EE1101 Signals and Systems JAN—MAY 2018 Tutorial 5 Solutions

1) (a) In this case the period $T_0 = 1$ and the fundamental frequency $f_0 = \frac{1}{T_0} = 1$ Hz, and $\omega_0 = \frac{2\pi}{T_0} = 2\pi$. Therefore,

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

$$a_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt \quad = A\int_0^1 te^{-jn2\pi t}dt$$

$$= \frac{Aj}{2\pi n} \qquad \text{for } n \neq 0$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt = \frac{A}{2}$$

The magnitude and phase spectra are presented in Fig. 1. For these plots we assume $A = 1$.

(b) The given function is $y(t) = A + x(-t + 0.5)$. Fourier coefficients of

$$x(t) \longrightarrow a_n$$
$$x(t + 0.5) \longrightarrow e^{jn\pi}a_n$$
$$x(-t + 0.5) \longrightarrow e^{-jn\pi}a_{-n}$$

Therefore, Fourier coefficients of $y(t)$

$$a_n = \frac{-Aj}{2\pi n}e^{-jn\pi}$$
$$a_0 = \frac{3A}{2}$$

The magnitude and phase spectra are presented in Fig. 2. For these plots we assume $A = 1$.

2)

$$a_0 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)dt$$

$$a_n = \frac{1}{T_0}\int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} x(t)e^{-jn\omega_0 t}dt$$

$$x_1(t) = A, \qquad -\frac{d}{2} < t < \frac{d}{2}$$

$$a_0 = \frac{1}{T_0}\int_{-\frac{d}{2}}^{\frac{d}{2}} Adt = \frac{Ad}{T_0}$$

$$a_n = \frac{1}{T_0}\int_{\frac{-d}{2}}^{\frac{d}{2}} Ae^{-jn\omega_0 t}dt$$

$$a_n = \frac{-A}{jn\omega_0 T_0}\left(e^{-jn\omega_0\frac{d}{2}} - e^{jn\omega_0\frac{d}{2}}\right)$$

$$a_n = \frac{2A}{n\omega_0 T_0}\left(\sin\left(n\omega_0\frac{d}{2}\right)\right), \quad \text{for } n \neq 0.$$

Phase spectrum of $x_1(t)$ is either zero or π for all n because there is no imaginary part in a_n. The magnitude and phase spectra are shown in Fig. 3. For these plots, we assume $A = 1, d = 1, T_0 = 2, \omega_0 = \pi$.

$$x_2(t) = A\sin(\omega_0 t), \qquad 0 < t < \frac{T_0}{2}$$

$$a_0 = \frac{A}{T_0}\int_0^{\frac{T_0}{2}} \sin(\omega_0 t)d(t)$$

$$a_0 = \frac{-A}{\omega_0 T_0}\,(\cos(\pi) - 1) = \frac{A}{\pi}$$

$$a_n = \frac{A}{T_0}\int_0^{\frac{T_0}{2}} \sin(\omega_0 t)e^{-jn\omega_0 t}dt$$

$$= \frac{A}{2jT_0}\int_0^{\frac{T_0}{2}} \left(e^{j\frac{2\pi t}{T_0}} - e^{-j\frac{2\pi t}{T_0}}\right)e^{-jn\omega_0 t}dt$$

$$= \frac{-A}{2T_0}\left(\frac{e^{j(1-n)\pi} - 1}{(\frac{(1-n)2\pi}{T_0})} - \frac{e^{-j(1+n)\pi} - 1}{-(\frac{(1+n)2\pi}{T_0})}\right)$$

$$= \frac{-A}{4\pi}\left(\frac{e^{j(1-n)\pi} - 1}{(1-n)} + \frac{e^{-j(1+n)\pi} - 1}{(1+n)}\right)$$

$$a_n = \frac{A}{2\pi}\left(\frac{2}{1-n^2}\right), \quad n = \text{even}$$

$$a_n = 0, \quad n = \text{odd},\ n \neq 1,\ n \neq -1$$

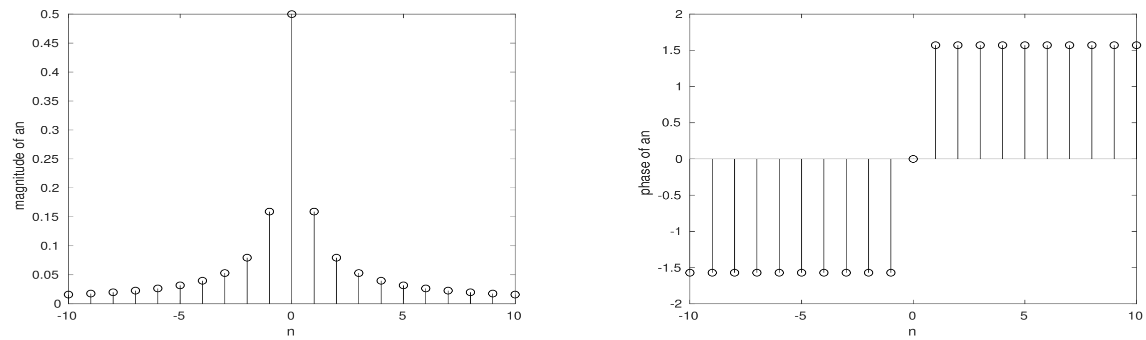

Fig. 1: Q1.a) Magnitude and phase spectra of $x(t)$ for $A = 1$.

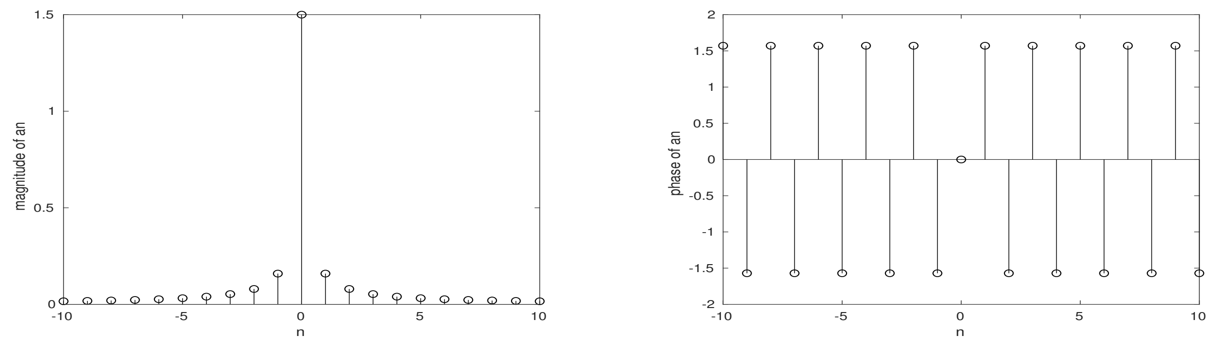

Fig. 2: Q1.b) Magnitude and phase spectra of $y(t)$ for $A = 1$.

$$a_1 = \frac{A}{T_0}\int_0^{\frac{T_0}{2}} \sin(\omega_0 t)e^{-j\omega_0 t}dt = \frac{-Aj}{4}$$

$$a_{-1} = \frac{A}{T_0}\int_0^{\frac{T_0}{2}} \sin(\omega_0 t)e^{j\omega_0 t}dt = \frac{Aj}{4}$$

Phase spectrum of $x_2(t)$ for $n = 1$ is $-\pi/2$, for $n = -1$ is $\pi/2$. The magnitude and phase spectra are shown in Fig. 4. For these plots, we assume $A = 1, d = 1, T_0 = 2, \omega_0 = \pi$.

3) We have

$$x(t) = \sum_{n=-\infty}^{+\infty} a_n e^{jnt}$$

where

$$a_o = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt$$

$$= \frac{1}{2\pi}\left(\int_0^A \frac{t}{A}dt + \int_A^\pi dt\right)$$

Therefore,

$$a_o = \frac{1}{2\pi}\left(\pi - \frac{A}{2}\right)$$

and

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-jnt}dt$$

$$= \frac{1}{2\pi}\left(\int_0^A \frac{t}{A}e^{-jnt}dt + \int_A^\pi e^{-jnt}dt\right)$$

Integrating by parts, we have

$$\int_0^A \frac{t}{A}e^{-jnt}dt = j\frac{e^{-jnA}}{n} - \frac{1-e^{-jnA}}{An^2}$$

and

$$\int_A^\pi e^{-jnt}dt = -\frac{j}{n}\left(e^{-jnA} - (-1)^n\right)$$

Therefore,

$$a_n = \frac{1}{2\pi}\left(j\frac{(-1)^n}{n} - \frac{1-e^{-jnA}}{An^2}\right).$$

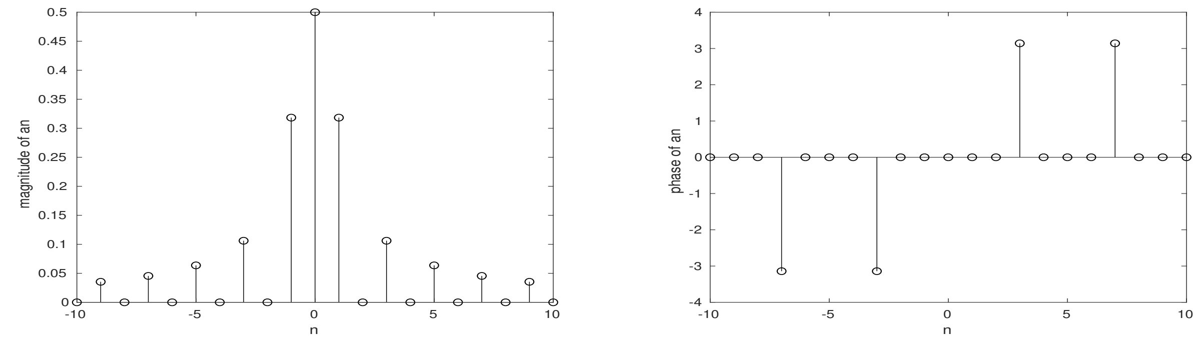

Fig. 3: Q2.a) Magnitude and phase spectra of $x_1(t)$ for $A = 1, d = 1, T_0 = 2, \omega_0 = \pi$.

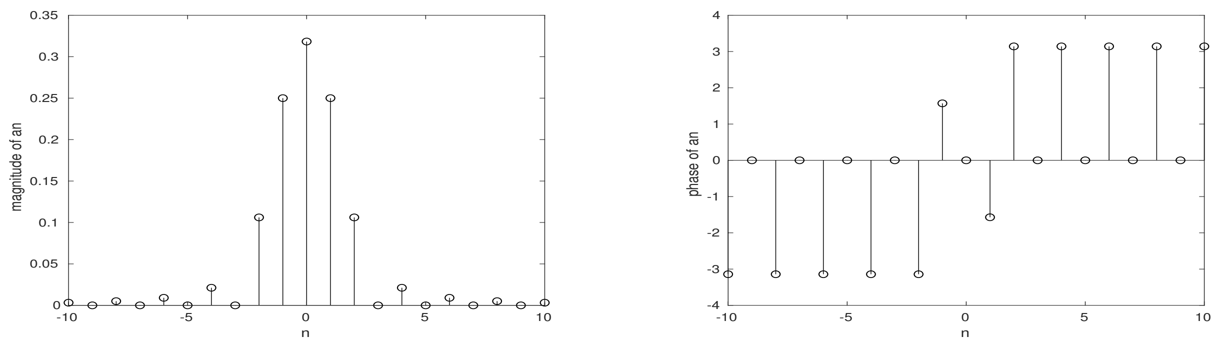

Fig. 4: Q2.b) Magnitude and phase spectra of $x_2(t)$ for $A = 1, d = 1, T_0 = 2, \omega_0 = \pi$.

4) $d_k = jk,\ |k| < 3$

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t} \\ x(t) &= j\Big(-2e^{j(-2)\frac{2\pi}{4}t} - 1e^{j(-1)\frac{2\pi}{4}t} + 1e^{j\frac{2\pi}{4}t} \\ &\quad + 2e^{j(2)\frac{2\pi}{4}t}\Big) \\ x(t) &= (-1)\left(4\sin(\pi t) + 2\sin(\frac{\pi}{2}t)\right). \end{aligned}$$

5) Given $x(t)$ is a periodic signal with fundamental period T and Fourier series coefficients a_k

(a) $x(t - t_0) + x(t + t_0)$

By time shifting property,

$$x(t - t_0) \rightarrow a_k e^{-jk\omega_0 t_0}$$

where $\omega_0 = 2\pi/T$. Therefore, Fourier coefficients of $x(t - t_0) + x(t + t_0)$ are

$$a_k e^{-jk\omega_0 t_0} + a_k e^{jk\omega_0 t_0} = 2a_k \cos(k\omega_0 t_0)$$

(b) Even $\{x(t)\}$ = $\dfrac{x(t) + x(-t)}{2}$

$$x(t) \rightarrow a_k$$
$$x(-t) \rightarrow a_{-k}$$

Therefore, Fourier coefficients of Even $\{x(t)\}$ are $\dfrac{a_k + a_{-k}}{2}$

(c) Real $\{x(t)\}$ = $\dfrac{x(t) + x^*(t)}{2}$

$$x(t) \rightarrow a_k$$
$$x^*(t) \rightarrow a^*_{-k}$$

Therefore, Fourier coefficients of Real $\{x(t)\}$ are $\dfrac{a_k + a^*_{-k}}{2}$

6) The exponential Fourier Series expansion of a periodic signal $x(t)$ is

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

Let the Fourier Series coefficients of $\cos(t)$ be C_n and that of $\cos(3t)$ be D_n. Then, by the periodic convolution property of Fourier series, the FS coefficients of $\cos t\, *$

$\cos(3t)$ will be TC_nD_n where T is the period. Now,

$$\cos(t) = \frac{e^{jt}+e^{-jt}}{2} \quad \text{and} \quad \cos(3t) = \frac{e^{j3t}+e^{-j3t}}{2}.$$

Thus we see that only C_1, C_{-1}, D_3 and D_{-3} are non-zero. Therefore the FS coefficients of $\cos t * \cos(3t)$ will all be 0.

Verification in time-domain

$$\begin{aligned}\int_T \cos(t-\tau)\cos(3\tau)d\tau &= \int_T \big(\cos(t)\cos(\tau)\cos(3\tau) \\ &\quad + \sin(t)\sin(\tau)\cos(3\tau)\big)d\tau \\ &= 0.\end{aligned}$$

Thus the periodic convolution yields 0, whose FS coefficients will then be 0. This verifies our previous result.

7) (a) The plots for signals $x(t)$, $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ are shown in Fig. 5.

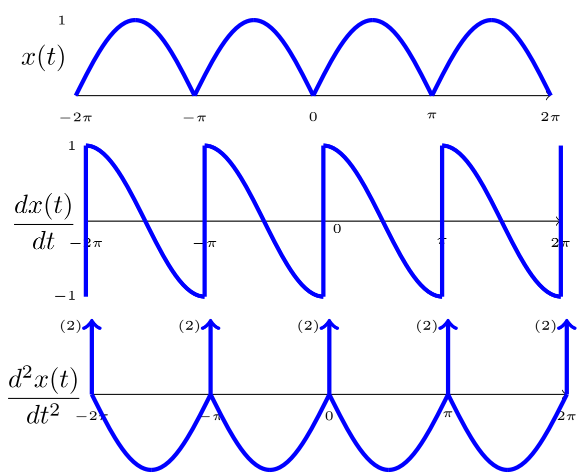

Fig. 5: Q7.(a)

(b) The plot for $x(t) + \frac{d^2x(t)}{dt^2}$ is shown in Fig. 6.

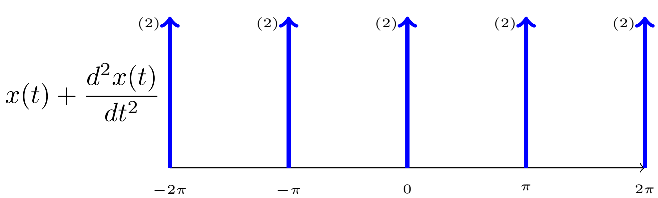

Fig. 6: Q7.(b)

$$\begin{aligned}x(t) + \frac{d^2x(t)}{dt^2} &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{jk(2)t}\end{aligned}$$

where

$$\begin{aligned}a_k &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{p=-\infty}^{\infty} 2\delta(t-p\pi)e^{-jk2t}dt \\ &= \frac{1}{\pi}\sum_{p=-\infty}^{\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\delta(t-p\pi)e^{-jk2t}dt\end{aligned}$$

For the given range, $p = 0$

$$a_k = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\delta(t)e^{-jk2t}dt = \frac{2}{\pi}$$

(c)

$$x(t) \leftrightarrow c_k \qquad \text{and} \qquad \frac{d^2x(t)}{dt^2} \leftrightarrow (jk\omega_0)^2\, c_k$$

$$x(t) + \frac{d^2x(t)}{dt^2} \leftrightarrow \big(1+(jk\omega_0)^2\big)c_k$$

$$\begin{aligned}a_k &= \big(1+(jk\omega_0)^2\big)c_k \\ c_k &= \frac{a_k}{\big(1+(jk\omega_0)^2\big)} = \frac{2}{\pi(1-4k^2)}\end{aligned}$$

8) Since x(t) is a real signal, $a_k = a^*_{-k}$. It is given that $a_k = 0$ for $k > 2$. This implies that $a_{-k} = a^*_k = 0$ for $k > 2$.
Also it is given that $a_0 = 0$. Therefore the only non-zero Fourier coefficients are a_1, $a_{-1} = a_1^*$, a_2 and $a_{-2} = a_2^*$. It is also given that a_1 is positive real number. Therefore $a_{-1} = a_1$. Thus we have,

$$\begin{aligned}x(t) &= a_1\big(e^{j\frac{2\pi}{T}t} + e^{-j\frac{2\pi}{T}t}\big) + a_2 e^{j\frac{4\pi}{T}t} + a_2^* e^{-j\frac{4\pi}{T}t} \\ &= 2a_1 \cos\frac{2\pi}{T}t + a_2 e^{j\frac{4\pi}{T}t} + a_2^* e^{-j\frac{4\pi}{T}t}\end{aligned}$$

It is given that $T = 6$. This gives

$$= 2a_1\cos\frac{\pi}{3}t + a_2 e^{j\frac{2\pi}{3}t} + a_2^* e^{-j\frac{2\pi}{3}t}.$$

Since $e^{j\frac{2\pi}{3}t}$ and $e^{-j\frac{2\pi}{3}t}$ are both periodic with period 3, we have

$$x(t-3) = -2a_1\cos\frac{\pi}{3}t + a_2 e^{j\frac{2\pi}{3}t} + a_2^* e^{-j\frac{2\pi}{3}t}.$$

It is given that $x(t) = -x(t-3)$, which implies that

$$2\big(a_2 e^{j\frac{2\pi}{3}t} + a_2^* e^{-j\frac{2\pi}{3}t}\big) = 0.$$

Therefore we have,

$$x(t) = 2a_1\cos\frac{\pi}{3}t.$$

Finally, it is given that

$$\frac{1}{6}\int_{-3}^{3} \mid x(t) \mid^2 \, dt = \frac{1}{2}$$
$$\implies \frac{4}{6}\int_{-3}^{3} a_1^2 \cos^2\left(\frac{\pi}{3}t\right) dt = \frac{1}{2}$$
$$\implies a_1 = \frac{1}{2}$$

Therefore, $x(t) = \cos\frac{\pi}{3}t$ and the constants $A = 1$, $B = \frac{\pi}{3}$ and $C = 0$.