EE1101 Signals and Systems JAN—MAY 2018
Tutorial 4 Solutions

1) Let $y(t)$ be the response to the input $i(t)$. Then, $y(t) = i(t) * h(t) = y(t) = \int_{-\infty}^{\infty} h(\tau)i(t-\tau)d\tau$. Fig. [1] shows $i(t-\tau)$ and $h(\tau)$ respectively.

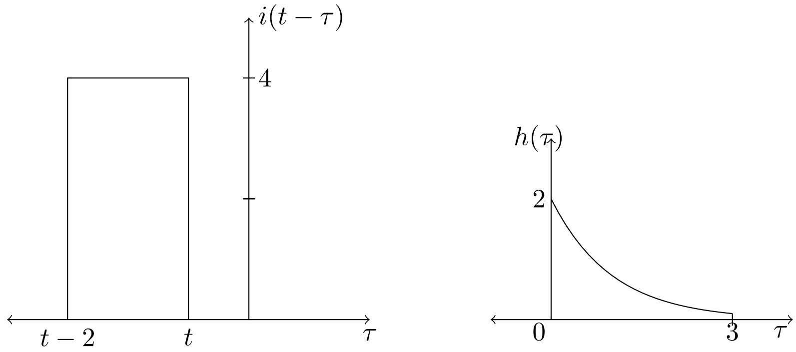

Figure 1: Plot showing variation of $i(t-\tau)$ and $h(\tau)$ as a function of τ for some given real value t.

Case 1: For $t < 0$, there is no overlap between the non-zero regions of $i(t-\tau)$ and $h(\tau)$. Hence, $y(t) = 0$, $\forall t < 0$.

Case 2: Suppose $t \geq 0$ and $t - 2 < 0$, i.e., $0 \leq t < 2$.
Then,

$$y(t) = \int_{0}^{t} (4)(2e^{-\tau})d\tau = 8\left(1 - e^{-t}\right).$$

Case 3: For $t < 3$ and $t - 2 \geq 0$, i.e., $2 \leq t < 3$, we have,

$$y(t) = 8\int_{t-2}^{t} e^{-\tau}d\tau = 8\left(e^{2-t} - e^{-t}\right).$$

Case 4: When $t \geq 3$, but still $t - 2 < 3$, i.e., $3 \leq t < 5$, we get,

$$y(t) = 8\int_{t-2}^{3} e^{-\tau}d\tau = 8\left(e^{2-t} - e^{-3}\right).$$

Case 5: Finally, if $t - 2 \geq 3$, i.e., $t \geq 5$, due to non-overlapping of the non-zero portions of $h(\tau)$ and $i(t-\tau)$, $y(t)$ becomes zero.

The signal $y(t)$ is,

$$y(t) = \begin{cases} 0, & t < 0 \\ 8(1 - e^{-t}), & 0 \leq t < 2 \\ 8(e^{2-t} - e^{-t}), & 2 \leq t < 3 \\ 8(e^{2-t} - e^{-3}), & 3 \leq t < 5 \\ 0, & t \geq 5. \end{cases}$$

2) (a) Given that $f(t) * g(t) = y(t)$. Hence,

$$y(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\, d\tau.$$

Let us consider $f(t-T_1) * g(t-T_2)$, and by using the definition of convolution, we have

$$f(t-T_1) * g(t-T_2) = \int_{-\infty}^{\infty} f(\tau - T_1)g(t-\tau-T_2)d\tau = \int_{-\infty}^{\infty} f(\tau - T_1)g(t-T_2-\tau)d\tau.$$

Denote $\tau' = \tau - T_1$, note that the limits and derivative does not change.

$$\begin{aligned} f(t-T_1) * g(t-T_2) &= \int_{-\infty}^{\infty} f(\tau')g\big(t-T_2-(\tau'+T_1)\big)d\tau' = \int_{-\infty}^{\infty} f(\tau')g\big(t-(T_1+T_2)-\tau'\big)d\tau' \\ &= y\big(t-(T_1+T_2)\big) \text{ [On comparing with the first equation].} \end{aligned}$$

(b) If $u(t) * u(t) = r(t)$, then

$$\begin{aligned} &\big(u(t+1)-u(t-2)\big) * \big(u(t-3)-u(t-4)\big) \\ &= u(t+1) * u(t-3) - u(t+1) * u(t-4) + u(t-2) * u(t-4) - u(t-2) * u(t-3) \\ &= r(t-2) - r(t-3) + r(t-6) - r(t-5). \end{aligned}$$

The last equality is a consequence of the result obtained in (a). We now sketch $r(t-2)-r(t-3)+r(t-6)-r(t-5)$ in Fig. [2].

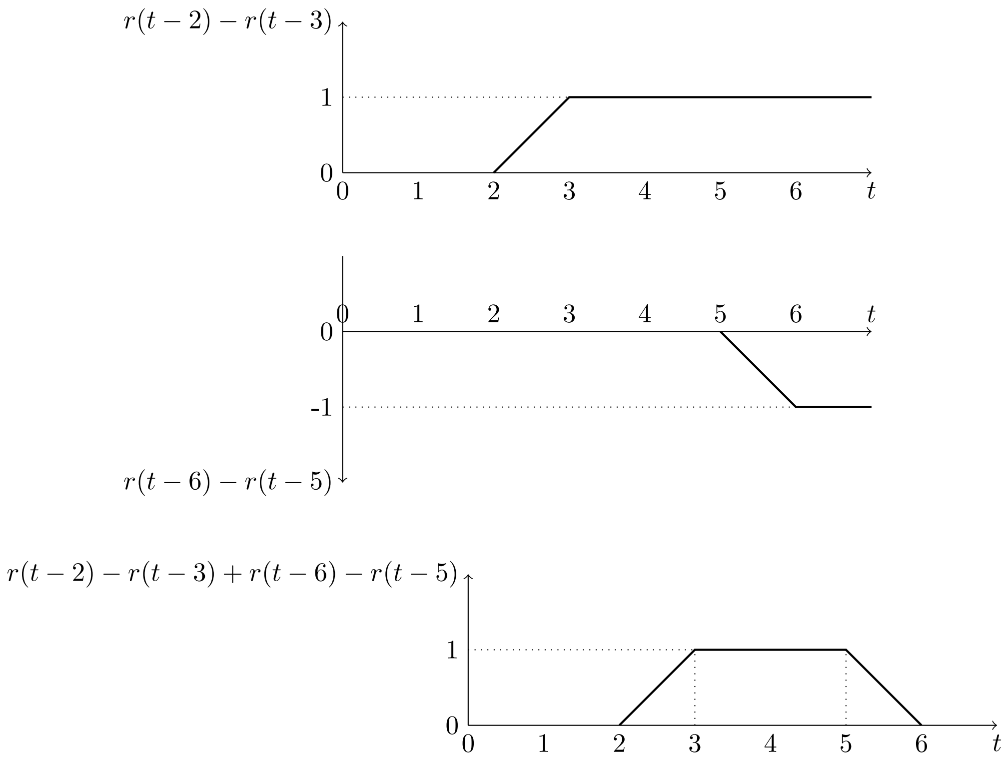

Figure 2: Sketch of the signal $\big(u(t+1)-u(t-2)\big) * \big(u(t-3)-u(t-4)\big)$ using the distributive and shift property of convolution.

One can verify the result by performing convolution of pulses $\big(u(t+1)-u(t-2)\big)$ and $\big(u(t-3)-u(t-4)\big)$, shown in Fig. [3].

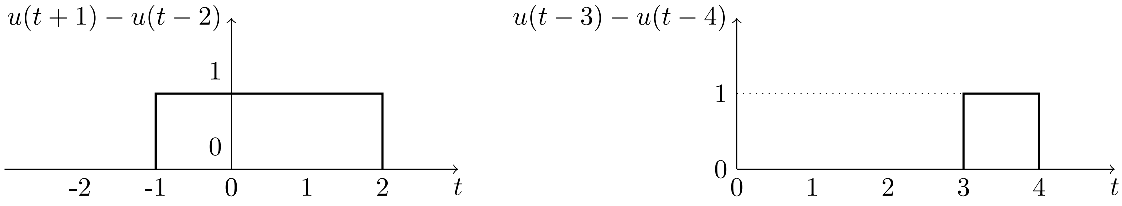

Figure 3: Signals $x(t)=\big(u(t+1)-u(t-2)\big)$ and $y(t)=\big(u(t-3)-u(t-4)\big)$.

Hence, if $x(t)=\Big(u(t+1)-u(t-2)\Big)$ and $y(t)=u(t-3)-u(t-4)$, then $x(t)*y(t)$ is given by,

$$x(t)*y(t)=\begin{cases}0 & t<2\\ t-2 & 2\leq t<3\\ 1 & 3\leq t\leq 5\\ 6-t & 5\leq t\leq 6\\ 0 & t>6\end{cases}$$

3) Given: $y(t)=f(t)*g(t)$. Now, consider the following:

$$y(t)=\int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau \Longrightarrow y(ct)=\int_{-\infty}^{\infty}f(\tau)g(ct-\tau)d\tau$$

At the same time,

$$f(ct)*g(ct)=\int_{-\infty}^{\infty}f(c\tau)g(ct-c\tau)d\tau$$

Case 1: Let $c>0$. Then, $c=|c|$. Let $\tau'=c\tau=|c|\tau \Longrightarrow d\tau=\frac{d\tau'}{|c|}$. Hence,

$$f(ct)*g(ct)=\int_{-\infty}^{\infty}f(\tau')g(ct-\tau')\frac{d\tau'}{|c|}=\frac{1}{|c|}\int_{-\infty}^{\infty}f(\tau')g(ct-\tau')d\tau'=\frac{1}{|c|}y(ct).$$

Case 2: Suppose $c<0$, then $c=-|c|$. In which case, let $\tau'=c\tau=-|c|\tau \Longrightarrow d\tau'=-\frac{d\tau}{|c|}$.

$$f(ct)*g(ct)=\int_{\infty}^{-\infty}f(\tau')g(ct-\tau')\Big(-\frac{d\tau'}{|c|}\Big)=\frac{1}{|c|}\int_{-\infty}^{\infty}f(\tau')g(ct-\tau')d\tau'=\frac{1}{|c|}y(ct).$$

Therefore, if $y(t)=f(t)*g(t)$, then $f(ct)*g(ct)=\frac{1}{|c|}y(ct)$, for all $c\neq 0$.

Fig. [4] shows $f(t)$ and $g(t)$.

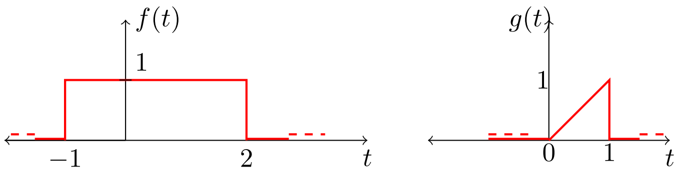

Figure 4: $f(t)$

Then, $y(t)=\int\limits_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau$ and $g(t-\tau)$, as a function of τ, will be non-zero from $\tau=t-1$ to $\tau=t$, as shown in figure [5].

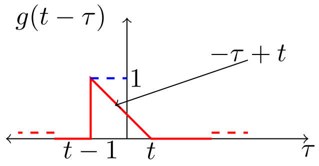

Figure 5: $g(t-\tau)$

Case 1: If $t<-1$. Then, $f(\tau)g(t-\tau)=0$ in this range. Hence, $y(t)=0, \forall t<-1$.

Case 2: If $t \geq -1$ but $t-1<-1$, i.e., $-1 \leq t < 0$.

Here, $y(t)=\int\limits_{-1}^{t}(-\tau+t)\,d\tau=\frac{t^2}{2}+t+\frac{1}{2}$.

Case 3: If $t<2$ but $t-1 \geq -1$, i.e., $0 \leq t < 2$.

Then, $y(t)=\int\limits_{t-1}^{t}(-\tau+t)\,d\tau=0.5$.

Case 4: If $t \geq 2$ but $t-1<2$, i.e., $2 \leq t < 3$.

Now, $y(t)=\int\limits_{t-1}^{2}(-\tau+t)\,d\tau=-\frac{t^2}{2}+2t-\frac{3}{2}$.

Case 5: If $t-1 \geq 2$, $f(\tau)g(t-\tau)=0 \Longrightarrow y(t)=0$, $\forall t \geq 3$.

Using the result derived initially, we get,

$$f(2t)*g(2t)=\begin{cases}0, & t<-0.5\\ t^2+t+0.25, & -0.5 \leq t < 0\\ 0.25, & 0 \leq t < 1\\ -t^2+2t-0.75, & 1 \leq t < 1.5\\ 0, & t \geq 1.5\end{cases}$$

4) A continuous-time LTI system is stable if and only if, the impulse response is absolutely integrable

$$\int_{-\infty}^{\infty}|h(\tau)|d\tau<\infty.$$

A continuous-time LTI system is causal if,

$$h(t)=0 \quad \text{for } t<0,$$

A continuous-time LTI system is instantaneous/memoryless if, $h(t)=c\delta(t)$, where c is a nonzero scaling factor.

(a) $h(t)=e^{-(t+2)}u(t)$
This system is stable as impulse response is absolutely integrable i.e, $\int_0^\infty e^{-(\tau+2)}d\tau=e^{-2}$, causal as $u(t)=0$ for $t<0$ and, not instantaneous as $h(t)\neq c\delta(t)$.

(b) $h(t)=e^{-|t|}$ is stable as $\int_{-\infty}^{\infty}e^{-|\tau|}d\tau=2$. Since the impulse response is two sided i.e, $h(t)=e^{t}\ \forall\ t<0$ and $h(t)=e^{-t}\ \forall\ t\geq 0$, the system is non-causal and, not instantaneous as $h(t)\neq c\delta(t)$.

(c) $h(t)=\delta(t)+\delta(t-3)$ is stable as $\int_{-\infty}^{\infty}(\delta(t)+\delta(t-3))d\tau=2$, causal as $h(t)=0\ \forall\ t<0$ and not instantaneous as $h(t)\neq c\delta(t)$.

5) a) The output of S_1 is, $w(t)=e^{-4t}u(2t)$. Further, the output of S_2 is $y(t)=w(t)*h(t)=\int_0^t e^{-4\tau}e^{-(t-\tau)}d\tau$. Thus, $y(t)=\frac{1}{3}e^{-t}(1-e^{-3t})$, $\forall t\geq 0$.

b) Here, the output $p(t)$ is given by, $p(t)=x(t)*h(t)=\int_0^t e^{-2\tau}e^{-(t-\tau)}d\tau=e^{-t}(1-e^{-t})$. Thus, $p(t)=e^{-t}(1-e^{-t})$, $\forall t\geq 0$. Now, the final output $z(t)$ will be,

$$z(t)=e^{-2t}(1-e^{-2t}),\ \forall t\geq 0.$$

The final outputs are not same as S_1 is not time-invariant system.

6) (a) The response is given by,

$$y(t) = x_1(t) * h(t) = 5 \int\limits_{-\infty}^{\infty} e^{-2(t-\tau)} u(t-\tau) u(\tau) d\tau$$
$$= 5 \int\limits_{0}^{t} e^{-2(t-\tau)} d\tau \quad \textit{for} \quad 0 \le \tau \le t$$
$$= 2.5 e^{-2t}(e^{2t} - 1) \quad \textit{for} \quad t \ge 0.$$

$$\textit{Hence,} \quad y(t) = \begin{cases} \frac{5}{2}(1 - e^{-2t}), \forall t \ge 0 \\ 0, \quad t < 0 \end{cases}$$

(b) Given: $x_2(t) = \cos(4\pi t) = \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})$. Then, we obtain,

$$y(t) = x_2(t) * h(t) = \int\limits_{-\infty}^{\infty} h(\tau) x_2(t-\tau) d\tau = \frac{1}{2} \int\limits_{0}^{\infty} e^{j4\pi t} e^{-j4\pi\tau - 2\tau} + e^{-j4\pi t} e^{j4\pi\tau - 2\tau} \mathrm{d}\tau$$
$$= \frac{1}{2}\Big(e^{j4\pi t} \int\limits_{0}^{\infty} e^{-(j4\pi+2)\tau} d\tau + e^{-j4\pi t} \int\limits_{0}^{\infty} e^{-(2-j4\pi)\tau} \mathrm{d}\tau\Big)$$
$$= \frac{1}{2}\Big(\frac{e^{j4\pi t}}{2+j4\pi} + \frac{e^{-j4\pi t}}{2-j4\pi}\Big) = \frac{2\cos(4\pi t) + 4\pi \sin(4\pi t)}{4 + 16\pi^2}.$$

7) (a) True. If $h(t)$ periodic and nonzero, then

$$\int_{-\infty}^{\infty} |h(t)| dt = \sum_{n=-\infty}^{+\infty} \int_{(n-1)T}^{nT} |h(t)| dt.$$

Since each summand is positive, the infinite sum is unbounded. Thus $h(t)$ is unstable.

(b) False. For instance, suppose that the inverse of $h[n] = \delta[n - n_0]$ is $g[n]$. Then,

$$\Rightarrow h[n] * g[n] = \delta[n]$$
$$\Rightarrow \sum_{k=-\infty}^{\infty} \delta[k - n_0] g[n-k] = g[n - n_0] = \delta[n]$$
$$\Rightarrow g[n] = \delta[n + n_0]$$

which is noncausal.

(c) False. For example $h[n] = u[n]$ implies that

$$\sum_{-\infty}^{\infty} |h[n]| = \infty.$$

This is an unstable system.

(d) True. Assuming that $h[n]$ is bounded in the range $n_1 \le n \le n_2$,

$$\sum_{k=n_1}^{n_2} |h[k]| < \infty.$$

This implies that the system is stable.

(e) False. For example, $h(t) = tu(t)$ is causal. However, $\int_{-\infty}^{\infty} |h(t)| dt = \int_{0}^{\infty} e^t dt = \infty$.

(f) False. For example, the cascade of a causal system with impulse response $h_1[n] = \delta[n-1]$ and a non-causal system with impulse response $h_2[n] = \delta[n+1]$ leads to a system with overall impluse response given by $h[n] = h_1[n] * h_2[n] = \delta[n]$.

(g) False. For example, if $h(t) = e^{-t}u(t)$, then $s(t) = e^{-t}u(t) * u(t) = \int_{\tau=-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau)d\tau$ $= \int_{\tau=0}^{t} e^{-\tau}u(\tau)d\tau = (1-e^{-t})u(t)$ and

$$\int_0^\infty |1-e^{-t}|dt = t + e^{-t}|_0^\infty = \infty.$$

Although the system is stable, the step response is not absolutely integrable.

(h) True. We may write $u[n] = \Sigma_{k=0}^{\infty}\delta[n-k]$. Therefore,

$$s[n] = \Sigma_{k=-\infty}^{\infty} u[k]h[n-k] = \Sigma_{k=0}^{\infty} h[n-k].$$

If $s[n] = 0$ for $n < 0$, then $h[n] = 0$ for $n < 0$ and the system is causal.

8) Given: System A is LTI and system B is inverse of A. Let $y_1(t)$ and $y_2(t)$ be outputs of system A for inputs $x_1(t)$ and $x_2(t)$ respectively. Combining these informations, we get,

$$x_1(t) \xrightarrow{\text{A}} y_1(t) \xrightarrow{\text{B}} x_1(t). \tag{1}$$

And,

$$x_2(t) \xrightarrow{\text{A}} y_2(t) \xrightarrow{\text{B}} x_2(t). \tag{2}$$

(a) **To prove system B is linear**. Assume that system B is not linear. From equations (1) and (2), we observe that an input $ax_1(t) + bx_2(t)$ to system A can generate $ay_1(t) + by_2(t)$, i.e.,

$$ax_1(t) + bx_2(t) \xrightarrow{\text{A}} ay_1(t) + by_2(t).$$

This is due to linearity property of the system A. Our assumption that system B is not linear implies that the output of B for the input $ay_1(t) + by_2(t)$ is not $ax_1(t) + bx_2(t)$, i.e., in (1) and (2) the outputs of system B does not add up linearly even if the inputs combine linearly. Therefore, we arrive at a situation which is as follows:

$$ax_1(t) + bx_2(t) \xrightarrow{\text{A}} ay_1(t) + by_2(t) \overset{\text{B}}{\not\rightarrow} ax_1(t) + bx_2(t).$$

This contradicts the fact that B is inverse of A. Hence, our assumption is incorrect, and so, system B is linear.

(b) **To prove system B is time-invariant**. Assume system B is time variant. By time-invariant property of system A, we have,

$$x_1(t-\tau) \xrightarrow{\text{A}} y_1(t-\tau).$$

When this output of system A, is fed to system B, we must not expect its response to be $x_1(t-\tau)$ because of our assumption. So, we land up in a situation where,

$$x_1(t-\tau) \xrightarrow{\text{A}} y_1(t-\tau) \overset{\text{B}}{\not\rightarrow} x_1(t-\tau).$$

This contradicts the fact that B is the inverse of A. So, the assumption about system B is incorrect. Therefore, system B is also time-invariant.