End Semester Examination - Solution Key
EE1101 Signals and Systems - Jan-May 2018

Duration:3 Hrs **Max Marks : 80**

Instructions

- ***Qn 1-10*** *:* ***2 marks each****. Answer briefly, with appropriate justification/counter example(s).*
- ***Qn 11-17*** *- Marks are shown in brackets. Show all working steps neatly.*
- *Do not write anything in the question paper. Attach the question paper with your answersheet.*

1. Does a two-sided signal exist such that the RoC of its Laplace Transform is a left half-plane ?

   Solution: Yes, for example $x(t) = u(-t) + e^{-t^2}$. Binary grading, full credit only if the example is correct.

2. Find the RoC of the Laplace transform of the signal $x(t) = \cos \pi t$.

   Solution: RoC is empty because the LT integral does not converge absolutely for any $s \in \mathbb{C}$. Binary grading.

3. When a causal LTI system is cascaded with a non-causal LTI system, then the resultant system must be non-causal. Is this statement true or false ?

   Solution: FALSE : When a LTI causal system is cascaded with a non-causal system, then resultant system need NOT be non-causal. Consider $y_1(t) = x(t-1)$ which is causal and $y_2(t) = x(t+1)$ which is non-causal. The resultant system is $y(t) = x(t)$, which is causal.

   Binary grading; credit for correct explantion.

4. For the LTI systems shown in Fig.1a., $y_1[n] = y_2[n]$. Is this statement true or false ?

   Solution : True; convolution property. Binary grading.

5. Find whether pole-zero plot shown in Fig. 1b could correspond to an even function in time ? If yes- find the required ROC. (Given that the transfer function is rational.)

   Solution: Given that $H(s)$ is rational and the zeros are at $\pm j$ and the poles are at ± 1; let $H(s) = \frac{A(s-j)(s+j)}{(s-1)(s+1)}$. Now, it is evident that $H(-s) = H(s)$. Thus, $H(s)$ is even. Since $H(s)$ is even, $h(t)$ can be even. However, if $h(t)$ is even, it should be either two sided or it has to be bounded and symmetric. For these to be satisfied, the ROC has to be a strip or the entire s-place. Since there are poles in the ROC, the ROC has to be a strip, defined by $-1 < Re\{s\} < 1$.

   1 mark for identifying $H(s)$ is even, and 1 mark for identifying the ROC ; marks to be given only if the supporting arguments are correct.

6. Sketch the magintude and phase response of a system that performs differentiation.

   Solution : See figure; 1 mark for the correct magnitude response, 1 mark for phase response - axes should be marked for full credit.

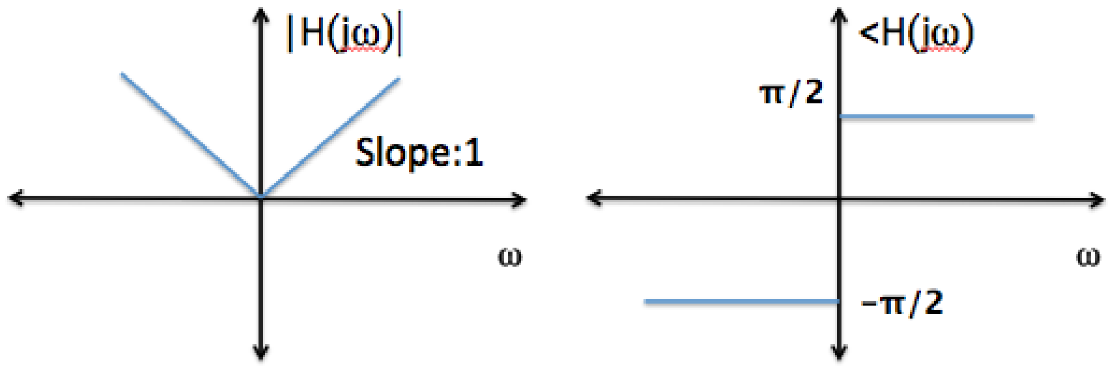


Figure 1: Solution to Qn 6

7. The output of a linear system for a step input is $t^2e^{-t}u(t)$. Find the impulse response, $h(t)$.

   Solution : If $u(t)$ represents the step response, the impulse response is given by, $h(t) = \frac{du}{dt} = (-t^2e^{-t} + 2te^{-t})u(t) = (te^{-t}(2-t))u(t)$

   1 mark to identify that impulse response is the derivative of step response, 1 mark for correct answer. Pl check for alternate solutions.

8. Given that $X(s) = \frac{2s}{s^2+2s+1}$, find the value of $x(t)$ just after $t = 0$.

   Solution : From initial value theorem, $x(t)_{t=0^+} = sH(s)$ evaluated as $s \to \infty$ - (1 mark) $sH(s) = \frac{2s^2}{(s^2+2s+1)} = \frac{2}{(1+\frac{2}{s}+\frac{1}{s^2})} = 2$; when $s \to \infty$. (1 mark).

9. What are the necessary conditions on α , β and n_0 for the system described below to be causal and stable : $y[n] = \alpha y[n-1] + \beta x[n-n_0]$.

   Solution : For x bounded, β should be finite and $-1 \leq \alpha \leq +1$ for stability. For causality, $n_0 \geq 0$.

   1 mark for stability condition, 1 mark for causality conditon.

10. Consider a system with input $x[n]$ and output, $y[n] = cos\big(\frac{\pi}{4}x[n]\big)$ If the input to the system is $\big(\frac{n^2}{2}\big)$ find whether the output is periodic; if periodic, find its fundamental period.

    Solution : $y[n] = cos\big(\frac{\pi}{4}\frac{n^2}{2}\big)$ If $y[n]$ is periodic with period N, $y[n+N] = y[n]$

    $cos\big(\frac{\pi}{4}\frac{(N+n)^2)}{2}] = cos\big(\frac{\pi}{4}\frac{n^2}{2}\big)$

    $\frac{\pi}{8}(n+N)^2 = \frac{\pi}{8}(n)^2 \pm 2k\pi$; where k is an integer.

    $N(N+2n) = \pm 16k$

    N has to be an even number since the product $N(N+2n)$ is even.

    Let $N = 2M$, where M is an integer.

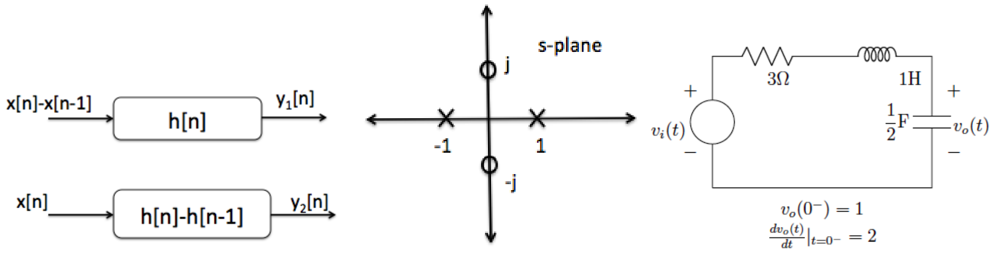
    $i.e, 2M(2M+2n) = 16k$

    or, $M(M+n) = 4k$

    Now the projduct, $M(M+n)$ is even; $M+n$ could be even or odd and hence should correspond to k; and hence M must be equal to 4.

Therefore, the fundamental period, N must be 8.

1 mark for correct procedure, 1 mark for correct answer.



(a) Qn 4 (b) Qn 5 (c) Qn 11

Figure 2: Figures for Questions 4, 5 and 11

11. Determine the differential equation relating $v_i(t)$ and $v_o(t)$ for the RLC circuit shown in Fig 1c. Suppose that $v_i(t) = e^{-3t}u(t)$. Using the unilateral Laplace transform, determine $v_o(t)$ for $t > 0$. (8)

Solution : (a) The differential equation relating $v_i(t)$ and $v_o(t)$ can be expressed as

$$LC\frac{d^2v_o(t)}{dt^2} + RC\frac{dv_o(t)}{dt} + v_o(t) = v_i(t)$$

$$\frac{d^2v_o(t)}{dt^2} + \frac{R}{L}\frac{dv_o(t)}{dt} + \frac{1}{LC}v_o(t) = \frac{1}{LC}v_i(t)$$

Substituting the values of R, L and C, we get

$$\frac{d^2v_o(t)}{dt^2} + 3\frac{dv_o(t)}{dt} + 2v_o(t) = 2v_i(t)$$

(b) Taking the unilateral Laplace transform of the above differential equation, we get

$$s^2V_o(s) - sv_o(0^-) - v_o'(0^-) + 3sV_o(s)$$
$$- 3v_o(0^-) + 2V_o(s) = 2V_i(s)$$

Since, $v_i(t) = e^{-3t}u(t)$,

$$V_i(s) = \frac{1}{s+3}, Re\{s\} > -3$$

Substituting this along with the initial conditions, we get

$$V_o(s) = \frac{(s^2 + 8s + 17)}{(s+1)(s+2)(s+3)}$$

The partial fraction expansion of $V_o(s)$ is

$$V_o(s) = \frac{5}{s+1} - \frac{5}{s+2} + \frac{1}{s+3}$$

Taking the inverse Laplace transform, we get

$$v_o(t) = (5e^{-t} - 5e^{-2t} + e^{-3t})u(t)$$

Marking Scheme : Writing the correct time domain eqn, with values of components subsituted (2)

Writing the correct eqn in LT domain. (2)

Substituting boundary conditions and solving for V(s) (1)

Partial Fraction Expansion (2)

ILT to find V(t) (1)

12. Given two real signals $x(t)$ and $y(t)$.

    The convolution of $x(t)$ and $y(t)$ is given by $x(t) * y(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$.

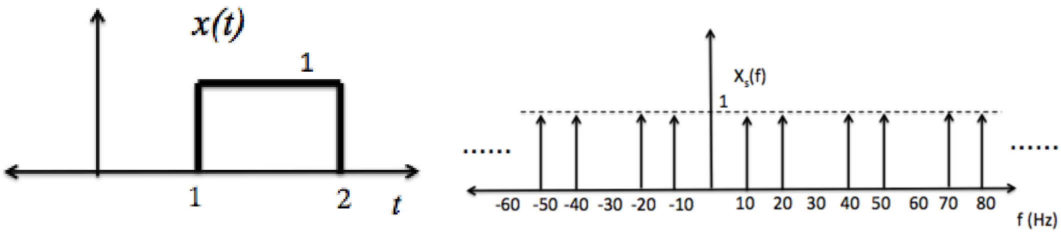
    The correlation between $x(t)$ and $y(t)$ is given by $x(t) \star y(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)y(\tau - t)d\tau$.

    For the $x(t)$ given in the Fig.2a, sketch (i) $x(t) * x(t)$ and (ii) $x(t) \star x(t)$. (4+4=8)

13. Let $x(t) = 60\cos(2\pi f_0 t)$. This signal is sampled at $30Hz$, by multiplying it by an impulse train given as
    $\sum_{m=-\infty}^{\infty} \delta(t - \frac{m}{30})$, to obtain the signal, $x_s(t)$.
    The corresponding Fourier transform, $X_s(f)$, is shown in Fig 2b.

    (a) What is the frequency f_0 of the sinusoid? (2)

    (b) Is there a unique solution for f_0 in the above problem? If yes, please justify. If not, find at least two other values. (4)



(a) Qn 12 (b) Qn 13

Figure 3: Figures for Questions 12 and 13

14. Use the properties of Fourier and Laplace transforms to solve the following :

    (a) Find the Fourier transform of $x_1(t) = \frac{t}{(1+t^2)^2}$, $t \in \mathbb{R}$. (5)

    (b) Find the inverse Laplace transform of $X_2(s) = \ln \frac{s+a}{s+b}$. (Assume that $a, b > 0$ and that $x_2(t)$ is a causal signal). (5)

    Solution:

    (a) We know that $\frac{1}{1+t^2} \longleftrightarrow \pi e^{-|\omega|}$. (See Example 4.13 O & W). Using the differentiation property, we differentiate in time and write

$$\frac{-2t}{(1+t^2)^2} \longleftrightarrow j\omega\pi e^{-|\omega|}$$

from which

$$\frac{t}{(1+t^2)^2} \longleftrightarrow \frac{-j\omega\pi}{2} e^{-|\omega|}$$

.

(b) From LT property, $\frac{d}{ds}X_2(s) \longleftrightarrow -tx_2(t)$. But, $\frac{d}{ds}X_2(s) = \frac{1}{s+a} - \frac{1}{s+b} \longleftrightarrow \left(e^{-at} - e^{-bt}\right)u(t)$, since the signal is causal. Thus, combining the two, we get

$$x_2(t) = \frac{1}{t}\left(e^{-bt} - e^{-at}\right)u(t).$$

For both these questions, students may choose an alternate method to solve, 2.5 marks for the correct approach, 2.5 marks for the correct answer.

(c) A causal LTI system with impulse response $h(t)$ has the following properties:

(i) When the input is $x(t) = 2e^{2t}$, the output is $y(t) = \frac{1}{3}e^{2t}$.

(ii) The impulse response satisfies $\frac{d}{dt}h(t) + 2h(t) = e^{-4t}u(t) + bu(t)$, where b is an unknown constant.

Determine the output of this system, when the input signal is $\cos 3t$. There should be no unknown constants in your answer. (8)

Solution: First observe that (i) implies $H(2) = 1/6$, using the eigenfunction property. Next, taking the BLT of the differential equation, we get $H(s)(s+2) = \frac{1}{s+4} + \frac{b}{s}$. Putting $s = 2$ in the above gives $b = 1$. Thus,

$$H(s) = \frac{2}{s(s+4)}.$$

The response to $\cos 3t$ again using the eigenfunction property, is $\frac{H(3j)e^{3jt} + H(-3j)e^{-3jt}}{2} = -\frac{je^{3jt}}{3(4+3j)} + \frac{je^{-3jt}}{3(4-3j)}$.

2 marks for finding H(2); 3 marks for finding b; 3 marks for finding the output of cos(3t)

(d) Consider an ideal lowpass filter with cutoff frequency 500 Hz as an LTI system H_l whose frequency response is given in Fig 3a. (3+7=10)

i. What is the response of this filter to the input signal $x(t) = \cos(1500\pi t) + \sin(3000\pi t)$?

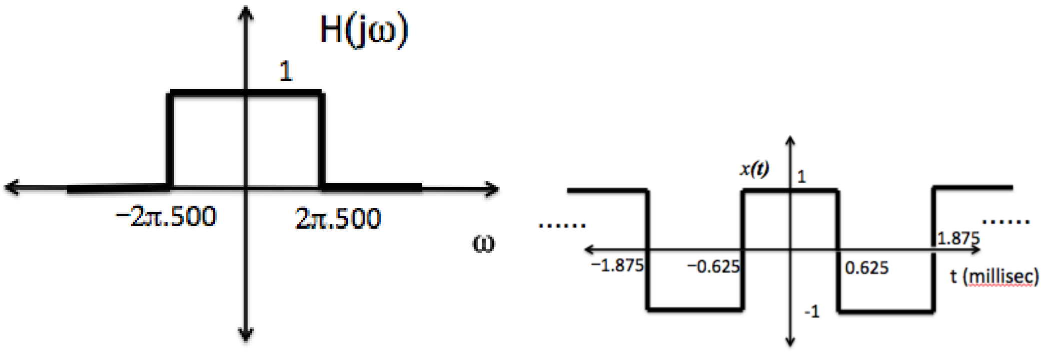
ii. What is the output $y(t)$ of this filter for an input, $x(t)$, which is a periodic square wave with period 2.5 ms, as shown in Fig 3b.

(e) The impulse response $h(t)$ of a causal continuous-time LTI system is known to be real-valued. Due to some practical constraints, suppose only the *real part* of the frequency response of the system is available to us.

i. Express the inverse Fourier transform of $\text{Re}\{H(j\omega)\}$ in terms of $h(t)$.

ii. From your answer to the previous part, show how we can determine $h(t)$ completely, although only the real part of the frequency response is available to us. (10)

i. $\text{Re}\{H(j\omega)\} = \frac{H(j\omega)+H^*(j\omega)}{2} = \frac{H(j\omega)+H(-j\omega)}{2}$, since $h(t)$ is real. Next, note that $H(-j\omega) \longleftrightarrow h(-t)$. Thus it follows that $\text{Re}\{H(j\omega)\} \longleftrightarrow \frac{h(t)+h(-t)}{2}$, so that the IFT of $\text{Re}\{H(j\omega)\}$ is simply the even part of $h(t)$.



(a) (b)

Figure 4: Figures for Questions 16

ii. $h(t)$ can be recovered from the IFT of $\mathrm{Re}\{H(j\omega)\}$ by multiplying the result (in the previous part) by $2u(t)$. To see this, note that since the system is causal, the impulse response satisfies $h(t)u(t) = h(t)$ and $h(-t)u(t) = 0$, for all t. Thus, to recover $h(t)$, we take the IFT of $\mathrm{Re}\{H(j\omega)\}$, and multiply the result by $2u(t)$.

End

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2. Find the RoC of the Laplace transform of the signal $x(t) = \cos \pi t$.
3. When a causal LTI system is cascaded with a non-causal LTI system, then the resultant system must be non-causal. Is this statement true or false ?
4. For the LTI systems shown in Fig.1a., $y_1[n] = y_2[n]$. Is this statement true or false ?
5. Find whether pole-zero plot shown in Fig. 1b could correspond to an even function in time ? If yes- find the required ROC. (Given that the transfer function is rational.)
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7. The output of a linear system for a step input is $t^2e^{-t}u(t)$. Find the impulse response, $h(t)$.
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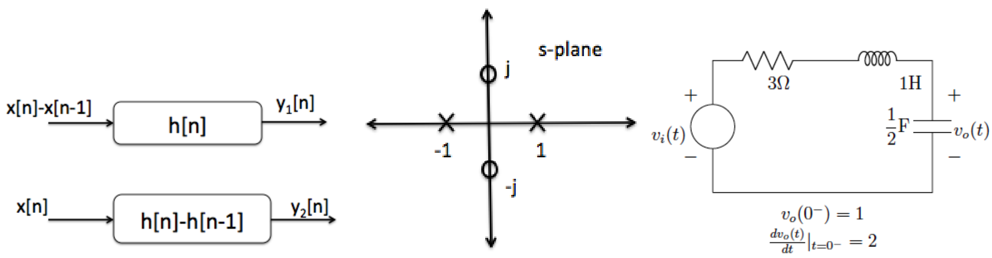


(a) Qn 4 (b) Qn 5 (c) Qn 11

Figure 1: Figures for Questions 4, 5 and 11

11. Determine the differential equation relating $v_i(t)$ and $v_o(t)$ for the RLC circuit shown in Fig 1c. Suppose that $v_i(t) = e^{-3t}u(t)$. Using the unilateral Laplace transform, determine $v_o(t)$ for $t > 0$. (8)

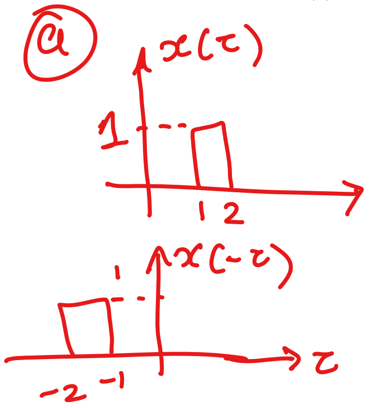
12. Given two real signals $x(t)$ and $y(t)$.

The convolution of $x(t)$ and $y(t)$ is given by $x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$.

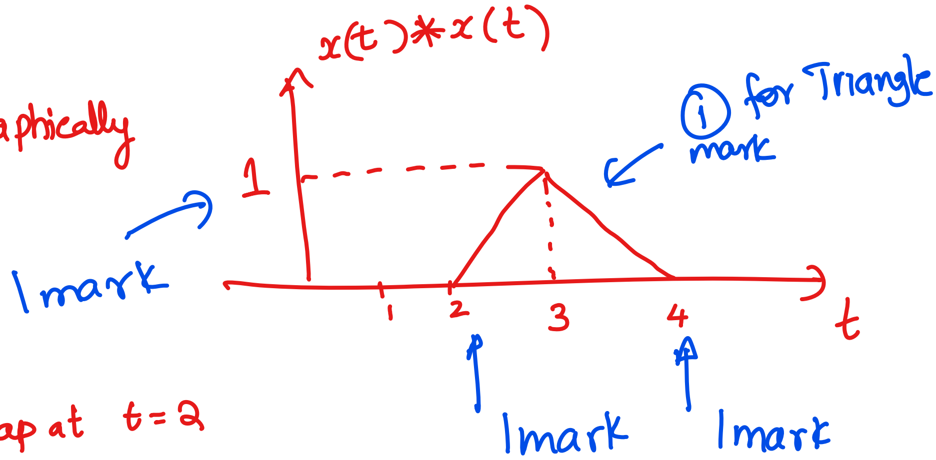
The correlation between $x(t)$ and $y(t)$ is given by $x(t) \star y(t) = \int_{-\infty}^{\infty} x(\tau)y(\tau - t)d\tau$.

For the $x(t)$ given in the Fig.2a, sketch (i) $x(t) * x(t)$ and (ii) $x(t) \star x(t)$. (4+4=8)

(a)



} Graphically

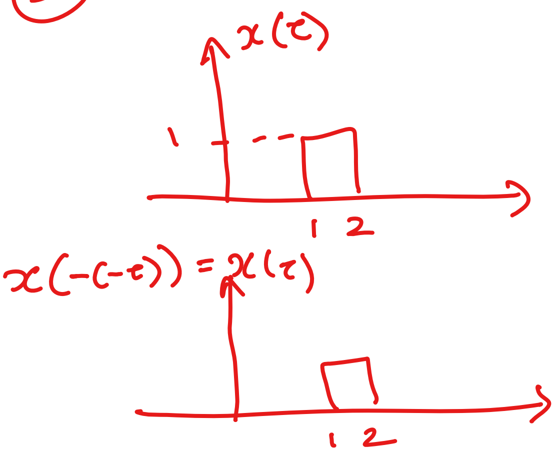


① First non-zero overlap at $t=2$

② Complete overlap $t=3 \Rightarrow$ Area $= 1 \times 1 = 1$

③ Final overlap at $t=4$

(b) Correlation: $\underbrace{x(\tau) * x(-\tau)}_{\text{conv.}} = x(t) \underbrace{\star}_{\text{Correlation}} x(t)$



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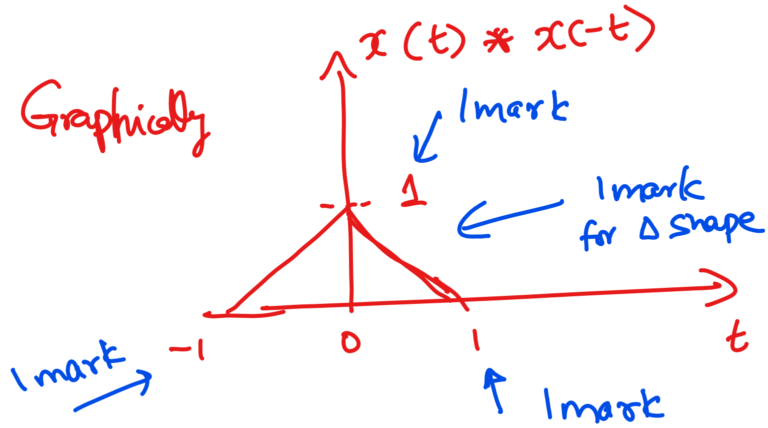


① Complete overlap at $t=0$ } Area of overlap $= 1 \times 1 = 1$

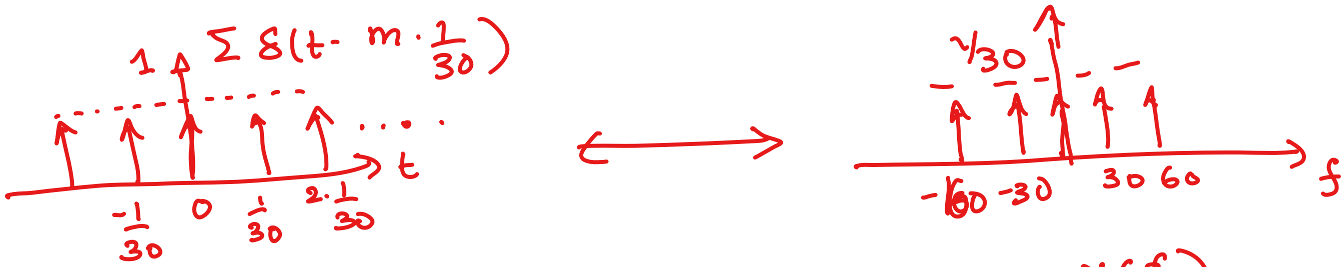
② To the left last overlap $t=-1$

③ To the right last overlap $= +1$

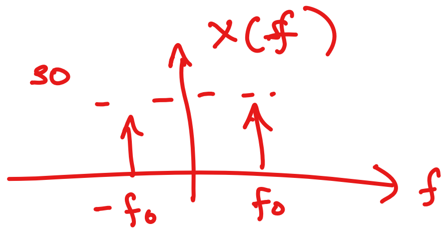
13. Let $x(t) = \cancel{2}\,60\cos(2\pi f_0 t)$. This signal is sampled at $30Hz$, by multiplying it by an impulse train given as
$\sum_{m=-\infty}^{\infty} \delta(t - \frac{m}{30})$, to obtain the signal, $x_s(t)$.
The corresponding Fourier transform, $X_s(f)$, is shown in Fig 2b.

(a) What is the frequency f_0 of the sinusoid? (2)

(b) Is there a unique solution for f_0 in the above problem? If yes, please justify. If not, find at least two other values. (4)



$x(t) = 60\cos(2\pi f_0 t)$ $\longleftrightarrow$



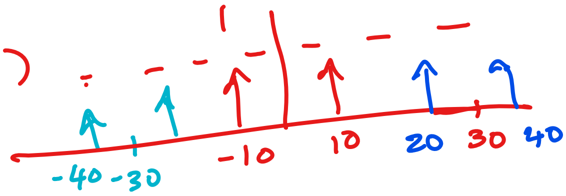
$$X_s(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f - k \cdot 30)$$

No Aliasing

① If $f_0 = 10$ Hz

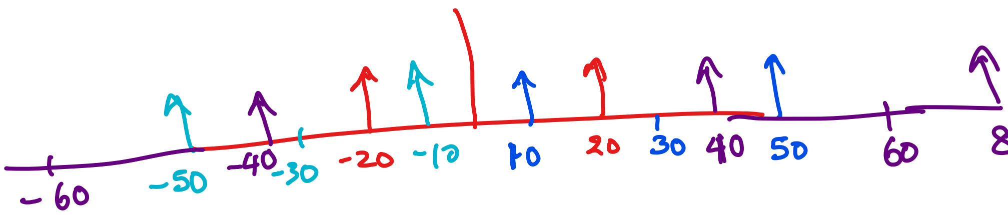
2 marks

$\Rightarrow X_s(f)$



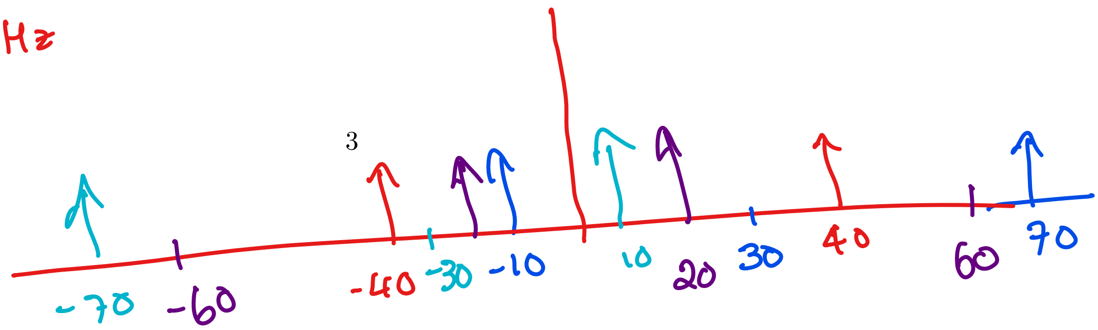
Aliasing

② If $f_0 = 20$ Hz



2 marks

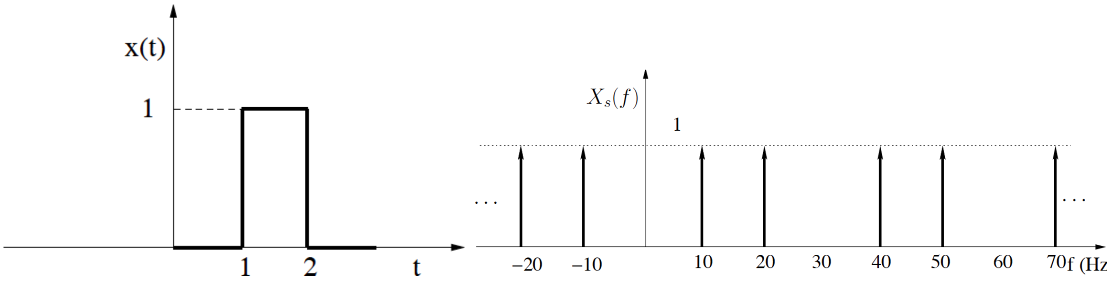
③ If $f_0 = 40$ Hz



2 marks

Similarly $f_0 = 50$Hz is also okay

Note : Give marks only if they explain using the figures or $\frac{1}{T}\sum_{k=-\infty}^{\infty} X(f-k\cdot 30)$



(a) Qn 12

(b) Qn 13

Figure 2: Figures for Questions 12 and 13

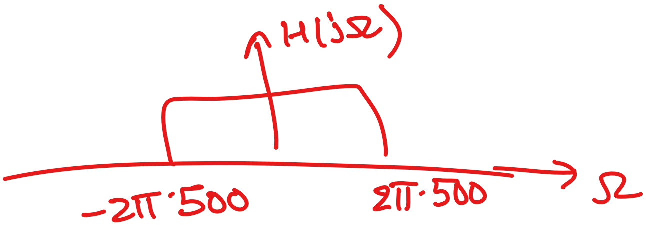
14. Use the properties of Fourier and Laplace transforms to solve the following :
    (a) Find the Fourier transform of $x_1(t) = \frac{t}{(1+t^2)^2}$, $t \in \mathbb{R}$. (5)
    (b) Find the inverse Laplace transform of $X_2(s) = \ln\frac{s+a}{s+b}$. (Assume that $a, b > 0$ and that $x_2(t)$ is a causal signal). (5)

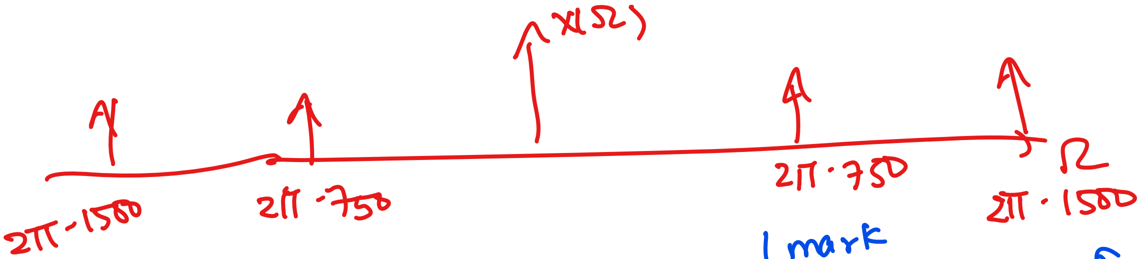
15. A causal LTI system with impulse response $h(t)$ has the following properties:
    (i) When the input is $x(t) = 2e^{2t}$, the output is $y(t) = \frac{1}{3}e^{2t}$.
    (ii) The impulse response satisfies $\frac{d}{dt}h(t) + 2h(t) = e^{-4t}u(t) + bu(t)$, where b is an unknown constant.

    Determine the output of this system, when the input signal is $\cos 3t$. There should be no unknown constants in your answer. (8)

16. Consider an ideal lowpass filter with cutoff frequency 500 Hz as an LTI system H_l whose frequency response is given in Fig 3a. (3+7=10)
    (a) What is the response of this filter to the input signal $x(t) = \cos(1500\pi t) + \sin(3000\pi t)$?
    (b) What is the output $y(t)$ of this filter for an input, $x(t)$, which is a periodic square wave with period 2.5 ms, as shown in Fig 3b.

(a)





1 mark

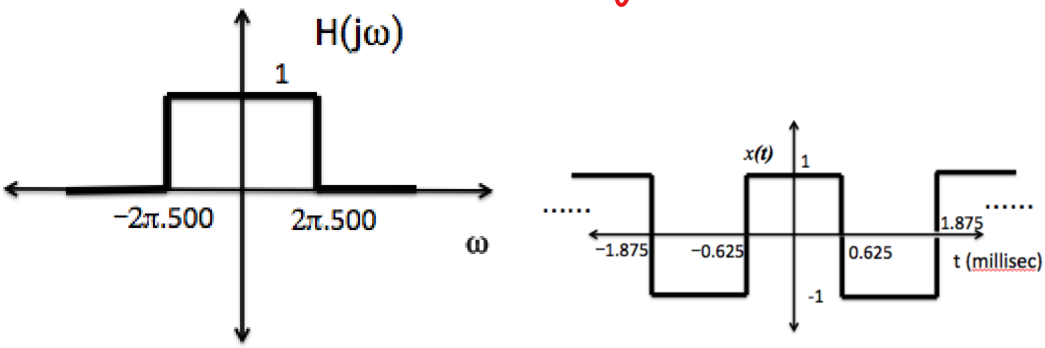
1 mark

The response is zero ⇒ 1 mark

(b) Note over one period area of $x(t)=0$

$\Rightarrow a_0 = \frac{1}{T}\int_T x(t)\,dt = 0$ → 2 mark secondly: $T = 2.5\ ms \Rightarrow F_0 = \frac{1}{2.5 \times 10^{-3}} = 400\ Hz$ (1) mark

only $\left.\begin{array}{l} a_1 \\ a_{-1} \end{array}\right\} \Rightarrow$ Component will pass through the low pass filter (1) mark $\quad \sum a_k e^{jk \cdot 2\pi f_0 t}$



(a) (b)

Figure 3: Figures for Questions 16

17. The impulse response $h(t)$ of a causal continuous-time LTI system is known to be real-valued. Due to some practical constraints, suppose only the *real part* of the frequency response of the system is available to us.

    (a) Express the inverse Fourier transform of $\text{Re}\{H(j\omega)\}$ in terms of $h(t)$.

    (b) From your answer to the previous part, show how we can determine $h(t)$ completely, although only the real part of the frequency response is available to us. (10)

End

(b) Hence $a_{-1} e^{-j 1.2\pi F_0 t} + a_1 e^{j 1.2\pi F_0 t}$

But since $x(t)$ is real and even

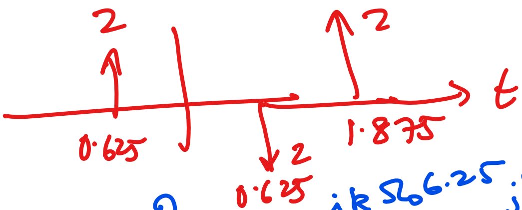
$\Rightarrow a_1 = a_{-1} \Rightarrow$ real and even

2 mark

$y(t) = 2a_1 \cos(2\pi f_0 t) = 2a_1 \cos(2\pi \cdot 400 t)$

1 mark $\quad a_1 = \frac{1}{2.5}\left[\int_{-0.625}^{0.625} 1 \cdot e^{j 1.2\pi f_0 t} + \int_{0.625}^{1.875} (-1)\, e^{j \cdot 1.2\pi f_0 t}\right]$

$\Rightarrow$ Easier way $\quad g(t) = \frac{d}{dt} x(t)$



$\Rightarrow a_1 = \frac{1}{j 2\pi \cdot 400} b_1$

$b_k = \frac{2}{2.5}\left[e^{jk\Omega_0 0.625} - e^{jk\Omega_0 0.5}\right.$