DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II Problem Set 5 (Solutions) 2019

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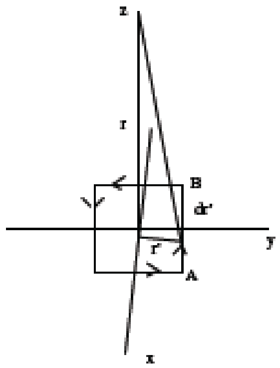
1. Use Biot-Savart law:

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int d\mathbf{r}' \times \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3},$$

where $d\mathbf{r}'$ is an element along the current. For the segment designated AB, (see figure) :



$$\mathbf{r}' = x\hat{e}_x + y\hat{e}_y = x\hat{e}_x + \frac{L}{2}\hat{e}_y.$$

, and the element

$$d\mathbf{r}' = dx\hat{e}_x.$$

Thus,

$$d\mathbf{r}' \times (\mathbf{r}-\mathbf{r}') = -\left(z\hat{e}_z - x\hat{e}_x - \frac{L}{2}\hat{e}_y\right) \times dx\hat{e}_x = -zdx\hat{e}_y - \frac{L}{2}\hat{e}_z.$$

From symmetry, one can argue that there are no components in the $x-y$ plane of the current. Thus, only the $\hat{e}_z$ component survives.

Thus, the magnetic field in the $\hat{e}_z$ due to the line segment designated AB is given by:

$$B_{\mathbf{AB}} = \frac{\mu_0 I}{4\pi}\int_{\mathbf{L}/2}^{-\mathbf{L}/2} \frac{-\frac{\mathbf{L}}{2}dx}{(\mathrm{L}^2/4 + z^2 + x^2)^{3/2}}.$$

The net field due to all the segments is thus $4\times$ the above result. Hence, we get the net field

$$B_z = \frac{\mu_0 IL}{\pi}\int_0^{\mathbf{L}/2} \frac{dx}{(\mathrm{L}^2/4 + z^2 + x^2)^{3/2}}.$$

Performing the integral we get

$$\mathbf{B} = \frac{4\mu_0 IL^2}{\pi\,(4z^2+L^2)\sqrt{4z^2+2L^2}}\hat{e}_z.$$

2. The force on the two sides cancel each other. Thus, one needs to look at the force contributions from the bottom and top segments of the square loop: At the bottom, the field is

$$B = \frac{\mu_0 I}{2\pi s}\hat{e}_z.$$

. To find the force $F = I\int d\ell \times \mathbf{B}$ gives

$$F_{\text{lower}} = \frac{\mu_0 I^2}{2\pi s}\int_0^a \left(dx\,(-\hat{e}_x)\times\hat{e}_z\right) = \frac{\mu_0 I^2 a}{2\pi s}\hat{e}_y$$

. The force on the upper line segment, F_{upper} of the loop can also be found in similar fashion. However in this case the magnetic field felt by the upper segment of the square is

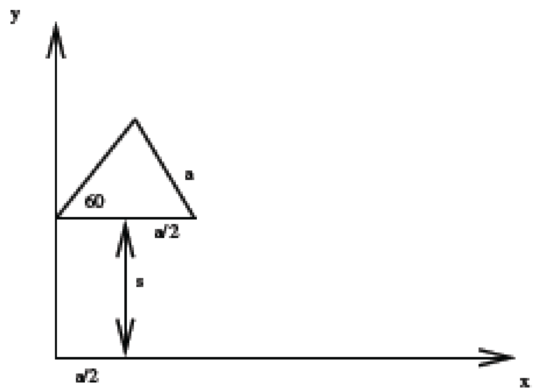
$$B_{\text{upper}} = \frac{\mu_0 I}{2\pi\,(s+a)}\hat{e}_z.$$

Once again,

$$F_{\text{upper}} = \frac{\mu_0 I^2}{2\pi\,(s+a)}\int_0^a \left(dx\,(\hat{e}_x)\times\hat{e}_z\right) = -\frac{\mu_0 I^2 a}{2\pi\,(s+a)}\hat{e}_y$$

. Thus, the net force felt on the square loop is thus

$$F = \frac{\mu_0 I^2 a}{2\pi}\left(\frac{1}{s} - \frac{1}{s+a}\right)\hat{e}_y = \frac{\mu_0 I^2 a^2}{2\pi s(s+a)}\hat{e}_y.$$

3. 

For the lower line segment of the triangle we can borrow the result from the previous problem. Thus we have

$$F_{\text{lower}} = \frac{\mu_0 I^2 a}{2\pi s}\hat{e}_y.$$

Now, we need to find the force on the other two segments. To do so, we let us look at the incremental force $dF = d\ell \times B$ felt by a segment $d\ell$:

$$dF = I\,(d\ell \times B) = I\,(dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z) \times \frac{\mu_0 I}{2\pi y}\hat{e}_z = \frac{\mu_0 I^2}{2\pi y}\left(-dx\hat{e}_y + dy\hat{e}_x\right).$$

The wire on the left hand side has an x-component of the force that cancels exactly with the x-component coming from the leg of the triangle placed on the left hand side. Thus, the force on the left leg is

$$F_{\text{left}} = \frac{\mu_0 I^2}{2\pi}\int_0^{a/2}\frac{-dx}{y}\hat{e}_y.$$

Please note y is dependent on x. From the figure this can easily be worked out to be $y = \sqrt{3}x + s$. Thus, we get

$$F_{\text{left}} = \frac{\mu_0 I^2}{2\pi}\int_0^{a/2}\frac{-dx}{\sqrt{3}x+s}\hat{e}_y = \frac{\mu_0 I^2}{2\pi}\left(\frac{-\hat{e}_y}{\sqrt{3}}\ln\frac{s+a\sqrt{3}}{s}\right)$$

.

Now, for the right leg we follow the same strategy. Thus, we get

$$F_{\text{left}} = \frac{\mu_0 I^2}{2\pi}\int_{a/2}^{a}\frac{-dx}{y}\hat{e}_y.$$

However in this case the constraint equation connecting y and x is given by $y = -\sqrt{3}x + s + a\sqrt{3}$

$$F_{\text{right}} = \frac{\mu_0 I^2}{2\pi} \int_{a/2}^{a} \frac{-dx}{-\sqrt{3}x + s + a\sqrt{3}} \hat{e}_y = \frac{\mu_0 I^2}{2\pi} \left(\frac{\hat{e}_y}{\sqrt{3}} \ln \frac{s}{s + a\sqrt{3}/2} \right)$$

. Thus, the net force $F = F_{\text{lower}} + F_{\text{left}} + F_{\text{right}}$ which gives

$$F = \frac{\mu_0 I^2}{2\pi} \left(\frac{a}{s} + \frac{2}{\sqrt{3}} \ln \left(\frac{s}{s + a\sqrt{3}/2} \right) \right) \hat{e}_y.$$

4. First indicate the magnetic field **B** due to a long straight cylindrical conductor with uniform current density **J**, parallel to the axis. For a circular path of radius ρ ($\rho < R$) in a plane normal to the axis of the conductor, the current enclosed is $\pi \rho^2 J$. **B** is tangential to the circle everywhere and of the same magnitude because of the cylindrical symmetry. (Recall the argument for the case of magnetic field due to an infinitely long wire, done in class) Therefore, for $\rho < R$

$$\begin{aligned} \oint \mathbf{B} \cdot \mathbf{dl} &= B(\rho)\ (2\pi\rho) \\ &= \mu_0 \pi \rho^2 J \\ \implies B(\rho) &= \frac{\mu_0 J \rho}{2} \end{aligned}$$

In vector form,

$$\mathbf{B}(\rho < R) = \frac{\mu_0}{2} (\mathbf{J} \times \boldsymbol{\rho}) \ .$$

For $\rho > R$

$$\begin{aligned} \oint \mathbf{B} \cdot \mathbf{dl} &= \mu_0 \pi R^2 J \\ \implies \mathbf{B}(\rho \geq R) &= \frac{\mu_0 R^2}{2\rho^2} (\mathbf{J} \times \boldsymbol{\rho}) \ . \end{aligned}$$

Now, we consider the case of a cavity in this cylinder. This cavity has zero current density. C is the center of the cavity.

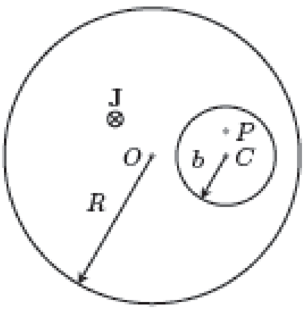


Figure 1: Hole in a cylindrical conductor

B (at P) if the entire cylinder had a current density is

$$\frac{\mu_0}{2}\ (\mathbf{J} \times \mathbf{OP}) \ .$$

B (at P) if the cavity alone had a current density **J** is

$$\frac{\mu_0}{2}\ (\mathbf{J} \times \mathbf{CP}) \ .$$

Therefore,

$$\text{net } \mathbf{B} \text{ (at P)} = \frac{\mu_0}{2} \left(\mathbf{J} \times (\mathbf{OP} - \mathbf{CP})\right)$$

$$\boxed{\mathbf{B} = \frac{\mu_0}{2} \left(\mathbf{J} \times \mathbf{OC}\right) \ .}$$

The dependence on P has totally disappeared and $\mathbf{B}$ is uniform within the hole at all points and its direction is normal to $\mathbf{OC}$.

5. This problem is a simple application of Biot-Savart's law. The equation of the spiral coil is

$$\varrho(\varphi) = (b-a)\tfrac{\varphi}{2\pi N} + a \ , \quad \varphi \in [0, 2\pi N] \ .$$

An infinitesimal element has $\mathbf{dl} = d\varrho(\varphi)\ \hat{e}_\varrho + \varrho(\varphi) d\varphi\ \hat{e}_\varphi$. At the centre of the coil, one has

$$\mathbf{dB} = \frac{\mu_0}{4\pi}\, i\mathbf{dl} \times \frac{-\hat{e}_\varrho}{\varrho^2} = \frac{\mu_0}{4\pi}\, \frac{id\varphi}{\varrho} \hat{e}_z \ .$$

Integrating over the coil, we get

$$\mathbf{B} = \frac{\mu_0 i}{4\pi}\, \hat{e}_z \int_0^{2\pi N} \frac{d\varphi}{\varrho(\varphi)}$$

$$\boxed{\mathbf{B} = \frac{\mu_0 i N}{2(b-a)}\ \ln \tfrac{b}{a}\ \hat{e}_z \ .}$$

Alternate Solution

This problem can be solved using the expression for the magnetic field of a single loop (carrying a steady current), at the center of the loop. Let us determine the (more general case of the) magnetic field at a distance z along the axis of a circular current carrying loop.

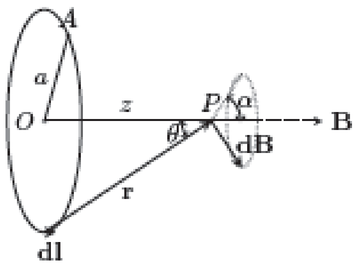


Figure 2: The magnetic field due to a circular loop

The magnetic field, $\mathbf{dB}$, the magnetic field due to the element $\mathbf{dl}$ at P will clearly be perpendicular to $\mathbf{dl}$ and $\mathbf{r}$. The angle between $\mathbf{dl}$ and $\mathbf{r}$ is $\pi/2$. Biot-Savart's law gives

$$dB = \frac{\mu_0}{4\pi} \frac{i\ dl\ r}{r^3} = \frac{\mu_0}{4\pi} \frac{i\ dl}{(z^2 + a^2)},$$

where a is the radius of the loop.

To obtain $\mathbf{B}$, we integrate $\mathbf{dl}$ over the loop and correspondingly $\mathbf{dB}$ sweeps out a cone as shown, with semi-vertical angle $\alpha = (\frac{\pi}{2} - \theta)$. Further, only the component of $\mathbf{B}$ along the z axis remains, as the other component cancels out. $(\cos\alpha = \sin\theta = \frac{a}{\sqrt{a^2+z^2}})$

$$\therefore |\mathbf{B}| = \frac{\mu_0 i}{4\pi} \int \frac{\mathbf{dl}}{r^2} \cos\alpha = \frac{\mu_0 i}{4\pi r^2} \cos\alpha\ (2\pi a) = \frac{\mu_0 i a^2}{2(a^2 + z^2)^{3/2}}$$

$$\boxed{\mathbf{B} = \frac{\mu_0 i a^2}{2(a^2 + z^2)^{3/2}}\ \hat{e}_z \ .} \qquad (1)$$

The field at the center of the loop has magnitude $\frac{\mu_0 i}{2a}$.

Now, in this problem, the contribution from one turn of the planar spiral to the field at the center is $\frac{\mu_0 i}{2\rho}$ (ρ is the radius of the turn). Therefore, the field at the center from all N turns is

$$\int \frac{\mu_0 i}{2\varrho}\, dN = \int \frac{\mu_0 i}{2\varrho} \frac{N}{(b-a)} d\varrho,$$

where dN is the number of the turns that lie in the interval $(\varrho, \varrho + d\varrho)$.

$$\begin{aligned} \therefore \mathbf{B} &= \frac{\mu_0 i}{2} \frac{N}{(b-a)} \int_a^b \frac{d\varrho}{\varrho} \\ &= \frac{\mu_0 i}{2} \frac{N}{(b-a)} \ln\left(\frac{b}{a}\right)\, \hat{e}_z \end{aligned}$$

3[rd] method-Using surface charge density $\vec{B} = \frac{\mu_0}{4\pi} \int \left(\frac{NI}{b-a} \hat{e}_\varphi \;\times\; \frac{-\hat{e}_\rho}{\rho^2} \right)\, da$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{NI}{b-a} \hat{e}_z \int \left(\frac{1}{\rho^2} \right) \rho\, d\rho\;\, d\varphi$$

6. **Determining the trajectory of a charged particle**

$$\mathbf{B} = B\hat{e}_z \quad \text{and} \quad \mathbf{E} = E\hat{e}_y = \frac{\phi_0}{d}\hat{e}_y$$

Therefore, the force is along the y axis (qE) due to the electric field and is in the xy plane ($\mathbf{v} \times \mathbf{B}$), due to $\mathbf{B}$. The particle has a velocity

$$\mathbf{v} = \dot{x}\ \hat{e}_x + \dot{y}\ \hat{e}_y \tag{2}$$

$$\mathbf{F} = m\ddot{x}\ \hat{e}_x + m\ddot{y}\ \hat{e}_y \tag{3}$$

Equate $\mathbf{F}$ to $q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ component wise.

$$\ddot{x} = \omega\dot{y} \tag{4}$$

$$\ddot{y} = -\omega\dot{x} + \frac{qE}{m} \tag{5}$$

with $\omega = \frac{qB}{m}$ being the cyclotron frequency. Differentiate equations (4) and (5) to get the uncoupled equations

$$\dddot{x} = -\omega^2\dot{x} + \frac{\omega qE}{m} \tag{6}$$

$$\dddot{y} = -\omega\ddot{x} = -\omega^2\dot{y} \tag{7}$$

Impose boundary conditions and solve. At $t = 0$,

$$y = 0, \quad \dot{y} = 0, \quad \ddot{y} = \frac{qE}{m}\ .$$

The solution for $y(t)$ is $y(t) = \frac{E}{\omega B}(1 - \cos(\omega t))$.

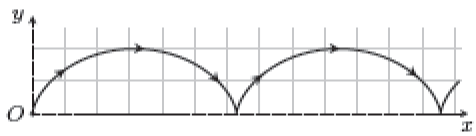


Figure 3: Cycloid

Therefore,

$$x = \frac{E}{\omega B}(\omega t - \sin\omega t) \ , \tag{8}$$

$$\text{and } y = \frac{E}{\omega B}(1 - \cos(\omega t)) \ . \tag{9}$$

Equations (8) and (9) are the parametric equations for a cycloid, shown below in Figure 3.

7. (a) Using the expression for the magnetic field along the axis of a single coil given in Eq. (1) after suitably accounting for the direction of the current as well as the number of turns, we obtain

$$\mathbf{B}_1 \text{ at } P \text{ due to coil } 1 = -\frac{\mu_0 N I}{4\pi}\frac{2\pi a^2}{[a^2+(d+z)^2]^{3/2}}\hat{e}_z$$

$$\mathbf{B}_2 \text{ at } P \text{ due to coil } 2 = -\frac{\mu_0 N I}{4\pi}\frac{2\pi a^2}{[a^2+(d-z)^2]^{3/2}}\hat{e}_z$$

$$\therefore \mathbf{B}(z) = \mathbf{B}_1 + \mathbf{B}_2 = -\frac{\mu_0 N I a^2}{2}\hat{e}_z\left(\frac{1}{[a^2+(d+z)^2]^{3/2}} + \frac{1}{[a^2+(d-z)^2]^{3/2}}\right)$$

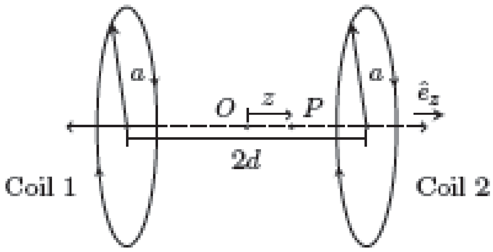


Figure 4: Helmholtz coils

(b) Expand $\mathbf{B}(z)$ in terms of $\mathbf{B}(0)$ as a Taylor series in z (for small z), we get

$$\mathbf{B}(z) = \mathbf{B}(0)\left(1 + \frac{3(-a^2+4d^2)}{2(a^2+d^2)^2}\ z^2 + \frac{15(a^4-12a^2d^2+8d^4)}{8(a^2+d^2)^4}\ z^4 + \mathcal{O}(z^6)\right) \ .$$

The odd power of z do not appear as the magnetic field is an even function of z. When $4d^2 = a^2$ or $d = \frac{a}{2}$, the term proportional to z^2 vanishes and we obtain

$$\mathbf{B}(z)\big|_{d=\frac{a}{2}} = \mathbf{B}(0)\left(1 - \frac{144z^4}{125a^4} + \mathcal{O}(z^6)\right) \ .$$

Therefore, when $d = \frac{a}{2}$, $\mathbf{B}$ is uniform to the third power of z. This setup is called *Helmholtz coils* and is used to provide an approximately uniform magnetic field $\mathbf{B}$ at O.

8. (a) The expression for $|\mathbf{B}|$ at a distance x away from the sheet has already been obtained in class. The direction of $|\mathbf{B}|$ is along $\hat{e}_y$ and has a magnitude $\frac{\mu_0 K}{2}$. $\mathbf{B}$ is therefore uniform.

(b) Using superposition, we see that In between the sheets,

$$\mathbf{B} = \frac{\mu_0 K}{2}\hat{e}_y - \frac{\mu_0 K}{2}\hat{e}_y = \mathbf{0},$$

whereas for the region outside the sheets, $\mathbf{B}$ has magnitude $\mu_0 K$.

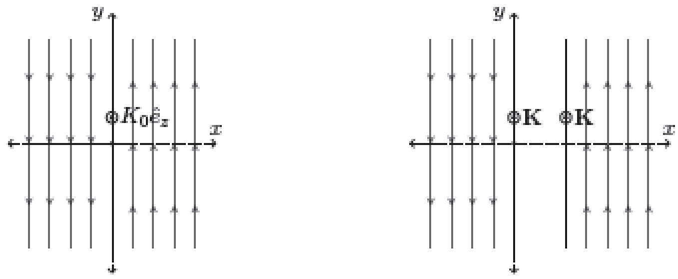


Figure 5: To the left is the magnetic field lines due to a single sheet and to the right is the magnetic field lines due to two parallel sheets.