Department of Mathematics, Indian Institute of Technology Madras

MA1102 Series and Matrices Quiz-2

March 27, 2018 Tuesday 8:00-8:50 Maximum Marks: 20

Answer all the five questions.

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1. Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^n\, n\, x^n.$$

Also, find the function to which the power series converges. [4 marks]

Solution:

With $a_n = (-1)^n\, n$, $\lim\limits_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right| = \lim\limits_{n\to\infty}\dfrac{n}{n+1} = 1$.

Hence radius of convergence is 1.

At $x = 1$, the series is $-1+2-3+4-\cdots$ which diverges.

At $x = -1$, the series is $1+2+3+\cdots$ which diverges.

Hence, interval of convergence is $(-1, 1)$.

$$\sum_{n=1}^{\infty}(-1)^n n x^n = x\sum_{n=1}^{\infty}(-1)^n n x^{n-1} = x\sum_{n=1}^{\infty}(-1)^n \frac{dx^n}{dx}$$

$$= x\frac{d}{dx}\Big(\sum_{n=1}^{\infty}(-1)^n x^n\Big) = x\frac{d}{dx}\Big[\frac{1}{1+x} - 1\Big] = \frac{-x}{(1+x)^2}.$$

2. Determine the Taylor series of the function $f(x) = \sin x$ about the point $x = \dfrac{\pi}{2}$.

Also, determine all values of x where the Taylor series converges. [4 marks]

Solution:

$f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x,\ f^{(3)}(x) = -\cos x,\ f^{(4)}(x) = \sin x.$

Then it repeats.

$f(\pi/2) = 1,\ f'(\pi/2) = 0,\ f''(\pi/2) = -1,\ f^{(3)}(\pi/2) = 0,\ f^{(4)}(\pi/2) = 1, \cdots$

The Taylor series is $1 - \dfrac{(x-\pi/2)^2}{2!} + \dfrac{(x-\pi/2)^4}{4!} - \cdots$

$\lim\limits_{n\to\infty}|R_n(x)| \le \lim\limits_{n\to\infty}\dfrac{|x-\pi/2|^{n+1}}{(n+1)!} = 0$ for each real x.

Hence the series converges everywhere.

3. Find the Fourier series of the function $f(x) = \begin{cases} x & \text{for } -\pi \le x < 0 \\ 1 & \text{for } 0 \le x < \pi. \end{cases}$

Also, find the sum of the Fourier series at $x = 0$. [5 marks]

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} t\,dt + \frac{1}{\pi}\int_{0}^{\pi} dt = 1 - \frac{\pi}{2}.$$

$$a_n = \tfrac{1}{\pi}\int_{-\pi}^{0} t\cos nt\,dt + \tfrac{1}{\pi}\int_0^{\pi}\cos nt\,dt = \tfrac{1}{\pi}\Big[\tfrac{t\sin nt}{n}\Big]_{-\pi}^{0} - \tfrac{1}{\pi}\int_{-\pi}^{0}\tfrac{\sin nt}{n}\,dt + \tfrac{1}{\pi}\int_0^{\pi}\cos nt\,dt$$

$$= \frac{1}{\pi}\Big[\frac{\cos nt}{n}\Big]_{-\pi}^{0} = \frac{1}{\pi n^2}(1 - (-1)^n) = \begin{cases} 2/(\pi n^2) & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

$b_n = \frac{1}{\pi}\int_{-\pi}^{0} t\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}\sin nt\,dt = \frac{1}{\pi}\left[\frac{-t\cos nt}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\int_{-\pi}^{0}\frac{\cos nt}{n}\,dt + \frac{1}{\pi}\int_{0}^{\pi}\sin nt\,dt$
$= -\frac{1}{\pi}\left(\frac{\pi}{n}(-1)^n\right) + \frac{1}{\pi}\left[\frac{\sin nt}{n^2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-\cos nt}{n}\right]_{0}^{\pi} = \frac{(-1)^{n+1}(\pi+1)+1}{n\pi}$
$= \begin{cases} (\pi+2)/(n\pi) & n \text{ odd} \\ -1/n & n \text{ even.} \end{cases}$
The Fourier series is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$.
At $x = 0$, The Fourier series sums to $\frac{1}{2}(f(0-)+f(0+)) = \frac{1}{2}$.

4. Let $A = \begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix}$. Determine whether A is (a) normal (b) unitary.

[3 marks]

Solution:
$A = \begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix}$, $A^* = \begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix}$.
$AA^* = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. $A^*A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.
Hence A is normal but not unitary.

5. Find the row reduced echelon form of the matrix $\begin{bmatrix} 1 & -3 & 2 & -2 \\ 5 & 2 & -3 & 1 \\ 3 & 1 & -1 & 5 \end{bmatrix}$. [4 marks]

(Simplify the fractions but do not convert them to decimals.)

Solution:

$$\begin{bmatrix} \boxed{1} & -3 & 2 & -2 \\ 5 & 2 & -3 & 1 \\ 3 & 1 & -1 & 5 \end{bmatrix} \xrightarrow{E_{-5}[2,1],\, E_{-3}[3,1]} \begin{bmatrix} \boxed{1} & -3 & 2 & -2 \\ 0 & 17 & -13 & 11 \\ 0 & 10 & -7 & 11 \end{bmatrix}$$

$$\xrightarrow{E_{1/17}[2]} \begin{bmatrix} \boxed{1} & -3 & 2 & -2 \\ 0 & \boxed{1} & -13/17 & 11/17 \\ 0 & 10 & -7 & 11 \end{bmatrix} \xrightarrow{E_3[1,2],\, E_{-10}[3,2]} \begin{bmatrix} \boxed{1} & 0 & -5/17 & -1/17 \\ 0 & \boxed{1} & -13/17 & 11/17 \\ 0 & 0 & 11/17 & 77/17 \end{bmatrix}$$

$$\xrightarrow{E_{17/11}[3]} \begin{bmatrix} \boxed{1} & 0 & -5/17 & -1/17 \\ 0 & \boxed{1} & -13/17 & 11/17 \\ 0 & 0 & \boxed{1} & 7 \end{bmatrix} \xrightarrow{E_{5/17}[1,3],\, E_{13/17}[2,3]} \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 6 \\ 0 & 0 & \boxed{1} & 7 \end{bmatrix}.$$