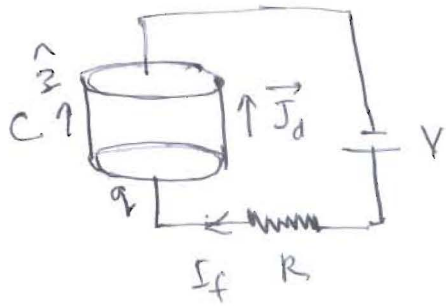
Tutorial Sheet 9: Solutions

(1)

$$Q = CV = C d E_z$$



$$\Rightarrow E_z = \frac{Q}{Cd} = \frac{V}{d}\left(1 - e^{-t/RC}\right)$$

$$\vec{D} = \epsilon \vec{E} \Longrightarrow D_z = \epsilon_0 k E_z$$

For an Amperian loop about z-axis

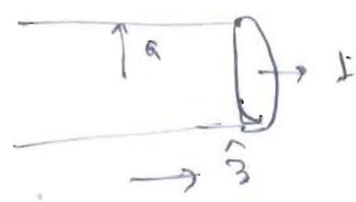
$$\oint \vec{H} \cdot d\vec{l} = \int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

$$2\pi \rho H_\phi = \frac{\partial D_z}{\partial t} \pi \rho^2 = \frac{\epsilon_0 k V}{d}\left(\frac{e^{-t/RC}}{RC}\right) \pi \rho^2$$

$$\Rightarrow H_\phi = \frac{V \epsilon_0 k \rho}{2RCd} e^{-t/RC} \qquad \vec{H} = H_\phi \hat{e}_\phi$$

(2)

$$\vec{B} = \frac{\mu_0 I}{2\pi a} \hat{e}_\phi$$



$$\vec{E} = \frac{\vec{J}}{\sigma} = \frac{I}{\pi a^2 \sigma} \hat{e}_z$$

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{I^2}{2\pi^2 a^3 \sigma} \hat{e}_z \times \hat{e}_\phi = -\frac{I^2}{2\pi^2 a^3 \sigma} \hat{e}_\rho$$

For a length L of the conductor, energy flux

$$= \int \vec{S} \cdot d\vec{S} \quad L = -\frac{I^2}{2\pi^2 a^3 \sigma} 2\pi a L = -\frac{I^2 L}{\pi a^2 \sigma}$$

energy flows into the conductor at a rate $\frac{I^2 L}{\pi a^2 \sigma}$

this is also simply $I^2 R$ where $R = \frac{L}{\pi a^2 \sigma}$

Q.3 $\vec{E}(z,t) = E_0 \hat{e}_x \cos(kz) \cos(\omega t)$

$$\vec{\nabla} \times \vec{E} = -k E_0 \hat{e}_y \sin(kz) \cos(\omega t) = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \vec{B} = \frac{k E_0}{\omega} \hat{e}_y \sin(kz) \sin(\omega t) + \vec{F}(\vec{r})$$

$\vec{F}(\vec{r})$ = Is independent of time

at $t = 0$, $\vec{B} = 0 \Rightarrow \vec{F}(\vec{r}) = 0$

$$\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{k E_0}{\mu_0 \omega} \sin(kz) \sin(\omega t) \hat{e}_y$$

$$\vec{S} = \vec{E} \times \vec{H} = \frac{k E_0^2}{4 \mu_0 \omega} \sin(2kz) \sin(2\omega t) \hat{e}_z$$

$$\langle \vec{S} \rangle = \text{average } \vec{S} = \frac{\int_0^{2\pi/\omega} \vec{S}\, dt}{2\pi/\omega} = 0$$

Q.4 (a) 1st wave $\vec{E}_1 = |E_0| \hat{e}_x \cos(kz - \omega t + \delta_1)$

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} = \frac{k|E_0|}{\omega} \hat{e}_y \cos(kz - \omega t + \delta_1)$$

2nd wave $\vec{E}_2 = |E_0| \hat{e}_y \cos(kz - \omega t + \delta_2)$

$$\vec{B}_2 = -\frac{|\vec{E}_0| k}{\omega} \hat{e}_x \cos(kz - \omega t + \delta_2)$$

(b) $$w = \frac{1}{2}\epsilon_0 \vec{E}^2 + \frac{1}{2\mu_0}\vec{B}^2 = \epsilon_0 \vec{E}^2 = \epsilon_0 E_0^2 \cos^2(kz-\omega t+\delta_1)$$

$$\frac{\partial w}{\partial t} = 2\epsilon_0 E_0^2 \omega \cos(kz-\omega t+\delta_1)\sin(kz-\omega t+\delta_1)$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{kE_0^2}{\mu_0\omega}\hat{e}_z \cos^2(kz-\omega t+\delta_1)$$

$$\vec{\nabla}\cdot\vec{S} = -2\frac{k^2E_0^2}{\mu_0\omega}\cos(kz-\omega t+\delta_1)\sin(kz-\omega t+\delta_1)$$

$$\frac{k^2}{\mu_0\omega} = \frac{k^2}{\mu_0\omega}\frac{\omega}{\omega} = \left(\frac{k^2}{\omega^2}\right)\frac{\omega}{\mu_0} = \frac{\omega}{\mu_0 c^2} = \frac{\mu_0\epsilon_0\omega}{\mu_0} = \epsilon_0\omega$$

$$\Rightarrow \frac{\partial w}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0$$ For each wave

(c) (i) Left circularly polarized

$$\delta_2 = \delta_1 + \pi/2$$

superimposing two waves

$$\vec{E} = |E_0|\left[\hat{e}_x\cos(kz-\omega t+\delta_1) - \hat{e}_y\sin(kz-\omega t+\delta_1)\right)$$

$\vec{E}$ vector has a constant magnitude and rotates anticlockwise when seen down (towards origin) along the z-axis (direction of propagation)

(ii) Right circularly polarized

$$\delta_2 = \delta_1 - \pi/2 \Rightarrow \vec{E} = |E_0|\left[\hat{e}_x\cos(kz-\omega t+\delta_1) + \hat{e}_y\sin(kz-\omega t+\delta_1)\right]$$

clockwise rotation

$$\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} = \frac{k|E_0|}{\omega}\left[\hat{e}_y \cos(kz-\omega t-\delta_1) \pm \hat{e}_x \sin(kz-\omega t-\delta_1)\right]$$

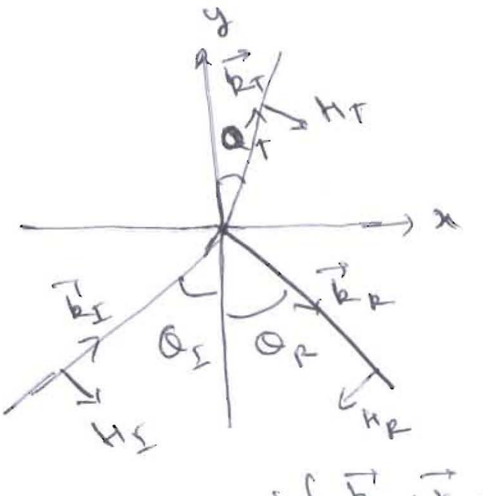
$+$: Left ⊙ polarized

$-$: Right ⊙ polarized.

Q. 5(a) → See Section 9.3.2 of Griffiths

Q. 5(b) and Q. 6 } → See Section 9.3.3 of Griffiths

Q. 7 Consider the interface at $y=0$ (xz plane)



just as in the case when polarization lies in the plane of incidence, we get

$\theta_I = \theta_R$ and $n_1 \sin\theta_I = n_2 \sin\theta_T$

$$\vec{E}_I = \vec{E}_{I0}\, e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}$$
$$\vec{E}_R = \vec{E}_{R0}\, e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}$$
$$\vec{E}_T = \vec{E}_{T0}\, e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}$$

all the vectors $\vec{E}_{I0}$, $\vec{E}_{R0}$, $\vec{E}_{T0}$ are $\perp$r to the plane of incidence, i.e., $\hat{e}_z$ direction

then $\vec{H}$ field as indicated lies on the plane of incidence

Since frequencies are equal, we have

$k_I = k_R$ $\quad k_T v_2 = k_1 v_1$ or $k_T = k_I \frac{n_2}{n_1}$

continuity of $E^{\parallel}$ gives at the interface

$$\vec{E}_{I0} + \vec{E}_{R0} = \vec{E}_{T0} \quad \text{or} \quad E_{I0} + E_{R0} = E_{T0} \qquad -①$$

As $\perp r$ component of $\vec{H}$ is continuous

$$(\vec{H}_I)_x + (\vec{H}_R)_x = (\vec{H}_T)_x$$

$$H_I \cos\theta_I - H_R \cos\theta_R = H_T \cos\theta_T$$

$$\theta_I = \theta_R \quad \text{and} \quad \vec{H} = \frac{\vec{B}}{\mu} = \frac{\vec{k}\times\vec{E}}{\mu\omega} \Rightarrow H = \frac{kE}{\mu\omega} = \frac{E}{\mu v}$$

$$\Rightarrow (E_{0I} - E_{0R})\frac{\cos\theta_I}{\mu_1 v_1} = \frac{E_{0T}\cos\theta_T}{\mu_2 v_2}$$

$$\text{or} \quad E_{0I} - E_{0R} = E_{0T}\,\alpha\beta \quad -② \quad \text{where} \quad \beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

From ① and ②

$$\frac{E_{0I}}{E_{0T}} = \frac{1+\alpha\beta}{2} \quad \& \quad \frac{E_{0R}}{E_{0I}} = \frac{1-\alpha\beta}{1+\alpha\beta}$$

$$\alpha\beta = \frac{\mu_1 n_2}{\mu_2 n_1}\frac{\cos\theta_T}{\cos\theta_I} = \frac{\mu_1}{\mu_2}\frac{\sin\theta_I}{\sin\theta_T}\frac{\cos\theta_T}{\cos\theta_I} = \frac{\mu_1}{\mu_2}\frac{\tan\theta_I}{\tan\theta_T}.$$

as $\mu_1 \approx \mu_2 \approx \mu_0$ and $\theta_T \neq \theta_I$; $\alpha\beta \neq 1$, no Brewsters angle.