Solutions for Problem sheet-1

PROBLEM SHEET – I

① A line of force is a directed curve such that the forward drawn tangent at any point has the direction of the electric field. If $d\vec{s}$ is our element of this curve, then $d\vec{s} \propto E$. i.e. $d\vec{s} = \lambda E$ (λ scalar factor)

We need the lines of force in the plane $z = 0$

The field lines of $\vec{E}$ are given by

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}$$

$$\frac{dx}{\alpha y^2} = \frac{dy}{2\alpha xy} \implies 2x\,dx - y\,dy = 0$$

$$\implies x^2 - \frac{y^2}{2} = c$$

Note: For $c > 0$, these curves represent hyperbola. Different lines of force are obtained for different values of c.

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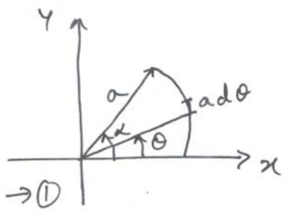
②



$$F_x = kq \int_{\theta=0}^{\alpha} \frac{Q\,a\,d\theta}{a\alpha} \frac{\cos\theta}{a^2} \quad \left(k = \frac{1}{4\pi\epsilon_0}\right)$$

$$= \frac{kqQ}{a^2\alpha}\left[\sin\theta\right]_0^{\alpha} = \frac{kqQ}{a^2\alpha}\sin\alpha \rightarrow (1)$$

$$F_y = kq \int_{\theta=0}^{\alpha} \frac{Q\,a\,d\theta}{a\alpha}\,\frac{1}{a^2}\sin\theta = \frac{kqQ}{a^2\alpha}(1-\cos\alpha) \rightarrow (2)$$

From (1) & (2)

$$|\vec{F}| = \frac{kqQ}{a^2\alpha}\left[\sin^2\alpha + (1-\cos\alpha)^2\right]^{1/2} = \frac{2kqQ}{a^2\alpha}\sin\frac{\alpha}{2}$$

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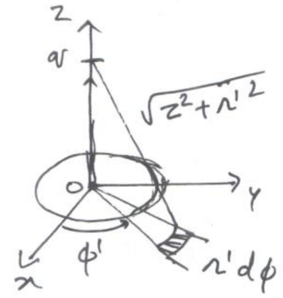

(3) The field point $\vec{r} = z\,\hat{e}_z$

The source point

$$(r', \phi') = r'\cos\phi'\,\hat{e}_x + r'\sin\phi'\,\hat{e}_y = \vec{r}\,'$$

$$\therefore \vec{r}-\vec{r}\,' = z\,\hat{e}_z - r'\cos\phi'\,\hat{e}_x - r'\sin\phi'\,\hat{e}_y$$

Element of charge $dq' = \sigma r'\,dr'\,d\phi'$

$$\text{The force on } q = \frac{q}{4\pi\epsilon_0}\int_a^\infty\int_0^{2\pi}\frac{\sigma(z\,\hat{e}_z - r'\cos\phi'\,e_x - r'\sin\phi'\,\hat{e}_y)\,r\,dr'\,d\phi'}{(z^2+r'^2)^{3/2}}$$

Integration over $\cos\phi'$ and $\sin\phi'$ vanish.

$$\text{The force on } q = \vec{F}_q = \frac{q}{4\pi\epsilon_0}\,2\pi\int_a^\infty\frac{\sigma\,z\,\hat{e}_z}{(z^2+r'^2)^{3/2}}\,r'\,dr'$$

$$= -\frac{q}{2\epsilon_0}\frac{\sigma z}{\sqrt{z^2+r'^2}}\hat{e}_z\Bigg|_{r'=a}^{\infty} = \frac{q\sigma z\,\hat{e}_z}{2\epsilon_0\sqrt{z^2+a^2}}$$

At large distance $\vec{F}_q = \frac{q\sigma}{2\epsilon_0}\hat{e}_z$, which is exactly same as corresponding to the situation in which the hole is absent. So, at large distances q does not see the hole.

For $z << a$ and $q < 0$

$$\vec{F}_q = \frac{q\sigma z\,\hat{e}_z}{2\epsilon_0 a\left(1+\frac{z^2}{a^2}\right)^{1/2}} = \frac{q\sigma z}{2\epsilon_0 a}\,\hat{e}_z$$

The motion of charge is simple harmonic

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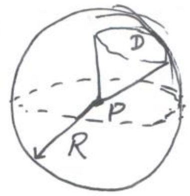
(4) Radius of sphere is R and center is at P.



On the surface of sphere

$$d\vec{s} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{e}_r = R^2 \, d\Omega \, \hat{e}_r$$

Electric field due to q at P,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \hat{e}_r$$

$$\vec{E}\cdot d\vec{s} = \frac{q}{4\pi\epsilon_0} d\Omega$$

Electric flux through D,

$$\int_D \vec{E}\cdot d\vec{s} = \frac{q}{4\pi\epsilon_0} \int_D d\Omega = \frac{q}{\epsilon_0} \frac{\Omega_D}{4\pi}$$

where Ω_D is the solid angle subtended by D at P

Flux through the face of the cube (centered at P)

$$= \frac{q}{4\pi\epsilon_0} \text{(Solid angle subtended by face)}$$

$$= \frac{q}{4\pi\epsilon_0} \cdot \frac{4\pi}{6} = \frac{q}{6\epsilon_0}$$

(i) The answer does not depend on the size ~~or~~ orientation of the cube as the solid angle for the face subtended at P is always $\frac{1}{6} \cdot 4\pi$

(ii) It does depend on the cube being centered at P – only then does each face subtend the same solid angle at P.

(iii) More generally, we see that any surface S not intersecting the charge, we find:

$$\oint \vec{E}\cdot d\vec{s} = \begin{cases} 0 & \text{if the surface does enclose } q \\ \frac{q}{\epsilon_0} & \text{if the surface encloses } q \end{cases}$$

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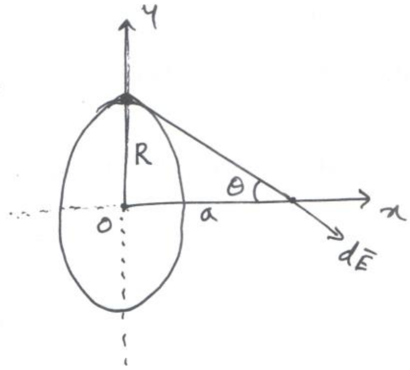
(5)

The x-component of the force on the charge is



$$F(x) = -Q\,\vec{E}(x)\cdot\hat{e}_x$$

$$= (-Q)\,\frac{2\pi R\lambda}{4\pi\epsilon_0}\,\frac{\cos\theta}{(x^2+R^2)}$$

where we see that x component of $\vec{E}$ for each infinitesimal segment of the ring add up with the Y and Z components cancelling out.

$$F(x) = -\frac{QR\lambda}{2\epsilon_0}\,\frac{x}{(x^2+R^2)^{3/2}} \simeq -\left(\frac{Q\lambda}{2\epsilon_0 R^2}\right)x$$

This is in the form of $F(x) = -K(x)$ $[K>0]$

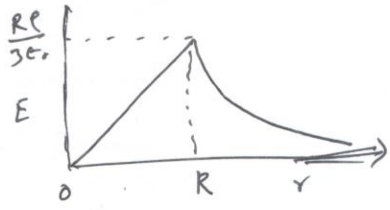
$$\therefore\ T = 2\pi\sqrt{\frac{2m\epsilon_0 R^2}{Q\lambda}} \qquad T = 2\pi\sqrt{\frac{m}{K}}$$

(6) Electric field $r \leq R$ (Using Gauss's Law)

$$\oint \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0}\int \rho\, dv \qquad \left|\ dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r^2 \sin\theta\, d\theta\, d\phi\, dr\right.$$

$$\Rightarrow \hat{E}\cdot 4\pi r^2 = \frac{4}{3}\,\frac{\pi r^3}{\epsilon_0}\,\rho$$

$$\Rightarrow \vec{E} = \frac{\rho r}{3\epsilon_0}\,\hat{r} \qquad (r\leq R)$$



|||ly $r \geq R$

$$E\,4\pi r^2 = \frac{4}{3}\,\frac{\pi R^3}{\epsilon_0}\,\rho$$

$$\therefore\ \vec{E} = \frac{\rho\rho^3}{3\epsilon_0 r^2}\,\hat{r} \qquad \cancel{(r \geq R)}\ (r\geq R)$$

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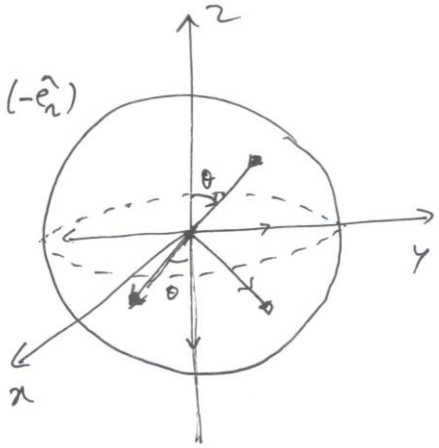
(7)

$$\sigma = \vec{k}\cdot\vec{r} = kR\cos\theta$$



$$d\vec{E} = \frac{1}{4\pi\epsilon_0 R^2}(kR\cos\theta)(R^2\sin\theta\, d\theta\, d\phi)(-\hat{e}_r)$$

$$= \frac{kR\cos\theta}{4\pi\epsilon_0 R^2}(R^2\sin\theta\, d\theta\, d\phi)(-\cos\theta\,\hat{e}_z)$$

x, y components of the electric field vanish.

$$d\vec{E}\Big|_{net} = \frac{2kR\cos\theta}{4\pi\epsilon_0 R^2}(R^2\sin\theta\, d\theta\, d\phi)(-\cos\theta)\,\hat{e}_z$$

$$\vec{E} = \int d\vec{E}\Big|_{net} = \int_0^{2\pi}\int_0^{\pi/2}\frac{(-2kR\cos^2\theta)(R^2\sin\theta\, d\theta\, d\phi)(\hat{e}_z)}{4\pi\epsilon_0 R^2}$$

$$= \frac{2\pi}{4\pi\epsilon_0}(-2kR)\int_0^{\pi/2}\cos^2\theta\sin\theta\, d\theta\,(\hat{e}_z)$$

$$\vec{E} = -\frac{kR}{\epsilon_0}\hat{e}_z\int_0^{\pi/2}\cos^2\theta\, d(-\cos\theta)$$

$$E = -\frac{kR}{3\epsilon_0}\hat{e}_z$$