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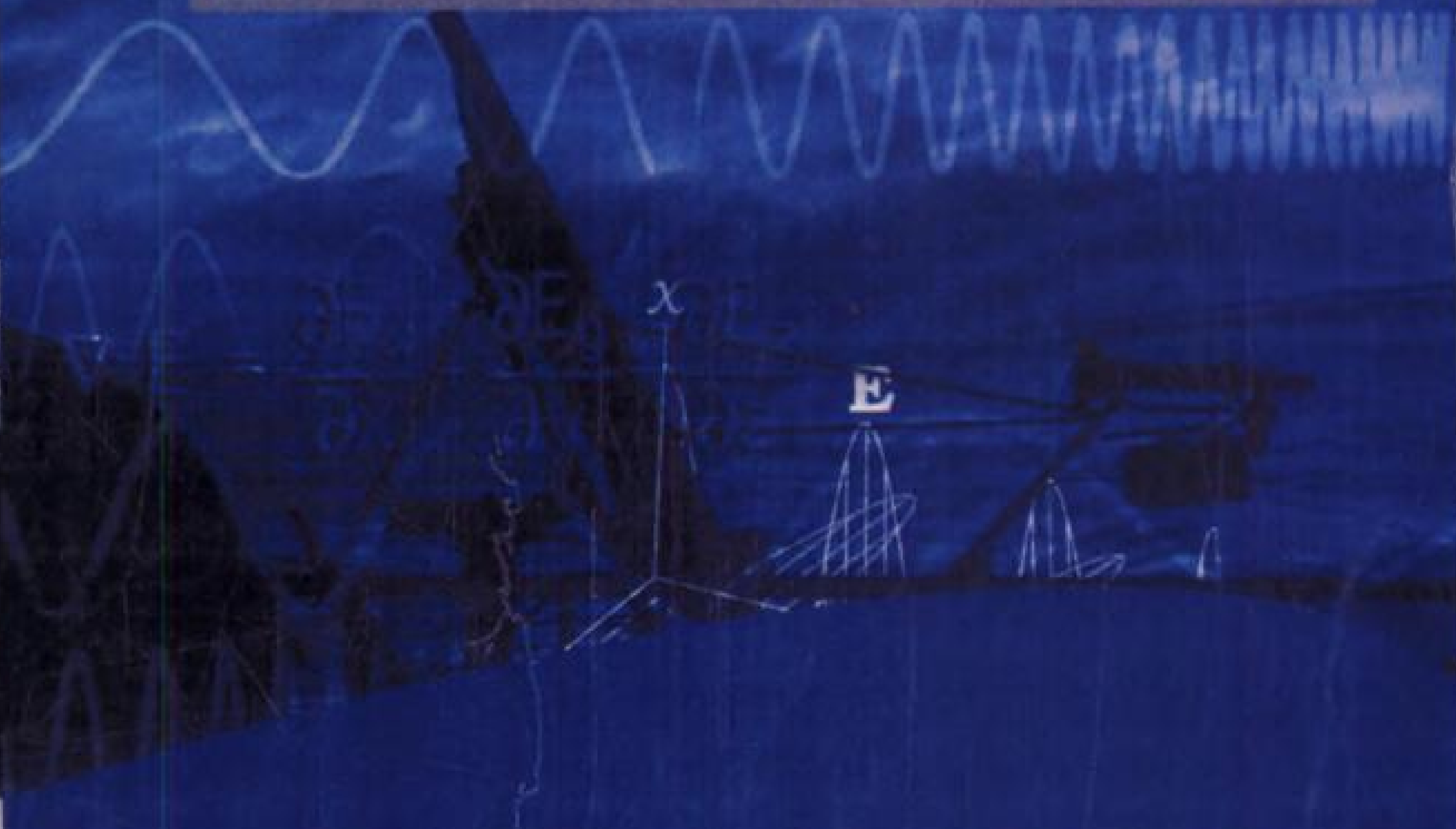

ELECTROMAGNETIC WAVES

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Preface

THE BACKGROUND

Electromagnetics is one of the foundation subjects of electrical engineering and physics. My liking towards the subject of electromagnetic waves dates back to my undergraduate days. Of course that time the subject was taught in a fashion to make it appear like a vector calculus course. Later when I studied the subject of antenna and propagation during my graduate years from an excellent teacher, I was fascinated by it. My association with radio astronomy further increased my understanding of electromagnetics. The more I understood the subject the more I realized that electromagnetics is rather conceptual and not mathematical. Unfortunately, the conceptual nature of the subject is not emphasized at the undergraduate level. A course on electromagnetics then appears as a vector algebra course. The students, thus, quickly loose interest in the subject. In absence of any conceptual understanding of the subject they find no relevance of electromagnetics in the rest of the engineering curriculum. They perceive the course as a formula substitution exercise, which is certainly not the true nature of the course. The true nature of the course is conceptual. The course appears difficult at the outset because it requires a fair amount of imagination in visualizing the vector fields in three-dimensional space. However, once these fields are properly visualized, the subject becomes more of a fun than a burden. When I adopted the conceptual approach to the course in my classes, the students got more interested in the subject. They started enjoying the concepts of electromagnetics and finding their applications in real life problems.

THE APPROACH

At the undergraduate level the challenge is to make the subject interesting without compromising on mathematical rigor. This book, *Electromagnetic Waves*, is an attempt to make the subject exciting, conceptually sound and useful to the students of electrical engineering.

Having adapted the conceptual approach to electromagnetics, the availability of proper textbooks became a major concern. All major textbooks on electromagnetics follow the vector-algebra-dominated approach. Most of the books start with vector algebra and calculus followed by Maxwell's equations and their solutions. Before junior level, where the subject of electromagnetic waves is usually taught, the students acquire a good knowledge of circuit theory. A subject that starts with three-dimensional vector calculus does not immediately gel with the previous circuit-oriented courses. As a result, the students find an abrupt change in electrical engineering concepts. However, if a gradual transition from the circuit theory to the field theory is made, students find it more digestible. The subject of transmission line is precisely this transition. While analyzing transmission lines we still retain circuit terminology, like voltage, current, resistance, capacitance, etc. but introduce the concept of space. An electromagnetic wave related terminology is slowly introduced as the discussion on transmission line progresses. By the time the discussion on transmission lines is completed, the students already have some idea about the electromagnetic waves, albeit in one dimension. Extending the concepts from there to three dimensions is not very difficult.

The book has a large number of solved and unsolved problems of practical nature that would be useful in enhancing the analytical skills of the students. The problems are designed to test the conceptual understanding of the students rather than their algebraic manipulation capabilities. The review questions can help them in self-evaluating their understanding of the subject. For proper understanding of electromagnetic waves, the students are expected to solve all the solved and unsolved problems to the last detail and to answer all the review questions.

ORGANISATION OF THE BOOK

The book starts with a brief introduction to Electromagnetic Waves and their applications in Chapter 1. Chapter 2 is on transmission lines. In this chapter the limitations of circuit approach in analyzing high frequency circuits are highlighted and the concept of distributed element is introduced. It is shown that the natural solution for voltage and current is a wave type solution. Concepts of reflection, impedance transformation, impedance mismatch are discussed in detail and finally the applications of transmission line are explained. Chapter 3 is on Maxwell's equations. It is assumed that the students are familiar with basic vector operations like dot product, cross product, curl, divergence, etc. If not, the basics of vector algebra and calculus are given in an appendix at the end of the book. Maxwell's equations are derived in integral and differential form from the basic laws of electromagnetics. Further the boundary conditions are derived from the integral form of Maxwell's equations. At this stage there is no undue usage of different coordinate systems, etc. However, basics of the three coordinate systems have been given at the beginning of the chapter. Maxwell's equations are directly derived for the time varying fields which could be reduced to those for the static fields by putting time derivatives to zero.

In Chapter 4, the solution of Maxwell's equation for the time varying fields in an unbound medium is derived. Hence the concept of uniform plane wave is introduced. Ample references are made to transmission lines to show the basic similarity between the two cases. Polarization of a wave has been discussed in detail highlighting its importance in real life situations. Further propagation of an EM wave in a conducting medium is investigated and the concept of skin depth, complex dielectric constant, surface current, etc are introduced in this chapter. The chapter concludes with the derivation of the Poynting theorem and introduction of the Poynting vector.

Chapter 5 discusses propagation of EM waves across media interfaces. Using physical reasoning, the Snell's law is established. Also the important concept of phase and group velocity is introduced in this chapter. An in-depth understanding to total internal reflection is developed at this stage. The reflection of EM waves from conducting boundaries is discussed in such a way that the reader can almost foresee the parallel plane waveguide.

Chapter 6 and 7 are on waveguides. The concept of modal propagation is developed in these chapters. Chapter 6 mainly discusses metallic waveguides in parallel plane and rectangular form. Chapter 7 is on dielectric waveguides that are important in optical technology, like thin film optical devices and optical fibers.

Chapter 8 and 9 are on antennas and antenna arrays. First, the basic philosophy of radiation is established and the concept of magnetic vector potential is introduced. The relation between the current and the magnetic vector potential is logically developed. Radiation characteristics of Hertz dipole and other dipoles are discussed further. Limitations of simple antennas are pointed out and a case is made for antenna arrays. After basic analysis of linear uniform arrays, the final Fourier transform relationship between the current distribution and the antenna radiation pattern has been established. Students who are already familiar with the Fourier transform find this relationship extremely exciting.

Chapter 10 is on propagation of EM waves for radio wave communications. Different transmission modes like ground waves, space waves and sky waves are discussed in this chapter. Propagation of an EM wave in the Ionosphere is discussed in great detail. Dielectric constant for ionized medium, with and without magnetization is derived and new interesting features of EM waves are discussed. A flavor of scattering of EM waves from the ionospheric irregularities is provided at the end.

All the chapters have numerous solved and unsolved problems besides conceptual review questions at the end of each chapter. Unnecessary complex formulae and their derivations are avoided in the book. Also there are no problems that would merely test the memory of a student. Formula substitution type problems are also avoided in the book. The problems are essentially designed to test the conceptual understanding of the subject and not the computational and algebraic skills of the students.

It is sincerely hoped that the presentation in the book and the problems will help the students in developing conceptual understanding of the subject without getting lost in the mathematical manipulations, and the students will start seeing EM wave related phenomena around them. If this book helps in making the subject of electromagnetic waves enjoyable, the whole exercise is worth the efforts.

WHO WILL BENEFIT

The book is targeted at junior students of electrical engineering in general, and electronics and telecommunication engineering in particular. A semester long course can be offered using Chapters 3–8 for general electrical/electronics engineering students. For telecommunication students, two-semester long courses can be offered using the whole content of the book. Depending upon the background of the students, the vector calculus can be covered before the Maxwell's.

The book will also be useful as reference material to practicing engineers working in the area of RF communication, radio broadcasting, microwave engineering and radar, optical and satellite communication, mobile communication, telemetry, remote sensing, radio astronomy, etc.

ACKNOWLEDGEMENTS

A large number of people have contributed directly or indirectly to this book. The first and the foremost are the wonderful teachers that I was fortunate enough to have during my student days. Prof. N.C. Mathur, Prof. D.K. Paul, Prof. A. Paul, and Prof. S. Mahaptra have played a very important role in igniting my curiosity about electromagnetics. Later during my professional career I interacted with great personalities like Prof. V. Radhakrishnan, Prof. Ch. V. Sastry, Prof. G. Swarup, who not only provided an in-depth understanding of electromagnetics and antenna, but also inspired me in many ways. Professional colleagues like Dr. Avinash Deshpande, Dr. K.S. Dwarakanath, Prof. B.N. Dwivedi, Prof. S.K. Jain, Prof. S.V. Kulkarni and many others who through their unending discussions polished my understanding of electromagnetics. I am indebted to all my teachers and colleagues for making me what I am today.

My understanding of the subject has been under constant scrutiny of the students of the electrical engineering department at IIT, Bombay. The students through their very intriguing questions force me to think deeper, which further refines my understanding of the subject. Every time I teach this subject I learn something new and exciting. I am indeed grateful to the students for making the teaching of this subject enjoyable.

My thanks are also due to the following referees of the manuscript whose comments positively contributed in improving the presentation.

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My family has contributed both, directly and indirectly, to this book. My wife, Sarita, who herself has doctorate in electrical engineering with specialization in microwaves and microwave photonics, has been a regular critic of this work. Her comments and critical suggestions have greatly improved the contents and presentation of the book. I am indebted to her for all the professional sacrifice she has made for my progress. My daughter, Kamini (who also helped in preparing part of the manuscript) and son, Chaitanya have been a constant source of energy needed to take up such a mammoth task. The love and affection bestowed on me by my parents and my family is beyond words.

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This book is a tribute to all those who taught me something in life.

R K SHEVGAONKAR

CHAPTER 1

Introduction

1.1 WHY STUDY ELECTROMAGNETIC WAVES?

Electromagnetic waves, a subject so classical and at the same time so modern. In ancient times people used to investigate things like

- why do small paper clips get attracted to an amber rod rubbed with silk,
- what causes lightening,
- why does a magnetic needle deflect when it is kept close to current carrying conductor,
- why do stars twinkle but the planets do not,
- how does the light reach the earth from the sun when there is no medium in between,
- why do light rays get focused when passed through a curved mirror or a lens
- why do colors get separated when a light beam passes through a prism, and so on.

While in the modern days people ask questions like

- how do we receive TV and radio signals,
- how do transmitted signals get affected while propagating through the earth's atmosphere,
- why do we not receive radio stations inside a train compartment,
- why is the radio reception good in some corner of the room but not in the other,
- why do medium wave radio stations not show any temporal variation in their signal strengths but the short wave radio stations do,
- why do the cell phones have signal fluctuation,
- why are TV antennas mounted on the top of the building but the radio antennas are not,
- why do some things get heated in a microwave oven and others not,

- why is there a disturbance on a radio but not on a TV when a motor cycle is started in the vicinity,
- why does a printed circuit board which was working satisfactorily at low frequencies start malfunctioning at high frequencies, and so on.

The list is unending. All these phenomena, which appear to come from different areas, have a common thread running through them and that is the *Electromagnetism.*

Initially the subject of electricity and magnetism was a matter of intellectual curiosity. Also in early days electricity and magnetism were considered unrelated phenomena. As time progressed the relationship between electricity and magnetism became evident and word 'Electromagnetism' emerged. Today, we hardly find any electrical or electronic device around us, which does not work on the principles of electromagnetism, or is not influenced by it. The things which were considered science fictions a hundred years ago have become a reality due to tremendous progress of electromagnetics and its engineering employment.

Today, the applications of electromagnetics can be broadly divided into two categories (i) low frequency but high power (ii) high frequency but low power. There are a few applications which have high frequency and high power. For low frequency applications, the analysis of static fields like electrostatics and magnetostatics is adequate to investigate the electromagnetic phenomenon. However, as the frequency increases, electric and magnetic fields get more strongly coupled and we always observe a composite phenomenon of electric and magnetic fields. One can then conveniently divide the subject of electromagnetics into two parts, the static electromagnetics and the time varying electromagnetics. As will be clear subsequently, the time varying electric and magnetic fields always constitute a wave phenomenon called the electromagnetic wave which is the prime subject of discussion of this book. (In this book, the discussion is focused on the time varying electric and magnetic fields, their inter relationship, their spatial and temporal characteristics, etc.)

The phenomenon of electromagnetism in totality is governed by the four Maxwell's equations, which can be derived from the physical laws like the Gauss Law, the Ampere's law and the Faraday's law of electromagnetic induction. Generally, we find two schools of thought regarding Maxwell's equations. Some people believe that the laws of electricity and magnetism were first established through experimentation and the mathematical representation of these experimental laws led to Maxwell's equations. People following this line of thought consider the origin of the electromagnetism in experimentation and therefore seek stronger physical reasoning to explain an electromagnetic phenomenon. The people belonging to the other school of thought, take the four Maxwell's equations as mathematical postulates and the physical laws like the Gauss law, etc. as the experimental verification of the postulates. Both approaches have their merits. The first approach, which has a foundation in experimentation, prevents people from getting lost in mathematical manipulations. This approach keeps reminding an investigator that electromagnetism is a physical phenomenon and therefore one should develop a habit of finding a physical picture for every mathematical manipulation one carries out. Consequently, one should not lose touch with the physical phenomenon. The second approach is more accurate and elegant as it is free from experimental errors. With only experimental measurements one could never establish such exact relationships like inverse square law and so on.

The electromagnetic theory is the generalization of the circuit theory, or the circuit theory is rather a special case of the electromagnetic theory. The circuit theory deals with quantities like voltage, current, resistance, inductance and capacitance which are scalar in nature. The electromagnetic theory on the other hand deals with quantities like the electric and magnetic fields which are vector quantities. The complexity of the electromagnetic theory is several times higher compared to the circuit theory. Although every phenomena of electricity and magnetism can be analyzed in the frame work of electromagnetic theory, at low frequencies the circuit approach is adequate. As the frequency increases the inadequacy of the circuit approach is evident and one is forced to follow the electromagnetic field approach. For example, sending electrical signals through free space can never be visualized using circuit approach. Similarly, it is difficult to visualize a current flow in a wire which is connected to a source at one end but is left open at the other end. The correct approach therefore would be to apply the simpler circuit approach as far as it can be applied and make a transition to the field approach when the circuit model tends to break down.

1.2 APPLICATIONS OF ELECTROMAGNETIC WAVES

Time varying electric and magnetic fields, in general, constitute a wave phenomenon. Some of the applications where electromagnetic waves can be encountered are given in the following sections:

1.2.1 Transmission Lines

At low frequencies, an electrical circuit is completely characterized by the electrical parameters like resistance, inductance, etc. and the physical size of the electrical components plays no role in the circuit analysis. Simple Kirchhoff's laws are adequate for analyzing a circuit. However, as the frequency increases the size of the components becomes important and the space starts playing a role in the performance of the circuit. The voltage and currents exist in the form of waves. Even a change in the length of a simple connecting wire may alter the behavior of the circuit. The circuit approach then has to be re-investigated with inclusion of the space into the analysis. This approach is then called the transmission line approach. Two conductor transmission media like the coaxial cable, flat ribbon cable, etc. are the examples of transmission lines. Although, the primary objective of a transmission line is to carry electromagnetic energy efficiently from one location to other, they find wide applications in high frequency circuit designs.

1.2.2 High Frequency and Microwave Circuits

As the frequency increases, any discontinuity in the circuit path leads to electromagnetic radiation. Also at high frequencies the transit time of the signals cannot be ignored. In the era of high speed computers, where data rates are approaching to a few Gb/sec, the phenomena related to the electromagnetic waves, like the bit distortion, signal reflection, impedance matching play a vital role in high speed communication networks.

1.2.3 Antennas

An antenna is a device which can launch and receive electromagnetic waves efficiently. An antenna which appears as a passive looking device in a communication system, is one of the most important devices. But for the large antennas, the communication between an earth station and a satellite is practically impossible. The communication which can be established with a few watts of power, would need few MW of power in the absence of proper antennas. Many types of antennas have been in use over several decades. However, the antenna research is still very active. With recent advances in mobile communication, design of compact, efficient, multi-frequency antennas have received a new impetus in the last decade.

1.2.4 Fiber Optic Communication

Fiber optic communication is the most modern form of guided wave communication. Electromagnetic theory is used to investigate propagation of light inside the optical fibers. The phenomenon of dispersion which has a direct bearing on the speed of communication, is due to the difference in speed of light for different modes of the optical fiber. The modal propagation inside an optical fiber is a direct consequence of the wave nature of light. Electromagnetic wave theory is also important in the analysis of lasers and photo detectors. By employing the complex phenomena of electromagnetic waves, a variety of fiber optic devices have been developed for efficient high speed long haul communication.

1.2.5 Mobile Communication

Knowledge of electromagnetic wave propagation plays a vital role in understanding the radio environment. Depending upon the variation of the signal strength as a function of distance, different frequency reuse schemes can be employed in a cellular system. Fading is one of the important aspects of mobile communication. Efficient signal processing algorithms need the knowledge of the radio environment to correctly predict the fading behavior.

1.2.6 Electromagnetic Inference (EMI) and Compatibility

An electrical circuit, which especially switches heavy current, tends to give electromagnetic radiation. This radiation interferes with the other circuit elements affecting the overall performance of the circuit. Switch mode power supply and high speed digital circuits create a substantial amount of EMI. To protect the circuits from EMI, shielding techniques are employed. Designing of proper EMI shields needs a thorough knowledge of electromagnetics.

1.2.7 Radio Astronomy

This is a branch of astronomy in which observations of the sky are carried out at radio frequencies. The radio signals received from the sky are very weak in nature. State of art communication receivers and antennas are required to detect these signals. Therefore radio astronomy is a combination of electronics engineering and physics. This is one of the fields where a thorough understanding of electromagnetic waves is necessary. In fact, almost all aspects of electromagnetic waves in some form or the other, are employed in radio astronomy.

These are some of the major areas where knowledge of electromagnetic waves is profoundly used. There are many more applications of electromagnetic waves in addition to these.

1.3 SUMMARY

In this chapter the importance of electromagnetic waves in electrical engineering has been highlighted. A wide application of electromagnetic waves have been metioned. The understanding of electromagnetic waves is useful in invertigating many classical and modern engineering problems. Many, apparently unrelated, fields like optical astronomy and wireless communication share the same principles of electromagnetic waves. In the following chapters we develop understanding of various aspects like, propagation, confinement, radiation, etc, of the electromagnetic waves.

CHAPTER 2

Transmission Lines

2.1 INTRODUCTION

The phenomenon of electromagnetic waves, in principle, is associated with the time varying electric or magnetic fields although its presence becomes more visible at higher frequencies. Like any other wave, the electromagnetic wave is a composite phenomenon of space and time. At low frequencies since the size of the circuit elements is negligible compared to the wavelength, the effect of space is generally ignored. A circuit is characterized by its temporal response and the circuit components are described by their electrical parameters. That is to say that at low frequencies we get correct circuit performance if the circuit elements have the correct electrical values irrespective of their physical size. This however is not true at frequencies beyond few hundreds of kHz. In fact, even the lengths of simple connecting wires alter the circuit performance. Naturally, there is a conceptual change as we increase the frequency of operation and an electrical circuit can no longer be analysed without taking the 'space' into consideration. This is due to the fact that the signal requires a finite time to travel along an electric circuit which is no more negligible compared to the time period of the signal. This effect is called the 'transit time effect'. Therefore at high frequencies the transit time effect has to be included in the circuit analysis.

In general, the electromagnetic wave phenomenon is described by behavior of the electric and the magnetic fields in the three dimensional space as a function of time. However, to develop the basic concepts let us first analyse a simpler but important case of electromagnetic waves, the *transmission line*. In the anlysis of a transmission line, we continue to use the electrical quantities like voltage, current, resistance, capacitance, etc. as used in low frequency circuits, but appropriately incorporate the effect of finite transit time. We then obtain the voltage and current which have wave type behavior. Analysis of a tranmission line therefore helps in understanding some of the fundamentals of the electromagnetic waves.

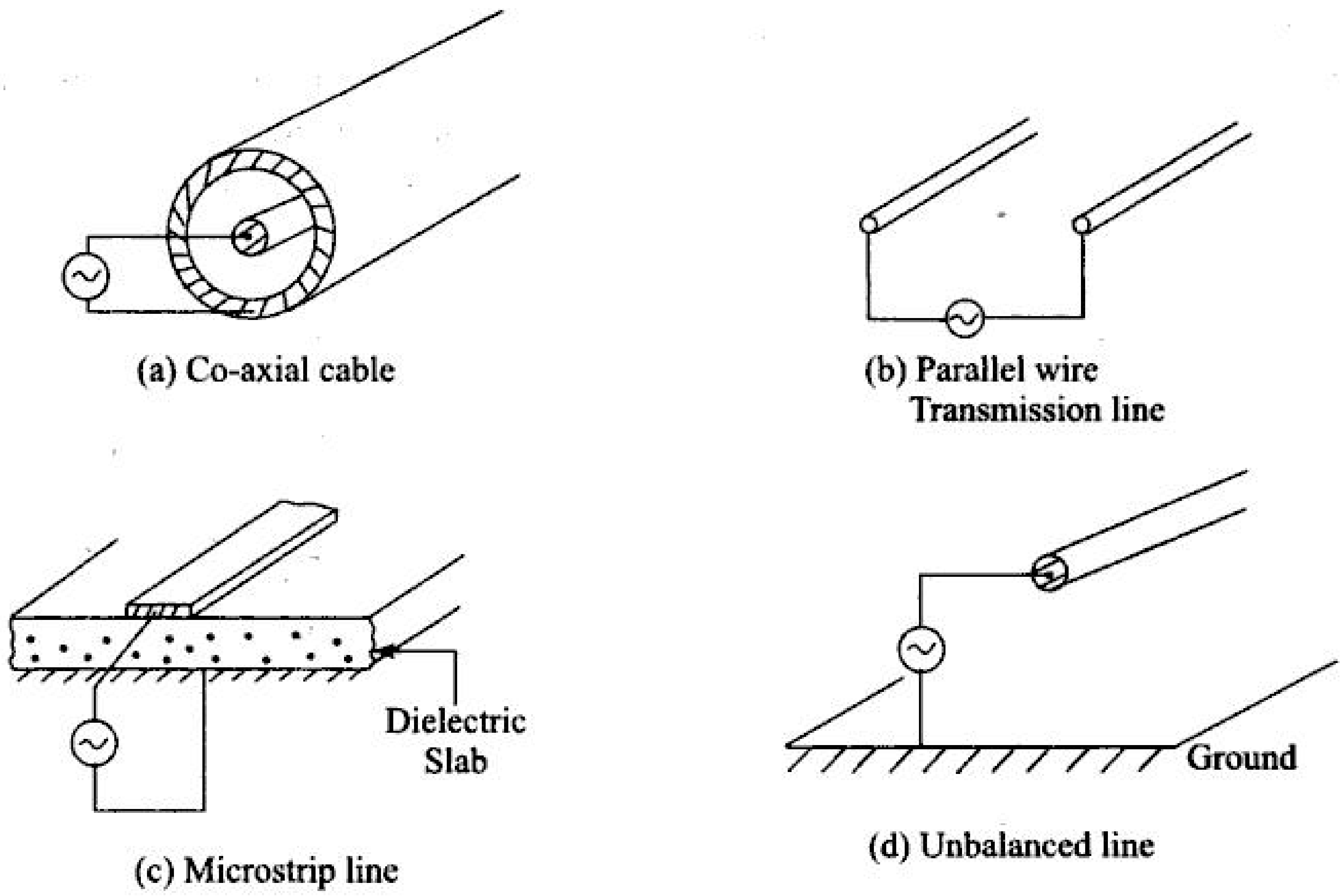


Fig. 2.1 *Transmission line configurations.*

2.2 CONCEPT OF DISTRIBUTED ELEMENTS

Figure 2.1 shows various types of transmission lines used in practice. However, generically a transmission line is a two-conductor system one end of which is connected to a source and the other end is connected to a load as shown in Fig. 2.2. Let the length of the transmission line be l and let it be excited by a voltage source at frequency f. Realizing that no signal can travel with infinite speed, let the speed of the signal on the tranmission line be v. Now, let us suppose that at some instant of time, the voltage corresponding to point P on the voltage waveform (say V_p) is connected to the transmission line at AA′. Due to finite speed of the signal the V_p voltage will not appear at BB′ instantaneously but will be delayed by the transit time, $t_r = l/v$. Obviously during this time the input signal does not remain at point P but changes to point Q. That means when the voltage is passing through a point Q at AA′, it is passing through point P at BB′. In other words there is a phase difference between the voltages at AA′ and BB′. It should be realized that there will be a phase difference between any two locations on the transmission line, and the phase difference will increase as the seperation between the locations increases. This will be true even when the conductors used in the transmission line are ideal. It is then interesting to find that even when the resistance of a conductor is zero there is a potential difference between its ends. With a little thinking, one would find that although the resistive voltage drop across an ideal conductor is zero, the reactive drop will be finite for any non-zero angular frequency, ω. As the frequency increases the inductive reactance (ωL) and the capacative susceptance (ωC) increase proportionately making reactive effect more observable. From transit time view-point, an increase in frequency reduces the signal time period, making the transit time a measureable fraction of the signal period. Therefore, the dominance of the circuit reactance at high frequencies and the effect of finite transit time are the two faces of the same phenomenon.

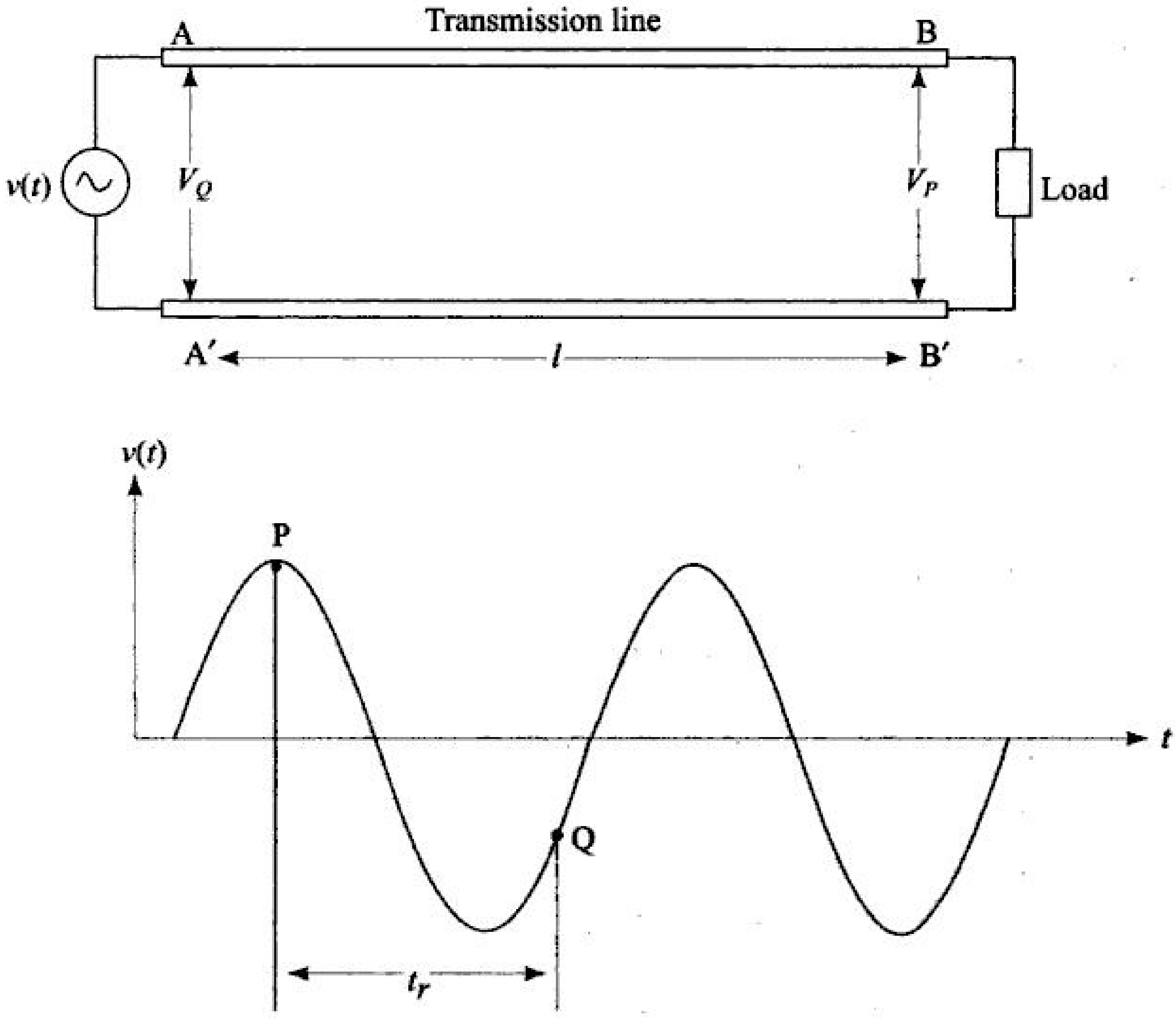


Fig. 2.2 *Voltage on a transmission line.*

The lumped element circuit analysis, at low frequencies, is valid provided the signal transit time is negligible compared to the signal period, i.e. if $l/v << T$, where $T = 1/f$. One can, therefore, model a transmission line by lumped elements by dividing it into small sections such that over each sub-section the transit time effect is negligible. It is clear that for any finite length of the subsection, the transit time effect will be appreciable beyond certain frequency, no matter how small the subsection is. If the lumped element model be applicable at all frequencies, the length of the subsection must tend to zero and the analysis must be carried out in some form of limit. The circuit elements now cannot be defined for the entire transmission line but has to be defined for a subsection. It is, therefore, appropriate to define circuit parameters for unit length of transmission line. The circuit elements are not located at a particular location of the line but are distributed all along its length. The high frequency analysis of a transmission line can be carried out by using the concept of distributed elements.

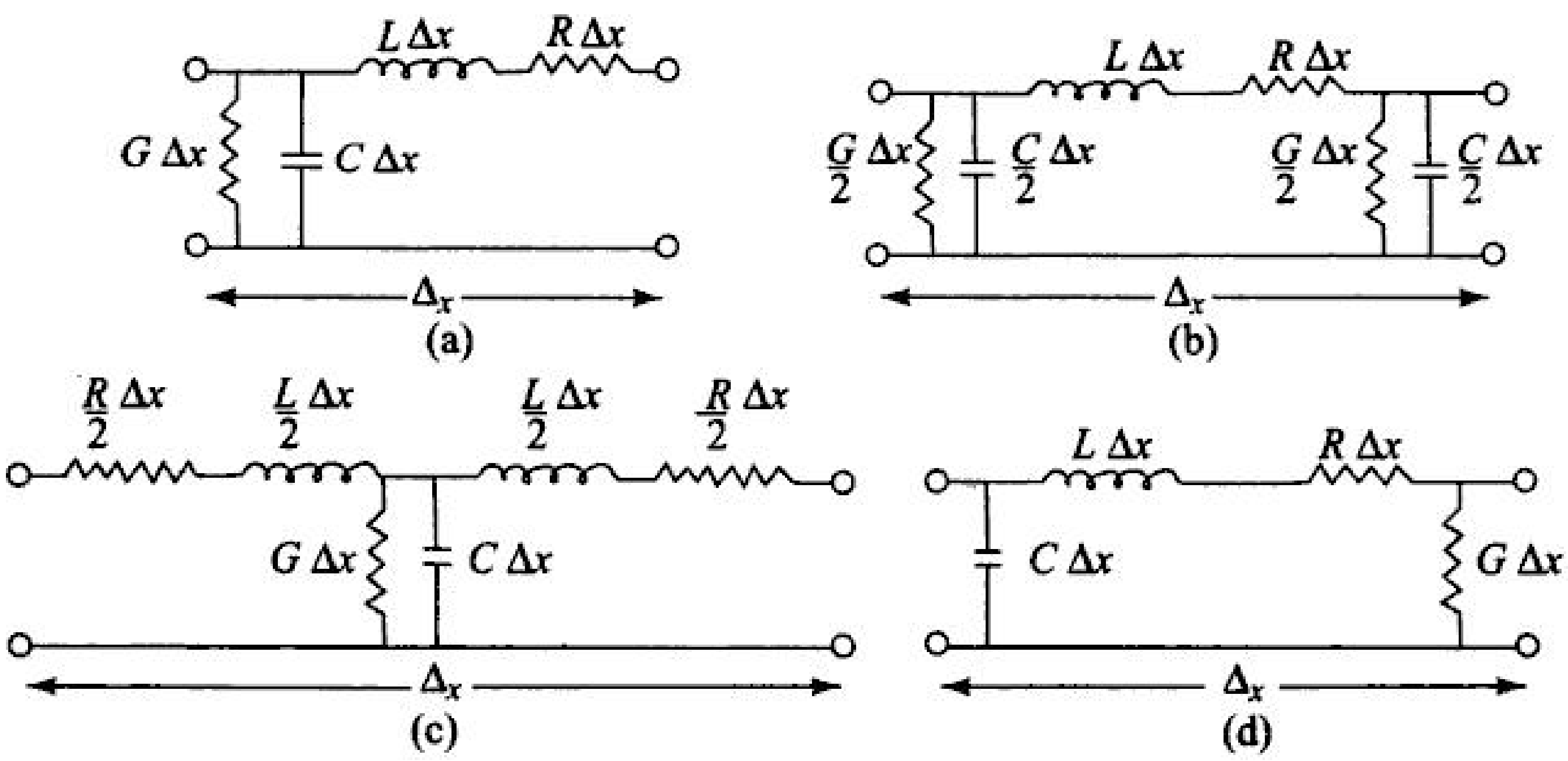


Fig. 2.3 *Lumped circuit equivalent for infinitesimal section of a transmission line.*

EXAMPLE 2.1 A two conductor transmission line is 10 cm long. A sinusoidal signal of 1V peak amplitude is applied to one end of the line. If the signal travels on the line with a speed of 2×10^8 m/sec, find (a) transit time on the line (b) frequency at which the transit time is 10% of the signal period (c) signal voltage on the other end of the line at an instant when the input signal is passing through +ve maximum for a frequency of 500 MHz.

Solution:

(a) $$\text{Transit time } t_r = \frac{\text{Length of the line}}{\text{Velocity}} = \frac{0.1\text{ m}}{2 \times 10^8\text{ m/sec}} = 0.5\text{ nsec}$$

(b) Transit time should be 10% of the period T, i.e.

$$\text{The transit time } t_r = 0.1\text{ T}$$

$$\Rightarrow \qquad T = \frac{t_r}{0.1} = 10\, t_r = 5\text{ nsec}$$

$$\text{Frequency} = \frac{1}{T} = \frac{1}{5 \times 10^{-9}} = 200\text{ MHz}$$

(c) Let the signal voltage be given as

$$v(t) = A\cos(2\pi f t)\text{V}$$

$$\text{Given that } A = 1,\ f = 500\text{ MHz}.$$

We, therefore, get

$$v(t) = \cos(\pi \times 10^9 t)$$

Taking the instant as $t = 0$, when the input voltage $v(t) = 1$ V, the signal at the other end of the line will correspond to $t = -t_r$. The voltage at the other end of the line therefore is

$$\cos(-\pi \times 10^9 t_r) = \cos\left(-\frac{\pi}{2}\right) = 0 \text{ V}.$$

2.3 EQUATIONS OF VOLTAGE AND CURRENT

Consider an infinitesimally small section of a transmission line of length Δx. Assuming that the conductor and the dielectrics are non-ideal, any two conductor system can be represented by the four characteristic circuit parameters (called the 'primary constants' of the line) namely, the series resistance, the series inductance, the shunt capacitance and the shunt conductance. Let R, L, C and G represent these quantities respectively per unit length of the line. The choice of unit length is arbitary but in MKS system it is one meter. The equivalent lumped circuit for the infinitesimal section of the transmission line can be shown as in Fig. 2.3. All the circuits in Fig. 2.3 are equivalent for $\Delta x \to 0$. The resistance, inductance, conductance and capacitance of the infinitesimal section of the line are $R\Delta x$, $L\Delta x$, $G\Delta x$, and $C\Delta x$ respectively.

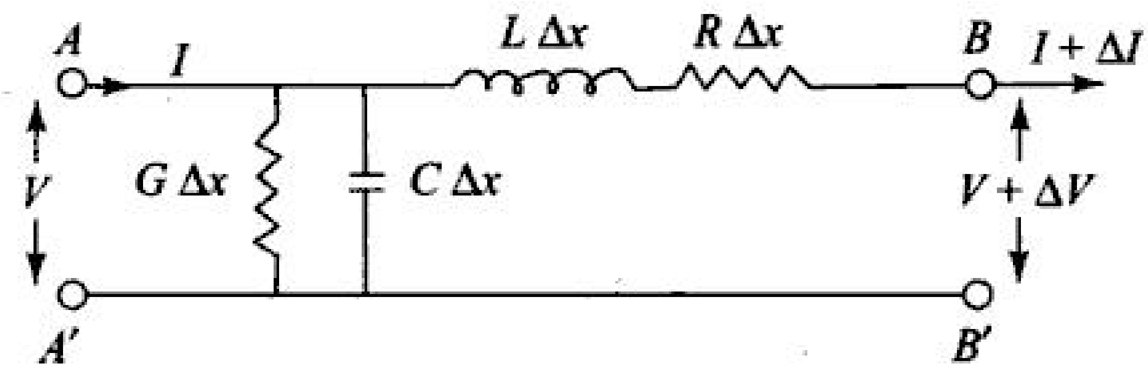


Fig. 2.4 *Voltage and current on an infinitesimal section of a transmission line.*

Let a sinusoidal voltage of angular frequency ω be applied between AA′, and let a current I flow into the terminal A (refer to Fig. 2.4). Now due to voltage drop in the series elements, R and L, the voltage at BB′ will not be same as that at AA′. Similarly, since some part of the input current will be bypassed through the shunt elements, C and G, the output current at point B will not be same as that at A. Let the current and the voltage at BB′ be $I + \Delta I$ and $V + \Delta V$ respectively. We can then write,

$$\Delta V = -(R\Delta x + j\omega L\Delta x)I \tag{2.1}$$

$$\Delta I = -(G\Delta x + j\omega C\Delta x)V \tag{2.2}$$

The negative sign indicates that the voltage and current at BB′ are less than their respective values at AA′.

Equations (2.1), (2.2) can be re-written as

$$\frac{\Delta V}{\Delta x} = -(R + j\omega L)I \tag{2.3}$$

$$\frac{\Delta I}{\Delta x} = -(G + j\omega C)V \tag{2.4}$$

As discussed earlier, if the lumped analysis has to be valid at all frequencies, the length of the section Δx must tend to zero. Taking the limit of Eqns (2.3) and (2.4) for $\Delta x \to 0$, we get

$$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} \equiv \frac{dV}{dx} = -(R + j\omega L)I \tag{2.5}$$

$$\lim_{\Delta x \to 0} \frac{\Delta I}{\Delta x} \equiv \frac{dI}{dx} = -(G + j\omega C)V \tag{2.6}$$

Therefore, we find that the voltage and current on a transmission line are governed by two coupled first order differential Eqns (2.5) and (2.6). Differentiating Eqn (2.5) with respect to x, we get

$$\frac{d^2V}{dx^2} = -(R + j\omega L)\frac{dI}{dx} \tag{2.7}$$

Now substituting for dI/dx from Eqn (2.6) we have

$$\frac{d^2V}{dx^2} = -(R + j\omega L)[-(G + j\omega C)V] \tag{2.8}$$

$$= (R + j\omega L)(G + j\omega C)V \tag{2.9}$$

Similarly, if we differentiate Eqn (2.6) with respect to x and substitute for dV/dx from Eqn (2.5), we get

$$\frac{d^2I}{dx^2} = (R + j\omega L)(G + j\omega C)I \tag{2.10}$$

Let us now define a quantity called *propagation constant*, γ of the transmission line as

$$\gamma^2 = (R + j\omega L)(G + j\omega C) \tag{2.11}$$

The physical interpretation of γ will be discussed later. However, one thing is clear that γ is some characteristic parameter of the line similar to the primary constants except that it depends upon the operating frequency also. With the introduction of γ the Eqns (2.9) and (2.10) can be compactly written as

$$\frac{d^2V}{dx^2} = \gamma^2 V \tag{2.12}$$

$$\frac{d^2I}{dx^2} = \gamma^2 I \tag{2.13}$$

It is interesting to note that both voltage and current are governed by the same differential equation.

EXAMPLE 2.2 For a transmission line the per unit length parameters are $0.1\Omega/\text{m}$, 0.2 μH/m, 10 pF/m and 0.02 ℧/m. Find the complex propagation constant at (a) 1 MHz (b) 1 GHz.

Solution:

(a) The propagation constant

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$R = 0.1\ \Omega/\text{m}, \qquad L = 0.2 \times 10^{-6}\text{H/m}$$

$$G = 0.02\ \mho/\text{m}, \qquad C = 10 \times 10^{-12}\text{F/m}$$

$$\omega = 2\pi f = 2\pi \times 10^6 \text{rad/sec}$$

$$\Rightarrow \gamma = \sqrt{(0.1 + j2\pi \times 10^6 \times 0.2 \times 16^{-6})(0.02 + j2\pi \times 10^6 \times 10^{-11})}$$

$$= 0.117 + j0.108\,/\text{m}$$

(b) At 1 GHz, $\omega = 2\pi \times 10^9$ rad/s,

$$\gamma = 1.4 + j9\,/\text{m}$$

Since, for a given operating frequency γ is constant, Eqns (2.12) and (2.13) are homogeneous equations with constant coefficients, and their solutions can be written as

$$V = V^+ e^{-\gamma x} + V^- e^{+\gamma x} \tag{2.14}$$

$$I = I^+ e^{-\gamma x} + I^- e^{+\gamma x} \tag{2.15}$$

Where, V^+, V^-, I^+, I^- are arbitrary constants which are to be evaluated by using appropriate boundry conditions. These constants in general are complex and their phases represent the temporal phases from some reference time. In the above equations the time harmonic function is implicit. If we want to write the instantaneous values of the voltage and current we have to multiply Eqns (2.14) and (2.15) by $e^{j\omega t}$ to arrive at

$$V(t) = V^+ e^{j\omega t - \gamma x} + V^- e^{j\omega t + \gamma x} \tag{2.16}$$

$$I(t) = I^+ e^{j\omega t - \gamma x} + I^- e^{j\omega t + \gamma x} \tag{2.17}$$

Now, from Eqn (2.11) we see that the propagation constant γ is a complex quantity and we can write,

$$\gamma = \alpha + j\beta \tag{2.18}$$

Substituting for γ from Eqn (2.18) into Eqns (2.16) and (2.17) we obtain,

$$V(t) = V^+e^{-\alpha x}e^{j\omega t - j\beta x} + V^-e^{\alpha x}e^{j\omega t + \beta x} \tag{2.19}$$

$$I(t) = I^+e^{-\alpha x}e^{j\omega t - j\beta x} + I^-e^{\alpha x}e^{j\omega t + \beta x} \tag{2.20}$$

It can be noted that all the terms in Eqns (2.19) and (2.20) are complex in general and their phases are functions of both space (x) and time (t). For example, take the first term of $V(t)$. The phase of this term is $\phi = \omega t - \beta x + phase(V^+)$. We can then see the contribution which the space and time make to the phase of this term as follows.

If we freeze time by making $t = constant$, the phase ϕ changes linearly as a function of distance x. It means if we simultaneously look at the whole transmission line, the phase will vary linearly from one end of the line to the other end of the line. On the other hand, if we freeze space, it means observe the phase at a given point on the transmission line, the phase will linearly vary as a function of time t. If we, therefore, plot the function $Re\left\{e^{(j\omega t - j\beta x)}\right\} = cos(\omega t - \beta x)$ in space and time, the function will appear like a sinusoidally corrugated surface as shown in Fig. 2.5. Let us consider a point P on the surface at $t = 0$. As we now increase t, the point moves towards right in space, that is towards $+x$ direction. Since, this is true for every point on the surface, the whole pattern apears to move in the $+x$ direction as a function of time. This behavior is nothing but a 'travelling wave'. The first term on the RHS of Eqn (2.19) hence represents a voltage wave travelling in the $+x$ direction.

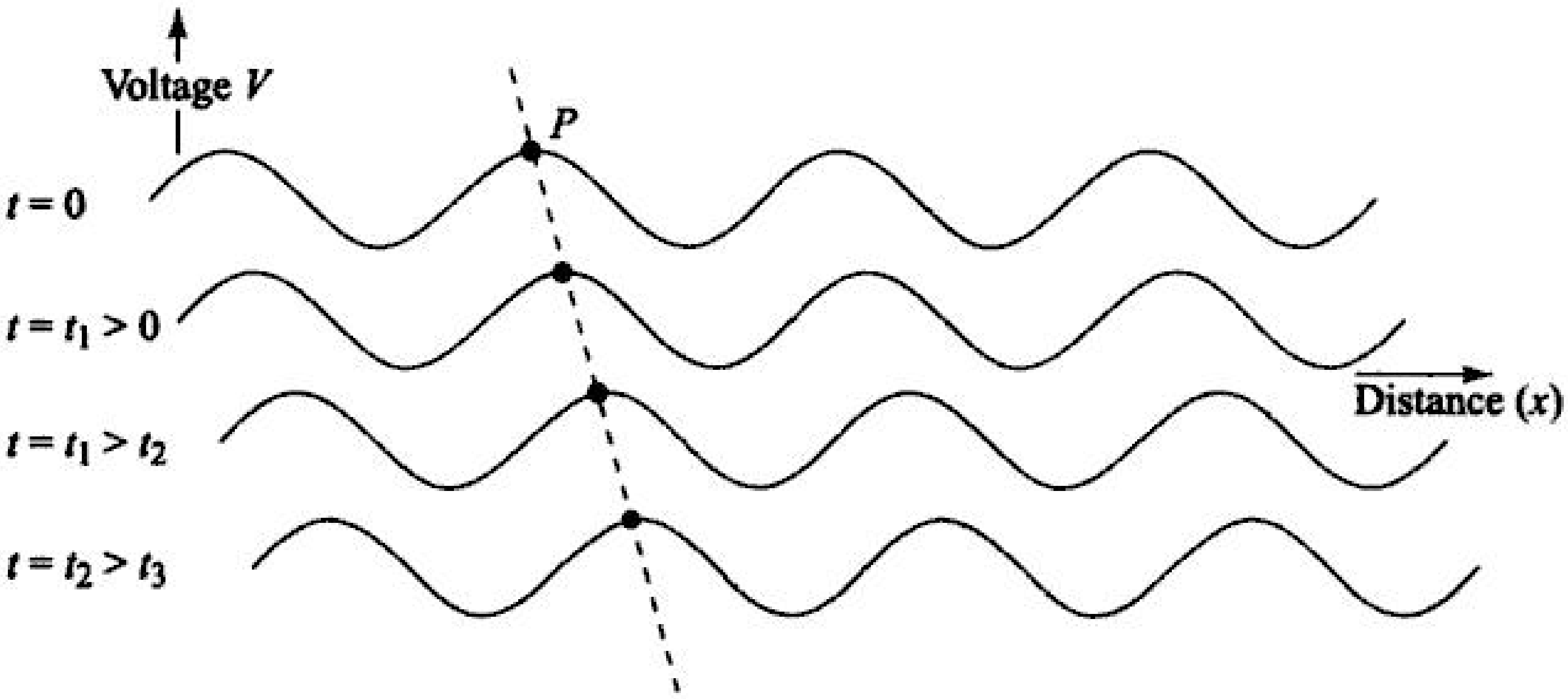


Fig. 2.5 *Voltage as a Function of x for different time*

Similarly, if we plot the $Re\left\{e^{j\omega t + j\beta x}\right\} = \cos(\omega t + \beta x)$ corresponding to the second term on the RHS of Eqn (2.19), we will get a corrugated surface similar to that in the previous case, but now a point P will move towards the left that is towards $-x$ direction as a function of time as shown in Fig. 2.6. In other words, this term represents a voltage wave travelling in $-x$ direction.

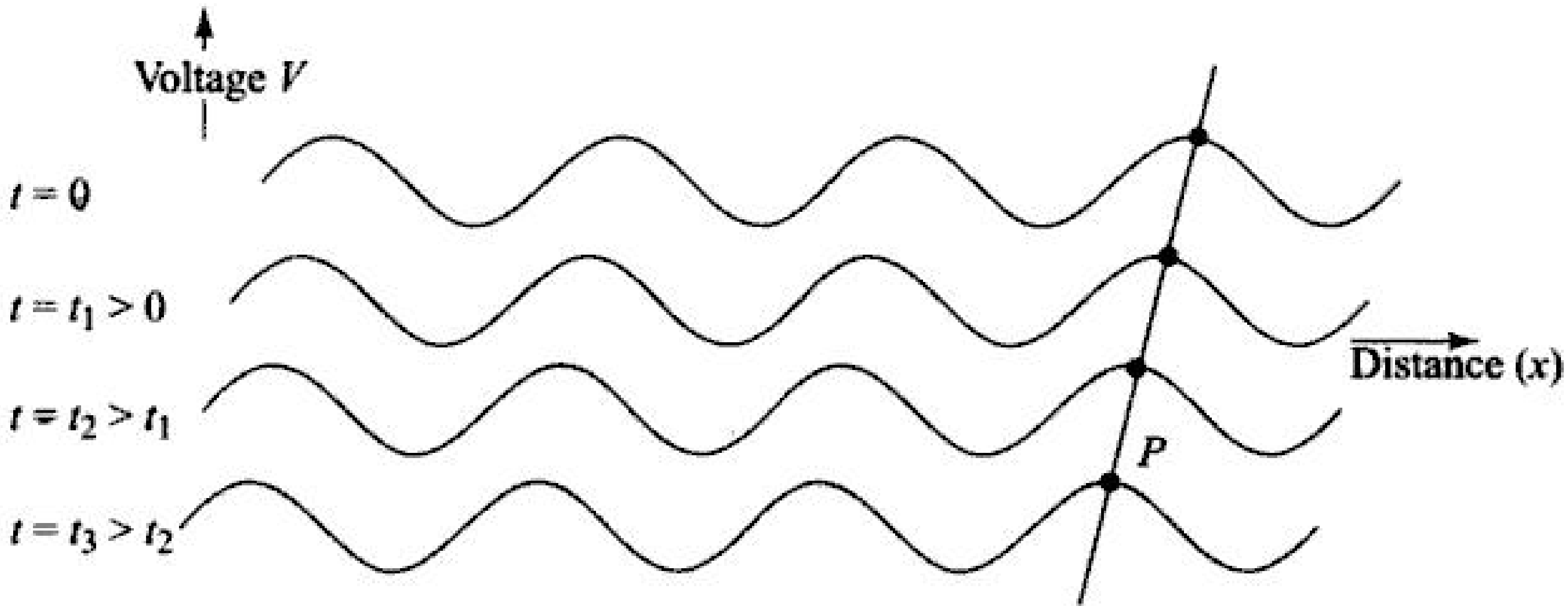


Fig. 2.6 *Voltage as a function of x for different time.*

On identical lines one can show that the first term of $I(t)$ represents a current wave travelling in the $+x$ direction, and the second term represents a current wave travelling in $-x$ direction.

It is clear from the above discussion that as the effect of finite transit time is introduced, that is, as we make use of the distributed elements, the natural solution which we get for voltage and current is a wave type solution. In general, we can therefore say that in an electric circuit the time varying voltage and current exist in the form of waves.

EXAMPLE 2.3 A voltage wave at 1 GHz is travelling on a transmission line in $+x$ direction. The primary constants of the line are, $R = 0.5$ Ω/m, $L = 0.2$ μH/m, $G = 0.1$ ℧/m, $C = 100$ pF/m. The wave has 30° phase at $t = 0$ and $x = 0$. Find the phase of the wave at $x = 50$ cm and $t = 1\,\mu$sec.

Solution:

$$\omega = 2\pi \times 10^9 \text{rad/m}$$

The propagation constant:

$$\gamma = \sqrt{(R + j\omega l)(G + j\omega C)}$$

$$= \sqrt{(0.5 + j2\pi \times 10^9 \times 0.2 \times 10^{-6})(0.1 + j2\pi \times 10^9 \times 100 \times 10^{-12})}$$

$$\gamma = 2.23 + j28.2 \ \ 1/\text{m}$$

$$\beta = 28.2 \text{ rad/m}$$

$$\text{Phase of the wave} = \text{Initial Phase} + \omega t - \beta x$$

$$= 30^\circ + 2\pi \times 10^9 \times 10^{-6}\text{rad} - 28.2 \times 0.5 \text{ rad}$$

$$= 6269.608 \text{ rad} = 35922.129^\circ$$

2.3.1 Phase and Attenuation Constants

Having understood the nature of the voltage and current on a transmission line, we can now assign some physical meaning to the propagation constant γ. Since βx is the phase of the wave as a function of x, β represents the phase change per unit length of the transmission line for a travelling wave. Since β defines the phase variation along the transmission line, it is called the 'phase constant' of the transmission line, and it has the unit of *radians per meter*. Now, since for a wave the phase change over a wavelength λ is 2π, by definition we have, $\beta\lambda = 2\pi$. The wavelength of a wave on the transmission line therefore is $\lambda = 2\pi/\beta$. Note, γ and therefore β depends upon the primary constants, R, L, G, C and the frequency, and consequently the wavelength of the wave on the transmission line is also a function of the primary constants and the frequency, in general.

Let us now look at the amplitudes of the two terms of $V(t)$. Amplitude of the first term, i.e. $|V^+|e^{-\alpha x}$ represents the amplitude of the wave travelling in the $+x$ direction at location x on the line. For a positive α, the amplitude exponentially decreases as a function of x. The quantity α then represents attenuation of the wave on the transmission line, and consequently called the 'attenuation constant' of the line. The unit of α is nepers per meter. It can be easily seen that if a line has an attenuation constant of 1 neper/m, a unit amplitude voltage wave will reduce to e^{-1} over a distance of 1 meter. The attenuation of the wave in dB is $-20\log\ (e^{-1}) = 8.68$ dB. We can then say that 1 neper is equivalent to 8.68 dB. The attenuation constant (α) of a line therefore can be given either in neper/m or dB/m. It should however be remembered that the use of α in any of the voltage or current equations, has to be in the unit of neper/m.

EXAMPLE 2.4 For the wave in Example 2.3, calculate attenuation constant in nepers/m and dB/m. If the voltage of the forward travelling wave at $t = 0$ and $x = 0$ is 8.66 V, find the voltage at $x = 1$m at time $t = 100$ nsec. What is the peak voltage at $x = 1$m?

Solution:

From Example 2.3, $\gamma = 2.23 + j28.2$ per meter.

$$\Rightarrow \qquad \alpha = 2.23 \text{ nepers/m} = 8.68 \times 2.23 = 19.356 \text{ dB/m}$$

The voltage for the forward travelling wave is

$$v(t) = Re\{V^+ e^{-\alpha x} e^{j(\omega t - \beta x)}\}$$

Taking $V^+ = |V^+|e^{j\phi}$, where ϕ is the initial phase, we get

$$v(t) = |V^+|e^{-\alpha x}\cos(\phi + \omega t - \beta x)$$

At $x = 0$ and $t = 0$, it is given that $v(t) = 8.66V$. Therefore

$$8.66 = |V^+| \cos 30^o$$

$$\Rightarrow \qquad |V^+| = 10\text{V}$$

The instantaneous voltage at $x = 1\text{m}$ and $t = 100$ nsec is

$$v(t) = 10e^{(-2.23\times 1)} \cos(30^\circ + 2\pi \times 10^9 \times 100 \times 10^{-9}(\text{rad}) - 28.2 \times 1(\text{rad}))$$
$$= -0.8887\text{V}$$

Peak voltage at $x = 1$ m is

$$V_p = 10\, e^{-2.23} = 1.0753\text{V}$$

EXAMPLE 2.5 If the wave in Example 2.4 is travelling in $-x$ direction with all other things same, find the instantaneous voltage at the same time and at the same location.

Solution:

The voltage for a wave travelling in $-x$ direction is given as

$$v(t) = Re\{V^- e^{\alpha x} e^{j(\omega t + \beta x)}\}$$

Taking $V^- = |V^-| e^{j\phi}$, we have

$$v(t) = |V^-| e^{\alpha x} \cos(\phi + \omega t + \beta x)$$

At $x = 0$ and $t = 0$, $v(t) = 8.66$ V (given). Therefore

$$8.66 = |V^-| \cos 30^\circ$$

$$\Rightarrow \qquad |V^-| = 10\text{ V}$$

The voltage at $x = 1$ m and $t = 100$ nsec will be

$$v(t) = 10\, e^{(2.23\times 1)} \cos(30^o + 2\pi \times 10^9 \times 100 \times 10^{-9}(\text{rad}) + 28.2 \times 1(\text{rad}))$$
$$= -83.77\text{ V}$$

2.3.2 Evaluation of Arbitrary Constants

To evaluate the arbitrary constants V^+, V^-, I^+, I^- in Eqns (2.19) and (2.20), we have to apply appropriate boundary conditions. However, before we do that, we can reduce the number of arbitrary constants by making use of the fact that the voltage and current given by Eqns (2.19) and (2.20) must satisfy the basic differential Eqns (2.5) and (2.6) at every point on the transmission line.

Substituting for V and I from Eqns (2.19) and (2.20) in (2.5) we get

$$\frac{d}{dx}\{V^+e^{-\gamma x} + V^-e^{\gamma x}\} = -(R + j\omega L)\{I^+e^{-\gamma x} + I^-e^{\gamma x}\} \quad (2.21)$$

$$\Rightarrow \ -\gamma V^+e^{-\gamma x} + \gamma V^-e^{\gamma x} = -(R + j\omega L)\{I^+e^{-\gamma x} + I^-e^{\gamma x}\} \quad (2.22)$$

Although, Eqn (2.22) appears as one equation, there are two equations embedded in it. As seen earlier, the two terms $e^{-\gamma x}$ and $e^{\gamma x}$ represent two travelling waves in opposite directions (one in $+x$ direction and one in $-x$ direction). Therefore, if Eqn (2.22) is to be satisfied at every point on the line, that is, for every value of x, the individual waves must satisfy the equation. In other words, the coefficients of $e^{-\gamma x}$ and $e^{\gamma x}$ on the two sides of the equality sign must be separately equated. We, therefore, get

$$\text{Coefficient of } e^{-\gamma x}: \qquad -\gamma V^+ = -(R + j\omega L)I^+ \qquad (2.23)$$

$$\text{Coefficient of } e^{\gamma x}: \qquad \gamma V^- = -(R + j\omega L)I^- \qquad (2.24)$$

Since, $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$ (see Eqn (2.11)), we get from (Eqns 2.23) and (2.24) respectively

$$\frac{V^+}{I^+} = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad (2.25)$$

$$\frac{V^-}{I^-} = -\frac{R + j\omega L}{\gamma} = -\sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad (2.26)$$

It can be seen easily that since the quantity $\sqrt{(R + j\omega L)/(G + j\omega C)}$ has the dimensions of impedance (it is a ratio of voltage and current) and is a function of the primary constants of the line and the operating frequency, it is a characteristic parameter of the line. It is therefore called the 'characteristic impedance' of the transmission line and is normally denoted by Z_0.

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad (2.27)$$

The parameters γ and Z_0 are called secondary parameters of the transmission line. We will subsequently see that for the analysis of a transmission line, the knowledge of secondary parameters is adequate and generally there is no necessity of going to the primary constants.

EXAMPLE 2.6 A transmission line has primary constants $R = 0.1\Omega$/m, $G = 0.01$ ℧/m, $L = 0.01\mu$H/m, $C = 100$ pF/m. Find the characteristic impedance of the line at 2 GHz.

Solution:

$$\omega = 2\pi \times 2 \times 10^9 = 4\pi \times 10^9 \text{ rad/sec}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$= \sqrt{\frac{0.1 + j4\pi \times 19^9 \times 0.01 \times 10^{-6}}{0.01 + j4\pi \times 10^9 \times 100 \times 10^{-12}}}$$

$$= 10 + j0.0358\ \Omega$$

From Eqns (2.25), (2.26) we note that the ratio of V^+ and I^+ is Z_0 whereas the ratio of V^- and I^- is $-Z_0$. Recalling that the V^+ and I^+ are the respective amplitudes of the voltage and current waves travelling in $+x$ direction, we can conclude that the ratio of the voltage and current for a forward travelling wave at every point on the line is constant and is equal to the characteristic impedance Z_0. Said differently, a forward travelling wave always sees the characteristic impedance irrespective of the other boundary conditions on the transmission line. Similarly, we can conclude that the backward travelling wave (the wave travelling in $-x$ direction) always sees an impedance which is $-Z_0$. It should be pointed out that this $-Z_0$ does not represent a negative resistance in the conventional sense but is rather a manifestation of the direction of the wave travel.

Substituting for I^+ and I^- from Eqns (2.25) and (2.26) into Eqns (2.14) and (2.15) we get

$$V = V^+ e^{-\gamma x} + V^- e^{\gamma x} \tag{2.28}$$

$$I = \frac{V^+}{Z_0} e^{-\gamma x} - \frac{V^-}{Z_0} e^{\gamma x} \tag{2.29}$$

Equations (2.28) and (2.29) represent the generalized voltage and current on a transmission line which has propagation constant γ and characteristic impedance Z_0.

EXAMPLE 2.7 For the transmission line in Example 2.6, there are two waves travelling in opposite directions. At $x = 0$ and $t = 0$, the phase of the forward wave is zero and its amplitude is 2 V, whereas the phase of the backward wave is $\pi/3$ and its amplitude is 0.5 V. (i) What is the instantaneous voltage and current at $x = 50$ cm and $t = 1$ nsec (ii) What is the peak voltage and peak current at $x = 1$ m.

Solution:

Angular frequency $\quad \omega = 4\pi \times 10^9$ rad/sec

(i) The voltage of the forward travelling wave is

$$v_f(t) = |V^+| e^{-\alpha x} \cos(\phi^+ + \omega t - \beta x)$$

Given: $\phi^+ = 0$ and $v_f(t) = 2$ V at $x = 0, t = 0$. Therefore we get $|V^+| = 2$
Similarly for the backward travelling wave

$$v_b(t) = |V^-|e^{\alpha x}\cos(\phi^- + \omega t + \beta x)$$

Given: $\phi^- = \pi/3$, and $v_b(t) = 0.5$ V at $x = 0$ and $t = 0$. Therefore we have

$$0.5 = |V^-|\cos\pi/3$$

$$\Rightarrow \qquad |V^-| = 1.0 \text{ V}$$

Now, the propagation constant of the line is

$$\gamma = \sqrt{(0.1 + j\omega 0.01 \times 10^{-6})(0.01 + j\omega \times 10^{-10})}$$
$$= 0.055 + 12.566 \quad \text{per meter}$$
$$\Rightarrow \alpha = 0.055 \quad \text{nepers/m}$$
$$\beta = 12.566 \quad \text{rad/m}$$

Voltage on the line is superposition of the forward and backward wave voltages giving

$$v(t) = Re\left\{|V^+|e^{j\phi^+}e^{-\alpha x}e^{j(\omega t - \beta x)} + |V^-|e^{j\phi^-}e^{\alpha x}e^{j(\omega t + \beta x)}\right\}$$
$$= Re\left\{2e^{-0.055x}e^{j(\omega t - \beta x)} + 1e^{j\pi/3}e^{0.055x}e^{j(\omega t + \beta x)}\right\}$$

The current on the line is

$$i(t) = Re\left\{\frac{|V^+|e^{j\phi^+}}{Z_0}e^{-\alpha x}e^{j(\omega t - \beta x)} - \frac{|V^-|e^{j\phi^-}}{Z_0}e^{\alpha x}e^{j(\omega t + \beta x)}\right\}$$
$$= Re\left\{\frac{2}{Z_0}e^{-0.055x}e^{j(\omega t - \beta x)} - \frac{1e^{j\pi/3}}{Z_0}e^{0.055x}e^{j(\omega t + \beta x)}\right\}$$

Therefore, at $x = 50$ cm $= 0.5$ m, and $t = 10$ nsec $= 10^{-9}$ sec, we get

$$v(t) = 2.42 \quad \text{V}$$
$$i(t) = 0.148 \quad \text{A}$$

(ii) Finding the maximum of $|v(t)|$ the peak voltage at $x = 0.5$ m will be $V_p = 2.6$ V, and finding the maximum of the current $i(t)$ the peak current will be 0.168 Amp.
Note that the maximum voltage occurs when $\omega t \approx 2.8$ rad, whereas the maximum current occurs when $\omega t \approx 0.5$ rad.

2.4 STANDING WAVES AND IMPEDANCE TRANSFORMATION

We have seen earlier that on a transmission line the voltage and current are represented by superposition of two waves travelling in the opposite directions.

Since the two waves have same frequency and the propagation constant, the resultant is a stationary wave or a standing wave as shown in Fig. 2.7. Depending on the relative amplitudes of the two waves one can either get a full standing wave or a partial standing wave.

To obtain the voltage and current distribution on the line we have to supply the boundary conditions on the line. To evaluate both the arbitary constants V^+ and V^- we need two boundary conditions. However, if we want to analyse only the standing wave behavior on the line one boundary condition is adequate. This boundary condition is represented in terms of the impedance at any point on the transmission line. Generally this 'any point' is taken as the load end of transmission and all distances on the line are measured from this point towards the generator and are denoted by say 'l'. Then the positive direction of l is opposite to that of x. Replacing x by $-l$ in Eqns (2.28) and (2.29) we get voltage and current on the line as

$$V(l) = V^+e^{\gamma l} + V^-e^{-\gamma l} \tag{2.30}$$

$$I(l) = \frac{V^+}{Z_0}e^{\gamma l} - \frac{V^-}{Z_0}e^{-\gamma l} \tag{2.31}$$

From Fig. 2.7 it is clear that the energy source is connected to the line at AA′ and therefore a voltage or current wave going away from the source, i.e. in $+x$ direction is understandable. But what is the origin of the wave which is travelling in the $-x$ direction? There is no energy source at the other end of the line at BB′. One can then argue that if at all there is a wave travelling in the $-x$ direction , it must be because of the reflection of the forward wave from the load point. It is then important to investigate how the terminating load impedance Z_L (Fig. 2.7) affects the reflection of the wave. Before we do that, however, let us first define a parameter which gives the relative amplitudes of the two waves at any point on the line. This parameter is called the 'Reflection Coefficient' and is defined as

$$\Gamma(l) = \frac{V^-e^{-\gamma l}}{V^+e^{\gamma l}} = \frac{V^-}{V^+}e^{-2\gamma l} \tag{2.32}$$

Γ in general is a complex quantity. As the name suggests, $\Gamma(l)$ defines the complex relation between the reflected voltage wave and the forward or incident voltage wave at any location l on the line. Higher value of Γ indicates more reflection from the load end.

Substituting from Eqn (2.32) into Eqns (2.30) and (2.31) the voltage and current can be written as

$$V(l) = V^+e^{\gamma l}\,[1 + \Gamma(l)] \tag{2.33}$$

$$I(l) = \frac{V^+}{Z_0}e^{\gamma l}\,[1 - \Gamma(l)] \tag{2.34}$$

The impedance seen at any point on the line then is

$$Z(l) \equiv \frac{V(l)}{I(l)} = Z_0\frac{1 + \Gamma(l)}{1 - \Gamma(l)} \tag{2.35}$$

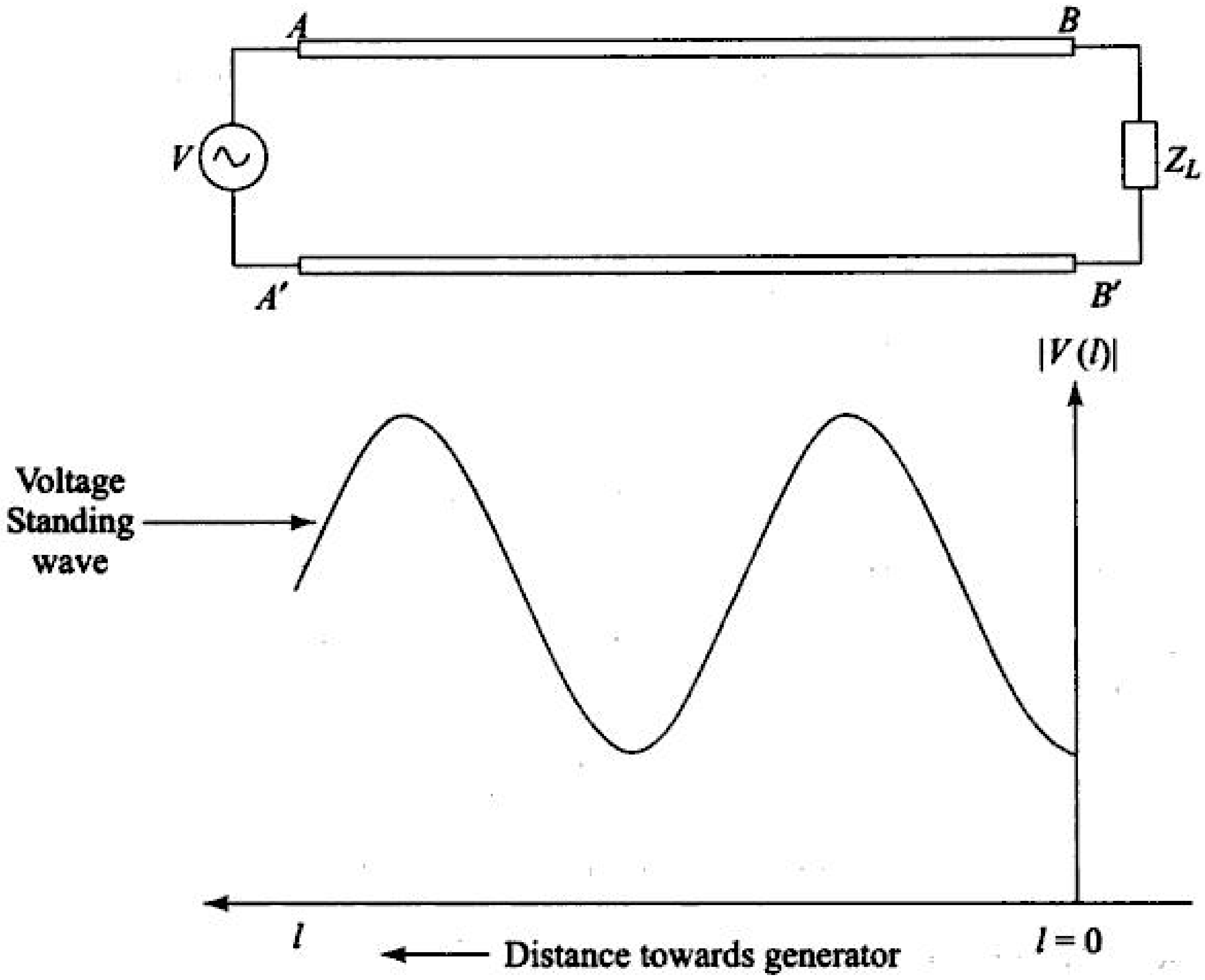


Fig. 2.7 *Standing wave on a transmission line.*

Inverting Eqn (2.35) we get

$$\Gamma(l) = \frac{Z(l) - Z_0}{Z(l) + Z_0} \tag{2.36}$$

Without losing generality let us now take the load end of the line as reference point and measure all distances from this point. Then at $l = 0$, the impedance $Z(l)$ is equal to the load impedance, Z_L.

Substituting $Z(l) = Z_L$ in Eqn (2.36) we get

$$\Gamma(0) \equiv \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{2.37}$$

Γ_L denotes the reflection coefficient at the load-end and from Eqn (2.32) we have

$$\Gamma_L = \frac{V^-}{V+} = \frac{\text{Reflected voltage at the load-end}}{\text{Incident voltage at the load-end}} \tag{2.38}$$

EXAMPLE 2.8 The transmission line in Example 2.6 is connected to a load impedance $10 + j20\ \Omega$ at 2 GHz. Find the reflection coefficient (i) at the load-end of the line (ii) at a distance of 20 cm from the load.

Solution:

(i) From Example 2.6 we have $Z_0 = 10 + j0.0358\ \Omega$.
The reflection coefficient at the load-end of the line is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(10 + j20) - (10 + j0.0358)}{(10 + j20) + (10 + j0.0358)}$$
$$= 0.499 + 0.498j$$

(ii) As solved in Example 2.7, $\gamma = 0.055 + j12.566$ per meter. Therefore

$$\Gamma(l = 20\text{cm}) = \Gamma_L e^{-2\gamma l}$$
$$= (0.499 + j0.498)e^{-2(0.055+j12.566)0.2}$$
$$= -0.3127 + 0.6149j$$

The voltage and current at any location on transmission line can then finally be written as

$$V(l) = V^+ e^{\gamma l}\{1 + \Gamma_L e^{-2\gamma l}\} \tag{2.39}$$

$$I(l) = \frac{V^+}{Z_0} e^{\gamma l}\{1 - \Gamma_L e^{-2\gamma l}\} \tag{2.40}$$

and the impedance at any point on the line is

$$Z(l) = Z_0 \left[\frac{1 + \Gamma_L e^{-2\gamma l}}{1 - \Gamma_L e^{-2\gamma l}}\right] \tag{2.41}$$

Substituting for Γ_L from Eqn (2.37) into Eqn (2.41) and taking $e^{-\gamma l}$ common from numerator and denominator, we get

$$Z(l) = Z_0 \left[\frac{e^{\gamma l} + \frac{Z_L - Z_0}{Z_L + Z_0} e^{-\gamma l}}{e^{\gamma l} - \frac{Z_L - Z_0}{Z_L + Z_0} e^{-\gamma l}}\right] \tag{2.42}$$

$$= Z_0 \left[\frac{(Z_L + Z_0)e^{\gamma l} + (Z_L - Z_0)e^{-\gamma l}}{(Z_L + Z_0)e^{\gamma l} - (Z_L - Z_0)e^{-\gamma l}}\right] \tag{2.43}$$

Rearranging terms of Z_L and Z_0, we get

$$Z(l) = Z_0 \left[\frac{Z_L(e^{\gamma l} + e^{-\gamma l}) + Z_0(e^{\gamma l} - e^{-\gamma l})}{Z_L(e^{\gamma l} - e^{-\gamma l}) + Z_0(e^{\gamma l} + e^{-\gamma l})}\right] \tag{2.44}$$

Since $(e^x + e^{-x})/2 = \cosh x$ and $(e^x - e^{-x})/2 = \sinh x$, the impedance at a distance l from the load can finally be written as

$$Z(l) = Z_0 \left[\frac{Z_L \cosh \gamma l + Z_0 \sinh \gamma l}{Z_L \sinh \gamma l + Z_0 \cosh \gamma l}\right] \tag{2.45}$$

Equation (2.45) is the impedance transformation equation. It shows that if a line is terminated in an impedance Z_L, the impedance seen at a distance l from it is not Z_L but is $Z(l)$. Or in other words, a length l of the transmission line, transforms the impedance from Z_L to $Z(l)$.

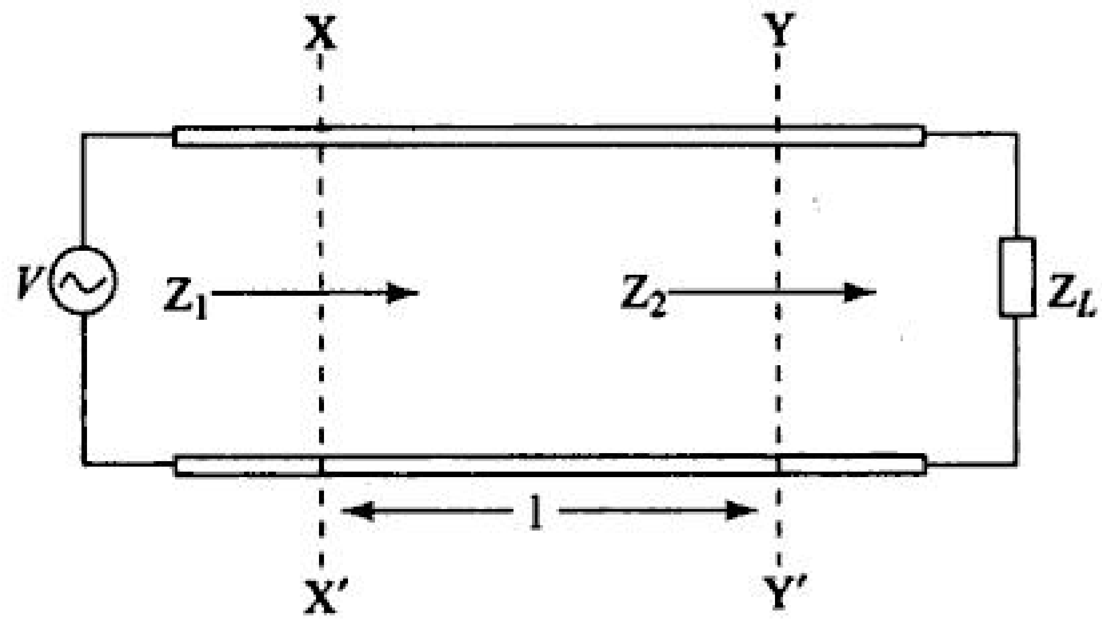


Fig. 2.8 *Impedance transformation on a transmission line.*

It should be noted here that although Eqn (2.45) gives the transformation of the load impedance, there is nothing special about the load impedance. We obtained transformation of load impedance because we defined $l = 0$ at the load end. If we define $l = 0$ at some other location on the line we get transformation of the impedance from that point. We can therefore generalise the impedance transformation equation for any two points on the transmission line.

Let the impedances measured at two locations on the line be Z_1 and Z_2 respectively as shown in Fig. 2.8.

Then from Eqn (2.45) we get

$$Z_1 = Z_0 \left[\frac{Z_2 \cosh \gamma l + Z_0 \sinh \gamma l}{Z_2 \sinh \gamma l + Z_0 \cosh \gamma l} \right] \tag{2.46}$$

If we invert Eqn (2.46) we get

$$Z_2 = Z_0 \left[\frac{Z_1 \cosh \gamma l - Z_0 \sinh \gamma l}{-Z_1 \sinh \gamma l + Z_0 \cosh \gamma l} \right] \tag{2.47}$$

Since, $\sinh(-\gamma l) = -\sinh \gamma l$ and $\cosh(-\gamma l) = \cosh \gamma l$, Eqn (2.47) can be re-written as

$$Z_2 = Z_0 \left[\frac{Z_1 \cosh(-\gamma l) + Z_0 \sinh(-\gamma l)}{Z_1 \sinh(-\gamma l) + Z_0 \cosh(-\gamma l)} \right] \tag{2.48}$$

From Eqns (2.46) and (2.48) it is evident that the two expressions are identical except the sign of l. In Eqn (2.46), l has positive sign whereas in Eqn (2.48) it has negative sign. Therefore it is clear that if the impedance at YY′ (Z_2) is known we can compute the impedance at XX′ by taking l positive in Eqn (2.46) and if the impedance at XX′ (Z_1) is known we can calculate impedance at YY' by again using Eqn (2.46) but with l negative. Equation (2.46) hence represents a generalized impedance transformation from any point on the line to any other point on the line with appropriate use of the sign for the length l. It should be remembered that all lengths measured towards the generator are positive and all lengths measured away from generator are negative. For analysis of transmission lines, therefore, the knowledge of location of the generator is crucial.

From Eqn (2.46) we can make one more important observation.

Let us re-arrange Eqn (2.46) to get

$$\frac{Z_1}{Z_0} = \frac{\frac{Z_2}{Z_0}\cosh \gamma l + \sinh \gamma l}{\frac{Z_1}{Z_0}\sinh \gamma l + \cosh \gamma l} \tag{2.49}$$

In Eqn (2.49) every impedance has been written as the ratio of the impedance and Z_0. It means that absolute values of impedances do not have any meaning in impedance transformation. The meaningful quantities are the normalized impedances with respect to the characteristic impedance of the line. Denoting a normalized impedance with a 'bar', Eqn (2.49) can be finally written as

$$\bar{Z}_1 = \frac{\bar{Z}_2 \cosh \gamma l + \sinh \gamma l}{\bar{Z}_1 \sinh \gamma l + \cosh \gamma l} \tag{2.50}$$

EXAMPLE 2.9 A transmission line has the propagation constant $\gamma = 0.1 + j10$ /m, and characteristic impedance $Z_0 = 50 + j5\ \Omega$. The line is terminated in an impedance $100 - j30\ \Omega$. Find the impedance at a distance of 1.5 m from the load.

Solution:

The impedance can be obtained using Eqn (2.45). For $l = 1.5$ m,

$$\cosh \gamma l = \cosh\{(0.1 + j10)1.5\} = -0.7683 + 0.0979j$$
$$\sinh \gamma l = \sinh\{(0.1 + j10)1.5\} = -0.1144 + 0.6576j$$

and the impedance is given by

$$Z(l) = Z_0 \left[\frac{Z_L \cosh \gamma l + Z_0 \sinh \gamma l}{Z_L \sinh \gamma l + Z_0 \cosh \gamma l}\right]$$
$$= 57.3175 + 38.6619j \quad \Omega$$

EXAMPLE 2.10 At some location on a long transmission line, the impedance is $100 + j50\ \Omega$. Find the impedance at 50 cm on either side of this location. The transmission line has $Z_0 = 50\ \Omega$ (almost real), and $\gamma = 0.1 + j20$ /m.

Solution:

Assume that the generator is on the left side of the line. Let the location at which the impedance is given, is denoted by O, the location 50 cm towards the generator be denoted by A and the location away from the generator be denoted by B. Then by sign convention for the distance $l = +50$ cm for A and $l = -50$cm for B. The impedance at A and B can be obtained using Eqn (2.45) as

$$\gamma l = 0.05 + 10j$$

$$Z(l) = Z_0 \left[\frac{Z_L \cosh \gamma l + Z_0 \sinh \gamma l}{Z_L \sinh \gamma l + Z_0 \cosh \gamma l} \right]$$

$$\Rightarrow \qquad Z_A = Z(l = +50\text{ cm}) = 77.73 - 47.70j\ \Omega$$

$$\Rightarrow \qquad Z_B = Z(l = -50\text{ cm}) = 29.36 + 38.37j\ \Omega$$

2.5 LOSS-LESS AND LOW-LOSS TRANSMISSION LINES

In the previous sections we developed basic voltage, current and impedance relations for a general tranmission line. However, in practice since the primary use of a tranmission line is to send signals efficiently from one point to another, every effort is made to minimize power loss on the line. Since the power is lost in the resistance of the two conductors and the conductance of the dielectric separating the two conductors, a loss-less tranmission line implies $R = 0$ and $G = 0$. Therefore, for a loss-less transmission line we get (from Eqns (2.11) and (2.27)).

$$\gamma = \sqrt{j\omega L\, j\omega C} = j\omega\sqrt{LC} \tag{2.51}$$

$$Z_0 = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}} \tag{2.52}$$

It should be noted that for a loss-less line, γ is purely imaginary and the characteristic impedance is purely real. Since $\gamma = \alpha + j\beta$, pure imaginary γ means $\alpha = 0$ which by definition is the condition for a loss-less line. The phase constant for a loss-less line is $\beta = \omega\sqrt{LC}$.

In practice we do not find a loss-less transmission line. What we however get is a low-loss transmission line. A line is called low-loss provided it has $R << \omega L$ and $G << \omega C$ at the frequency of operation. The line parameter γ for a low-loss line can be written as

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{2.53}$$

$$= \sqrt{j\omega L \left\{1 - j\frac{R}{\omega L}\right\} j\omega C \left\{1 - j\frac{G}{\omega C}\right\}} \tag{2.54}$$

$$= j\omega\sqrt{LC} \left\{1 - j\frac{R}{\omega L}\right\}^{1/2} \left\{1 - j\frac{G}{\omega C}\right\}^{1/2} \tag{2.55}$$

Since $R/\omega L$ and $G/\omega C$ are $<< 1$, expanding the brackets binomially and retaining only first order terms, we get,

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC} \left\{1 - j\frac{R}{2\omega L}\right\} \left\{1 - j\frac{G}{2\omega C}\right\} \tag{2.56}$$

Again the product $(R/2\omega L)(G/2\omega C)$ is a second order term and therefore can be neglected, giving

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}\left\{1 - j\frac{R}{2\omega L} - j\frac{G}{2\omega C}\right\} \tag{2.57}$$

Separating real and imaginary parts we get

$$\text{Attenuation constant: } \alpha = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}} \tag{2.58}$$

$$\text{Phase constant: } \beta = \omega\sqrt{LC} \tag{2.59}$$

From Eqns (2.51) and (2.59) it can be observed that the phase constant of a low-loss line is same as that of the loss-less line for given L and C. A low-loss transmission line therefore can be analyzed like a loss-less transmission line. The use of α is made only in those cases where specifically power loss calculations are to be carried out. In practice if the condition $\alpha << \beta$ is satisfied, we treat the line as a loss-less line. The condition $\alpha << \beta$ implies negligible reduction in the wave amplitude over one wavelength distance on the tranmission line. Until and unless it is specified one can take the liberty of choosing $\alpha = 0$ in the analysis of low-loss transmission lines. In the following examples we investigate the characteristics of a loss-less tranmission line.

EXAMPLE 2.11 A transmission line has $L = 0.25\mu$H/m, $C = 100$pF/m and $G = 0$. What should be the value of R for the line so that the line can be treated as low-loss line? The frequency of operation is 100 MHz.

Solution:

The phase constant of the low-loss line is

$$\beta \approx \omega\sqrt{LC} = 2\pi\sqrt{0.25 \times 10^{-6} \times 100 \times 10^{-12}} \times 10^{8}$$
$$= \pi$$

The attenuation constant from Eqn (2.58) is

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}} \qquad (\text{ since } G = 0\ )$$
$$= \frac{R}{2}\sqrt{\frac{10^{-10}}{0.25 \times 10^{-6}}} = 0.01R \text{ nepers/m}$$

For low-loss line we should have $\beta >> \alpha$. Taking $\alpha < 1\%$ of β, we get

$$0.01R < \frac{\beta}{100}$$
$$\Rightarrow \qquad R < \pi = 3.14\ \Omega/\text{m}.$$

Substituting $\gamma = j\beta$ in Eqns (2.30) and (2.31) we obtain the voltage and current on a loss-less transmission line as

$$V(l) = V^+e^{j\beta l} + V^-e^{-j\beta l} \tag{2.60}$$

$$I(l) = V^+e^{j\beta l} - V^-e^{-j\beta l} \tag{2.61}$$

The reflection coefficient from Eqn (2.32) will be

$$\Gamma(l) = \frac{V^-}{V^+}e^{-j2\beta l} \tag{2.62}$$

From Eqn (2.38) $V^-/V^+ = \Gamma_L$ and we get,

$$\Gamma(l) = \Gamma_L e^{-j2\beta l} \tag{2.63}$$

From Eqn (2.63) we can note that $|\Gamma(l)| = |\Gamma_L|$. It means that the magnitude of the reflection coefficient remains the same at every point on the line and only its phase changes as one traverses the transmission line.
Now let us write the voltage and current on the line in terms of the reflection coefficient as has been done in Eqns (2.39) and (2.40) as

$$V(l) = V^+e^{j\beta l}\left\{1 + \Gamma_L e^{-j2\beta l}\right\} \tag{2.64}$$

$$I(l) = \frac{V^+}{Z_0}e^{j\beta l}\left\{1 - \Gamma_L e^{-j2\beta l}\right\} \tag{2.65}$$

Since Γ_L in general is complex (refer Eqn (2.32)), let us assume that the magnitude of Γ_L is denoted by $|\Gamma_L|$ and its phase is denoted by ϕ. Then by definition

$$\Gamma_L = |\Gamma_L|e^{j\phi} \tag{2.66}$$

Substituting for Γ_L in Eqns (2.64) and (2.65) we get

$$V(l) = V^+e^{j\beta l}\left\{1 + |\Gamma_L|e^{j(\phi-2\beta l)}\right\} \tag{2.67}$$

$$I(l) = \frac{V^+}{Z_0}e^{j\beta l}\left\{1 - |\Gamma_L|e^{j(\phi-2\beta l)}\right\} \tag{2.68}$$

As one moves along the transmission line, the length l and therefore the phase $(\phi - 2\beta l)$ changes. Since the second term inside the bracket represents reflected wave, the phase of the reflected wave changes relative to the incident wave as a function of l. At locations where the phase $(\phi - 2\beta l)$ equals multiples of π the exponential term becomes ± 1. At this location the two terms in the bracket are real and they add or subtract directly. When $\exp(j(\phi - 2\beta l)) = +1$ the terms in Eqn (2.67) add whereas they subtract in Eqn (2.68) giving voltage maximum but current minimum. Similarly, when $\exp(j(\phi - 2\beta l)) = -1$ the terms in voltage expression subtract and they add in the current expression giving current maximum but voltage minimum. So we find that on a transmission line when the voltage is maximum the current is minimum and vice versa as shown in Fig. 2.9.

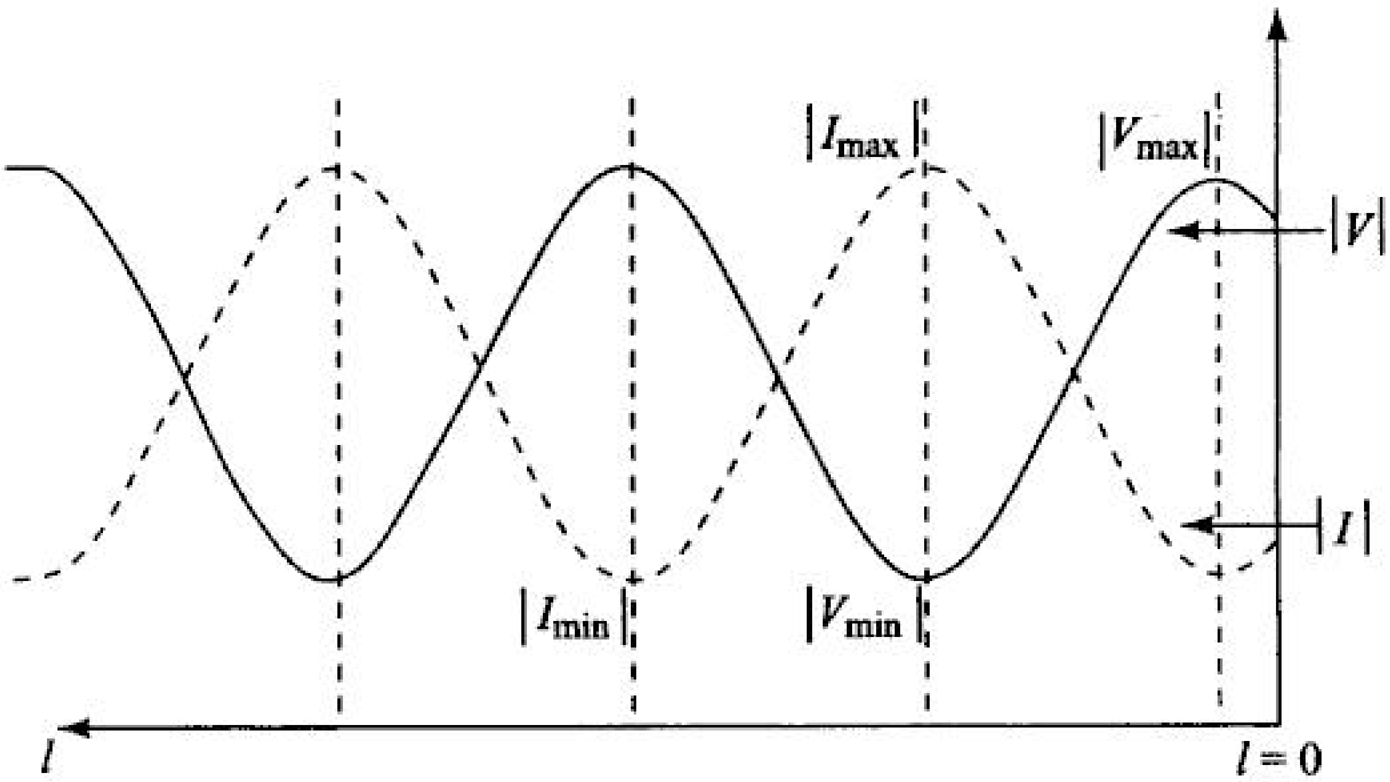


Fig. 2.9 *Voltage and current standing waves.*

The maximum and minimum magnitudes of the voltage are

$$|V|_{max} = |V^+|(1 + |\Gamma_L|) \tag{2.69}$$

$$|V|_{min} = |V^+|(1 - |\Gamma_L|) \tag{2.70}$$

Similarly, the maximum and minimum magnitudes of the current are

$$|I|_{max} = |\frac{V^+}{Z_0}|(1 + |\Gamma_L|) = \frac{|V|_{max}}{Z_0} \tag{2.71}$$

$$|I|_{min} = |\frac{V^+}{Z_0}|(1 - |\Gamma_L|) = \frac{|V|_{min}}{Z_0} \tag{2.72}$$

Remember here that for a loss-less line the characteristic impedance Z_0 is real and hence, $|Z_0| = Z_0$.

At this stage, we can make one more important observation from Eqns (2.71) and (2.72) and that is, at points where voltage or current is maximum or minimum, since the quantity inside the bracket is real, the phase difference between $V(l)$ and $I(l)$ is zero irrespective of the phase of V^+. It should be noted that this phase difference is temporal at a specific location in space. Therefore, the ratio $V(l)/I(l)$ which is the impedance at location 'l', is real. In other words, we can say that at locations where voltage or current is maximum or minimum, the impedance measured on the line is purely resistive (real impedance).

The plot of $|V|$ or $|I|$ as a function of distance l is called the voltage or current standing wave pattern. A characteristic parameter of the standing wave pattern is called the *voltage standing wave ratio* (written in short as VSWR) and is denoted by ρ. The VSWR is defined as

$$\rho = \frac{|V|_{max}}{|V|_{min}} \tag{2.73}$$

VSWR is an easily measurable parameter as it does not require any measurement of phase. It is worthwhile to mention here that the measurement of phase at high frequency is rather a tedious task and at times becomes unreliable.

Substituting from Eqns (2.69) and (2.70) into Eqn (2.73) we have

$$\rho = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \tag{2.74}$$

$$\text{or } |\Gamma_L| = \frac{\rho - 1}{\rho + 1} \tag{2.75}$$

The VSWR is an accurate indicator of the reflections on the transmission line. By definition, since $|V|_{\max}$ is greater than $|V|_{\min}$, ρ is always greater than 1. It could be as high as ∞ when $|V|_{\min}$ goes to zero. Since from Eqn (2.75) $\rho = 1$ corresponds to $|\Gamma_L| = 0$, it represents 'no-reflection' condition. Similarly $\rho = \infty$ corresponds to $|\Gamma_L| = 1$ meaning amplitude of reflected wave is equal to that of the incident wave, i.e. full reflection. Since reflected wave carries some power backward, full incident power does not get delivered to the load in the presence of reflection. For efficient power delivery to the load therefore $|\Gamma_L|$ and hence ρ should be as small as possible. VSWR of 1 corresponds to the maximum power transfer efficiency whereas, VSWR of ∞ represents no power delivery to the load.

EXAMPLE 2.12 A loss-less transmission line has 75 Ω characteristic impedance. The line is terminated in a load impedance of $50 - j100$ Ω. The maximum voltage measured on the line is 100 V. Find the maximum and minimum current, and the minimum voltage on the line. At what distance from the load the voltage and current are maximum?

Solution:

The reflection coefficient at the load is

$$\Gamma_L = \frac{50 - j100 - 75}{50 - j100 + 75}$$

$$= 0.2683 - 0.5854j$$

$$\Rightarrow \qquad |\Gamma_L| = 0.644$$

The maximum voltage on the line $|V_{\max}| = 100V$ (given). Therefore,

$$100 = |V^+|(1 + |\Gamma_L|)$$

$$\Rightarrow \qquad |V^+| = \frac{100}{1 + |\Gamma_L|} = \frac{100}{1 + 0.644} = 60.83$$

Maximum current $|I_{\max}| = |V_{\max}|/Z_0 = 100/50 = 2$ A.

Minimum current $|I_{\min}| = \frac{|V^+|}{Z_0}(1 - |\Gamma_L|) = 0.433$ A.

Minimum voltage $|V_{\min}| = |I_{\min}|Z_0 = 21.66$ V.

Maximum voltage occurs when

$$\phi - 2\beta l = \pm 2m\pi \qquad m = 0, 1, 2, 3...$$

We choose appropriate sign so that l is positive. Substituting $\beta = 2\pi/\lambda$ we get

$$2\beta l = 2m\pi + \phi$$

$$\text{i.e.} \qquad l = \frac{(2m\pi + \phi)\lambda}{4\pi} \qquad m = 0, 1, 2, 3...$$

Therefore the voltage maxima occur at

$$l = 0.4\lambda, 0.9\lambda, 1.4\lambda \, ...$$

Now, the voltage maxima and the current minima occur at the same locations. Therefore, the current minima also occur at $l = 0.4\lambda, 0.9\lambda, 1.4\lambda$ ...
The voltage minima (also current maxima) are staggered by $\lambda/4$ with respect to the voltage maxima (also current minima). Hence, the voltage minima (current maxima) occur at

$$l + \frac{\lambda}{4} = 0.16\lambda, 0.66\lambda, 1.16\lambda...$$

2.5.1 Impedance Variation on Loss-less Transmission Line

The impedance at any point on the transmission line is

$$Z(l) = \frac{V(l)}{I(l)} = Z_0 \left[\frac{1 + \Gamma_L e^{-j2\beta l}}{1 - \Gamma_L e^{-j2\beta l}}\right] \tag{2.76}$$

Substituting for Γ_L from Eqn (2.37) we can write

$$Z(l) = \frac{V(l)}{I(l)} = Z_0 \left[\frac{1 + \frac{Z_L - Z_0}{Z_L + Z_0} e^{-j2\beta l}}{1 - \frac{Z_L - Z_0}{Z_L + Z_0} e^{-j2\beta l}}\right] \tag{2.77}$$

Rearranging terms of Z_L and Z_0 and noting that $e^{j\beta l} + e^{-j\beta l} = 2\cos\beta l$ and $e^{j\beta l} - e^{-j\beta l} = 2j\sin\beta l$ we get

$$Z(l) = Z_0 \left[\frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l}\right] \tag{2.78}$$

Or in terms of normalized impedances,

$$\overline{Z}(l) = \left[\frac{\overline{Z}_L \cos\beta l + j\sin\beta l}{\cos\beta l + j\overline{Z}_L \sin\beta l}\right] \tag{2.79}$$

Where, $\overline{Z}(l) = Z(l)/Z_0$, and $\overline{Z}_L = Z_L/Z_0$.

The maximum impedance occurs where the voltage is maximum and current is minimum and its value is

$$[Z(l)]_{\max} = \frac{V_{\max}}{I_{\min}} = Z_0 \left[\frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}\right] = R_{\max}(\text{say}) \tag{2.80}$$

Noting that the quantity inside the square bracket is the VSWR, we get

$$R_{\max} = Z_0 \rho \tag{2.81}$$

Similarly, the minimum impedance occurs at a location where the voltage is minimum and the current is maximum, and its value is

$$[Z(l)]_{\min} = \frac{V_{\min}}{I_{\max}} = Z_0 \left[\frac{1 - |\Gamma_L|}{1 + |\Gamma_L|}\right] = R_{\min}(\text{say}) \tag{2.82}$$

$$\Rightarrow R_{\min} = Z_0/\rho \tag{2.83}$$

The impedance on a line therefore varies between Z_0/ρ and $Z_0\rho$. This means the impedance value at some point on the line is greater than Z_0 and at other point it is less than Z_0.

EXAMPLE 2.13 A 50 Ω transmission line is connected to a parallel combination of a 100 Ω resistance and a 1 nF capacitance. Find the VSWR on the line at a frequency of 2 MHz. Also find the maximum and minimum resistance seen on the line.

Solution:

The load impedance Z_L is a parralel combination of $R = 100\Omega$ and $C = 1$ nF. Therefore,

$$Z_L = \frac{R(1/j\omega C)}{R + (1/j\omega C)} = \frac{R}{1 + j\omega RC}$$

$$= \frac{100}{1 + j2\pi \times 2 \times 10^6 \times 100 \times 10^{-9}}$$

$$= \frac{100}{1 + 0.4\pi j} = 38.77 - j48.72$$

The reflection coefficient at the load-end of the line is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = 0.134 - j0.475$$

$$|\Gamma_L| = 0.494$$

The VSWR $\rho = \dfrac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = 2.95$

The maximim resistance on the line is $R_{\max} = \rho Z_0 = 147.53$ kΩ

The minimum resistance on the line is $R_{\min} = Z_0/\rho = 16.94$ Ω

2.5.2 Important Characteristics of a Loss-less Line

A loss-less transmission line exhibits following important characteristics.

1. Line characteristics repeat every λ/2 As we have seen earlier, the propagation constant $\beta = 2\pi/\lambda$. Let the impedance at some point l on the line be $Z(l)$. The impedance at a distance of $\lambda/2$ from this location from Eqn (2.78) will be

$$Z(l+\lambda/2) = Z_0\left[\frac{Z_L\cos\frac{2\pi}{\lambda}(l+\lambda/2)+jZ_0\sin\frac{2\pi}{\lambda}(l+\lambda/2)}{Z_0\cos\frac{2\pi}{\lambda}(l+\lambda/2)+jZ_L\sin\frac{2\pi}{\lambda}(l+\lambda/2)}\right] \quad (2.84)$$

$$= Z_0\left[\frac{-Z_L\cos\beta l - jZ_0\sin\beta l}{-Z_0\cos\beta l - jZ_L\sin\beta l}\right] \quad (2.85)$$

$$\Rightarrow \quad Z(l+\lambda/2) = Z(l) \quad (2.86)$$

Equation (2.86) tells us that the impedance on a line repeats over every $\lambda/2$ distance. In other words, a study of a $\lambda/2$ section of a line is sufficient to understand the behavior of the whole line.

2. Normalized impedance inverts every $\lambda/4$ Suppose the normalized impedance at a location l is $\overline{Z}(l)$. Then the normalized impedance at a distance of $\lambda/4$ from it from Eqn (2.79) will be

$$\overline{Z}(l+\lambda/4) = \frac{\overline{Z}_L\cos\frac{2\pi}{\lambda}(l+\lambda/4)+j\sin\frac{2\pi}{\lambda}(l+\lambda/4)}{\cos\frac{2\pi}{\lambda}(l+\lambda/4)+j\overline{Z}_L\sin\frac{2\pi}{\lambda}(l+\lambda/4)} \quad (2.87)$$

$$= \frac{-\overline{Z}_L\sin\beta l + j\cos\beta l}{-\sin\beta l + j\overline{Z}_L\cos\beta l} \quad (2.88)$$

$$\Rightarrow \quad \overline{Z}(l+\lambda/4) = \frac{1}{\overline{Z}(l)} \quad (2.89)$$

The normalized impedance on the line therefore inverts every $\lambda/4$ distance. Note that it is the normalized impedance and not the actual impedance. The actual impedance at location l will be $Z_0\overline{Z}(l)$ and at $(l+\lambda/4)$ it will be $Z_0/\overline{Z}(l)$.

3. For load impedance $Z_L = Z_0$, the impedance at any point on the line is Z_0 Suppose the line is terminated in an impedance equal to the characteristic impedance, i.e. $Z_L = Z_0$. Then the impedance at a distance l from the load is

$$Z(l) = Z_0\left[\frac{Z_0\cos\beta l + jZ_0\sin\beta l}{Z_0\cos\beta l + jZ_0\sin\beta l}\right] \quad (2.90)$$

$$= Z_0 \quad (2.91)$$

This indicates that if a line is terminated in the characteristic impedance Z_0, the impedance at every point on the line is also Z_0. This condition is called the 'matched load' condition. If the load impedance is matched to the characteristic impedance i.e. $Z_L = Z_0$, the reflection coefficient Γ is zero everywhere on the line (see Eqn (2.37)). In other words, for the matched load condition there is only forward wave (and no reflected wave) on the line. As discussed earlier, the forward wave always sees the characteristic impedance, and consequently the impedance at every location on the line is Z_0. We can now use this property to give a physical meaning to the characteristic impedance.

Definition of the Characteristic Impedance: *The characteristic impedance of a line is that impedance with which when the line is terminated, the impedance measured on any point on the line is same as the terminating impedance.*

2.6 POWER TRANSFER ON A TRANSMISSION LINE

Consider a loss-less transmission line with characteristic impedance Z_0. Let the line be terminated in a complex load impedance $Z = R + jX \neq Z_0$. Since the load impedance is not equal to the characteristic impedance, there is reflection on the line, and the voltage and the current on the line can be given as

$$V(l) = V^+ e^{j\beta l}\left\{1 + \Gamma_L e^{-j2\beta l}\right\} \tag{2.92}$$

$$I(l) = \frac{V^+}{Z_0} e^{j\beta l}\left\{1 - \Gamma_L e^{-j2\beta l}\right\} \tag{2.93}$$

Since, the reference point $l = 0$ is at the load end, the power delivered to the load is

$$P_L = \frac{1}{2}\,\mathrm{Re}(VI^*)\ \text{at}\ l = 0 \tag{2.94}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{\left[V^+(1+\Gamma_L)\right]\left[\frac{V^+}{Z_0}(1-\Gamma_L)\right]^*\right\} \tag{2.95}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{\frac{|V^+|^2}{Z_0}\left[1 - |\Gamma_L|^2 + (\Gamma_L - \Gamma_L^*)\right]\right\} \tag{2.96}$$

Since, the difference of any complex number and its conjugate is purely imaginary, $(\Gamma_L - \Gamma_L^*)$ is a purely imaginary quantity. Therefore, the power delivered to the load is

$$P_L = \frac{|V^+|^2}{2Z_0}\left\{1 - |\Gamma_L|^2\right\} \tag{2.97}$$

One could have arrived at the same expression using a different argument. The wave which travels towards the load has an amplitude V^+. As discussed earlier, a travelling wave always sees the characteristic impedance irrespective of the terminating load. The power carried by the wave travelling towards the load (also called the incident power) is

$$P_{inc} = \frac{1}{2} Re\{V^+(I^+)^*\} = \frac{1}{2} Re\{V^+(\frac{V^+}{Z_0})^*\} \tag{2.98}$$

$$= \frac{|V^+|^2}{2Z_0} \qquad \text{Note: } Z_0 \text{ is real for a loss-less line} \tag{2.99}$$

Now there is a reflected wave on the line, and its amplitude is $V^- = \Gamma_L V^+$. The reflected wave also sees characteristic impedance and therefore the power taken back by the reflected wave is

$$P_{ref} = \frac{|\Gamma_L V^+|^2}{2Z_0} \tag{2.100}$$

$$= \frac{|V^+|^2}{2Z_0}|\Gamma_L|^2 \tag{2.101}$$

Since P_{inc} is the power travelling towards the load, and P_{ref} is the power travelling back from the load, the difference of the two powers is the power delivered to the load. We therefore, have

$$P_L = P_{inc} - P_{ref} \tag{2.102}$$

$$P_L = \frac{|V^+|^2}{2Z_0}\{1 - |\Gamma_L|^2\} \tag{2.103}$$

Equation (2.103) is same as Eqn (2.97). We hence see that one can either use circuit concept or wave concept to calculate the power delivered to the load. Further, it is interesting to note, that since the line is loss-less, the resistive power calculated at any point on the line must be same as that delivered to the load. Let us verify this.

The complex power at any point on the line is

$$P(l) = \frac{1}{2}\{V(l)[I(l)]^*\} \tag{2.104}$$

Substituting for $V(l)$ and $I(l)$ from Eqns (2.92) and (2.93) we get

$$P(l) = \frac{1}{2}\left\{ V^+ e^{j\beta l}(1 + \Gamma_L e^{-j2\beta l})\right\}\left\{\frac{V^+}{Z_0} e^{j\beta l}(1 - \Gamma_L e^{-j2\beta l})\right\}^* \tag{2.105}$$

$$= \frac{|V^+|^2}{2Z_0}\left\{1 - |\Gamma_L|^2 + \text{Im}(\Gamma_L e^{-j2\beta l})\right\} \tag{2.106}$$

Separating real and imaginary parts, we get

$$\text{Re}\{P(l)\} = \frac{|V^+|^2}{2Z_0}\left\{1 - |\Gamma_L|^2\right\} = P_L \tag{2.107}$$

$$\text{Im}\{P(l)\} = \frac{|V^+|^2}{2Z_0} Im(\Gamma_L e^{-j2\beta l}) \tag{2.108}$$

It should be noted that the real power is independent of l and is equal to the power delivered to the load, whereas, the imaginary (reactive) power is a function of l and therefore varies along the transmission line. This variation is due to the variation of voltage and current along the line which consequently varies the reactive fields and hence the energy stored along the line. It is important that we clearly differentiate between the energy flow and the energy storage at different locations on the line. The energy flow is same everywhere on the line whereas the energy stored is not.

EXAMPLE 2.14 A 50Ω loss-less transmission line is connected to a load of $50 + j50\ \Omega$. The maximum voltage measured on the line is 50 V. Find the power delivered to the load and the peak voltage at the load-end of the line.

Solution:

The magnitude of the reflection coefficient

$$|\Gamma_L| = \left|\frac{Z_L - Z_0}{Z_L + Z_0}\right| = \left|\frac{50 + j50 - 50}{50 + j50 + 50}\right|$$
$$= |0.2 + 0.4j| = 0.4472$$

The VSWR $\rho = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = 2.618$

Since the line is loss-less, the power loss at any point on the line is same as the power delivered to the load. We can, therefore, conviniently choose a point on the line where the voltage is maximum (so the current is minimum). The impedance at this location is real and its value is $\rho Z_0 = 130.9$. The power loss at that point (which is same as power delivered to the load) is

$$P_L = \frac{1}{2}\frac{|V_{\max}|^2}{\rho Z_0} = 9.55 \quad \text{W}$$

Now, the power loss in the load can also be written in terms of the voltage across the load as

$$P_L = \frac{1}{2}\frac{|V_L|^2}{R_L}$$

$\Rightarrow$ Peak voltage at the load$|V_L| = 30.9 \quad \text{V}$

2.6.1 Evaluation of V^+

In the earlier section, we have derived expressions for voltage, current and power in terms of the parameter V^+, the amplitude of the incident voltage wave. In other words, we assumed that the amplitude of the incident voltage wave is known a priori. The question is—how to find the incident wave amplitude. In practice, we get a line which is connected to a voltage or a current generator at one end, and to a load at the other, and we will have to find out the complex V^+ in terms of the generator voltage/current, the line parameters and the load impedance. Let us consider a transmission line of character impedance Z_0 and length l. Let the line be connected to a voltage source, having peak voltage V_s and internal impedance Z_s, as shown in Fig. 2.10(a). Let the line be connected to a load impedance Z_L at other end.

It is clear by now, that the voltage source does not see Z_L connected across it but sees the transformed impedance between terminals AA′. If the transformed

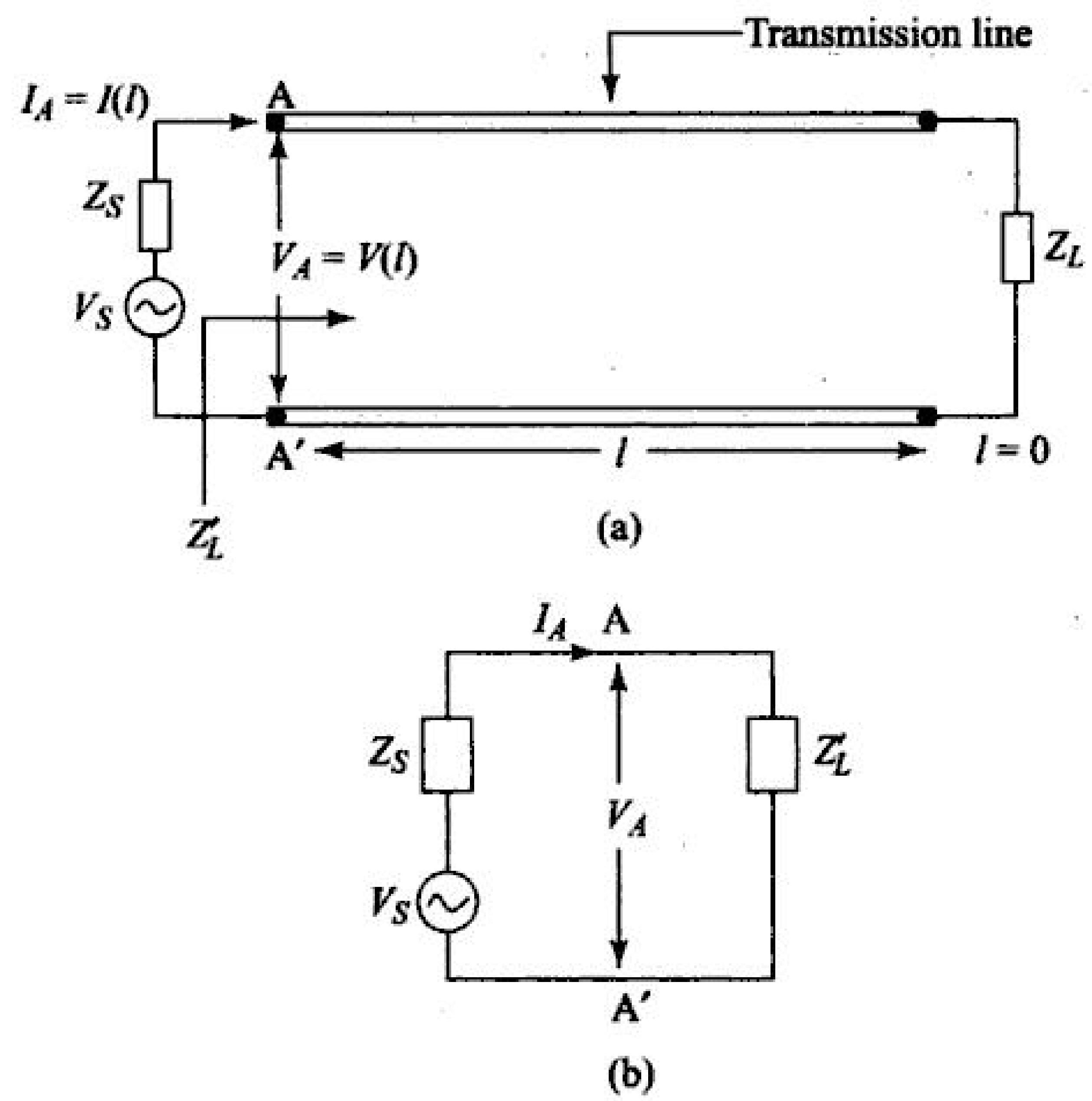


Fig. 2.10 *(a) Transmission line connected to a generator and a load (b) Lumped circuit at the generator-end of the line.*

impedance is denoted by Z_L', we have

$$Z_L' = Z_0 \left[\frac{Z_L \cos \beta l + jZ_0 \sin \beta l}{Z_0 \cos \beta l + jZ_L \sin \beta l} \right] \tag{2.109}$$

The lumped circuit at AA′ will be as shown in Fig. 2.10(b), and the current and voltage at terminals AA′ can be obtained as

$$I_A = \frac{V_s}{Z_S + Z_L'} \tag{2.110}$$

$$V_A = Z_s' I_s = \frac{Z_L' V_s}{Z_S + Z_L'} \tag{2.111}$$

Let us now write the expressions for voltage and current at AA′ in terms of V^+ using transmission line equations, as done in the previous sections.

$$V(l) = V^+ e^{j\beta l}\{1 + \Gamma_L e^{-j2\beta l}\} \tag{2.112}$$

$$I(l) = \frac{V^+}{Z_0} e^{j\beta l}\{1 - \Gamma_L e^{-j2\beta l}\} \tag{2.113}$$

where, $\Gamma_L = (Z_L - Z_0)/(Z_L + Z_0)$.

From Fig. 2.10 (a), it is clear that V_A and $V(l)$ represent the same voltage. Similarly, I_A and $I(l)$ represent the same current. Equating (2.111) and (2.112) we get

$$V^+e^{j\beta l}\{1+\Gamma_L e^{-j2\beta l}\} = \frac{Z'_L V_s}{Z_S + Z'_L} \tag{2.114}$$

Solving for V^+ we can obtain the amplitude of the incident voltage wave as

$$V^+ = \frac{Z'_L V_s e^{-j\beta l}}{(Z_s + Z'_L)(1+\Gamma_L e^{-j2\beta l})} \tag{2.115}$$

One can verify that the Eqn (2.115) could have been obtained by equating (2.110) and (2.113) as well.

Since the line is loss-less, the power delivered to the load Z_L is same as the power delivered to the transformed load $Z'_L \equiv R' + jX'$, and is given by

$$P = \frac{1}{2}\mathrm{Re}(V_A I_A^*) \tag{2.116}$$

$$= \frac{1}{2}R'|\frac{V_s}{Z'_L + Z_s}|^2 \tag{2.117}$$

For maximum power transfer to take place from the source to the load, obviously, we should have $Z_s = Z'^*_L$. Since Z'_L is a function of l and the frequency f, for a given load and source impedance, the maximum power transfer takes place only for a specific frequency and length of the transmission line. The only exception is when both source and load impedances are equal to the characteristic impedance Z_0, in which case there is maximum power transfer irrespective of the length of the line. Since, neither the load nor the source impedance is in our control, usually we use the so called 'matching networks' in front of source or load impedances. The matching networks transform the source and load impedances to the characteristic impedance and the the power transfer becomes independent of the length of the transmission line and the frequency.

It is clear from the above discussion that in high frequency circuits, the characteristic impedance plays a very important role. It is, therefore, desirable to have some standardization of the characteristic impedance to make different circuits compatible with each other. For a co-axial transmission line the standard characteristic impedances are 50 Ω and 75 Ω, and for parallel wire lines (flat ribbon cables) the standard characteristic impedances are 300 Ω and 600 Ω.

EXAMPLE 2.15 A 2.5 m long 75Ω co-axial cable is connected to a generator at one end, and to a load of $75 + j25$ Ω at the other end. The generator has an open-circuit rms voltage of 10 V, and an internal resistance of 50 Ω. Find the power delivered to the load. The frequency of operation is 150 MHz and the velocity of the wave on the cable is 2×10^8 m/sec.

Solution:

Here we have $V_s = 10$ V, and $Z_s = 50$ Ω.
The wavelength on the cable is

$$\lambda = \frac{v}{f} = \frac{2 \times 10^8}{150 \times 10^6} = 1.333 \text{ m}$$

$$\Rightarrow \qquad \beta = \frac{2\pi}{\lambda} = 1.5\pi \text{ rad/m}$$

$$\Rightarrow \qquad \beta l = 1.5\pi \times 2.5 = 11.781 \text{ rad}$$

The transformed impedance at the generator-end is

$$Z'_L = Zo\left(\frac{Z_L \cos\beta l + jZ_o \sin\beta l}{Z_o \cos\beta l + jZ_L \sin\beta l}\right)$$
$$= 54 + 2.96j$$

Now, from the equivalent lumped circuit at the generator-end we can calculate the power supplied to the input of the line. However, since the line is loss-less, power supplied by the generator to the line, i.e. the power delivered to the impedance Z'_L, is same as the power delivered to the load. Using Eqn 2.117 we get

$$P_L = Re\{V_A I_A^*\} = 0.4988 \text{ W}$$

(Note that there is no factor $1/2$, since V_A and I_A are rms values.)

2.7 ANALYSIS OF TRANSMISSION LINE IN TERMS OF ADMITTANCES

Many times, especially while analyzing transmission lines connected in parralel, analysis in terms of admittance becomes simpler compared to the analysis in terms of impedance. Therefore we here analyze transmission line characteristics in terms of admittances.

To start with, we define the characteristic admittance Y_0 which is the reciprocal of the characteristic impedance Z_0, i.e.

$$Y_0 = \frac{1}{Z_0} \tag{2.118}$$

The characteristic admittance, therefore, is a ratio of I^+ and V^+ by definition. For a loss-less line since the characteristic impedance Z_0 is real, so is the characteristic admittance Y_0.

Now converting every impedance to the corresponding admittance i.e. $Y_L = 1/Z_L$, $Y(l) = 1/Z(l)$, etc. we can re-write Eqn (2.36) to get reflection coefficient on any location on the line as

$$\Gamma(l) = \frac{1/Y(l) - 1/Y_0}{1/Y(l) + 1/Y_0} = \frac{Y_0 - Y(l)}{Y_0 + Y(l)} \tag{2.119}$$

Similarly, we can write the admittance at any point on the line (using Eqn (2.78)) as

$$Y(l) = Y_0 \left[\frac{Y_L \cos\beta l + jY_0 \sin\beta l}{Y_0 \cos\beta l + jY_L \sin\beta l} \right] \tag{2.120}$$

Note that Eqn (2.120) is identical to Eqn (2.78). That is to say that the impedances and admittances are governed by the same transformation relation.

EXAMPLE 2.16 A line of 300Ω charateristic impedance is terminated in an admittance $0.01 + j0.02f$. Find (i) The reflection coefficient at the load-end (ii) reflection coefficient at a distance of 0.2λ from the load-end (iii) impedance at a distance of 0.2λ from the load-end.

Solution:

The characteristic impedance of the line $Z_0 = 300\Omega$. Therefore the characteristic admittance is

$$Y_0 = \frac{1}{Z_0} = \frac{1}{300} = 3.33 \times 10^{-3}\ \mho$$

(i) The reflection coefficient at the load-end is

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L} = -0.8462 - 0.2308j$$

(ii) Reflection coefficient at a distance of 0.2λ towards the generator is

$$\Gamma(l) = \Gamma_L e^{-j2\beta l} = \Gamma_L e^{-j2\times\frac{2\pi}{\lambda}\times 0.2\lambda}$$

$$= 0.5489 + 0.684j$$

(iii) Impedance at location $l = 0.2\lambda$ on the line is

$$Z(l) = Z_0 \frac{1 + \Gamma(l)}{1 - \Gamma(l)}$$

$$= 103.11 + 611.3j\ \Omega$$

EXAMPLE 2.17 A 50 Ω co-axial cable terminated in a 50 Ω resistance, forms the bus of a network. A network peripheral unit having impedance 75 Ω is connected to the bus at some location through a 50 Ω cable of length 0.3 λ. Find the impedance at a distance of 0.2 λ from the junction. Find the VSWR on the cable.

Solution:

In Fig. 2.11 at the junction we have two impedances Z_1 and Z_2 in parallel. Since the cable has been terminated in its characteristic impedance, Z_1 will be same as the characteristic impedance 50 Ω. Z_2 however will be transformed version of 75 Ω impedance. Hence, we have

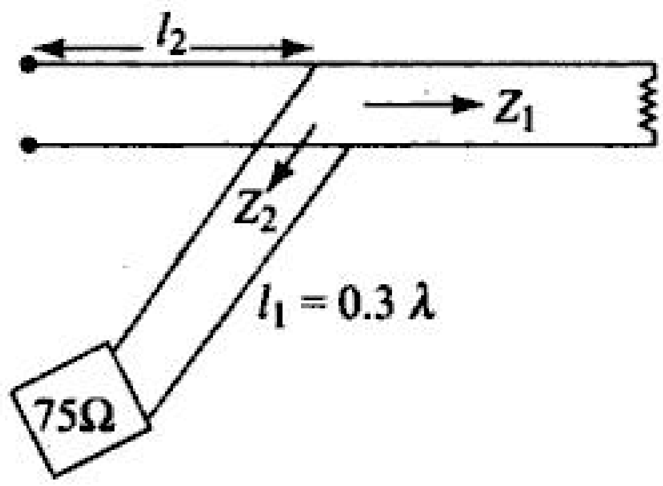


Fig. 2.11 *Bus network.*

$$Z_2 = Z(l_1) = Z_0 \frac{75\cos\beta l_1 + j50\sin\beta l_1}{50\cos\beta l_1 + j75\sin\beta l_1}$$

and $\beta l_1 = 2\pi/\lambda(0.3\lambda) = 0.6\pi = 108^\circ$, giving

$$Z_2 = Z(l_1) = 50\frac{75\cos 108^\circ + j50\sin 108^\circ}{50\cos 108^\circ + j75\sin 108^\circ}$$
$$= 35.2008 + 8.621j\ \Omega$$

Since, at the junction, the two impedances are connected in parallel, the impedance Z is

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2} = 20.9549 + 2.9389j\ \Omega$$

The impedance at a distance of l_2 from the junction is

$$Z(l_2) = Z_0\frac{Z\cos\beta l_2 + j50\sin\beta l_2}{50\cos\beta l_2 + jZ\sin\beta l_2}$$

and $\beta l_2 = 2\pi/\lambda(0.2\lambda) = 0.4\pi = 72^\circ$, we get

$$Z(l_2) = 50\frac{(20.9549 + 2.9389j)\cos 72^\circ + j50\sin 72^\circ}{50\cos 72^\circ + j(20.9549 + 2.9389j)\sin 72^\circ}$$
$$= 94 + j43.47\ \Omega$$

The magnitude of the reflection coefficient on the line is

$$|\Gamma| = \left|\frac{Z - Z_0}{Z + Z_0}\right| = 0.41$$

$$\Rightarrow \qquad \text{VSWR on the line, } \rho = \frac{1 + |\Gamma|}{1 - |\Gamma|} = 2.396$$

2.8 GRAPHICAL REPRESENTATION OF A TRANSMISSION LINE

In the previous sections we derived analytical expressions for voltage, current, reflection coefficient, impedance, admittance, etc. It is well known that an image or a graph creates a long lasting impression on mind than text or an equation. A graphical representation of a transmission line helps in pictorial visualization of some of the basic concepts. One can use the graphical means for solving transmission line problems and at times they are also easier as compared to analytical means. However, that is not the whole purpose of graphical representation. A graphical representation provides a good account of tranmission line characteristics in a compact manner. Even while solving transmission line problems analytically, a qualitative cross-check with the graphical model is always helpful in avoiding conceptual mistakes.

2.9 IMPEDANCE SMITH CHART

The graphical representation describes the impedance/admittance characteritics of a transmission line. For the time being let us confine our analysis to passive impedances only. Later we will extend it to the admittances. As we have seen earlier, all impedance expressions can be written in terms of normalized impedances. Let us therefore carry out the analysis in terms of

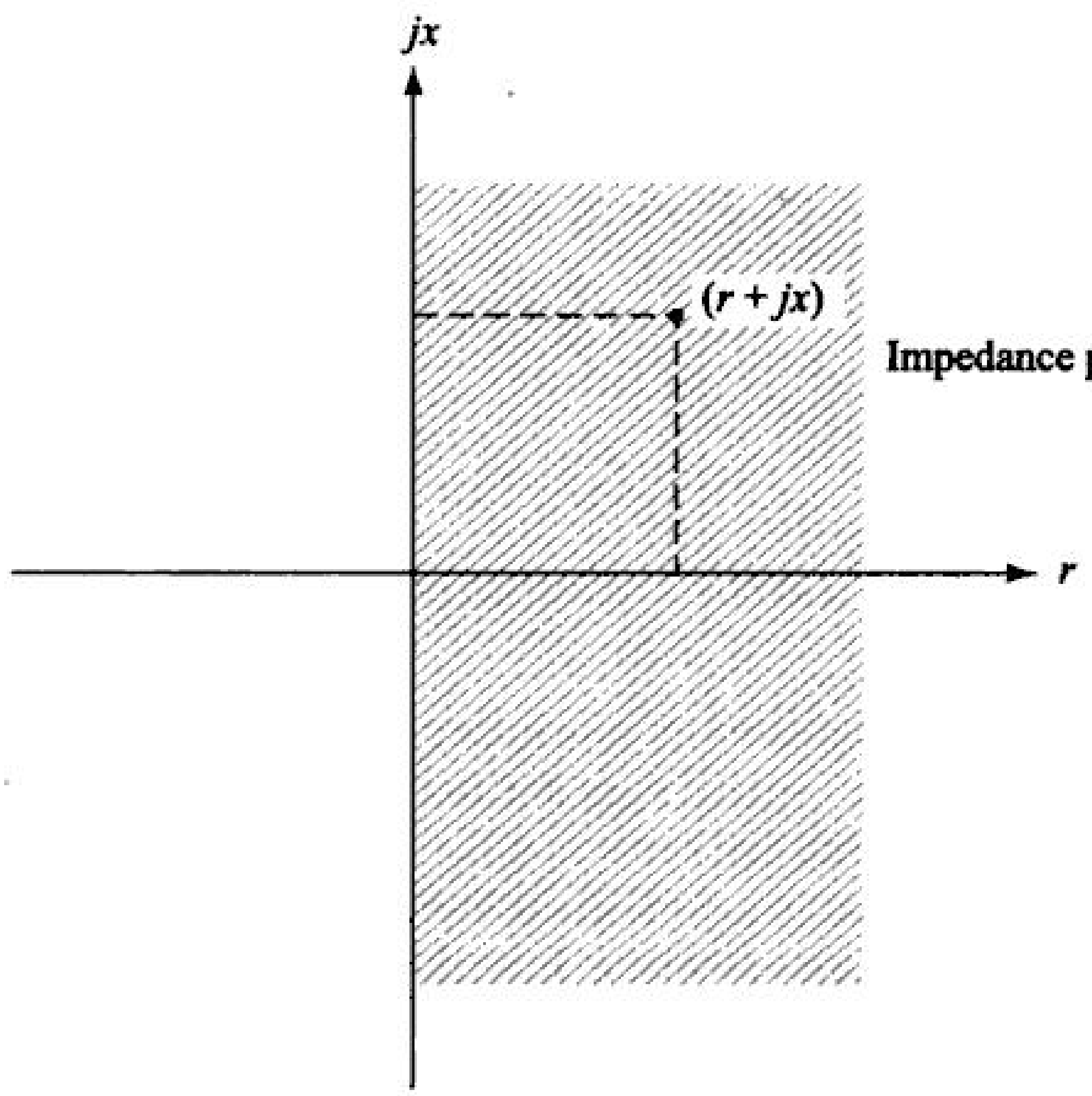


Fig. 2.12 *Complex impedance plane.*

normalized impedances. An impedance $Z = R + jX$ when normalized with the characteristic impedance Z_0, is denoted by $\overline{Z} = r + jx$ where $\overline{Z} = Z/Z_0$, $r = R/Z_0$ and $x = X/Z_0$. The reflection coefficient for a normalized impedance $\overline{Z}$ is then given as

$$\Gamma = \frac{Z - Z_0}{Z + Z_0} = \frac{\overline{Z} - 1}{\overline{Z} + 1} \tag{2.121}$$

$$= \frac{r + jx - 1}{r + jx + 1} = \frac{(r - 1) + jx}{(r + 1) + jx} \tag{2.122}$$

For passive loads, r lies between 0 and ∞ and x lies between $-\infty$ and $+\infty$. A passive load $r + jx$ therefore can be represented by a point in the right half of the complex plane including the imaginary axis as shown in Fig. 2.12.

The reflection coefficient Γ also is complex in general and can be written as

$$\Gamma \equiv u + jv \equiv Re^{j\theta} \tag{2.123}$$

We can note from Eqn (2.121) that there is one-to-one correspondance between $\overline{Z}$ and Γ, and for a normalized impedance $r + jx$ we obtain a unique complex reflection coefficient Γ. Moreover, the magnitude of the reflection coefficient $|\Gamma|$ is always less than or equal to unity. The possible values of Γ are therefore confined within the unit circle in the complex Γ plane as shown in Fig. 2.13.

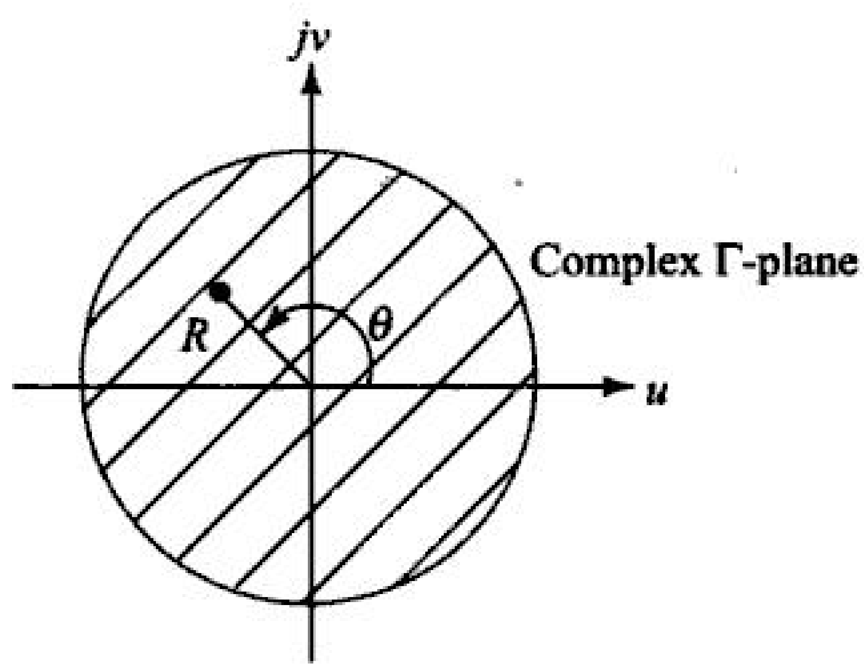


Fig. 2.13 *Complex reflection coefficient plane.*

We, therefore, see that the semi-infinite impedance plane is mapped to the area within the unit circle in the Γ-plane with one to one correspondence between the points in the two planes. One can show that the transformation from $\overline{Z}$-plane to the Γ- plane and vice-versa is a conformal transformation. Let us map a point $\overline{Z} = r + jx$ onto the $\Gamma = u + jv$ plane. Inverting Eqn (2.121) we have

$$\overline{Z} = \frac{1 + \Gamma}{1 - \Gamma} \tag{2.124}$$

$$\Rightarrow \qquad r + jx = \frac{1 + (u + jv)}{1 - (u + jv)} \tag{2.125}$$

Separating the real and the imaginary parts we get two equations of u and v, one in terms of r and other in terms of x, as follows.

$$u^2 - 2\left(\frac{r}{r+1}\right)u + v^2 + \left(\frac{r-1}{r+1}\right) = 0 \tag{2.126}$$

$$u^2 + v^2 - 2u - \left(\frac{2}{x}\right) v + 1 = 0 \qquad (2.127)$$

It is interesting to note that both Eqns (2.126) and (2.127) represent circles in the Γ-plane. Since r = constant represents a vertical line in the Z-plane, Eqn (2.126) transforms vertical lines in the Z-plane into circles in the Γ-plane. These circles are called constant resistance circles. Similarly the lines x = constant map into circles in the Γ-plane as given by Eqn (2.127). These are called the constant reactance circles. Remember that only those portions of the circles are of relevance which lie within the unit circle in the Γ-plane. Let us now analyse the characteristics of the transformed circles.

2.9.1 Constant Resistance Circles

The constant resistance circles have their centres at $(\frac{r}{r+1}, 0)$ and radii $(\frac{1}{r+1})$. Figure 2.14 shows the constant resistance circles for different values of r ranging between 0 and ∞.

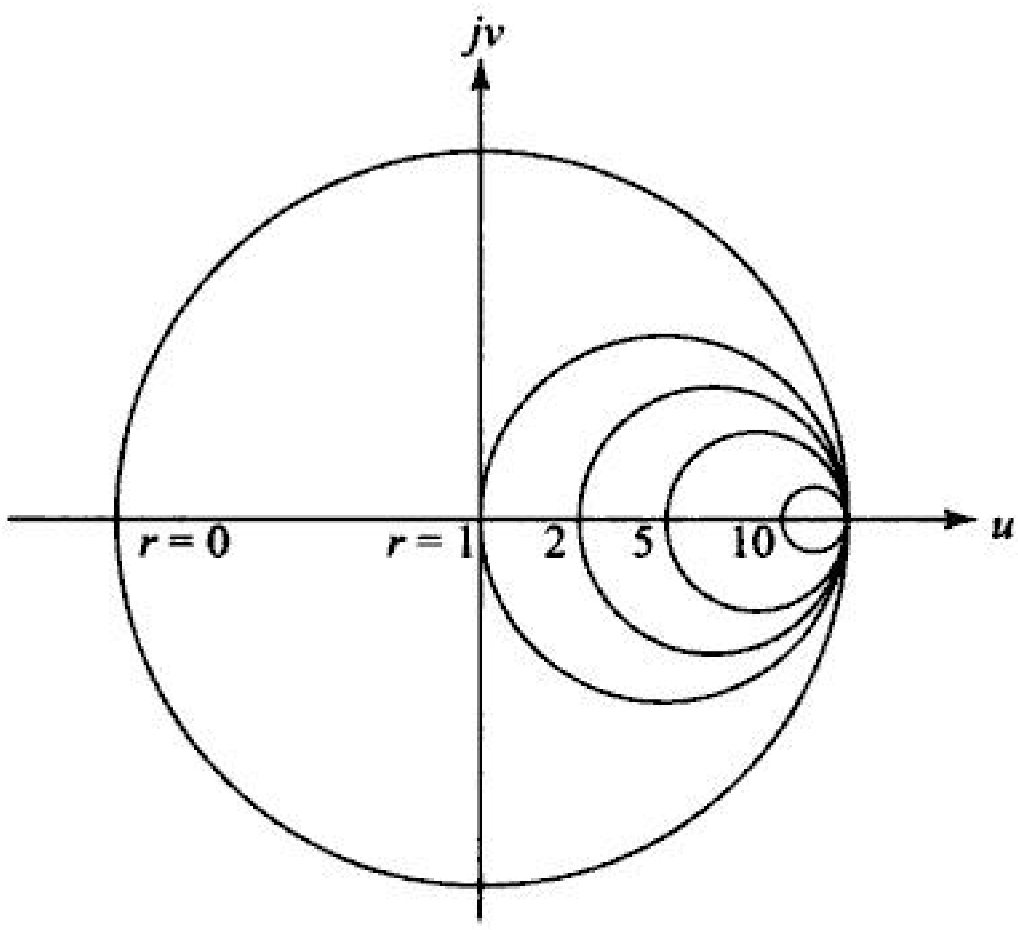


Fig. 2.14 *Constant resistance circles in the complex Γ-plane.*

We can note following things about the constant resistance circles.

(a) The circles always have centres on the real Γ-axis (u-axis).

(b) All circles pass through the point$(1, 0)$ in the complex Γ plane.

(c) For $r = 0$ the center lies at the origin of the Γ plane and it shifts to the right as r increases.

(d) As r increases the radius of the circle goes on reducing and for $r \to \infty$ the radius approaches zero, i.e. the circle reduces to a point.

(e) The outermost circle with center (0, 0) and radius unity, corresponds to $r = 0$ or in other words represents reactive loads only.

(f) The right most point on the unit circle $(1, 0)$ represents $r = 0$ as well as $r = \infty$.

2.9.2 Constant Reactance Circles

The constant reactance circles have their centers at $(1, \frac{1}{x})$ and radii $(\frac{1}{x})$. The centres for these circles lie on a vertical line passing through $(1, 0)$ point in the Γ-plane. The constant reactance circles are shown in Fig. 2.15 for different values of x.

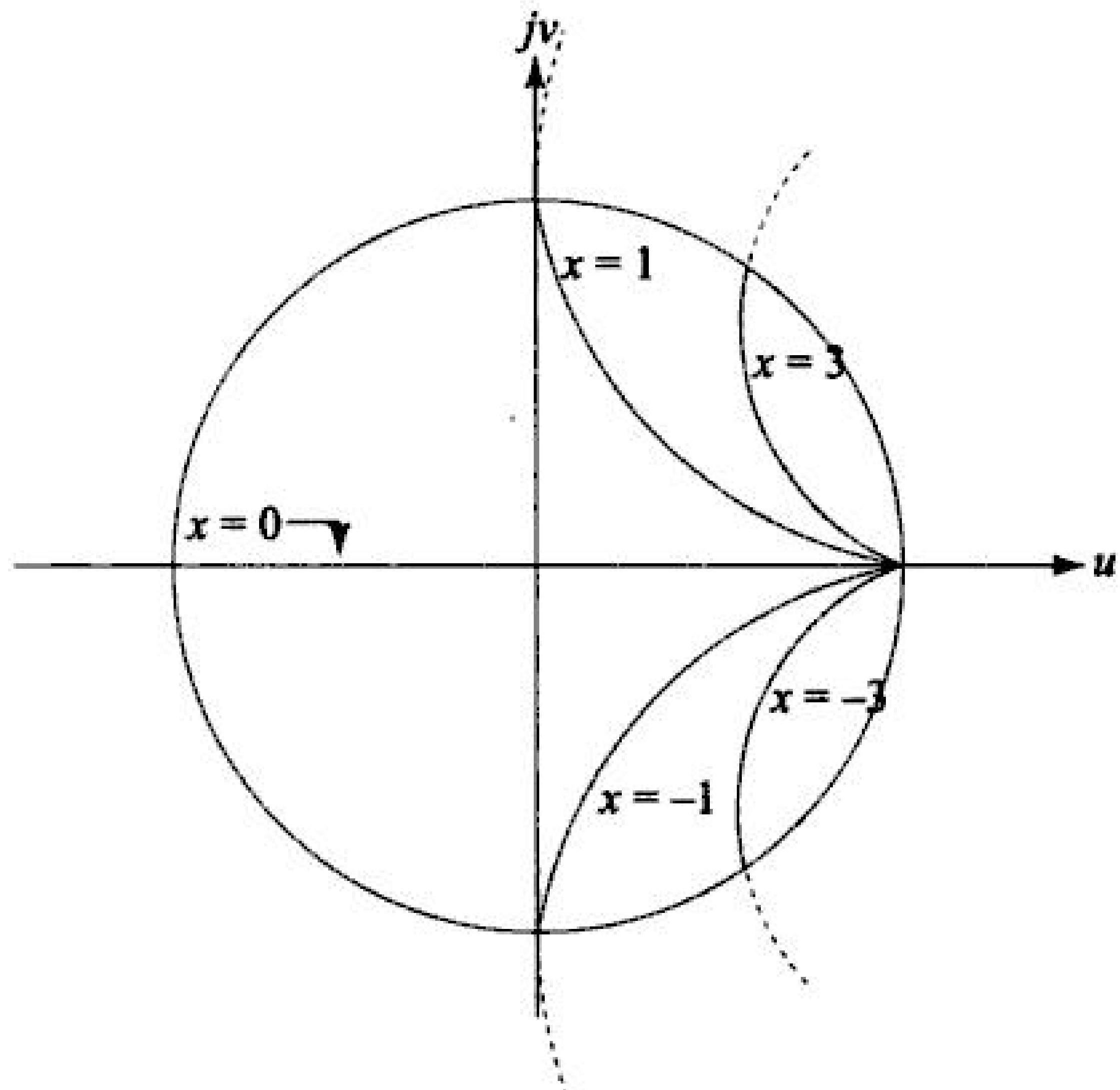


Fig. 2.15 *Constant reactance circles in the complex Γ-plane.*

Note again that only those portions of the circles are of significance which lie within the unit circle in the Γ-plane. The curves shown dotted do not correspond to any passive load impedance. We can note the following points about constant reactance circles:

(a) These circles have their centers on a vertical line passing through point $(1, 0)$.

(b) For positive x the center lies above the real Γ-axis and for negative x, the center lies below the real Γ-axis.

(c) For $x = 0$, the center is at $(1, \pm\infty)$ and radius is ∞. This circle therefore represents a straight line.

(d) As the magnitude of the reactance increases the center moves towards the real Γ-axis and it lies on the real Γ-axis at (1,0) for $x = \pm\infty$.

(e) As the magnitude of the reactance increases, the radius of the circle $(\frac{1}{x})$ decreases and it approaches zero as $x \to \pm\infty$.

(f) All circles pass through the point $(1, 0)$.

(g) The real Γ-axis (u-axis) corresponds to $x = 0$ and therefore represents real load impedances, i.e. purely resistive impedances.

(h) The right most point on the unit circle, $(1, 0)$, corresponds to $x = 0$ as well as $x = \pm\infty$.

2.9.3 The Smith Chart

The Smith chart is a graphical figure which is obtained by superposing the constant resistance and the constant reactance circles within the unit circle in the complex Γ-plane. A Smith chart is shown in Fig. 2.16. Since we have mapped here the impedances to the Γ-plane, let us call this the Impedance Smith chart.

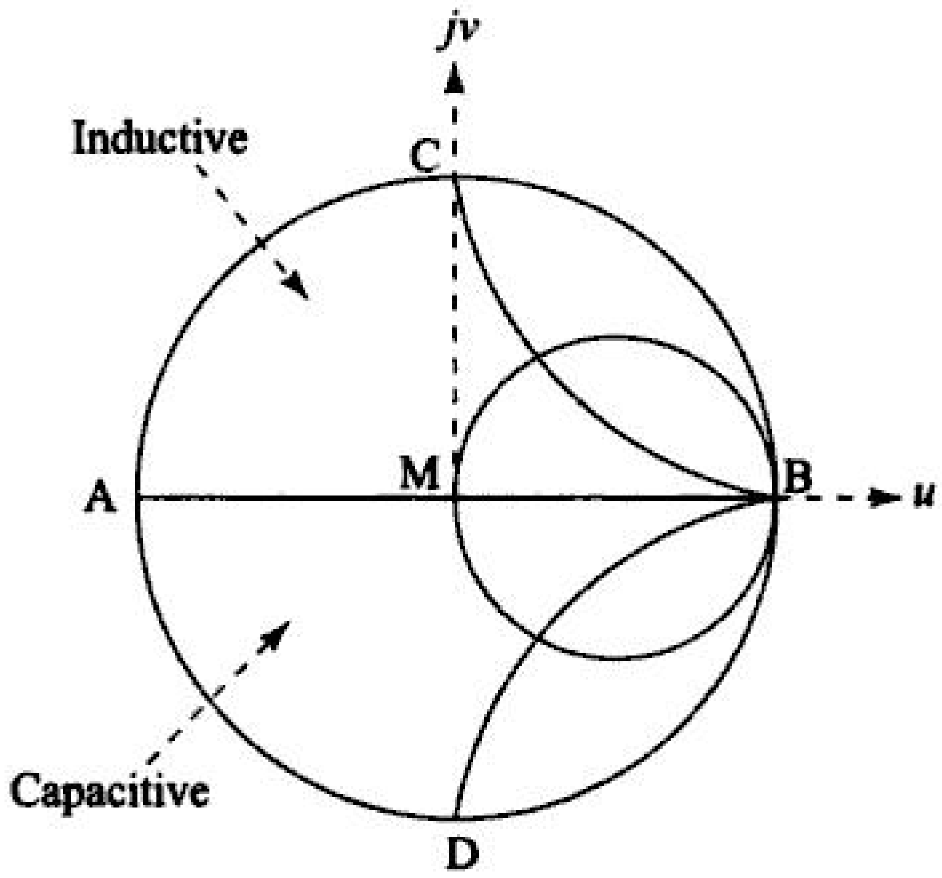


Fig. 2.16 *Smith chart: Superposition of constant resistance and constant reactance circles in the complex Γ-plane.*

Generally the u, v axes are not drawn on the Smith chart. However one should not forget that the Smith chart is a figure which is drawn on the complex Γ-plane with its center as origin. The intersection of constant resistance and constant reactance circles uniquely defines a complex load impedance in the Γ-plane. Let us identify some special points on the Smith Chart.

(a) The left most point A on the Smith chart corresponds to $r = 0, x = 0$ and therefore represents ideal short-circuit load.
(b) The right most point B on the Smith chart corresponds to $r = \infty, x = \infty$ and therefore represents ideal open circuit load.
(c) The center of the Smith chart M , corresponds to $r = 1, x = 0$ and hence represents the matched load.
(d) Line AB represents pure resistive loads and the outermost circle passing through A and B represents pure reactive loads.
(e) The upper most point C represents a pure inductive load of unity reactance and the lower most point D represents a pure capacitive load of unity reactance.

(f) In general the upper half of the Impedance Smith chart represents the complex inductive loads and the lower half represents the complex capacitive loads.

2.9.4 Constant VSWR Circles

We have seen earlier that the voltage reflection coefficient at any location l from the load is given as (see Eqn (2.63))

$$\Gamma(l) = \Gamma_L e^{-j2\beta l} \tag{2.128}$$

where Γ_L is the complex reflection coefficient at the load and is given by Eqn (2.37). Let the Γ_L be represented in the polar form as

$$\Gamma_L = |\Gamma_L| e^{j\theta_L} \tag{2.129}$$

Then the reflection coefficient $\Gamma(l)$ will be

$$\Gamma(l) \equiv Re^{j\theta} = |\Gamma_L| e^{j(\theta_L - 2\beta l)} \tag{2.130}$$

The magnitude of the reflection coefficient is

$$R \equiv |\Gamma(l)| = |\Gamma_L| \tag{2.131}$$

and the phase of the reflection coefficient is

$$\theta \equiv \theta_L - 2\beta l \tag{2.132}$$

As we change the value of l, i.e. as we move along the tranmission line the magnitude of $\Gamma(l)$ remains same (R = constant) but its phase varies linearly. As l increases, i.e. as we move towards the generator, the phase of the reflection coefficient $\Gamma(l)$ becomes more negative. In the complex Γ-plane therefore a point $Re^{j\theta}$ moves clockwise on a circle with center at (0,0) and radius $|\Gamma_L|$ as shown in Fig. 2.17.

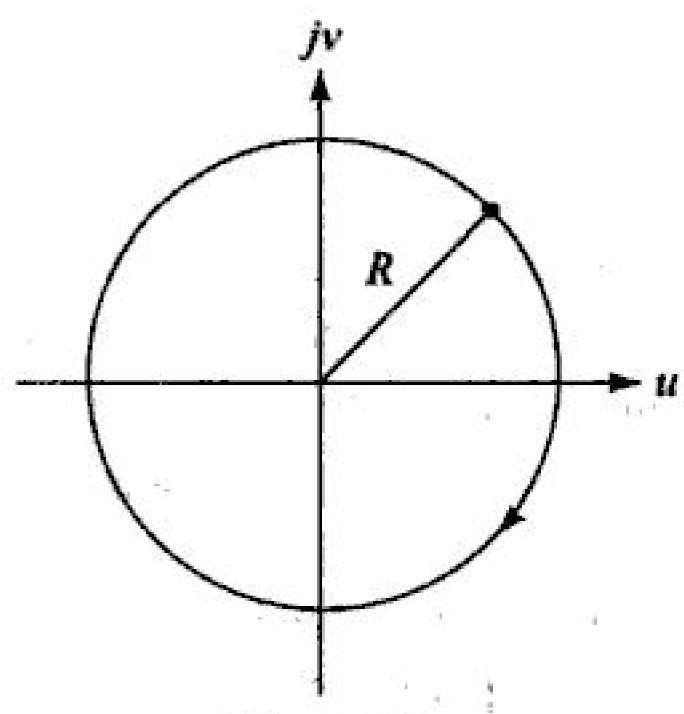


Fig. 2.17

All points on this circle have same $|\Gamma| = |\Gamma_L|$. Now since the VSWR on the line is

$$\rho = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \tag{2.133}$$

all the points on this circle have same VSWR. Hence, these circles are called the 'constant VSWR circles'. The constant VSWR circles are shown in Fig 2.18.

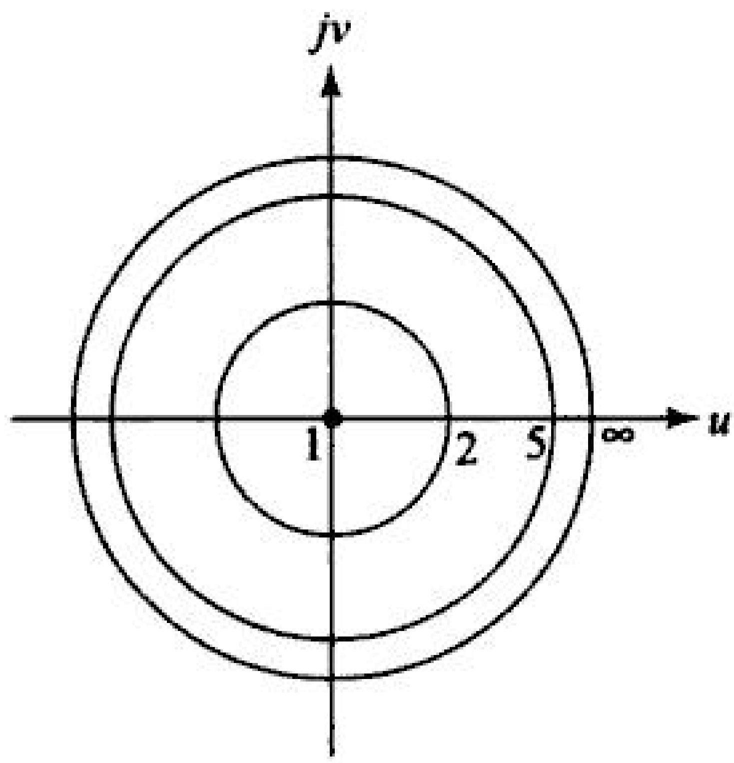


Fig. 2.18 *Constant VSWR circles drawn in the complex Γ-plane.*

We can make following observations about the constant VSWR circles:

(a) All the circles have same center, the origin of the complex Γ-plane.

(b) The origin in the Γ-plane represents $|\Gamma_L| = 0$ or $\rho = 1$. As we move radially outwards the $|\Gamma_L|$ and hence ρ increases monotonically and for the outermost unity circle, $|\Gamma_L| = 1$ and $\rho = \infty$.

(c) The origin corresponds to the condition $|\Gamma_L| = 0$, i.e. no reflection on the line. This point represents the best matching of the load as there is no reflected power on the line. For the outer most circle, since $|\Gamma_L| = 1$, we get the worst impedance matching as the entire power is reflected on the transmission line. We can therefore make a general statement that closer is the point to the origin of the Γ-plane, (i.e. the center of the Smith chart) better is the impedance matching.

(d) As we have defined earlier, the $+l$ indicates a distance towards the generator. As l becomes more positive, θ decreases and the point moves clockwise on the constant VSWR circle. If we move away from the generator, l becomes negative and then the point on the circle moves in the anticlockwise direction.

For analysing a transmission line problem graphically, the constant VSWR circles are to be superimposed or drawn on the Smith chart. For the sake of visual clarity the constant VSWR circles are not permanently drawn on the Smith chart. As and when required the user draws an appropriate constant VSWR circle on the Smith chart.

EXAMPLE 2.18 Draw following on the Smith chart. The normalizing impedance is 50Ω. (a) $50 + j75\Omega$ (b) $10 + j0\Omega$ (c) $0 - j80\Omega$ (d) $\Gamma = 0.3\angle 60^o$ (e) Constant VSWR circle for $\rho = 2.5$ (f) Minimum resistance point on the constant VSWR circle for $\rho = 1.5$.

Solution:

The points are marked on the Smith chart in Fig. 2.19.

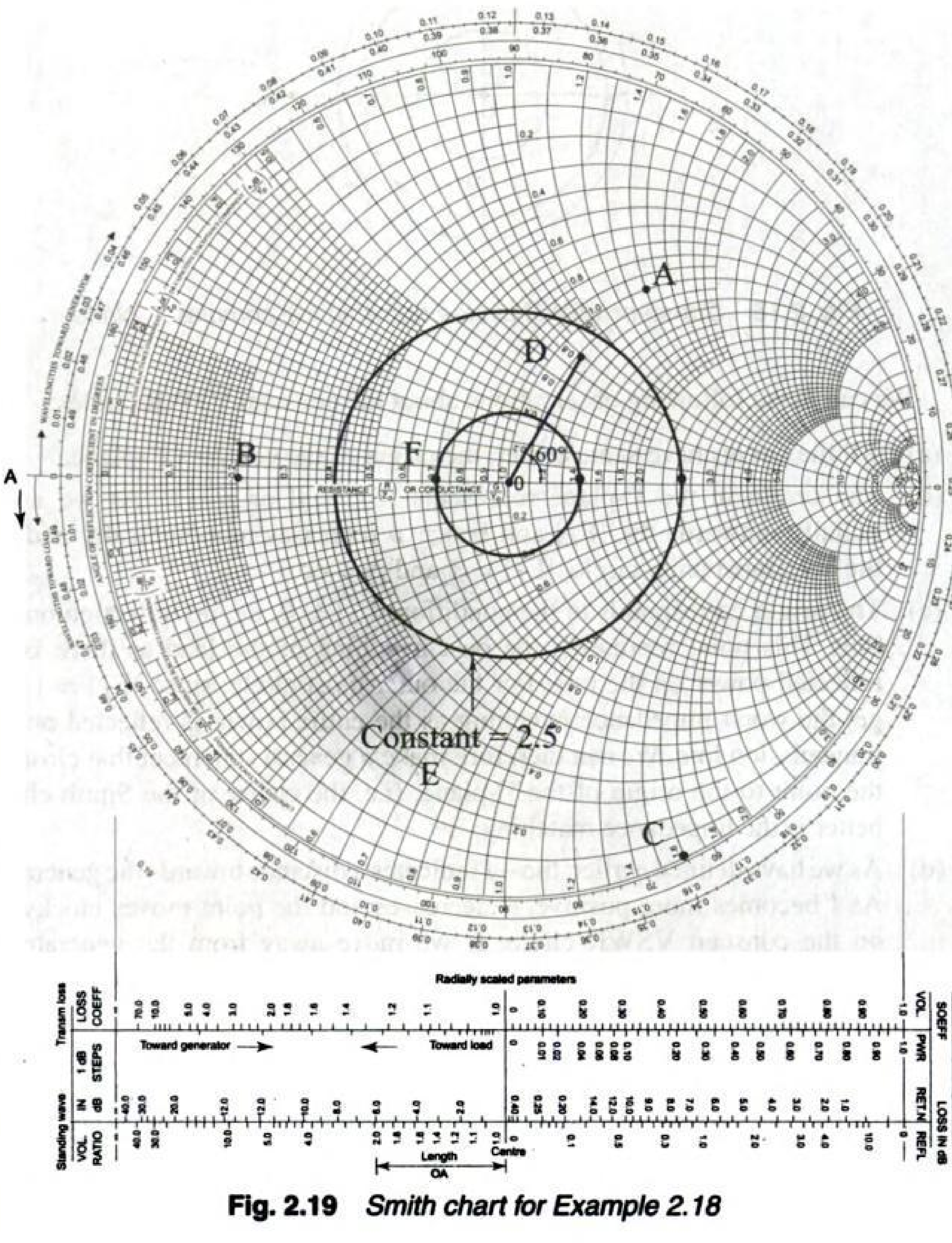


Fig. 2.19 *Smith chart for Example 2.18*

A

2.10 TRANSMISSION LINE CALCULATIONS WITH THE HELP OF THE SMITH CHART

The Smith chart is readily availaible in the printed form without constant VSWR circles drawn on it (see Fig. 2.20 as an Extended sheet). Generally the circumference of the Smith chart is marked with the length normalized to the wavelength. Arrows indicating 'towards generator' and 'towards load' are also normally indicated. However, even if this information is not printed, one can workout the direction from the first principles. As stated in the previous section, a clockwise rotation on the Smith chart indicates movement 'towards generator' and an anti-clockwise rotation indicates a movement 'away from the generator'. One complete rotation around the Smith chart corresponds to a phase change of $2\beta l = 2\pi$. Since $\beta = \frac{2\pi}{\lambda}$, one rotation is equal to a distance of $\frac{\lambda}{2}$, i.e. half the wavelength. One can verify the transmission line characteristic which states that impedance characteristics repeat every $\frac{\lambda}{2}$ distance on the line, because after one rotation on the Smith chart the point retraces the same circle and the impedance values repeat.

Smith chart is a very useful tool for solving transmission line problems. A variety of calculations can be carried out using the Smith chart without getting into complex computations. The Smith chart finds application in obtaining following quantities on a transmission line.

(a) Load Reflection Coefficient Let us find the reflection coefficient for a load impedance $R + jX$. First normalize the impedance with the characteristic impedance Z_0 to get $r + jx \equiv \frac{R+jX}{Z_0}$. Identify the constant resistance and the constant reactance circles corresponding to r and x respectively. Intersection of the two circles, marks the load impedance $r + jx$ on the Smith chart as point P (see Fig. 2.21). Measure the radial distance of P from the centre of the chart M. This is the magnitude of the load reflection coefficient $|\Gamma_L|$. The angle which the radius vector MP makes with the $+u$-axis is the phase of the load reflection coefficient θ_L. Note here that the Smith chart should be placed in such a way that the most clusterred portion of the chart lies on the right side. The horizontal line towards right then indicates the real $+u$-axis.

(b) Reflection Coefficient at a Distance from the Load Let the transmission line be terminated in a load impedance Z_L. Let us find the reflection coefficient at a distance l from the load using the Smith chart.

First, mark the load impedance as described in (a). Draw the constant VSWR circle passing through P. Rotate the radius vector MP by an angle $2\beta l$ in the clockwise direction to get point Q (see Fig. 2.22). Radial distance MQ gives the magnitude of the reflection coefficient, $|\Gamma(l)|$. Angle which the radius vector MQ makes with the $+u$-axis gives the phase of $\Gamma(l)$, θ.

(c) Transformed Impedance at a Distance from the Load Suppose, we have to obtain the transformed impedance at a distance l from the load, carry out the steps in (a) and (b) to get to point Q. Now instead of measuring reflection coefficient, identify the constant resistance and the constant reactance

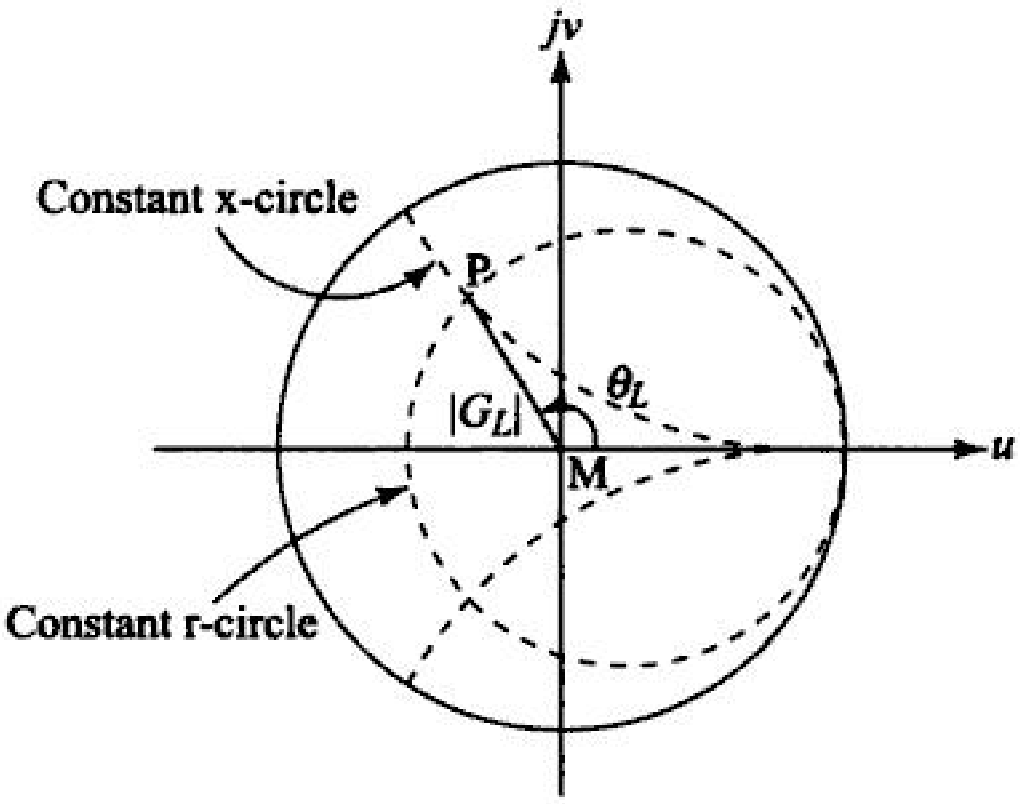


Fig. 2.21 *Smith chart as point P*

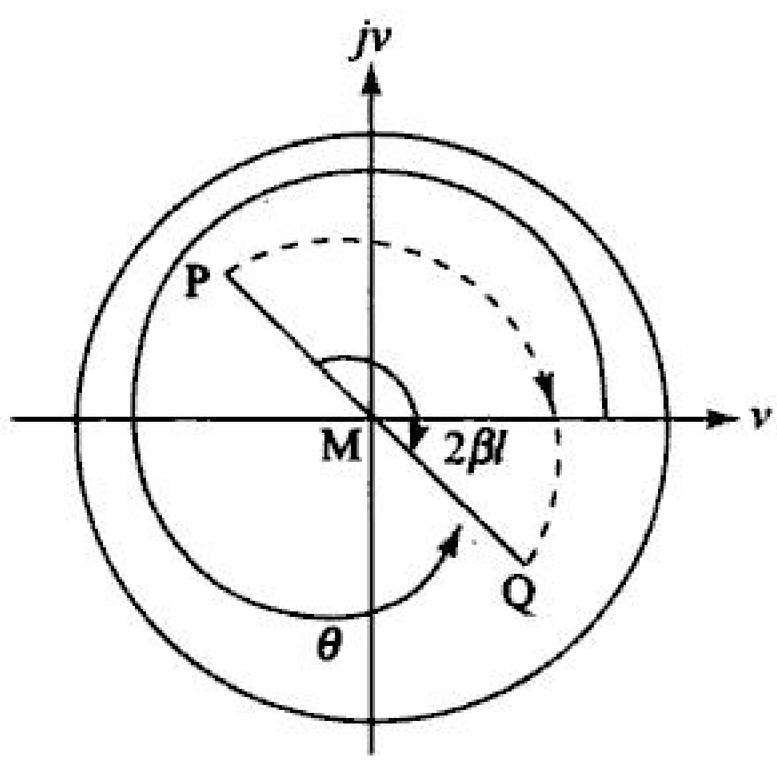


Fig. 2.22

circles passing through Q. They provide the transformed normalized resistance and reactance $r(l)$ and $x(l)$ respectively. Multiplying by Z_0 we get the transformed impedance $Z(l) = Z_0(r(l) + jx(l))$. The same procedure is used for transforming impedance from any point on the line to any other point. However if the distance l is away from the generator, it should be treated negative and hence the rotation of the radius vector must be by $2\beta l$ in the anti-clockwise direction.

(d) VSWR on the Line As we have seen earlier if the load impedance Z_L is not equal to Z_0 there is a standing wave on the transmission line. We also note from Eqn (2.81) that the maximum impedance seen on the line $R_{\max} = Z_0\rho$. This means that the maximum normalized impedance $r_{\max}(= R_{\max}/Z_0)$ measured on the line is nothing but the VSWR, ρ. The task of finding ρ is then very simple. Mark the normalized load impedance $\overline{Z}_L = r + jx$ on the Smith chart. Draw the constant VSWR circle passing through the marked point. The impedance corresponding to the intersection point of the constant VSWR circle and the $+u$-axis is $r_{\max} = \rho$

(see point T in Fig. 2.23). So, just by reading the value r_{max} from the chart we obtain the VSWR on the line.

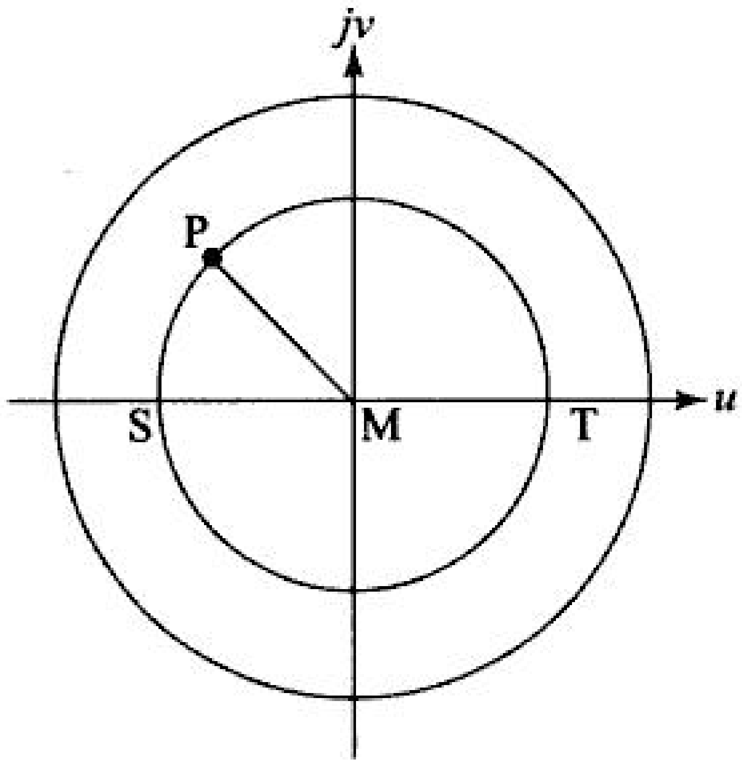


Fig. 2.23

(e) Location of Voltage Maximum or Minimum As we have discussed earlier, at the location of voltage maximum the impedance is maximum (R_{max}), and at the location of voltage minimum the impedance is minimum (R_{min}). Hence point T indicates the location of voltage maximum and point S indicates the location of voltage minimûm in Fig. 2.23. To find distance of these points from the load, measure the angle between the load point P and points T and S respectively in the clockwise direction.The angle PMT in the clockwise direction when divided by 2β gives the distance of the voltage maximum from the load, l_{max}. Similarly, one can obtain distance of voltage minimum, l_{min} by measuring angle PMS in clockwise direction from P to S. Alternatively one can make use of the fact that the maximum and minimum voltage are seperated by a distance of $\lambda/4$.

(f) Identifying the Type of Lc ad From the above discussion we can make one more observation. If the load is inductive, point P lies in the upper half of the Smith chart. Then while moving clockwise on the constant-VSWR circle, we first meet point T and then S. In other words, for inductive loads, voltage maximum is closer to the load point than the voltage minimum. Exactly the opposite occurs for capacitive loads i.e. the voltage minimum is closer to the load than the maximum. We can therefore quickly identify the load looking at the standing wave pattern. If the pattern is like the one given in Fig. 2.24(a), that is the voltage drops towards the load, the load is inductive. Similarly if the pattern is like that in Fig. 2.24(b), that is, the voltage rises towards load, the load is capacitive. Consequently if the voltage is maximum or minimum at the load, the load impedance is purely resistive (Fig. 2.24(c)). For the purely reactive loads the point P in Fig. 2.3 will lie on the outermost circle making $|\Gamma_L| = 1$ and $V_{min} = 0$. The pattern for pure inductive load will be like that in Fig. 2.24(d) and that for the pure capacative load will be like that in Fig. 2.24(e).

The above explanation clearly demonstrates the effective use of the Smith chart for the transmission line calculations. The important thing is, most of the impedances and the standing wave quantities can be calculated without any major computation.

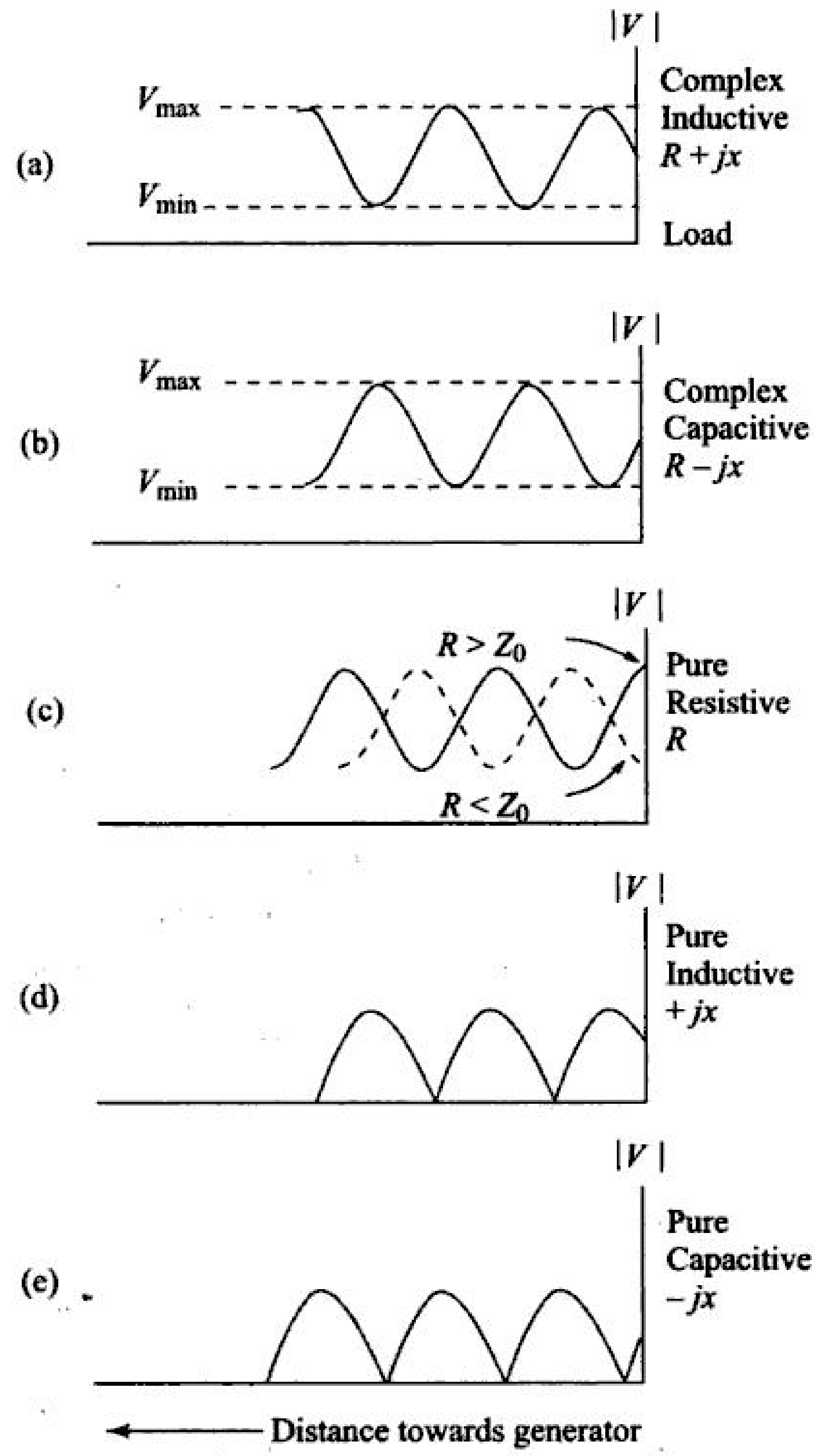


Fig. 2.24 *Voltage standing wave patterns for various types of load impedances.*

EXAMPLE 2.19 A 50Ω line is terminated in a load impedance 25 + j35Ω. With the help of the Smith chart find (i) Reflection coefficient in cartesian and polar form (ii) Reflection coefficient and impedance at a distance of 0.2λ from the load-end of the line (iii) VSWR on the line.

Solution:

(i) The normalized impedance is

$$\overline{Z} = r + jx = \frac{25 + j35}{50} = 0.5 + j0.7$$

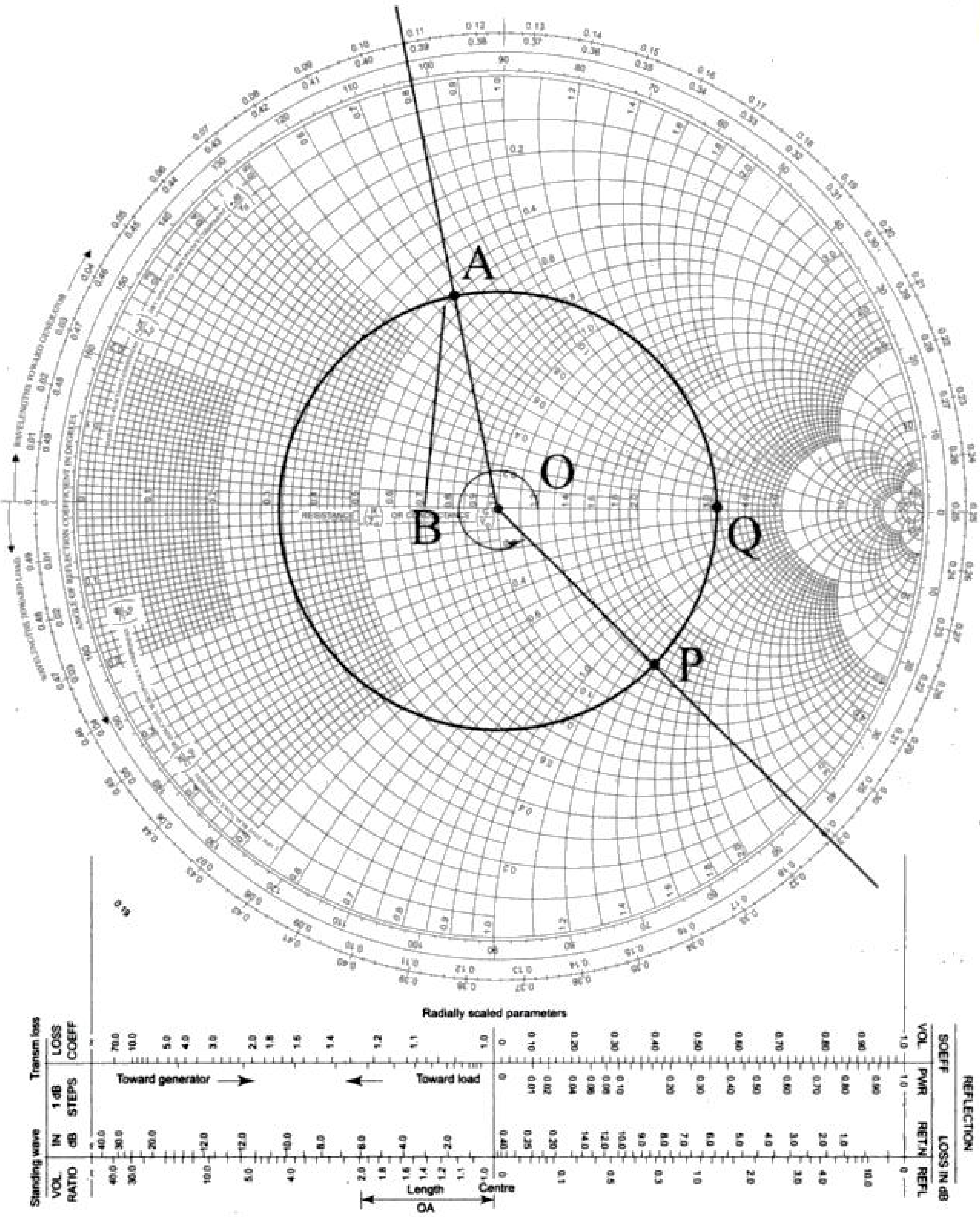


Fig. 2.25

Refer to Fig. 2.25. $\overline{Z}$ is denoted by point A on the chart. All distances on the Smith chart are normalized with respect to the radius of the chart. $|\Gamma_L|$ = distance OA = 0.52.

The complex reflection coefficient is

$$\text{Polar form: } \Gamma_L = |OA|e^{j\theta_L} = 0.52e^{j100.52^\circ}$$

$$\text{Cartesian form: } \Gamma_L = OB + jAB = -0.095 + j0.51$$

(ii) Draw constant VSWR circle passing through point A. The distance 0.2λ towards the generator corresponds to $\theta = 2\beta l = 2\frac{2\pi}{\lambda}0.2\lambda = 0.8\pi = 144^\circ$ in the clockwise direction. Moving on the constant VSWR circle by 144° from point A, we reach to point P. The reflection coefficient corresponding to point P is

$$\Gamma = |OP|e^{j\theta_1} = 0.51e^{j326^o} = 0.42 - j0.28$$

The impedance at point P is, $Z = 50 \times (1.4 - j1.37) = 70 - j68.5$
(iii) We know that, the VSWR is the r_{max} seen on the line. The r_{max} and hence the VSWR on the line corresponds to point Q. Therefore

$$\text{VSWR}, \rho = 3.1$$

2.11 ADMITTANCE SMITH CHART

In the previous section we developed the impedance Smith chart by mapping complex normalized impedances to the complex reflection coefficient plane. In many applications when the transmission lines and the impedances are connected in parallel, the admittance analysis turns out to be simpler compared to the impedance analysis.

In the following sections, we develop the admittance Smith chart by mapping normalized admittances to the complex Γ-plane. Normalization of every admittance is done with the characteristic admittance Y_0 of the transmission line. An admittance $Y = G + jB$ when normalized with Y_0 is noted by

$$\overline{Y} = g + jb = \frac{G}{Y_0} + j\frac{B}{Y_0}$$

Writing Eqn (2.121) in the normalized admittance form we get

$$\Gamma = \frac{1/Y - 1/Y_0}{1/Y + 1/Y_0} = \frac{1 - \overline{Y}}{1 + \overline{Y}} \tag{2.134}$$

To obtain the admittance Smith chart we can carryout the mapping of points $g + jb$ to the Γ-plane on the line identical to that given in the previous section. However, without going through the same algebraic steps, let us use the basic concepts to derive the admittance Smith chart from the impedance Smith chart.

Equation (2.134) can be written as

$$\Gamma = -\frac{\overline{Y} - 1}{\overline{Y} + 1} = \frac{\overline{Y} - 1}{\overline{Y} + 1}\angle\pi \tag{2.135}$$

Now, if we take normalized impedance $\overline{Z}$ equal to $\overline{Y}$ i.e. $r = g$ and $x = b$, we get Γ for $\overline{Z}$ which is 180^o out of phase with respect to the Γ for $\overline{Y}$. It means that for same numerical values, if the normalized load is impedance we get some point P on the Γ plane and if the load is admittance we get point P′ which is diagonally opposite to P on the Γ-plane (see Fig. 2.26). P′ is obtained by rotating P by 180° around the origin of the Γ plane. Since, this is true for every $\overline{Z}$ and $\overline{Y}$, all constant resistance and constant reactance circles when rotated by 180° around the origin of the Γ-plane give corresponding constant conductance (constant-g) and constant susceptance (constant-b) circles respectively.

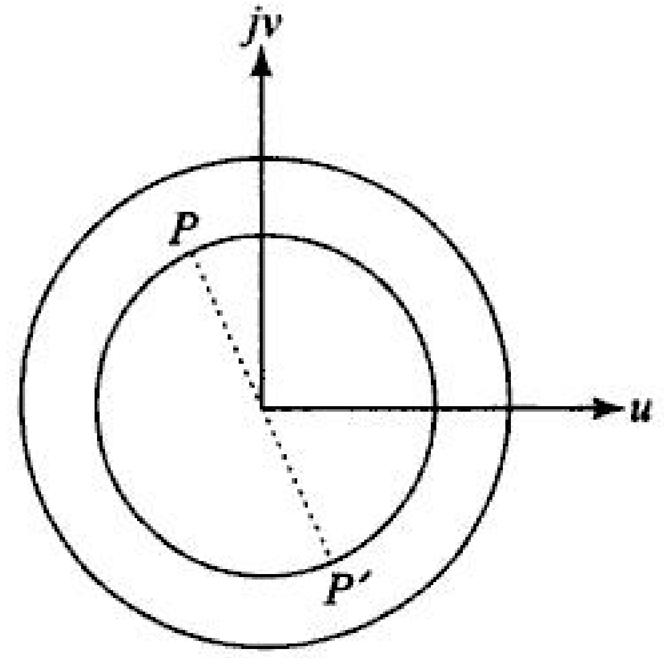


Fig. 2.26 *Impedance to admittance conversion.*

The Admittance Smith chart therefore appears as shown in Fig. 2.27.

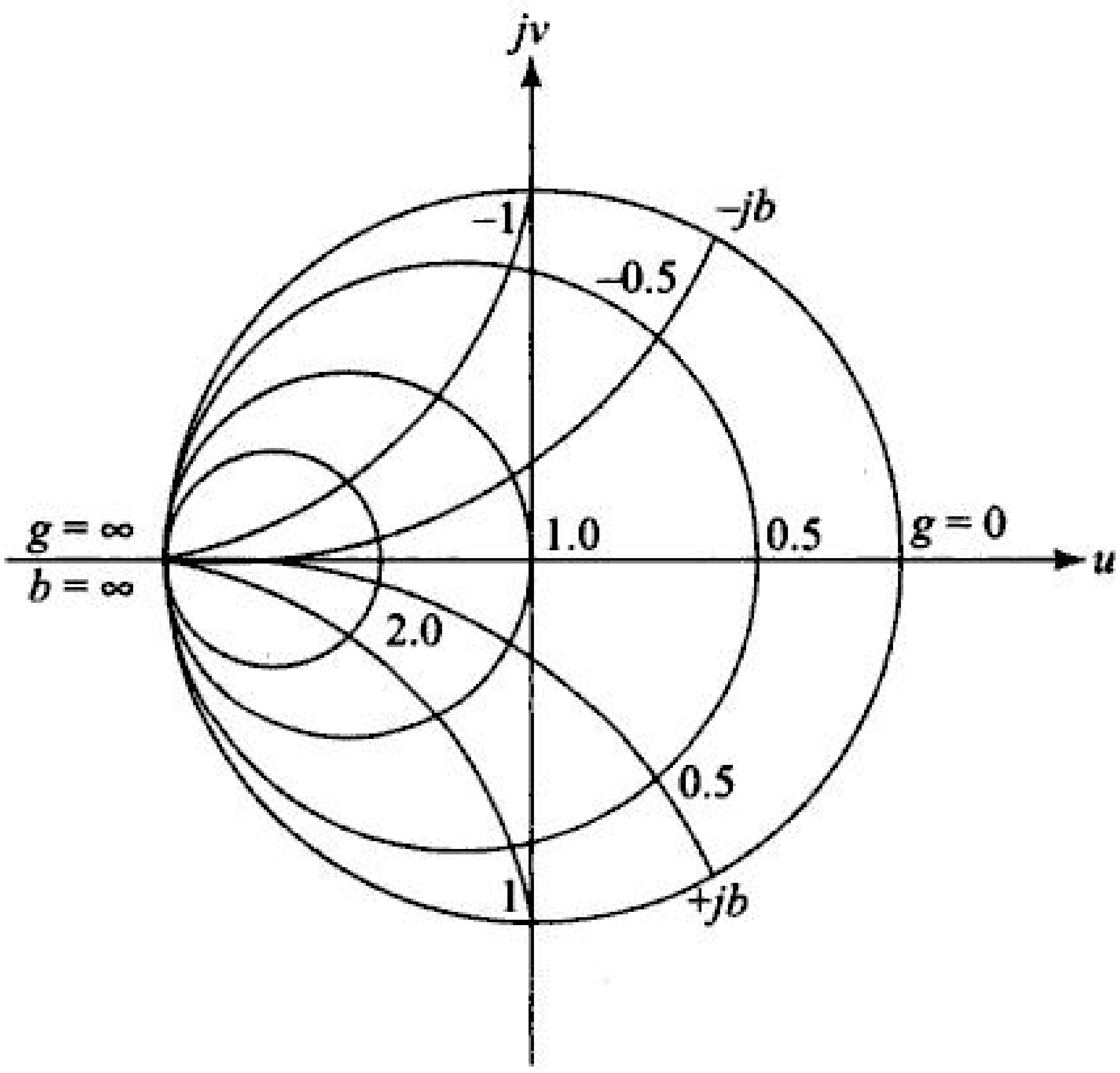


Fig. 2.27 *Constant conductance and constant susceptance circles drawn on the complex Γ-plane.*

The admittance Smith chart therefore is obtained by rotating the impedance Smith chart by 180° and replacing r by g and x by b. Since it is just a matter of rotation, there is no need to have separate Smith charts for impedance and admittance. Generally the Smith chart is used with orientation as shown in Fig. 2.20, and while using it as admittance chart, the u and v axes are rotated by 180°. In Fig. 2.18 therefore the u-axis will point to left and jv-axis will point downwards if the chart is used as the admittance chart. The admittance chart then would be as shown in Fig. 2.28.

Although Figs 2.20 and 2.28 appear identical, the following points should be kept in mind while making their use for transmission line calculations.

(a) While calculating phase of the reflection coefficient from the admittance Smith chart the phase must be measured from the rotated u-axis as shown in Fig. 2.28.

(b) Although r and x can be interchanged with g and b respectively and a point (r, x) and (g, b) will have the same spatial location on the Smith chart for $r = g$ and $x = b$, physical conditions corresponding to the two will not be identical. Let us analyse some specific examples.

- Upper half of the Smith chart with $+jx$ represents inductive loads whereas $+jb$ represents capacitive loads.
- Point A in Fig. 2.16 is $r = 0, x = 0$ as well as $g = 0, b = 0$. But $r = 0, x = 0$, represents short circuit load whereas, $g = 0, b = 0$, represents an open circuit load. The point A therefore represents the short circuit in the impedance chart whereas it represents the open circuit in the admittance chart.
- Similarly Point B in Fig. 2.16 represents the open circuit for the impedance chart and in admittance chart it represents the short circuit.
- In Fig. 2.23, point T corresponds to the voltage maximum if the chart is the impedance chart, and a voltage minimum if the chart is the admittance chart. The opposite is true for point S. Now, since the voltage maximum coincides with the current minimum and vice versa, the point T in admittance Smith chart represents the location of the current maximum and point S represents location of the current minimum. So we find that the voltage standing wave pattern and the impedance have the same relationship as the current standing wave pattern and the admittance.
- As we have seen, the reflection coefficients for same normalized impedance and admittance values are 180° out of phase. Therefore any normalized impedance can be converted to normalized admittance and vice-versa by taking a diagonally opposite point on the constant VSWR circle. In Fig. 2.26, P′ gives normalized admittance corresponding to the normalized impedance at P. We can therefore switch between admittance and impedance Smith charts freely without any additional computation.

All transmission calculations using admittance Smith chart are identical to that with the impedance Smith chart (described in the previous section) except the modifications mentioned earlier.

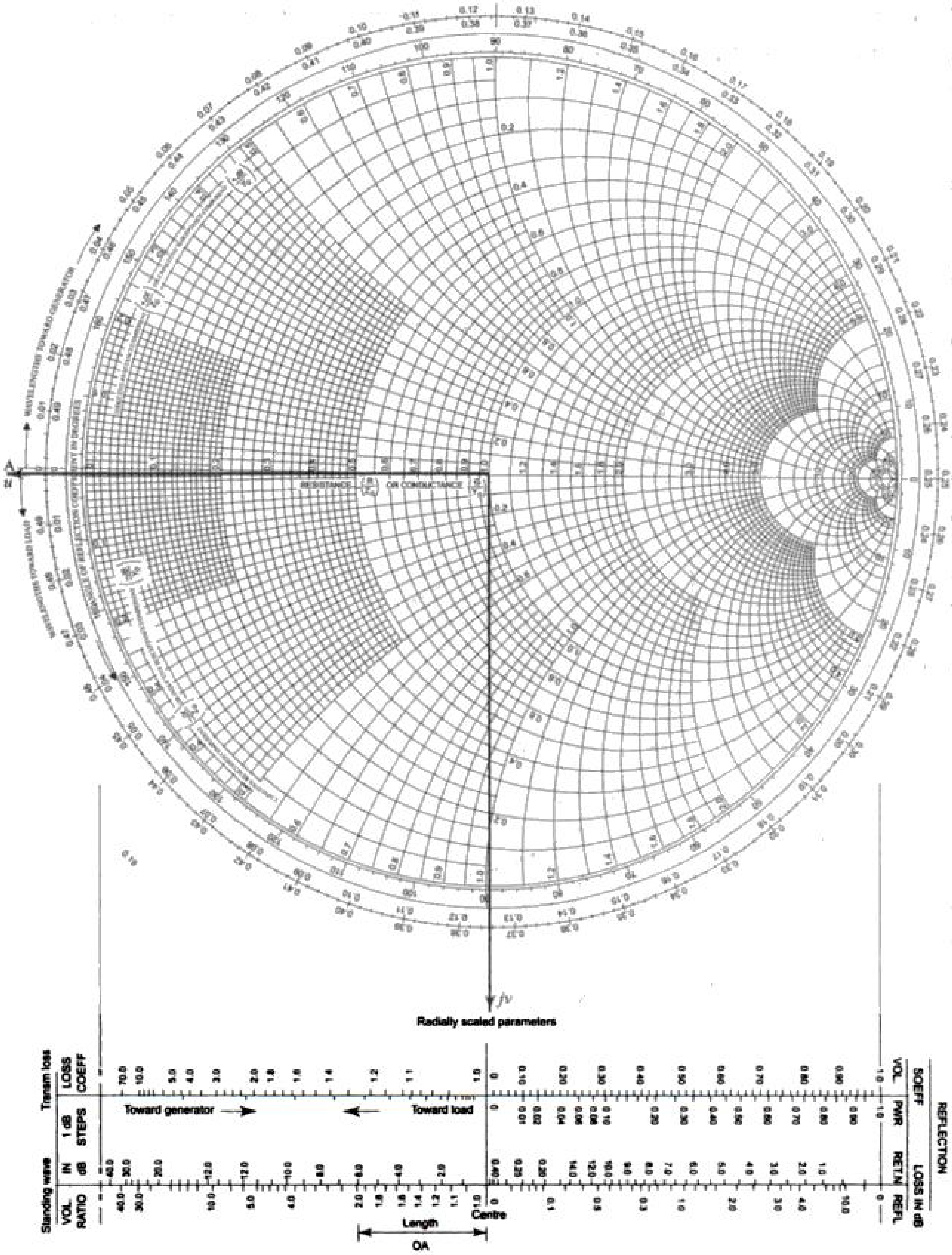


Fig. 2.28 *For Admittance Smith chart, the coordinate axes of the complex* Γ*-plane is rotated by* 180°.

EXAMPLE 2.20 A line is terminated in a normalized admittance $0.2 - j0.5$. Find the location of the voltage maximum from the load-end. Also find the reflection coefficient, normalized admittance, and normalized impedance at a distance of $0.12\ \lambda$ from the load.

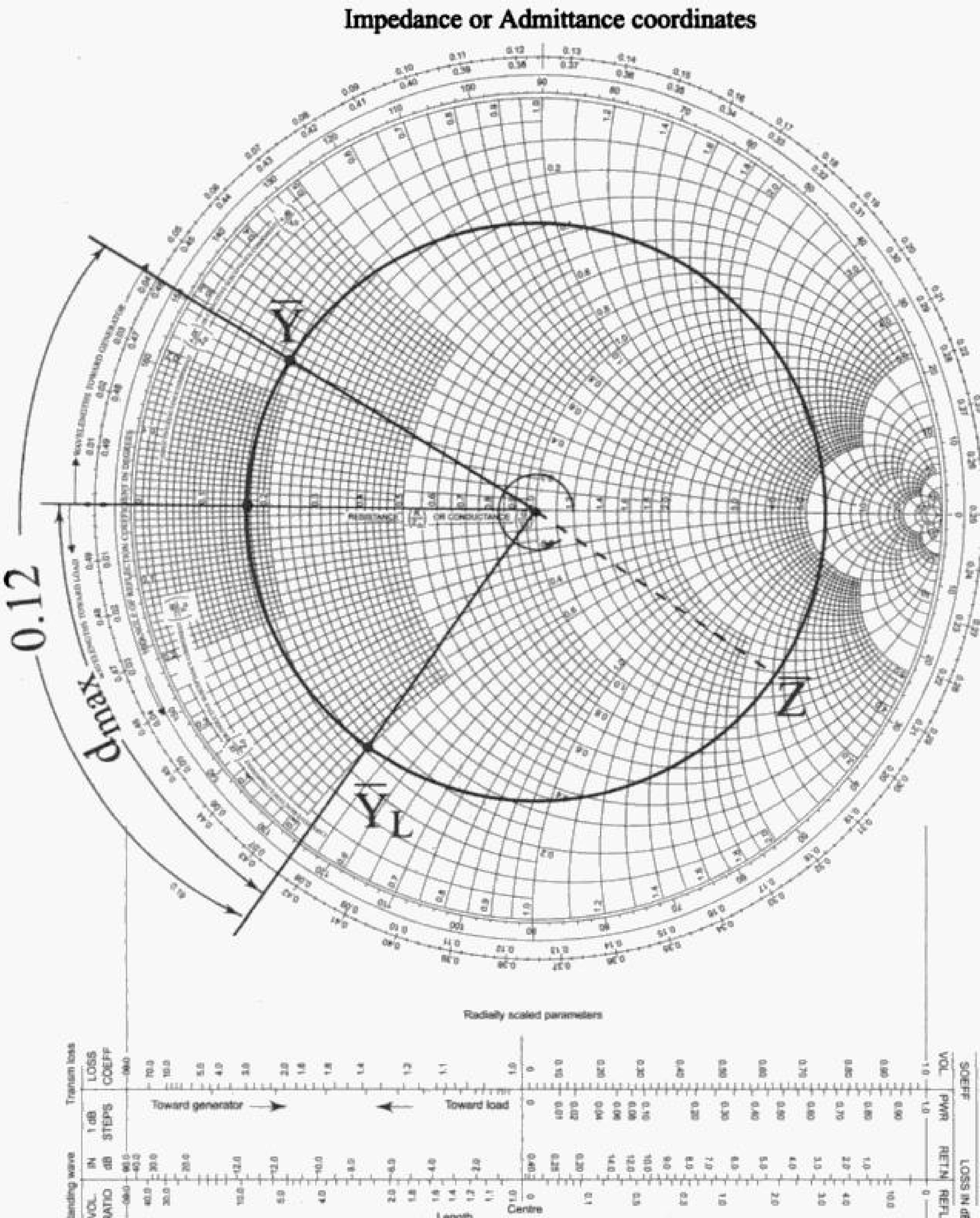


Fig. 2.29 *Smith chart for Example 2.20*

Solution:

Refer to the Smith chart in Fig. 2.29.

First take the Smith chart as the admittance chart. Mark the load admittance $\overline{Y}_L = 0.2 - j0.5$ on the chart. Draw the constant VSWR-circle passing through $\overline{Y}_L$.

Note that the voltage maximum occurs where the admittance is minimum. The left most point on the constant VSWR-circle corresponds to the voltage maximum. The distance of the voltage maximum is $d_{\text{max}} = 0.076\ \lambda$.

Following the procedure identical to that in Example 2.19 and using the chart as the admittance chart, we get at a distance of $0.12\ \lambda$,

$$\overline{Y} = 0.18 + j0.28$$

$$\Gamma = 0.706\angle 328^\circ$$

The normalized impedance $\overline{Z}$ is the diagonally oppsite point to $\overline{Y}$ on the constant VSWR circle. Hence, the normalized impedance is

$$\overline{Z} = 1.6 - j2.6$$

2.12 APPLICATIONS OF TRANSMISSION LINE

In the previous sections, we developed the understanding of basic characteristics of a transmission line. The impedance transformation property which a transmission line has, can be utilized in a variety of applications. The transmission line is therefore, not merely used for transporting power from one point to another but has some important applications as discussed in the following sections.

2.12.1 Measurement of Unknown Impedance

At high frequencies the measurement of an impedance is rather a difficult task. This is due to the fact that as the frequency increases, direct measurement of the signal phase becomes more and more difficult and at times impossible. A transmission line can be of great use in this situation. As we have seen earlier, the characteristics of the load impedance are uniquely manifested in the standing wave pattern on the line. By measuring the standing wave pattern, (which is only a measurement of the signal amplitude) one can indirectly obtain the phase of the complex load impedance.

For measuring the standing wave pattern, a special type of transmission line called the slotted transmission line is used. In this transmission line there is a voltage probe which moves along the length of the line and measures the magnitude of the voltage. A plot of probe output as a function of distance gives the standing wave pattern.

The unknown impedance which is to be measured is connected at the end of the transmission line as shown in Fig. 2.30. The transmission line is excited with a source of desired frequency ω. From the standing wave pattern three quantities, namely the maximum voltage $|V|_{max}$, minimum voltage $|V|_{min}$, and the distance of the voltage minimum from the load is measured. The ratio of $|V|_{max}$ and $|V|_{min}$ gives the VSWR on the line.

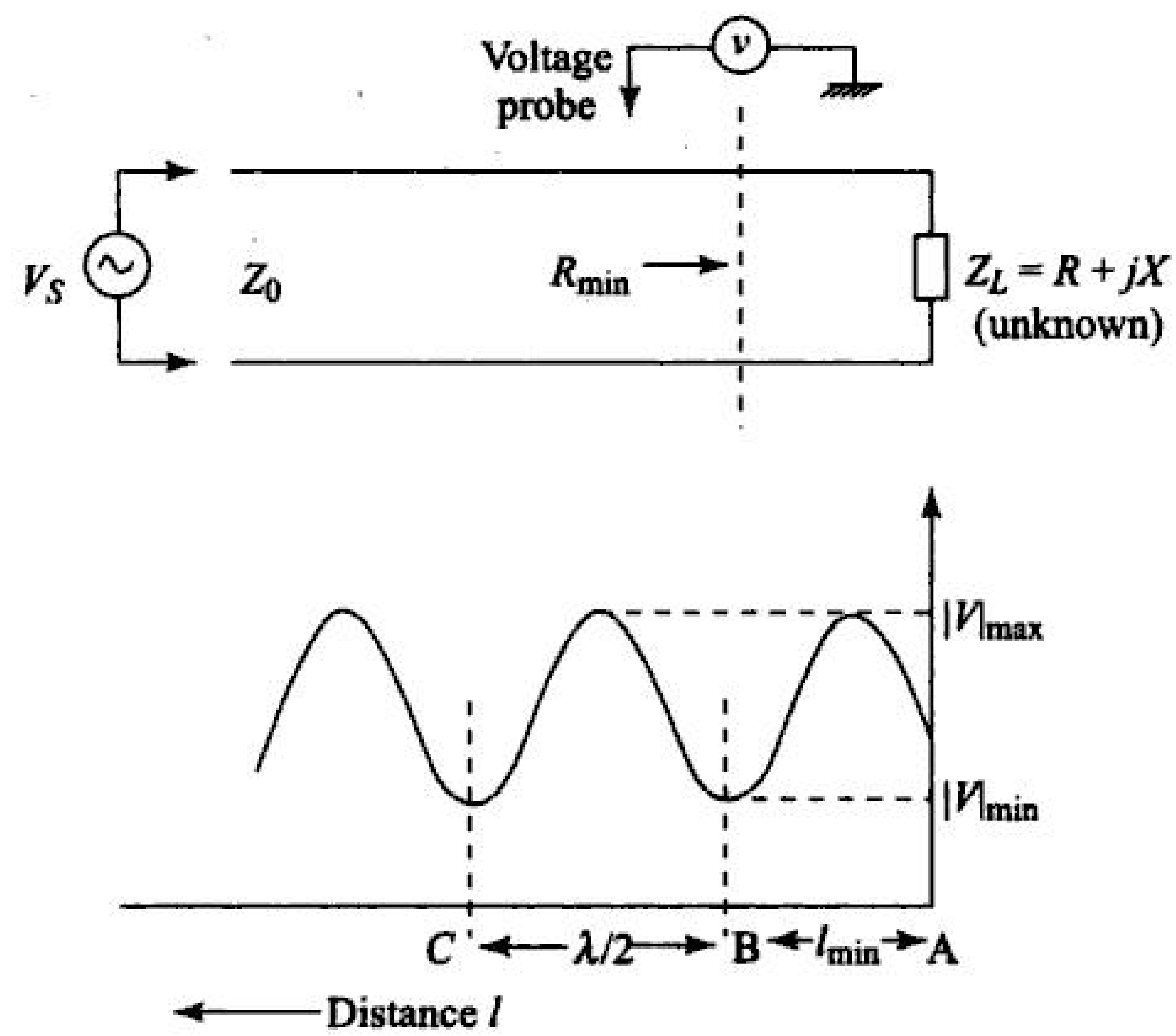


Fig. 2.30 *Transmission line with voltage measuring probe and the voltage standing wave pattern.*

$$\rho = \frac{|V|_{max}}{|V|_{min}} \tag{2.136}$$

We know that at point B on the transmission line where the voltage is minimum, the impedance is real and its value is $R_{min} = Z_0/\rho$. The impedance R_{min} is nothing but the transformed value of the load impedance Z_L. We can therefore obtain the unknown impedance Z_L by transforming back R_{min} from point B to point A. Let the distance of the voltage minimum from the load be l_{min}. Since the transformation from B to A is away from the generator, the distance BA is negative. The unknown impedance therefore is

$$Z_L \equiv R + jX = Z_0 \left[\frac{R_{min} \cos\beta(-l_{min}) + jZ_0 \sin\beta(-l_{min})}{Z_0 \cos\beta(-l_{min}) + jR_{min} \sin\beta(-l_{min})} \right] \tag{2.137}$$

Substituting for $R_{min} = Z_0/\rho$, we get

$$Z_L \equiv R + jX = Z_0 \left[\frac{\frac{Z_0}{\rho} \cos \beta l_{\min} - jZ_0 \sin \beta l_{\min}}{Z_0 \cos \beta l_{\min} - j\frac{Z_0}{\rho} \sin \beta l_{\min}} \right] \tag{2.138}$$

$$= Z_0 \left[\frac{1 - j\rho \tan \beta l_{\min}}{\rho - j \tan \beta l_{\min}} \right] \tag{2.139}$$

Separating real and imaginary parts we get

$$R = \frac{\rho(1 + \tan^2 \beta l_{\min})}{\rho^2 + \tan^2 \beta l_{\min}} \tag{2.140}$$

$$X = \frac{(1 - \rho^2) \tan \beta l_{\min}}{\rho^2 + \tan^2 \beta l_{\min}} \tag{2.141}$$

So, we see that the complex impedance is measured without directly measuring phase of any quantity. Generally, the value of β which is required in Eqns (2.140) and (2.141) is not supplied in advance and one has to obtain it from the standing wave pattern itself. Since, the distance between any two consecutive voltage minima or maxima is equal to $\lambda/2$, and β is equal to $2\pi/\lambda$, measurement of β from the standing wave pattern is a trivial task. Generally, voltage minima are preferred because they can be located with higher precision compared to the voltage maxima. While practically implementing the above scheme one would also notice that invariably the location of unknown impedance Z_L is not precisely defined. As a result, the measurement of $l_{\min}$ may have some error which in turn will result into an error in the load impedance.

To overcome this problem the measurement is carried out in two steps. First, the standing wave pattern is obtained with the unknown load as explained earlier. Now replace the unknown impedance by an ideal short-circuit and obtain the standing wave pattern again. The two standing wave patterns are shown in Fig. 2.31.

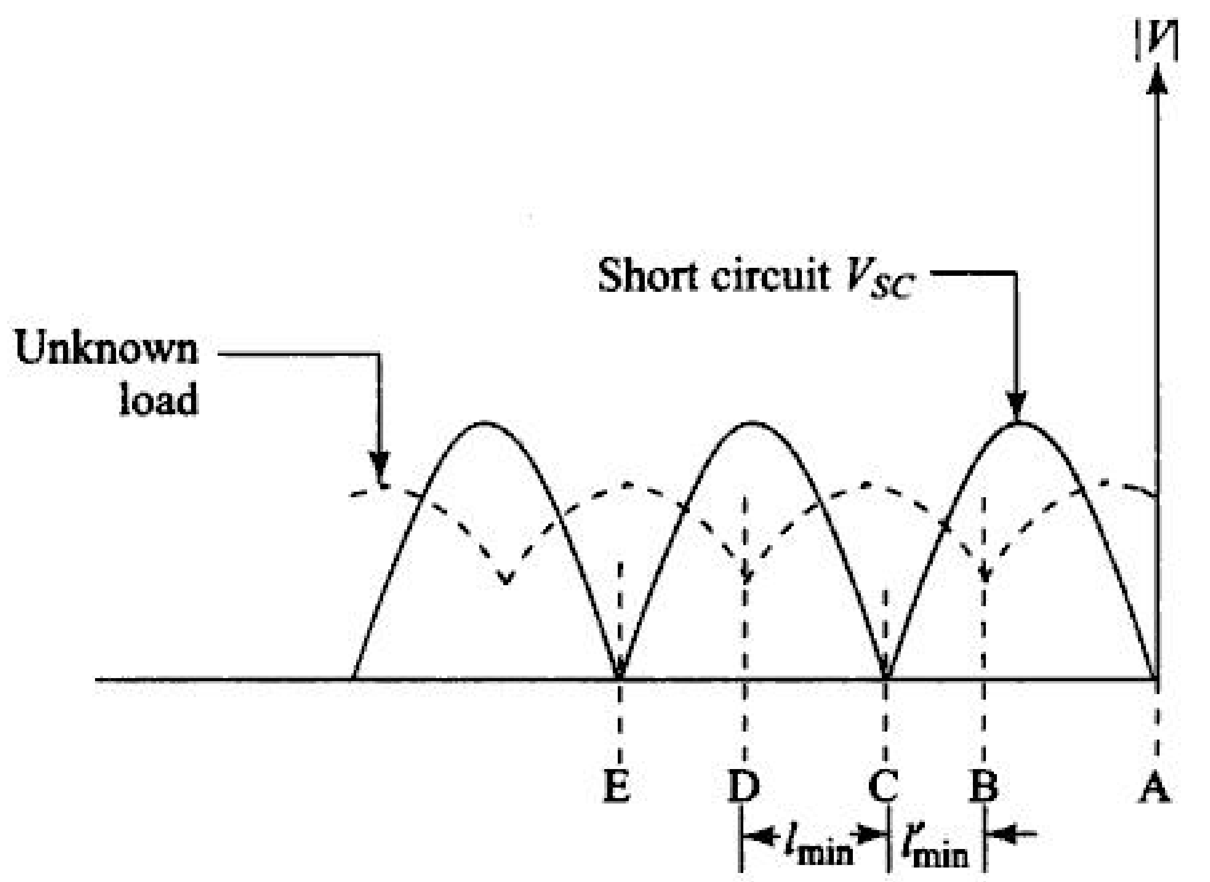


Fig. 2.31 *Voltage standing wave patterns for a short circuit and an unknown load.*

At the short circuit point (which is also the location of the unknown impedance) the voltage V_{sc} is zero. The voltage is also zero at points which are multiple of $\lambda/2$ away from it, i.e. at points C and E, etc. The points C and E, etc. represent impedance conditions identical to that at A, that is, the impedance at C or E is equal to the unknown impedance. The unknown impedance, therefore, can be obtained by transforming impedance Z_0/ρ at B or D to point C. If impedance is transformed from D to C the distance $l_{\min}$ is negative, whereas, if the transformation is made from B to C the distance $l'_{\min}$ is positive. The unknown impedance, therefore, can be evaluated as

$$Z_L = R + jX = Z_0\left[\frac{1 - j\rho\tan\beta l'_{\min}}{\rho - j\tan\beta l'_{\min}}\right] = Z_0\left[\frac{1 + j\rho\tan\beta l_{\min}}{\rho + j\tan\beta l_{\min}}\right] \quad (2.142)$$

One can note here that, $l_{\min} + l'_{\min} = \lambda/2$. In the impedance calculation either of l_{min} or l'_{min} can be used. As long as the sign of the distance is taken correctly it does not matter which of the minima is taken for impedance transformation.

EXAMPLE 2.21 A 50 Ω line is terminated in an unknown impedance Z. The distance of the voltage maximm from the load is 6 cm, and the distance between two consecutive maxima is 30 cm. The VSWR on the line is measured to be 3. Find the unknown impedance Z.

Solution:

The distance between two consecutive maxima $= \lambda/2 = 30$ cm i.e., $\lambda = 60$ cm. The phase constant on the line is

$$\beta = \frac{2\pi}{\lambda} = \frac{\pi}{30} \text{ rad/cm}$$

Refer Fig. 2.32. Draw VSWR = 3 circle and mark $V_{\min}$ point.
Since the distance between adjacent voltage maximum and minimum is $\lambda/4 =$ 15 cm, the distance of the voltage minimum from the load is

$$l_{\min} = 15 + 6 = 21 \text{ cm}$$

$$\beta l_{\min} = \frac{\pi}{30} \times 21 \text{ rad} = 126^\circ$$

Move on constant VSWR circle in anticlockwise direction to reach to point Z. Read the impedance value at Z.

$$Z = 50 \times (1.15 - j1.23) = 57.5 - j61.5\ \Omega$$

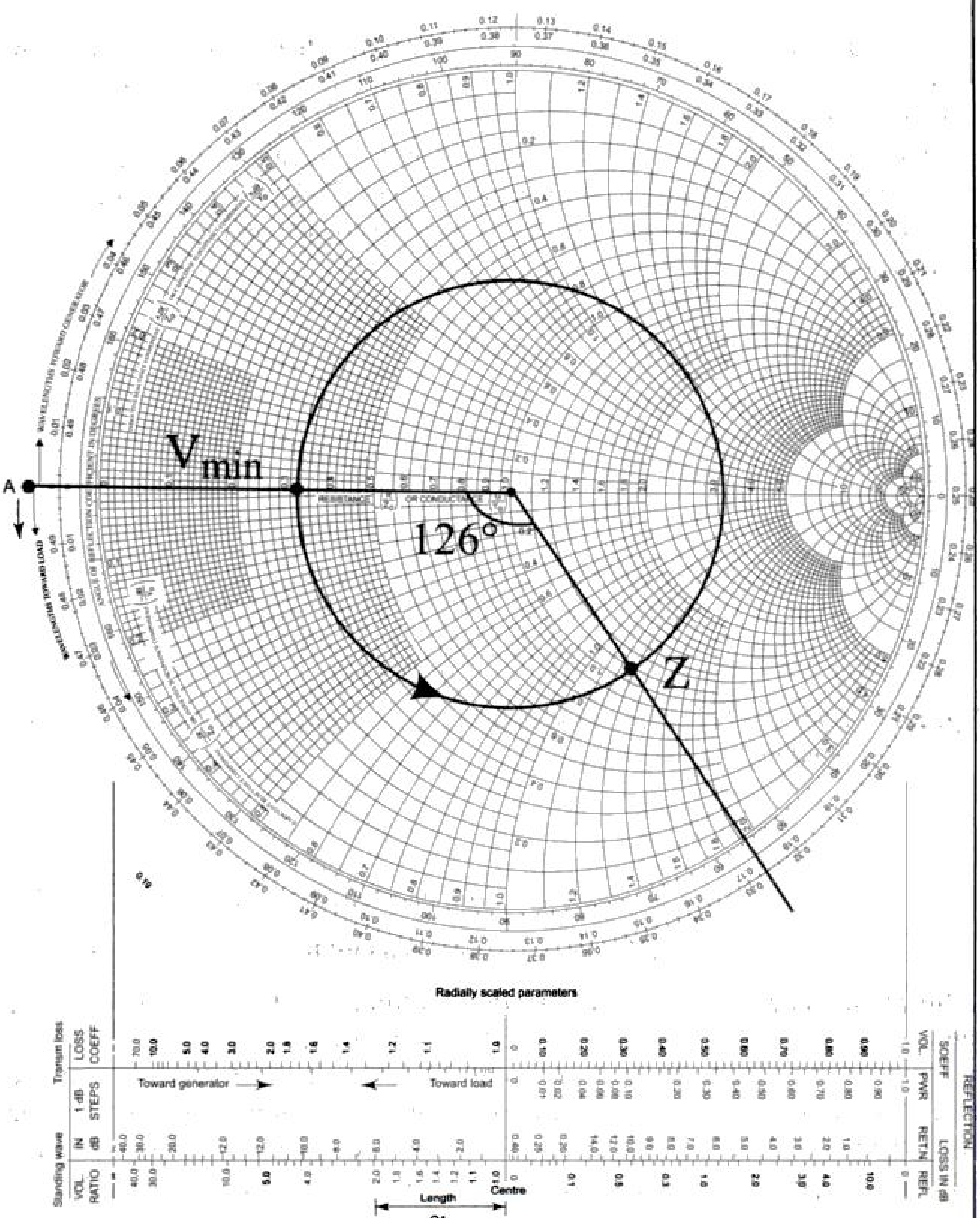


Fig. 2.32

2.12.2 Transmission Line as a Circuit Element

As the frequency increases, the realization of lumped reactive elements becomes more and more difficult. For example, if we wind a coil to get a certain inductance, the distributed capacitance of the coil may be so large that the coil, instead of showing inductive reactance may show capacitive reactance. Similarly, at times the lead inductance of a capacitor may be large enough to nullify the capacitive reactance.

At frequencies of hundreds and thousands of MHz where lumped elements are hard to realize, the use of sections of transmission line as reactive elements may be more convenient.

From the impedance relation we can see that if a line of length l is terminated in a short circuit or open circuit (see Fig. 2.33) the input impedance of the transmission line is purely reactive. The input impedance of a loss-less line can be written as

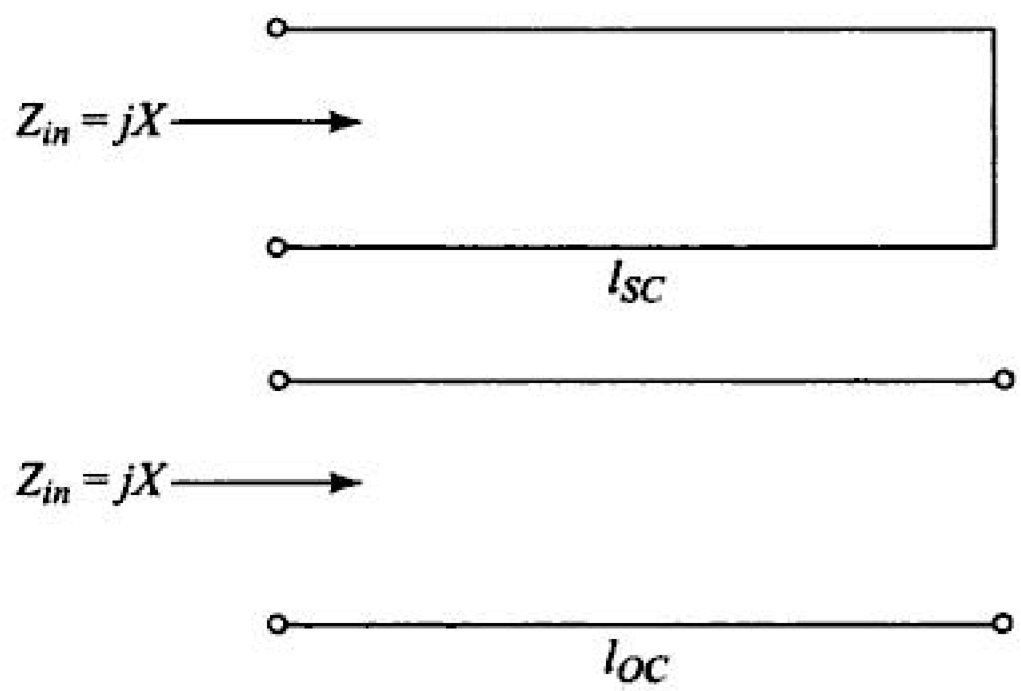


Fig. 2.33 *Short and open circuited section of a transmission line as a reactive element.*

$$Z_{\text{in}} = jZ_0 \tan \beta l \qquad \text{for short circuit load} \tag{2.143}$$

$$= -jZ_0 \cot \beta l \qquad \text{for open circuit load} \tag{2.144}$$

Since, the range of tan and cot functions is from $-\infty$ to $+\infty$, any reactance can be realized by proper choice of l. Moreover, any reactance can be realized by either open or short circuit termination. This is a very useful feature because depending upon the transmission line structure, terminating one way may be easier than other. For example, for a microstrip type line (see in later sections), realizing an open circuit is easier as short circuit would require drilling a hole in the substrate.

Now, if a reactance X is to be realized in a high frequency circuit one can use a short circuited line of length l_{sc} or an open circuited line of length l_{oc} where l_{sc} and l_{oc} can be obtained by inverting Eqns (2.143) and (2.144) to give

$$l_{sc} = \frac{1}{\beta} \tan^{-1}\left(\frac{X}{Z_0}\right) \tag{2.145}$$

$$l_{oc} = \frac{1}{\beta} \cot^{-1}\left(\frac{-X}{Z_0}\right) \tag{2.146}$$

Alternatively, one can use the Smith chart to find l_{sc} or l_{oc} as follows:

We know that the pure reactive impedances lie on the outermost circle of the Smith chart. Mark the reactance X to be realized on the Smith chart to get point 'X' in Fig. 2.34. Now, the length l_{sc} is the distance of the short circuit point from point X, away from the generator. If we, therefore, move in anticlockwise direction from point X to the short circuit (SC) point on the Smith chart we get l_{sc} (see Fig. 2.34). Similarly, l_{oc} is the distance of open circuit (OC) from X in the anticlockwise direction as indicated in Fig. 2.34. Note here that, instead of reactance if we had to realize a susceptance B, the procedure is identical except that SC and OC points are interchanged.

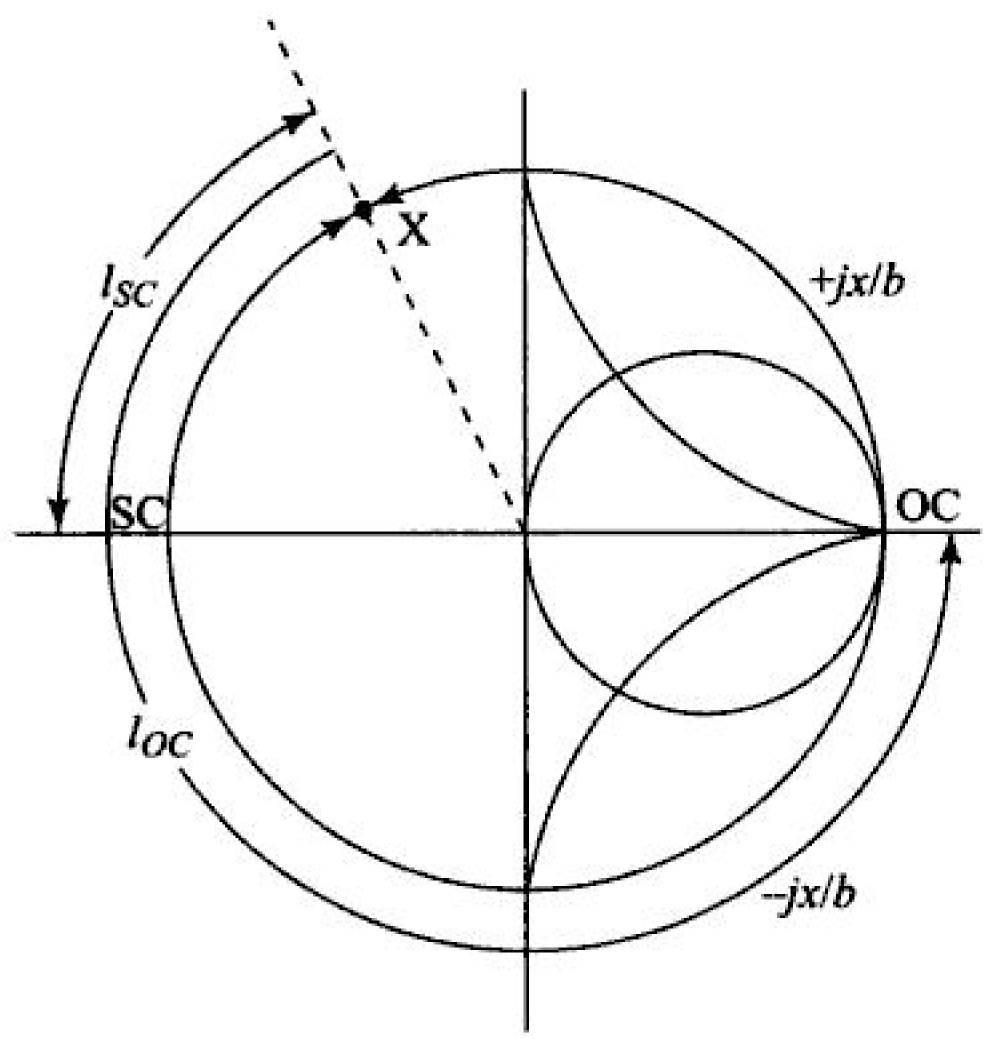
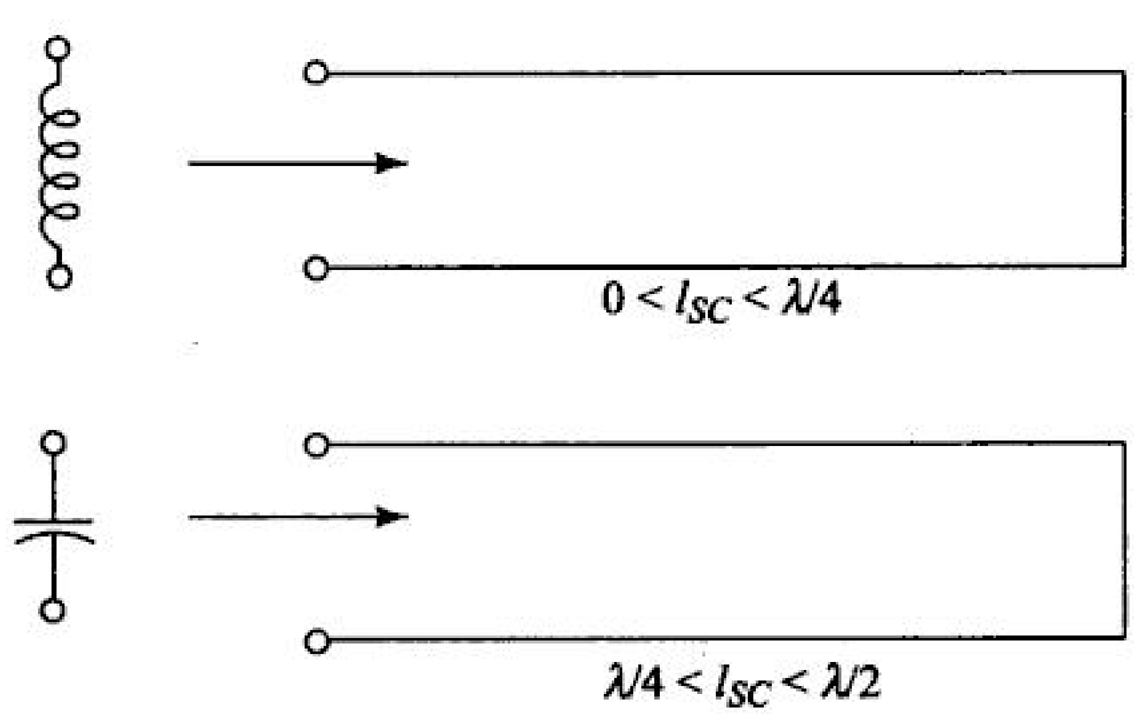


Fig. 2.34 *Calculation of length of a transmission line section for realizing a reactance using the Smith chart.*

Figure 2.35 shows the range of tranmission line lengths and the corresponding reactances which can be realized at the input terminals of the line.



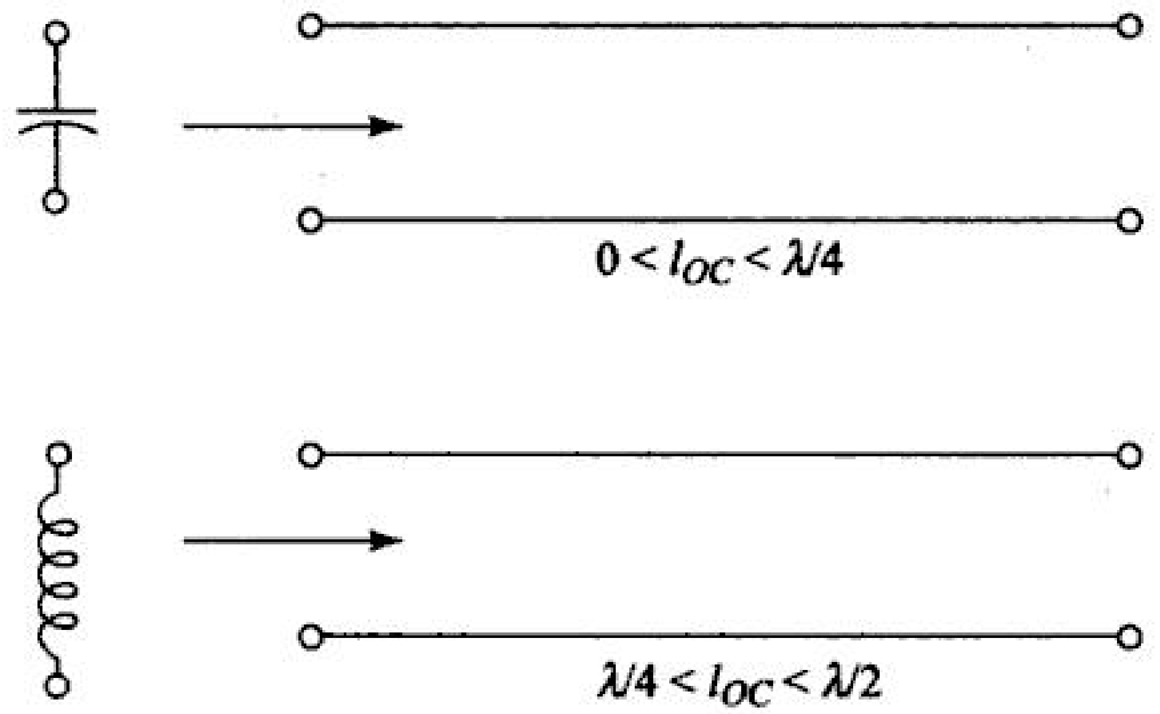


Fig. 2.35 *Transmission line sections for realizing inductance and capacitance at high frequencies.*

EXAMPLE 2.22 In a printed circuit, an inductance of 0.01 μH is to be realized at 6 GHz using a section of a transmission line. The wavelength of the signal on the PCB is 4 cm. Design the transmission line section as a reactive element.

Solution:

The reactance to be realized is

$$X = \omega L = 2\pi \times 6 \times 10^9 \times 0.01 \times 10^{-6} = 377 \ \Omega$$

Generally, in a PCB it is difficult to provide a short circuit since a hole has to be drilled through the substrate. So, it is convinient to use open circuited section of a transmission line. The length of line, therefore, is (see Eqn (2.146))

$$l_{oc} = \frac{1}{\beta}\cot^{-1}\left(\frac{-X}{Z_0}\right) = \frac{\lambda}{2\pi}\cot^{-1}\left(\frac{-X}{Z_0}\right)$$

Now, first we have to choose suitable Z_0 for the line. Let us take $Z_0 = 150\ \Omega$. (Choose Z_0 such that the ratio X/Z_0 is not too large. The $\cot x$-function is very steep around $x = 0, \pi/2$). We then get

$$l_{oc} = \frac{4.0}{2\pi}\cot^{-1}\left(\frac{-377}{150}\right) = 1.76 \ \text{cm}$$

2.12.3 Transmission Lines as Resonant Circuits

We have seen in the previous section that a short circuited line behaves like an inductor if $0 < l_{sc} < \lambda/4$, and it behaves like a capacitor if $\lambda/4 < l_{sc} < \lambda/2$. If the length is exact multiple of $\lambda/4$ the imput impedance of the line is zero or ∞. Let us plot the input impedance as a function of frequency 'f', for a given length of transmission l and a given termination (short circuit or open circuit).

Figure 2.36 shows the variation of reactance as a function of frequency for open and short circuited sections of a transmission line. It is clear that around frequencies f_1, f_2, f_3, f_4.., for which the length l is an integer multilple of $\lambda/4$, the impedance variation is identical to an L-C resonant circuit. The impedance characteristics of a series and a parallel resonant circuit are shown in Fig. 2.37.

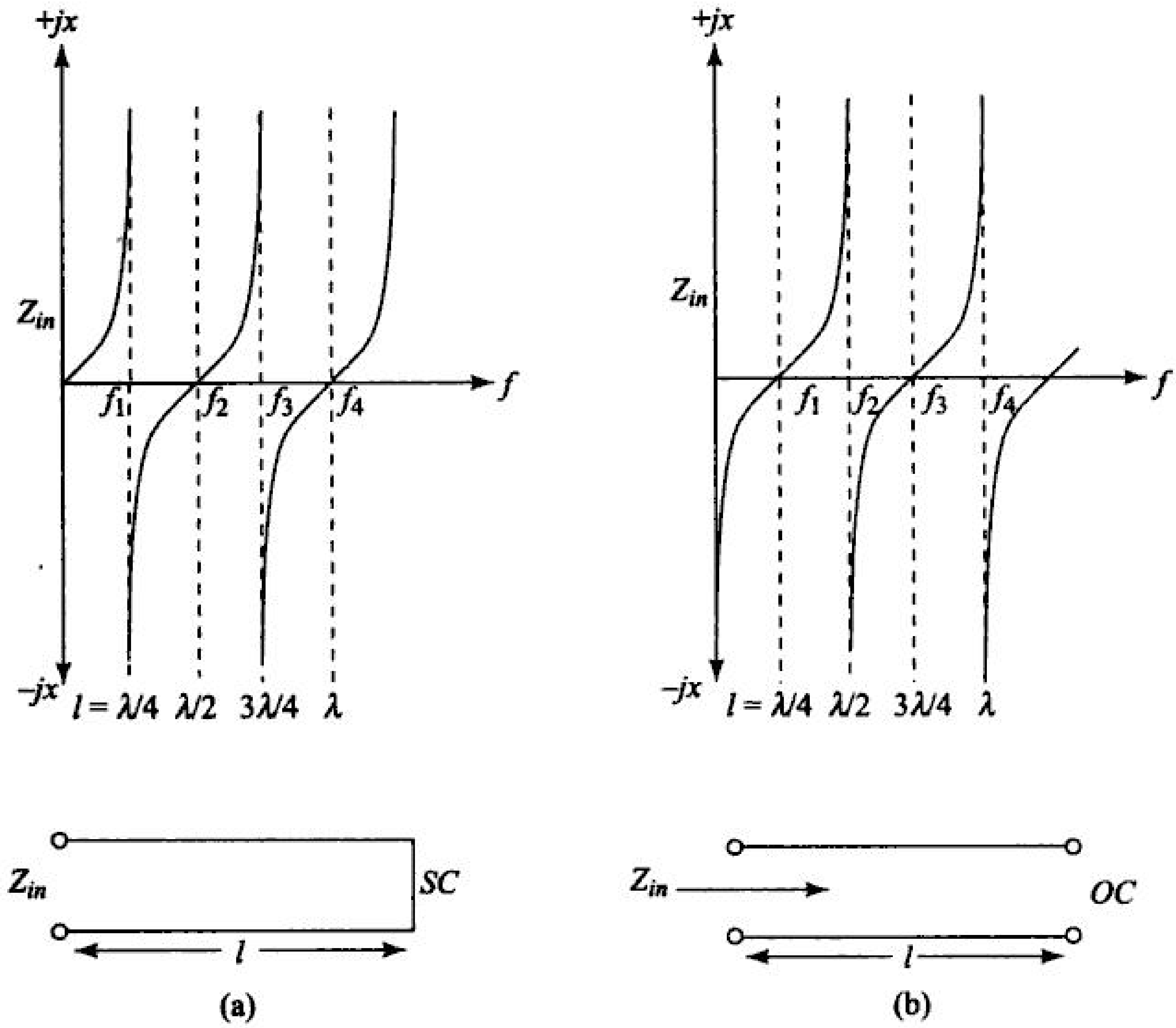
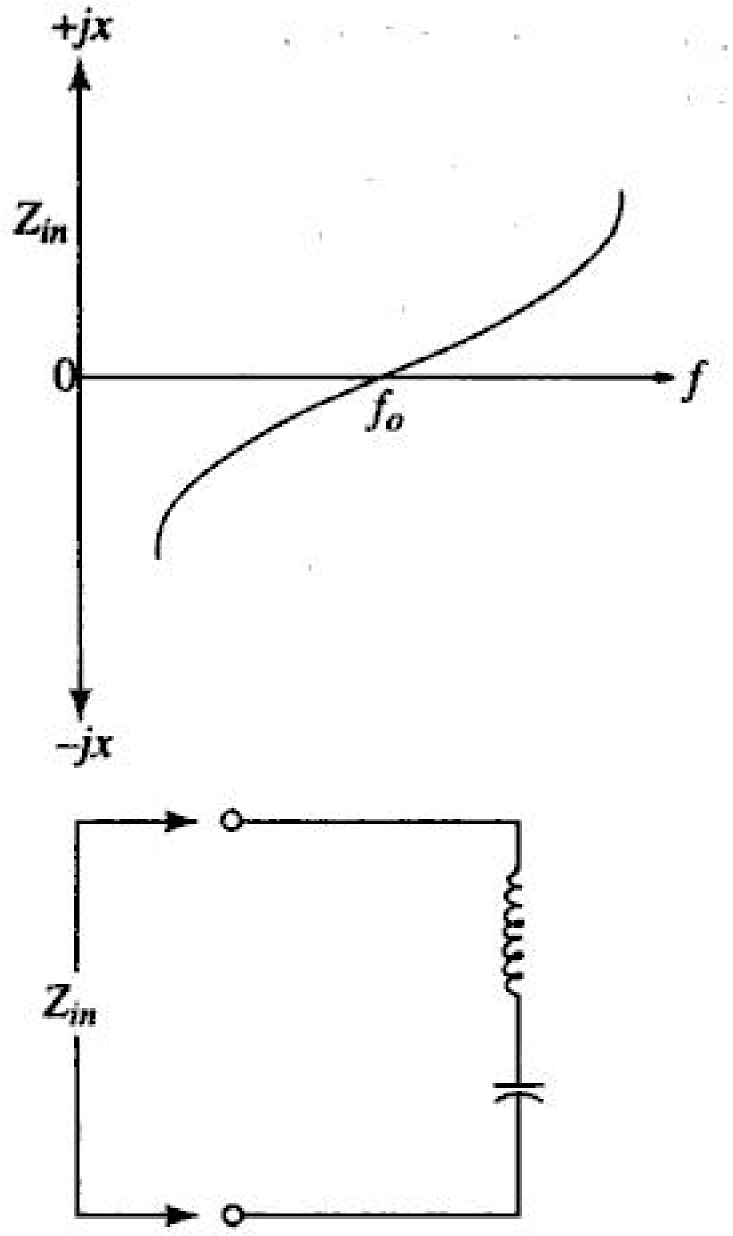


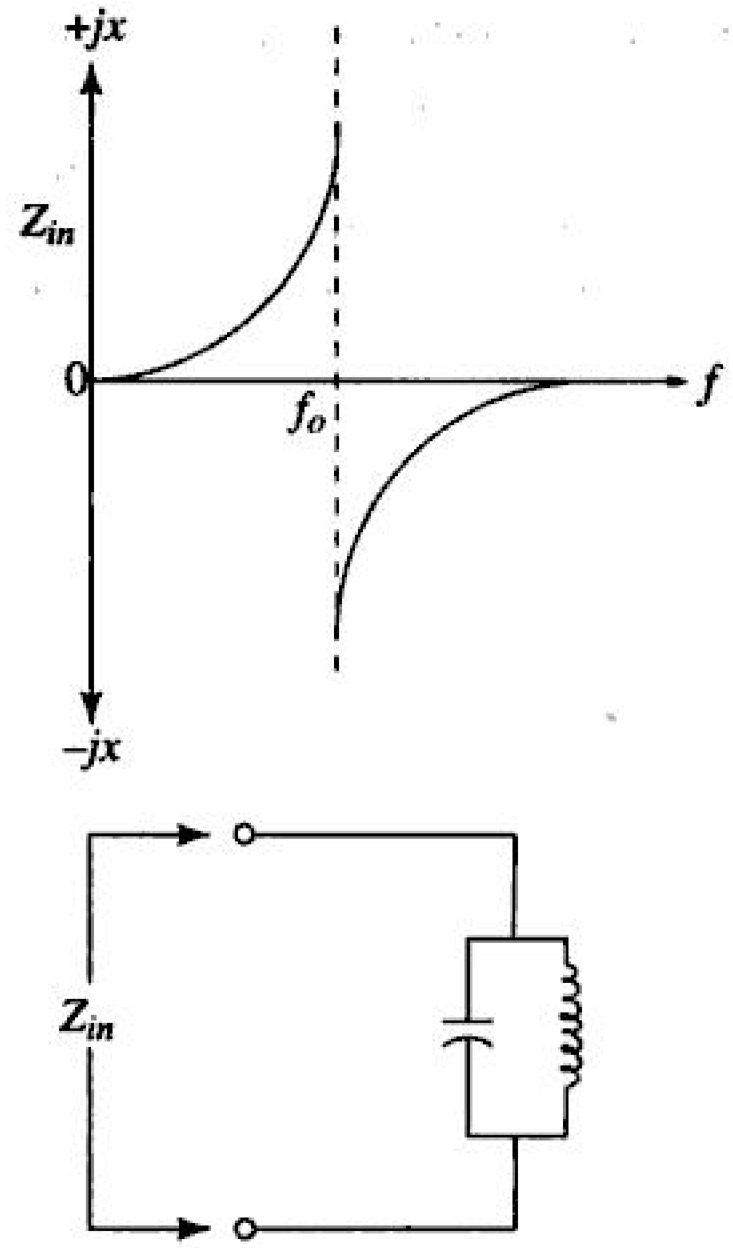
Fig. 2.36 *Variation of input impedance of a transmission line section as function of frequency.*

Comparing Fig. 2.36 with Fig. 2.37, one can observe that a short circuited line behaves like a parallel resonant circuit around frequencies f_1 and f_3, whereas around f_2 and f_4 its behavior is like a series resonant circuit. In general then we can say that a short circuited section of a line having length equal to odd multiples of $\lambda/4$ (i.e. $\lambda/4$, $3\lambda/4$, $5\lambda/4$, etc) is equivalent to a parallel resonant circuit. Similarly, if the length is equal to even multiples of $\lambda/4$ (i.e. $\lambda/2$, λ, $3\lambda/2$, etc.) the line is equivalent to a series resonant circuit. A converse is true for an open circuited section of a line i.e. if the length of the line is equal to odd multiples of $\lambda/4$, the line behaves like a series resonant circuit, and if the length of the line is equal to even multiple of $\lambda/4$, the line behaves like a parallel resonant circuit.

Since the section of transmission line having length equal to integer multiples of $\lambda/4$ is equivalent to a resonant circuit, it is worthwhile to ask what is its quality



Series resonant circuit
(a)



Parallel resonant circuit
(b)

Fig. 2.37 *Frequency response of series and parallel resonance circuit.*

factor Q. The Q of a LCR resonant circuit is defined as

$$Q = 2\pi \frac{\text{Energy stored in a circuit}}{\text{Enegy lost per cycle}} \tag{2.147}$$

As the Q is related to the loss in the circuit, the Q by definition is ∞ for an ideal loss-less line. However, for a low-loss line the Q is finite. In this particular case where one is explicitly investigating the loss, the assumption which we have been making so far that is, a low-loss line can be treated as loss-less line, is not justified. No matter how small the loss is, the propagation constant γ can not be taken to be purely imaginary. Taking γ to be complex, i.e. $\gamma = \alpha + j\beta$ ($\alpha << \beta$ for low loss lines) one has to re-investigate the input impedance of a resonant line. The input impedance of a short or open circuited line having propagation constant γ can be obtained from Eqn (2.45) as

$$Z_{sc} = Z_0 \tanh \gamma l \qquad \text{for short circuit} \tag{2.148}$$

$$Z_{oc} = Z_0 \coth \gamma l \qquad \text{for open circuit} \tag{2.149}$$

Note that although γ has been taken complex for a low-loss transmission line, Z_0 is almost real. Substituting for $\gamma = \alpha + j\beta$, we get

$$Z_{sc} = Z_0 \tanh(\alpha + j\beta)l = Z_0 \left[\frac{\tanh \alpha l + \tanh (j\beta)l}{1 + \tanh \alpha l \tanh (j\beta)l} \right] \tag{2.150}$$

For a low-loss line, taking $\alpha l << 1$, we have $\tanh \alpha l \approx \alpha l$. Also, $\tanh (j\beta)l = j \tan \beta l$. Hence, we get

$$Z_{sc} \approx Z_0 \left[\frac{\alpha l + j \tan \beta l}{1 + j\alpha l \tan \beta l} \right] \tag{2.151}$$

Similarly, for an open circuited line we get

$$Z_{oc} \approx Z_0 \left[\frac{1 + j\alpha l \tan \beta l}{\alpha l + j \tan \beta l} \right] \tag{2.152}$$

For resonant lines, l is integer multiples of $\lambda/4$ i.e. $\beta l (= 2\pi l/\lambda)$ is integer multiples of $\pi/2$. If, we take odd multiples of $\lambda/4$, $\tan \beta l = \infty$, and we get

$$Z_{sc} \approx \frac{Z_0}{\alpha l} \quad \text{Parallel resonance} \tag{2.153}$$

$$Z_{oc} \approx Z_0 \alpha l \quad \text{Series resonance} \tag{2.154}$$

On the other hand, if we take even multiples of $\lambda/4$, $\tan \beta l = 0$, giving

$$Z_{sc} \approx Z_0 \alpha l \quad \text{Series resonance} \tag{2.155}$$

$$Z_{oc} \approx \frac{Z_0}{\alpha l} \quad \text{Parallel resonance} \tag{2.156}$$

It is now clear that a short-circuited line having length equal to odd multiples of $\lambda/4$, and an open circuited line having length equal to even multiples of $\lambda/4$ is equivalent to a parallel resonant circuit. Similarly, a short-circuited line having length equal to even multiples of $\lambda/4$, and an open-circuited line of length equal to odd multiples of $\lambda/4$ is equivalent to a series resonant circuit. From Eqns (2.153) to (2.156) we conclude that a parallel resonant section of a line has an impedance $Z_0/\alpha l$ and a series resonant section has an impedance $Z_0 \alpha l$. One can cross-check the result with that of an ideal loss-less line. In the absence of any loss, the parallel resonant circuit shows infinite impedance and a series resonant circuit shows zero impedance at the resonance. The above result is indeed consistent with this.

For calculation of Q let us re-write Eqn (2.147) as

$$Q = 2\pi f_0 \frac{\text{Energy stored in the line}}{\text{Energy lost per second}} \tag{2.157}$$

Here, 'Energy lost per cycle' is written as 'Energy lost per second divided by the resonant frequency f_0'.

Let us now consider a short circuited section of a line having length equal to odd multiples of $\lambda/4$. This line is equivalent to a parallel resonant circuit. Let the line be applied with a voltage V_0 between its input terminals as shown in Fig. 2.38.

The voltage and current standing wave patterns on the line are shown in Fig. 2.38(b,c). The voltage is zero at the short-circuit-end of the line and is maximum at the input end of the line. Similarly, current is maximum at the short-circuit end and minimum at the input end of the line.

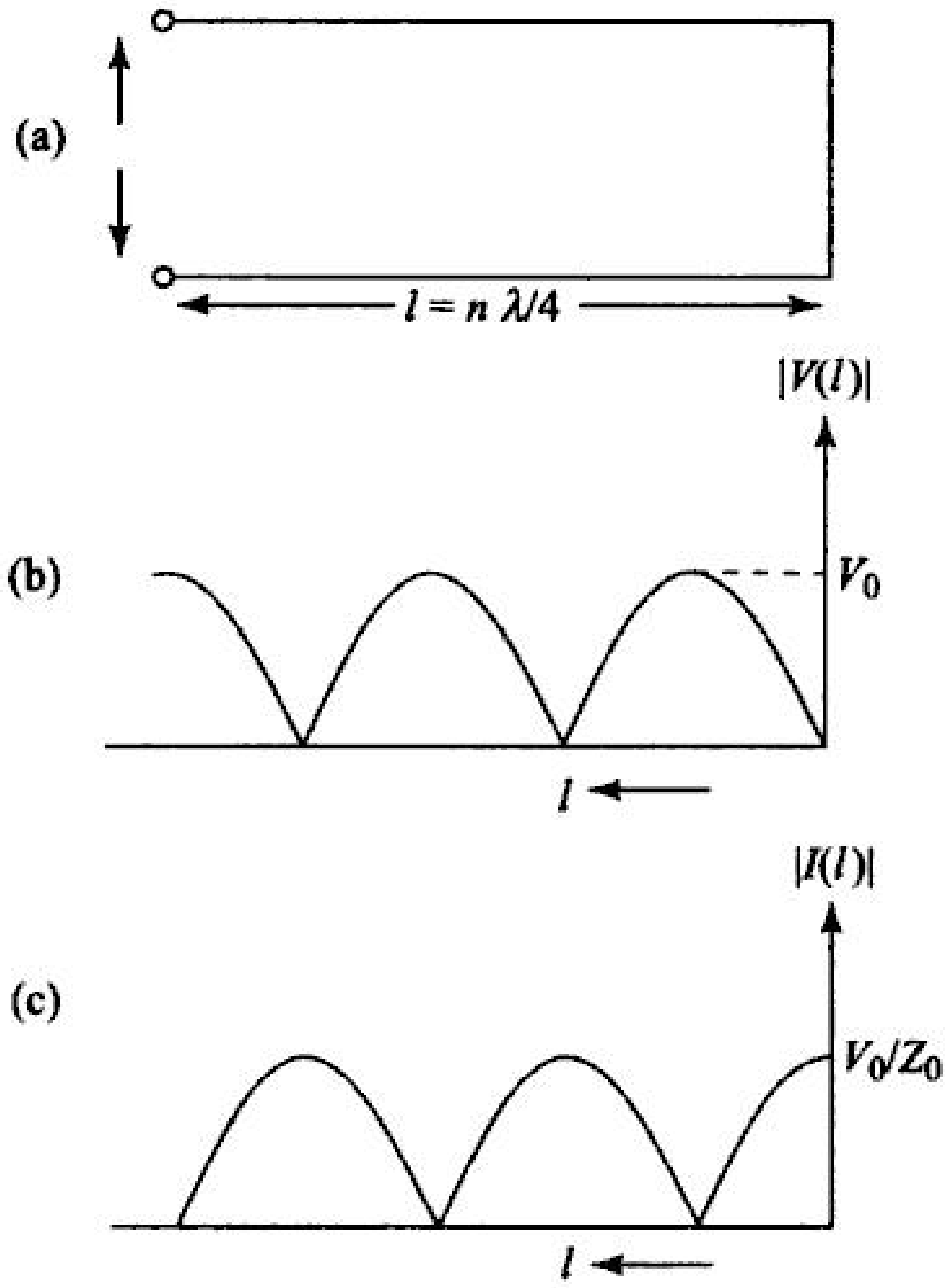


Fig. 2.38 *Voltage and current variation on a resonant section of transmission line.*

The maximum value of the voltage on the line is V_0 and maximum value of current is V_0/Z_0 (see Eqn (2.71)). For a short-circuited line the voltage and current on the line are given as

$$V(l) = |V_0 \sin \beta l| \tag{2.158}$$

$$I(l) = |\frac{V_0}{Z_0} \cos \beta l| \tag{2.159}$$

The energy stored in a $n\lambda/4$ long section of the line is

$$U = \frac{1}{2}C\int_0^{n\lambda/4} [V(l)]^2 dl + \frac{1}{2}L\int_0^{n\lambda/4} [I(l)]^2 dl \tag{2.160}$$

$$= \frac{1}{2}C\int_0^{n\lambda/4} [V_0 \sin \beta l]^2 dl + \frac{1}{2}L\int_0^{n\lambda/4} \left[\frac{V_0}{Z_0}\cos \beta l\right]^2 dl \tag{2.161}$$

$$= \frac{1}{4}CV_0^2\left(\frac{n\lambda}{4}\right) + \frac{1}{4}L\frac{V_0^2}{Z_0^2}\left(\frac{n\lambda}{4}\right) \tag{2.162}$$

Since $Z_0 = \sqrt{L/C}$, we have $L/Z_0^2 = C$. Substituting in Eqn (2.162) we note that the two terms on RHS of Eqn (2.162) are equal, i.e. the inductive and capacitive energies are equal, and the total energy is

$$U = \frac{1}{2} C V_0^2 \left(\frac{n\lambda}{4}\right) \tag{2.163}$$

The energy lost per second is nothing but the power loss in the line. At resonance the line effectively appears like a resistance of value $Z_0/\alpha l$ (see Eqn (2.153)). The power loss in the line therefore is ($l = n\lambda/4$)

$$P_{loss} = \frac{V_0^2}{(Z_0/\alpha l)} = \frac{V_0^2}{Z_0}.\alpha.\frac{n\lambda}{4} \tag{2.164}$$

Substituting from Eqns (2.163) and (2.164) in Eqn (2.157) we get the quality factor of the line as

$$Q = 2\pi f_0 \frac{Z_0 C}{2\alpha} \tag{2.165}$$

Again noting that $Z_0 C = (\sqrt{L/C})C = \sqrt{LC}$, and $2\pi f_0 = \omega$, the numerator in Eqn (2.165) is equal to $\omega\sqrt{LC}$, which is nothing but β (refer Eqn (2.59)). We therefore get

$$Q = \frac{\beta}{2\alpha} \tag{2.166}$$

One can note here that Q is independant of the length of the line as long as the loss is small. Equation (2.166) also suggests that to achieve high Q, α should be $<< \beta$. In practice generally the lines have loss low enough to give a Q of few hundred very easily. Since, the 3 dB-bandwidth of a resonant circuit is f_0/Q, higher value of Q implies highly tuned circuits. The transmission line sections therefore act as excellent frequency selective circuits at high frequencies.

EXAMPLE 2.23 A 75 Ω low-loss transmission line has a loss of 1.5 dB/m. The velocity of the voltage wave on the line is 2×10^8 m/sec. A section of the line is used to make a series resonant circuit at 1 GHz. Find the input impedance of the line, its quality factor and the 3 dB bandwidth of the resonant circuit.

Solution:

The wavelength on the line is

$$\lambda = \frac{v}{f} = \frac{2 \times 10^8}{10^9} = 0.2 \text{ m}$$

The phase constant is

$$\beta = \frac{2\pi}{\lambda} = 10\pi \text{ rad/m}$$

If we take open circuited section of the line, for series resonance, the length of the section would be

$$l_{oc} = \frac{\lambda}{4}, \frac{3\lambda}{4}....$$
$$= 0.05, 0.15, ...\text{m}$$

If we take short circuited section of the line, for series resonance, the length of the section would be

$$l_{sc} = \frac{\lambda}{2}, \frac{2\lambda}{2}....$$
$$= 0.1, 0.2, ...\text{m}$$

The loss of the line is $\alpha = 1.5$ dB/m $= 1.5/8.68 \doteq 0.173$ nepers/m
For series resonance, the input impedance is

$$Z_{in} = Z_0 \alpha l = 12.95\, l$$

where $l = l_{oc}$ or l_{sc}. The quality factor is

$$Q = \frac{\beta}{2\alpha} = \frac{10\,\pi}{2 \times 0.173} = 90.95$$

The 3-dB bandwidth is

$$BW = \frac{f_0}{Q} = \frac{1\text{ GHz}}{90.95} = 10.99 \quad \text{MHz}$$

2.12.4 Voltage or Current Step-up Transformer

In the previous application, we studied the impedance behavior of a resonant transmission line. Here, let us investigate the voltage and current that exist on a resonant section of a transmission line. Let us take a resonant transmission line of length $\lambda/4$. The line is open circuited at one end and short circuited at the other as shown in Fig. 2.39.

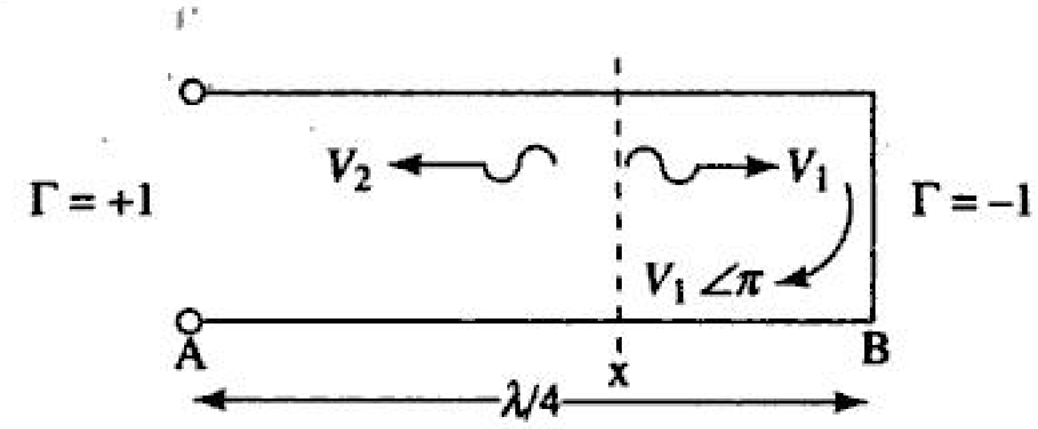


Fig. 2.39 *Growth of voltage waves on a resonant section of a transmission line. Transmission line as a step-up transformer.*

Let us say there is a voltage source which induces a voltage in the line at some point X. This induced voltage will send two voltage waves V_1 and V_2 with equal amplitudes. Consider now one of the waves, say V_1. This wave travels upto point B to encounter a short circuit. Since the reflection coefficient for a short circuit is

-1, the wave gets fully reflected with a phase reversal. The wave after travelling a distance BA reachs to the open circuited end of the line and again gets fully reflected but with no phase reversal as $\Gamma = +1$ for the open circuit. After one round trip, therefore, when the wave V_1 reaches point X, its amplitude is same as its original value but its phase is changed by 2π, π due to reflection at point B and π due to propagation of a round trip distance of $\lambda/2$. This wave, therefore, adds up with the induced voltage in phase and the added up wave travels on the transmission line. The process is regenerative and the amplitude of the voltage wave V_1 goes on increasing. Exactly identical thing happens with the other wave V_2. Since, the two waves travelling in the opposite directions identically grow in amplitude, the result is a continiously growing standing wave on the line with appropriate voltage maximum at A and voltage minimum at B. If the coupling of voltage is sustained, and the line is loss-less, there is no limit on the voltage and current and the voltage and current eventually would grow to ∞. However, if the line has a loss (no matter how small), then of course the voltage and current stabilize at some finite values. As the voltage/current increases, the ohmic loss also increases and when the power lost in the line just equals the power supplied by the coupling source, the voltage/current stabilizes. It should be noted, however, that the maximum stabilized voltage or current on the line could be much higher than the coupling voltage or current. This suggests a possibility of using a resonant transmission line as a step-up voltage or current transformer.

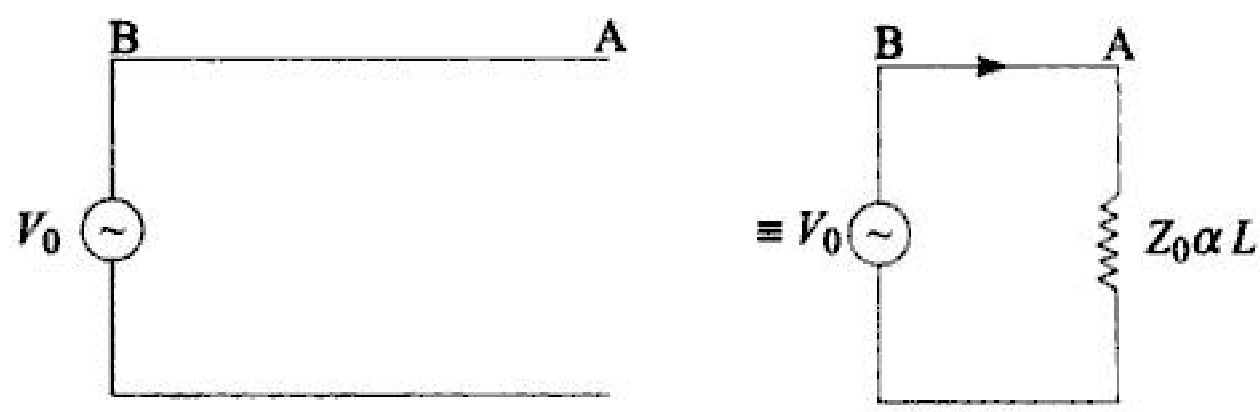


Fig. 2.40 *Resonant section of a low-loss transmission line.*

As an illustrative example let us take a resonant section of a line same as that in Fig. 2.39, and instead of putting a short circuit put an ideal voltage source at point B as shown in Fig. 2.40. The open circuit at point A appears as almost short (for a low loss line) at point B. The impedance seen by the voltage source is $Z_0\alpha l$ and a current $V_0/Z_0\alpha l$ flows in terminal B. Since point B is a voltage minimum and current maximum, the source current $V_0/Z_0\alpha l$ is equal to the maximum current on the line $I_{\max}$. The maximum voltage on the line then is $Z_0 I_{\max}$ and it appears at point A. We therefore have

$$V_A = Z_0 \frac{V_0}{Z_0\alpha l} = \frac{V_0}{\alpha l} \tag{2.167}$$

Since, $\alpha l << 1$ for a low-loss line, we get $V_A >> V_0$. That is, the voltage at the open-circuited end of the resonant line is much higher compared to the excitation voltage V_0. A resonant section of a transmission line therefore can be used as a step-up transformer. It should be kept in mind, however, that a large step-up ratio is possible only in the no-load condition. Any loading at point A reduces the voltage V_{max}, i.e. the voltage amplification ratio.

The voltage step-up ratio is

$$\frac{V_A}{V_0} = \frac{1}{\alpha l} \tag{2.168}$$

Taking $l = n\lambda/4$ (where n is an odd integer), and substituting $\lambda = 2\pi/\beta$, the voltage step-up ratio can be written as

$$\frac{V_A}{V_0} = \frac{2\beta}{n\pi\alpha} \tag{2.169}$$

$$\Rightarrow \quad \text{Volatge Amplification Ratio} = \frac{4Q}{n\pi} \tag{2.170}$$

Since, Q is typically few hundreds for a low-loss transmission line, a voltage amplification of few hundreds may be expected in a resonant line.

2.13 IMPEDANCE MATCHING USING TRANSMISSION LINES

While discussing characteristics of a transmission line, we have seen that if a line is terminated in the characteristic impedance Z_0, the impedance at every point on the line becomes Z_0. Also, in this case there is no reflection on the line and the power is maximally transferred to the load. In practice, however, it is not always possible to design a circuit whose input or output impedance is matched to the adjacent circuits. For example, there may be a circuit which is to be connected to a signal generator of 50 Ω output impedance but the input impedance of the circuit is not 50 Ω. When this connection is made, not only the maximum power is not being transferred by the generator to the load, but the reflected wave might enter the generator and may alter its characteristics like frequency, etc. It is, therefore, essential to devise a technique which can avoid reflections from the circuits.

Transmission lines can be used for matching two impedances. Due to low-loss the transmission line provides impedance matching with negligible loss of power. In the following sections we discuss various impedance matching techniques using transmission line sections.

2.13.1 Quarter-Wavelength Transformer

This technique is generally used for matching two resistive loads, or for matching a resistive load to a transmission line, or for matching two transmission lines with unequal characteristic impedances (see Fig. 2.41). All cases are identical in principle as all require matching between two purely resistive impedances.

The principle here is very simple. We introduce a section of a transmission line (transformer) between two resistances to be matched, such that the transformed impedances perfectly match at either end of the transformer section. That is,

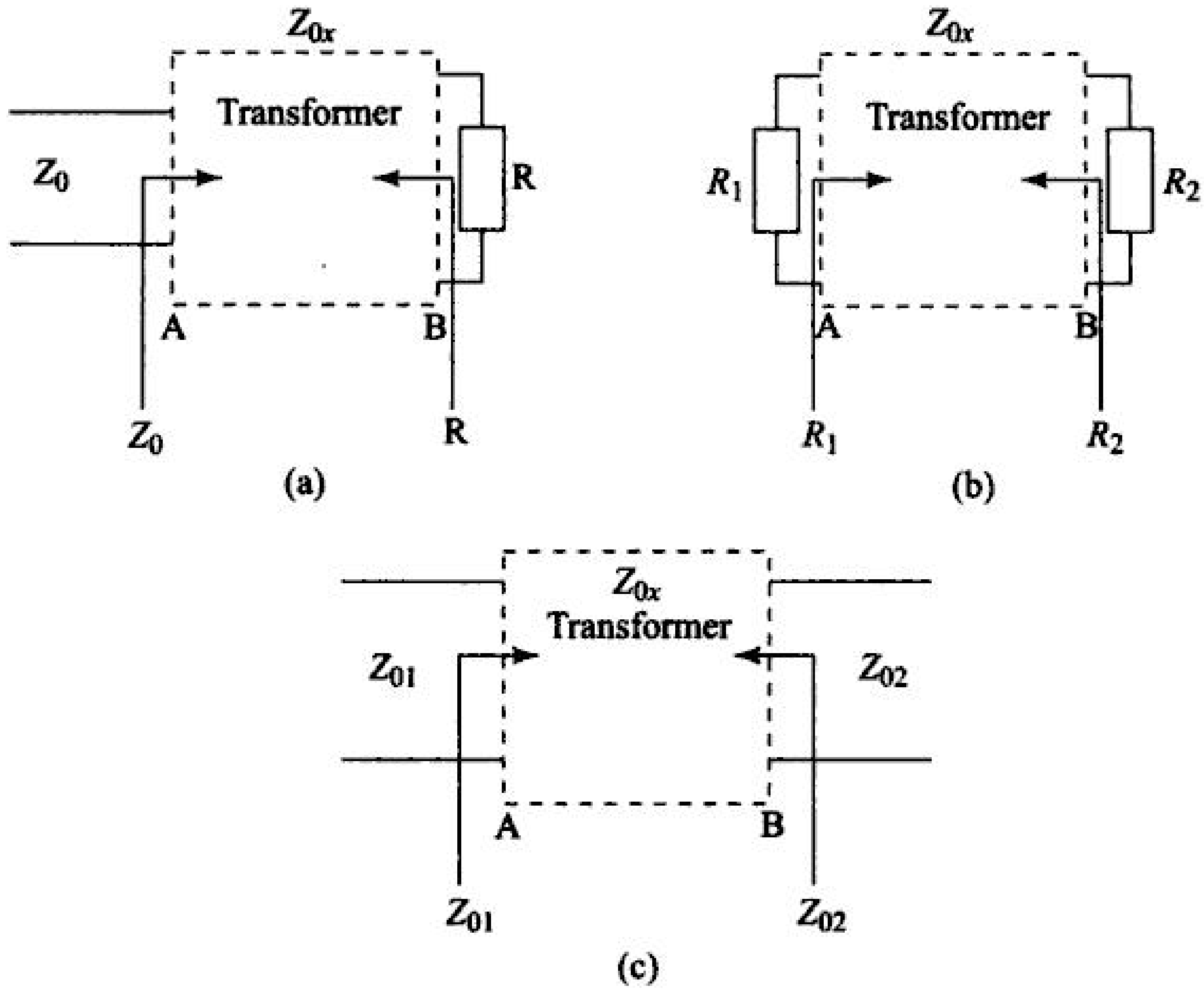


Fig. 2.41 *Impedance matching using quarter wavelength transformer. (a) Matching of a resistance to a line (b) Matching of two resistances (c) Matching between two long lines.*

in Fig. 2.41(a) say, the impedance seen towards right at A appears to be Z_0, and impedance seen towards left at B appears to be R. So, when seen from transmission line side it appears to be terminated in Z_0, and when seen from load resistance side it appears to be connected to a conjugately matched load R. Similar is true for Figs 2.41(b,c).

Since, pure resistances are to be matched here, the impedance transformation in the transformer has to be from a resistive impedance to a resistive impedance. This is possible only in the following two cases.

(a) when length of the transforming line is $\lambda/2$
(b) when the length of the transforming line is $\lambda/4$

A $\lambda/2$ section of a transmission line transforms an impedance into itself and hence does not serve any purpose for matching. So, the only possibility is that the transformer must be $\lambda/4$ long.

Let us assume that the characteristic impedance of the transformer section is Z_{0x}. For $\lambda/4$ length, the transformer inverts the normalized impedance (see Eqn (2.89)). Therefore, the impedance seen at A towards right in Fig. 2.41(a) would be

$$Z_A = \frac{1}{(R/Z_{0x})}.Z_{0x} = \frac{Z_{0x}^2}{R} \tag{2.171}$$

For matching at A, Z_A should be equal to Z_0, i.e.

$$\frac{Z_{0x}^2}{R} = Z_0 \tag{2.172}$$

$$\Rightarrow Z_{0x} = \sqrt{RZ_0} \tag{2.173}$$

For Fig. 2.41(b) $Z_{0x} = \sqrt{R_1 R_2}$ and for Fig. 2.41(c), $Z_{0x} = \sqrt{Z_{01} Z_{02}}$.

So, in general we can say that two resistive impedances can be matched by a section of a transmission line which is quarter-wavelength long and has characteristic impedance equal to the geometric mean of the two resistances.

In first look, it appears from the above discussion that a quarter wavelength transformer can be used to match only purely resistive impedances. However, if we see carefully, we find that it is not true. This is due to the fact that we can always transform a complex impedance into a purely real one by adding an appropriate section of a transmission line. Let us consider matching of a complex impedance $R + jX$ to the characteristic impedance of a transmission line Z_0 . To transform impedance $R + jX$ to some real value let an extra length 'L' of a transmission line be added between the quarter wavelength transformer and the impedance as shown in Fig. 2.42. The characteristic impedance of the extra length of the line be say Z_{01} (one can use a line of characteristic impedance Z_0 as well). The length L should be chosen such that the transformed impedance Z' seen towards right at B is purely real. This can be done easily by using the Smith chart.

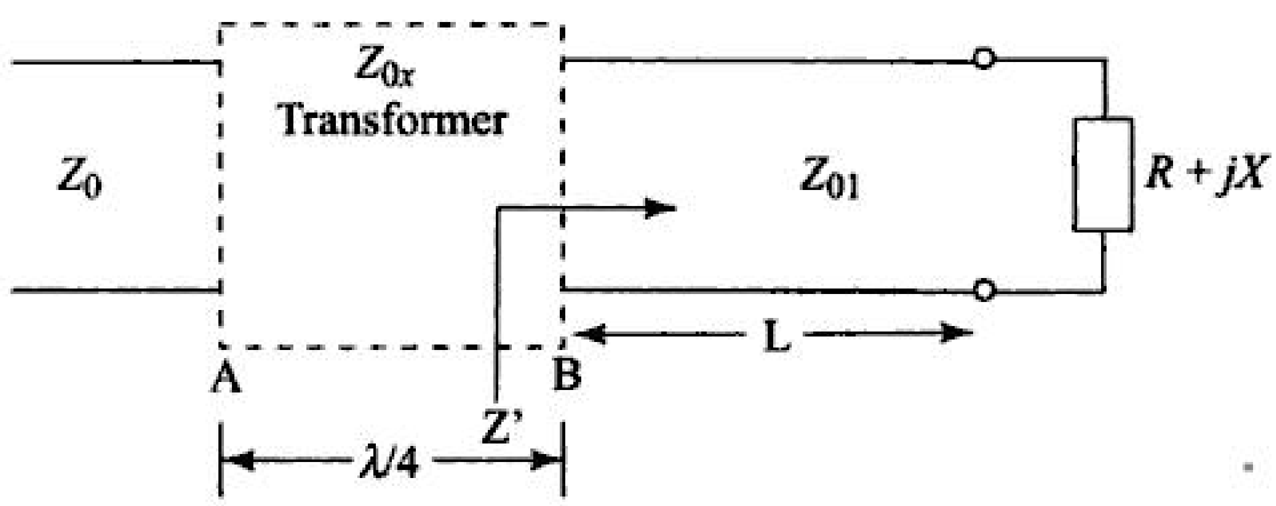


Fig. 2.42 *Matching of a complex impedance to a line using quarter wavelength transformer.*

First the impedance $Z = R + jX$ is normalized with respect to Z_{01} to give $\bar{Z} = (R + jX)/Z_{01} \equiv r + jx$. Let this point be denoted by P (see Fig. 2.43). Draw the constant VSWR circle through P. The circle will intersect the real axis at points S and T. These points represent location on the transmission line where the impedance is purely resistive. At point T the normalized impedance is r_{max} and at point S the normalized impedance is r_{min}.

If we take $L = L_{max}$ then impedance $Z' = Z_{01} r_{max}$ and then for matching we get

$$Z_{0x} = \sqrt{Z_0 Z_{01} r_{max}} \tag{2.174}$$

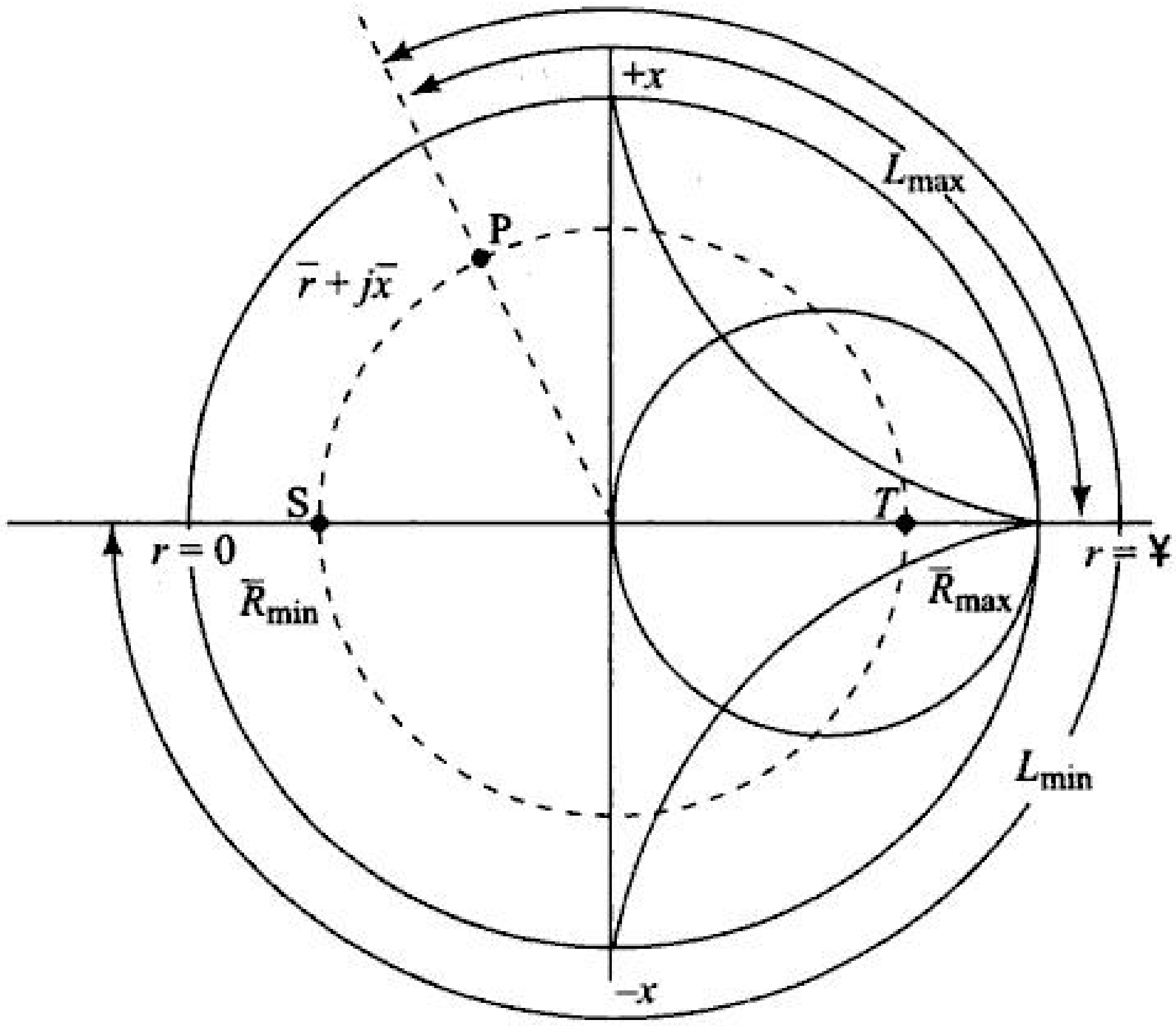


Fig. 2.43 *Matching of complex impedance using quarter wavelength transformer using the Smith chart.*

Similarly, if we take $L = L_{\min}$ then $Z' = Z_{01}r_{\min}$ and we get

$$Z_{0x} = \sqrt{Z_0 Z_{01} r_{\min}} \tag{2.175}$$

Theoretically, both solutions are equally acceptable and only their numerical values will decide which is practically more realizable.

The quarter-wavelength transformer although can match any impedance, has a very serious draw-back. For every impedance to be matched, one needs a line with different Z_{0x}. Since the characteristic impedance of a line is decided by the physical structure of the line like conductor size, dielectric constant, separation between conductors etc. for every impedance Z_{0x}, one would need a special transmission line. Realizing a line with a particular Z_{0x} may not be always possible in practice.

To overcome this drawback, the stub-matching techniques have been proposed. These techniques make use of the standard tranmission line sections for matching arbitrary impedances.

EXAMPLE 2.24 Two very long loss-less cables of characteristic impedances 50Ω and 100Ω respectively are to be joined for reflection-less transmission. Find the suitable matching transformer.

Solution:

The quarter wavelength transformer is appropriate for this case as we have to match two real impedances. The characteristic impedance of the tranformer section is

$$Z_{0x} = \sqrt{50 \times 100} = 70.7\ \Omega$$

The length of the transformer should be odd multiples of $\lambda/4$.

EXAMPLE 2.25 A load impedance $75 - j35\Omega$ is to be matched to 50Ω using quarter wave transformer. Design the matching set-up.

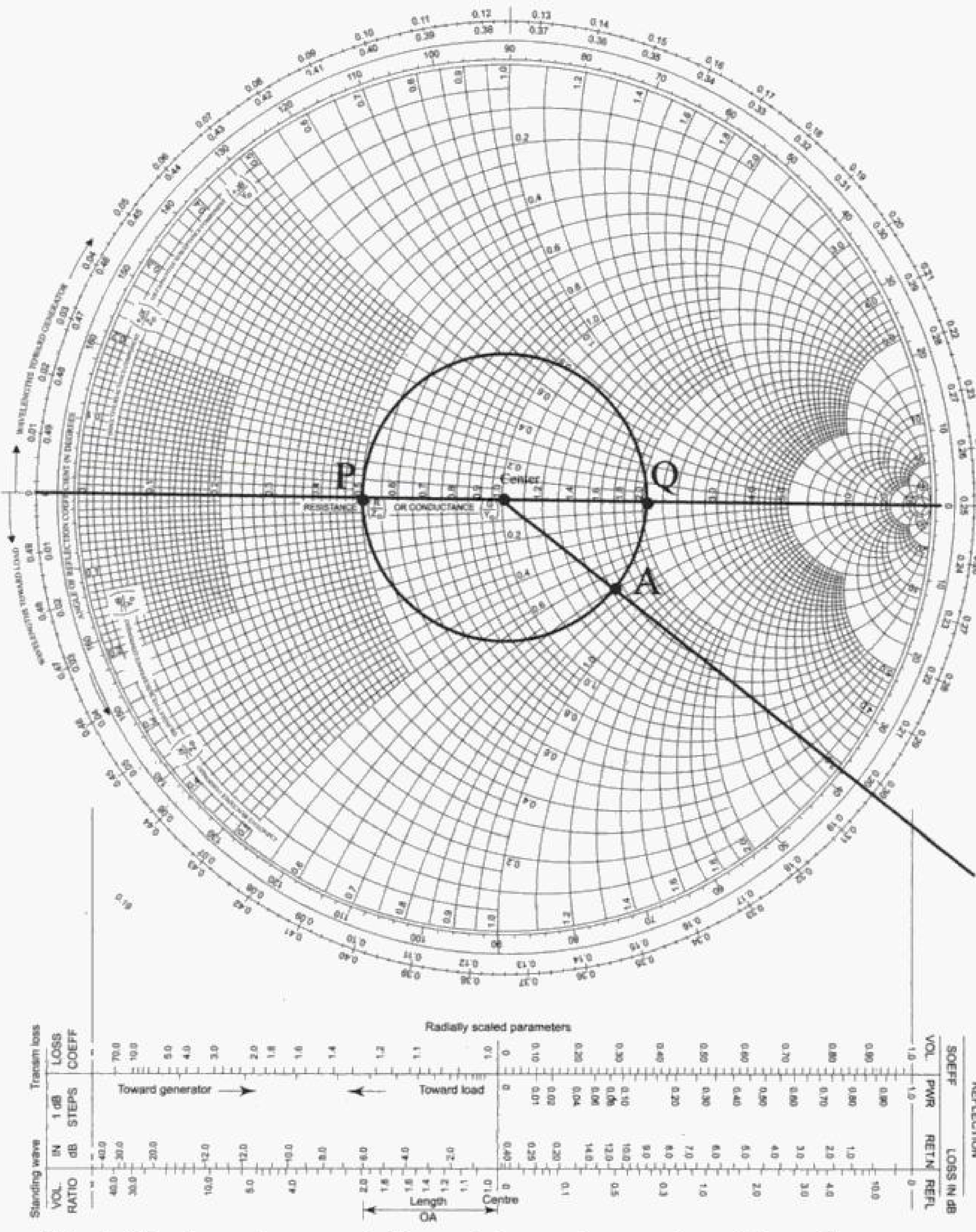


Fig. 2.44 *Impedance matching using quarter wavelength transformer*

Solution:

The normalized load impedance is

$$\overline{Z} = \frac{75 - j35}{50} = 1.5 - j0.7$$

Since, the quarter wave transformer matches real impedances, first connect a cable between the transformer and the load such that the transformer sees a real impedance. Let the cable used be having 50 Ω characteristic impedance. We can find the length of the cable, l, with the help of the Smith chart. The length l would correspond to arc AP or AQ measured clockwise (see Fig. 2.44).

If we take l corresponding to arc AP, we get $Z' = R_P = 25\ \Omega$ and the transformer characteristic impedance will be $Z_{0x} = \sqrt{50R_P} = 35.34\ \Omega$.

If, we, however, take l corresponding to arc AQ, we get $Z' = R_Q = 100\ \Omega$ and the transformer characteristic impedance will be $Z_{0x} = \sqrt{50R_Q} = 70.7\ \Omega$.

Both solutions are valid solutions and only other practical constraints may favour one over the other.

2.13.2 Single-Stub Matching Technique

A stub is a short-circuited section of a transmission line connected in parallel to the main transmission line. A stub of appropriate length is placed at some distance from the load such that the impedance seen beyond the stub is equal to the characteristic impedance.

Suppose, we have a load impedance Z_L connected to a transmission line with characteristic impedance Z_0 (Fig. 2.45). The objective here is that no reflection should be seen by the generator. In other words, even if there are standing waves in the vicinity of the load Z_L, the standing waves must vanish beyond certain distance from the load. Conceptually, this can be achieved by adding a stub to the main line such that the reflected wave from the short-circuit end of the stub and the reflected wave from the load on the main line completely cancel each other at point B to give no net reflected wave beyond point B towards the generator. In terms of impedances, the philosophy of the single stub matching is as follows:

We note that, any movement along the transmission line changes both the real and imaginary parts of the impedance seen on the line. Therefore, first make the real part of the impedance equal to the characteristic impedance by moving along the line. At that location add a reactance to nullify the reactance of the transformed impedance.

In the first look, manipulation of reflected waves might appear a tedious task and infact if one tries to solve the problem by analytical means it is a rather tedious task. However, with the help of the Smith chart the problem can be solved very easily.

As the Smith chart always uses normalized impedances, let us normalize the impedance Z with Z_0 to give $\overline{Z} = r + jx$. In terms of normalized impedances Fig. 2.45 can be re-drawn as Fig. 2.46.

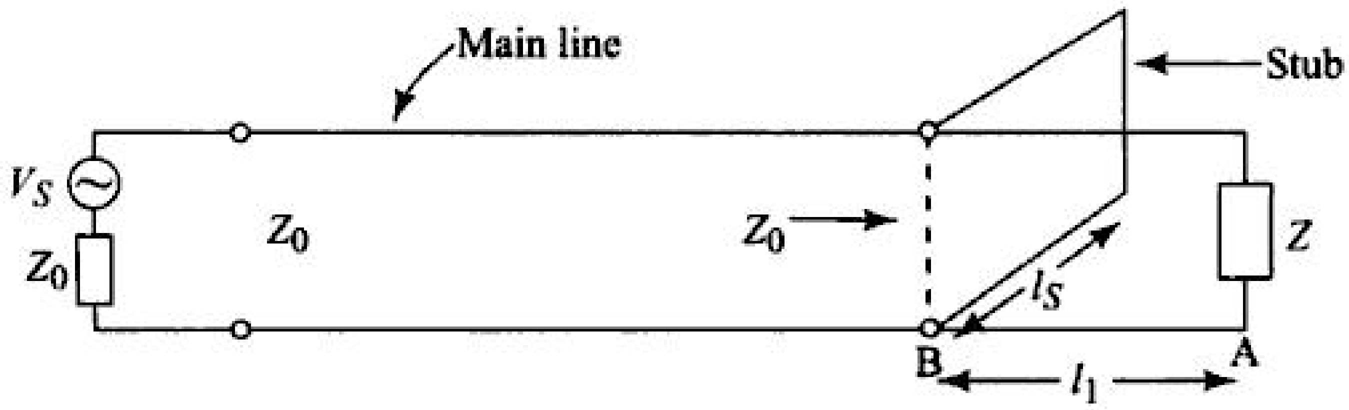


Fig. 2.45 *A stub connected to the main transmission line for matching the load.*

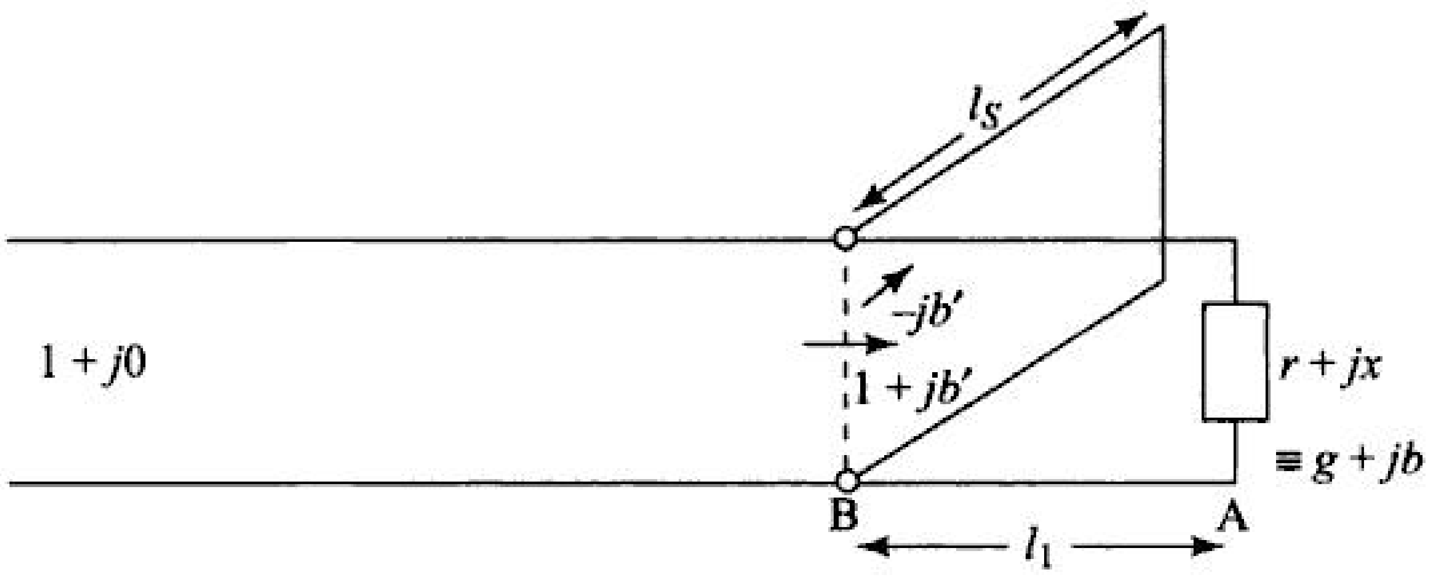


Fig. 2.46 *Main transmission line and the stub with normalized admittances.*

Since, we are having a parallel connection of transmission lines, it is more convenient to solve the problem using admittances rather than impedances. To convert the impedance into admittance also we make use of the Smith chart and avoid any analytical calculation.

First, taking the Smith chart as the impedance chart mark the normalized load impedance $r + jx$ as point P (Fig. 2.47) and draw the constant VSWR circle through P. A diagonally opposite point A then represents the normalized load admittance $g + jb$ corresponding to the load impedance $r + jx$. The admittance seen at point A (Fig. 2.47) is therefore $Y_A = g + jb$. Now onwards treat the Smith chart as the admittance chart.

We can note in Fig. 2.47 that the constant VSWR circle intersects the g = 1 circle at two points B and D. This means, if we move along the transmission line by a distance corresponding to arc AB or AD in the clockwise direction, the real part of the transformed admittance would become unity. Let us choose point B. The arc AB corresponds to the distance of the stub from the load, l_1. Let the admittance at B be $Y_B = 1 + jb'$ (say). Now if the stub susceptance at B is $-jb'$, the total admittance at B would be $1 + j0$.

Finding the length of a short circuited line which would give a desired susceptance (reactance) at its input terminals, has already been discussed in the Section 2.12.2. Mark the $-jb'$ point on the Smith chart say Q. Then on the stub if we move away from the generator we should reach the short circuit point. The arc QS in the anticlockwise direction therefore provides the length of the stub, l_s.

One may note that, instead of point B if we had chosen point D, the transformed admittance would have been $Y_D = 1 - jb'$. Then the susceptance of the stub would have been $+jb'$ and the stub length would have been $(\lambda/2 - l_s)$.

The admittance flow diagram is shown by arrows in Fig. 2.47. The admittance moves from A to B and then to M (center of the chart).

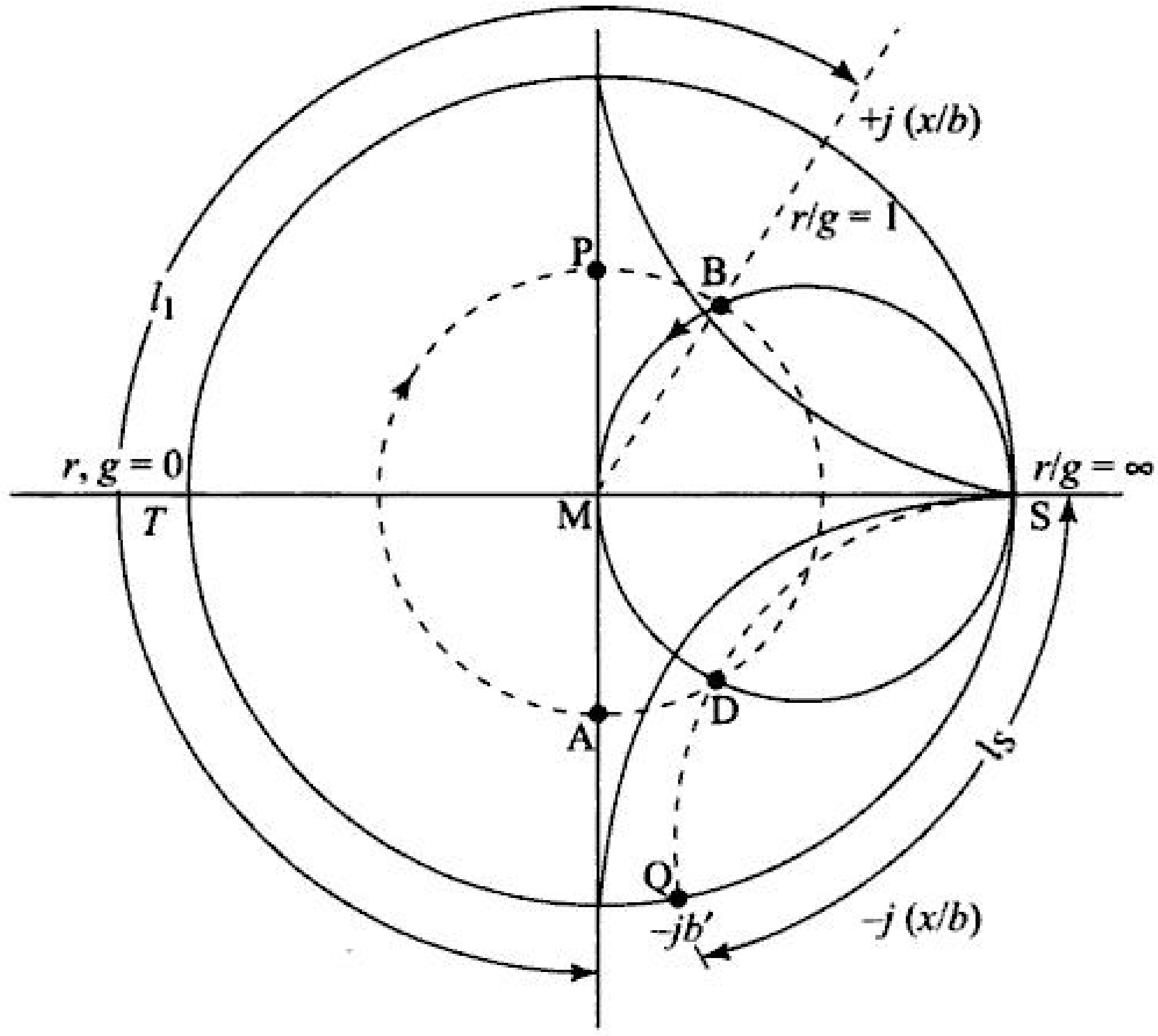


Fig. 2.47 *Single-stub matching using the Smith chart.*

The single-stub matching technique is superior to the quarter wavelength transformer as it makes use of only one type of transmission line for the main line as well as the stub. This technique also in principle is capable of matching any complex load to the characteristic impedance/admittance. The single stub matching technique is quite popular in matching fixed impedances at microwave frequencies.

The single stub matching technique although has overcome the drawback of the earlier technique, it still is not suitable for matching variable impedances. A change in load impedance results in a change in the length as well as the location of the stub. Even if changing length of a stub is a simpler task, changing the location of a stub may not be easy in certain transmission line configurations. For example, if the transmission line is a co-axial cable, the connection of a stub would need drilling of a hole in the outer conductor.

For a variable impedance, the matching is achieved using the double-stub matching technique.

EXAMPLE 2.26 A load impedance $90 - j25$ is to be matched to 50Ω using single stub matching. Find the length and location of stub.

Solution:

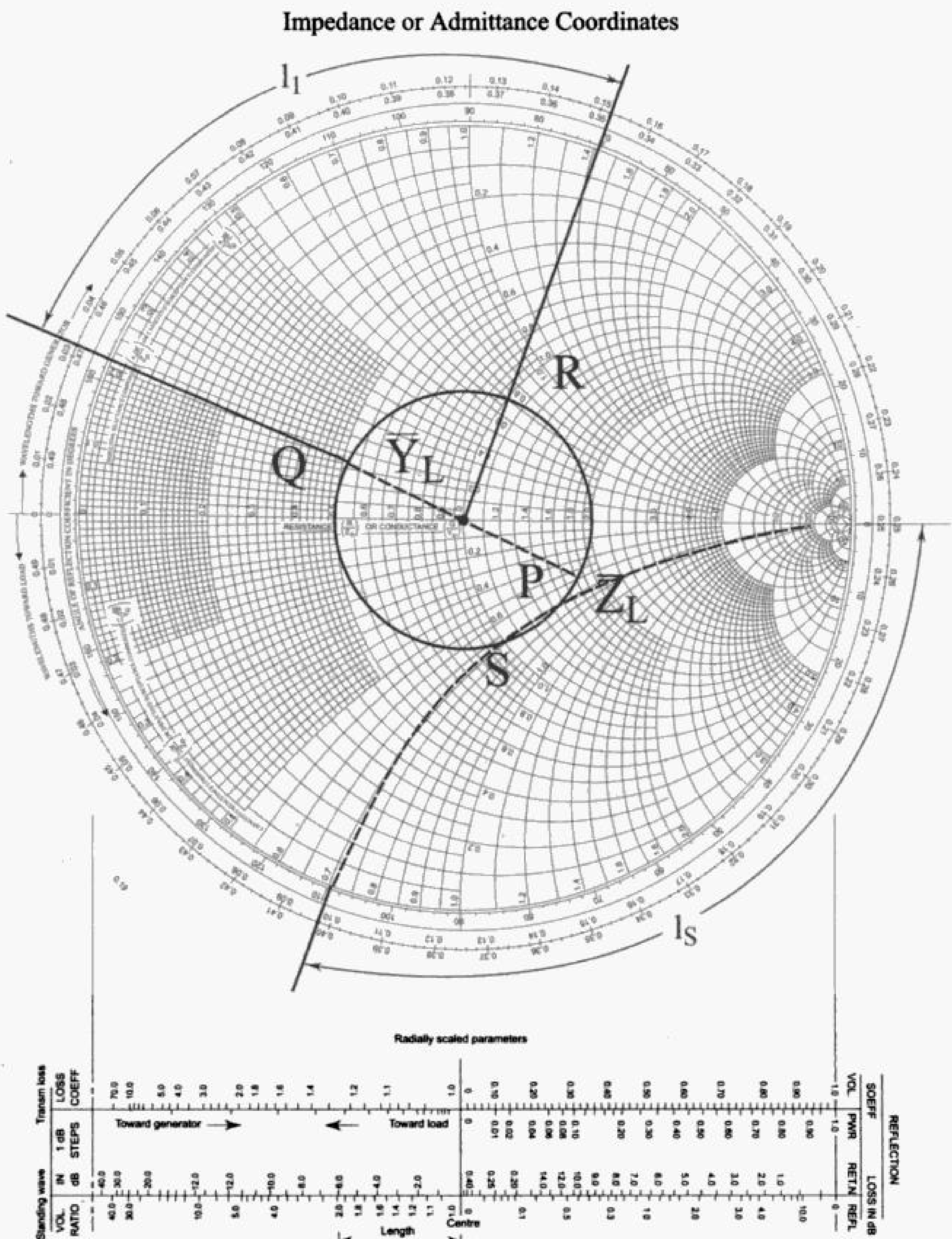


Fig. 2.48 *Single stub matching using the Smith chart.*

$$\bar{Z}_L = \frac{90 - j25}{50} = 1.8 - j0.5$$

Refer to Fig. 2.48 for length and location of the stub.

2.13.3 Double-Stub Matching Technique

To overcome the drawbacks of the single-stub matching technique, the double-stub matching technique is employed. The technique uses two stubs with fixed locations. As the load changes only the lengths of the stubs are adjusted to achieve matching.

Let us assume that a normalized admittance $g + jb$ is to be matched using the double stub matching technique. The first stub is located at a convenient distance from the load say l_1 (Fig. 2.49). The second stub is located at a distance of $3\lambda/8$ from the first stub. Although there is nothing unique about this distance of $3\lambda/8$, we will see later that it has certain advantages. The impedance matching philosophy can be understood as follows:

We have already seen in the single-stub matching technique that at the location where a stub is connected if the admittance seen towards the load is of the form $1 + jb$ then the stub can cancel the reactive part of the admittance and we can get matching. In the present case, if the admittance at point C is of the form $1 + jb$, the matching can be achieved. This means at location B the admittance must be a transformed version of $1 + jb$ by a distance $3\lambda/8$ away from the generator. Since the admittance $1 + jb$ is represented by $g = 1$ constant conductance circle, the impedance at B must lie on a circle which is generated by rotating every point on the $g = 1$ circle by $270^\circ = \beta\frac{3\lambda}{8}$ around the center of the Smith chart in the anticlockwise direction. The first stub essentially helps in bringing the admittance to lie on the rotated circle.

Working backwards now, one can write the steps involved in the double stub matching as follows (see Fig. 2.49):

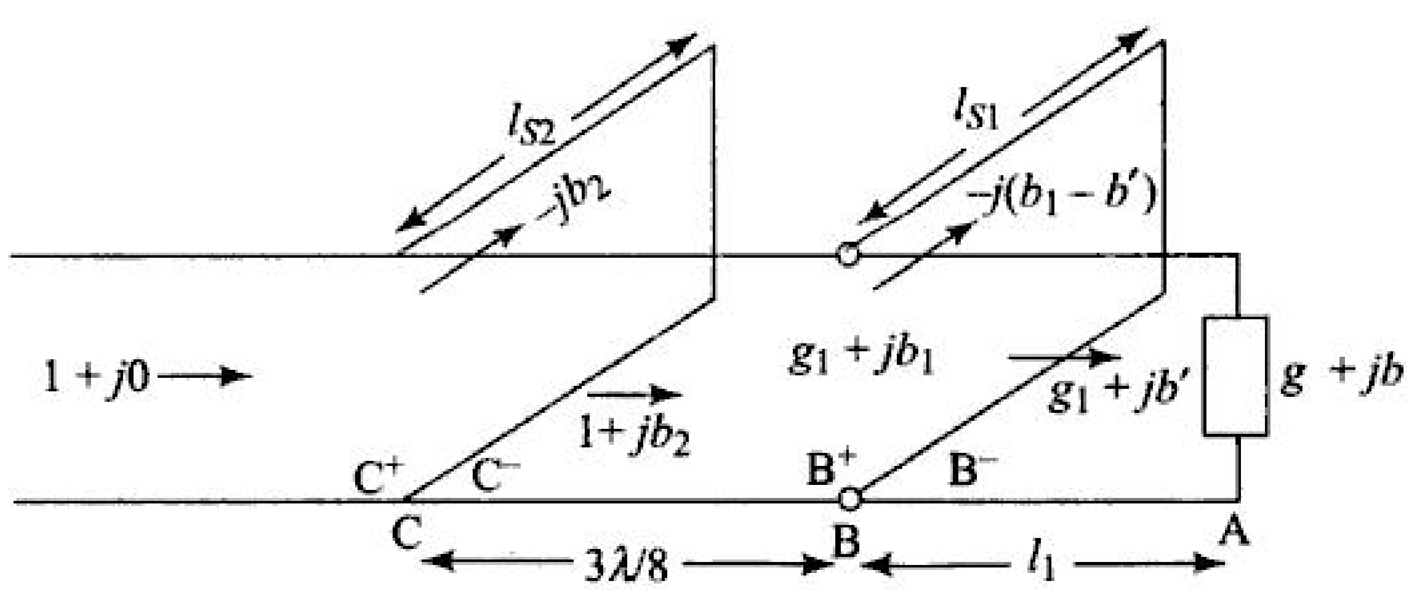


Fig. 2.49 *Double-stub matching configuration.*

1. Mark the admittance $g + jb$ on the Smith chart (Point A).
2. Move on constant VSWR circle passing through A by a distance l_1 to reach B^-. Let the admittance be $g_1 + jb'$.

3. Move along the constant-conductance (constant-g) circle to reach $B^+(g_1 + jb_1)$ (a point on the rotated $g = 1$ circle). Note that a stub at B will change only the reactive part and therefore we move on a circle which keeps the real part of $g_1 + jb'$ same while going from B^- to B^+.
4. Transform admittance $g_1 + jb_1$ at B^+ to C^- by moving a distance of $3\lambda/8$ on a constant VSWR circle passing through B^+. The point C^- must be lying to the $g = 1$ circle. Let the transformed admittance at point C^- be $1 + jb_2$.
5. Add a stub to give susceptance $-jb_2$ at location C so as to move the point C^- to C^+ which is the matched point.
6. To calculate the length of the first stub l_{s1} we note that this stub must provide a susceptance which is the difference between the susceptances at B^+ and B^-. That is, the stub susceptance b_{s1} is equal to $(b_1 - b')$. Mark the susceptance $j(b_1 - b')$ on the chart to get point S_1. Distance from S_1 to S in the anticlockwise direction gives the length l_{s1} of the first stub.
7. The second stub should have a susceptance of $-jb_2$. To get the length l_{s2} of the second stub the procedure is same as that used in the single stub matching. That is, mark $-jb_2$ on the Smith chart to get point S_2. Measure distance S_2S in anti-clockwise direction to give l_{s2}.

From the above discussion, it might appear that the double stub matching is the ultimate solution for impedance matching as it has overcome the drawbacks of the single-stub matching technique. However, if we observe carefully we will note that there is some problem with this technique. The whole matching process relies on the fact that by moving along a constant conductance circle one can go from point B^- to B^+. (B^+ lies on the rotated $g = 1$ circle). If this step is not realizable then the whole matching process is unrealizable. Knowing the nature of the constant-g circles, that they lie one inside another we note that if the point B^- lies in the hatched region (Fig. 2.50), a movement along a constant-g circle can never bring the point outside the hatched region and consequently can never reach the rotated $g = 1$ circle. In this situation, the impedance matching can not be obtained. The hatched region is the region enclosed by the constant-g circle which is externally tangential to the rotated $g = 1$ circle. The hatched region is called the forbidden region. So, we find that the double-stub matching technique cannot match all the load admittances as it was possible by the single-stub matching technique. However, the important thing to note here is that as such there is no constraint on the load admittance, $g + jb$. The load $g + jb$ may lie anywhere on the Smith chart but the admittance $g_1 + jb'$ should not lie in the forbidden region. Since $g_1 + jb'$ is a function of l_1 (location of the first stub), one can always vary l_1 so as not to get $g_1 + jb'$ in the forbidden region. In brief, if l_1 can be varied appropriately the double stub matching also can provide matching for all load admittances. One would then wonder, if the location of the stub, l_1, is to be varied, in what way is this technique superior to the single stub technique! The two situations are not quite identical. In previous case l_1 had to be precisely

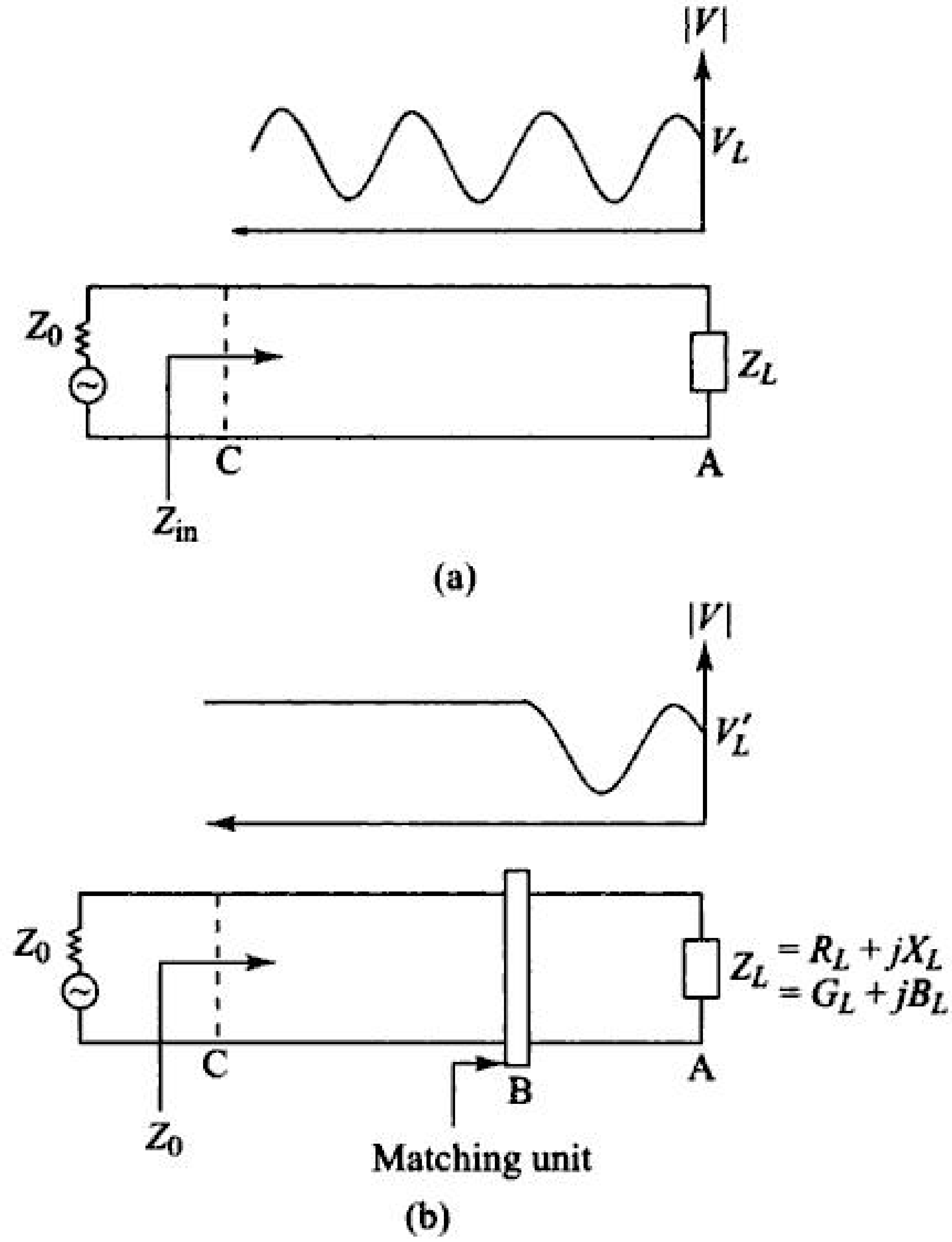


Fig. 2.52 *(a) Voltage standing waves without the impedance matching unit (b) Voltage standing waves with the impedance matching unit.*

$$P_{\text{out}} = \frac{V_0^2}{4Z_0} = Re\left[V_L'\left(\frac{V_L'}{Z_L}\right)^*\right] \tag{2.177}$$

$$\Rightarrow \quad |V_L'|^2 = \frac{V_0^2}{4Z_0G_L} \tag{2.178}$$

$$\Rightarrow \quad |V_L'| = \frac{V_0}{2}\sqrt{\frac{1}{Z_0G_L}} \tag{2.179}$$

Suppose, for sake of argument if we assume that the length of the line is integer multiples of $\lambda/2$, then without matching unit the input impedance Z_{in} will be same as Z_L. The magnitude of the voltage at point C and also at point A then will be

$$|V_C| = |\frac{V_0 Z_L}{Z_0 + Z_L}| \tag{2.180}$$

Since, the voltage at point A is same as the voltage V_L across the load Z_L, we get

$$|V_L| = \frac{V_0 R_L}{|(R_L + Z_0) + jX_L|} \tag{2.181}$$

From Eqns (2.179) and (2.181) one can observe that $|V_L|$ is not equal to $|V_L'|$. One can verify that $|V_L'|$ is always greater than $|V_L|$. The conclusion, therefore, is with the matching unit, the amplitude of the standing wave pattern between the load and the matching unit always increases. More reactive the load is, higher is the amplitude of the standing wave. In the limit when the load becomes purely reactive, the standing wave amplitude becomes infinite. This is due to the fact that, the power delivered by the generator is independent of the load as after the matching unit the generator always sees an impedance equal to the charactristic impedance. To absorb the power a highly reactive load must have a very high voltage across it or very high current through it. In the limit, therefore, the voltage across the purely reactive load tends to infinity.

2.14 LOSSY TRANSMISSION LINES

Till now we assumed the loss on the transmission line to be negligible and therefore we assumed the propagation constant of the line to be purely imaginary ($\gamma = j\beta$). In situations where the attenuation constant α becomes comparable to β, the line characteristics have to be re-visited. In the following we develop modifications to the analysis of the loss-less lines to take into account the loss.

If we take the line to be very lossy, i.e. $R >> \omega L$ and $G >> \omega C$, we get the characteristic impedance

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \approx \sqrt{\frac{R}{G}} \text{ (real)} \tag{2.182}$$

and the propagation constant

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \approx \sqrt{RG} \text{ (real)} \tag{2.183}$$

Comparing this with that of a loss-less case we note that Z_0 is real (resistive) in both cases but γ is purely imaginary for a loss-less line but purely real for a very lossy line. The almost resistive nature of the characteristic impedance therefore does not guarantee the line to be loss less. In general, complex Z_0 ($\equiv R_0 + jX_0$) and complex γ ($= \alpha + j\beta$) represent moderately lossy lines.

2.14.1 Standing Waves on a Lossy Line

The voltage on a lossy transmission line can be written as (see Eqn (2.28))

$$V = V^+ e^{-(\alpha + j\beta)x} + V^- e^{+(\alpha + j\beta)x} \tag{2.184}$$

The voltage amplitude of the incident wave is $V^+ e^{-\alpha x}$ and that of the reflected wave is $V^- e^{+\alpha x}$. The forward (incident) wave, therefore, exponentially dies down as it travels away from the generator (Remember $+x$ indicates away from generator). Similarly, the reflected wave grows away from the generator or decays towards the generator. Since the reflected wave originates from the load end, its exponentially decaying behavior with distance is identical to that of the incident wave (Fig. 2.53)

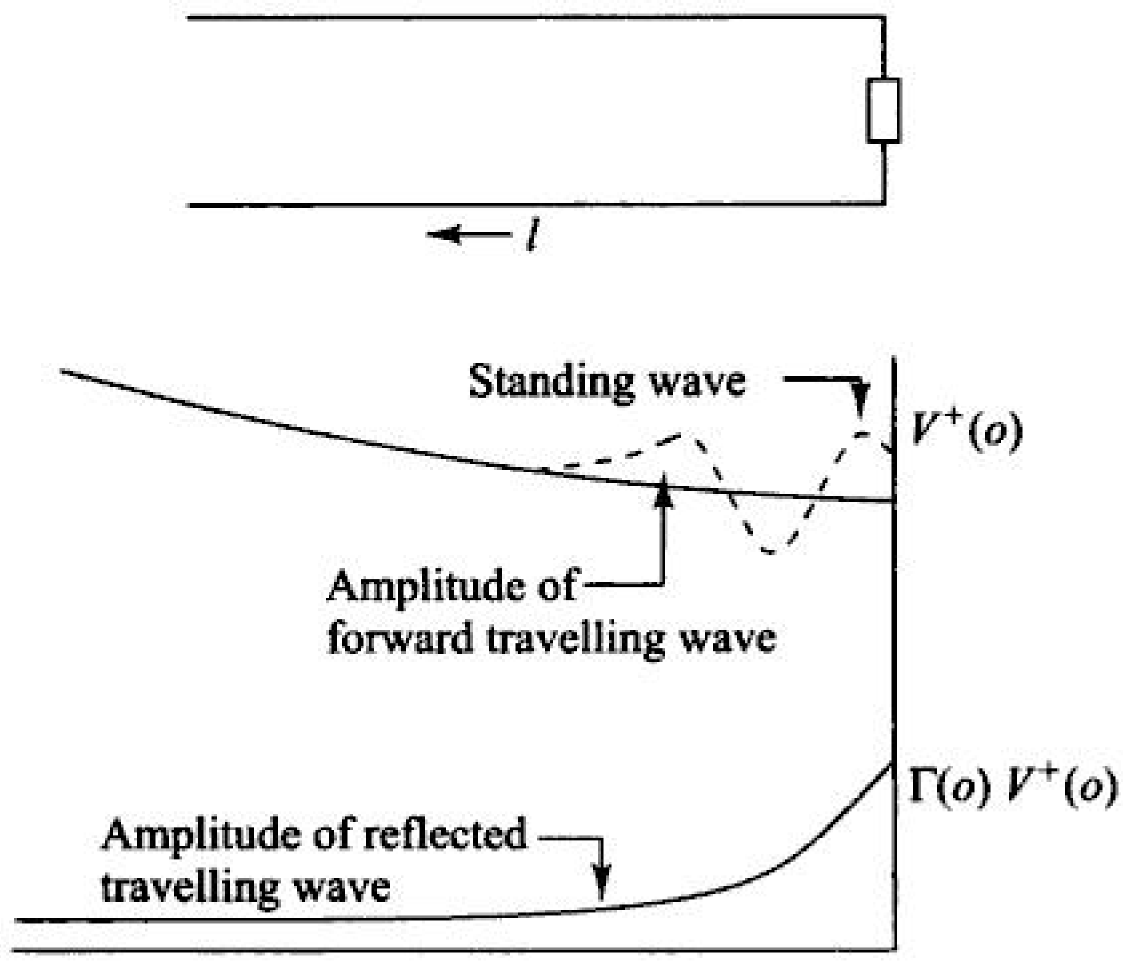


Fig. 2.53 *Amplitude variation of voltage waves on a lossy transmission line.*

The reflection coefficient at any point on the line is (see Eqns (2.32) and (2.37))

$$\Gamma(l) = \Gamma(0)e^{-2\gamma l} = \left(\frac{Z_L - Z_0}{Z_L + Z_0}\right) e^{-2\gamma l} \tag{2.185}$$

Substituting for $\gamma = \alpha + j\beta$ we get

$$|\Gamma(l)| = |\Gamma(0)|e^{-2\alpha l} \tag{2.186}$$

Since α is positive and l is positive towards the generator, $\exp(-2\alpha l)$ becomes smaller and smaller as we move towards the generator. In other words the reflection coefficient which the generator sees is always less than its value at the load. If $2\alpha l >> 1$ then $\Gamma(l) \rightarrow 0$ and the generator does not see any reflected wave. Consequently, from the generator end the system appears matched and the generator delivers maximum power to the line input. The power, however, does not get delivered to the load but is lost in the line itself. If the power is not a limitation, one can then use a lossy line for the matching purposes.

The standing wave pattern for a lossy line is as shown in Fig. 2.53. The standing wave slowly dies down to merge with the exponential function of the forward wave. In this case it is not quite correct to define the VSWR as the ratio of the V_{max} and V_{min} on the line as one would not get the same value for the VSWR at different locations on the line. If at all VSWR is to be defined it should be defined as a function of distance as

$$\rho(l) = \frac{1+|\Gamma(l)|}{1-|\Gamma(l)|} = \frac{1+|\Gamma_L|e^{-2\alpha l}}{1-|\Gamma_L|e^{-2\alpha l}} \tag{2.187}$$

In practice one may use adjacent voltage maximum and minimum to obtain the value of $\rho(l)$. One can however, note that the distance between two adjacent voltage maxima or minima is not exactly equal to $\lambda/2$ as in case of the loss-less line.

2.14.2 Use of Smith Chart for Lossy Lines

Since the Smith chart is a set of circles corresponding to normalized resistances and reactances, it as such does not change for a lossy line. However, since the VSWR is no more constant along the length of the line, the constant VSWR circles have to be reinvestigated. From Eqn (2.185) we have

$$\Gamma(l) = \Gamma(0)e^{-2(\alpha+j\beta)l} \tag{2.188}$$

Since $\Gamma(0) = \Gamma_L \equiv |\Gamma_L|e^{j\theta_L}$, (see Eqn (2.129)), we can re-write Eqn (2.188) as

$$\Gamma(l) = |\Gamma_L|e^{-2\alpha l}\, e^{j(\theta_L - 2\beta l)} \tag{2.189}$$

It is clear that as we move on the line towards the generator, the phase of $\Gamma(l)$ decreases similar to that in the case of a loss-less line. But in this case even the magnitude of $\Gamma(l)$ decreases. The plot of $\Gamma(l)$ as a function of l, therefore, is not a circle but a converging spiral as shown in Fig. 2.54.

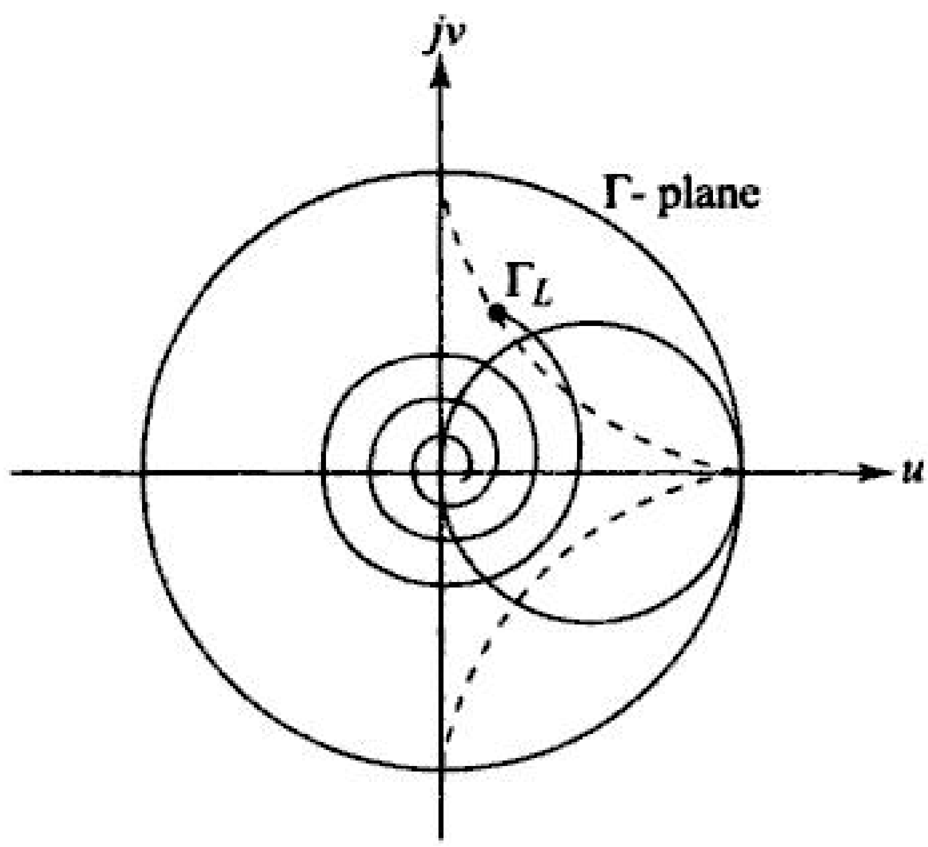


Fig. 2.54 *Variation of reflection coefficient on a lossy transmission line.*

In this case, therefore, there are no constant VSWR circles. For any impedance transformation, ideally one has to draw the Γ-spiral. For moderately low-loss lines however, one can make some approximations. One can assume that even if α is not negligibly small it is much smaller compared to β. Physically, it means that the reduction in the amplitude of a wave due to the loss is small over a distance

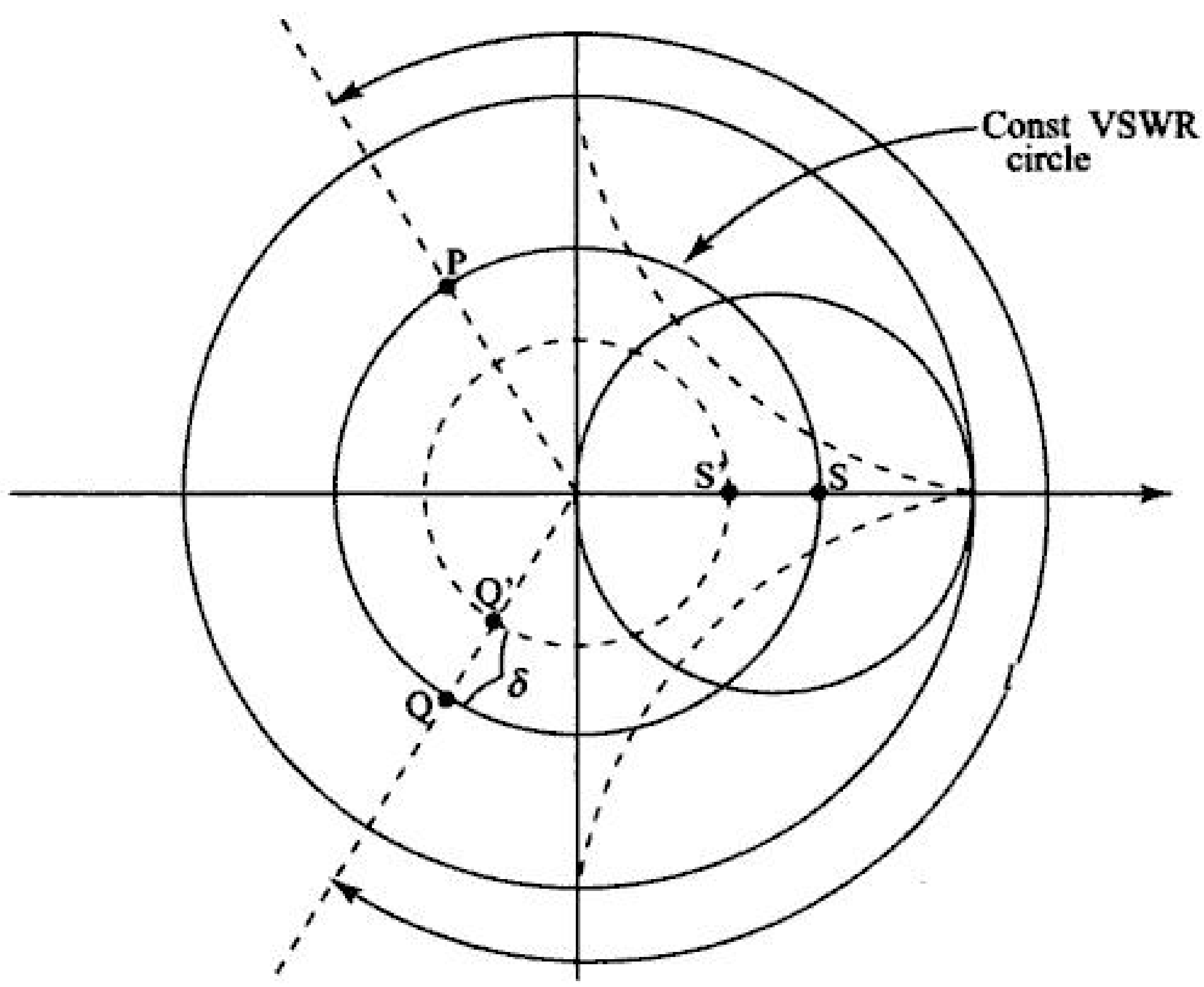


Fig. 2.56 *Use of Smith chart for impedance calculations on a lossy transmission line.*

2.15 MEASUREMENT OF LINE PARAMETERS

We have seen in the beginning of this chapter that, a transmission line can be either characterized by its primary constants R, L, G and C or by its secondary constants γ and Z_0. Usually, the knowledge of γ and Z_0 is sufficient for transmission line calculations and one rarely needs the values of the primary constants. The secondary constants γ and Z_0 of a line are evaluated by conducting short-circuit and open circuit tests on any arbitrary length of the line.

Take any arbitrary length of the line say, L, and measure the input impedance Z_{oc} and Z_{sc}(technique for measuring impedance has been described earlier) of the line with its other end open and short respectively. From Eqn (2.45) we have

$$Z_{oc} = Z_0 \coth \gamma L \tag{2.191}$$

$$Z_{sc} = Z_0 \tanh \gamma L \tag{2.192}$$

Multipying Eqns (2.191) and (2.192) we get

$$Z_0^2 = Z_{sc} Z_{oc} \tag{2.193}$$

$$\Rightarrow \qquad Z_0 = \sqrt{Z_{sc} Z_{oc}} \tag{2.194}$$

Dividing Eqn (2.192) by Eqn (2.191) we get

$$\tanh \gamma L = \sqrt{\frac{Z_{sc}}{Z_{oc}}} \equiv A \text{ (say)} \tag{2.195}$$

where A is a complex number.

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Let us say we conduct the short and open circuit test on a line of length L at frequency f_1 and get A. Then from Eqn (2.200) we have

$$\beta_1 = \frac{1}{2L}\angle\left(\frac{1+A}{1-A}\right) + 2m\pi \tag{2.204}$$

Now, increase the frequency slowly till you get Z_{oc} and Z_{sc} same as that at f_1. Let this frequency be f_2. (Note: It has been assumed here that the loss of the cable is same at the two frequencies. This is more or less true if the two frequencies are not widely separated.) Now, since the input impedances are same at the two frequencies f_1 and f_2, the electrical lengths of the line at the two frequencies must be differing by $\lambda/2$, or m must have been increased by 1. The phase constant, would therefore, be

$$\beta_2 = \frac{1}{2L}\angle\left(\frac{1+A}{1-A}\right) + 2(m+1)\pi \tag{2.205}$$

Subtracting Eqn (2.204) from Eqn (2.205) we get

$$\beta_2 - \beta_1 = \frac{\pi}{L} \tag{2.206}$$

Using Eqn (2.203) for β_1 and β_2, we get

$$\frac{2\pi f_2}{v} - \frac{2\pi f_1}{v} = \frac{\pi}{L} \tag{2.207}$$

$$\Rightarrow \qquad v = 2L(f_2 - f_1) \tag{2.208}$$

The phase constant can be then obtained as

$$\beta = \frac{\pi f}{L(f_2 - f_1)} \tag{2.209}$$

Although, the above measurement procedure appears very straight forward theoretically, in practice it encounters certain problems. The problems are mainly due to difficulties in realizing ideal open and short-circuit at the line end. An open circuit generally has some fringing capacitance and a short-circuit has a small looping inductance. Multiple measurements on lines of different lengths can help in removing errors due to nonideal short or open circuits at the line ends.

2.16 VARIOUS TYPES OF TRANSMISSION LINES

In practice we find a variety of transmission lines, like the coaxial lines, parallel wire lines, etc. Although, structures like waveguides and optical fibres (explained in later sections) also fall in the catagory of transmission line, we confine our discussion to only those transmission lines which have two conductors separated by a dielectric. In the following sections we describe some commonly used transmission lines and also give expression for their characteristic impedances.

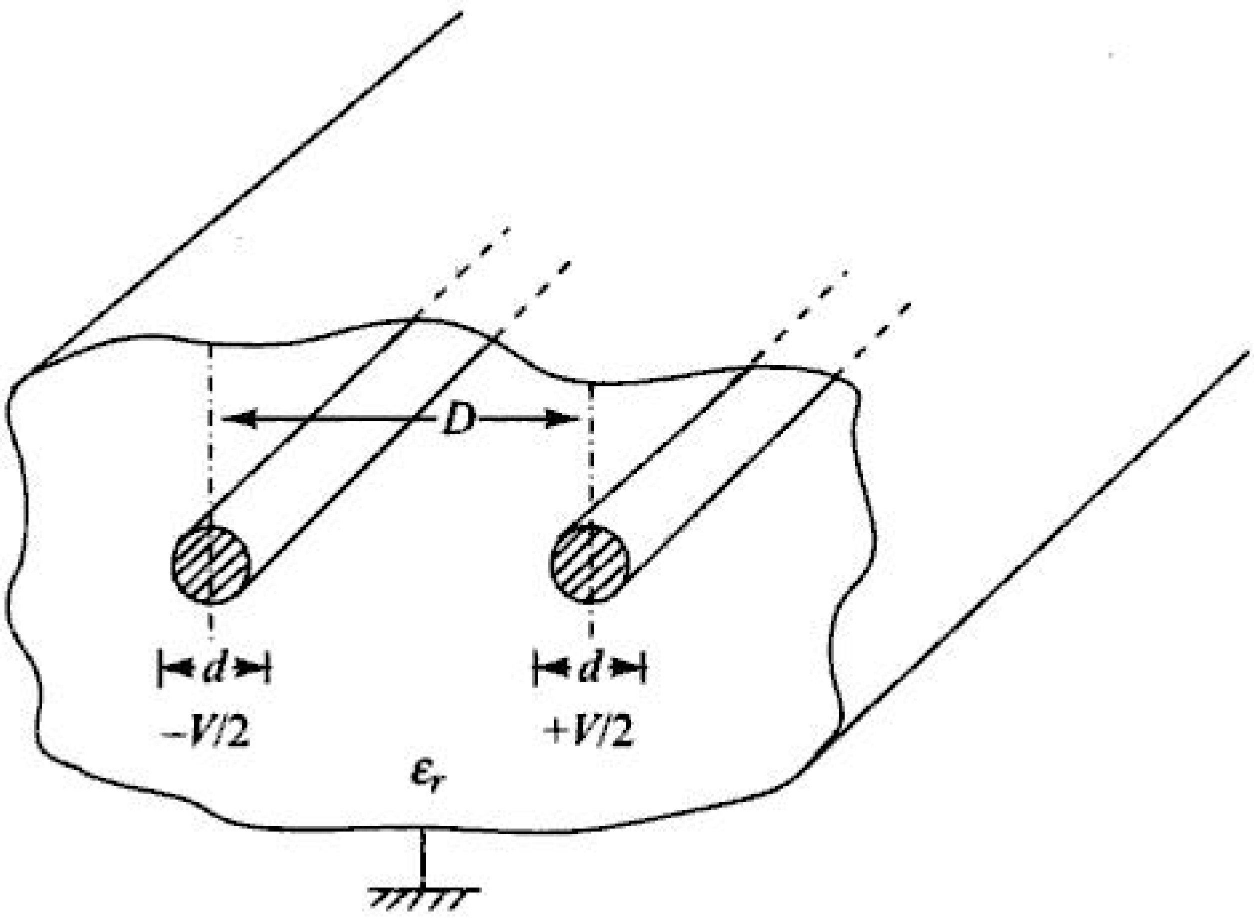

Fig. 2.58 *Parallel wire transmission line.*

2.16.3 Microstrip Transmission Line

This transmission line consists of an infinitely large conducting plane and a flat metal strip placed at a distance from it (Fig. 2.59). The region between the conducting plane and the strip may be filled with a dielectric.

The characteristic impedance for this line is approximately given as

$$Z_0 \approx \frac{377}{\sqrt{\epsilon_r}[(W/h)+2]} \tag{2.212}$$

W and h are shown in Fig. 2.59.

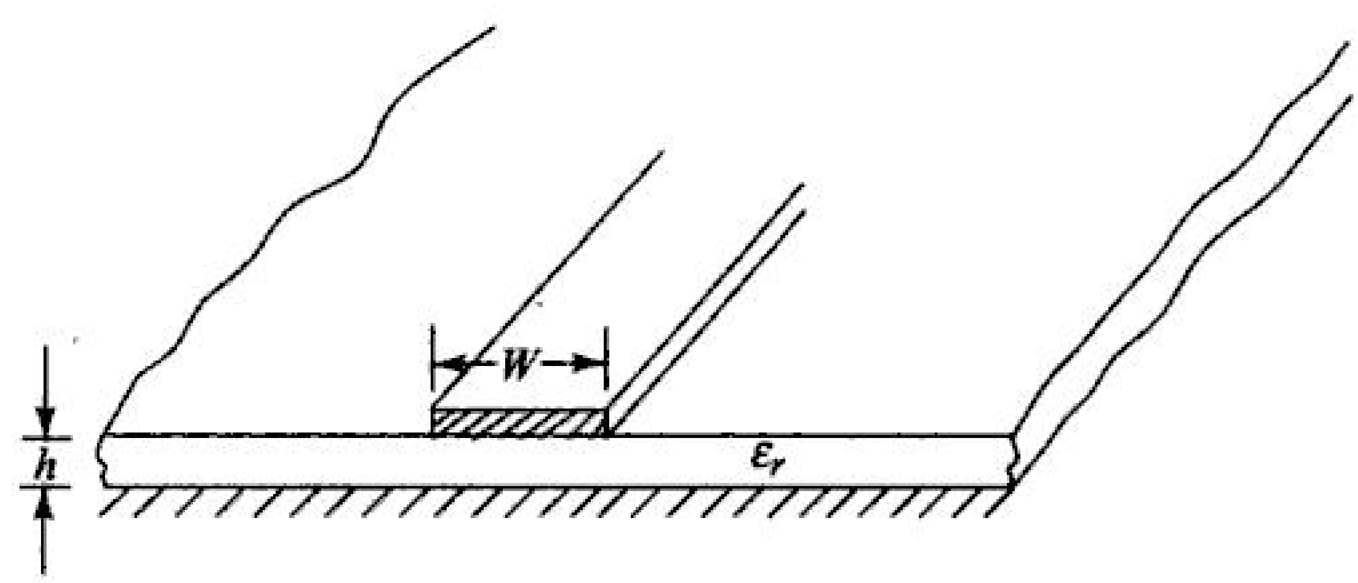

Fig. 2.59 *Micro-strip transmission line.*

This type of transmission lines are encountered in high frequency printed circuits. The characteristic impedance generally lies in a range which is compatible with the co-axial cables.

2.17 SUMMARY

In this chapter, we developed a special but important case of electromagnetic waves, the transmission line. The transmission lines form the foundation of the high frequency circuits. For frequencies beyond a few MHz electrical circuits can not be correctly analyzed without using the concepts of transmission line. Today, when computer and data network speeds are approaching GHz, the knowledge of transmission line is vital to the electrical and computer engineers. A thorough understanding of transmission line also helps in understanding more complex phenomenon of electromagnetic waves in the three dimensional space. In the following chapters, we discuss the general phenomenon of electromagnetic waves, however, we make frequent references to the results of the transmission lines derived and discussed in this chapter.

Review Questions

2.1 What is transit time effect?

2.2 When does transit time effect become appreciable?

2.3 What is a distributed element?

2.4 Why should high frequency circuits be analyzed with distributed element approach?

2.5 What is a transmission line?

2.6 What equations govern voltage and current on a transmission line?

2.7 What is a travelling wave ?

2.8 What is a propagation constant, attenuation constant and phase constant?

2.9 What are the units of attenuation and phase constants?

2.10 What is characteristic impedance? What does it signify?

2.11 If the ratio of voltage and current for a travelling wave is negative. What does it signify?

2.12 What is a standing wave?

2.13 Why do we get standing waves on a transmission line?

2.14 Are standing waves desirable on a transmission line? Why?

2.15 What are full and partial standing waves?

2.16 What is voltage reflection coefficient?

2.17 How is reflection coefficient related to the local impedance?

2.18 What is impedance transformation?

2.19 If impedance at any point on a transmission line is known, can we find impedance uniquely at any point on the line?

2.20 What are loss-less and low-loss transmission lines?

2.21 What is the characteristic impedance of a loss-less transmission line?

2.22 How does reflection coefficient vary along a lossy and loss-less transmission line?

2.23 Over what distance does the impedance repeat along a transmission line?

2.24 What is voltage standing wave ratio?

2.25 What is the range of VSWR?

2.26 How is VSWR related to the reflection coefficient?

2.27 What is normalized impedance and what is its importance?

2.28 What is the impedance at voltage maximum and voltage minimum?

2.29 What is the maximum and minimum impedance seen on a line which is terminated in a particular load?

2.30 What is matched load?

2.31 Why is matching of load impedance important?

2.32 If impedance at some point on a line is Z_0 what will be the load impedance?

2.33 On a loss-less transmission line if the reflection coefficient is Γ, what % of incident power is delivered to the load?

2.34 On a loss-less line the real power is same at every location on the line but the imaginary power varies? Why?

2.35 What is the Smith chart?

2.36 Why is the graphical representation of a transmission line like the Smith chart important?

2.37 In what curves the constant resistance and constant reactance lines in the impedance plane are mapped to in the reflection coefficient plane?

2.38 What is the speciality of constant resistance and constant reactance circle in the Γ-plane?

2.39 As we move along a transmission line, how does a point move on the Smith chart?

2.40 What is constant VSWR circle?

2.41 Identify the load impedances in the following cases:

(a) Partial standing wave and the voltage drops towards the generator at the load point.

(b) Fully standing wave and the voltage rises on the line as we move from load towards generator.

(c) No standing wave.

(d) Partial standing wave with VSWR = 3 and voltage is minimum at the load?

(e) Fully standing wave and voltage is maximum at the load point.

2.42 What is the difference between impedance and admittance Smith charts?

2.43 Name applications of transmission lines.

2.44 Why at high frequencies should the impedance be measured using transmission lines ?

2.45 Can a short circuited section of a transmission line be used for realizing an inductance?

2.46 Under what condition does a short circuited or open circuited section of a transmission lines behave like a series and parallel resonant circuit?

2.47 What is the Q of a section of a loss-less transmission line?

2.48 Can a section of transmission line be used as a voltage or current transformer?

2.49 What is a quarter wavelength transformer? What impedances can be matched using this transformer?

2.50 What is the single stub matching technique? What is its advantage over the quarter wavelength transformer?

2.51 What are the drawbacks of the single stub matching technique?

2.52 What is double stub matching technique ? What is its advantage over the single stub matching technique ?

2.53 What is the limitation of the double stub matching technique and how is it overcome?

2.54 What is a lossy transmission line?

2.55 How does VSWR vary along a lossy transmission line?

Problems

2.1 Find the transit time over a parallel wire transmission line of length 50 cm. A sinusoidal signal of 1.5 GHz is applied to one end of the line. What is the phase difference between the signals at the two ends of the line?

2.2 For a transmission line the primary constants are R = 0.2 Ω/m, G = 0, L = 0.3 μ H/m and C = 15 pF/m. Find the phase and attenuation constants of the line at 900 MHz. Also, find the characteristic impedance of the line.

2.3 In Question 2.2 what should be the value of G so that the line becomes distortionless? What will be the characteristic impedance of the distortionless line?

2.4 On a 50 Ω loss less line two waves travel in opposite directions. At some location on the line and at some instant the voltage and current are respectively 25 V and 2 A. Find the voltage and current at a distance of $\pm\lambda/3$ from the point at the same instant.

2.5 A transmission line has complex propagation constant γ = 1 + j10 per meter at 1 GHz. Find the distance over which the amplitude of a traveling wave will decrease by 20 %. What phase change will the wave undergo over that distance?

2.6 A transmission line has Z_0 = 65 + j5Ω and γ = 1 + j20 per meter. Is the line loss less? Find the primary constants of the line at 900 MHz. For these primary constants, over what frequency range can the line be treated as a low loss line ? (Note: for a low loss line $\alpha << \beta$, say $\alpha < 1\%$ of β).

2.7 A coaxial cable has 10 dB loss per 100 m length. A 10V-3A signal is connected to one end of the 50 m long cable and the other end of the cable is connected to a matched load. Find the power loss in the cable and the power delivered to the load.

2.8 A line has $Z_0 \approx 70$ Ω and $\alpha = 5$ neper per meter. A 100 V/50 Ω generator is connected to one end of a long piece of the line. What is the power supplied by the generator? Over what length of the line about 99 % of the input power will be absorbed by the line?

2.9 A transmission line with Z_0 = 50 -j5 and γ = 0.2 + j 2.5 per meter is connected to a load impedance of 100 + J 50 Ω. Find the reflection coefficient at the load end of the line. Also, find the reflection coefficient and the impedance at a distance of 3 m from the load.

2.10 On a 50 Ω line the reflection coefficient at the load is $0.7\angle 30^\circ$. If the propagation constant of the line is $\gamma = 20\angle 89^\circ$ per meter. Find the impedance at a distance of 4 m form the load.

2.11 A $\lambda/3$ long 50 Ω loss-less line is connected to a load at one end, and to $100\angle 0$V/50Ω generator at the other end. If the voltage at the input terminal of the line is $75\angle 30^\circ$, find the load impedance and the power delivered to the load.

2.12 A 300 Ω parallel wire line is connected to an antenna of 75 + j35 Ω input impedance. Find the VSWR on the line. What is the maximum and minimum impedance seen on the line?

2.13 On a loss-less transmission line connected to a load the maximum voltage and current are 100 V and 4 A respectively. If 25% of the incident power is reflected by the load find the minimum and maximum impedance seen on the line.

2.14 A 300 Ω loss-less transmission line connected to a load has maximum and minimum currents of 20 A and 12 A respectively. What is the power delivered to the load?

2.15 A 75 Ω loss-less coaxial cable is connected to an antenna of impedance 100 + j35 Ω at one end and to a generator at the other end. The internal impedance of the generator is 50 Ω. What should be the length of the cable so that maximum power is delivered to the antenna?

2.16 A 50 Ω line of length $3\lambda/5$ is connected to an admittance of 0.03 − j0.01 ℧ at one end, and a 50 V − 75 Ω generator at the other end. What are the amplitudes of the forward voltage and current travelling waves on the line? Find the complex powers at the input and load ends of the line.

2.17 A constant VSWR circle is mapped onto the impedance plane. Find the equation of the transformed curve. Identify the curve and plot it for VSWR = 1, 5, ∞.

2.18 Transform impedances which have same phase angle onto the complex reflection coefficeint plane. Draw the transformed curves for phase angles of 0°, 30°, 45°, 60°, 90°.

2.19 What will be the equation of the curve in the complex Γ-plane corresponding to $|Z|$ = constant ?

Smith chart based problems (2.20 to 2.26)

2.20 Mark following points on the smith chart.

(i) $\Gamma = 0.3\angle -45^\circ$
(ii) $\Gamma = 0.2 + j0.5$
(iii) $\bar{z} = 3 + j2$
(iv) $\bar{y} = 0.5 - j2$
(v) short and open circuit on the impedance chart
(vi) short and open circuit on the admittance chart
(vii) g = 1, b = 0.6 on the admittance chart
(viii) VSWR = 2, $\angle\Gamma = 120^\circ$
(ix) Z = 3 and x = 2r
(x) Z = 2 and maximum phase angle of Z

2.21 An impedance of 75 − j30 Ω is connected to a 100 Ω line. Find complex reflection coefficient at the load, the complex reflection coefficent and the impedance at a distance of 0.35 λ from the load. What is the VSWR measured on the line?

2.22 On a line the maximum and minimum voltages measured are 25 V and 15 V respectively. The distance between adjacent maximum and minimum voltages is 20 cm. If minimum voltage occurs at a distance of 25 cm from the load, find the load admittance. The characteristic impedance of the line is 100 Ω.

2.23 Maximum and minimum impedances seen on line are 150 Ω and 50 Ω. What is the load impedance if maximum current occurs at a distance of 0.15 λ from the load?

2.24 A transmission line of 50 Ω characteristic impedance is terminated in a load impedance of 80 − j 30 Ω. At a distance of 0.3 λ a short circuited stub made of 75 Ω line is connected in parallel to the main line such that the impedance is real at the junction. What is the length of the stub? What will be the VSWR on the main line beyond the junction towards the generator?

2.25 An inductive reactance of 100 Ω is to be realized using an open circuited section of a 50 Ω line at 2 GHz. Find the length of section. If 10 % variation in the reactance is acceptable, over what bandwidth will the reactance realization be satisfactory?

2.26 A 300 Ω line is connected in a load impedance of 200 -j100 Ω. A short circuted section of the transmission line is connected in parallel with the load impedance. What should be the length of the section so as to get minimum reflection from the load?

2.27 A $\lambda/2$ section of line has 0.05 dB loss. If the line is open circuited at one end, find the impedance at the other end of the line. The characteristic impedance of the line is 50 Ω. What will be the quality factor of the section?

2.28 A 1.25 λ long section of a 75 Ω line is short circuited at one end and open circuited at the other. The voltage measured at the mid point of the line is 40 V. If the loss in the line is 0.2 dB per meter and the wavelength of the signal is 5 m, find the energy stored and energy dissipated on the line. Hence, find the quality factor of the section of the line. Assume that the line has a velocity factor 0.66. (velocity factor is the ratio of the velocity of a wave on the line to the velocity of the light in vacuum).

2.29 A short circuited section of a line is to be used as a parallel resonant circuit at 2.4 GHz. Find the length of the section. If the line has 3dB/m loss, what is the 3-dB bandwidth of the resonant circuit?

2.30 Two long cables with characteristic impedances 100 Ω and 200 Ω are to be joined through a quarter wavelength transform at 900 MHz. If the velocity factor for the transformer section is 0.8, find the length and the characteristic impedance of the transformer.

2.31 Two transmission lines with characteristic impedances Z_1 and Z_2 are connected through a quarter wavelength transformer to get a perfect match at frequency f_0. Derive an expression for power transfer efficiency at the junction as a function of frequency deviation Δf around f_0 ($\Delta f \ll f_0$). For Problem 2.30 if the power transfer efficiency is to be greater than 90 %, find the frequency range over which the transformer can be used.

2.32 An impedance of 50 + j50 Ω is to be matched to 50 Ω using a quarter wavelength transformer. Find the location and the characteristic impedance of the quarter wavelength transformer.

100 m. If a 1 MHz square wave of 2 V pp amplitude is launched on the line, find voltage waveform across the load. Use computer program if necessary. Plot the voltage waveform to the scale.

2.45 For lossy transmission line the characteristic impedance $Z_0 = R_0 + jX_0$. If R,L,G,C are resistance, inductance, conductance and capicitance/unit length of the line show that

$$\alpha = \frac{R}{2R_0} + \frac{G|Z_0|^2}{2R_0}$$

$$\beta = \frac{\omega L}{2R_0} + \frac{\omega C|Z_0|^2}{2R_0}$$

2.46 A transmission line of $Z_0 = 50\ \Omega$ is terminated in a load to give a VSWR of 3. The first voltage maximum is at 5 cm from the load and the next is at 25 cm from the load. Find the minimum value of VSWR that can be obtained by placing a stub in parallel with the line at the load.

2.47 Find an expression for the quality factor for a resonant section of a transmission line using frequency response of the circuit.

2.48 A lossy line has a characteristic impedance of 50Ω. About 100 m length of this line is terminated with an impedance 200 + j 50 Ω. The magnitude of the measured input impedance of the line is within 10 percent of the characteristic impedance. Calculate the value of the attenuation factor in dB/m.

2.49 If d is the distance between two points on either side of any voltage minimum at which the magnitude of the voltage is $\sqrt{2}$ times the V_{min}, show that

$$VSWR = \frac{\sqrt{1 + \sin^2(\beta d/2)}}{\sin(\beta d/2)}$$

2.50 A 50 Ω line is terminated in a load impedance of 75 - j 69 ohms. The line is 3.5 m long and is excited by a source at 50 MHz. Velocity of propagation is 3×10^8 m/sec. Find the input impedance, the complex input reflection coefficeint, the VSWR and the position of the voltage minimum.

2.51 The VSWR of an ideal 70 ohm line is measured as 3.2 and a voltage minimum is observed 0.23 wavelength in front of the load. Find the load impedance. Determine the maximum phase angle for the impedance at any point on the line.

2.52 A 70 ohm line is terminated in an impedance of 50 + j 10 ohms. Find the position and length of a short circuited stub required for matching if the stub is to be added in (a) series (b) parallel.

2.53 A 50 ohms transmission line is terminated with a load of 200 + j 300 ohms. A double stub tuner consisting of a pair of short circuited 50 ohm lines connnected in shunt to the main line at points spaced by 0.25 λ is located with one stub at 0.2 λ from the load. Find the lengths of the stubs.

2.54 A transmission line is terminated in a load which can attain any value of admittance given by the lower half of the smith chart. Can these loads be matched by a double stub tuner with stub spacing of a $5\lambda/16$. If yes, design the tuner. If no, give the range of loads which cannot be matched.

CHAPTER 3

Maxwell's Equations

In the previous chapter, we discussed the limitations of the lumped element analysis for high frequency circuits and introduced the concept of distributed elements. Apart from the normal independent parameter 'time', 'space' was also introduced in the circuit analysis. The electrical quantities however were still voltages and currents which were scalar in nature. One can quickly realize that the concept of voltage and current is rather suitable for circuits having wires, resistances, capacitances, etc. If one goes to non-circuit like configurations, use of voltage or current becomes rather inappropriate. For example if one has to investigate the unbound medium like free space, the use of voltage and current appears unattractive. In this situation, one then naturally has to fall back on the more fundamental quantities like electric and magnetic fields. Due to vector nature of these fields, the analysis becomes relatively more involved compared to that with voltages and currents. However, we will see that some of the concepts developed with voltage and current can be extended to the field analysis and it would be a good practice to validate the generalized field concepts with the special examples of transmission line.

In this chapter, we develop the basic vector equations which govern the electric and magnetic fields. These equations are called the Maxwell's equations. In general, we develop the equations for the time varying electric and magnetic fields, which can be reduced to their static form by appropriate substitution for the time varying function. Before we get into the derivation of the Maxwell's equations some basic definations regarding co-ordinate systems, a brief discussion on the vector operations, and a few vector theorems are in order. It is essential that a consistent set of conventions be followed throughout so as not to commit any mistake in defining the vector directions. The basics of vector algebra and vector calculus are given in Appendices B and C.

It is an interesting phenomenon that the most fundamental of the electromagnetic fields is the charge. A stationary charge produces an electric field. However, when the same charge is kept in uniform motion, it constitutes a

The quantity ϵ is a characteristic parameter of the medium surrounding the charge Q, and is called the permittivity of the medium. Even no medium like vacuum has a finite permittivity, and is generally denoted by ϵ_0. The permittivity of the free-space (vacuum) is

$$\epsilon_0 = \frac{1}{36\pi} \times 10^{-9}\ \mathrm{F/m} \tag{3.3}$$

Generally, other media have permittivities ϵ higher than that of vaccuum. The ratio of ϵ and ϵ_0 is called the relative permittivity or the dielectric constant of the medium and is denoted by ϵ_r.

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} \tag{3.4}$$

We observe here that, since, the permittivity is different for different media, the same charge Q produces different electric fields at the same distance r from it. A quantity which remains independent of the medium characteristics is called the electric displacement density, $\mathbf{D}$, and is defined as the product of the electric field and the medium permittivity. Hence,

$$\mathbf{D} = \epsilon\mathbf{E} = \hat{\mathbf{r}}\frac{Q}{4\pi r^2} \tag{3.5}$$

If the medium is isotropic, meaning properties of the medium are not direction dependent, then the permittivity ϵ is a scalar quantity. For this medium it can be seen from Eqn (3.5) that the direction of $\mathbf{D}$ is same as that of $\mathbf{E}$. On the other hand, for an anisotropic medium (a medium for which the properties are direction dependent) the permittivity in general is a tensor and then Eqn (3.5) has to be re-written as

$$\mathbf{D} = \bar{\bar{\epsilon}} : \mathbf{E} \tag{3.6}$$

where $\bar{\bar{\epsilon}}$ is a 3×3 matrix.

In the cartesian co-ordinate system this can be written as

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{3.7}$$

For an anisotripic medium, the direction of $\mathbf{D}$ is not same as that of $\mathbf{E}$. As can be seen from Eqn (3.7) D_x is not produced by only E_x but also by E_y and E_z. Same is true for D_y and D_z. Depending upon the elements of the permittivity tensor, $\mathbf{D}$ can have any arbitrary direction with respect to $\mathbf{E}$. The isotropic case is a special case of Eqn (3.7) where $\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \epsilon$ and other off diagonal elements $\epsilon_{xy}, \epsilon_{xz}, \epsilon_{yx}, \epsilon_{yz}, \epsilon_{zx}, \epsilon_{zy} \equiv 0$. For isotropic medium Eqn (3.7) reduces to

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{3.8}$$

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$$= \frac{100}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left\{ \frac{\sin 3\theta - \sin\theta}{2} \right\}$$

$$= \frac{50}{r^2 \sin\theta} \{3\cos 3\theta - \cos\theta\}$$

The charge enclosed in the region is

$$Q = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{r=1}^{2} \frac{50}{r^2 \sin\theta} \{3\cos 3\theta - \cos\theta\} r^2 \sin\theta d\theta d\phi dr$$

$$= 2\pi(50) \int_1^2 dr \int_0^{\pi/2} (3\cos 3\theta - \cos\theta) d\theta$$

$$= 100\pi [\sin 3\theta - \sin\theta]_0^{\pi/2}$$

$$= -200\pi \text{ C}$$

3.2.2 Gauss's Law for Magnetic Flux Density

As we write the Gauss's law for the electric displacement and the electric charges, we can identically write the law for the magnetic flux density and the magnetic charges. *The total magnetic flux coming out of a closed surface is equal to the total magnetic charge (poles) inside the surface.* However, there are no isolated magnetic monopoles. The magnetic poles are always found in pairs with opposite polarity. As a result, there are always equal number of north and south poles inside any closed surface making net magnetic charge identically zero inside a volume. The total outward magnetic flux from any closed surface must therefore be identically equal to zero. Writing mathematically,

$$\oint_S \mathbf{B} \cdot \mathbf{da} = 0 \tag{3.37}$$

where $\mathbf{B}$ is the magnetic flux density. Applying the Divergence theorem to Eqn (3.37) (as done in Eqn (3.34)) we get

$$\int\int\int_V (\nabla \cdot \mathbf{B}) dv = 0 \tag{3.38}$$

or

$$\nabla \cdot \mathbf{B} = 0 \tag{3.39}$$

Equation (3.37) or Eqn (3.39) is essentially a mathematical statement of non-existence of the free magnetic monopoles. In future at some stage if one invents free magnetic charges the right hand side of Eqn (3.39) has to be modified by magnetic charge density.

3.2.3 Ampere's Circuital Law

The Ampere's circuital law states that *the total magnetomotive force along a closed loop is equal to the net current enclosed by the loop.* The magnetomotive

The magnetic field is

$$H = \frac{0.105}{2\pi r} = \frac{1.67 \times 10^{-2}}{r} \text{ A/m}$$

3.2.4 Faraday's Law of Electromagnetic Induction

The Biot Savart law tells us that the magnetic field is produced by a current. It therefore seems logical to find out whether the reverse is true, i.e. whether the magnetic fields would produce electricity. Faraday's experiments demonstrated that the static magnetic fields produce no electric current, but a time varying magnetic field produces an electromotive force in a closed loop which causes a current flow. According to Faraday's law, *the net electromotive force (EMF) in a closed loop is equal to the rate of change of magnetic flux* (Φ) *enclosed by the loop*.

Now, the current due to induced EMF will produce a magnetic field. Therefore there will be a magnetic field induced by the current which will be enclosed by the loop. There are two possibilities (i) the magnetic field due to the induced current is in same direction as the original field (ii) the magnetic field due to the induced current is in the opposite direction to the original field. In the first case, the two magnetic fields will add to enhance the net flux enclosed by the loop. This will increase the current and hence the magnetic flux enclosed. The process will be regenerative as there is no stabilizing element in this process. In the second case on the other hand, the induced magnetic field opposes the original magnetic field and therefore, the original magnetic field feels an opposition from the induced current. This process is a more appropriate physical process. We, therefore, find that the EMF in the loop is produced in such a way that the magnetic field due to the induced current is in opposite direction to that of the original field. This is called the Lenz's law.

Consider a loop in Fig. 3.13. The length segment **dl** and the area segment **da** are chosen by the right hand rule. Now, to satisfy the Lenz's law the current direction in the loop must be opposite to the direction of the length segment **dl**.

Mathematically the Faraday's law can be written as

$$V = \oint \mathbf{E} \cdot \mathbf{dl} = -\frac{\partial \Phi}{\partial t} \tag{3.45}$$

where V is the net EMF around the loop, and the negative sign is due to the Lenz's law.

Now if the loop has magnetic flux density $\mathbf{B}$, the total flux enclosed by the loop is

$$\Phi = \int \int_S \mathbf{B} \cdot \mathbf{da} \tag{3.46}$$

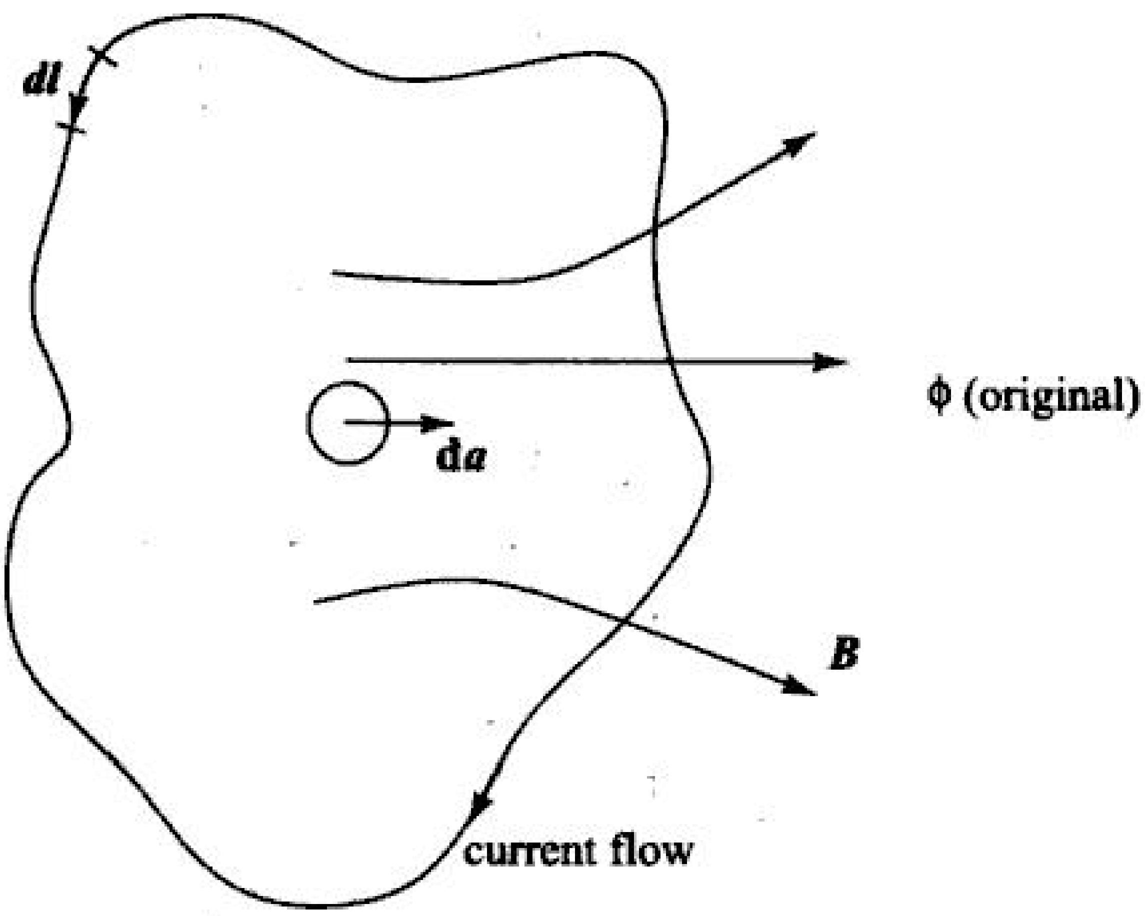

Fig. 3.13 *Schematic for Faraday's law of electromagnetic induction*

The Faraday's law Eqn (3.45) then becomes

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{da} \tag{3.47}$$

From Fig. 3.13 we can note that the magnetic flux enclosed by the loop can be varied in time in three ways

(i) keeping loop area stationary and varying the magnetic flux density with time,
(ii) keeping magnetic flux density static but changing the loop area,
(iii) changing the loop area in a time varying magnetic field.

Since, here we are interested in time varying fields we consider the first case only, that is, the area of the loop is stationary and the magnetic flux is time varying. In this case, we can take the time derivative in Eqn (3.48) inside the integral giving

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\int \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{da} \tag{3.48}$$

Applying the Stoke's theorem (C.29) to Eqn (3.47) we get

$$\oint \mathbf{E} \cdot \mathbf{dl} = \int \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{da} = -\int \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{da} \tag{3.49}$$

$$\Rightarrow \qquad \int \int_S \left\{ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right\} \cdot \mathbf{da} = 0 \tag{3.50}$$

Again, using the argument that Eqn (3.50) has to be valid for any arbitrary area, we get

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.51}$$

This is the Faraday's law of electromagnetic induction in differential form.

Let the observation point be P.
Total electric field at point P (refer Fig. 3.14) is

$$E_A = \frac{10\cos\omega t}{4\pi\epsilon_0 r}$$

$$E_B = -\frac{10\cos\omega t}{4\pi\epsilon_0 r}$$

$$E_{T1} = E_A\cos\theta + E_B\cos\theta = 0$$

$$E_{T2} = E_A\sin\theta - E_B\sin\theta = \frac{20\cos\omega t}{4\pi\epsilon_0 r}\sin\theta$$

Now, $\sin\theta = 5/10 = 1/2$.

$$\Rightarrow \quad E_{T2} = \frac{10\cos\omega t}{4\pi\epsilon_0 r}$$

Therefore, the vector electric field at point P is

$$\mathbf{E} = -\frac{10\cos\omega t}{4\pi\epsilon_0 r}\hat{\mathbf{z}}$$

Now, from Eqn (3.51) we have

$$\mathbf{H} = \frac{1}{j\omega\mu_0}(\nabla\times\mathbf{E}) = \frac{1}{j\omega\mu_0}\left\{\frac{\partial E_z}{\partial y}\hat{\mathbf{x}} - \frac{\partial E_z}{\partial x}\hat{\mathbf{y}}\right\}$$

Now, since $r = \sqrt{x^2+y^2}$, we get
$\partial/\partial x(1/r) = -x/r^3$ and $\partial/\partial y(1/r) = -y/r^3$.
The magnetic field therefore is

$$\mathbf{H} = \frac{1}{j\omega\mu_0}\left(-\frac{10\cos\omega t}{4\pi\epsilon_0}\left\{-\frac{y}{r^3}\hat{\mathbf{x}} + \frac{x}{r^3}\hat{\mathbf{y}}\right\}\right)$$

At point P, $x = \sqrt{(10)^2-(5)^2} = \sqrt{75}$ and $y = 0$, giving

$$\mathbf{H} = -\frac{1}{j\omega\mu_0}\frac{10\cos\omega t}{4\pi\epsilon_0}\left\{\frac{\sqrt{75}}{1000}\hat{\mathbf{y}}\right\}$$

$$= j\frac{\sqrt{3}}{80\pi}\frac{\cos\omega t}{\omega\epsilon_0\mu_0}\hat{\mathbf{y}}$$

3.3 MAXWELL'S EQUATIONS

Maxwell's equations are essentially a compilation of the mathematical equations derived from the basic laws of electromagnetics like the Gauss law, the Ampere's circuital law and the Faraday's law. While compiling the results however, Maxwell encountered a difficulty which was due to inconsistency in the Ampere's law.

Since Eqn (3.56) again has to be valid for any arbitrary volume, the integrand should be identically zero, giving

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{3.57}$$

Equation (3.57) is called the 'continuity equation'.

The difficulty faced by Maxwell was precisely this that the Ampere's law is not consistent with the continuity equation. Let us see how?

The Ampere's law in differential form is given by Eqn (3.44). If we take divergence on both sides of Eqn (3.44), we get

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} \tag{3.58}$$

The divergence of a curl of any vector ($\nabla \cdot (\nabla \times \mathbf{H})$) is identically zero. In other words, according to the Ampere's law, the divergence of $\mathbf{J}$ is identically zero, i.e.

$$\nabla \cdot \mathbf{J} \equiv 0 \tag{3.59}$$

This is inconsistent with the continuity Eqn (3.57) for a general time varying case. The correct modification to the Ampere's law was suggested by Maxwell.

Substituting for ρ from Eqn (3.36) in Eqn (3.57) we get

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{D})$$

$$\Rightarrow \qquad \nabla \cdot \left(J + \frac{\partial \mathbf{D}}{\partial t}\right) = 0 \tag{3.60}$$

Equation (3.60) can be put in integral form by integrating over a volume and applying the divergence theorem, as

$$\oint_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathbf{da} = 0 \tag{3.61}$$

So, in Ampere's law if we regard $J + \frac{\partial D}{\partial t}$ as the total current enclosed by the loop, the law becomes consistent with the continuity equation. The term $\frac{\partial \mathbf{D}}{\partial t}$ obviously has dimension same as the current density $\mathbf{J}$ and is called the 'Displacement current density'. The displacement current is a notion of time varying electric field. For a static field, there is no displacemet current. This is the current which flows even in a medium, which does not have free charges.

At this point, one can distinguish between the current due to movement of free charges, $\mathbf{J}$, and the curent due to time varying fields. The quantity $\mathbf{J}$ which is due to conduction of charges, is called the 'conduction current density', whereas, the rate of change of the electric displacement is called the 'displacement current density'.

In view of above discussion, the Ampere's law can be restated as:

The net magnetomotive force around a closed loop is equal to the total current, which is a combination of conduction and the dispalcement current.

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and $E_y = E_{y0}$, and the electric field will reach to its maximum value E_{max}. In the next quarter period, the field will reduce to zero and in the following half period the field direction would reverse and would go through the same maximum to zero variation.

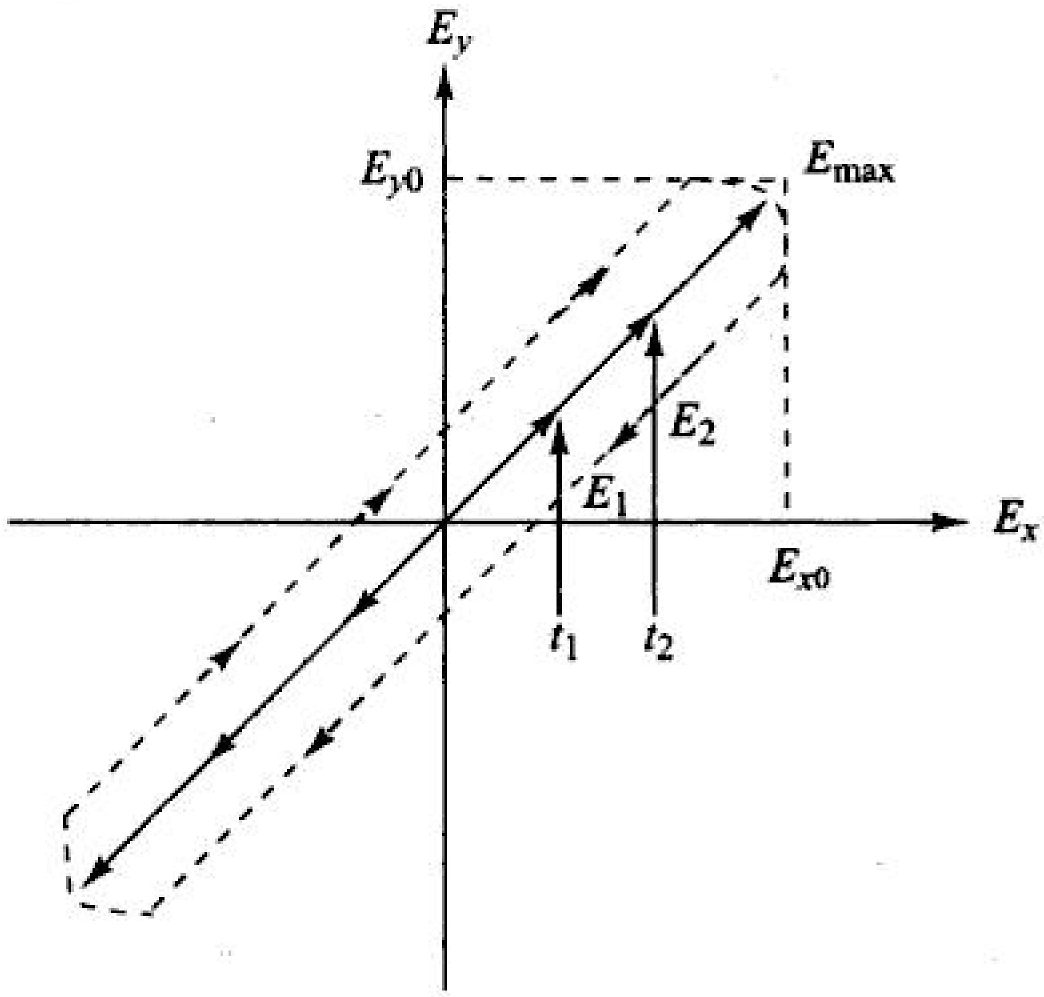

Fig. 4.6 *Linear polarization.*

Depending upon the ratio of E_{y0} and E_{x0} the slope of the line changes.

1. If $E_{x0} = 0$, the line becomes vertical and the wave is called 'vertically polarized wave', or the wave is said to have vertical polarization.
2. If $E_{y0} = 0$, the line becomes horizontal giving rise to a horizontally polarized wave, i.e. horizontal polarization.
3. If $E_{x0} = E_{y0}$, the wave is said to be linearly polarized with 45° polarization angle.

EXAMPLE 4.3 A uniform plane wave travelling in $+z$ direction has two components of the electric field $E_x = 5$ V/m and $E_y = 10\angle 30°$ V/m. At some instant say $t = 0$, the E_x component is maximum. Find the magnitude and direction of the field at $t = 0$ and at $t = 0.1$ nsec. Frequency of the wave is 2 GHz.

Solution:

The E_y component leads the E_x component by $30° = \frac{\pi}{6}$ rad.
The two components of the field can be written as

$$E_x = 5\cos(\omega t)$$

$$E_y = 10\cos(\omega t + \frac{\pi}{6})$$

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Also we get, $\epsilon = \cot^{-1}(+\frac{5}{3}) = 0.54$ rad
The point on the Poincare' sphere has longitude and latitude $(2\tau, 2\epsilon) = (\pi, 1.08)$.

EXAMPLE 4.6 A LHE polarized wave is to be generated using x and y polarized waves. The tilt angle and axial ratio of the ellipse of polarization is 30° and 3 respectively. Find the amplitude and phase of the x and y polarized waves. Assume the wave to be propagating in the $+z$-direction.

Solution:

For the polarization ellipse we have $\tau = 30^\circ = \frac{\pi}{6}$ and $\epsilon = \cot^{-1}(+3) = 18.4^\circ$
Using Eqns (4.89) and (4.90) we obtain

$$\cos 2\gamma = \cos 2\epsilon \cos 2\tau = \cos(36.8^\circ)\cos(60^\circ)$$

$$\Rightarrow \gamma = 33.2^\circ$$

and $$\tan\phi = \frac{\tan 2\epsilon}{\sin 2\tau} = \frac{\tan(36.8^\circ)}{\sin(60^\circ)} = 0.8658$$

$$\Rightarrow \quad \phi = 40.88^\circ$$

If we assume that the wave is propagating in the +ve z direction, and the amplitudes of the x and y polarized waves are E_1 and E_2 respectively, we get

$$\frac{E_2}{E_1} = \tan(\gamma) = 0.65$$

and the phase difference between E_y and E_x is 40.88°

As will be seen in the later sections, every antenna system has a state of polarization, i.e. it maximally responds to a particular state of polarization. If now an electromagnetic wave with different state of polarization falls on the antenna, the power transfer efficiency from the wave to the antenna depends upon how close their states of polarization are. Closer the states of polarization higher the power transfer efficiency. If the two states are completely non-interacting (orthogonal), no power transfer takes place between the wave and the antenna. The antenna and the wave become transparent to each other.

Two orthogonal states of polarization can be used to double the signal transmission capacity of a communication channel. Since the orthogonal states of polarization have no transfer of power between them, one may transmit two waves of same frequency but orthogonal polarizations simultaneously and can increase the information transmission capacity by a factor of two.

The closeliness of the two states of polarization is directly related to the distance between the corresponding points on the Poincare' sphere. If the two states are denoted by points M and M' on the Poincare' sphere respectively (see Fig. 4.14),

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One can find the change over frequency f_T at which the medium behavior changes from conductor to dielectric or vice-versa by making the conduction and the displacement currents equal. At $f = f_T$ we have

$$\omega\epsilon_0\epsilon_r = \sigma \tag{4.102}$$

$$\Rightarrow \qquad f_T = \frac{\sigma}{2\pi\epsilon_0\epsilon_r} \tag{4.103}$$

For copper, taking $\epsilon_r \approx 1$ and $\sigma = 5.6 \times 10^7$ ℧/m, f_T is $\approx 10^{18}$ Hz. For sea water on the other hand, taking $\epsilon_r \approx 80$, and $\sigma = 10^{-3}$ ℧/m, f_T is ≈ 225 kHz.

The concept of complex dielectric constant is quite useful in analysing electromagnetics problems. For a non-ideal dielectric medium, one can first analyse the problem assuming the dielecric to be ideal with dielectric constant ϵ_r. The results for non-ideal dielectric medium can then be obtained by replacing the dielectric constant ϵ_r by ϵ_{rc}.

To analyse the plane wave propagation in a dielectric medium with finite conductivity, we use the concept of the complex dielectric constant. For a conducting medium, the wave Eqns (4.31) and (4.32) respectively, become

$$\nabla^2\mathbf{E} = -\omega^2\mu_0\mu_r\epsilon_0\epsilon_{rc}\mathbf{E} = \sqrt{j\omega\mu_0\mu_r(\sigma + j\omega\epsilon_0\epsilon_r)}\ \mathbf{E} \tag{4.104}$$

$$\nabla^2\mathbf{H} = -\omega^2\mu_0\mu_r\epsilon_0\epsilon_{rc}\mathbf{H} = \sqrt{j\omega\mu_0\mu_r(\sigma + j\omega\epsilon_0\epsilon_r)}\ \mathbf{H} \tag{4.105}$$

Here, we have explicitly written $\mu = \mu_0\mu_r$ and $\epsilon = \epsilon_0\epsilon_{rc}$. μ_0 and ϵ_0 are free space permeability and permittivity respectively. Since, here, we are primarily interested in non-magnetic media, we can take $\mu_r = 1$ in our further discussion of the wave propagation.

The Eqn (4.51) of a plane wave travelling in z direction now becomes

$$\frac{\partial^2 E_x}{\partial z^2} = -\omega^2\mu_0\epsilon_0\epsilon_{rc}E_x \equiv \gamma^2 E_x \tag{4.106}$$

The propagation constant of the wave therefore is

$$\gamma = \sqrt{-\omega^2\mu_0\epsilon_0\epsilon_{rc}} = +j\omega\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon_{rc}} \tag{4.107}$$

Substituting for ϵ_{rc} from Eqn (4.99), the propagation constant of the wave is

$$\gamma = j\omega\sqrt{\mu_0\epsilon_0}\left\{\epsilon_r - j\frac{\sigma}{\omega\epsilon_0}\right\}^{1/2} \tag{4.108}$$

The propagation constant for a conducting medium, therefore, is complex and can be written as

$$\gamma = \alpha + j\beta \tag{4.109}$$

where

$$\alpha = Re(\gamma) = \omega\sqrt{\frac{\mu_0\epsilon_0\epsilon_r}{2}}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon_0^2\epsilon_r^2}} - 1\right]^{1/2} \tag{4.110}$$

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etc at the conductor surface, let us assume that after all transient adjustments have taken place, the field just inside the conductor surface has some value E_0. Due to high attenuation constant α, the field decreases rapidly as the wave propagates deeper in the conductor. The wave amplitude reduces to $1/e$ of its value at the surface, over a distance of $1/\alpha$ and within a distance of few times $1/\alpha$ the field reduces practically to zero. Figure 4.19 shows the amplitude of a wave as a function of depth inside a conductor. The field is, therefore, effectively confined to a layer which is $\sim 1/\alpha$ deep below the surface of the conductor. The thickness of the layer ($\sim 1/\alpha$) decreases as ω and σ increase. At a frequency of tens of MHz, and for conductivity of $\sim 10^7$ ℧/m (good conductor), the thickness lies in the range of μm. The field confinement then is just in the skin of the conductor. This effect is therefore called the 'skin effect', and the thickness of the layer is called the 'skin depth' or depth of penetration. The skin depth is given as

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu_0\sigma}} = \sqrt{\frac{1}{\pi f \mu_0 \sigma}} \tag{4.127}$$

The skin effect can be wisely exploited for shielding an electromagnetic wave. If a region with electromagnetic radiation is covered with a metal sheet of thickness much larger than the skin depth the fields outside the metal cover would be of negligibly small value.

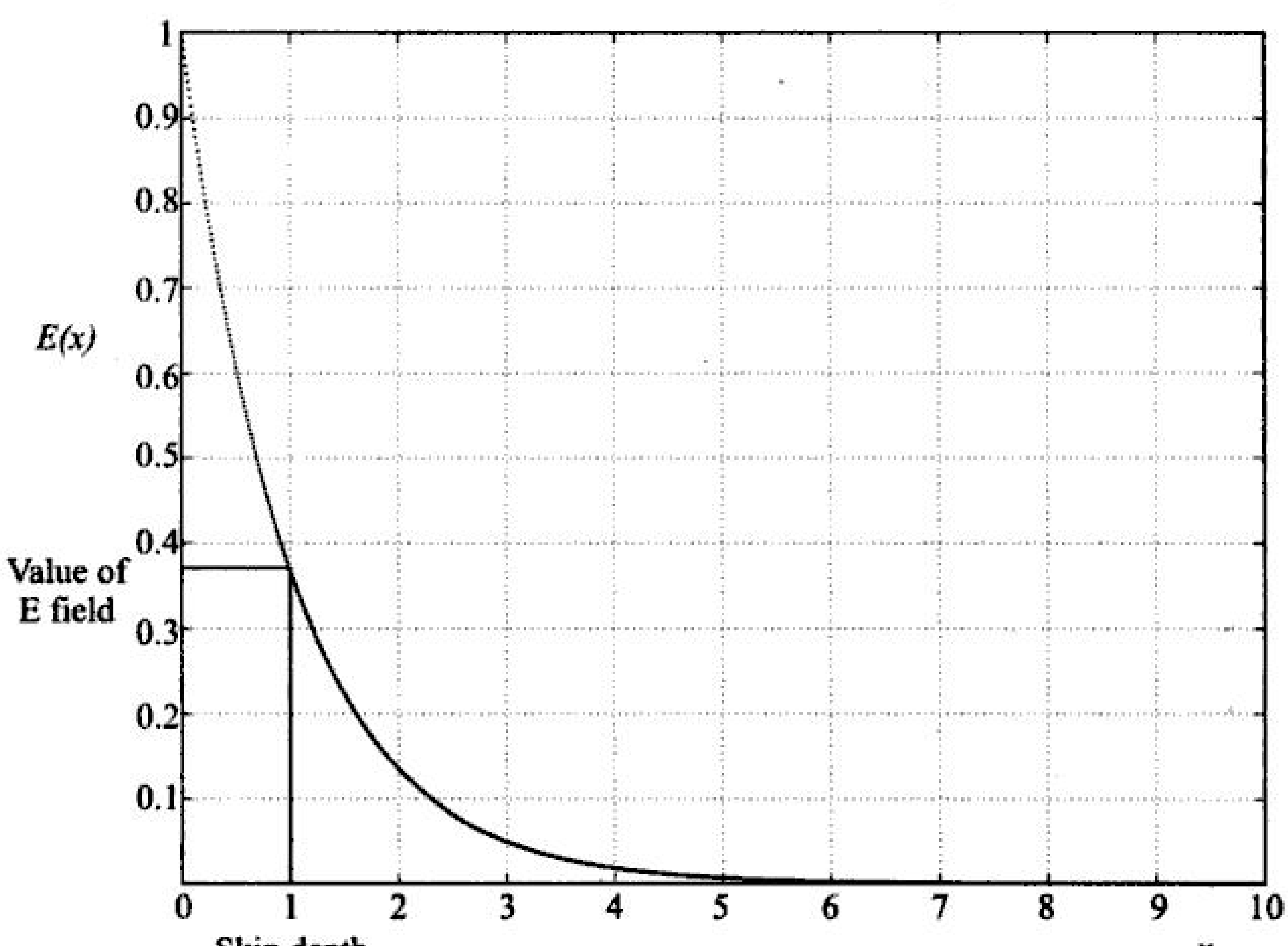

Fig. 4.19 *A decaying electrical field inside a good conductor.*

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to be a 'dispersive medium'. One can then say that a loss-less dielectric is non-dispersive, whereas a lossy medium is dispersive and the dispersion increases with the loss in the medium. For a low-loss medium however, the dispersion is generally small.

4. Good Conductor: For a good conductor $\sigma >> \omega\epsilon$ and the phase constant is (Eqn (4.126))

$$\beta = \sqrt{\frac{\omega\mu_0\sigma}{2}} \tag{4.143}$$

$$\Rightarrow \qquad v_p = \frac{\omega}{\sqrt{\frac{\omega\mu_0\sigma}{2}}} = \sqrt{\frac{2\omega}{\mu_0\sigma}} \tag{4.144}$$

Multiplying numerator and denominator within the square root sign of Eqn (4.144) by $\omega\epsilon_0$ and re-arranging we get

$$v_p = \frac{1}{\sqrt{\mu_0\epsilon_0}}\sqrt{\frac{2\omega\epsilon_0}{\sigma}} = c\sqrt{\frac{2\omega\epsilon_0}{\sigma}} \tag{4.145}$$

For a good conductor $\omega\epsilon_0/\sigma << 1$ and therefore $v_p << c$. The electromagnetic wave therefore slows down considerably in a conductor. As can be seen from the example, the phase velocity of an electromagnetic wave inside copper is few hundred meter/s which is of the order of the velocity of sound in copper.

Compared to a lossy dielectric, the conductor is much dispersive since v_p varies as $\sqrt{\omega}$. It should be noted however, that the dispersion decreases with σ and for an ideal conductor ($\sigma = \infty$) the dispersion is zero. Figures 4.20 and 4.21 show velocity and dispersion as a function of frequency and the conductivity of the medium.

EXAMPLE 4.11 Just outside a train compartment the field strength of a radio station is 0.1 V/m. What will be the approximate field strength inside the compartment? Assume the compartment to be a closed box of metal having conductivity 5×10^6 ℧/m. The thickness of the compartment wall is 5 mm. Frequency of radio station is 600 kHz.

Solution:

The attenuation constant for the compartment material (assuming the compartment material non-magnetic)

$$\alpha = \sqrt{\pi f \mu_o \sigma} = \sqrt{\pi \times 600 \times 10^3 \times 4\pi \times 10^{-7} \times 5 \times 10^6} = 3441.44 \text{ napers/m.}$$

The wave amplitude after passing through the compartment wall will be

$$E = E_o e^{-\alpha z} = 0.1 e^{-17.2} = 3.366 \times 10^{-9} \text{ V/m}$$

The wave, therefore, is attenuated by a factor $\sim 3 \times 10^7$.

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The average power density is $150\sqrt{2}\ \mathrm{W/m^2} = 212.132\ \mathrm{W/m^2}$ and the power flows in the direction $\frac{(\hat{\mathbf{x}}+\hat{\mathbf{z}})}{\sqrt{2}}$, i.e. in the xz plane at an angle of 45° with respect to x and z axes.

4.8.2 Power Density of a Uniform Plane Wave

For a uniform plane wave, **E** and **H** are perpendicular to each other and the ratio of their magnitudes is equal to the intrinsic impedance of the medium, η. Without losing generality let us take the **E**-field oriented along the $+x$ direction and the **H**-field oriented along the $+y$ direction as

$$\mathbf{E} = E_0 e^{j\omega t}\hat{\mathbf{x}} \tag{4.176}$$

$$\mathbf{H} = H_0 e^{j\omega t}\hat{\mathbf{y}} = \frac{\mathbf{E_0}}{\eta}\mathbf{e}^{j\omega t}\hat{\mathbf{y}} \tag{4.177}$$

The average power density of the wave is

$$\mathbf{P_{av}} = \frac{1}{2}Re\{\mathbf{E}\times\mathbf{H}^*\} \tag{4.178}$$

$$= \frac{1}{2}Re\{E_0 e^{j\omega t}[\frac{E_0}{\eta}e^{j\omega t}]^*\}\hat{\mathbf{x}}\times\hat{\mathbf{y}} \tag{4.179}$$

$$= \frac{1}{2}Re\{\frac{|E_0|^2}{\eta^*}\}\hat{\mathbf{z}} = \frac{1}{2}Re\{\eta|H_0|^2\}\hat{\mathbf{z}} \tag{4.180}$$

$$\Rightarrow \qquad P_{av} = \frac{|E_0|^2}{2}Re\{\frac{1}{\eta^*}\} \equiv \frac{|H_0|^2}{2}Re\{\eta\} \tag{4.181}$$

(a) For a **loss-less dielectric medium** $\eta = \sqrt{\frac{\mu}{\epsilon}}$ = Real number
The average power density of the wave is

$$P_{av} = \frac{1}{2}\frac{|E_0|^2}{\eta} = \frac{1}{2}|E_0|^2\sqrt{\frac{\epsilon}{\mu}} \tag{4.182}$$

(b) For a **lossy medium** however, η is complex, and **E** and **H** are not in time phase. Consequently, P_{av} has to be calculated using exact Eqn (4.181).

(c) For a **good conductor** $\sigma >> \omega\epsilon$, and $\eta \approx \sqrt{\frac{\omega\mu}{2\sigma}} + j\sqrt{\frac{\omega\mu}{2\sigma}}$. The phase angle between **E** and **H** is approximately 45°, and the average power density is

$$P_{av} = \frac{1}{2}|E_0|^2 Re\{\frac{1}{\eta^*}\} = \frac{1}{2}\frac{|E_0|^2}{|\eta|^2}Re\{\eta\} \tag{4.183}$$

$$= \frac{1}{2}|E_0|^2\sqrt{\frac{\sigma}{2\omega\mu}} \tag{4.184}$$

The resistance of the slab is

$$dR = \frac{\rho l}{A} = \frac{1}{\sigma dz} \tag{4.197}$$

The ohmic loss in the slab is

$$dW = \frac{1}{2}|I(z)|^2 dR \tag{4.198}$$

Substituting for $I(z)$ from Eqn (4.189) we get

$$dW = \frac{1}{2}|\sigma E_0 e^{-\gamma z} dz|^2 \frac{1}{\sigma dz} \tag{4.199}$$

$$= \frac{1}{2}\sigma |E_0|^2 e^{-2\alpha z} dz \tag{4.200}$$

The total loss per unit area of the conductor surface can be obtained by integrating Eqn (4.200) from $z = 0$ to ∞ as

$$W = \frac{1}{2}\int_0^{\infty} \sigma |E_0|^2 e^{-2\alpha z} dz \tag{4.201}$$

$$= \frac{1}{2}\sigma |E_0|^2 \left[\frac{e^{-2\alpha z}}{-2\alpha}\right]_0^{\infty} \tag{4.202}$$

$$\Rightarrow \quad W = \frac{1}{2}\frac{\sigma |E_0|^2}{2\alpha} = \frac{1}{2}\frac{\sigma}{2\alpha}\frac{|\gamma|^2}{\sigma^2}|\mathbf{J}_s|^2 \tag{4.203}$$

Substituting for γ and α from Eqn (4.186) we get

$$W = \frac{1}{2}\frac{|\mathbf{J}_s|^2 \omega \mu_0 \sigma}{2\sigma\sqrt{\omega\mu_0\sigma/2}} \tag{4.204}$$

$$= \frac{1}{2}|\mathbf{J}_s|^2 \sqrt{\frac{\omega\mu_0}{2\sigma}} \tag{4.205}$$

$$= \frac{1}{2}R_s |\mathbf{J}_s|^2 \tag{4.206}$$

The power loss, therefore, is proportional to the surface resistance ($R_s = \sqrt{\omega\mu_0/2\sigma}$) which increases with frequency and decreases with the conductivity. It is then interesting to see that as the conductivity increases, the wave attenuates rapidly inside the conductor but this attenuation is not due to the ohmic loss. This case, therefore, is not similar to a lossy transmission line where the power is lost in the heating of the line due to ohmic loss. This means that, as the conductivity increases, the energy finds it difficult to enter the conducting surface. For ideal conductor, i.e. for $\sigma = \infty$, there is no penitration of the wave. The current flows only on the conductor surface and there is no power loss.

EXAMPLE 4.15 The magnetic field at the surface of a good conductor is 2 A/m. The frequency of the field is 600 MHz. If the conductivity of the

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therefore, necessary to formulate the wave function for a wave which is travelling at an arbitrary angle with respect to the coordinate axes.

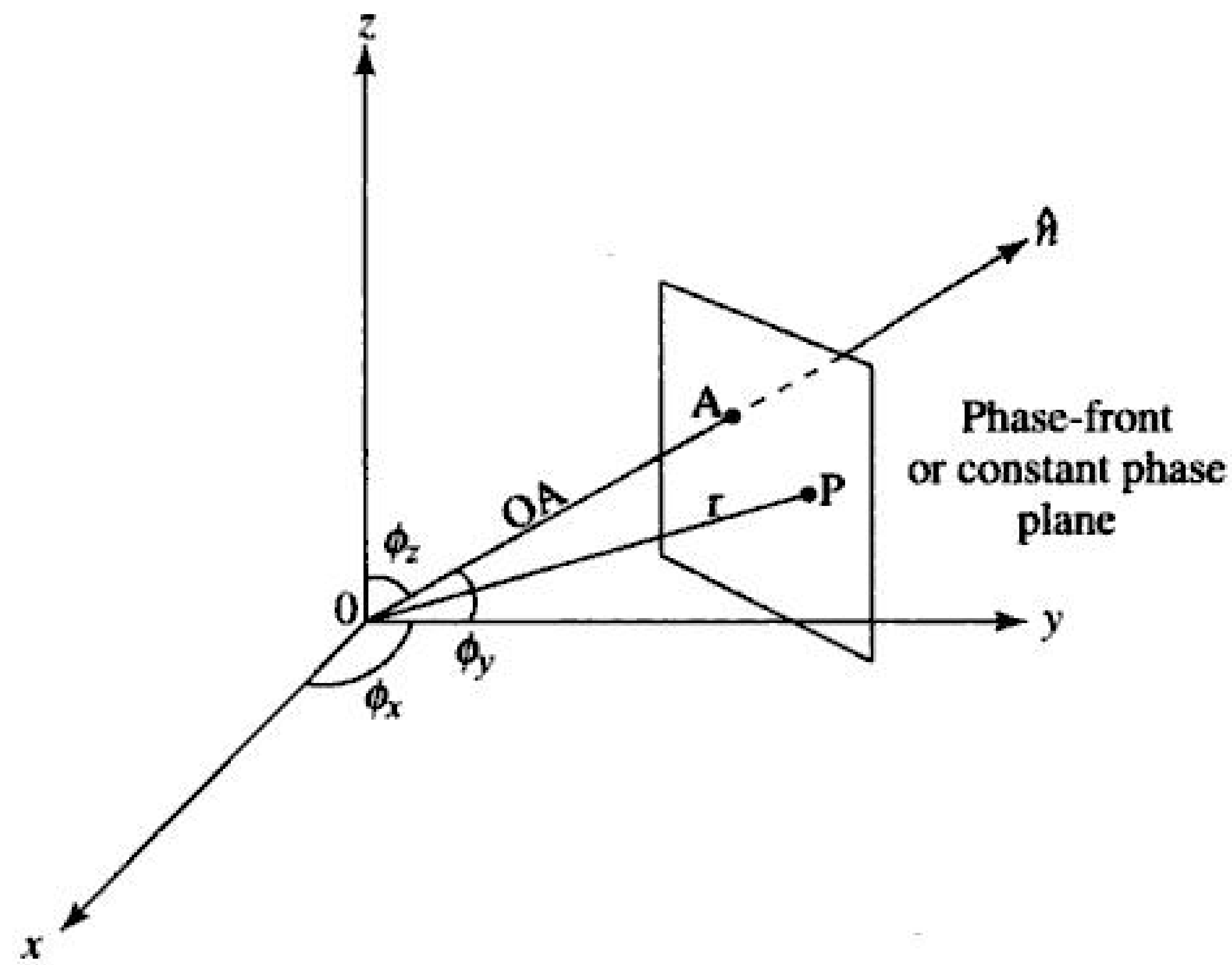

Fig. 5.1 *Wave travelling in an arbitrary direction with respect to the coordinate axes.*

5.1 PLANE WAVE IN ARBITRARY DIRECTION

As we have seen earlier, a plane wave is described by a phase front which is a plane perpendicular to the direction of the wave motion. As shown in Fig. 5.1, let us consider a wave travelling in some arbitrary direction, and let the unit vector in the direction of wave motion be denoted by $\hat{\mathbf{n}}$. If the unit vector $\hat{\mathbf{n}}$ makes angles ϕ_x, ϕ_y, ϕ_z respectively with the three axes x, y, z we have

$$\hat{\mathbf{n}} = \cos\phi_x \hat{\mathbf{x}} + \cos\phi_y \hat{\mathbf{y}} + \cos\phi_z \hat{\mathbf{z}} \tag{5.1}$$

where $\cos\phi_x$, $\cos\phi_y$ and $\cos\phi_z$ are called the direction cosines of the vector $\hat{\mathbf{n}}$.

The planes perpendicular to $\hat{\mathbf{n}}$ are then the phase fronts or the constant phase planes. Let us consider one of the phase fronts as shown in Fig. 5.1, and let any point P on the plane have coordinates (x, y, z). The vector **OP** can be written as

$$\mathbf{OP} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \equiv \mathbf{r} \tag{5.2}$$

From Fig. 5.1 we can see that the dot product

$$\hat{\mathbf{n}} \cdot \mathbf{OP} = \hat{\mathbf{n}} \cdot \mathbf{r} = |\mathbf{OA}| \tag{5.3}$$

$$= \text{Normal distance of the plane from the origin}$$

Equation (5.3) is valid for any point on the phase front and hence, we can write the equation of the phase front as

$$\hat{\mathbf{n}} \cdot \mathbf{OP} = \hat{\mathbf{n}} \cdot \mathbf{r} = \text{constant} \tag{5.4}$$

$$= \mathbf{E_0} e^{-j\frac{8\pi}{3}(\frac{x}{2}+\frac{y}{\sqrt{2}}+\frac{z}{2})}$$

$\mathbf{E_0}$ is a constant vector perpendicular to the wave normal.

$$\mathbf{E_0} = E_{0x}\hat{\mathbf{x}} + E_{0y}\hat{\mathbf{y}} + E_{0z}\hat{\mathbf{z}}$$

Since, the wave is linearly polarized E_{0x}, E_{0y} and E_{0z} should be in phase. Therefore, without losing generality, let us assume them to be real.
Since, for uniform plane wave $\mathbf{E_0}$ is perpendicular to $\hat{\mathbf{n}}$, we have

$$\hat{\mathbf{n}} \cdot \mathbf{E_0} = \frac{E_{0x}}{2} + \frac{E_{0y}}{\sqrt{2}} + \frac{E_{0z}}{2} = 0 \tag{1}$$

It is given that,

$$E_{0x} = 2E_{0y} \tag{2}$$

and the peak amplitude $= \sqrt{E_{0x}^2 + E_{0y}^2 + E_{0z}^2} = 10\ \text{V/m}$ (3)
Solving (1), (2) and (3), we get,

$$E_{0x} = 4.9\ \text{V/m}$$
$$E_{0y} = 2.45\ \text{V/m}$$
$$E_{0z} = -8.37\ \text{V/m}$$

and the vector electric field is given as

$$\mathbf{E} = (4.9\hat{\mathbf{x}} + 2.45\hat{\mathbf{y}} - 8.37\hat{\mathbf{z}})e^{-j\frac{8\pi}{3}(\frac{x}{2}+\frac{y}{\sqrt{2}}+\frac{z}{2})}$$

Using Eqn (5.15), the vector magnetic field can be obtained as

$$H = \frac{1}{\omega\mu_0}(\mathbf{k} \times \mathbf{E}) = \frac{\beta}{\omega\mu_0}(\hat{\mathbf{n}} \times \mathbf{E_0})e^{-j\beta\hat{\mathbf{n}}\cdot\mathbf{r}}$$

$$= \sqrt{\frac{\epsilon_0}{\mu_0}} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ 1.5496 & 0.7748 & -2.6454 \end{vmatrix} e^{j\beta\hat{\mathbf{n}}\cdot\mathbf{r}}$$

$$= \frac{1}{120\pi}(-7.14\hat{\mathbf{x}} + 6.63\hat{\mathbf{y}} - 2.24\hat{\mathbf{z}})e^{-j\frac{8\pi}{3}(\frac{x}{2}+\frac{y}{\sqrt{2}}+\frac{z}{2})}\ \text{A/m}$$

[Note: The intrinsic impedance of the free-space is is $\sqrt{\mu_0/\epsilon_0} = 120\pi$.]

5.1.1 Phase Velocity and Wavelength

As seen in the previous section, the choice of coordinate system is generally guided by the media boundaries, etc. and the wave travels in an arbitrary direction. It is, however, useful to find the velocity of the wave along the principal coordinate axes. As will be seen in the following sections, the phase velocity of a wave along the principal axes is not a simple vector resolution of the phase velocity of the

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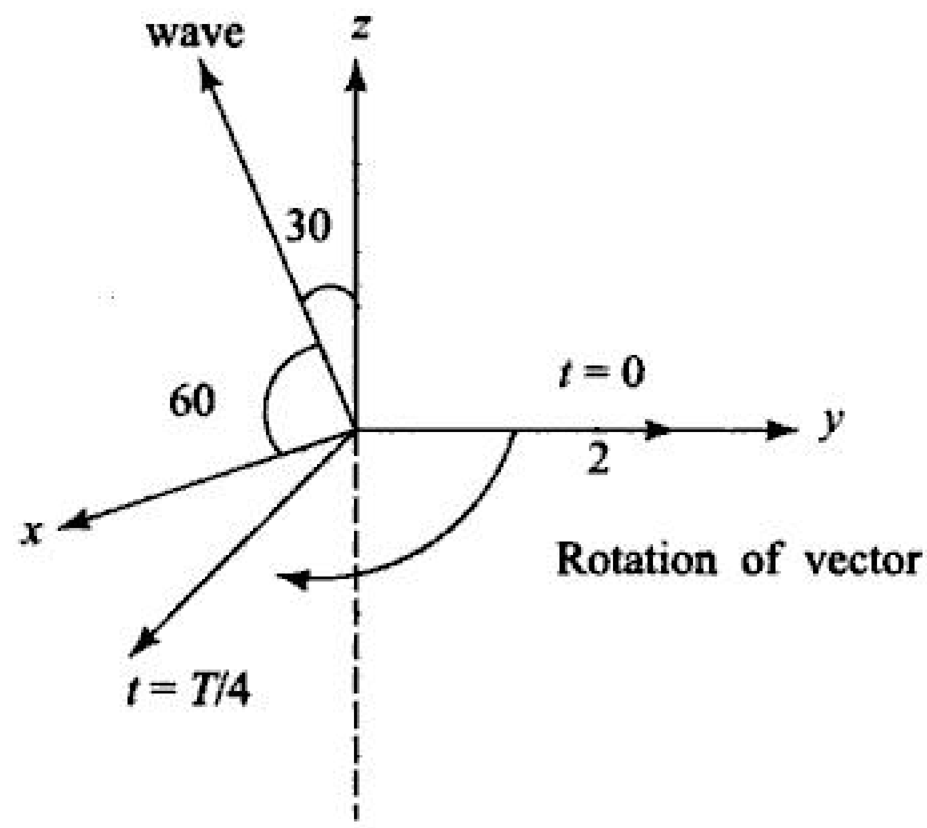

Fig. 5.5 *Figure for Example 5.2*

$$\Rightarrow \quad \hat{\mathbf{n}} = \frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{z}}$$

$$\mathbf{E}\cdot\mathbf{H} = (-j3 + j4 - j) = 0$$

$$\mathbf{E}\cdot\hat{\mathbf{n}} = (-j\frac{\sqrt{3}}{2} + j\frac{\sqrt{3}}{2}) = 0$$

$$\mathbf{H}\cdot\hat{\mathbf{n}} = \frac{1}{30\pi}(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}) = 0$$

The fields in Fig. 5.5, therefore, represent fields of a uniform plane wave. Now taking the coefficient of $-j$ in the exponent, we get,

$$\beta\hat{\mathbf{n}}\cdot\mathbf{r} = 0.02\pi(x + \sqrt{3}z) = 0.02\pi(\hat{\mathbf{x}} + \sqrt{3}\hat{\mathbf{z}})(x\hat{\mathbf{x}} + z\hat{\mathbf{z}})$$

$$\Rightarrow \beta\hat{\mathbf{n}} = 0.02\pi(\hat{\mathbf{x}} + \sqrt{3}\hat{\mathbf{z}}) = 0.04\pi\left(\frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{z}}\right)$$

$$\Rightarrow \beta = 0.04\pi$$

The components of **E** and **H** are not in time phase and consequently the fields are not linearly polarized. For the fields the z-component leads the y-component by $\frac{\pi}{2}$ and as the x-component lags the y-component by $\pi/2$. At some instant when the y-component is at its positive peak, the x-component and the z-component are zero. At that instant we have

$$\mathbf{E} = 2\hat{\mathbf{y}} \Rightarrow |\mathbf{E}| = 2$$

and

$$\mathbf{H} = \frac{1}{30\pi}(\sqrt{3}\hat{\mathbf{x}} - \hat{\mathbf{z}}) \qquad \Rightarrow |H| = \frac{2}{30\pi}$$

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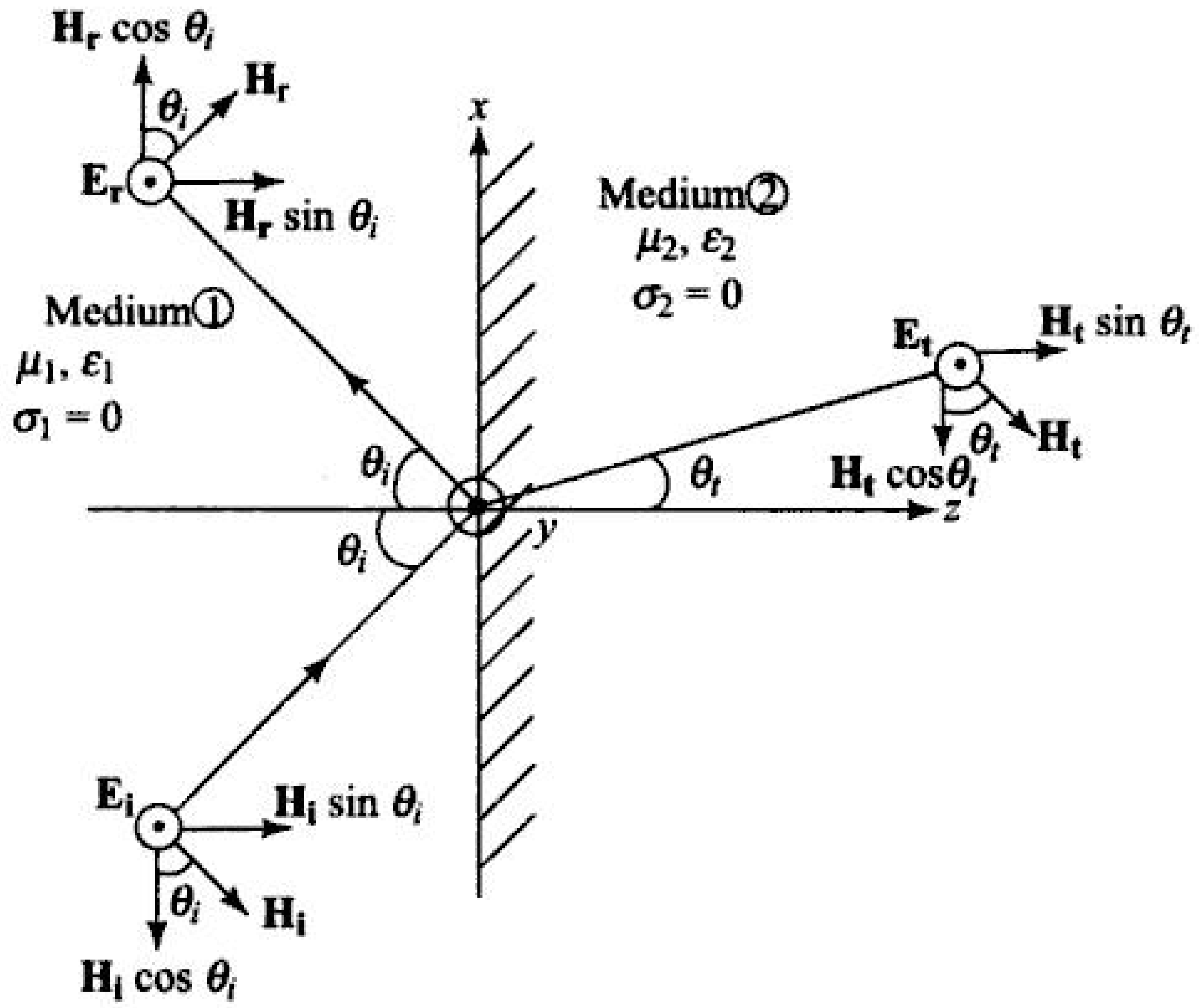

Fig. 5.11 *Uniform plane wave with perpendicular polarization at a media interface.*

the paper), and the angle of reflection is same as the angle of incidence. One may further argue that since the tangential component of **E** should be continuous at the interface and since, the incident wave has only y-directed electric field, the reflected and transmitted waves must also have y-directed **E** field only. Of course, the fields might point into the plane of paper or point out of it. However, the direction reversal can easily be accommodated by assigning a +ve or -ve sign to a field. Therefore, without losing generality we assume that both fields $\mathbf{E_r}$ and $\mathbf{E_t}$ are along $+y$ direction, i.e. pointing outwards from the plane of the paper. If any of them or both of them were directed in the opposite direction (along -y-direction) their signs would come negative automatically.

Now for each wave, the Poynting vector $\mathbf{E} \times \mathbf{H}$ should give the direction of the wave vector. Since, the directions of the incident, reflected and transmitted waves are known, the directions of the magnetic field for the three waves $\mathbf{H_i}$, $\mathbf{H_r}$ and $\mathbf{H_t}$ can be easily obtained as shown in Fig. 5.11. Note that, the incident and transmitted waves are travelling from left to right upwards and consequently the magnetic fields point downwards. Whereas, the reflected wave is travelling from right to left upwards and hence its magnetic field is oriented upwards.

We can write the electric fields for the three waves as (see Eqns (5.31), (5.33) and (5.34))

$$\text{Incident wave:} \quad \mathbf{E_i} = \mathbf{E_{i0}} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \tag{5.43}$$

$$\text{Reflected wave:} \quad \mathbf{E_r} = \mathbf{E_{r0}} e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)} \tag{5.44}$$

$$\text{Transmitted wave:} \quad \mathbf{E_t} = \mathbf{E_{t0}} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \tag{5.45}$$

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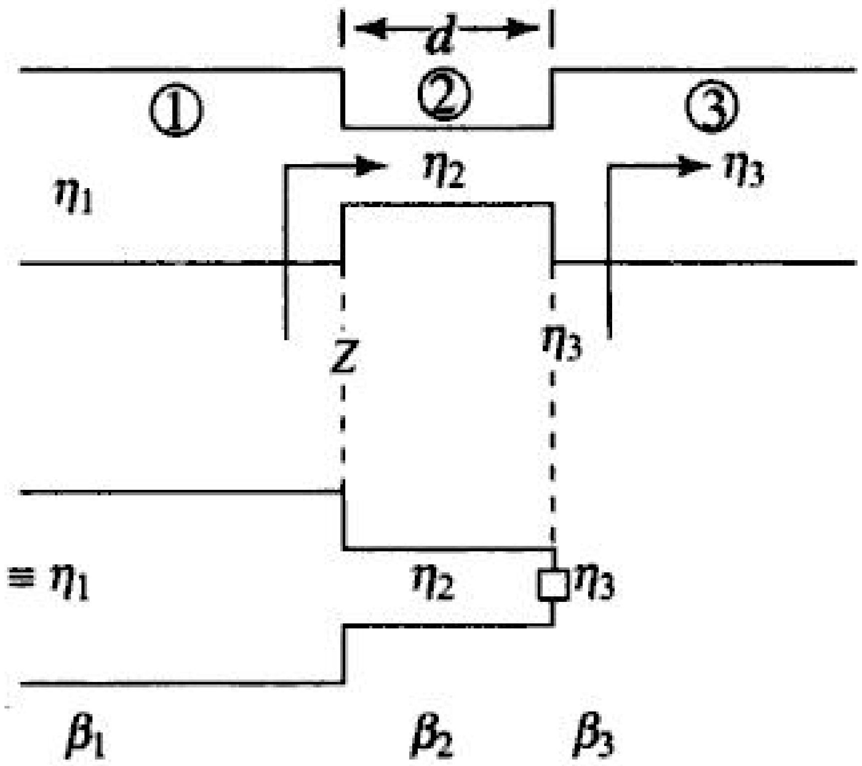

Fig. 5.17 *Transmission line analogy for a layered medium.*

an impedance (say) Z at (1/2) junction as

$$Z = \eta_2 \left[\frac{\eta_3 \cos \beta_2 d + j\eta_2 \sin \beta_2 d}{\eta_2 \cos \beta_2 d + j\eta_3 \sin \beta_2 d} \right] \tag{5.90}$$

Note that the characteristic impedance of the transforming line 2 is η_2.

The reflection coefficient on the line 1 then can be written as

$$\Gamma = \frac{Z - \eta_1}{Z + \eta_1} \tag{5.91}$$

$$= \frac{\eta_2(\eta_3 \cos \beta_2 d + j\eta_2 \sin \beta_2 d) - \eta_1(\eta_2 \cos \beta_2 d + j\eta_3 \sin \beta_2 d)}{\eta_2(\eta_3 \cos \beta_2 d + j\eta_2 \sin \beta_2 d) - \eta_1(\eta_2 \cos \beta_2 d + j\eta_3 \sin \beta_2 d)} \tag{5.92}$$

Equations (5.87) and (5.92) although look algebraically different, are infact the same. It is only a matter of algebraic manipulation to reduce one to the other.

The magnitude of the transmission coefficient can be obtained by applying conservation of power.

The electric field E_i in medium 1 has a power density of $|E_i|^2/\eta_1$. The power density of the reflected wave will be $|\Gamma E_i|^2/\eta_1$. Since, all the media are lossless, the difference of the two power densities is equal to the power density of the transmitted wave in medium 3, $|E_t|^2/\eta_3$, giving

$$\frac{|E_i|^2}{\eta_1} - \frac{|\Gamma E_i|^2}{\eta_1} = \frac{|E_t|^2}{\eta_3} = \frac{|\tau E_i|^2}{\eta_3} \tag{5.93}$$

$$\Rightarrow \qquad |\tau| = \sqrt{\frac{|E_t|^2}{|E_i|^2}} = \sqrt{\frac{\eta_3}{\eta_1}(1 - |\Gamma|^2)} \tag{5.94}$$

The above analysis clearly shows that the problem of normal incidence on a multi-layer medium can be elegently solved using the transmission line concepts.

The analysis of layered medium finds many practical applications. For example, one may be interested in sending electromagnetic energy efficiently from one medium to another, or one may be interested in providing a protective sheet

the geometric mean of the intrinsic impedances of the two media. This technique is frequently used in realizing anti-reflecting coatings in optical components.

EXAMPLE 5.6 A uniform plane wave having 10 W/m^2 power density is normally incident on a 5 cm thick dielectric sheet with $\epsilon_r = 9$. If the frequency of the wave is 1 GHz. Find the power density of the wave transmitted through the sheet.

Solution:

Intrinsic impedance of the dielectric is

$$\eta_d = \frac{\eta_0}{\sqrt{9}} = \frac{\eta_0}{3}$$

$$\Gamma_{12} = \frac{\eta_d - \eta_0}{\eta_d + \eta_0} = \frac{\frac{\eta_0}{3} - \eta_0}{\frac{\eta_0}{3} + \eta_0} = -\frac{1}{2}$$

$$\Gamma_{23} = -\Gamma_{12} = \frac{1}{2}$$

$$\tau_{12} = \frac{2\eta_d}{\eta_d + \eta_0} = \frac{1}{2}$$

$$\tau_{23} = \frac{2\eta_0}{\frac{\eta_0}{3} + \eta_0} = \frac{3}{2}$$

$$\beta_2 = \omega\sqrt{\mu_0\epsilon_0\epsilon_r} = \frac{2\pi \times 10^9}{3 \times 10^8} \times \sqrt{9} = 20\pi \text{ rad/m}$$

$$\beta_2 d = 20\pi(0.05) = \pi$$

The transmission coefficient from Eqn (5.89) is

$$\tau = \frac{(\frac{1}{2})(\frac{3}{2})e^{-j\pi}}{1 - (-\frac{1}{2})(\frac{1}{2})e^{-j2\pi}} = \frac{-\frac{3}{4}}{\frac{5}{4}} = -\left(\frac{3}{5}\right)$$

Power density of the transmitted wave

$$= |\tau|^2 \times \text{Power density of the incident wave}$$

$$= \frac{9}{25} \times 10 = \frac{18}{5} \text{ W/m}^2$$

5.5 TOTAL INTERNAL REFLECTION

Let us now investigate an interesting case of the wave propagation known as the 'total internal reflection (TIR)'.

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the two polarizations separately and combine them. Let us, therefore, write the incident electric field as

$$\mathbf{E_i} = \mathbf{E_{i\parallel}} + \mathbf{E_{i\perp}} e^{j\phi} \tag{5.116}$$

Note that locally the coordinate system is oriented such that $\mathbf{E_{i\parallel}}$ is along x'-axis and $\mathbf{E_{i\perp}}$ is along y'-axis and z' is the direction of the wave vector (see Fig. 5.19).

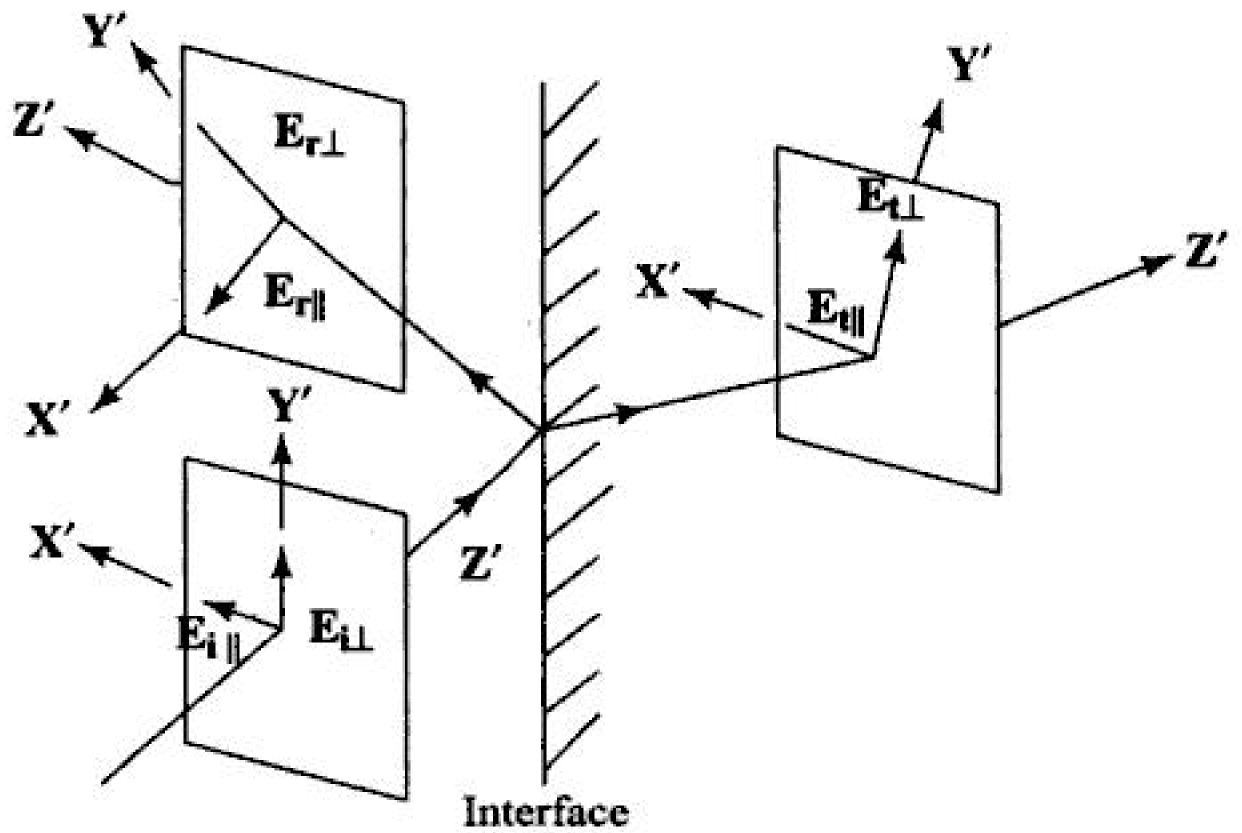

Fig. 5.19 *Polarization of incident, reflected and transmitted plane waves at media interface.*

If the reflection and transmission coefficients for parallel and perpendicular polarizations are denoted by $\Gamma_\parallel$, $\tau_\parallel$ and $\Gamma_\perp$, $\tau_\perp$ respectively, the reflected and transmitted electric fields are

$$\mathbf{E_r} = \mathbf{E_{r\parallel}} + \mathbf{E_{r\perp}} = \Gamma_\parallel \mathbf{E_{i\parallel}} + \Gamma_\perp \mathbf{E_{i\perp}} e^{j\phi} \tag{5.117}$$

$$\mathbf{E_t} = \mathbf{E_{t\parallel}} + \mathbf{E_{t\perp}} = \tau_\parallel \mathbf{E_{i\parallel}} + \tau_\perp \mathbf{E_{i\perp}} e^{j\phi} \tag{5.118}$$

since the reflection and transmission coefficients are different for parallel and perpendicular polarizations, we can say that in general the states of polarization of the reflected and transmitted waves are not same as that of the incident wave. However, it is worthwhile to investigate here a few simple but important cases.

5.6.1 Change in Polarization at Simple Reflection

1. Linearly Polarized Incident Wave If the incident wave is linearly polarized, $\mathbf{E_{i\parallel}}$ and $\mathbf{E_{i\perp}}$ are in phase, i.e. $\phi = 0$. For simple reflection (not total internal reflection) since the reflection coefficients $\Gamma_\parallel$ and $\Gamma_\perp$ are real (though they could be positive or negative), the components of the reflected wave, $\mathbf{E_{r\parallel}}$ and $\mathbf{E_{r\perp}}$ are either in phase or 180° out of phase (depending upon the sign of $\Gamma_\parallel$ and $\Gamma_\perp$). The polarization of the reflected wave hence remains linear. The orientation

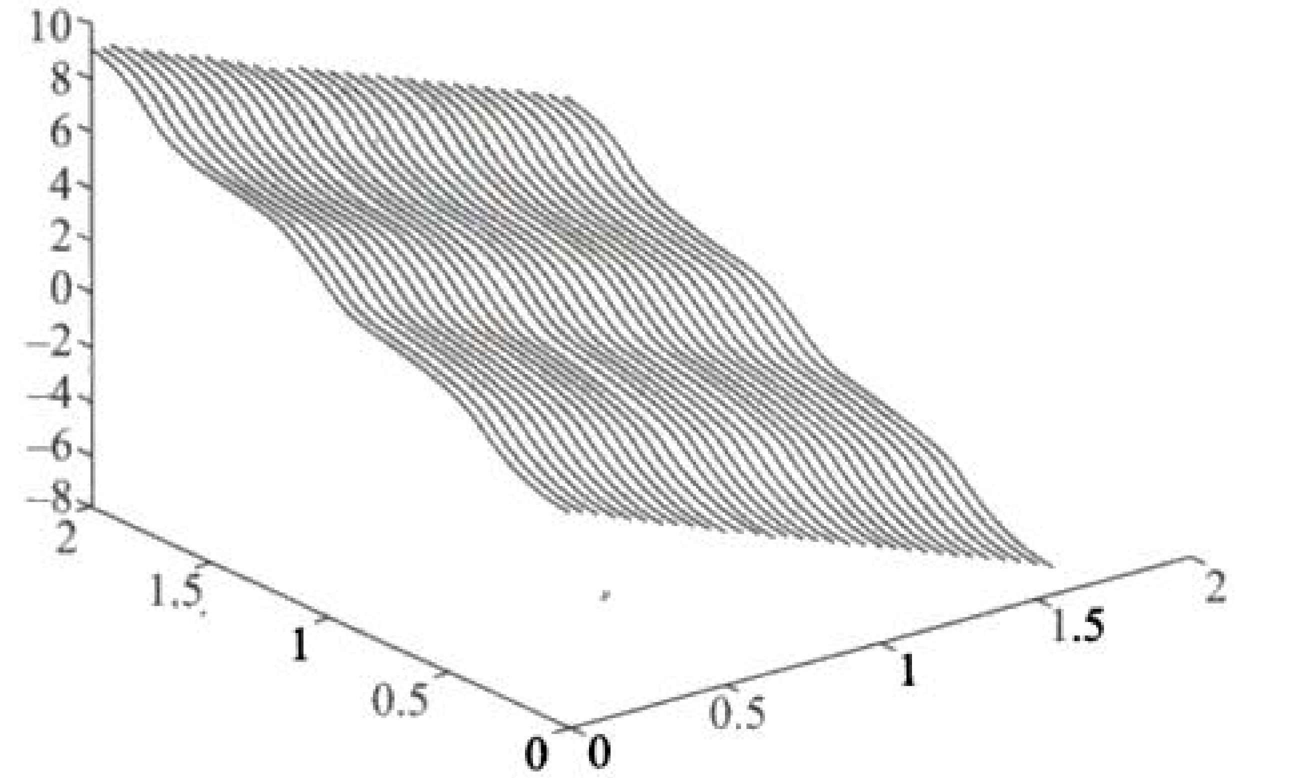

Fig. 5.23 *Spatial variation of phase for the waves at a media interface.*

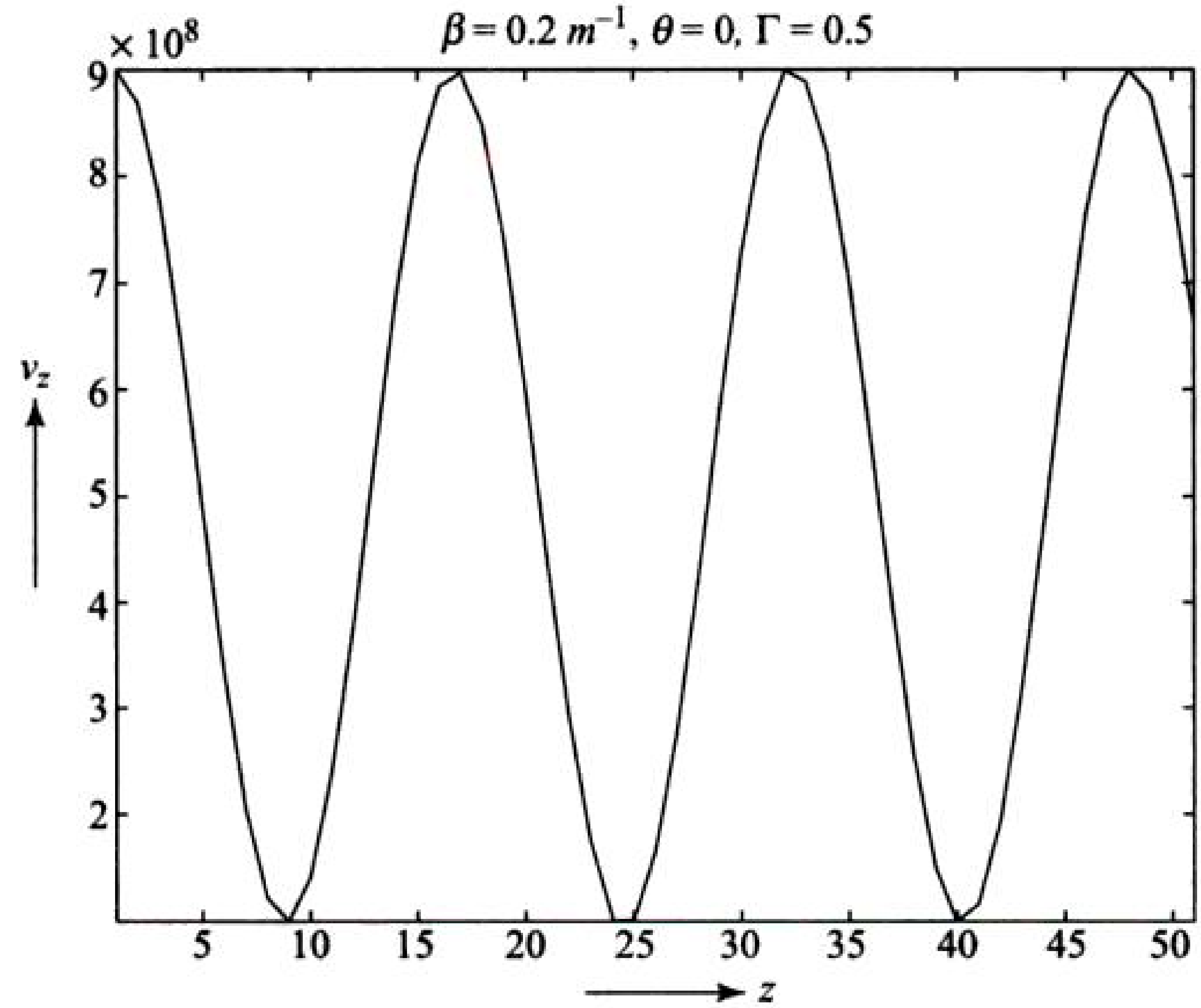

Fig. 5.24 *Variation of phase velocity in z-direction, v_z as a function of z.*

Medium 2 In medium 2 the fields are rather simple as there is only one travelling wave. The amplitude of the field is

$$|\mathbf{E_2}| = \tau_\perp |\mathbf{E_{i0}}| \tag{5.144}$$

and the phase of the wave (including the time phase) is

$$\phi_2 = \omega t - \beta_2 x \sin\theta_t + \beta_2 z \cos\theta_t \tag{5.145}$$

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4 respectively, find the vector electric and magnetic fields. What are the phase constants and phase velocities of the wave in the x, y and z-directions? Frequency of the wave is 10 MHz.

5.5 Two uniform plane waves of equal strengths, one travelling along the $+z$-direction and other travelling in the xz-plane at an angle of 60° with respect to the $+z$-axis, co-exist in a medium. The dielectric constant of the medium is 4. Find the phase velocity of the compositie wave along the z-direction. Also find the constant phase and constant electric field amplitude surfaces of the combined wave.

5.6 A 100 MHz plane wave is launched at an angle θ from the ground. The wavelength along the ground is measured to be 5 m. Find the value of θ and the phase velocity of the wave along the ground surface and in the vertical direction. What is the group velocity of the wave in the vertical direction?

5.7 An elliptically polarized TEM wave travels in vacuum in the xy-plane with direction cosines (0.8, 0.6, 0). The axial ratio of the ellipse is 2.5 and the major axis of the ellipse lies in the xy-plane. Write the expression for the electric and magnetic fields if the peak electric field is 20 V/m.

5.8 A uniform wave is incident from air on an infinitely thick medium at the angle of incidence of 35°. Find the angle of reflection and angle of transmission. The medium has $\mu_r = 49$ and $\epsilon_r = 6$. What is the phase velocity of the wave along the media interface?

5.9 A dielectric interface is along z = 0 plane. The medium for z > 0 is air and for z < 0 is water with refractive index 1.33. If the incident wave vector has direction cosines (0.4, 0.5, ℓ). Find the direction cosines of the wave vectors of the reflected and transmitted waves.

5.10 When a light beam enters a dielectric medium from air, its path is deviated by 20° and is slowed down by a factor 1.5. What is the phase velocity of the wave along the dielectric air interface?

5.11 A plane wave having peak electric field of 25 V/m is incident at a air dielectric interface with perpendicular polarization. The dielectric constant of the medium is 2.4 and the angle of incidence is 30°. Find the power density of the incident, reflected and transmitted waves.

5.12 Show that a circularly polarized wave cannot remain circularly polarized after reflecting from a dielectric interface.

5.13 A light beam is incident on a glass slab at an angle of incidence of 50°. The light is linearly polarized and the plane of polarization makes an angle of 30° with the plane of incidence. By what angle does the plane of polarization rotate when the beam gets inside the glass slab?

5.14 A plane wave is incident with parallel polarization at a dielectric interface. The two media have parameters $\mu_1 = \mu_0$, $\epsilon_{r1} = 3$, $\mu_2 = 10\mu_0$, $\epsilon_{r2} = 2$. Find the expressions for the incident, reflected and transmitted waves if the angle between the reflected and transmitted wave vectors is 90°. The peak incident electric field is 10 V/m.

5.15 A 1 GHz electromagnetic wave is normally incident on a 3 cm thick plastic slab of dielectric constant 5. What per cent of the incident power is transmitted through the slab?

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of the magnetic field has to be identically zero. Since $m = 0$ corresponds to $\theta = \pi/2$, it can be concluded that no wave can be launched between two conducting boundaries with its electric field parallel to them if $m = 0$. This can also be seen from the following argument.

Since $m = 0$ means no field variation in x-direction, the electric field is constant along the x-direction. However the electric field which is tangential to the boundary has to be zero at $x = 0$ and $x = d$. This is possible only if the field is identically zero everywhere between the boundaries.

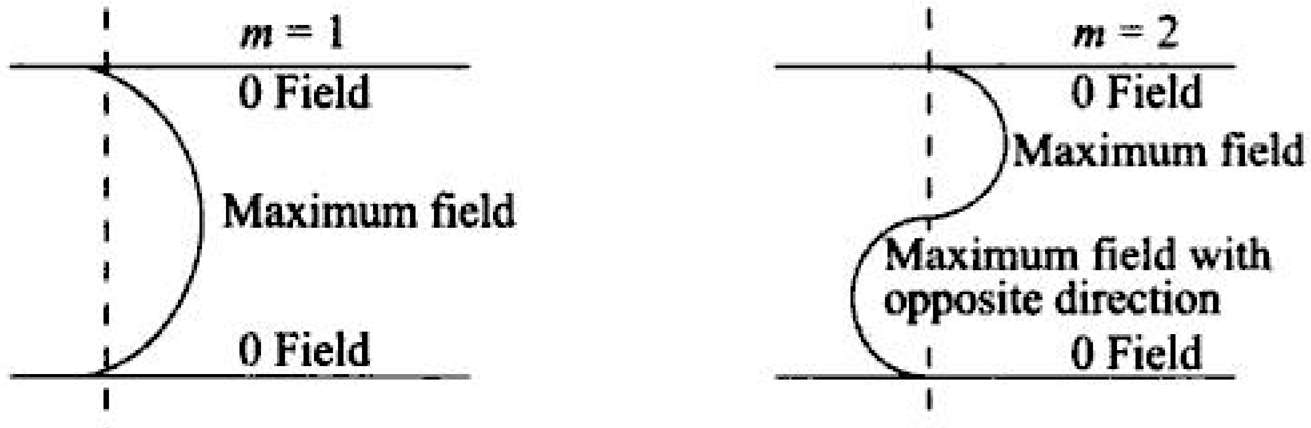

Fig. 6.2 *Spatial variation of electric field amplitude for different mode indices m.*

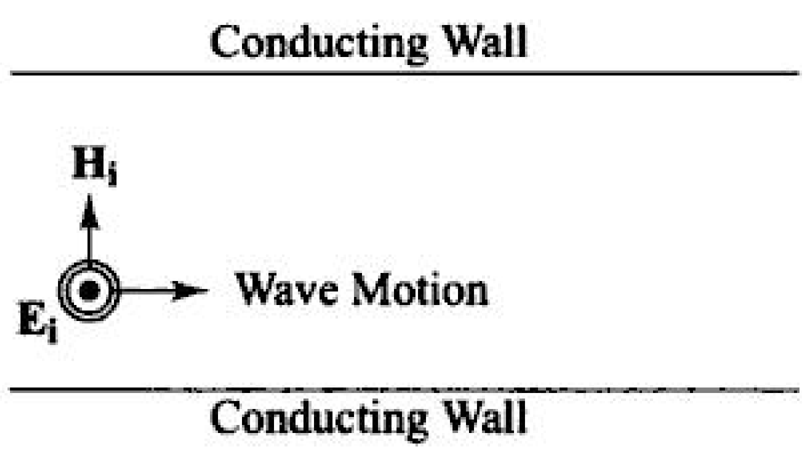

Fig. 6.3 *Electric field between two parallel conducting planes for $m = 0$.*

From Eqns (6.10) and (6.11) we note that the fields travel along $+z$-direction. The electric field which is in y-direction is transverse to the direction of wave propagation. The magnetic field has x as well as z-component, i.e. it has components parallel to as well as perpendicular to the direction of the wave propagation. The electric field therefore has a special nature, that it is transverse to the direction of the wave propagation and consequently, the mode is called the 'Transverse Electric or TE mode'. m is put as a suffix to TE to indicate the order of the mode. A TE_m mode therefore is a transverse electric mode with fields having m half cycle variations in the transverse plane. As discussed above no fields exist for $m = 0$ and consequently m should be $1, 2, \ldots$, etc.

EXAMPLE 6.1 An air filled parallel plane waveguide has 10 cm height. The maximum peak electric field measured inside the waveguide is 10 V/m. If the frequency of the wave is 3 GHz and if the waveguide is excited in TE_1 mode, find the expressions for the electric and the magnetic fields inside the waveguide.

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therefore, have fields which will reduce to the fields of parallel plane waveguide in the limit a or $b \to \infty$. In a parallel plane waveguide we have seen that the field variation in the transverse plane (xy-plane) is due to superposition of plane waves bouncing between the two planes. That is to say that the fields in transverse plane are of standing wave nature, whereas the fields have traveling wave nature in z-direction. We, therefore, expect from the understanding of the parallel plane waveguide, that the magnitude of the field in the transverse plane undergoes through maxima and minima, whereas the fields travel in the z-direction. The solutions to the Eqns (6.71), (6.72), (6.73) can be appropriately chosen as

$$X = C_1 \cos Ax + C_2 \sin Ax \text{............. Standing wave} \tag{6.74}$$

$$Y = C_3 \cos By + C_4 \sin By \text{........... Standing wave} \tag{6.75}$$

and $$Z = C_5 e^{-j\beta z} + C_6 e^{+j\beta z} \text{.......... Traveling waves} \tag{6.76}$$

where $C_1, C_2, \ldots, C_6$ are the arbitrary constants which are to be evaluated from the boundary conditions.

In general, there are two traveling modes, one travelling in $+z$-direction and other travelling in $-z$ direction. If we, however, assume that the waveguide is infinitely long, and there is only one mode travelling in $+z$-direction, we can choose

$$C_6 \equiv 0 \tag{6.77}$$

Substituting for X, Y and Z from Eqns (6.74) to (6.76) in Eqn (6.65) the field can be written as

$$E_z = C_5 (C_1 \cos Ax + C_2 \sin Ax)(C_3 \cos By + C_4 \sin By)\, e^{-j\beta z} \tag{6.78}$$

Applying boundary condition that the tangential component of the electric field should be zero at the conducting boundary, we get

$$E_z = 0 \quad \text{at} \quad x = 0, \quad x = a$$
$$y = 0, \quad y = b \tag{6.79}$$

Note that the field component E_z is tangential to all the four walls of the waveguide.

Applying Boundary condition at $x = 0$, we get

$$E_z|_{x=0} = C_5 (C_1 \cos Ax)(C_3 \cos By + C_4 \sin By)\, e^{-j\beta z} = 0$$

$$\Rightarrow \quad C_1 = 0 \tag{6.80}$$

Substituting $C_1 = 0$ in Eqn (6.78) and applying boundary condition at $y = 0$, we get

$$E_z|_{y=0} = C_5 (C_2 \sin Ax)(C_3 \cos By)\, e^{-j\beta z} = 0$$

$$\Rightarrow \quad C_3 = 0 \tag{6.81}$$

The cut off frequency of the mode is

$$\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{89}$$

$$\text{Noting that} \frac{1}{\sqrt{\mu\epsilon}} = 3 \times 10^8 \text{m/s}$$

$$\omega_c = 2.83 \times 10^9 \text{rad/s}$$

From the dispersion relation, we get

$$\text{Phase velocity} \quad v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2\mu\epsilon - 89}}$$

$$\text{Group velocity} \quad v_g = \frac{\partial\omega}{\partial\beta} = \frac{1}{\mu\epsilon}\frac{\beta}{\omega}$$

At $\omega = 2\omega_c$, we get

$$v_p = 3.46 \times 10^8 \text{ m/s}$$

$$v_g = 2.598 \times 10^8 \text{ m/s}$$

EXAMPLE 6.9 The cross section of a rectangular waveguide is 20 cm × 5 cm. Find 6 lowest order modes which will propagate on the waveguide and their cut-off frequencies.

Solution:

For the waveguide $a = 0.20$ m and $b = 0.05$ m
The cut-off frequency of a mode is given by

$$\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\}^{1/2}$$

$$\Rightarrow \qquad 2\pi f_c = 3 \times 10^8 \left\{\left(\frac{m}{0.2}\right)^2 + \left(\frac{n}{0.05}\right)^2\right\}^{1/2} \pi$$

$$\Rightarrow \qquad f_c = 1.5 \times 10^{10} \left\{\left(\frac{m}{20}\right)^2 + \left(\frac{n}{5}\right)^2\right\}^{1/2}$$

The cut-off frequency in ascending order will correspond to

$m = 1$	$n = 0$	$m = 4$	$n = 0$
$m = 2$	$n = 0$	$m = 0$	$n = 1$
$m = 3$	$n = 0$	$m = 1$	$n = 1$

Since, for TM modes m and n both have to be non zero we get the 6 lowest order modes as $TE_{10}, TE_{20}, TE_{30}, TE_{40}, TE_{01}, TE_{11}, TM_{11}$

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6.7 SURFACE CURRENT ON THE WAVEGUIDE WALLS

Let us assume that the waveguide walls are made of highly conducting material. The modal fields inside a waveguide then induce surface charges and surface currents on the waveguide walls. In other words, we may say that the modal fields are supported by the surface charges and currents on the inner walls of the waveguide. Since, the fields are time varying, the surface charges and currents also vary as a function of time.

The power flow inside the waveguide is normally related to these times varying charges and currents. The students, therefore, get a wrong impression that the power is carried by the charges and the currents and hence the charges and currents should flow in the direction of the power flow. It should be emphasized that the power inside the waveguide or for that matter along any guided structure, is carried by the fields and not by the charges. It is then possible that the currents in waveguide walls may flow in any direction along the surface of the walls whereas the power flows along the length of the waveguide only. The surface current distributions for TE_1 mode (Section 6.7.1) and TE_{10} mode (Section 6.7.2) clearly demonstrate this point.

The approach to finding surface current distribution is similar to that used in visualising the modal fields inside a waveguide. First we freeze time to get instantaneous surface current distribution along the waveguide walls, then we allow the distribution to drift along the waveguide with the phase velocity. Without losing generality, we, therefore, first obtain spatial surface current distribution at $t = 0$.

6.7.1 Surface Currents in a Parallel Plane Waveguide for TE_1 Mode

The surface current on a conducting boundary is related to the tangential component of the magnetic field at the conducting surface.

$$\mathbf{J_S} = \hat{\mathbf{n}} \times \mathbf{H} \tag{6.154}$$

where $\mathbf{H}$ is the magnetic field at the conducting surface and $\hat{\mathbf{n}}$ is the outward normal at the surface.

From Eqn (6.11) the magnetic field at the waveguide plates is

$$\mathbf{H} = \frac{2E_{i0}}{\eta_1}\frac{\beta}{\beta_1}e^{-j\beta z}\hat{\mathbf{z}} \qquad \text{at} \qquad x = 0 \quad \text{wall} \tag{6.155}$$

$$= \frac{-2E_{i0}}{\eta_1}\frac{\beta}{\beta_1}e^{-j\beta z}\hat{\mathbf{z}} \qquad \text{at} \qquad x = d \quad \text{wall} \tag{6.156}$$

At $x = 0$ wall, $\mathbf{n}$ is oriented in $+x$ direction and at $x = d$ wall, $\hat{\mathbf{n}}$ is oriented in $-x$ direction (Fig. 6.20). The surface current along $x = 0$ wall flows along $\hat{\mathbf{n}} \times \mathbf{H} \to \hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}}$ direction, and along $x = d$ wall it flows along $-\hat{\mathbf{x}} \times (-\hat{\mathbf{z}}) = -\hat{\mathbf{y}}$ direction.

Now, taking the real part of $\mathbf{H}$ we get at $t = 0$ time.

$$\mathbf{H} = \frac{2E_{i0}}{\eta_1}\frac{\beta}{\beta_1}\cos(\beta z)\hat{\mathbf{z}} \qquad x = 0 \quad \text{wall} \tag{6.157}$$

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The total power flow through the waveguide is

$$W = \int_{x=0}^{a}\int_{y=0}^{b} \frac{1}{2} Re(\mathbf{E}\times\mathbf{H}^*)_z dxdy$$

$$= -\int_{x=0}^{a}\int_{y=0}^{b} \frac{1}{2} Re\left\{\left[\frac{j\omega\mu a}{\pi} D\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}\right]\left[\frac{j\beta a}{\pi} D\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}\right]^*\right\} dxdy \quad (6.194)$$

(the -ve sign is used since $E_y\hat{\mathbf{y}} \times H_x\hat{\mathbf{x}} = -\hat{\mathbf{z}}E_yH_x$)

$$W = \int_{x=0}^{a}\int_{y=o}^{b} \frac{\omega\mu\beta a^2}{2\pi^2}|D|^2\sin^2\left(\frac{\pi x}{a}\right)dxdy$$

$$= \frac{\omega\mu\beta a^2 b|D|^2}{2\pi^2}\int_{x=0}^{a}\sin^2\left(\frac{\pi x}{a}\right)dx$$

$$W = \frac{\omega\mu\beta a^3 b|D|^2}{4\pi^2} \quad (6.195)$$

To determine the power loss in the waveguide walls we need surface current on the four walls of the waveguide. One can easily note that the surface current on the vertical wall has only y component whereas on the horizontal walls (*i.e.* $y = 0$, $y = b$) the surface current has x and z components. The magnitude of the surface current on the vertical walls is

$$|\mathbf{J}_s(x=0)| = |\mathbf{J}_s(x=a)| = |H_z(x=0)| = |D| \quad (6.196)$$

and consequently the loss per unit length of vertical walls is

$$W_{L_{ver}} = 2\int_{y=0}^{b}\int_{z=0}^{1}\frac{1}{2}R_s|D|^2 dydz$$

$$= R_s|D|^2 b \quad (6.197)$$

The magnitude of the surface current on horizontal walls

$$|\mathbf{J}_s(y=0)| = |\mathbf{J}_s(y=b)| = |\mathbf{H}(y=0)| \quad (6.198)$$

$$\Rightarrow \quad |\mathbf{J}_s(y=0)|^2 = |\mathbf{J}_s(y=b)|^2 = |\mathbf{H}(y=0)|^2$$

$$= H_x^2(y=0) + H_z^2(y=0) \quad (6.199)$$

The loss per unit length of the horizontal walls is

$$W_{L_{hor}} = 2\int_{x=0}^{a}\int_{z=0}^{1}\frac{1}{2}R_s\left(|H_x(y=0)|^2 + |H_z(y=0)|^2\right)dxdz$$

$$= R_s\int_{x=0}^{a}\left[\left(\frac{\beta a}{\pi}\right)^2|D|^2\sin^2\left(\frac{\pi x}{a}\right) + |D|^2\cos^2\left(\frac{\pi x}{a}\right)\right]dx \quad (6.200)$$

$$= \frac{R_s|D|^2 a}{2}\left[\left(\frac{\beta a}{\pi}\right)^2 + 1\right] \quad (6.201)$$

The cutoff frequencies of the modes are

$$f_{cTE_{01}} = \frac{3 \times 10^8}{2\pi}\left(\frac{\pi}{0.04}\right) = 3.75 \text{ GHz}$$

$$f_{cTE_{02}} = \frac{3 \times 10^8}{2\pi}\left(\frac{2\pi}{0.04}\right) = 7.5 \text{ GHz}$$

$$f_{cT!E_{10}} = \frac{3 \times 10^8}{2\pi}\left(\frac{3\pi}{0.06}\right) = 7.5 \text{ GHz}$$

6.9 CAVITY RESONATOR

In the previous sections, we discussed propagation of modes in a rectangular waveguide. In the direction of the modal propagation the waveguide is assumed to be of infinite length and consequently we assume only one travelling wave along $+z$ direction along the waveguide. Let us now investigate what will happen if we take a section of a waveguide and close it from both sides with metal plates?

By placing a metal plate at the end of the waveguide the propagation of the electromagnetic wave is blocked and the wave is reflected. Obviously, in this situation then we have two waves travelling in opposite directions. The two waves together have to satisfy the boundary conditions at the two ends of the waveguide. For the sake of clarity let us investigate the fields for the TE_{mn} modes inside this structure.

For the TE_{mn} mode the longitudinal magnetic fields for the travelling wave is +z direction is given as (see Eqn (6.102)).

$$H_z = D\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z} \tag{6.205}$$

For the reflected TE_{mn} wave let the amplitude be D'. The longitudinal magnetic field for the reflected wave is

$$H_z' = D'\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z} \tag{6.206}$$

The total longitudinal magnetic field inside the waveguide then is

$$H_Z = H_z + H_z' \tag{6.207}$$

Now, since, the magnetic field component H_z is perpendicular to the closing plates at $z = 0$ and $z = d$, its value should be zero.
Making $H_z = 0$ at $z = 0$ we get

$$D = -D' \tag{6.208}$$

Equation (6.207) can be written as

$$H_z = D\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\left(e^{-j\beta z} - e^{j\beta z}\right)$$

6.10 SUMMARY

In this Chapter a very important structure called the 'waveguide' has been discussed. A variety of waveguides are found in practice. However, the rectangular metallic waveguides are the most common ones. To physically understand the wave propagation inside a rectangular waveguide, first the parallel plane waveguides have been investigated. A general approach for analysing waveguides has been developed subsequently. The concept of modal propagation has been introduced and some typical modes of parallel plane and rectangular waveguides are discussed. A three dimensional visualization of the modal fields and the surface currents on the waveguide walls have been developed. The power loss calculations for a practical waveguide have been presented towards the end of the chapter. In the next chapter, we investigate another important waveguiding structures, the dielectric waveguides.

Review Questions

6.1 What is a waveguide?

6.2 Can we have modal propagation in an unbound medium?

6.3 What is a parallel plane waveguide?

6.4 What are TE and TM modes?

6.5 At what angle does a uniform plane wave be launched inside a parallel plane waveguide so as to have sustained propagation?

6.6 As the height of a parallel plane waveguide increases, what happens to the number of modes?

6.7 What does the modal index 'm' signify?

6.8 How does the phase velocity of a modal field varies as a function of frequency?

6.9 For a mode how does the phase velocity varies as a function of the waveguide height?

6.10 What is the cut-off frequency?

6.11 If frequency is less than the cut-off frequency, what kind of fields are excited inside the waveguide?

6.12 What is the phase velocity if the frequency is equal to cut-off frequency?

6.13 Can TEM mode exist inside a parallel plane waveguide? Explain why.

6.14 Can TEM mode be dispersive? Why?

6.15 What are the boundary conditions on the surface of a waveguide?

6.16 Can TE_{mn} and TM_{mn} modes exist inside a rectangular waveguide for any value of m and n? Explain your answer.

6.17 Why does TEM mode not exist inside a rectangular waveguide?

6.18 What is the dominant mode of a parallel plane waveguide?

6.19 What is the dominant mode of a rectangular waveguide?

CHAPTER 7

Dielectric Waveguide

Dielectric waveguide is a dielectric structure which can guide electromagnetic waves. At frequencies beyond millimeter wavelengths, the metallic waveguides have excessive conductor loss. The dielectric waveguides are, therefore, more promising in sub-millimeter and optical wavelength range. The dielectric waveguides have played a prominent role in advancements in semiconductor lasers and optical communication. In fact, almost all the guided wave and integrated optics devices work on the principle of dielectric waveguide.

There are two types of dielectric waveguides namely, slab waveguides and cylindrical waveguides. Slab waveguides find applications in thin film and integrated optical devices, whereas, optical fibers are cylindrical dielectric waveguides. Due to the impetus received in the last few decades, the dielectric waveguides have become a rather important subject in electromagnetics.

In this chapter, we investigate the slab as well as cylindrical dielectric waveguides. Since, the light is an electromagnetic wave, the analysis presented in the following sections is applicable to optical propagation as well as to the millimeter or microwave propagation.

7.1 DIELECTRIC SLAB WAVEGUIDE

The slab waveguides are generally seen in thin film technology. A thin dielectric layer is deposited on another dielectric slab called 'substrate'. The deposited dielectric layer has a higher dielectric constant compared to that of the substrate and has much less propagation loss at the frequency of operation. Therefore, in general, the slab dielectric waveguide is an asymmetric structure since it has substrate on one side and air on the other. A dielectric slab waveguide is schematically shown in Fig. 7.1.

To make the analysis simple, it is assumed that the substrate is infinitely thick. It will become clear later that this assumption is justified since the fields decay rapidly away from the waveguide-substrate interface.

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Using vector identity, $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$, and rearranging, Eqn (8.12) becomes

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu\mathbf{J} + \mu\epsilon\frac{\partial}{\partial t}\left\{-\frac{\partial \mathbf{A}}{\partial t} - \nabla V\right\} \tag{8.13}$$

Equations (8.11) and (8.13) are coupled equations for **A** and V, and therefore are quite difficult to solve. Moreover, the equations do not have a unique solution as **A** itself is not completely defined. This is due to the Helmholtz theorem, which states that a vector field is uniquely defined if and only if its both divergence and curl are specified. We have defined the curl of **A** which is the magnetic flux density (Eqn (8.5)) but the divergence of **A** is still undefined. Obviously, depending upon the selection of the divergence of **A**, we will get different solutions to the equations. At this point, we may re-state the type of behavior we are expecting for the fields and hence for the potentials.

We know from the source-free analysis of the Maxwell's equations that we should get wave type of solution to the fields if we make all sources equal to zero. With this guide line the choice of divergence of **A** gets highly restricted. Examination of Eqns (8.11) and (8.13) suggest that if we define the divergence of **A** as

$$\nabla \cdot \mathbf{A} = -\mu\epsilon\frac{\partial V}{\partial t} \tag{8.14}$$

The Eqns (8.11) and (8.13) not only get decoupled but also reduce to the wave equations as

$$\nabla^2\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J} \tag{8.15}$$

and

$$\nabla^2 V - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{8.16}$$

In the absence of sources like ρ and **J**, Eqns (8.15), (8.16) reduce to the source-free wave equations discussed in Chapter 4. Also for non-time varying potentials, the Eqns (8.15), (8.16) reduce to static equations like Poisson's equation as

$$\nabla^2\mathbf{A} = -\mu\mathbf{J} \tag{8.17}$$

$$\nabla^2 V = -\frac{\rho}{\epsilon} \tag{8.18}$$

These arguments, therefore, suggest that the selection of divergence of **A** through Eqn (8.14) is indeed correct. Equation (8.14) is known as the *Laurentz gauge condition.*

Few important observations on Eqns (8.15) and (8.16) are in order at this point. Firstly, the electric scalar potential is related to the charges only whereas the magnetic vector potential is related to the current only. Secondly, in Eqn (8.15) since ∇^2 operator is scalar, for a given current density, the direction of the vector potential everywhere in the space is same and it is same as the direction of the current. Later, we will observe that the same behavior is not true for the electric

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$$A_\theta = A_y \cos\theta \sin\phi = \frac{10^{-8}}{\sqrt{14}}\frac{3}{\sqrt{14}} \cdot \frac{2}{\sqrt{5}} e^{-j2\pi\sqrt{14}} e^{j\omega t} \text{ Wb/m}$$

$$= \frac{3 \times 10^{-8}}{7\sqrt{5}} e^{-j2\pi\sqrt{14}} e^{j\omega t} \text{ Wb/m}$$

$$A_\phi = A_y \cos\phi = \frac{10^{-8}}{\sqrt{14}} \cdot \frac{1}{\sqrt{5}} .e^{-j2\pi\sqrt{14}} e^{j\omega t} \text{ Wb/m}$$

Since, the radiation is investigated in the spherical coordinate system first we have to convert **A** in (r, θ, ϕ) components. From Fig. 8.4 we get

$$A_r = A_z \cos\theta \tag{8.35}$$

$$A_\theta = -A_z \sin\theta \tag{8.36}$$

$$A_\phi = 0 \tag{8.37}$$

Substitution of **A** in (8.6) yields the magnetic field as

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \tag{8.38}$$

$$= \frac{1}{\mu}\frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \tag{8.39}$$

For the Hertz dipole we have symmetry in ϕ giving $\frac{\partial}{\partial\phi} \equiv 0$. Substitution of Eqns (8.35)–(8.37) in Eqn (8.39) yields

$$\mathbf{H} = \frac{1}{\mu r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & 0 \\ A_z\cos\theta & -rA_z\sin\theta & 0 \end{vmatrix} \tag{8.40}$$

$$\Rightarrow \qquad H_r = 0 \tag{8.41}$$

$$H_\theta = 0 \tag{8.42}$$

$$H_\phi = \frac{1}{\mu}\frac{1}{r}\left\{\frac{\partial}{\partial r}(-rA_z\sin\theta) - \frac{\partial}{\partial\theta}(A_z\cos\theta)\right\} \tag{8.43}$$

Substitution of A_z from Eqn (8.34) in Eqn (8.43) yields

$$H_\phi = -\frac{I_0 dl e^{j\omega t}}{4\pi r}\left\{\frac{\partial}{\partial r}(e^{-j\beta r}\sin\theta) + \frac{\partial}{d\theta}(\frac{e^{-j\beta r}\cos\theta}{r})\right\} \tag{8.44}$$

$$= \frac{I_0 dl e^{j\omega t}}{4\pi r}\left\{j\beta e^{-j\beta r}\sin\theta + \frac{e^{-j\beta r}}{r}\sin\theta\right\} \tag{8.45}$$

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$$\mathbf{H} = \frac{-I_0 dl}{4\pi}\left\{\frac{j\beta r + 1}{r^2}\right\} e^{-j\beta r} e^{j\omega t}\{-\cos\theta\hat{\mathbf{x}} + \sin\theta\cos\phi\hat{\mathbf{z}}\}$$

$$\Rightarrow \qquad H_x = \frac{I_0 dl cos\theta}{4\pi}\left\{\frac{j\beta r + 1}{r^2}\right\} e^{-j\beta r} e^{j\omega t} \text{ Amp/m}$$

$$H_y = 0$$

$$H_z = \frac{-I_0 dl \sin\theta\cos\phi}{4\pi}\left\{\frac{j\beta r + 1}{r^2}\right\} e^{-j\beta r} e^{j\omega t} \text{ Amp/m}$$

Now from Appendix C we have,

$$\begin{bmatrix} H_r \\ H_\theta \\ H_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

$$\begin{aligned} H_r &= H_x \sin\theta\cos\phi + H_z\cos\theta \\ &= \frac{I_0 dl}{4\pi}\left\{\frac{j\beta r + 1}{r^2}\right\} e^{-j\beta r} e^{j\omega t}\{\sin\theta\cos\theta\cos\phi - \sin\theta\cos\theta\cos\phi\} \\ &= 0 \end{aligned}$$

$$\begin{aligned} H_\theta &= H_x\cos\theta\cos\phi - H_z\sin\theta \\ &= \frac{I_0 dl}{4\pi}\left\{\frac{j\beta r + 1}{r^2}\right\} e^{-j\beta r} e^{j\omega t}\left\{\cos^2\cos\phi + \sin^2\theta cos\phi\right\} \\ &= \frac{I_0 dl}{4\pi}\left\{\frac{j\beta r + 1}{r^2}\right\} e^{-j\beta r} e^{j\omega t}\cos\phi \end{aligned}$$

$$\begin{aligned} H_\phi &= -H_x\sin\phi \\ &= -\frac{I_0 dl}{4\pi}\left\{\frac{j\beta r + 1}{r^2}\right\} e^{-j\beta r} e^{j\omega t}\cos\theta\sin\phi \end{aligned}$$

(ii) Knowing A_y we can find A_r, A_θ, A_ϕ as

$$A_r = A_y\sin\theta\sin\phi = \frac{\mu I_0 dl}{4\pi r} e^{-j\beta r} e^{j\omega t}\cdot\sin\theta\sin\phi$$

$$A_\theta = A_y\cos\theta\sin\phi = \frac{\mu I_0 dl}{4\pi r} e^{-j\beta r} e^{j\omega t}\cos\theta\sin\phi$$

$$A_\phi = A_y\cos\phi = \frac{\mu I_0 dl}{4\pi r} e^{-j\beta r} e^{j\omega t}\cos\phi$$

We, therefore, have the magnetic field as

$$\mathbf{H} = \frac{1}{\mu}\nabla\times A = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

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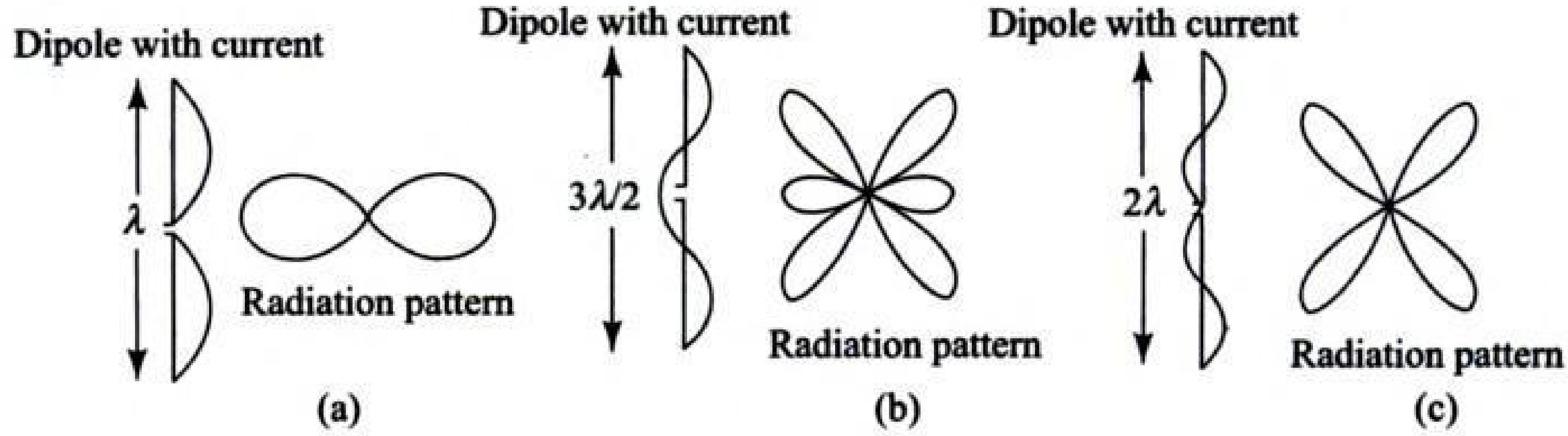


Fig. 8.20 *Current Distributions and E-plane radiation patterns for dipoles of different lengths.*

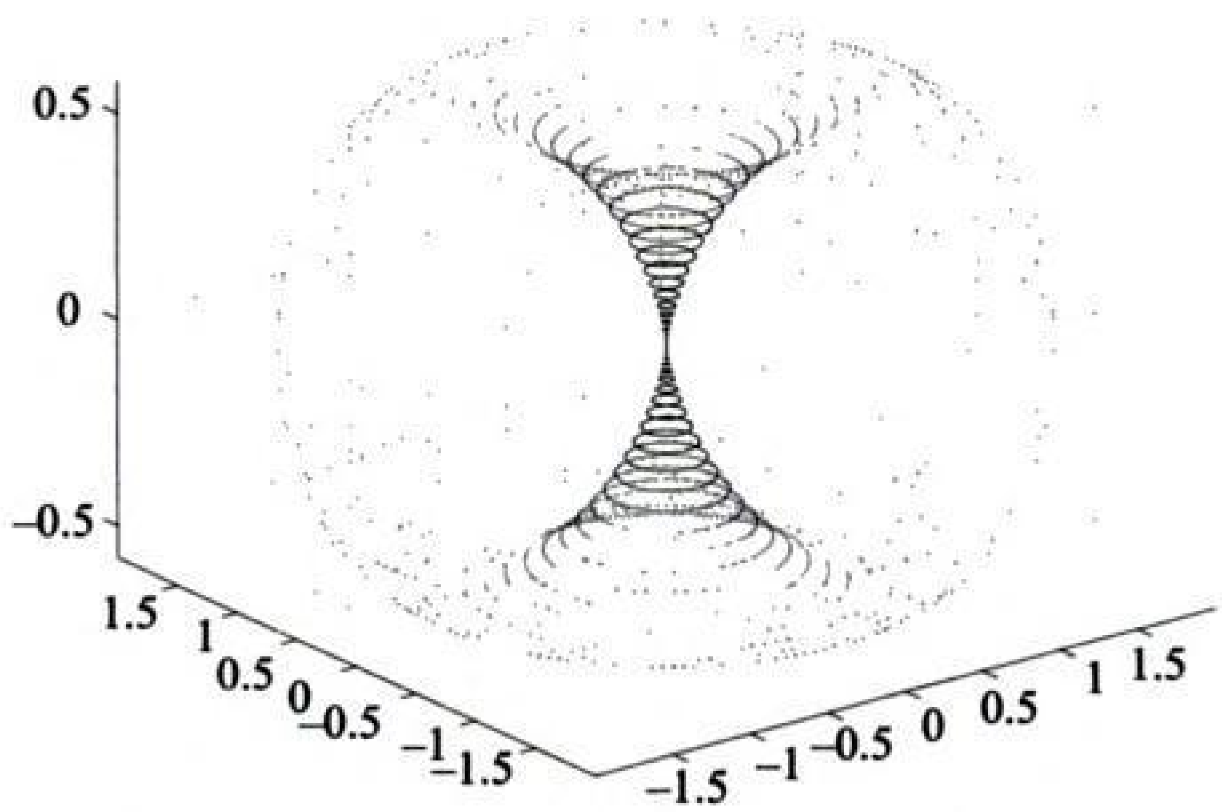
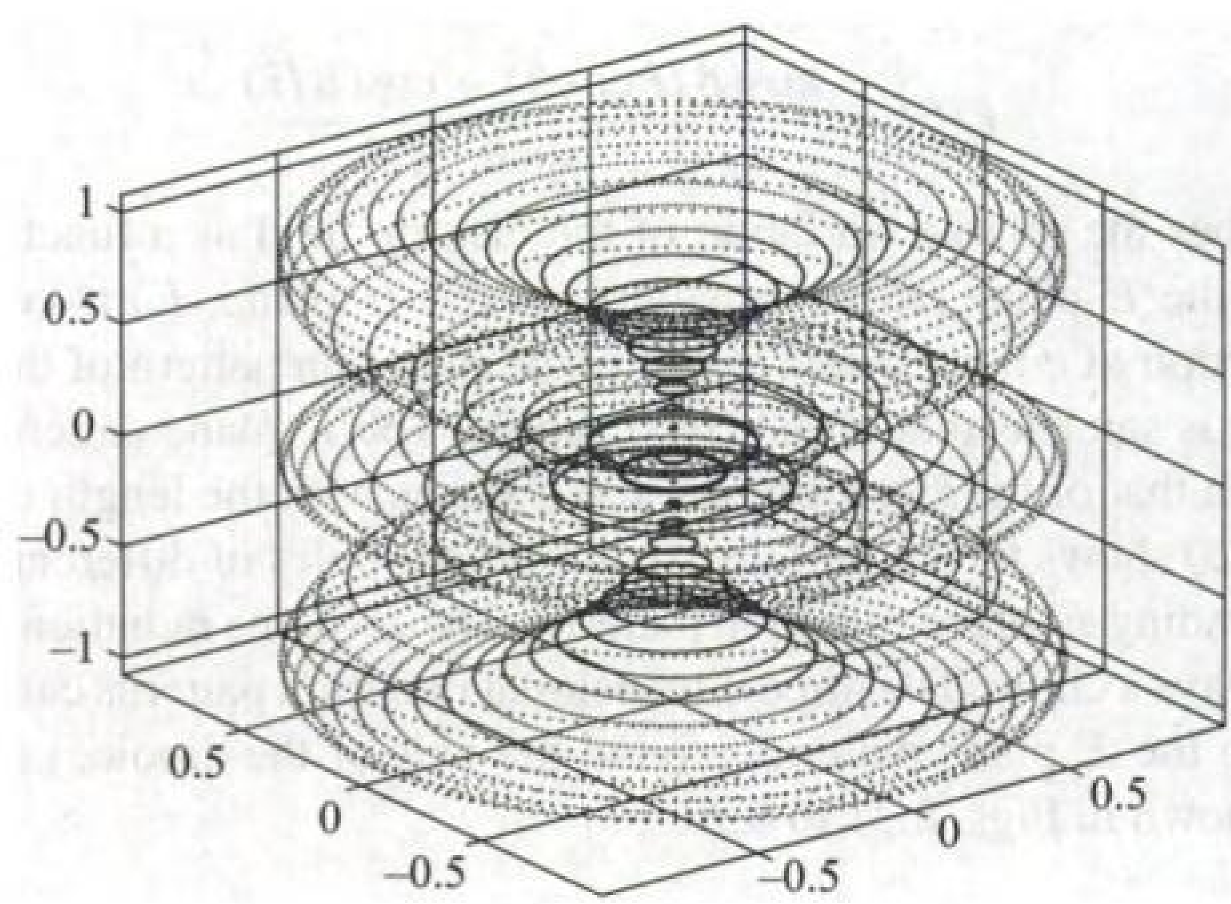


Fig. 8.21 *Three dimensional radiation pattern for a λ long dipole.*



H = (3/4) * lambda

Fig. 8.22 *Three dimensional radiation pattern for a 1.5λ long dipole.*

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characteristics without worrying about its radiation pattern. The radiation pattern can be controlled independently by adjusting the array parameters. The arrays, therefore, provide tremendous flexibility in shaping the radiation characteristics of a radiating system.

An antenna array is a spatial distribution of basic antenna elements excited with voltages or currents having definite amplitude and phase relationship. For an array, in principle, it is neither necessary to have identical antenna elements nor their uniform distribution. However, in practice, not having identical antenna elements, does not provide any advantage. On the contrary, identical elements guarantee same impedance characteristics for all antennas and also same radiation characteristics like polarization, individual radiation patterns, etc. Hence, the antenna arrays generally consist of identical antenna elements.

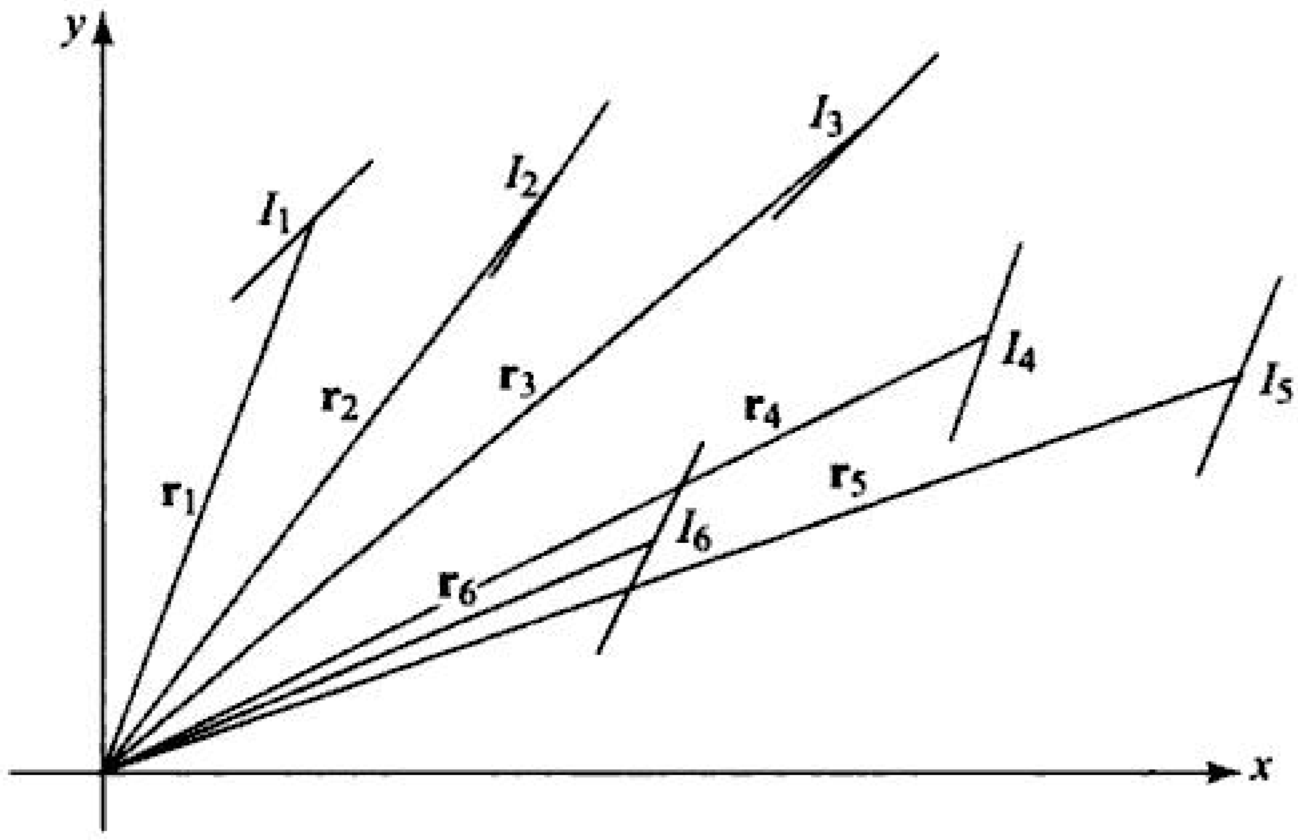

Fig. 9.1 *General array configuration.*

A general array of dipoles is shown in Fig. 9.1. The terminal currents for the elements are $I_1, I_2, I_3, \ldots$ etc. and their special location are $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}, \ldots$ etc. The terminal currents $I_1, I_2, \ldots$ are in general complex, i.e. different antennas have different current amplitudes and different relative phases. Since, the radiation field is directly proportional to the antenna current, the fields due to different elements also have relative amplitudes and phases same as that of the antenna currents. The total field at any point in space can then be obtained by superposition of the fields of the individual antennas. It should be noted that while applying superposition we assume that the characteristics of the individual antennas do not get affected by the presence of other antenna elements. This assumption is valid only if the spacing between the elements is not very small. For small spacing between the elements, the phenomenon called 'mutual coupling' affects the behavior of the neighbouring antennas. The effect of mutual coupling is manageably small if the element spacing is larger than about $\frac{\lambda}{2}$. The radiation pattern of an array can be controlled by controlling either one or some or all of the following parameters.

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EXAMPLE 10.7 A circular antenna dish pointed vertically upwards has the effective beam width of 90° at 12 MHz. An ionospheric layer has electron density of $10^{12}/m^3$ and is at 500 km height. Find the region on the ground over which there will be reflected wave.

Solution:

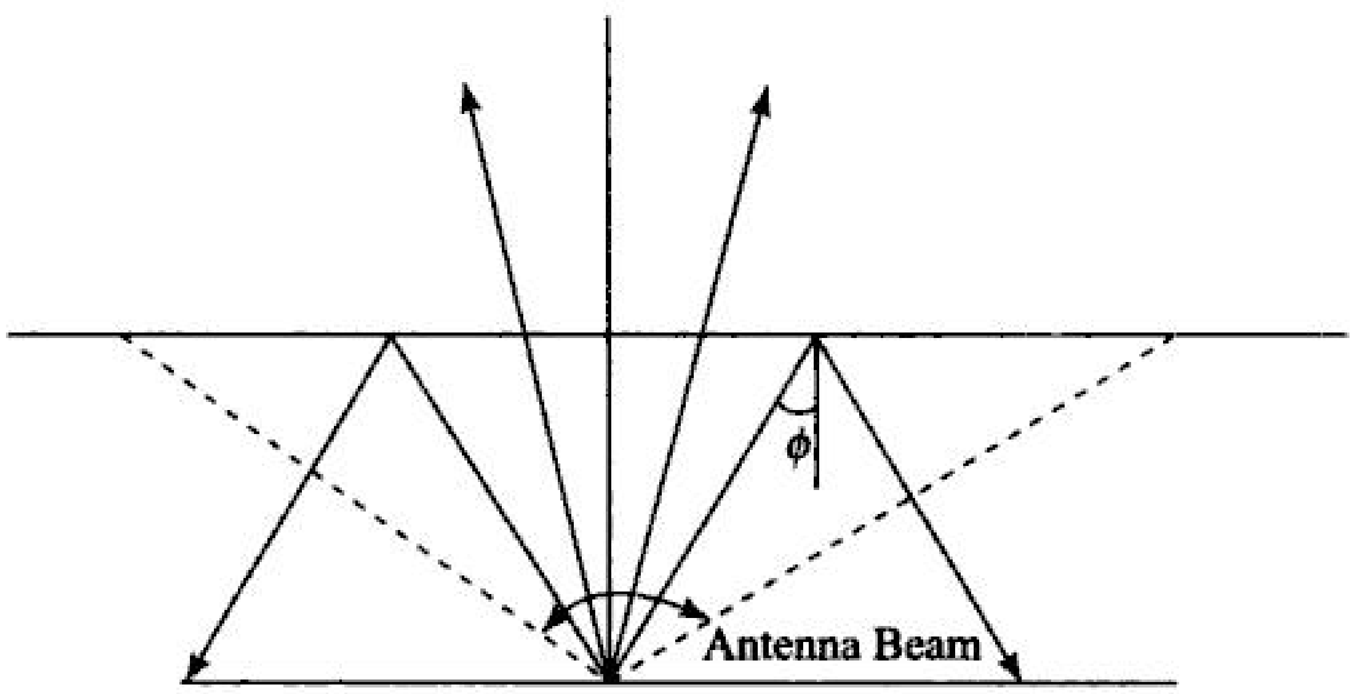

Fig. 10.17 *Reflected and refracted waves from the ionosphere.*

Plasma frequency of the layer

$$f_p = 9\sqrt{10^{12}} = 9 \text{ MHz}$$

Minimum angle of incidence ϕ is given by

$$f_p \sec\phi_{min} = f$$

$$\phi_{min} = \sec^{-1}\left(\frac{12}{9}\right) = 41.4^\circ$$

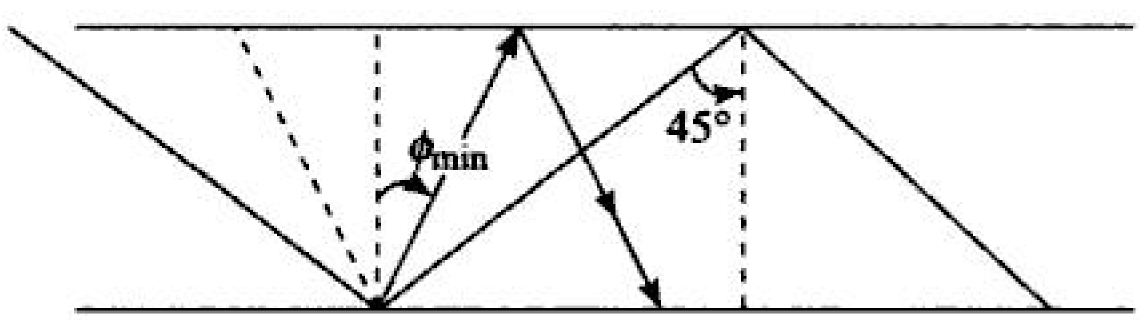

Fig. 10.18 *Maximum and minimum angle of the reflected waves.*

Maximum angle of incidence is decided by the beam of the antenna i.e. $\phi_{max} = 45^\circ$

Radiation going within $\phi_{min} = \pm 41.4^\circ$ is not reflected. We then have a circular region around the antenna where there is no reflection. The radiation is reflected within a circular annular region formed by radii

$$r_{min} = 2h\tan\phi_{min} = 2 \times 500\tan(41.4^\circ) = 881.9 \text{ km}$$

$$r_{max} = 2h\tan 45^\circ = 1000 \text{ km}$$

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at both ends of the line talk simultaneously, their signals are superimposed on the wire pair and can be heard at the other end of the line.

Transmission over long distances like between two switching offices, is best implemented with two unidirectional transmissions on different pairs of wires. This is due to the fact that the long distance transmission invariably requires amplifiers which are unidirectional. The long distance transmission, therefore, needs a four-wire system. At some point in long distance connection, hence, it is essential to convert from a two-wire-to-four-wire system. The conversion device is called a 'hybrid'. Figure 11.6 shows a two-wire-to-four-wire and back four-wire-to-two-wire connection using two hybrids.

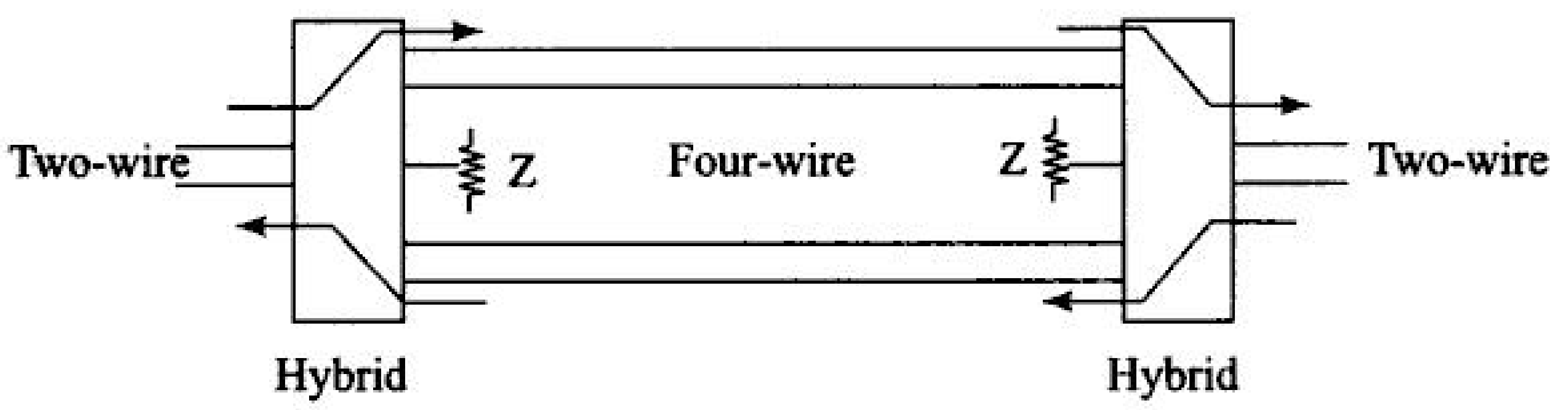

Fig. 11.6 *Two-wire-to-four-wire connections in telephone lines using hybrids.*

In earlier days, the hybrid circuits were implemented with interconnected transformers. In modern days, however, electronic hyrbrids have been developed. The impedance matching network Z is adjusted to get perfect isolation between signals travelling in opposite directions. However, in a switching environment the two wire line has variable impedance and therefore gives impedance mismatch. This impedance mismatch causes reflections on the line resulting in an echo. The amplitude of the echo depends upon the degree of impedance mismatch. Minimization of echoes using adaptive signal processing techniques is an interesting subject in telephone systems.

One more aspect of the telephone lines is the distortion of the voice signals. Invariably, the practical telephone lines have substantial loss in the conductors and relatively less loss in the dielectric separating the conductors. For distortionless transmission, the line must satisfy the condition

$$\frac{R}{L} = \frac{G}{C} \tag{11.10}$$

where R, L, G, C are the primary constants of the line defined in Chapter 2. Since, for practical lines $G \ll R$, L has to be increased to achieve the condition given by Eqn (11.10). The loading coils were introduced to artificially increase the inductance of the line so as to realize Eqn (11.10) over the length of the line.

11.4 RADAR

The word 'Radar' is an acronym for 'radio detection and ranging'. Radar is an instrument used to detect and locate objects like air crafts, ships, etc. using radio

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