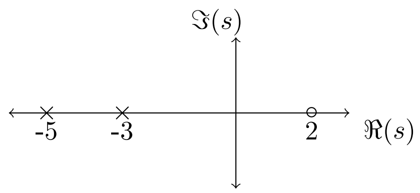
EE1101: Signals and Systems JAN — MAY 2018

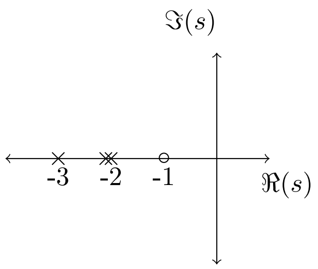
Tutorial 10 Solutions

Solution 1

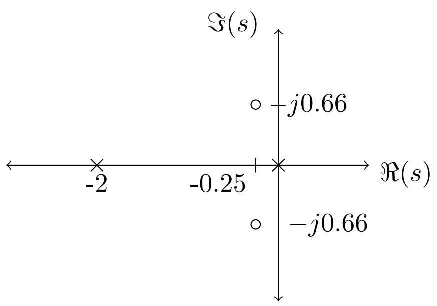
(a) Poles at s = -5, -3
Zero at s = 2
The system is BIBO stable, as all the poles are in the left half plane.



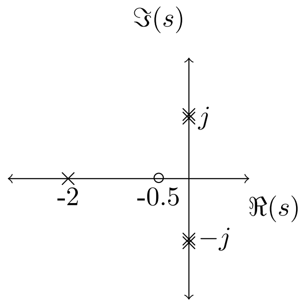
(b) Poles at s = -3, -2, -2
Zeros at s = -1
The system is BIBO stable, as all the poles are in the left half plane.



(c) Poles at s = -2, 0
Zeros at s = -0.25 + j0.66, -0.25 - j0.66
The system is not BIBO stable, as there is a pole at origin.



(d) Poles at s = -2, +j, +j, -j, -j
Zeros at s = -0.5
The system is not BIBO stable, as there are poles are at +j and -j.



Solution 2

For a system with rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole. An LTI system is stable iff the ROC of its system function $H(s)$ includes the $j\omega-$axis.

a) **Non-causal** as the ROC is not to the right to the rightmost pole $s = -2$, BIBO **unstable** as the ROC does not include the $j\omega-$axis.

b) **Non-causal** as the ROC is not to the right to the rightmost pole $s = 3$, BIBO **unstable** as the ROC does not include the $j\omega-$axis.

c) **Causal** as the ROC is to the right to the rightmost pole $s = -1$, BIBO **stable** as ROC includes the $j\omega-$axis.

d) **Non-causal** as the ROC is not to the right to the rightmost pole $s = -1$, BIBO **unstable** as the ROC does not include the $j\omega-$axis.

Solution 3

(a) $H(s) = \dfrac{s+3}{(s+2)^3} = \dfrac{s+2+1}{(s+2)^3} = \dfrac{1}{(s+2)^2} + \dfrac{1}{(s+2)^3}$. Therefore , the impulse response $h(t)$ is

$$h(t) = te^{-2t}u(t) + \frac{t^2}{2}e^{-2t}u(t)$$

(b) The Laplace transform of the output $y(t)$ of the system to the input $x(t)$ can be expressed as

$$Y(s) = H(s)X(s)$$

Given $x(t) = 10u(t) \implies X(s) = \dfrac{10}{s}$

$$\implies Y(s) = \frac{10(s+3)}{s(s+2)^3}$$

Using the final value theorem, the final value of $y(t)$ would be

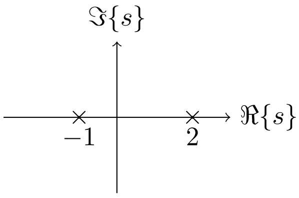
$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

$$\implies \lim_{t\to\infty} y(t) = \lim_{s\to 0} \frac{10(s+3)}{(s+2)^3} = 3.75$$

Solution 4

(a)

$$\begin{aligned} s^2Y(s) - sY(s) - 2Y(s) &= X(s) \\ H(s) = \frac{1}{s^2-s-2} &= \frac{1}{(s-2)(s+1)} \end{aligned}$$



(b)

$$H(s) = \frac{1}{(s-2)(s+1)} = \frac{1}{3(s-2)} - \frac{1}{3(s+1)}$$

(a) System is stable $\implies$ The ROC should contain the $j\omega$ axis.

$\therefore$ ROC : $-1 < Re\{s\} < 2$

$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t)$

(b) System is causal $\implies$ The ROC should be to the right of the rightmost pole

$\therefore$ ROC : $Re\{s\} > 2$

$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t) = \frac{1}{3}(e^{2t} - e^{-t})u(t)$

(c) System is neither causal nor stable $\implies$ ROC : $Re\{s\} < -1$

$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t) = \frac{1}{3}(e^{-t} - e^{2t})u(-t)$

Solution 5

Given that $x(t) = e^{-|t|}$,$-\infty < t < \infty$
Since the system is LTI, the output is $x(t) * h(t)$ or in s-domain, by L.T properties

$Y(s) = X(s)H(s)$

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\ &= \int_{-\infty}^{0} e^{t}e^{-st}dt + \int_{0}^{\infty} e^{-t}e^{-st}dt \\ &= \int_{-\infty}^{0} e^{t-st}dt + \int_{0}^{\infty} e^{-t-st}dt \\ &= \frac{1}{1-s} + \frac{1}{1+s};\ \ Re\{s\} < 1 \cap Re\{s\} > -1 \\ &= \frac{-2}{(s-1)(s+1)}\ \ ROC : -1 < Re\{s\} < 1 \end{aligned}$$

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \frac{s+1}{s^2+2s+2}X(s) \\ &= \frac{-2}{(s-1)(s^2+2s+2)} \\ &= \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+2} \end{aligned}$$

Solving for A,B,C we get

$A = \frac{-2}{5}$ $\quad B = \frac{2}{5}$ $\quad C = \frac{6}{5}$. Hence

$$\begin{aligned} Y(s) &= \frac{-2}{5(s-1)} + \frac{2s+6}{5(s^2+2s+2)} \\ &= \frac{-2}{5(s-1)} + \frac{2(s+1)}{5((s+1)^2+1))} + \frac{4}{5((s+1)^2+1))} \\ &\qquad ROC : -1 < Re\{s\} < 1 \end{aligned}$$

In time domain:

$$\begin{aligned} y(t) &= \frac{2}{5}e^{t}u(-t) + \frac{2}{5}e^{-t}\cos tu(t) + \frac{4}{5}e^{-t}\sin tu(t) \\ &= \frac{2}{5}e^{t}u(-t) + \frac{2}{5}e^{-t}(\cos t + 2\sin t)u(t) \end{aligned}$$

Solution 6

Given that $H(s)$ is a rational function, we may take it to be $H(s) = \dfrac{a(s)}{b(s)}$, where $a(s)$ and $b(s)$ are polynomials in s.
The response is absolutely integrable for the signal $u(t)$, whose Laplace transform is $\dfrac{1}{s}$. Therefore, $\dfrac{H(s)}{s}$ has no poles at $Re(s) \geq 0$. Therefore, $s = 0$ must be a root of $a(s)$. Take $a(s) = sa_1(s)$.
The response to $tu(t)$ (Laplace transform $\dfrac{1}{s^2}$) is not absolutely integrable. This implies, there cannot be a

repeated root for $a(s)$ at $s = 0$.
If a signal with a rational Laplace transform is of finite duration, then its denominator is a constant polynomial.
The Laplace transform of the signal $\frac{d^2h}{dt^2} + 2\frac{dh}{dt} + 2h(t)$ is $(s^2 + 2s + 2)H(s)$. This is given to be of finite duration in time domain. Hence, $b(s) = \frac{1}{K}(s^2 + 2s + 2)$ for some constant K.
The number of zeros at infinity is $deg(b(s)) - deg(a(s)) = 1$. Since $deg(b(s)) = 2$, $deg(a(s)) = 1$. Therefore $a(s) = K_1 s$.
Therefore $H(s) = KK_1 \frac{s}{s^2 + 2s + 2}$. Using the fact that $H(1) = 0.2$, we find that $KK_1 = 1$. The required impulse response is

$$H(s) = \frac{s}{s^2 + 2s + 2}$$

The poles of the system are $-1 \pm j$. Since the system is causal, the region of convergence is $Re(s) > -1$.

Solution 7

(a) Given that

$$\phi_{xx}(t) = \int_{-\infty}^{\infty} x(\tau)x(t+\tau)d\tau$$

If we want $\phi_{xx}(t)$ to be the output of the system when $x(t)$ is the input, then using the convolution integral

$$\phi_{xx}(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Let $p = -\tau$, then $d\tau = -dp$

$$\implies \phi_{xx}(t) = \int_{\infty}^{-\infty} h(-p)x(t+p)(-dp)$$

$$\implies \phi_{xx}(t) = \int_{-\infty}^{\infty} h(-p)x(t+p)dp$$

Now replace p as τ

$$\implies \phi_{xx}(t) = \int_{-\infty}^{\infty} h(-\tau)x(t+\tau)d\tau$$

Comparing this with the given definition of $\phi_{xx}(t)$, we get

$$\implies \quad h(t) = x(-t)$$

(b) Since $\phi_{xx}(t) = x(t) * x(-t)$,

$$\Phi_{xx}(s) = X(s)X(-s)$$

and

$$\Phi_{xx}(j\omega) = X(j\omega)X(-j\omega)$$

If $x(t)$ is real, $X^*(j\omega) = X(-j\omega)$, hence

$$\Phi_{xx}(j\omega) = |X(j\omega)|^2$$

(c) If $X(s)$ has ROC $Re\{s\} > -1$, then $X(-s)$ would have ROC $Re\{s\} < 1$. Now $\Phi_{xx}(s)$ will include the poles of both $X(s)$ and $X(-s)$. Further, its ROC would be the intersection of the ROCs of $X(s)$ and $X(-s)$. Hence the ROC of $\Phi_{xx}(s)$ is $-1 < Re\{s\} < 1$

Solution 8

The unilateral Laplace transform of a signal $x(t)$ is given as

$$X(s) = \int_0^{\infty} x(t)e^{-st}dt$$

(a) $x(t) = u(t-2)$

$$\begin{aligned} X(s) &= \int_0^{\infty} u(t-2)e^{-st}dt \\ &= \int_2^{\infty} e^{-st}dt \\ &= \frac{e^{-2s}}{s} \quad ; \quad Re\{s\} > 0 \end{aligned}$$

(b) $x(t) = u(t+2)$

$$\begin{aligned} X(s) &= \int_0^{\infty} u(t+2)e^{-st}dt \\ &= \int_0^{\infty} e^{-st}dt \\ &= \frac{1}{s} \quad ; \quad Re\{s\} > 0 \end{aligned}$$

(c) $x(t) = e^{3t}u(t)$

$$\begin{aligned} X(s) &= \int_0^{\infty} e^{3t}e^{-st}dt \\ &= \int_0^{\infty} e^{-(s-3)t}dt \\ &= \frac{1}{s-3} \quad ; \quad Re\{s\} > 3 \end{aligned}$$

(d) $x(t) = te^{t}u(t)$

$$\begin{aligned} X(s) &= \int_0^{\infty} te^{t}e^{-st}dt \\ &= \int_0^{\infty} te^{-(s-1)t}dt \\ &= \frac{1}{(s-1)^2} \quad ; \quad Re\{s\} > 1 \end{aligned}$$

(e) $x(t) = \sin(t)u(t)$

$$\begin{aligned} X(s) &= \int_0^{\infty} \sin(t)e^{-st}dt \\ &= \int_0^{\infty} \frac{e^{jt} - e^{-jt}}{2j}e^{-st}dt \\ &= \frac{1}{s^2+1} \quad ; \quad Re\{s\} > 0 \end{aligned}$$

Solution 9

(a) The differential equation relating $v_i(t)$ and $v_o(t)$ can be expressed as

$$LC\frac{d^2v_o(t)}{dt^2} + RC\frac{dv_o(t)}{dt} + v_o(t) = v_i(t)$$

$$\frac{d^2v_o(t)}{dt^2} + \frac{R}{L}\frac{dv_o(t)}{dt} + \frac{1}{LC}v_o(t) = \frac{1}{LC}v_i(t)$$

Substituting the values of R, L and C, we get

$$\frac{d^2v_o(t)}{dt^2} + 3\frac{dv_o(t)}{dt} + 2v_o(t) = 2v_i(t)$$

(b) Taking the unilateral Laplace transform of the above differential equation, we get

$$s^2V_o(s) - sv_o(0^-) - v_o'(0^-) + 3sV_o(s)$$

$$- 3v_o(0^-) + 2V_o(s) = 2V_i(s)$$

Since, $v_i(t) = e^{-3t}u(t)$,

$$V_i(s) = \frac{1}{s+3}, Re\{s\} > -3$$

Substituting this along with the initial conditions, we get

$$V_o(s) = \frac{(s^2 + 8s + 17)}{(s+1)(s+2)(s+3)}$$

The partial fraction expansion of $V_o(s)$ is

$$V_o(s) = \frac{5}{s+1} - \frac{5}{s+2} + \frac{1}{s+3}$$

Taking the inverse Laplace transform, we get

$$v_o(t) = (5e^{-t} - 5e^{-2t} + e^{-3t})u(t)$$