Indian Institute of Technology Madras
PH1020, 2018
Tutorial-6

Question-1:

Consider a plane boundary between two media of permeability μ_1 and μ_2, as shown in Figure 1. Find the relation between the angles θ_1 and θ_2. Assume that the media are linear with $\vec{B}$ and $\vec{H}$ in the same direction.

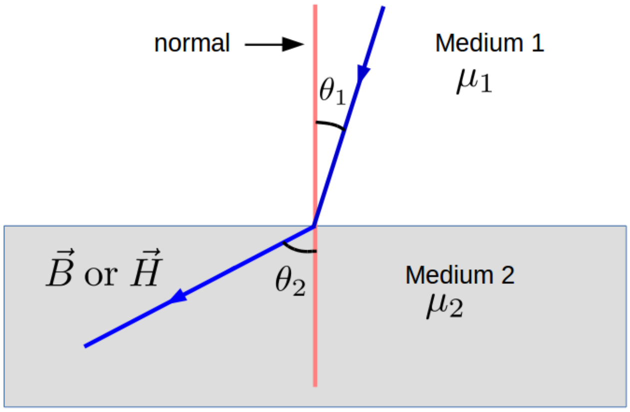


Figure 1

Solution:-

Boundary conditions $B_{n1} = B_{n2}$, $H_{t1} = H_{t2}$.
From Figure 1, $B_{n1} = B_1 cos\theta_1$, $B_{n2} = B_2 cos\theta_2$ and $H_{t1} = H_1 sin\theta_1$, $H_{t2} = H_2 sin\theta_2$

$$\frac{H_1\ sin\theta_1}{B_1\ cos\theta_1} = \frac{H_2\ sin\theta_2}{B_2\ cos\theta_2}$$

$$\Rightarrow \quad \frac{tan\ \theta_1}{\mu_1} = \frac{tan\ \theta 2}{\mu_2}$$

$\therefore \qquad \frac{tan\theta_1}{tan\theta_2} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}}$ where μ_r is relative permeability.

Question-2:

A sphere of a linear magnetic material of susceptibility χ_m is placed in an otherwise uniform magnetic field $\vec{B}_0$. Determine the new field inside the sphere.

Solution:-

$\vec{B}_0$ magnetizes the sphere and the magnetization is given by $\vec{M} = \chi_m \vec{H}_0 = \frac{\chi_m \vec{B}_0}{\mu} = \frac{\chi_m \vec{B}_0}{\mu_0(1+\chi_m)}$
This magnetization sets up a filed within the sphere

$$\vec{B}_1 = \tfrac{2}{3}\mu_0 \vec{M}_0 = \tfrac{2}{3}\frac{\chi_m}{(1+\chi_m)}\vec{B}_0 = \tfrac{2}{3}k\vec{B}_0 \qquad \left[\text{where } k = \frac{\chi_m}{1+\chi_m}\right]$$

$\vec{B}$ magnetizes the sphere by an additional amount given by $\vec{M} = k\frac{\vec{B}_1}{\mu_0}$. This in turn, sets up an additional field in the sphere given by

$$\vec{B}_2 = \tfrac{2}{3}\mu_0 \vec{M}_1 = \tfrac{2}{3}k\vec{B}_1 = \left(\tfrac{2}{3}k\right)^2 \vec{B}_0 \quad \text{so on ......}$$

Thus,

$$\vec{B} = \vec{B}_0 + \tfrac{2}{3}k\vec{B}_0 + \left(\tfrac{2}{3}k\right)^2 \vec{B}_0 \quad .... = \frac{\vec{B}_0}{1-\frac{2}{3}k}$$

$$\frac{1}{1-\frac{2}{3}k} = \frac{3}{3-2k} = \frac{3}{3-2\frac{\chi_m}{1+\chi_m}} = \frac{1+\chi_m}{1+\frac{\chi_m}{3}}$$

This problem illustrates the method of successive approximation, in a situation existing in dielectrics as well.

Question-3:

A long straight wire of radius a is made of a homogeneous, linear magnetic material with a susceptibility χ_m. A uniformly distributed current I flows down the wire. Determine

(a) the magnetic filed at a distance s from the axis of the wire and

(b) the net bound current flowing in the wire.

Solution:-

(a) Ampere's law in integral form is $\oint \vec{H} \cdot \vec{dl} = H(2\pi s) = I_{fenc}$

$$I_{fenc} = \begin{cases} \frac{Is^2}{a^2}, & s < a. \\ I, & s > a. \end{cases}$$

Therefore,

$$H = \begin{cases} \frac{Is}{2\pi a^2}, & s < a. \\ \frac{I}{2\pi s}, & s > a. \end{cases}$$

and

$$B = \mu H = \begin{cases} \frac{\mu_0(1+\chi_m)Is}{2\pi a^2}, & s < a. \\ \frac{I\mu_0}{2\pi s}, & s > a. \end{cases}$$

(b) The volume current density in a linear material is proportional to the free current density.

$$\vec{J_b} = \vec{\nabla} \times \vec{M} = \vec{\nabla} \times (\chi_m \vec{H}) = \chi_m \vec{J_f}$$

$\vec{J_b} = \chi_m \vec{J_f}$ and $J_f = \frac{I}{\pi a^2} \Rightarrow$ $J_b = \frac{\chi_m I}{\pi a^2}$, in the same direction as I.

$$\vec{M} = \chi_m \vec{H} = \frac{Is}{2\pi a^2}\hat{\phi}$$

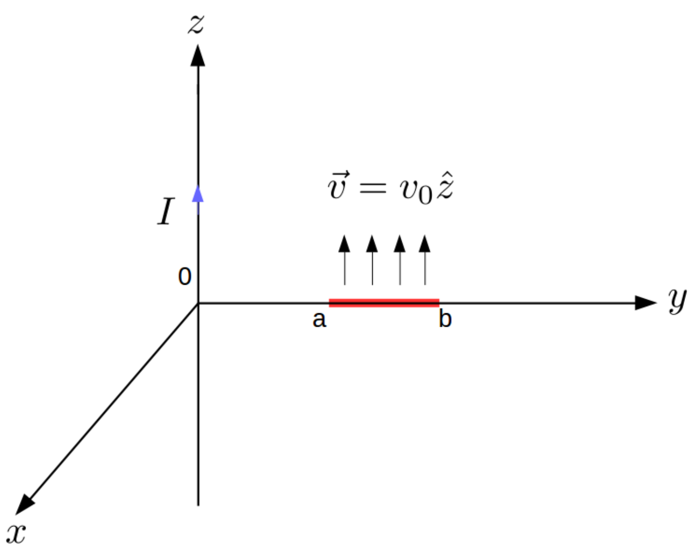
$$\vec{K} = \vec{M} \times \hat{s} = -\frac{\chi_m}{2\pi a} I\hat{z}$$

The total bound current $I_b = J_b(\pi a^2) + K_b(2\pi a) = 0$

Question-4:

An infinitely long straight wire located along the z-axis carries a steady current I in the positive z-direction. A copper rod is located on the y-axis, such that its ends are at $y = a$ and $y = b$. The rod moves with a constant velocity $\vec{v} = v_0\hat{z}$. Find the emf induced in the rod.

Solution:-



In the present case $\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi} = \frac{\mu_0}{2\pi y}(-\hat{x})$. The force per unit charge

$$\begin{aligned}\vec{f}_{\text{mag}} &= v_0\hat{z} \times B\ (-\hat{x}) \\ &= -v_0 B\ \hat{y}\end{aligned}$$

$$\begin{aligned}\text{The emf } &= \oint_C \vec{f}_{\text{mag}} \cdot \vec{dl} \\ &= -\int_a^b v_0 B\ dy \\ &= -\int_a^b v_0 \frac{\mu_0 I}{2\pi y}\ dy \\ \therefore \qquad \epsilon &= \tfrac{\mu_0 I v_0}{2\pi}\ \ln\left(\tfrac{a}{b}\right)\end{aligned}$$

Question-5:

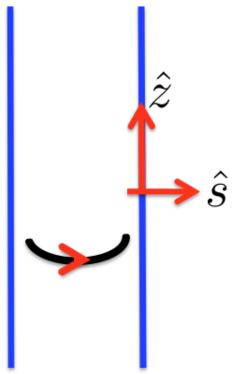
a) A long circular cylinder of radius 'a' carries a uniform magnetisation $\vec{M}$ parallel to its axis. Find the magnetic field due to $\vec{M}$ inside the cylinder.

b) A long circular cylinder of radius a carries a uniform magnetisation $\vec{M} = \frac{M_0 s^2}{a^2}\hat{\phi}$ where $M_0 > 0$ is a constant, s is the distance from the axis, and $\hat{\phi}$ is the usual azimuthal unit vector. Find $\vec{J_b}$, $\vec{K_b}$, and the magnetic field $\vec{B}$ due to $\vec{M}$ inside and outside of the cylinder.

Solution:-

a) $\vec{M} = M_0\hat{z}$. $\vec{J_b} = \vec{\nabla} \times \vec{M} = 0$ ("almost everywhere")

$$\vec{K_b}\Big|_{s=a} = \vec{M} \times \hat{s}\Big|_{s=a} = M_0\hat{\phi}$$



Hence, this configuration is just an infinite solenoid.

$$\vec{B}_{\text{inside}} = \mu_0 M_0\ \ \hat{z}, \quad \text{and} \quad \vec{B}_{\text{outside}} = 0.$$

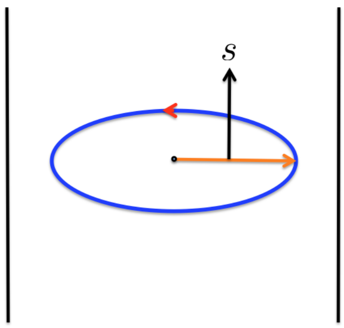
b)

$$\begin{aligned}\vec{M} &= M_0\left(\frac{s^2}{a^2}\right)\hat{\phi} \\ \vec{J_b} &= \vec{\nabla} \times \vec{M} = \frac{1}{s}\frac{\partial}{\partial s}(sM_\phi)\,\hat{z} \\ \vec{J_b} &= \frac{3M_0 s}{a^2}\hat{z} \qquad \text{(inside)} \\ \vec{J_b} &= 0 \qquad \text{(outside)}\end{aligned}$$

$$\vec{K_b}\Big|_{s=a} = \vec{M} \times \hat{s}\Big|_{s=a} = -M_0\hat{z}$$

$\vec{B}$ can be found from symmetry using Ampere's law: $\vec{B} = B(s)\hat{\phi}$

$\vec{B}_{\text{inside}}$:

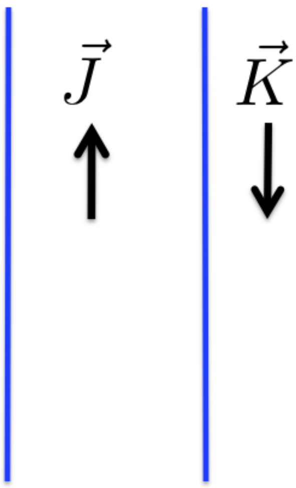


$$\vec{B}(s)(2\pi s) = \mu_0 \int_0^s \left(\frac{3M_0 s'}{a^2}\right) s' ds' d\phi'$$

$$B(s)s = \frac{\mu_0 M_0}{a^2} s^3$$

$$\therefore \qquad \vec{B}_{\text{inside}} = (\mu_0 M_0)\,\frac{s^2}{a^2}\,\hat{\phi}$$

$\vec{B}_{\text{outside}}$:



$$B(2\pi s) = \mu_0 \Big\{ \int_0^{2\pi} \int_0^a \left(\frac{3M_0 s'}{a^2}\right)(s' ds' d\phi') + \int_0^{2\pi} K(a d\phi) \Big\}$$

$$= \mu_0 \left[M_0 a - M_0 a\right] 2\pi$$

$$\therefore \qquad \vec{B}_{\text{outside}} = 0$$