EE2015 Electric Circuits and Networks - Tutorial 4
August 30 and September 3, 2019

1. Using time-domain methods, find the natural and forced response.

$$\frac{dy}{dt} + 3y(t) = 2e^{-2t}u(t), \quad y(0^+) = 1$$

Natural response

$$\frac{dy}{dt} + 3y_n(t) = 0$$
$$y_n(t) = ae^{-3t}; t \geq 0$$

Forced response

$$\frac{dy}{dt} + 3y(t) = 2e^{-2t}u(t) \qquad (1)$$
$$\implies y(t) \text{ is of the form } be^{-2t} \qquad (2)$$
$$(2) \text{ in } (1)$$
$$\implies -2be^{-2t} + 3be^{-2t} = 2e^{-2t}$$
$$\implies b = 2$$
$$\implies y_f(t) = 2e^{-2t}; t \geq 0$$

Total response

$$y(t) = y_n(t) + y_f(t)$$
$$y(t) = ae^{-3t} + 2e^{-2t}$$
$$\because y(0^+) = 1 \implies a = -1$$
$$\therefore y(t) = 2e^{-2t} - e^{-3t}; t \geq 0$$

2. Find the zero-state and zero-input response for the following system for $t \geq 0$. Solve it in the Laplace domain.

$$\frac{dy}{dt} + 2y(t) = x(t) + 2x(t-1)$$
$$x(t) = 4u(t), \quad y(0^-) = 2$$

Zero - state response

$$\frac{dy}{dt} + 2y(t) = x(t) + 2x(t-1)$$

Applying Laplace transform

$$sY(s) + 2Y(s) = X(s) + 2X(s)e^{-s}$$

$$Y(s) = \frac{X(s)}{s+2}[1 + 2e^{-s}]$$

$$\because x(t) = 4u(t) \Longrightarrow X(s) = \frac{4}{s}$$

$$\Longrightarrow Y(s) = \frac{4(1+2e^{-s})}{s(s+2)}$$

$$= 2\Big[\frac{1}{s} - \frac{1}{s+2}\Big] + 4\Big[\frac{e^{-s}}{s} - \frac{e^{-s}}{s+2}\Big]$$

Applying inverse Laplace transform

$$y(t) = 2(1 - e^{-2t})u(t) + 4(1 - e^{-2(t-1)})u(t-1)$$

Zero - input response

$$\frac{dy}{dt} + 2y(t) = 0$$

Applying Laplace transform

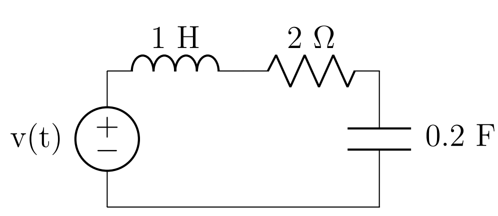
$$\Longrightarrow sY(s) - y(0^-) + 2Y(s) = 0$$

$$\Longrightarrow Y(s) = \frac{2}{s+2}$$

Applying inverse Laplace transform

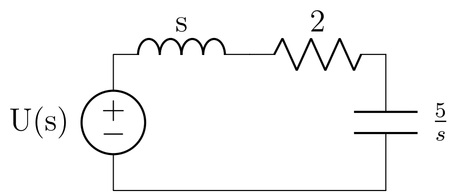
$$y(t) = 2e^{-2t}u(t)$$

3. For the circuit shown below, find the step and impulse response using Laplace transform techniques. The output is $v_c(t)$.



The circuit in s domain is given below.

$$\frac{V_c(s)}{U(s)} = \frac{\frac{5}{s}}{s + 2 + \frac{5}{s}}$$



$$H\left(s\right) = \frac{5}{s^2+2s+5}$$

$$H\left(s\right) = \frac{5*2}{2\left[\left(s+1\right)^2+2^2\right]}$$

Taking inverse Laplace Transform of H(s)

$$h\left(t\right) = 2.5\ e^{-t}\sin 2tu(t)$$

Step response

$$U\left(s\right) = \frac{1}{s}$$

Therefore,

$$V_c\left(s\right) = H\left(s\right)U\left(s\right)$$

$$V_c\left(s\right) = \frac{5}{s(s^2+2s+5)}$$

Doing partial fraction of above expression

$$V_c\left(s\right) = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

Comparing coefficients

$$A = 1,\ B = -1\ ,\ C = 2$$

$$V_c\left(s\right) = \frac{1}{s} - \frac{s+2}{s^2+2s+5}$$

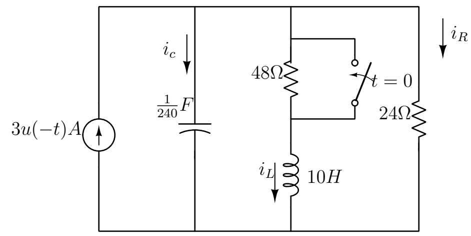
$$V_c\left(s\right) = \frac{1}{s} - \frac{s+1+1}{\left(s+1\right)^2+2^2}$$

$$V_c\left(s\right) = \frac{1}{s} - \frac{s+1}{\left(s+1\right)^2+2^2} - \frac{1}{\left[\left(s+1\right)^2+2^2\right]} * \frac{2}{2}$$

Taking inverse Laplace Transform of V$_c$(s)

$$v_c\left(t\right) = \left[1 - e^{-t}\cos 2t - 0.5e^{-t}\sin 2t\right]u(t)$$

4. After being open for a long time, the switch in the network closes at $t = 0$. Find (a) $i_L(0^-)$ (b) $v_C(0^-)$ (c) $i_R(0^+)$ (d)$i_C(0^+)$ and (e) $v_C(0.2)$.



The initial conditions for are calculated by making capacitor as open circuit and induct or as short circuit.

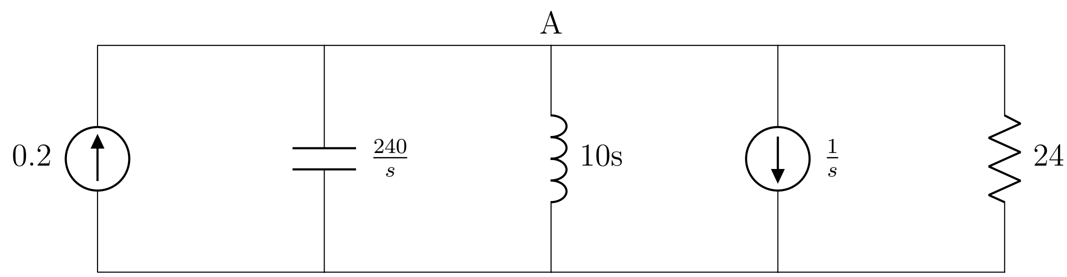
a.
$$i_L\left(0^-\right) = \frac{3 * 24}{48 + 24} = 1\ A$$

b.
$$v_C\left(0^-\right) = \ i_L\left(0^-\right) * 48 = 48\ V$$

c.
$$i_R\left(0^+\right) = i_R\left(0^-\right) = 2\ A since V_R = V_C$$

d.
$$\begin{aligned} i_C\left(0^+\right) &= -\left(i_R\left(0^+\right) + \ i_L\left(0^+\right)\right) \\ &= -2 - 1 = -3\ A \end{aligned}$$

e. Redrawing the above circuit at t=0^+ in s domain



Applying KCL at node A

$$-\left(0.2 - \frac{1}{s}\right) + \frac{V_c}{\frac{240}{s}} + \frac{V_c}{10s} + \frac{V_c}{24} = 0$$

$$V_c\left(\frac{s}{240} + \frac{1}{10s} + \frac{1}{24}\right) = \frac{0.2s - 1}{s}$$

$$V_c\left(s\right)=\frac{(0.2s-1)*240}{s^2+10s+24}$$

$$V_c\left(s\right)=\frac{(0.2s-1)*240}{(s+4)(s+6)}$$

Doing partial fraction

$$V_c\left(s\right)=\frac{(0.2s-1)*240}{\left(s+4\right)\left(s+6\right)}=\frac{A}{s+4}+\frac{B}{s+6}$$

By comparing coefficients we will get, A=-216 and B=264

$$V_c\left(s\right)=\frac{-216}{s+4}+\frac{264}{s+6}$$

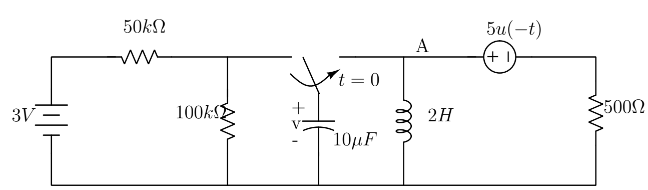
Taking Inverse Laplace Transform of Vc(s)

$$v_c\left(t\right)=264e^{-6t}-216e^{-4t}$$

At t=0.2

$$v_c\left(t\right)=-17.56\ V$$

5. The switch is in the left position for a long time and is moved to the right at $t=0$. Find (a) $\frac{dv}{dt}$ at $t=0^+$, (b) v at 1ms and (c) the first value of $t>0$ at which $v=0$.



Forming the differential equation using nodal law at node A at time t=0+

$$V=L\frac{dI}{dt}$$

$$C\frac{dV}{dt}+I+\frac{V}{R}=0$$

Substituting initial conditions, we get

$$\frac{dV}{dt}=-(\frac{2}{500}+\frac{5}{500})\frac{1}{C}$$

$$\frac{dV}{dt} = -1400V/sec$$

Further differentiating we get

$$C\frac{d^2V}{dt^2} + \frac{dI}{dt} + \frac{dV}{dt}(\frac{1}{R}) = 0$$

$$C\frac{d^2V}{dt^2} + \frac{V}{L} + \frac{dV}{dt}(\frac{1}{R}) = 0$$

Taking $V(t) = Ae^{at}$

$$Ca^2e^{at} + \frac{e^{at}}{L} + \frac{ae^{at}}{R} = 0$$

$$a^2 + 5 * 10^4 + 200a = 0$$

$$a = -100 \pm 200j$$

$$V(t) = A_1e^{(-100-200j)t} + A_2e^{(-100+200j)t}$$

Let A_1 be $a + jb$ and A_2 be $a - jb$ Substituting initial conditions

$$V(0) = A_1 + A_2 = 2 \implies \quad a = 1$$

$$\frac{dV}{dt} = -1400$$

$$\implies -100A_1 - 200jA_1 - 100A_2 + 200jA_2 = -1400$$

Substituting $a \pm jb$ we get,

$$b = -3$$

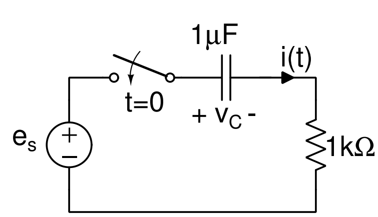
Hence,

$$V(t) = (1 + 3j)e^{(-100-200j)t} + (1 - 3j)e^{(-100+200j)t}$$

$$\implies V(t) = e^{-100t}(2cos(200t) - 6sin(200t))$$

First time zero when $t = \frac{\pi - tan^{-1}\frac{1}{3}}{200}$

6. In the circuit shown below $v_c(0^-) = 1V$ and $e_s(t) = e^{-2t}$ V is applied at $t = 0$. The output is $i(t)$. Find (a) the natural and forced response of the circuit and (b) zero state and zero input response using time domain techniques.



a. By writing loop equation'

$$\frac{1}{C}\int idt + iR = e^{-2t}$$

By differentiating on both sides

$$\frac{i(t)}{C} + R\frac{di(t)}{dt} + = -2e^{-2t}$$

$$\frac{di(t)}{dt} + \frac{i(t)}{10^{-3}} = -2 * 10^{-3}e^{-2t}$$

Natural Response

$$\frac{di(t)}{dt} + 1000i(t) = 0$$

$$i_N(t) = ae^{-1000t}$$

Forced response

$$\frac{di(t)}{dt} + \frac{i(t)}{10^{-3}} = -2 * 10^{-3}e^{-2t}$$

Here the forcing function is $-2 * 10^{-3}e^{-2t}$, so the solution must be in the form of $i_F = be^{-2t}$. By substituting this in differential equation.

$$-2be^{-2t} + 1000be^{-2t} = -2 * 10^{-3}e^{-2t}$$

By solving this

$$b = \frac{-2 * 10^{-3}}{998}$$

So,forced response

$$i_F(t) = \frac{-2 * 10^{-3}}{998}e^{-2t}$$

Now total response

$$i(t) = i_N(t) + i_F(t)$$

$$i(t) = ae^{-1000t} + \frac{-2 * 10^{-3}}{998}e^{-2t}$$

by applying initial conditions $i(0) = 0$

$$0 = a + \frac{-2 * 10^{-3}}{998}$$

$$a = \frac{2 * 10^{-3}}{998}$$

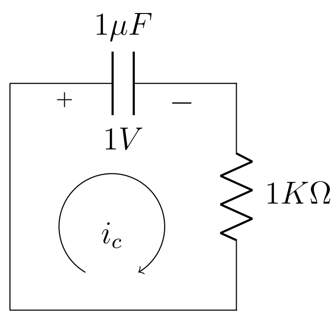
So

$$i_N(t) = \frac{2*10^{-3}}{998}e^{-1000t}$$

Finally total response

$$i(t) = \frac{2*10^{-3}}{998}e^{-1000t} - \frac{2*10^{-3}}{998}e^{-2t}$$

Zero input response
To calculate zero input response, the input function is nullified and the circuit is redrawn as shown below



Since $V_c(0^-) = V_c(0^+) = 1V$

$$i_c(0^+) = -\frac{1}{1000} = -1mA$$

For zero input response

Applying KVL in the circuit shown above

$$\frac{1}{10^{-6}}\int i_c dt + 1000 * i_c = 0$$

Differentiating

$$\frac{di_c}{dt} + 1000 * i_c = 0$$

$$\ln i_c = -1000t + A$$

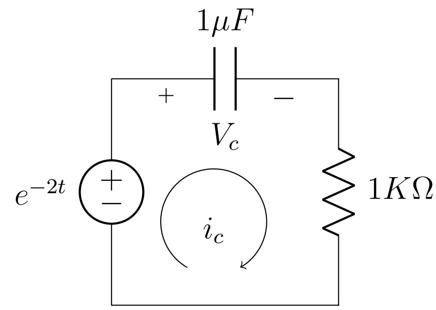
Using the initial condition,

$$A = \ln 10^{-3}$$

Substituting A,

$$i_c(ZIR) = -10^{-3} * e^{-1000t}$$

To calculate zero state response, the initial condition across the capacitor is nullified and the circuit is redrawn as shown below



For zero state response,

Applying KVL in the above circuit,

$$e^{-2t} - \frac{1}{10^{-6}} \int i_c dt - 1000 * i_c = 0$$

Differentiating

$$\frac{di_c}{dt} + 1000 * i_c = -2 * 10^{-3} * e^{-2t}$$

Here integration factor $= e^{1000t}$

Therefore,

$$e^{1000t} * i_c = -\int 2 * 10^{-3} * e^{-2t} * e^{1000t} dt + B$$

Using initial conditions,

$$B = 10^-3 + \frac{2 * 10^{-3}}{998}$$

Therefore,

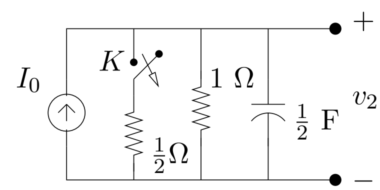
$$i_c(ZSR) = \frac{-2 * 10^-3 * e^{-2t}}{998} + \frac{e^{-1000t}}{998}$$

Therefore total response $= i_c(ZSR) + i_c(ZIR)$

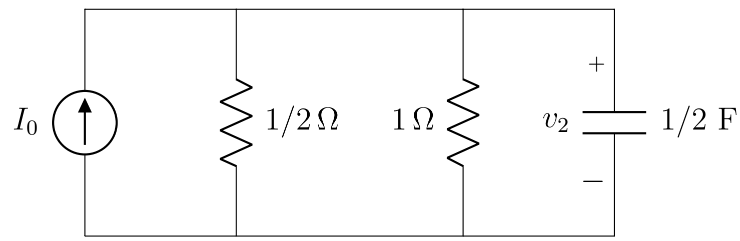
$$i_c(t) = \frac{-2 * 10^-3 * e^{-2t}}{998} + \frac{2 * 10^-3 * e^-1000t}{998}$$

For the following questions, use Laplace transform techniques to solve for the network

7. The network shown below consists of a constant current source of value I_0, two resistors and a capacitor. At $t = 0$, the switch K is opened. For the element values given on the figure, determine $v_2(t)$ for $t \geq 0$.



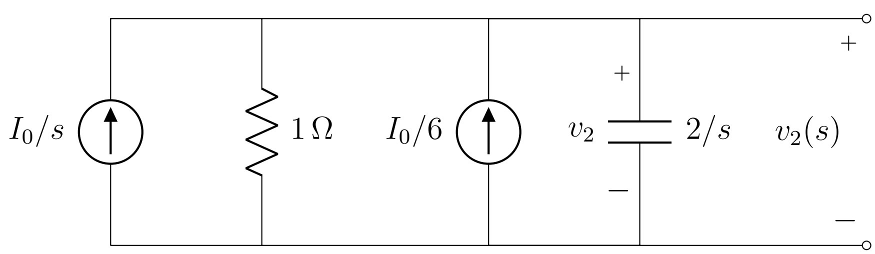
To determine the initial condition ($v_2(0^-)$), we need to analyze the circuit for $t < 0$. The circuit for $t < 0$ is given below.



Here, in steady state, capacitor is open and the voltage across it is the magnitude of current times equivalent resistance across capacitor.

$v_2 = I_0 \times 1/3 = I_0/3$

For $t > 0$, s domain representation of the circuit is as given below.



By applying KCL at v_2,

$$V_2(s)\left[1+\frac{s}{2}\right] = \frac{I_0}{s} + \frac{I_0}{6}$$

$$V_2(s) = \left[\frac{6+s}{6s}\right]\left[\frac{1}{2+s}\right]$$

$$V_2(s) = \left[\frac{6+s}{3s(2+s)}\right]$$

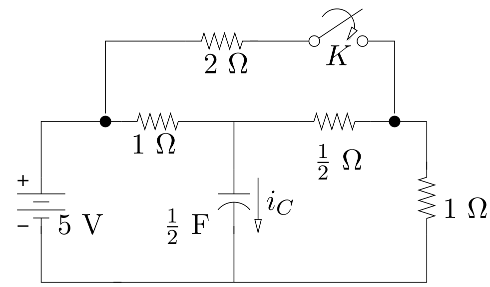
By doing partial fraction,

$$V_2(s) = I_0[\frac{1}{s} - \frac{2}{3(s+2)}]$$

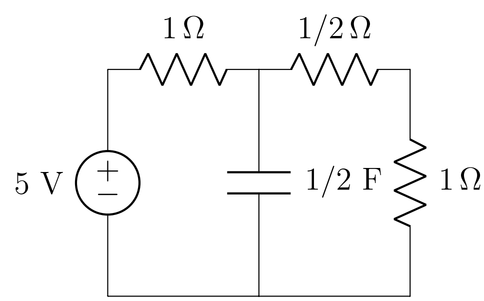
Now take the inverse Laplace transform to get

$$v_2(t) = I_0[1 - \frac{2}{3}e^{-2t}]u(t)$$

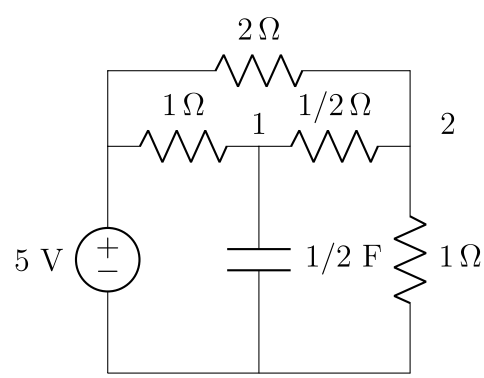
8. The network shown in the figure is in a steady state with the switch K open. At $t = 0$, the switch is closed. Find the current in the capacitor $i_C(t)$ for $t > 0$, sketch this waveform and determine the time constant.



To determine the initial condition ($v_c(0^-)$), we need to analyze the circuit for $t < 0$. The circuit for $t < 0$ is as given below.



Upon analysis we get $v_c(0^-) = v_c(0^+) = 3V$



Using nodal analysis at nodes 1 and 2 to obtain V_1 and V_2,we get

$$-2V_1 + 3.5V_2 = 2.5 - - - - (1)$$

$$3V_1 - 2V_2 + \frac{1}{2}\frac{dV_1}{dt} = 5 - - - - (2)$$

Replacing V_2 from equation (1) in (2), we get,

$$\frac{dV_1}{dt} + \frac{13}{3.5}V_1 = \frac{90}{7}$$

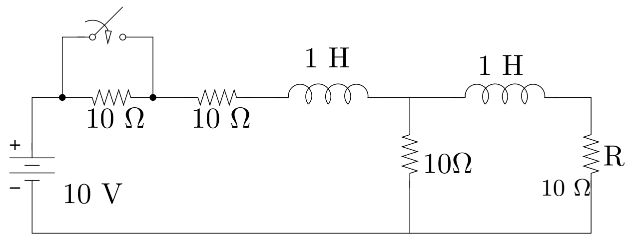
By solving the differential equation using the initial conditions, we get,

$$V_1 = -0.46 * e^{-\frac{26}{7}t} - 3.46$$

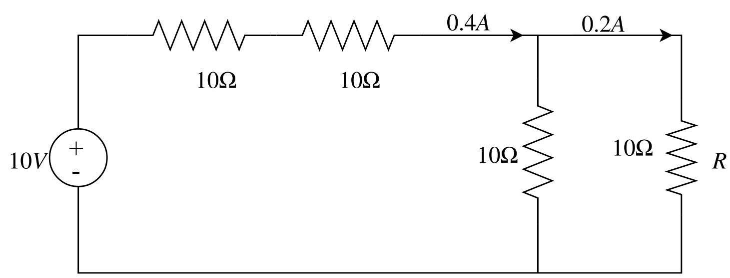
So current through the capacitor is

$$i_c(t) = C\frac{dV_1}{dt} = -0.85 * e^{-\frac{26}{7}t}$$

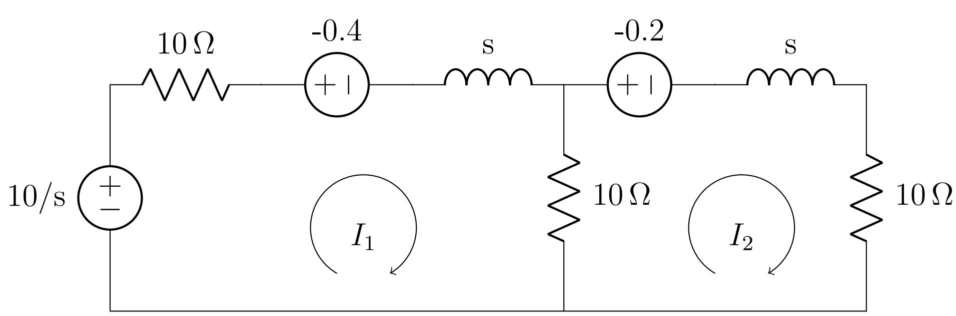
9. For the network shown below which is initially in steady-state, at time $t = 0$ the switch shorts the 10 Ω resistance. Find the current through the resistor R for $t \geq 0$ and sketch it.



For $t \leq 0$ (in steady state)



For $t > 0$
Representing the above circuit in 's' domain



using KVL, we get

$$\begin{aligned} (20 + s)\, I_1 - 10\, I_2 &= 0.4 + \frac{10}{s} \\ -10\, I_1 + (20 + s)\, I_2 &= 0.2 \end{aligned}$$

Solving above equations, we get

$$I_2 \;=\; \frac{1}{3} * \frac{1}{s} \;-\; \frac{1}{5} \;*\; \frac{1}{s+10} \;+\; \frac{1}{15} * \frac{1}{s+30}$$

Taking Inverse Laplace Transform

$$i_2(t) \;=\; \frac{1}{3} * u(t) \;-\; \frac{1}{5} \;*\; e^{-10t}u(t) \;+\; \frac{1}{15} * e^{-30t}u(t)$$

So, $i_2(t)$ will be obtained as

$$i_2(t) \;=\; [\frac{1}{3} \;-\; \frac{1}{5} \;*\; e^{-10t} \;+\; \frac{1}{15} * e^{-30t}]u(t)$$

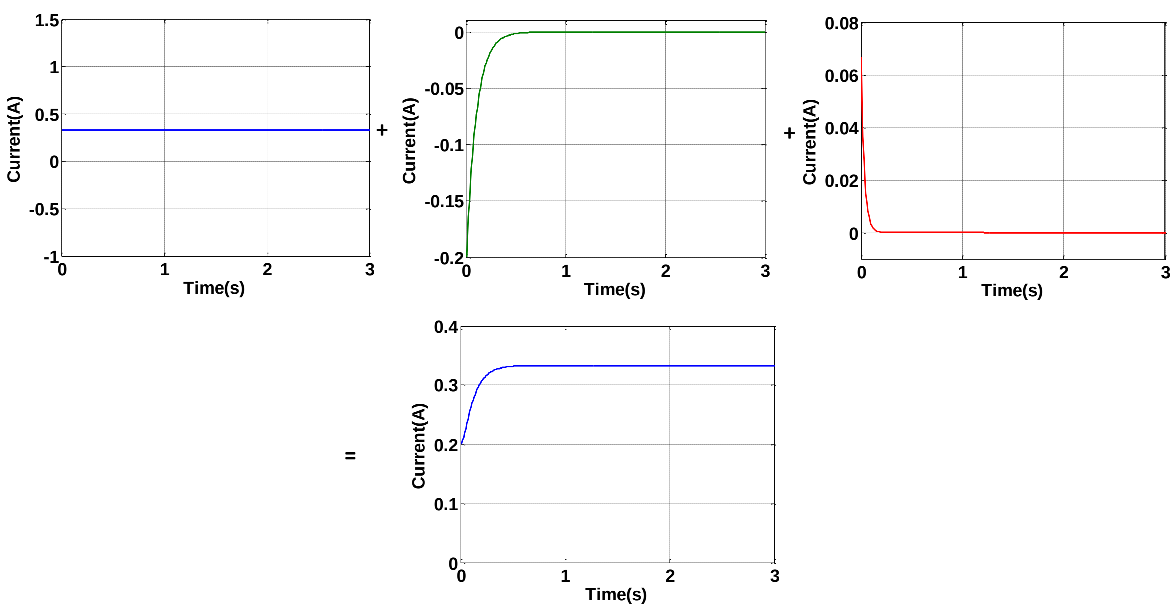
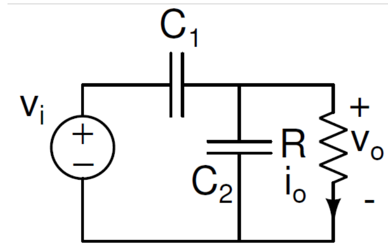


Figure 1: Three individual parts $i_2(t)$:$\frac{1}{3}$,$\frac{1}{5}$ $*$ e^{-10t} and $\frac{1}{15} * e^{-30t}$ and total $i_2(t)$

10. Evaluate and sketch the step response for the following circuits using time domain techniques. Write the differential equation and find the natural and forced response. Initial conditions are zero.

    1. All plots must be roughly to scale
    2. Key x and y axis values must be marked
    3. Time constant must be shown

    a.



applying KCL

$$i_0 = \frac{v_0}{R}$$

$$C_1\frac{d(v_i)}{dt} = (C_1 + C_2)\frac{dv_0}{dt} + \frac{v_0}{R}$$

Natural Response:

$$(C_1 + C_2)\frac{dv_0}{dt} + \frac{v_0}{R} = 0$$

Using separation of variable and integrating on both side, we obtain,

$$v_0 = ae^{\frac{-t}{(C_1+C_2)R}}$$

Forced Response:

$$C_1\frac{d(v_i)}{dt} = (C_1 + C_2)\frac{dv_0}{dt} + \frac{v_0}{R}$$

Since,

$$\frac{dV_i}{dt} = \delta t$$

So,

$$(C_1 + C_2)\frac{dv_0}{dt} + \frac{v_0}{R} = C_1\delta t$$

From this at steady state,

$$v_0 = 0$$

Therefore,
unit step response=forced response + natural response

$$= ae^{\frac{-t}{(C_1+C_2)R}}$$

after initial charge redistribution:

$$v_0(0^+) = \frac{C_1 v_i}{C_1 + C_2}$$
$$a = \frac{C_1}{C_1 + C_2}$$

$v_0 = \frac{C_1}{C_1+C_2} e^{\frac{-t}{(C_1+C_2)R}} u(t)$
$i_0 = \frac{C_1}{R(C_1+C_2)} e^{\frac{-t}{(C_1+C_2)R}} u(t)$

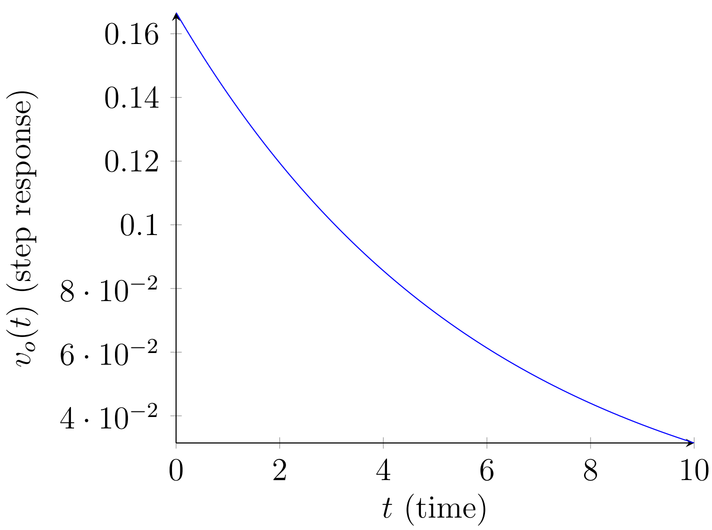
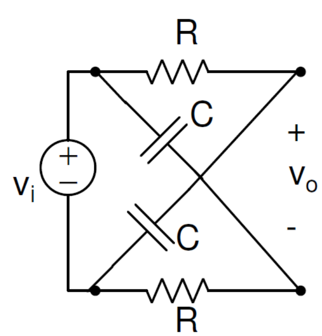
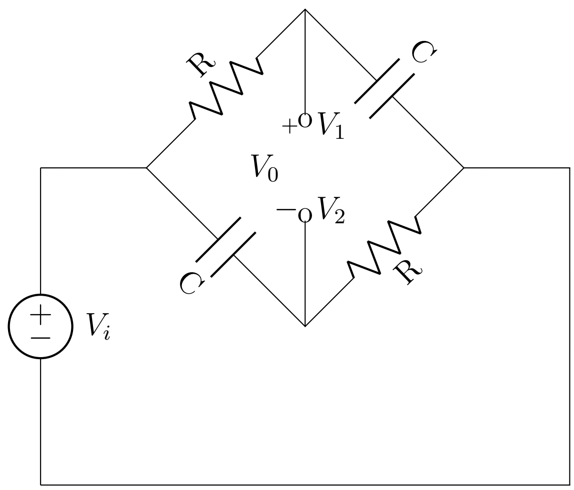


Figure 2: Step Response. $C_1 = 1F, C_2 = 5F, R = 1\Omega$

b.



The simplified circuit diagram is shown below



Now, writing the nodal equations at nodes 1 and 2, we get

$$C\frac{dV_1}{dt} = \frac{V_i - V_1}{R} - - - - (1)$$
$$C\frac{d(V_i - V_2)}{dt} = \frac{V_2}{R} - - - - (2)$$

And,

$$V_0 = V_1 - V_2$$

Adding 1 and 2,

$$\frac{dV_0}{dt} + \frac{V_0}{RC} = -\frac{dV_i}{dt} + \frac{V_i}{RC}$$

For natural response, $V_i = 0$

Therefore,

$$-RC\frac{dV_0}{dt} = V_0$$

$$V_0 = ae^{\frac{-t}{RC}}$$

For forced response, $V_i = u(t)$ Therefore

$$\frac{dV_0}{dt} + \frac{V_0}{RC} = -\delta(t) + \frac{u(t)}{RC}$$

At steady state,

The capacitors are fully charged. Therefore, as visible from the figure,

$$v_0(t) = v_i(t) = u(t)$$

Therefore,

Total response = natural response + forced response

$$V_0(t) = (1 + ae^{\frac{-t}{RC}})u(t)$$

At t = 0, with the initial conditions of the capacitor being zero,

$$V_0(t) = -V_i(t) = -1$$

Using the above condition, we get, $a = -2$

$$V_0(t) = (1 - 2e^{\frac{-t}{RC}})u(t)$$

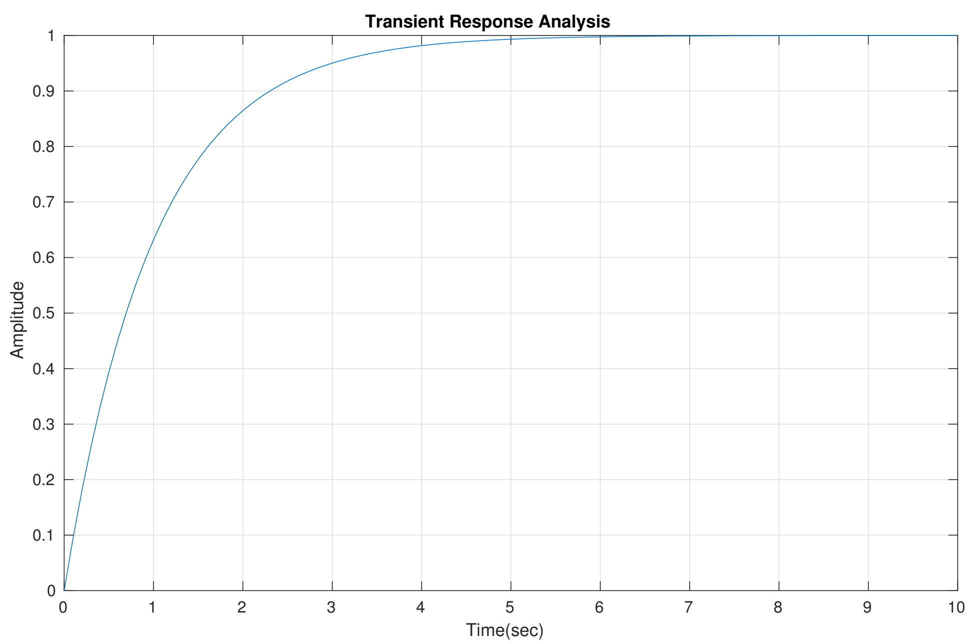
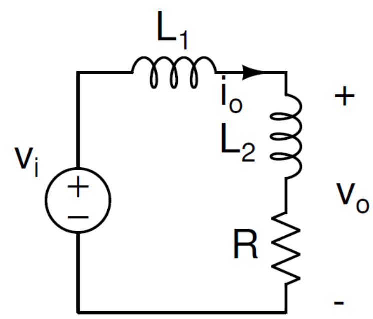


Figure 3: Voltage v(t), considering RC=1

c.



By applying KVL,

$$\begin{aligned} V_i &= L_1\frac{di_o}{dt} + L_2\frac{di_o}{dt} + i_oR \\ &= (L_1 + L_2)\frac{di_o}{dt} + i_oR \end{aligned}$$

$$\frac{di_o}{dt} + \frac{i_o}{\frac{(L_1+L_2)}{R}} = \frac{V_i(t)}{L_1 + L_2}$$

By substituting t = 0

$$A = i_0(0) - \frac{V_i(t)}{R}$$

Given that I.C.s are zero, we get:

$$\begin{aligned} i_o(t) - \frac{V_i(t)}{R} &= (0 - \frac{V_i(t)}{R})e^{-\frac{t*R}{L_1+L_2}} \\ i_o(t) &= \frac{V_i(t)}{R}(1 - e^{-\frac{t*R}{L_1+L_2}}) \end{aligned}$$

For Unit step response, input is a unit step

$$V_i(t) = u(t)$$

$$i_o(t) = \frac{u(t)}{R} - \frac{u(t)}{R}e^{-\frac{t*R}{L_1+L_2}}$$

Natural Response:

$$i_o(t) = -\frac{u(t)}{R}e^{-\frac{t*R}{L_1+L_2}}$$

Forced Response:

$$i_o(t) = \frac{u(t)}{R}$$

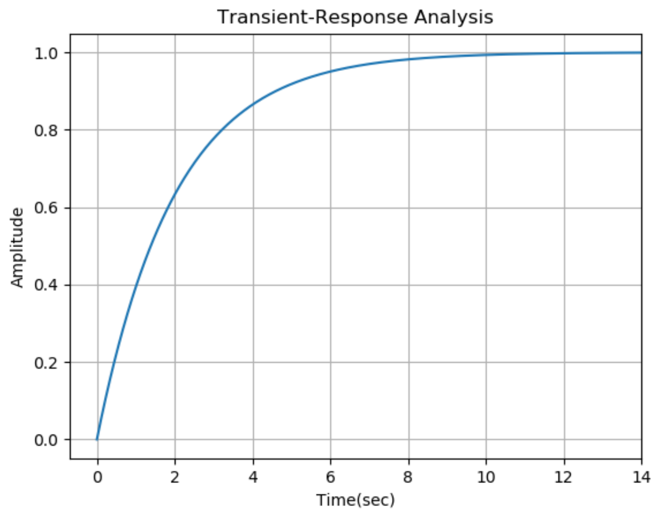


Figure 4: Current i(t), considering R=1 Ohm,L1=1H,L2=1H and V˙i=1

By KVL :

$$V_o(t) = V_i(t) - L_1\frac{di_o}{dt}$$
$$= V_i(t) - L_1\left(\frac{V_i(t)}{L_1+L_2} - \frac{t}{\tau}\right)$$
$$\text{where } \tau = \frac{L_1+L_2}{R}$$
$$= V_i(t)\left(1 - \frac{L_1}{L_1+L_2}e^{(}\frac{-t}{\tau})\right)$$

Unit Step Response:

$$V_i(t) = u(t)$$
$$V_o(t) = \left(1 - \frac{L_1}{L_1+L_2}e^{\frac{-t}{\tau}}\right)u(t)$$

Natural Response:

$$V_o(t) = -\frac{L_1}{L_1+L_2}e^{\frac{-t}{\tau}}u(t)$$

Forced Response:

$$V_o(t) = u(t)$$

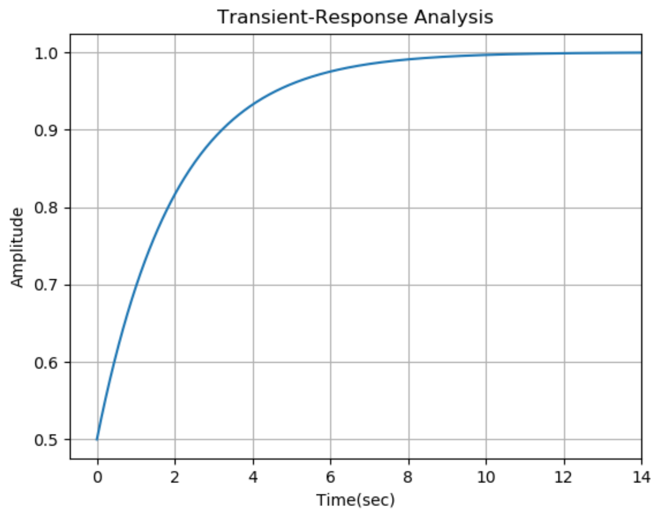
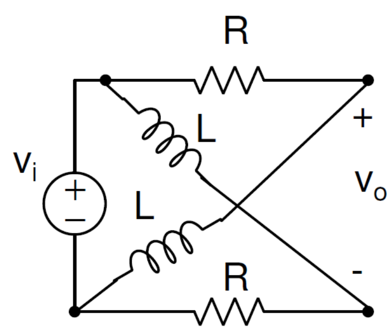
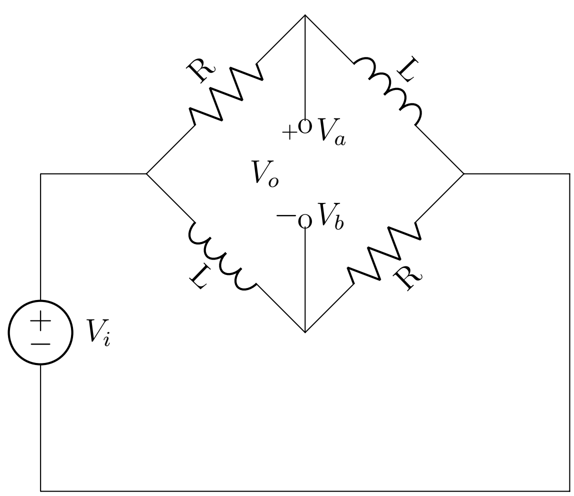


Figure 5: Voltage v(t), considering R=1 Ohm,L1=1H,L2=1H and V˙i=1

d.



The simplified circuit diagram is shown below



Now inspecting at nodes a and b,

$$V_o = V_a - V_b$$
$$V_a = V_i(t) - i_o R$$
$$V_b = V_i(t) - L\frac{di_o}{dt}$$
$$V_i = i_o R + L\frac{di_o}{dt}$$

By re-arranging the above eqn

$$\frac{di_o}{dt} + \frac{i_o R}{L} = \frac{V_i}{L}$$

let

$$i_i = i_o - \frac{V_i}{R}$$
$$\frac{di_1}{dt} + \frac{i_1 R}{L} = 0$$
$$i_1(t) = i_1(0)e^{-\frac{t}{\tau}}$$

$$\text{where } \tau = \frac{L}{R}$$

$$\begin{aligned} i_0(t) &= \frac{1}{R}(1 - e^{-\frac{t}{\tau}})V_i(t) \\ V_o &= L\frac{di_o}{dt} - i_o R \\ &= V_i(t) - 2i_o R \\ &= V_i(t) - 2R(\frac{1}{R}(1 - e^{-\frac{t}{\tau}})V_i(t)) \\ V_o(t) &= -u(t) + 2e^{-\frac{t}{\tau}}u(t) \end{aligned}$$

Unit Step Response:

$$V_o(t) = -u(t) + 2e^{-\frac{t}{\tau}}u(t)$$
$$i_0(t) = \frac{1}{R}(1 - e^{-\frac{t}{\tau}})u(t)$$

Natural Response:

$$V_o(t) = 2e^{-\frac{t}{\tau}}u(t)$$
$$i_0(t) = -\frac{1}{R}e^{-\frac{t}{\tau}}u(t)$$

Forced Response:

$$V_o(t) = -u(t)$$
$$i_0(t) = \frac{1}{R}u(t)$$

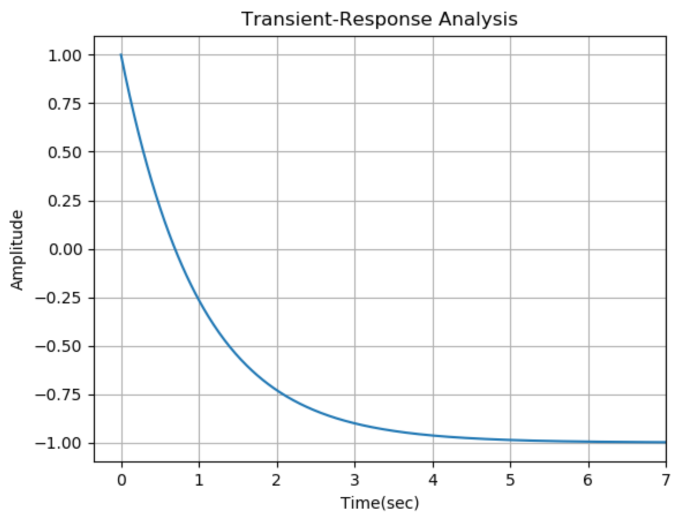


Figure 6: Voltage v(t), considering R=1 Ohm, L=1H and V˙i=1

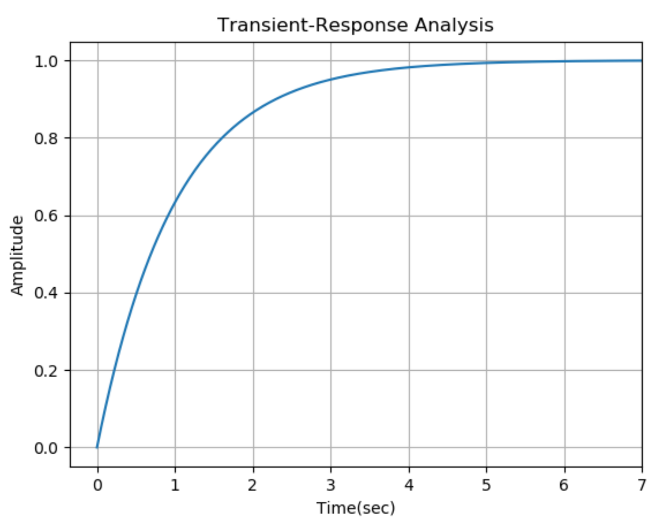
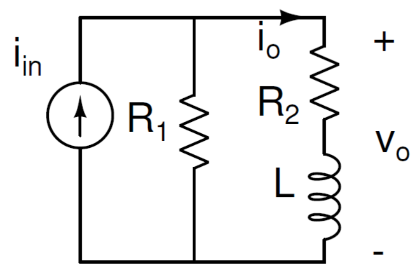


Figure 7: Current i(t), considering R=1 Ohm, L=1H and V˙i=1

e.



Applying KVL,

$$(i_{in} - i_o)R_1 = i_o R_2 + L\frac{di_o}{dt}$$
$$L\frac{di_o}{dt} + i_o(R_1 + R_2) = i_{in}R_1$$
$$\frac{di_o}{dt} + i_o\frac{R_1 + R_2}{L} = i_i n\frac{R_1}{L}$$

By making input zero:

$$\frac{di_o}{dt} + i_o\frac{R_1 + R_2}{L} = 0$$
$$\int \frac{di_o}{i_o} = -\int \frac{R_1 + R_2}{L}dt$$
$$i_o = K_1 e^{-\frac{R_1+R_2}{L}t}$$

when input is present:

$$\frac{di_o}{dt} + i_o\frac{R_1 + R_2}{L} = i_i n\frac{R_1}{L}$$

The above differential equation is of the form:

$$\frac{dy}{dt} + Py = Q$$

Therefore,

$$i_o e^{\frac{R_1+R_2}{L}t} = \int \frac{i_{in}R_1}{L}e^{\frac{R_1+R_2}{L}t}dt$$
$$i_o e^{\frac{R_1+R_2}{L}t} = \frac{i_{in}R_1}{R_1 + R_2}e^{\frac{R_1+R_2}{L}t}$$

Combining both:

$$
\begin{aligned}
i_o e^{\frac{R_1+R_2}{L}t} &= \int \frac{i_{in}R_1}{L} e^{\frac{R_1+R_2}{L}t} dt \\
i_o &= \frac{i_{in}R_1}{R_1+R_2} + K_1 e^{-\frac{R_1+R_2}{L}t}
\end{aligned}
$$

Given initial conditions are zero,

$$
\begin{aligned}
K_1 &= -i_{in}\frac{R_1}{R_1+R_2} \\
i_o &= i_{in}\frac{R_1}{R_1+R_2}(1-e^{-\frac{R_1+R_2}{L}t})
\end{aligned}
$$

Unit step response = Natural Response + Forced Response

$$
\begin{aligned}
i_o &= i_{in}\frac{R_1}{R_1+R_2}(1-e^{-\frac{R_1+R_2}{L}t}) \\
i_o &= \frac{R_1}{R_1+R_2}(1-e^{-\frac{R_1+R_2}{L}t})u(t) \\
V_o &= i_o R_2 + L\frac{di_o}{dt} \\
V_o &= i_{in}\frac{R_1R_2}{R_1+R_2}(1-e^{-\frac{R_1+R_2}{L}t}) + i_{in}R_1 e^{-\frac{R_1+R_2}{L}t} \\
V_o &= i_{in}\frac{R_1R_2}{R_1+R_2} + i_{in}R_1^2 e^{-\frac{R_1+R_2}{L}t} \\
V_o &= (\frac{R_1R_2}{R_1+R_2} + \frac{R_1^2}{R_1+R_2}e^{-\frac{R_1+R_2}{L}t})u(t)
\end{aligned}
$$

Natural Response:

$$
\begin{aligned}
(i_o)_n &= -\frac{R_1}{R_1+R_2}e^{-\frac{R_1+R_2}{L}t}u(t) \\
(V_o)_n &= \frac{R_1^2}{R_1+R_2}e^{-\frac{R_1+R_2}{L}t})u(t)
\end{aligned}
$$

Forced Response:

$$
\begin{aligned}
(i_o)_f &= \frac{R_1}{R_1+R_2}u(t) \\
(V_o)_f &= \frac{R_1R_2}{R_1+R_2}u(t)
\end{aligned}
$$

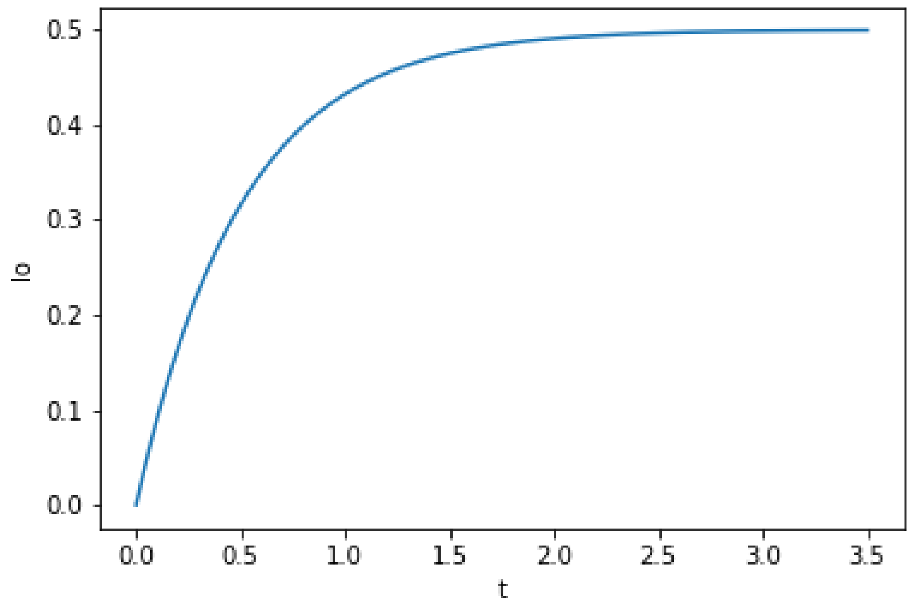


Figure 8: I(t) vs t ,considering R1=1,R2=1 L=1 and I˙in=1

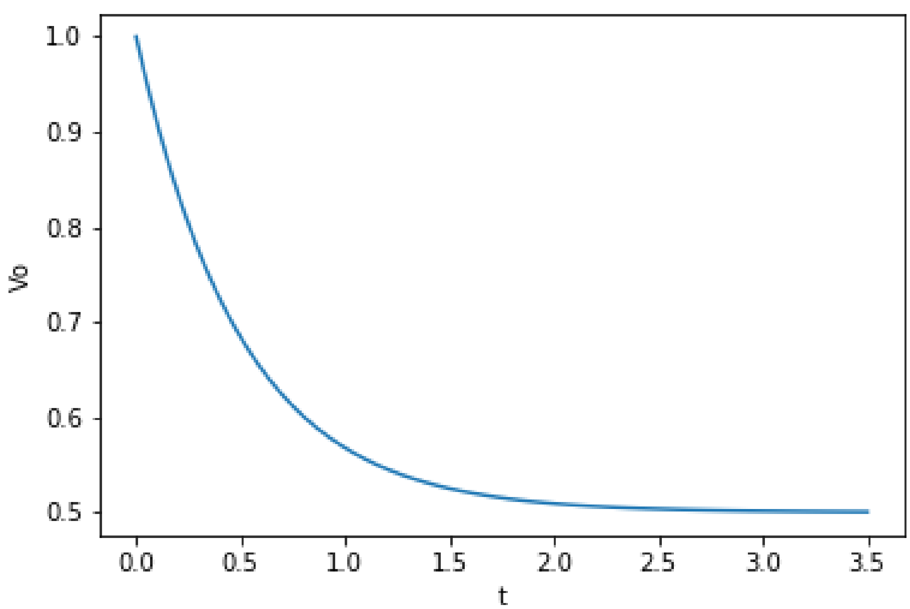


Figure 9: V(t) vs t ,considering R1=1,R2=1 L=1 and I˙in=1

11. (a) For the above circuits, find the impulse response by differentiating the step response
    (b) Verify your answer by finding the inverse Laplace transform of the transfer function.

(c) In each circuit, find the poles and zeros and plot it in the compex frequency plane.

a.

impulse response=$\frac{d(stepresponse)}{dt}$
impulse response of v_0:

$$h_0(t) = \frac{C_1}{C_1 + C_2}(\delta(t) - \frac{e^{\frac{-t}{(C_1+C_2)R}}u(t)}{(C_1 + C_2)R})$$

impulse response of i_0:

$$h_i(t) = \frac{C_1}{(C_1 + C_2)R}(\delta(t) - \frac{e^{\frac{-t}{(C_1+C_2)R}}u(t)}{(C_1 + C_2)R})$$

transfer function=s*step response
transfer function for v_0 vs v_i:

$$v_0(s) = \frac{C_1}{C_1 + C_2}\frac{s}{(s + \frac{1}{(C_1+C_2)R})}$$

transfer function for i_0 vs v_i:

$$i_0(s) = \frac{C_1}{(C_1 + C_2)R}\frac{s}{(s + \frac{1}{(C_1+C_2)R})}$$

Inverse Laplace transform of transfer function is Impulse response

poles and zeros for transfer function for v_0/v_i:

$$p = \frac{-1}{(C_1 + C_2)R}$$
$$z = 0$$

poles and zeros for transfer function for i_0/v_i:

$$p = \frac{-1}{(C_1 + C_2)R}$$
$$z = 0$$

b. Impulse Response:

$$h(t) = -\delta(t) + \frac{2}{RC}e^{-t/RC}u(t)$$

Transfer Function:

$$H(s) = \frac{1 - sRC}{1 + sRC}$$

Taking inverse laplace,

$$h(t) = -\delta(t) + \frac{2}{RC}e^{-t/RC}u(t)$$

Poles and zeros of transfer function:

$$p = -\frac{1}{RC}$$
$$z = \frac{1}{RC}$$

c. Impulse Response for i(o):

$$\begin{aligned} h_i(t) &= (\frac{1}{R} - \frac{e^{\frac{-t}{\tau}}}{R})\delta(t) + \frac{u(t)e^{\frac{-t}{\tau}}}{L_1 + L_2} \\ &= \frac{1}{L_1 + L_2}e^{-t\frac{R}{L_1+L_2}}u(t) \end{aligned}$$

Impulse Response for V(o):

$$h_vo(t) = \delta(t)(1 - \frac{L_1}{L_1 + L_2}e^{\frac{-t}{\tau}}) + \frac{L_1}{L_1 + L_2}\frac{1}{\tau}e^{\frac{-t}{\tau}}u(t)$$

Transfer Function for i(o) vs V(in):

$$\begin{aligned} \frac{I_o(s)}{V_{in}(s)} &= \frac{1}{R + s(L_1 + L_2)} \\ v_i(t) = \delta(t) \Rightarrow V_in(s) &= 1 \\ I_o(s) &= \frac{1}{R + s(L_1 + L_2)} \end{aligned}$$

Inverse Laplace Transform of Io(s):

$$h_i(t) = \frac{1}{L_1 + L_2}e^{-t\frac{R}{L_1+L_2}}u(t)$$

Transfer Function for V(o) vs V(in):

$$\begin{aligned} \frac{V_o(s)}{V_{in}(s)} &= \frac{R + sL_2}{R + s(L_1 + L_2)} \\ v_i(t) = \delta(t) \Rightarrow V_in(s) &= 1 \\ V_o(s) &= \frac{R + sL_2}{R + s(L_1 + L_2)} \end{aligned}$$

Inverse Laplace Transform of Vo(s):

$$h_o(t) = (1 - \frac{L_1}{L_1 + L_2} e^{\frac{-t}{\tau}})\delta(t) + \frac{L_1}{L_1 + L_2} \frac{1}{\tau} e^{\frac{-t}{\tau}} u(t)$$

Poles and zeros of transfer function for Io(s)/Vin(s):

$$p = -\frac{R}{L_1 + L_2}$$

$$\text{no zeros}$$

Poles and zeros of transfer function for Vo(s)/Vin(s):

$$p = -\frac{R}{L_1 + L_2}$$
$$z = -\frac{R}{L_2}$$

d. Impulse response by differentiating step response:

$$h_i(t) = \frac{1}{L} e^{-\frac{t}{\tau}} u(t) \text{ where } \tau = \frac{L}{R}$$
$$h_v(t) = \delta(t) - 2\frac{R}{L} e^{-\frac{t}{\tau}} u(t)$$

Transfer Function:

$$\frac{V_a(s)}{V_i(s)} = \frac{sL}{R + sL}$$
$$\frac{V_b(s)}{V_i(s)} = \frac{R}{R + sL}$$
$$\frac{V_o(s)}{V_i(s)} = \frac{sL - R}{R + sL}$$
$$zero = \frac{R}{L}; pole = -\frac{R}{L}$$
$$\frac{I_o(s)}{V_i(s)} = \frac{1}{R + sL}$$
$$pole = -\frac{R}{L}$$

Impulse response by taking inverse Laplace Transform :

$$V_o(s) = 1 - \frac{2R}{sL+R}$$
$$Vo(t) = \delta(t) - \frac{2R}{L}e^{-\frac{t}{\tau}}u(t) \text{ where, } \tau = \frac{L}{R}$$
$$I_o(s) = \frac{1}{L}\frac{1}{s+\frac{R}{L}}$$
$$i_o(t) = \frac{1}{L}e^{-\frac{t}{\tau}}u(t) \text{ where, } \tau = \frac{L}{R}$$

e. Impulse Response for i(o):

$$h_i(t) = \frac{R_1}{L}e^{-\frac{R_1+R_2}{L}t}u(t)$$

Impulse Response for V(o):

$$h_o(t) = (\frac{R_1R_2}{R_1+R_2} + \frac{R_1^2}{R_1+R_2})\delta(t) - \frac{R_1^2}{L}e^{-\frac{R_1+R_2}{L}t}u(t)$$
$$h_o(t) = R_1\delta(t) - \frac{R_1^2}{L}e^{-\frac{R_1+R_2}{L}t}u(t)$$

Transfer Function for i(o) vs i(in):

$$\frac{I_o(s)}{I_{in}(s)} = \frac{R_1}{L}(\frac{1}{s+\frac{R_1+R_2}{L}})$$

Inverse Laplace Transform:

$$h_i(t) = \frac{R_1}{L}e^{-\frac{R_1+R_2}{L}t}u(t)$$

Transfer Function for V(o) vs i(in):

$$\frac{V_o(s)}{I_{in}(s)} = R_1(\frac{s+\frac{R_2}{L}}{s+\frac{R_1+R_2}{L}})$$
$$\frac{V_o(s)}{I_{in}(s)} = R_1(\frac{s+\frac{R_1+R_2}{L}}{s+\frac{R_1+R_2}{L}} - \frac{\frac{R_1}{L}}{s+\frac{R_1+R_2}{L}})$$
$$\frac{V_o(s)}{I_{in}(s)} = R_1(1 - \frac{\frac{R_1}{L}}{s+\frac{R_1+R_2}{L}})$$

Inverse Laplace Transform:

$$h_o(t) = R_1\delta(t) - \frac{R_1^2}{L}e^{-\frac{R_1+R_2}{L}t}u(t)$$

Poles and zeros of transfer function for I˙o/I˙in:

$$p = -\frac{R_1 + R_2}{L}$$

Poles and zeros of transfer function for V˙o/I˙in:

$$p = -\frac{R_1 + R_2}{L}$$
$$z = -\frac{R_2}{L}$$

12. In the above questions find the steady state response by open circuiting the capacitor/short circuiting inductors. How are the values obtained related to the $v_o(t)$ and $i_o(t)$ you calculated earlier.

a. Steady state values:

$$v_0(\infty) = 0$$
$$i_0(\infty) = 0$$

b. Steady state values:

$$i_o(\infty) = 0$$
$$v_o(\infty) = 1$$

c. Steady state values:

$$i_o(\infty) = \frac{1}{R}$$
$$V_o(\infty) = 1$$

d. Steady state values: By short circuiting L;

$$V_o(\infty) = -V_i$$
$$i_o(\infty) = -\frac{V_i}{R}$$

e. Steady state values:

$$i_o(\infty) = i_{in}\frac{R_1}{R_1 + R_2}$$
$$V_o(\infty) = i_{in}\frac{R_1 R_2}{R_1 + R_2}$$

13. Using the initial and final value theorem, find $v_o(0^+)$ and $v_o(\infty)$. Use the transfer function you evaluated in question 11. and the Laplace transform of the input.

a. Initial value theorem:

$$v_0(0^+) = lim_{s->\infty} s v_0(s)$$
$$v_0(0^+) = lim_{s->\infty}\frac{C_1}{C_1 + C_2}\frac{s}{(s + \frac{1}{(C_1+C_2)R})}$$
$$v_0(0^+) = \frac{C_1}{C_1 + C_2}$$
$$i_0(0^+) = lim_{s->\infty} s i_0(s)$$
$$i_0(0^+) = lim_{s->\infty}\frac{C_1}{R(C_1 + C_2)}\frac{s}{(s + \frac{1}{(C_1+C_2)R})}$$
$$i_0(0^+) = \frac{C_1}{R(C_1 + C_2)}$$

Final value theorem:

$$v_0(\infty) = lim_{s->0} s v_0(s)$$
$$v_0(\infty) = lim_{s->0}\frac{C_1}{C_1 + C_2}\frac{s}{(s + \frac{1}{(C_1+C_2)R})}$$
$$v_0(\infty) = 0$$
$$i_0(\infty) = lim_{s->0} s i_0(s)$$
$$i_0(\infty) = lim_{s->0}\frac{C_1}{R(C_1 + C_2)}\frac{s}{(s + \frac{1}{(C_1+C_2)R})}$$
$$i_0(\infty) = 0$$

b. Initial Value theorem:

$$V_o(0) = lim_{s->\infty} s V_o(s)$$
$$v_o(0) = 0$$

Final Value theorem:

$$v_o(\infty) = lim_{s->0} s V_o(s)$$
$$v_o(\infty) = 1$$

c. Initial Value theorem:

$$\begin{aligned} V_o(0^+) &= lim_{s->\infty} s(V_o(s)V_i(s)) \\ &= \frac{L_2}{L_1 + L_2} \\ I_o(0^+) &= lim_{s->\infty} s(I_o(s)V_i(s)) \\ &= 0 \end{aligned}$$

Final Value theorem:

$$\begin{aligned} V_o(\infty) &= lim_{s->0} s(V_o(s)V_i(s)) \\ V_o(\infty) &= 1 \\ I_o(\infty) &= lim_{s->0} s(I_o(s)V_i(s)) \\ &= \frac{1}{R} \end{aligned}$$

d. Initial Value theorem:

$$\begin{aligned} V_o(0^+) &= lim_{s->\infty} \frac{sL - R}{R + sL} \\ &= 1 \\ I_o(0^+) &= lim_{s->\infty} s(I_o(s)V_i(s)) \\ &= 0 \end{aligned}$$

Final Value theorem:

$$\begin{aligned} V_o(\infty) &= lim_{s->0} \frac{sL - R}{R + sL} \\ &= -1 \\ I_o(\infty) &= lim_{s->0} s(I_o(s)V_i(s)) \\ &= \frac{1}{R + sL} \\ &= \frac{1}{R} \end{aligned}$$

e.

Initial Value theorem:

$$i_o(0) = lim_{s->\infty} sI_o(s)$$
$$i_o(0) = lim_{s->\infty} s\frac{R_1}{L}(\frac{1}{s(s+\frac{R_1+R_2}{L})})$$
$$i_o(0) = 0$$
$$V_o(0) = lim_{s->\infty} sR_1(\frac{s+\frac{R_2}{L}}{s(s+\frac{R_1+R_2}{L})})$$
$$V_o(0) = R_1$$

Final Value theorem:

$$i_o(\infty) = lim_{s->0} sI_o(s)$$
$$i_o(\infty) = lim_{s->0} s\frac{R_1}{L}(\frac{1}{s(s+\frac{R_1+R_2}{L})})$$
$$i_o(\infty) = \frac{R_1}{R_1+R_2}$$
$$V_o(\infty) = lim_{s->0} sR_1(\frac{s+\frac{R_2}{L}}{s(s+\frac{R_1+R_2}{L})})$$
$$V_o(\infty) = \frac{R_1R_2}{R_1+R_2}$$