EE1101: Signals and Systems JAN — MAY 2019

Tutorial 7 Solutions

Solution 1

We use the analysis equation to find the Fourier transform of signals $x_1(t)$ and $x_2(t)$.

a) Here,

$$x_1(t) = \begin{cases} 4, & 0 < t < 1 \\ 2, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} X_1(j\omega) &= \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt \\ &= \int_0^1 4e^{-j\omega t} dt + \int_1^2 2e^{-j\omega t} dt \\ &= \frac{4(e^{-j\omega} - 1)}{-j\omega} + \frac{2(e^{-j2\omega} - e^{-j\omega})}{-j\omega} \\ &= 4e^{-j\omega/2} \left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega} \right) + \\ &\quad 2e^{-j3\omega/2} \left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega} \right) \\ &= 4\frac{\sin(\pi(\omega/2\pi))}{\pi(\omega/2\pi)} e^{-j\omega/2} + \\ &\quad 2\frac{\sin(\pi(\omega/2\pi))}{\pi(\omega/2\pi)} e^{-j3\omega/2} \\ &= 4\,\mathrm{sinc}(\omega/2\pi) e^{-j\omega/2} + 2\,\mathrm{sinc}(\omega/2\pi) e^{-j3\omega/2} \end{aligned}$$

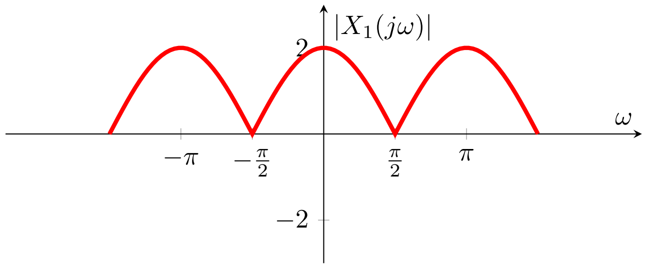
b) $x(t) = \begin{cases} \frac{|t|}{\tau} & |t| \le \tau \\ 0 & |t| > \tau \end{cases}$

$$\begin{aligned} X_2(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-\tau}^{0} \frac{(-t)}{\tau} e^{-j\omega t} dt + \int_0^{\tau} \frac{t}{\tau} e^{-j\omega t} dt \\ &= \int_0^{\tau} \frac{t}{\tau} e^{j\omega t} dt + \int_0^{\tau} \frac{t}{\tau} e^{-j\omega t} dt \\ &= \frac{2}{\tau} \int_0^{\tau} t \cos \omega t dt \\ &= \frac{2}{\tau} \left[\frac{t \sin \omega t}{\omega} + \frac{\cos \omega t}{\omega^2} \right]_0^{\tau} \\ &= 2 \left[\frac{\tau \sin \omega \tau}{\omega \tau} + \frac{\cos \omega \tau - 1}{\omega^2} \right] \\ &= 2\tau\,\mathrm{sinc}(\omega\tau/\pi) - \tau\,\mathrm{sinc}^2(\omega\tau/2\pi) \end{aligned}$$

Solution 2

a) Let $x_1(t) = \delta(t+1) + \delta(t-1)$, then the Fourier transform is given by,

$$\begin{aligned} X_1(j\omega) &= \int_{-\infty}^{\infty} [\delta(t+1) + \delta(t-1)]\, e^{-j\omega t} dt \\ &= e^{j\omega} + e^{-j\omega} = 2\cos\omega \end{aligned}$$

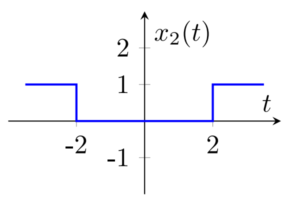


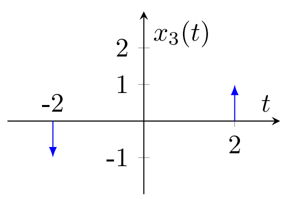
b)

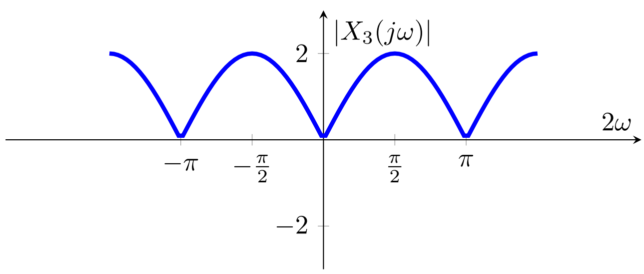
$$\begin{aligned} x_3(t) = \frac{dx_2(t)}{dt} &= \frac{d}{dt} [u(t-2) + u(-2-t)] \\ &= \frac{d}{dt} [u(t-2) + u(-(t+2))] \\ &= \delta(t-2) - \delta(-(t+2)) \\ &= \delta(t-2) - \delta(t+2) \end{aligned}$$

Therefore,

$$\begin{aligned} X_3(j\omega) &= \int_{-\infty}^{\infty} [\delta(t-2) - \delta(t+2)]\, e^{-j\omega t} dt \\ &= e^{-j2\omega} - e^{j2\omega} = -2j\sin(2\omega) \end{aligned}$$







Solution 3

a)

$$x(t) = e^{-\frac{|t|}{2}} = \begin{cases} e^{\frac{t}{2}}, & t < 0 \\ e^{-\frac{t}{2}}, & t \geq 0 \end{cases}$$

Fourier transform,

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{0} e^{\frac{t}{2}} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-\frac{t}{2}} e^{-j\omega t} dt \\ &= \frac{1}{(0.5 - j\omega)} \left[e^{(0.5 - j\omega)t} \right]_{-\infty}^{0} \\ &\quad - \frac{1}{(0.5 + j\omega)} \left[e^{-(0.5 + j\omega)t} \right]_{0}^{\infty} \\ &= \frac{1}{(0.5 - j\omega)} + \frac{1}{(0.5 + j\omega)} \\ X(j\omega) &= \frac{1}{0.25 + \omega^2} \end{aligned}$$

b) Given

$$x(t) = \sin(2\pi t) e^{-t} u(t) = \frac{(e^{j2\pi t} - e^{-j2\pi t})}{2j} e^{-t} u(t)$$

Method 1. Using the analysis equation,

$$\begin{aligned} X(j\omega) &= \int_{0}^{\infty} \frac{(e^{j2\pi t} - e^{-j2\pi t})}{2j} e^{-t} e^{-j\omega t} dt \\ &= \frac{1}{2j} \left\{ \int_{0}^{\infty} (e^{t(j2\pi - j\omega - 1)} dt - \int_{0}^{\infty} (e^{-t(j2\pi + j\omega + 1)} dt \right\} \\ &= \frac{1}{2j} \left\{ \left. \frac{(-e^{t(j2\pi - j\omega - 1)})}{(1 - j(2\pi - \omega))} \right|_{0}^{\infty} - \left. \frac{(-e^{-t(j2\pi + j\omega + 1)})}{(1 + j(2\pi + \omega))} \right|_{0}^{\infty} \right\} \\ &= \frac{1}{2j} \left\{ \frac{1 + j(2\pi + \omega) - 1 + j(2\pi - \omega)}{(1 + j\omega)^2 + 4\pi^2} \right\} \\ &= \frac{2\pi}{(1 + j\omega)^2 + 4\pi^2} \end{aligned}$$

Method 2. Using properties of FT,

We first find Fourier transform of $e^{-t}u(t)$

$$\begin{aligned} F(e^{-t}u(t)) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{0}^{\infty} e^{-t} e^{-j\omega t} dt = -\frac{1}{(1 + j\omega)} \left[e^{-(1 + j\omega)t} \right]_{0}^{\infty} \\ &= \frac{1}{(1 + j\omega)} \end{aligned}$$

By using Frequency shifting property, $x(t)e^{j\omega t} \longleftrightarrow X(j(\omega - \omega_0))$

Hence,

$$X(j\omega) = \frac{1}{2j} \left[\frac{1}{(1 + j(\omega - 2\pi))} - \frac{1}{(1 + j(\omega + 2\pi))} \right]$$

$$X(j\omega) = \frac{2\pi}{(1 + j\omega)^2 + 4\pi^2}$$

Solution 4

a) The $x(t)$ plot is as shown in figure 1.
Using the Fourier transform equation, we have

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(j\omega) = \int_{-1}^{1} \frac{(t+1)}{2} e^{-j\omega t} dt = \int_{-1}^{1} \left[\frac{te^{-jwt}}{2} + \frac{e^{-j\omega t}}{2} \right] dt$$

And integrate the first term by parts

$$\begin{aligned} &= \frac{1}{2} \left(\left[\frac{te^{-j\omega t}}{-j\omega} + \frac{e^{-j\omega t}}{\omega^2} \right]_{-1}^{1} + \left[\frac{e^{-jwt}}{-j\omega} \right]_{-1}^{1} \right) \\ &= \frac{e^{-j\omega}}{-j\omega} - \frac{\sin(\omega) 2j}{2\omega^2} \end{aligned}$$

$$X(j\omega) = \frac{e^{-j\omega}}{-j\omega} + \frac{\sin(\omega)}{j\omega^2}$$

b) Real part of $X(j\omega) = Real \left(\frac{e^{-j\omega}}{-j\omega} + \frac{\sin(\omega)}{j\omega^2} \right) = Real \left(\frac{1}{\omega} \sin(\omega) + \frac{j}{\omega} \cos(\omega) - \frac{j}{\omega^2} \sin(\omega) \right) = \frac{\sin \omega}{\omega}$

& Even part of $x(t)$ is given as $\Rightarrow \frac{x(t) + x(-t)}{2} = \frac{(t+1)/2 + (-t+1)/2}{2} = 0.5$, in the domain $t = [-1, 1]$.

The Even($x(t)$) plot is as shown in figure 2.

$$X(j\omega) = \int_{-1}^{1} 0.5 . e^{-j\omega t} d\omega = \frac{1}{2} \frac{e^{-j\omega t}|_{-1}^{1}}{-j\omega} = \frac{\sin \omega}{\omega}$$

And thus they are equal.

c) Odd part of $x(t)$ is given as $\Rightarrow \frac{x(t) - x(-t)}{2} = \frac{(t+1)/2 - (-t+1)/2}{2} = t/2$, in the domain $t = [-1, 1]$.
The Odd($x(t)$) plot is as shown in figure 3.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

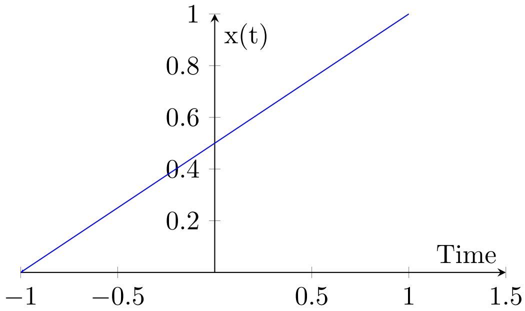
$$X(j\omega) = \int_{-1}^{1} \frac{t}{2} e^{-j\omega t} dt = \frac{-j \sin(\omega)}{\omega^2} + \frac{j \cos(\omega)}{\omega}$$

Property: The fourier transform of the odd part of $x(t)$ is the same as j times imaginary part of the answer to part (a). i.e

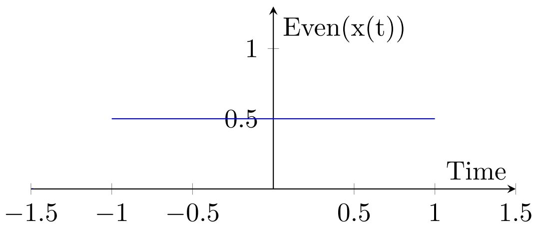
Let Y= $Im \left[\frac{e^{-j\omega}}{-j\omega} + \frac{\sin(\omega)}{j\omega^2} \right] = \frac{-\sin \omega}{\omega^2} + \frac{\cos \omega}{\omega}$

now $Y * j = \frac{-j \sin \omega}{\omega^2} + \frac{j \cos \omega}{\omega}$

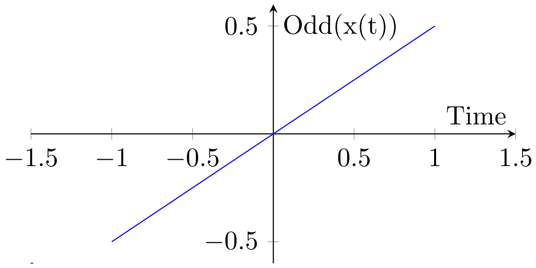
$x(t)$



$$\frac{x(t)+x(-t)}{2}$$



$$\frac{x(t)-x(-t)}{2}$$



Solution 5

a) $x(t) = \begin{cases} 1-|t|, & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$

F.T,

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-1}^{0} (1+t)e^{-j\omega t}dt + \int_{0}^{1} (1-t)e^{-j\omega t}dt \\
&= \int_{0}^{1} (1-t)e^{j\omega t}dt + \int_{0}^{1} (1-t)e^{-j\omega t}dt \\
&= 2\int_{0}^{1} (1-t)\cos\omega t dt \\
&= 2\left[\frac{(1-t)\sin\omega t}{\omega} - \frac{\cos\omega t}{\omega^2}\right]_0^1 \\
&= \frac{2}{\omega^2}(1-\cos\omega) = \frac{4\sin^2\left(\frac{\omega}{2}\right)}{\omega^2} \\
&= \operatorname{sinc}^2\frac{\omega}{2\pi}
\end{aligned}$$

b) Let $Let, y(t) = x(t+T) + x(t-T) = x_1(t) + x_2(t)$

F.T,

$$X_1(j\omega) = \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t+T)e^{-j\omega t}dt$$

Take $t + T = z \implies dt = dz$

$$X_1(j\omega) = \int_{-\infty}^{\infty} x(z)e^{-j\omega(z-T)}dz = X(j\omega)e^{j\omega T}$$

Similarly, $X_2(j\omega) = X(j\omega)e^{-j\omega T}$,

Therefore,

$$\begin{aligned}
Y(j\omega) &= X_1(j\omega) + X_2(j\omega) \\
&= 2X(j\omega)\left[\frac{e^{j\omega T} + e^{-j\omega T}}{2}\right] \\
&= 2X(j\omega)\cos(\omega T)
\end{aligned}$$

c)

$$y(t) = x(t+3) + x(t-3)$$

Therefore, using result from Qn.5(b),

$$Y(j\omega) = 2X(j\omega)\cos(3\omega)$$

Solution 6

a) If $x(t)$ is even in t, $x(t) = x(-t)$,

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} x(t)e^{-j\omega t}dt + \int_{0}^{\infty} x(t)e^{-j\omega t}dt
\end{aligned}$$

In first part replace $t \to -t \implies dt = -dt$

$$\begin{aligned}
&= \int_{\infty}^{0} x(-t)e^{j\omega t}(-dt) + \int_{0}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{0}^{\infty} x(-t)e^{j\omega t}dt + \int_{0}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{0}^{\infty} (x(-t)e^{j\omega t} + x(t)e^{-j\omega t})dt \\
&= \int_{0}^{\infty} (x(t)\left[e^{j\omega t} + e^{-j\omega t}\right])dt
\end{aligned}$$

$\because x(t) = x(-t)$

$$= 2\int_{0}^{\infty} x(t)\cos(\omega t)dt$$

b) If $x(t)$ is odd in t, $x(t) = -x(-t)$,

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} x(t)e^{-j\omega t}dt + \int_{0}^{\infty} x(t)e^{-j\omega t}dt \\
&\text{In first part replace } t \to -t \implies dt = -dt \\
&= \int_{\infty}^{0} x(-t)e^{j\omega t}(-dt) + \int_{0}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{0}^{\infty} x(-t)e^{j\omega t}dt + \int_{0}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{0}^{\infty} (x(-t)e^{j\omega t} + x(t)e^{-j\omega t})dt \\
&= \int_{0}^{\infty} (x(t)\left[-e^{j\omega t} + e^{-j\omega t}\right])dt \\
&\because x(t) = -x(-t) \\
&= -2j\int_{0}^{\infty} x(t)\sin(\omega t)dt
\end{aligned}
$$

Solution 7

a) The inverse fourier transform is

$x_1(t) = (1/2\pi)\int_{-\infty}^{\infty}[2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)]e^{j\omega t}d\omega$

$= (1/2\pi)[2\pi e^{j0t} + \pi e^{j4\pi t} + \pi e^{-j4\pi t}]$

$= 1 + (1/2)e^{j4\pi t} + (1/2)e^{-j4\pi t} = 1 + cos(4\pi t)$

b) $x_2(t) = (1/2\pi)\int_{-\infty}^{\infty} X_2(j\omega)e^{j\omega t}d\omega$

$= (1/2\pi)\int_0^2 2e^{j\omega t}d\omega + (1/2\pi)\int_{-2}^{0}(-2)e^{j\omega t}d\omega$

$= (e^{j2t} - 1)/(\pi jt) - (1 - e^{-j2t})/(\pi jt)$

$= 2(\cos(2t) - 1)/(\pi jt)$

$= -(4j\sin^2 t)/(\pi t)$