DEPARTMENT OF PHYSICS
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PH1020 Physics II

Problem Sheet 10 To be discussed on: (25.4.2018)

Question 1:

Calculate the total energy density of radiation in a blackbody at the following temperatures: (a) 300 K and (b) 2000 K. For each temperature calculate the wavelength at which the energy density is maximum.

Solution:

a) The energy density function for a given wavelength λ and a temperature T is

$$u(\lambda, T)d\lambda = \tfrac{8\pi hc}{\lambda^5}\tfrac{1}{\mathrm{e}^{\frac{hc}{\lambda k_B T}}-1}d\lambda$$

The total energy density

$$\begin{aligned} U &= \int_0^\infty u(\lambda, T)\ d\lambda \\ &= \int_0^\infty \frac{8\pi hc}{\lambda^5}\frac{1}{\mathrm{e}^{\frac{hc}{\lambda k_B T}}-1}\ d\lambda \\ &= \frac{8\pi}{(ch)^3}(KT)^4\int_0^\infty \frac{x^3}{e^x-1}dx, \qquad [x = \frac{hc}{\lambda KT}] \\ &= \frac{8\pi}{h^3c^3}(KT)^4\frac{\pi^4}{15} \end{aligned} \tag{1}$$

- Putting $T = 300K$ in equation 1, $U(300\ K) = 6.1 \times 10^{-6}\ Jm^{-3}$

b) For $T = 2000\ K$, $U(T = 2000) = 1.2 \times\ 10^{-2}\ Jm^{-3}$

The energy density function for a given wavelength and temperature is

$$u(\lambda, T)\ d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{\mathrm{e}^{\frac{hc}{\lambda k_B T}} - 1} d\lambda \tag{2}$$

Taking derivative equ 2 w.r.t λ and equating to zero we have,

$$\begin{aligned} \frac{du(\lambda, T)}{d\lambda} &= 0 \\ 5(e^{\frac{hc}{\lambda k_B T}} - 1) &= \frac{hc}{\lambda k_B T} e^{\frac{hc}{\lambda k_B T}} \end{aligned} \tag{3}$$

Let $x = \frac{hc}{\lambda k_B T}$. Now, from equation 3 we get,

$$e^{-x} = 1 - \tfrac{x}{5}$$

The solutions of this equation are $x = 0$ and $x \approx 4.956$.

$\therefore \quad \lambda = \frac{hc}{4.956 k_B T}$ for which the energy density is maximum.

- For $T = 300$ K, $\lambda = \frac{hc}{4.956 k_B \times 300} = 9.68 \times 10^{-6} \ \ m.$
- For $T = 2000$ K, $\lambda = \frac{hc}{4.956 k_B \times 2000} = 1.45 \times 10^{-6} \ \ m$

Question 2:

Consider a particle bound in the region $x > 0$. If its wavefunction in one dimension is given by $\Psi = e^{-x}(1 - e^{-x})$, then what is the probability to find the particle to the right of x = a and the expectation value $< x >$?

Solution:

The given wavefunction is $\Psi = e^{-x}(1 - e^{-x})$. We assume the normalized wavefunction is $\Psi = Ae^{-x}(1 - e^{-x})$.

$$\therefore \quad \left|A\right|^2 \int_0^\infty \Psi^* \Psi \ dx = 1$$

$$\left|A\right|^2 \int_0^\infty e^{-2x}(1 - e^{-x})^2 \ dx = 1$$

$$\left|A\right|^2 = 12$$

$$A = \sqrt{12}$$

The normalized wave function is $\Psi = \sqrt{12}e^{-x}(1 - e^{-x})$
The probability of finding the particle to the right side of $x = a$ is,

$$P = 12 \int_a^\infty e^{-2x}(1 - e^{-x})^2 \ dx$$

$$= 12 \left(\frac{e^{-2a}}{2} - \frac{2}{3}e^{-3a} + \frac{e^{-4a}}{4} \right)$$

The expectation value of x for the particle is

$$< \hat{x} > = \int_0^\infty \Psi^* x \Psi \ dx$$

$$= 12 \int_0^\infty x e^{-2x}(1 - e^{-x})^2 dx$$

$$= \frac{13}{12}$$

Question 3:

A particle, moving in one dimension, has a ground state wavefunction (not normalized and do not normalize) given by $\Psi_0(x) = e^{-\frac{\alpha^4 x^4}{4}}$ (where α is a real constant) belonging to the energy eigenvalue $E_0 = \frac{\hbar^2 \alpha^2}{m}$. Determine the potential in which the particle moves.

Solution:

The ground state wavefunction is $\Psi_0(x) = e^{-\frac{\alpha^4 x^4}{4}}$ and the corresponding ground state energy $E_0 = \frac{\hbar^2\alpha^2}{m}$.
From the time independent Schrodinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi_0(x)}{dx^2} + V(x)\Psi_0(x) = E_0\Psi_0(x) \tag{4}$$

1st term of the left side of equation 4,

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{d^2\Psi_0(x)}{dx^2} &= \frac{\hbar^2\alpha^2}{2m}\left[3x^2\alpha^2 - x^6\alpha^6\right] e^{-\frac{\alpha^4 x^4}{4}} \\ &= \frac{\hbar^2\alpha^2}{2m}\left[3x^2\alpha^2 - x^6\alpha^6\right] \Psi_0(x)\end{aligned}$$

Putting the value of $-\frac{\hbar^2}{2m}\frac{d^2\Psi_0(x)}{dx^2}$ and $E_0 = 2(\frac{\hbar^2\alpha^2}{2m})$ in equation 4 we get,

$$V(x) = \tfrac{\hbar^2\alpha^2}{2m}\left[2 - 3x^2\alpha^2 + x^6\alpha^6\right]$$

Question 4:

Consider a particle of mass m, in one dimension, confined between to infinetely hard walls at $x = -a$ and $x = a$. If the wavefunction of the particle is given by

$$\psi(x) = \frac{1}{\sqrt{2}}\cos\left(\frac{3\pi x}{2a}\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{\pi x}{a}\right)$$

then, what is the probability to find the particle to the right of x=0 and the expectation value of the energy of the particle?

Solution:

The normalized wavefunction is given by

$$\psi_N(x) = \frac{1}{\sqrt{a}}\left[\frac{1}{\sqrt{2}}\cos\left(\frac{3\pi x}{2a}\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{\pi x}{a}\right)\right]$$

$$< E >= \int_{-a}^{a} \psi_N^*(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi_N(x)\; dx$$

Since

$$\int_{-a}^{a} \sin(mx)\cos(nx)\ dx = 0$$

$$< E >= \frac{\hbar^2}{2m}\left[\int_{-a}^{a}\left(\frac{9\pi^2}{4a^2}\right)\frac{1}{a}\cos^2\left(\frac{3\pi x}{2a}\right) + \left(\frac{\pi^2}{a^2}\right)\frac{1}{a}\sin^2\left(\frac{\pi x}{a}\right)\right]\ dx$$

$$< E >= \frac{1}{2}.\frac{\hbar^2}{2m}.\left[\frac{9\pi^2}{4a^2} + \frac{\pi^2}{a^2}\right]$$

Probability to find the particle to the right of x = 0 is

$$P(x > 0) = \frac{1}{2}.\frac{1}{a}\int_{0}^{a}\left[\cos^2\left(\frac{3\pi x}{2a}\right) + \sin^2\left(\frac{\pi x}{a}\right) + 2.\cos\left(\frac{3\pi x}{2a}\right)\sin\left(\frac{\pi x}{a}\right)\right]\ dx$$

$$= \frac{1}{2}.\frac{1}{a}\left[\frac{a}{2} + \frac{a}{2} + 2.\frac{6a}{5\pi}\right] = \left[\frac{1}{2} + \frac{6}{5\pi}\right]$$

Note:

$$\int \sin^2 ax\ dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax + C$$

$$\int \cos^2 ax\ dx = \frac{x}{2} + \frac{1}{4a}\sin 2ax + C$$

$$\int \sin a_1 x \cos a_2 x\ dx = -\frac{\cos(a_1 - a_2)x}{2(a_1 - a_2)} - \frac{\cos(a_1 + a_2)x}{2(a_1 + a_2)} + C$$