Department of Mathematics, IIT Madras

MA1020 Series & Matrices

Assignment-2 Series Representation of Functions

1. Determine the interval of convergence for each of the following power series:

   (a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (b) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ (c) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

2. Determine the interval of convergence of the series $\frac{2x}{1} - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \cdots$.

3. Determine power series expansion of the following functions:

   (a) $\ln(1+x)$ (b) $\frac{\ln(1+x)}{1-x}$

4. The function $\frac{1}{1-x}$ has interval of convergence $(-1, 1)$. However, prove that it has power series representation around any $c \neq 1$.

5. Find the sum of the alternating harmonic series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$.

6. Give an approximation scheme for $\int_0^a \frac{\sin x}{x}\, dx$ where $a > 0$.

7. Give an example of an infinitely diifferentiable function which has a Taylor series expansion at a point but the Taylor series does not represent the function around that point.

8. Show that $\sin^{-1} x = x + \frac{1}{2} \cdot \frac{1}{3} x^3 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7} x^7 + \cdots$ for $-1 < x < 1$.

   Then, deduce that $1 + \frac{1}{2}\frac{1}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7} + \cdots = \frac{\pi}{2}$.

9. Find the Fourier series of $f(x)$ given by: $f(x) = 0$ for $-\pi \leq x < 0$; and $f(x) = 1$ for $0 \leq x \leq \pi$. Say also how the Fourier series represents $f(x)$. Hence give a series expansion of $\pi/4$.

10. Considering the fourier series for $|x|$, deduce that $\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

11. Considering the fourier series for x, deduce that $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

12. Considering the fourier series for $f(x)$ given by: $f(x) = -1$, for $-\pi \leq x < 0$ and $f(x) = 1$ for $0 \leq x \leq \pi$ deduce that $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

13. Considering $f(x) = x^2$, show that for each $x \in [0, \pi]$,

$$\frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} = \sum_{n=1}^{\infty} \frac{n^2\pi^2(-1)^{n+1} + 2(-1)^n - 2}{n^3\pi} \sin nx.$$

14. Represent the function $f(x) = 1 - |x|$ for $-1 \leq x \leq 1$ as a cosine series.