta = \frac{\alpha}{\omega_o} = \frac{1}{2Q}$
a) Amplitude of the envelope decay to 10%

$$\exp^{-\alpha t} = 0.1 \implies \ln 0.1 = -\alpha t = -\zeta\omega_o \frac{n\pi}{\omega_d} \implies$$

n=732 half cycles.
b) Amplitude of the envelope decay to 1%

$$\exp^{-\alpha t} = 0.01 \implies \ln 0.01 = -\alpha t = -\zeta\omega_o \frac{n\pi}{\omega_d} \implies$$

n=1464 half cycles.
c) Amplitude of the envelope decay to 0.1%

$$\exp^{-\alpha t} = 0.001 \implies \ln 0.001 = -\alpha t = -\zeta\omega_o \frac{n\pi}{\omega_d} \implies$$

n=2198 half cycles.

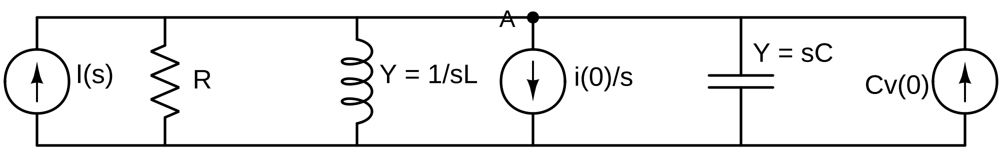
6. We have : C= 1F, $\omega = 10$ Rad/s, $Q = \frac{1}{2}$, $v_C(0) = 2V$, $i_L(0) = 5A$

$$\omega_o^2 = \frac{1}{LC}$$
$$\implies L = 0.01\text{H}$$

Similarly,

$$Q = \frac{\omega_o}{2\alpha}; \quad \alpha = \frac{1}{2RC}$$
$$\implies Q = \omega_o RC$$
$$\implies R = \frac{1}{20}\Omega$$

The circuit in s domain is :



By applying the nodal equation at point A, we get:

$$V_A\left[20 + \frac{100}{s} + s\right] = I(s) - \frac{5}{s} + 2$$

(a)

$$I(s) = 0$$
$$\implies V_A(s) = \frac{2s-5}{s^2+20s+100}$$
$$\implies v_A(t) = e^{(-10t)}(2-25t); \quad t \geq 0$$

(b)
For step response

$$I(s) = \frac{1}{S}$$
$$\implies V_A(s) = \frac{2s-4}{s^2+20s+100}$$
$$\implies v_A(t) = e^{(-10t)}(2-24t); \quad t \geq 0$$

For impulse response

$$I(s) = 1$$
$$\implies V_A(s) = \frac{3s-5}{s^2+20s+100}$$
$$\implies v_A(t) = e^{(-10t)}(3-35t); \quad t \geq 0$$

(c) In Zero state, we assume all initial values to be 0.

$$I(s) = \frac{s}{s^2+4}$$
$$\implies V_A(s) = \frac{s^2}{(s^2+4)(s^2+20s+100)}$$
$$\implies v_A(t) = \frac{5}{676}\cos(2t) - \frac{3}{169}\sin(2t) - \frac{5}{676}e^{-10t} + \frac{25e^{-10t}t}{26}; \quad t \geq 0$$

Now, as we know,

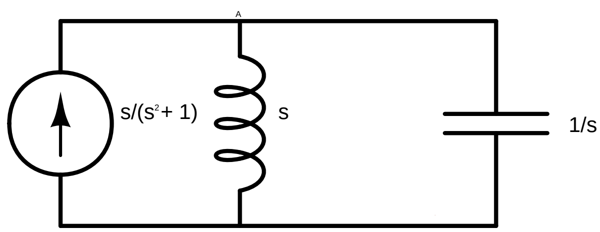
Total Response = Zero State Response + Zero Input Response

Therefore, Total response :

$$v_A(t) = \frac{5}{676}\cos(2t) - \frac{3}{169}\sin(2t) - \frac{5}{676}e^{-10t} + \frac{25e^{-10t}t}{26} + e^{(-10t)}.(2-25t)$$

(d) As we can see from the above expression for the Total response, there will be no transient if the the terms with the factor $[e^{-10t}]$ cancel each other. Thus we can set the initial values for the circuit so as to get the zero input response such that all the terms with transient state cancel each other, and hence it is possible that there is no transient.

7. For $t > 0$, the circuit in Laplace domain is shown below:



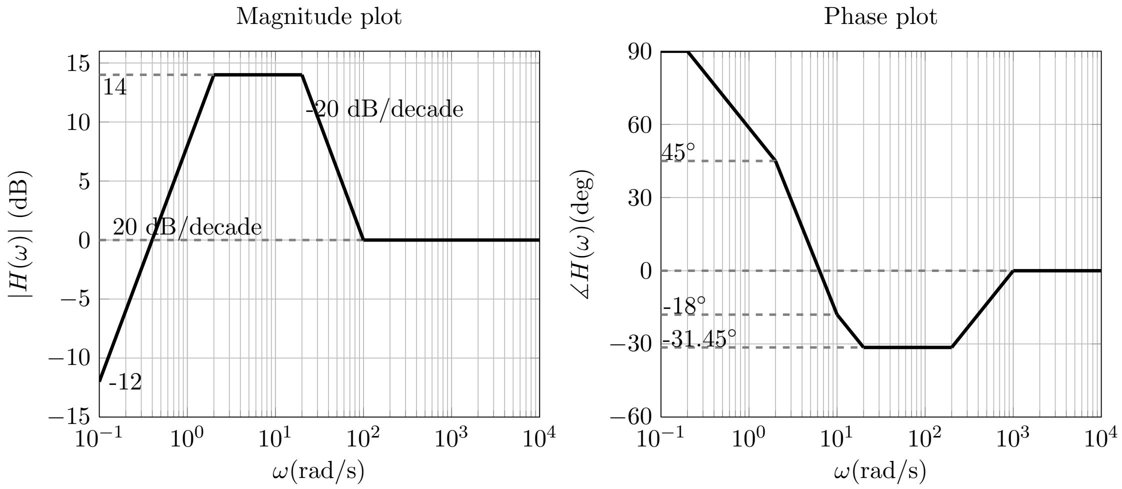
Q. 7

Now, applying Nodal equation at A,

$$\begin{aligned}
V(s)[\frac{1}{s} + s] &= \frac{s}{s^2+1} \\
V(s) &= \frac{s^2}{(s^2+1)^2} \\
&= \frac{s^2+1-1}{(s^2+1)^2} \\
&= \frac{1}{s^2+1} - \frac{1}{(s^2+1)^2} \\
&= \frac{1}{s^2+1} - \frac{1}{2}\Big[\frac{s^2+1^2-(s^2-1^2)}{(s^2+1^2)^2}\Big] \\
&= \frac{1}{s^2+1} - \frac{1}{2}\Big[\frac{s^2+1^2}{(s^2+1^2)^2} - \frac{(s^2-1^2)}{(s^2+1^2)^2}\Big] \\
\implies v(t) &= \frac{\sin t + t\cos t}{2}; \quad t > 0
\end{aligned}$$

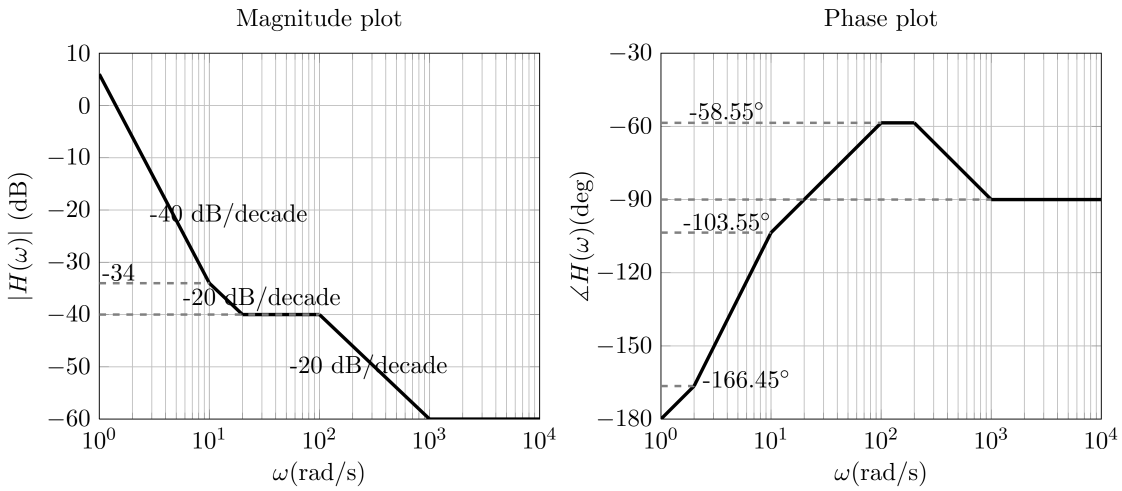
4. **(i)**

Bode plot of $\frac{s(s+100)}{(s+2)(s+20)}$



(ii)

Bode plot of $\frac{(s+10)(s+20)}{s^2(s+100)}$



(iii)

Bode plot of $\frac{(s+10)(s+200)}{(s+20)^2(s+1000)}$