Department of Mathematics, IIT Madras
MA1102 Series & Matrices
**Assignment-3 Matrix Operations**

1. Show that given any $n \in \mathbb{N}$ there exist matrices $A, B \in \mathbb{R}^{n\times n}$ such that $AB \neq BA$.

2. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Compute $A^n$.

3. Let $A \in \mathbb{F}^{m\times n}$; $B \in \mathbb{F}^{n\times k}$; $A_1, \ldots, A_m$ be the rows of $A$; $B_1, \ldots, B_k$ be the columns of $B$. Show that

   (a) $A_1B, \ldots, A_mB$ are the rows of $AB$. (b) $AB_1, \ldots, AB_k$ are the columns of $AB$.

4. Solve the following system by Gaussian elimination

$$\begin{array}{rrrrrl} x_1 & +x_2 & +x_3 & +x_4 & -3x_5 & = 6 \\ 2x_1 & +3x_2 & +x_3 & +4x_4 & -9x_5 & = 17 \\ x_1 & +x_2 & +x_3 & +2x_4 & -5x_5 & = 8 \\ 2x_1 & +2x_2 & +2x_3 & +3x_4 & -8x_5 & = 14 \end{array}$$

5. Let $A \in \mathbb{F}^{m\times n}$ have columns $A_1, \ldots, A_n$. Let $b \in \mathbb{F}^m$. Show the following:

   (a) The equation $Ax = 0$ has a non-zero solution iff $A_1, \ldots, A_n$ are linearly dependent.
   (b) The equation $Ax = b$ has at least one solution iff $b \in \operatorname{span}\{A_1, \ldots, A_n\}$.
   (c) The equation $Ax = b$ has at most one solution iff $A_1, \ldots, A_n$ are linearly independent.
   (d) The equation $Ax = b$ has a unique solution iff $\operatorname{rank} A = \operatorname{rank}[A|b] =$ number of unknowns.

6. Check if the system is consistent. If so, determine the solution set.

   (a) $x_1 - x_2 + 2x_3 - 3x_4 = 7,\ 4x_1 + 3x_3 + x_4 = 9,\ 2x_1 - 5x_2 + x_3 = -2,\ 3x_1 - x_2 - x_3 + 2x_4 = -2.$
   (b) $x_1 - x_2 + 2x_3 - 3x_4 = 7, 4x_1 + 3x_3 + x_4 = 9, 2x_1 - 5x_2 + x_3 = -2, 3x_1 - x_2 - x_3 + 2x_4 = -2.$

7. Using Gaussian elimination determine the values of $k \in \mathbb{R}$ so that the system of linear equations

$$x + y - z = 1,\ 2x + 3y + kz = 3,\ x + ky + 3z = 2$$

   has (a) no solution, (b) infinitely many solutions, (c) exactly one solution.

8. Determine linear independence of $\{(1,2,2,1),\ (1,3,2,1),\ (4,1,2,2),\ (5,2,4,3)\}$ in $\mathbb{C}^{1\times 4}$.

9. Let $u, v, w \in \mathbb{F}^{n\times 1}$. Show that $\{u, v, w\}$ is linearly independent iff $\{u+v, v+w, w+u\}$ is linearly independent.

10. Find a basis for the subspace $\{(a, b, c) : 2a + 3b - 4c = 0\}$ of $\mathbb{R}^{1\times 4}$.

11. Let $A \in \mathbb{R}^{3\times 3}$ satisfy $A(a, b, c)^t = (a + b, 2a - b - c, a + b + c)^t$. Determine $A$ and also its rank and nullity.

12. Determine a basis of the subspace $\{(a, b, c, d, e) : a = c = e, b + d = 0\}$ of $\mathbb{R}^{1\times 5}$.

13. Let $A \in \mathbb{F}^{m\times n}$ have rank $r$. Give reasons for the following:

    (a) $\operatorname{rank}(A) \le \min\{m, n\}$.
    (b) If $n > m$, then there exist $x, y \in \mathbb{F}^{n\times 1}$ such that $x \neq y$ and $Ax = Ay$.
    (c) If $n < m$, then there exists $y \in \mathbb{F}^{m\times 1}$ such that for no $x \in \mathbb{F}^{n\times 1}$, $Ax = y$.
    (d) If $n = m$, then as a map, $A$ is one-one iff $A$ is onto.

14. Convert the following matrices into their row echelon and row reduced echelon forms. Then determine their rank and nullity.

    (a) $\begin{bmatrix} 5 & 2 & -3 & 1 & 7 \\ 1 & -3 & 2 & -2 & 11 \\ 3 & 8 & -7 & 5 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 2 & -3 & 1 & 7 \\ 1 & -3 & 2 & -2 & 11 \\ 3 & 8 & -7 & 5 & 8 \end{bmatrix}$