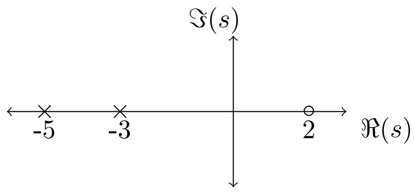
EE1101: Signals and Systems JAN — MAY 2019

Tutorial 10 Solutions

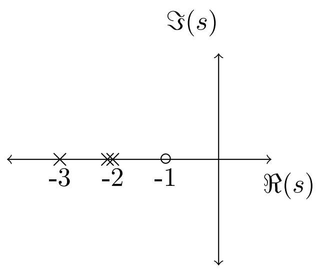
Solution 1

For a causal system, the ROC is towards the right of the rightmost pole in the pole-zero plot. Also for a system to be stable, the ROC should include the imaginary axis.

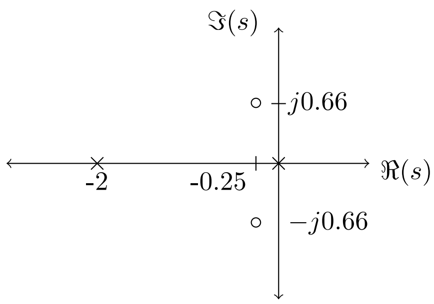
(a) Poles at s = -5, -3
Zero at s = 2
The system is BIBO stable, as all the poles are in the left half plane.



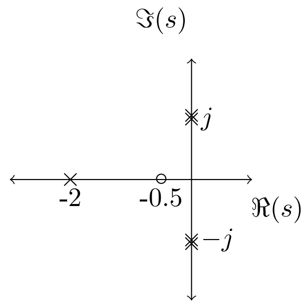
(b) Poles at s = -3, -2, -2
Zeros at s = -1
The system is BIBO stable, as all the poles are in the left half plane.



(c) Poles at s = -2, 0
Zeros at s = -0.25 + j0.66, -0.25 - j0.66
The system is not BIBO stable, as there is a pole at origin.



(d) Poles at s = -2, +j, +j, -j, -j
Zeros at s = -0.5
The system is not BIBO stable, as there are poles are at +j and -j.



Solution 2

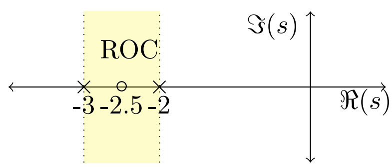
Stability: An LTI system is stable iff the ROC of its system function $H(s)$ includes the $j\omega-$axis.

Causality:

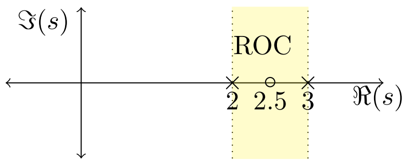
- If an LTI system is causal (with a right sided impulse response function $h(t) = 0$ for $t < 0$), then the ROC of its transfer function H(s) is a right sided plane.

- When H(s) is rational, then the system is causal if and only if its ROC is the right half plane to the right of the rightmost pole, and the order of numerator is no greater than that of the denominator.

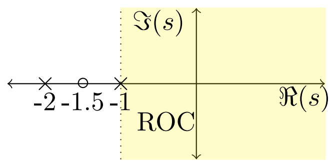
a) **Non-causal** as the ROC is not to the right to the rightmost pole $s = -2$, BIBO **unstable** as the ROC does not include the $j\omega-$axis.



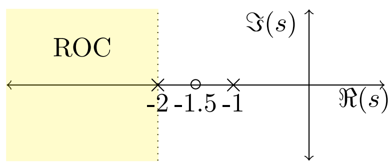
b) **Non-causal** as the ROC is not to the right to the rightmost pole $s = 3$, BIBO **unstable** as the ROC does not include the $j\omega-$axis.



c) **Causal** as the ROC is to the right to the right-most pole $s = -1$, BIBO **stable** as ROC includes the $j\omega-$axis.



d) **Non-causal** as the ROC is not to the right to the rightmost pole $s = -1$, BIBO **unstable** as the ROC does not include the $j\omega-$axis.



NOTE: Here we describe an example of a right sided ROC for a $H(s)$ which is **not causal**.

Consider $H(s) = \frac{e^s}{s+1}$ with the ROC as $Re(s) > -1$. We know that $\mathcal{L}(e^{-t}u(t)) = \dfrac{1}{s+1}$. Also, the e^s causes a time shift giving $h(t) = e^{-(t+1)}u(t+1)$. This is not causal as $h(t) \neq 0$ when $-1 < t < 0$. However, the ROC is a right half plane.

Only if $H(s)$ is a rational function with the order of denominator greater than the order of the numerator, will the ROC being a right-half plane imply that the system is causal.

Solution 3

(a) Applying partial fractions to $H(s)$,

$$H(s) = \frac{s+3}{(s+2)^3} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}$$
$$\implies s+3 = A(s^2+4s+4) + B(s+2) + C$$

Solving we get $A = 0, B = 1$ and $C = 1$

Alternatively we can observe that $H(s) = \dfrac{s+3}{(s+2)^3} = \dfrac{s+2+1}{(s+2)^3} = \dfrac{1}{(s+2)^2} + \dfrac{1}{(s+2)^3}$.

We know the following:

$$\mathcal{L}(u(t)) = \frac{1}{s}$$
$$\mathcal{L}(f(t)e^{-at}) = F(s+a)$$
$$\mathcal{L}(tf(t)) = -\frac{dF(s)}{ds}$$

where $\mathcal{L}(f(t) = F(s)$

Therefore , the impulse response $h(t)$ is

$$h(t) = te^{-2t}u(t) + \frac{t^2}{2}e^{-2t}u(t)$$

(b) The Laplace transform of the output $y(t)$ of the system to the input $x(t)$ can be expressed as

$$Y(s) = H(s)X(s)$$

Given $x(t) = 10u(t) \implies X(s) = \dfrac{10}{s}$

$$\implies Y(s) = \frac{10(s+3)}{s(s+2)^3}$$

Using the final value theorem, the final value of $y(t)$ would be

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

$$\implies \lim_{t\to\infty} y(t) = \lim_{s\to 0} \frac{10(s+3)}{(s+2)^3} = 3.75$$

To verify this, let us compute the signal $y(t)$ and then evaluating at the limit t tends to ∞. First we need to evaluate partical fractions of $Y(s)$.

$$Y(s) = \frac{10(s+3)}{s(s+2)^3} = \frac{A}{s} + \frac{Bs^2+Cs+D}{(s+2)^3}$$
$$\implies 10s+30 = A(s+2)^3 + (Bs^3+Cs^2+Ds)$$
$$\implies 10s+30 = (A+B)s^3 + (6A+C)s^2$$
$$+(12A+D)s + 8A$$

Solving we get $A = \frac{30}{8}, B = -\frac{30}{8}, C = -\frac{180}{8}, D = -15$

Now $\dfrac{Bs^2+Cs+D}{(s+2)^3} = -\dfrac{30}{8}\dfrac{s^2+6s+5}{(s+2)^3} = -\dfrac{30}{8}\left[\dfrac{s^2+4s+4}{(s+2)^3} + \dfrac{2s+4}{(s+2)^3} - \dfrac{1}{(s+2)^3}\right] = -\dfrac{30}{8}\left[\dfrac{1}{s+2} + \dfrac{2}{(s+2)^2} - \dfrac{1}{(s+2)^3}\right]$

Thus, $Y(s) = \dfrac{30}{8s} - \dfrac{30}{8}\left[\dfrac{1}{s+2} + \dfrac{2}{(s+2)^2} - \dfrac{1}{(s+2)^3}\right]$

Thus, $y(t) = \frac{30}{8}u(t) - \frac{30}{8}e^{-2t}u(t) - \frac{60}{8}te^{-2t}u(t) + \frac{30}{8}\frac{t^2}{2}e^{-2t}u(t)$

$$\lim_{t\to\infty} y(t) = \frac{30}{8} = 3.75$$

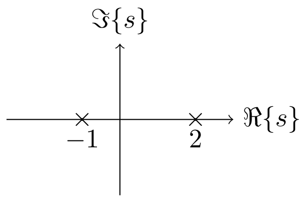
As

$$\lim_{t\to\infty} e^{-t} = 0, \lim_{t\to\infty} te^{-t} = 0, \lim_{t\to\infty} t^2e^{-t} = 0$$

Solution 4

(a)

$$\begin{aligned} s^2Y(s) - sY(s) - 2Y(s) &= X(s) \\ H(s) = \frac{1}{s^2 - s - 2} &= \frac{1}{(s-2)(s+1)} \end{aligned}$$



(b) The possible ROC's for the above system are

I. $Re\{s\} < -1$

II. $-1 < Re\{s\} < 2$

III. $Re\{s\} > 2$

$$H(s) = \frac{1}{(s-2)(s+1)} = \frac{1}{3(s-2)} - \frac{1}{3(s+1)}$$

(a) System is stable $\implies$ The ROC should contain the $j\omega$ axis.
$\therefore$ ROC : $-1 < Re\{s\} < 2$
$h(t) = -\frac{1}{3}e^{2t}u(-t) - \frac{1}{3}e^{-t}u(t)$

(b) System is causal $\implies$ The ROC should be to the right of the rightmost pole
$\therefore$ ROC : $Re\{s\} > 2$
$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t) = \frac{1}{3}(e^{2t} - e^{-t})u(t)$

(c) System is neither causal nor stable $\implies$ ROC : $Re\{s\} < -1$
$h(t) = -\frac{1}{3}e^{2t}u(-t) + \frac{1}{3}e^{-t}u(-t) = \frac{1}{3}(e^{-t} - e^{2t})u(-t)$

Solution 5

Given that $x(t) = e^{-|t|}$,$-\infty < t < \infty$
Since the system is LTI, the output is $x(t) * h(t)$ or in s-domain, by L.T properties

$$Y(s) = X(s)H(s)$$

Since the system is causal, the ROC will be the half plane right of the right most pole of $H(s)$. Since the poles of $H(s)$ are $-1 \pm j$, ROC is $Re(s) > -1$.

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\ &= \int_{-\infty}^{0} e^te^{-st}dt + \int_{0}^{\infty} e^{-t}e^{-st}dt \\ &= \int_{-\infty}^{0} e^{t-st}dt + \int_{0}^{\infty} e^{-t-st}dt \\ &= \frac{1}{1-s} + \frac{1}{1+s}; \quad Re\{s\} < 1 \cap Re\{s\} > -1 \\ &= \frac{-2}{(s-1)(s+1)} \quad ROC : -1 < Re\{s\} < 1 \end{aligned}$$

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \frac{s+1}{s^2+2s+2}X(s) \\ &= \frac{-2}{(s-1)(s^2+2s+2)} \end{aligned}$$

Taking partial fraction decomposition,

$$\begin{aligned} \frac{-2}{(s-1)(s^2+2s+2)} &= \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+2} \\ -2 &= A(s^2+2s+2) + (Bs+C)(s-1) \\ &= (A+B)s^2 + (2A-B+C)s + (2A-C) \end{aligned}$$

Solving for A,B,C we get

$$\begin{aligned} 0 = A + B &\implies B = -A \\ 0 = 2A - B + C &\implies C = -3A \\ -2 = 2A - C &\implies 5A = -2 \end{aligned}$$

$$A = \frac{-2}{5}, \quad B = \frac{2}{5}, \quad C = \frac{6}{5}.$$

Hence

$$\begin{aligned} Y(s) &= \frac{-2}{5(s-1)} + \frac{2s+6}{5(s^2+2s+2)} \\ &= \frac{-2}{5(s-1)} + \frac{2(s+1)}{5((s+1)^2+1))} + \frac{4}{5((s+1)^2+1))} \\ &\qquad ROC : -1 < Re\{s\} < 1 \end{aligned}$$

Notice that the pole at $s = -1$ in the ROC of Y(s) got cancelled with the zero at the same position, and the left limit of ROC at $s = -1$ came from the poles at $-1 \pm j$.
We know the following:

$$\begin{aligned} \mathcal{L}(e^{-at}\cos\omega t \cdot u(t)) &= \frac{s+a}{(s+a)^2 + \omega^2} \\ \mathcal{L}(e^{-at}\sin\omega t \cdot u(t)) &= \frac{\omega}{(s+a)^2 + \omega^2} \end{aligned}$$

Therefore, in time domain:

$$\begin{aligned} y(t) &= \frac{2}{5}e^tu(-t) + \frac{2}{5}e^{-t}\cos tu(t) + \frac{4}{5}e^{-t}\sin tu(t) \\ &= \frac{2}{5}e^tu(-t) + \frac{2}{5}e^{-t}(\cos t + 2\sin t)u(t) \end{aligned}$$

Solution 6

Given that $H(s)$ is a rational function, we may take it to be $H(s) = \frac{a(s)}{b(s)}$, where $a(s)$ and $b(s)$ are polynomials in s.

(a) The response is absolutely integrable for the signal $u(t)$, whose Laplace transform is $\frac{1}{s}$. Therefore, $\frac{H(s)}{s}$ has no poles at $Re(s) \geq 0$. Therefore, $s = 0$ must be a root of $a(s)$, which cancels the s in the denominator. Take $a(s) = sa_1(s)$.

(b) The response to $tu(t)$ (Laplace transform $\frac{1}{s^2}$) is not absolutely integrable. This implies, there cannot be a repeated root for $a(s)$ at $s = 0$.

(c) If a signal with a rational Laplace transform is of finite duration, then its denominator is a constant polynomial. The Laplace transform of the signal $\frac{d^2h}{dt^2} + 2\frac{dh}{dt} + 2h(t)$ is $(s^2 + 2s + 2)H(s)$. This is given to be of finite duration in time domain. Hence, $b(s) = \frac{1}{K}(s^2+2s+2)$ for some constant K.

(d) The number of zeros at infinity is $deg(b(s)) - deg(a(s)) = 1$. Since $deg(b(s)) = 2$, $deg(a(s)) = 1$. Therefore $a(s) = K_1 s$.
Therefore $H(s) = KK_1 \frac{s}{s^2 + 2s + 2}$.

(e) Using the fact that $H(1) = 0.2$, we find that $KK_1 = 1$.

The required impulse response is

$$H(s) = \frac{s}{s^2 + 2s + 2}$$

The poles of the system are $-1 \pm j$. Since the system is causal, the region of convergence is $Re(s) > -1$.

Solution 7

Taking unilateral Laplace transform of the given differential equation we get,

$$\begin{aligned} s^3Y(s) - s^2y(0^-) - sy'(0^-) - y''(0^-) \\ +6s^2Y(s) - 6sy(0^-) - 6y'(0^-) \\ +11sY(s) - 11y(0^-) + 6Y(s) = X(s) \end{aligned}$$

(a) For zero state response all the initial conditions are assumed to be zero. Laplace transform of $x(t)$ is given by,

$$X(s) = \frac{1}{s+4}, \qquad \mathbb{R}e(s) > -4$$

Using above equation we get,

$$Y(s)(s^3 + 6s^2 + 11s + 6) = \frac{1}{s+4}$$

Therefore,

$$\begin{aligned} Y(s) &= \frac{1}{(s^3 + 6s^2 + 11s + 6)(s+4)} \\ &= \frac{1}{(s+1)(s+2)(s+3)(s+4)} \\ &= \frac{1/6}{s+1} - \frac{1/2}{s+2} + \frac{1/2}{s+3} - \frac{1/6}{s+4} \end{aligned}$$

Taking inverse unilateral Laplace Transform we get,

$$y(t) = \frac{1}{6}e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t) - \frac{1}{6}e^{-4t}u(t)$$

(b) For zero input response, we assume that $X(s) = 0$, Substituting the initial conditions in the main equation 1 we get,

$$Y(s) = \frac{s^2 + 5s + 6}{s^3 + 6s^2 + 11s + 6} = \frac{1}{s+1}$$

Taking the unilateral Laplace transform of above equation we get,

$$y(t) = e^{-t}u(t)$$

(c) The total response is the sum of zero state response and the zero input response.

$$y(t) = \frac{7}{6}e^{-t}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t) - \frac{1}{6}e^{-4t}u(t)$$

Solution 8

The overall system shown can be treated as as two feedback systems connected in parallel. The system function of the upper feedback system is

$$H_1(s) = \frac{\frac{2}{s}}{1 + 4\frac{2}{s}} = \frac{2}{s+8}$$

. Similarly,the system function of the lower feedback system is

$$H_2(s) = \frac{\frac{1}{s}}{1 + 2\frac{1}{s}} = \frac{1}{s+2}$$

The system function of the overall system is

$$H(s) = H_1(s) + H_2(s) = \frac{3s + 12}{s^2 + 10s + 16}$$

. Since

$$H(s) = Y(s)/X(s)$$

,

$$Y(s)[s^2 + 10s + 16] = X(s)[3s + 12]$$

$$\frac{d^2y(t)}{dt^2} + 10\frac{dy(t)}{dt} + 16y(t) = 12x(t) + 3\frac{dx(t)}{dt}$$

Solution 9

The unilateral Laplace transform of a signal $x(t)$ is given as

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

(a) $x(t) = u(t-2)$

$$\begin{aligned} X(s) &= \int_0^\infty u(t-2)e^{-st}dt \\ &= \int_2^\infty e^{-st}dt \\ &= \frac{e^{-2s}}{s} \quad ; \quad Re\{s\} > 0 \end{aligned}$$

(b) $x(t) = u(t+2)$

$$\begin{aligned} X(s) &= \int_0^\infty u(t+2)e^{-st}dt \\ &= \int_0^\infty e^{-st}dt \\ &= \frac{1}{s} \quad ; \quad Re\{s\} > 0 \end{aligned}$$

(c) $x(t) = e^{3t}u(t)$

$$\begin{aligned} X(s) &= \int_0^\infty e^{3t}e^{-st}dt \\ &= \int_0^\infty e^{-(s-3)t}dt \\ &= \frac{1}{s-3} \quad ; \quad Re\{s\} > 3 \end{aligned}$$

(d) $x(t) = te^tu(t)$

$$\begin{aligned} X(s) &= \int_0^\infty te^te^{-st}dt \\ &= \int_0^\infty te^{-(s-1)t}dt \end{aligned}$$

Using integration by parts, take $u = t, v = \frac{e^{-(s-1)t}}{-(s-1)}$.

$$\begin{aligned} \int te^{-(s-1)t}dt &= \int u\ dv \\ &= uv - \int v\ du \\ &= \frac{te^{-(s-1)t}}{-(s-1)} - \int \frac{e^{-(s-1)t}}{-(s-1)}dt \\ &= \frac{te^{-(s-1)t}}{-(s-1)} - \frac{e^{-(s-1)t}}{(s-1)^2} \\ &= -\frac{(ts-t+1)e^{-(s-1)t}}{(s-1)^2} \end{aligned}$$

Using the above result,

$$\begin{aligned} X(s) &= \left[-\frac{(ts-t+1)e^{-(s-1)t}}{(s-1)^2}\right]_0^\infty \\ &= \frac{1}{(s-1)^2} \quad ; \quad Re\{s\} > 1 \end{aligned}$$

(e) $x(t) = \sin(t)u(t)$

$$\begin{aligned} X(s) &= \int_0^\infty \sin(t)e^{-st}dt \\ &= \int_0^\infty \frac{e^{jt}-e^{-jt}}{2j}e^{-st}dt \\ &= \frac{1}{s^2+1} \quad ; \quad Re\{s\} > 0 \end{aligned}$$

Solution 10

(a) To get the differential equation describing the input and output voltages, apply Kirchhoff's Voltage law to the given circuit.

$$Ri(t) + L\frac{di(t)}{dt} + v_o(t) = v_i(t)$$

Voltage across capacitor can be expressed as

$$v_o(t) = \frac{1}{C}\int_{-\infty}^t i(\tau)d\tau$$

Differentiating, we get, $i(t) = C\frac{dv_o(t)}{dt}$. Using the expression for $i(t)$ in the voltage law and dividing by LC, we get the differential equation relating $v_i(t)$ and $v_o(t)$:

$$\frac{d^2v_o(t)}{dt^2} + \frac{R}{L}\frac{dv_o(t)}{dt} + \frac{1}{LC}v_o(t) = \frac{1}{LC}v_i(t)$$

Substituting the values of R, L and C, we get

$$\frac{d^2v_o(t)}{dt^2} + 3\frac{dv_o(t)}{dt} + 2v_o(t) = 2v_i(t)$$

(b) We can solve the differential equation using the unilateral Laplace transform. The Laplace transform for the first derivative is

$$\mathcal{L}\left(\frac{dv_o}{dt}\right) = sV_o(s) - v_o(0^-).$$

We now derive the Laplace transform of the second derivative.

$$\begin{aligned} \mathcal{L}\left(\frac{d^2v_o}{dt^2}\right) &= \int_{0^-}^\infty \frac{d^2v_o}{dt^2}e^{-st}dt \\ &= \int_{0^-}^\infty \frac{d}{dt}\left(\frac{dv_o}{dt}\right)e^{-st}dt \\ &= \int_{0^-}^\infty d\left(\frac{dv_o}{dt}\right)e^{-st} \\ &= \frac{dv_o}{dt}e^{-st}\Big|_{0^-}^\infty + s\int_{0^-}^\infty \frac{dv_o}{dt}e^{-st}dt \\ &= -v_o'(0^-) + s(sV_o(s) - v_o(0^-)) \\ &= s^2V_o(s) - sv_o(0^-) - v_o'(0^-) \end{aligned}$$

Taking the unilateral Laplace transform of the differential equation, we get

$$s^2V_o(s) - sv_o(0^-) - v_o'(0^-) + 3sV_o(s)$$

$$- 3v_o(0^-) + 2V_o(s) = 2V_i(s)$$

Since, $v_i(t) = e^{-3t}u(t)$,

$$V_i(s) = \frac{1}{s+3}, Re\{s\} > -3$$

Substituting this along with the initial conditions, we get

$$V_o(s) = \frac{(s^2 + 8s + 17)}{(s+1)(s+2)(s+3)}$$

The partial fraction expansion of $V_o(s)$ is

$$V_o(s) = \frac{5}{s+1} - \frac{5}{s+2} + \frac{1}{s+3}$$

Taking the inverse Laplace transform, we get

$$v_o(t) = (5e^{-t} - 5e^{-2t} + e^{-3t})u(t)$$