DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II Problem Set 6 (Solutions) 2019

1. (a) We begin with the formula for the vector potential due to a magnetic dipole with moment $\mathbf{m}$.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{e}_r}{r^2} \ .$$

With no loss of generality, choose $\mathbf{m} = m\ \hat{e}_z = m(\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta)$. Then, one has

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\ \hat{e}_\varphi \ .$$

Now using the standard formula for the curl of a vector field in spherical polar coordinates[1], we find

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \frac{\mu_0 m}{4\pi r^3}\big(2\cos\theta\hat{e}_r + \sin\theta\hat{e}_\theta\big) \ ,$$

which can be rewritten (as in the case of an electric dipole) as

$$\boxed{\mathbf{B} = \frac{\mu_0}{4\pi r^3}\big(3(\mathbf{m}\cdot\hat{e}_r)\ \hat{e}_r - \mathbf{m}\big) \ .} \tag{1}$$

(b) **Derivation of the formula for the force:** The potential energy of a point magnetic dipole $\mathbf{m}_2$ in a magnetic field is given by $U = -\mathbf{m}_2\cdot\mathbf{B}$ from which the force can be obtained as

$$\boxed{\mathbf{F} = -\boldsymbol{\nabla}U = \boldsymbol{\nabla}(\mathbf{m}_2\cdot\mathbf{B}) = (\mathbf{m}_2\cdot\boldsymbol{\nabla})\ \mathbf{B} \ ,}$$

where the last term is obtained as follows:

$$F_i = \partial_i((m_2)_j B_j) = (m_2)_j(\partial_i B_j) = (m_2)_j(\partial_j B_i) = (\mathbf{m}_2\cdot\boldsymbol{\nabla})\ B_i \ ,$$

since $\boldsymbol{\nabla}\times\mathbf{B} = 0$ or $\partial_i B_j = \partial_j B_i$ in a region without currents.
Now using formula (1) for the magnetic field due to $\mathbf{m}_1 = m_1\ \hat{e}_z$ taken to be located at the origin, we get the force on $\mathbf{m}_2 = m_2\ \hat{e}_z$ located at $(0,0,z)$ due to the dipole $\mathbf{m}_1$ is

$$\begin{aligned}\mathbf{F} &= m_2\frac{\partial}{\partial z}\left(\frac{\mu_0 m_1\hat{e}_z}{2\pi z^3}\right)\bigg|_{z=d}\\ &= -\frac{\mu_0}{4\pi}\ \frac{6m_1m_2}{d^4}\ \hat{e}_z \ .\end{aligned}$$

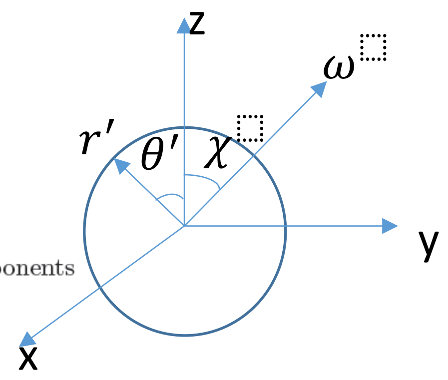
The negative sign indicates the attractive nature of the force. We leave it to the student to carefully check that the other components of the force vanish by choosing the second dipole at a more general point (x,y,z) before computing $(\mathbf{m}_2\cdot\boldsymbol{\nabla})\ \mathbf{B}$ and then setting $(x,y,z) = (0,0,d)$ at the end of the computation.

2. The rotating spherical shell has a surface current density $\mathbf{K}(\mathbf{r}') = \sigma(\boldsymbol{\omega}\times\mathbf{r}')$ for a point $\mathbf{r}'$ located on the sphere.

(a) Let P denote a point lying in the **interior** of the shell and we take it to lie on the z-axis with no loss of generality. The vector potential due to the surface current density at P in the

[1] It is acceptable if you need to look up the answer somewhere!

Coulomb gauge is given by



$$
\begin{aligned}
\mathbf{A}(\mathbf{r}=z\hat{e}_z) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')dS}{|\mathbf{r}-\mathbf{r}'|} \\
&= \frac{\mu_0\sigma\omega}{4\pi}\times\int \frac{\mathbf{r}'\ R^2 d\varphi' \sin\theta' d\theta'}{(z^2+R^2-2R\cos\theta'_{z})^{1/2}} \\
&= \frac{\mu_0\sigma\omega}{2}\times\int \frac{R\cos\theta'\hat{e}_z\ R^2\sin\theta' d\theta'}{(z^2+R^2-2\underset{z}{R}\cos\theta')^{1/2}}
\end{aligned}
$$

The last line follows since the integration over φ' makes the conribution from the components of $\mathbf{r}'$ along $\hat{e}_x$ and $\hat{e}_y$ vanish.

$$
\mathbf{A}(\mathbf{r}=z\hat{e}_z) = \frac{\mu_0\sigma(\boldsymbol{\omega}\times\hat{e}_z)}{2}\int_0^\pi \frac{\cos\theta'\ R^3\sin\theta' d\theta'}{(z^2+R^2-2R\underset{z}{\cos}\theta')^{1/2}}
$$

The integral can be carried out with following change of variables. Defining $s^2 = z^2+R^2-$
z $2R\cos\theta'$ with $s\in[R-z, R+z]$ leads to the simpler integral

$$
\begin{aligned}
\mathbf{A}(\mathbf{r}=z\hat{e}_z) &= \frac{\mu_0\sigma R}{4z^2}(\boldsymbol{\omega}\times\hat{e}_z)\int_{R-z}^{R+z}(R^2+z^2-s^2)\,\mathrm{ds} \\
&= \frac{\mu_0\sigma R}{3}(\boldsymbol{\omega}\times z\hat{e}_z)\ .
\end{aligned}
$$

Thus, we obtain

$$
\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0\sigma R}{3}(\boldsymbol{\omega}\times\mathbf{r})\ .}
$$

Using $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$, we obtain a constant magnetic field in the interior of the shell.

$$
\boxed{\mathbf{B}(\mathbf{r}) = \frac{2}{3}\mu_0\sigma R\ \boldsymbol{\omega}\ .}
$$

(b) The force between the two hemispheres can be obtained up to an overall numerical constant by dimensional analysis. Besides, M, L and T, we introduce Q as the physical dimension of charge. Thus, one has

$$
[\sigma]=QL^{-2}\quad,\quad[\omega]=T^{-1}\quad,\quad[R]=L\quad,\quad[B]=Q^{-1}MT^{-1}\ .
$$

Finally, using Biot-Savart's law, we can show that $[\mu_0]=Q^{-2}ML$. Let us assume that the magnitude of the force F is given by

$$
F\propto\omega^\alpha\sigma^\beta R^\gamma\mu_0^\delta\ ,
$$

for some constants $\alpha,\beta,\gamma,\delta$ to be fixed by dimensional analysis.

$$
[F]=MLT^{-2}=(T^{-1})^\alpha(QL^{-2})^\beta L^\gamma(Q^{-2}ML)^\delta\ ,
$$

which is solved by $\alpha=\beta=2$, $\gamma=4$ and $\delta=1$. Thus, we obtain

$$
\boxed{F=(\text{constant})\times\omega^2\sigma^2R^4\mu_0\ .}
$$

The constant turns to be $\frac{\pi}{4}$ from a direct computation.

3 (a) The electrical force on the charged particle is along the y-direction as long as the particle is between the plates of the capacitor i.e., $0\le x\le L$. Let the particle cross the line $x=L$ at time $t=t_1$. Since $\dot{x}=v_0$ is a constant, $x(t)=v_0t$ since $x(0)=0$. It is easy to see that $t_1=L/v_0$. Thus for $0\le t\le t_1$, one has

$$
\ddot{y}=\frac{qE_0}{m}\implies y(t)=\frac{qE_0}{m}\frac{t^2}{2}+at+b\ ,
$$

with initial conditions $y(0)=\dot{y}(0)=0$ which forces $a=b=0$. Thus, for $0\le t\le L/v_0$, the trajectory of the particle is the parabola

$$
\boxed{y=\frac{qEx^2}{2mv_0^2}\ .}
$$

For $t > t_1$, the particle moves in a straight line hitting the screen at time $t_2 = (D + (L/2))/v_0$. For $t_1 \le t \le t_2$, $x(t)$ remains equal to $v_0 t$. But $y(t)$ is determined by

$$\dot{y} = \frac{qE_0 t_1}{m} = \frac{qE_0 L}{mv_0} ,$$

which can be integrated to obtain

$$y(t) = y(t_1) + \frac{qE_0 L}{mv_0}(t - t_1) \text{ or } \boxed{y(t) = \frac{qE_0 L}{mv_0}\left(t - \frac{L}{2v_0}\right) .}$$

Thus at $t = t_2$, one has

$$\boxed{RR' = y(t_2) = \frac{qE_0 L}{mv_0}\left(\frac{D + (L/2)}{v_0} - \frac{L}{2v_0}\right) = \frac{qE_0 LD}{mv_0^2} .}$$

Thus, measuring RR' enables one to determine the ratio q/m. Thomson used this method to determine e/m.

(b) The path is a parabola till time $t = t_1 = L/v_0$ and a straight line thereafter, till the particle hits the screen at time t_2. The slope $dy/dx = \dot{y}\dot{x}$ remains continuous at $t = t_1$ when $x = L$ and $y = \frac{qEx^2}{2mv_0^2}$.

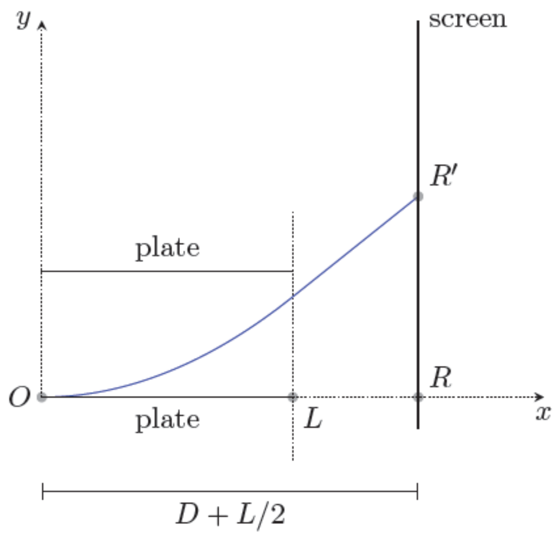


Figure 1: Thomson experiment: the trajectory is plotted in blue line

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(i) At the point (ϱ, φ, L), in cylindrical polar coordinates, lying on the planar top surface, one has $r^2 = \varrho^2 + L^2$, $\mathbf{r} = r\hat{e}_r = (\varrho\hat{e}_\varrho + L\hat{e}_z)$ and $\mathbf{dS} = \varrho d\varrho d\varphi \hat{e}_z$.

$$d\Phi_m = \mathbf{B} \cdot \mathbf{dS} = \frac{\mu_0 m}{4\pi r^3}\left(\frac{3L^2}{r^2} - 1\right) \varrho d\varrho d\varphi .$$

Carry out the integration over φ to obtain 2π and the change of variable, $\varrho = L\tan\theta$ ($\theta \in [0, \pi/4]$ in our problem) to obtain

$$
\begin{aligned}
\Phi_m &= \frac{\mu_0 m}{2L} \int_0^{\pi/4} (3\cos^2\theta - 1)\sin\theta d\theta \\
&= \frac{\mu_0 m}{4\sqrt{2}L}
\end{aligned}
$$

(ii) By symmetry, the flux through the bottom plane surface is equal in magnitude to the flux through the top plane surface but is of opposite sign. Since, $\oint \mathbf{B}\cdot\mathbf{dS} = 0$ over the entire closed cylindrical surface, it follows that flux through the curved surface **must** vanish as the two plane surfaces contribute zero flux to the integral.
Exercise: It is instructive to verify this conclusion by direct integration as well.

5. There are many ways of doing this problem: The current density is given by:

$$\mathbf{J} = \rho\mathbf{v} = \rho\omega r \sin\theta \hat{e}_z.$$

Consider that we actually decompose the sphere into infinitesimal rings. The rings are specified via their c-ordinates, (r,θ), and thickness dr, and $d\theta$. The volume of each of this current ring is

$2\pi r^2 \sin\theta d\theta dr$, the 2π arising out of integrating over the ϕ co-ordinate, $\left(\int_0^{2\pi} d\phi = 2\pi\right)$. Now the due to this slab is

$$dI = dq/T = \frac{\omega\rho d\tau}{2\pi} = \frac{2\pi\omega r^2\rho\sin\theta d\theta dr}{2\pi} = \rho\omega r^2 \sin\theta d\theta dr.$$

The magnetic dipole moment due to this element is $dm_z = dia$ where a is the area of the current slab. Thus,

$$dm_z = \rho\omega r^2 \sin\theta d\theta dr \left(\pi r^2 \sin^2\theta\right).$$

The total dipole moment can now be constructed by integrating such current labs over the entire sphere.

$$m_z = \pi\rho\omega \int_0^R r^4 dr \int_0^\pi \sin^3\theta d\theta.$$

The above integral can be calculated with ease to yield

$$\mathbf{m} = \frac{1}{5}Q\omega R^2$$

The angular momentum is

$$\mathbf{L} = I\omega = \frac{2}{5}MR^2\omega.$$

Therefore, the ratio of

$$\frac{|\mathbf{m}|}{|\mathbf{L}|} = Q/2m.$$