DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

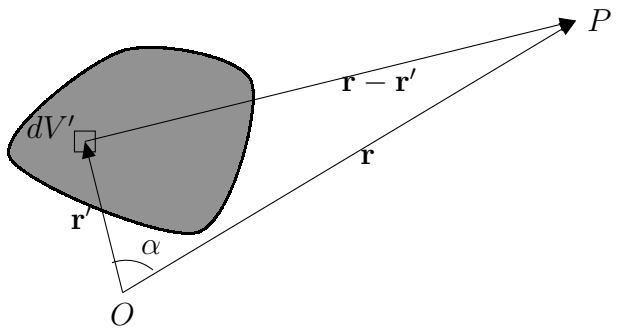
Multipole Expansion

This set is concerned with the multipole expansion of the electrostatic (or scalar) potential. In particular, we emphasize the *quadrupole term* in the expansion as you may have already encountered the monopole and dipole terms in some form. Please read the material below and work out the problems that follow.

As shown in class, consider a static charge distribution spread over a volume V with a density $\rho(\mathbf{r}')$ (we will use $\mathbf{r}'$ for the running coordinate of the points of the charge distribution). Then the electrostatic potential $\phi(\mathbf{r})$ at an arbitrary point, P, (see figure) with position vector $\mathbf{r}$, due to the charge distribution, is given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')\ dV'}{|\mathbf{r}-\mathbf{r}'|} \quad .$$

When $r \gg r'$ for all r' in V, we may expand $|\mathbf{r}-\mathbf{r}'|^{-1}$ in powers of r'/r to get



$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int_V \rho(\mathbf{r}')dV' + \frac{1}{r^2}\int_V r'\cos\alpha\ \rho(\mathbf{r}')\ dV' \right.$$
$$\left. + \frac{1}{r^3}\int_V r'^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\ dV' + \cdots\right]$$

Recall that α is the angle between $\mathbf{r}$ and $\mathbf{r}'$ (it therefore changes from point to point as $\mathbf{r}'$ takes on different values in V). Since $q_{\rm tot} = \int_V \rho(\mathbf{r}')dV'$ is the total charge in the distribution, and $\mathbf{p} = \int_V \rho(\mathbf{r}')\mathbf{r}'dV'$ is its dipole moment, we can write the first two terms in the expansion as

$$\boxed{\phi_{\rm monopole}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q_{\rm tot}}{r}} \quad \text{and} \quad \boxed{\phi_{\rm dipole}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{e}_r}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}} \qquad (1)$$

respectively. We would like to write the next term (the quadrupole term) in a similar fashion. Note that both q and $\mathbf{p}$ *are intrinsic properties of the charge distribution itself* (they involve integrations over $\mathbf{r}'$, the running variable), and do *not* depend on the observation or field point $\mathbf{r}$ at which the potential is being determined. We would like to have the quadrupole term expressed similarly in terms of some property of the charge distribution. The relevant quantity is called the *quadrupole moment* of the distribution.

Writing $\mathbf{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$ and $\mathbf{r}' = x'\hat{e}_x + y'\hat{e}_y + z'\hat{e}_z$, the integrand in the third term can be written as

$$\frac{r'^2}{r^3}\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) = \frac{3r'^2r^2\cos^2\alpha - r'^2r^2}{2r^5}$$
$$= \frac{3(xx' + yy' + zz')^2 - (x'^2 + y'^2 + z'^2)(x^2 + y^2 + z^2)}{2r^5} \quad .$$

The numerator is simplified as follows. Suppose we write the (components of) the position vector $\mathbf{r}$ in the form a column matrix, $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The transpose of this is the row matrix: $(x\ y\ z)$. Then the numerator can be written as the product of the three matrices below, namely,

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2x'^2 - y'^2 - z'^2 & 3x'y' & 3x'z' \\ 3y'x' & 2y'^2 - z'^2 - x'^2 & 3y'z' \\ 3z'x' & 3z'y' & 2z'^2 - x'^2 - y'^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Therefore, the quadrupole term can be written as

$$\boxed{\phi_{\text{quadrupole}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{r}\cdot\overleftrightarrow{Q}\cdot\mathbf{r}}{2r^5} \quad \text{or} \quad \frac{1}{4\pi\epsilon_0}\frac{x_i\ Q_{ij}\ x_j}{2r^5}\ ,} \tag{2}$$

where the symbol $\overleftrightarrow{Q}$ is called the *quadrupole moment*. It can also be written as a second-rank tensor[1] Q_{ij} with nine components given by

$$\begin{aligned} Q_{xx} &= \int_V \rho(\mathbf{r}')\ (2x'^2 - y'^2 - z'^2)\ dV' \\ Q_{yy} &= \int_V \rho(\mathbf{r}')\ (2y'^2 - z'^2 - x'^2)\ dV' \\ Q_{zz} &= \int_V \rho(\mathbf{r}')\ (2z'^2 - x'^2 - y'^2)\ dV' \\ Q_{xy} = Q_{yx} &= \int_V \rho(\mathbf{r}')\ (3x'y')\ dV' \\ Q_{yz} = Q_{zy} &= \int_V \rho(\mathbf{r}')\ (3y'z')\ dV' \\ Q_{zx} = Q_{xz} &= \int_V \rho(\mathbf{r}')\ (3z'x')\ dV' \ . \end{aligned}$$

[1]Recall that a second-rank tensor transforms under rotations as $Q'_{i_1i_2} = R_{i_1j_1}R_{i_2j_2}\ Q_{j_1j_2}$ where R_{ij} is the rotation matrix and a repeated index implies summation.

The monopole (charge) q is a scalar; the dipole moment $\mathbf{p}$ is a vector, with 3 components. The quadrupole moment $\overleftrightarrow{Q}$ is a tensor, with 9 components. However, since $Q_{xy} = Q_{yx}$, $Q_{yz} = Q_{zy}$, $Q_{zx} = Q_{xz}$ (symmetric) and further, $Q_{xx} + Q_{yy} + Q_{zz} = 0$ (traceless), $\overleftrightarrow{Q}$ only has 5 *independent* components. The quadrupole moment can also be written in the form of a real, symmetric (3×3) matrix with its trace equal to zero,

$$\begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{xy} & Q_{yy} & Q_{yz} \\ Q_{xz} & Q_{yz} & Q_{zz} \end{pmatrix} \ .$$

For a discrete distribution of n point charges q_i at positions $\mathbf{r}'_i$, the formulae above for the quadrupole moment are modified in obvious manner to

$$Q_{xx} = \sum_{i=1}^{n} q_i (2x_i'^2 - y_i'^2 - z_i'^2)$$

$$Q_{yy} = \sum_{i=1}^{n} q_i (2y_i'^2 - z_i'^2 - x_i'^2)$$

$$Q_{zz} = \sum_{i=1}^{n} q_i (2z_i'^2 - x_i'^2 - y_i'^2)$$

$$Q_{xy} = Q_{yx} = \sum_{i=1}^{n} q_i (3x'_i y'_i) \ , \quad Q_{yz} = Q_{zy} = \sum_{i=1}^{n} q_i (3y'_i z'_i)$$

$$Q_{zx} = Q_{xz} = \sum_{i=1}^{n} q_i (3z'_i x'_i) \ .$$

The quadrupole moment (of charge and/or mass distributions) makes its appearance in several branches of physics such as nuclear physics, solid state physics, plasma physics, geophysics and astrophysics, in different contexts.

Please scroll down for the next document...

MULTIPOLES

(1)

signs separated by a distance l. The electric dipole was treated in Example 2.9, and the potential was given in Eq. (2.44): $\Phi(\mathbf{r}) = \mathbf{p}\cdot\mathbf{r}/4\pi\varepsilon_0 r^3$, where $\mathbf{p} = q\boldsymbol{l}$. In the y-z plane, Φ reduces to $\Phi(r, \theta) = p\cos\theta/(4\pi\varepsilon_0 r^2)$ or $r(\theta) = \alpha(\cos\theta)^{1/2}$ where $\alpha = (p/4\pi\varepsilon_0\Phi)^{1/2}$. This relation between r and θ can now be plotted for different values of Φ, and hence different values of α, as shown in Fig. 2.23.

2.8 The Multipole Expansion

We turn now to the problems of characterizing the electrostatic potentials and fields of an arbitrary charge distribution, localized in a rather small region of space. One may think of this charge distribution as the charge distribution of a molecule, whose linear dimensions are of the order of 10^{-10} meter.

Consider Fig. 2.24, where we show a charge distribution that is localized in a volume V and characterized by a density ρ. We choose an origin, O, in or near this charge distribution. The displacement $\mathbf{r}'$ locates an element of charge relative to O. The displacement $\mathbf{r}$ locates a point in space outside the charge distribution, where we wish to determine the potential $\Phi(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{|\mathbf{r} - \mathbf{r}'|}$$

In fact, for any $\mathbf{r}'$, we shall assume that $r'/r \ll 1$. Then, we may expand the term in the integrand,

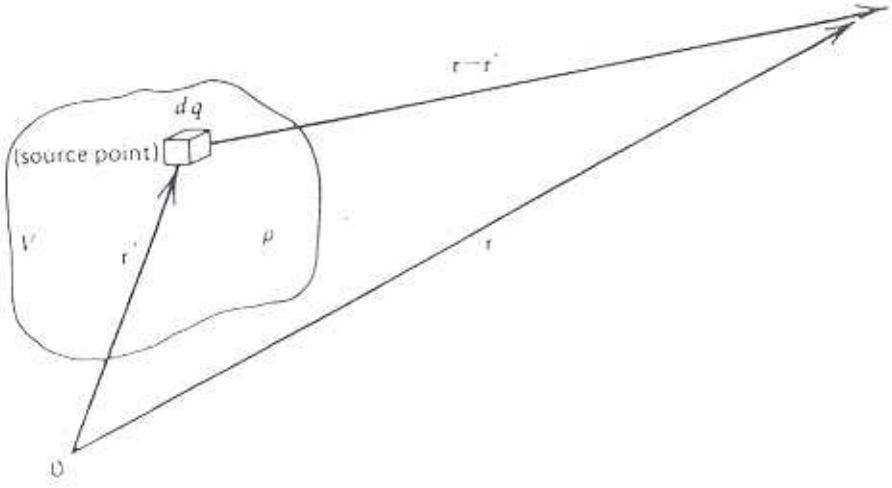
$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r\left[1 - \dfrac{2\mathbf{r}\cdot\mathbf{r}'}{\mathbf{r}^2} + \left(\dfrac{\mathbf{r}'}{\mathbf{r}}\right)^2\right]^{1/2}}$$

Now we apply the binomial expansion

$$(1 + x)^{-1/2} = \left[1 - \frac{1}{2}x - \left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^2}{2!} + \cdots\right]$$

valid for $|x| < 1$. Letting

$$x = \left(-\frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right)$$



we have

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \cong \frac{1}{r}\left[1 - \frac{1}{2}\left(-\frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right) - \frac{1}{2}\left(-\frac{3}{2}\right)\frac{1}{2!}\left(-\frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right)^2 + \cdots\right]$$

Grouping the powers of r'/r in ascending order, we find

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \cong \frac{1}{r}\left\{1 + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r} + \frac{1}{2}\left[3\left(\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r}\right)^2 - \left(\frac{r'}{r}\right)^2\right] + \cdots\right\}$$

Hence the potential becomes

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}\left\{1 + \hat{\mathbf{r}}\cdot\frac{\mathbf{r}'}{r} + \frac{1}{2}\left[3\left(\frac{\mathbf{r}'\cdot\hat{\mathbf{r}}}{r}\right)^2 - \left(\frac{r'}{r}\right)^2\right] + \cdots\right\}$$

We now rewrite the potential in a more revealing form by writing it as a sum of three integrals corresponding to three potential contributions $\Phi^{(0)}$, $\Phi^{(1)}$, and $\Phi^{(2)}$, respectively.

$$\begin{aligned}\Phi(\mathbf{r}) &= \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots \\ &= \frac{1}{4\pi\varepsilon_0 r}\int dq + \frac{1}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{r}}\cdot\int \mathbf{r}'\,dq + \frac{1}{4\pi\varepsilon_0 r^3}\int\left[\frac{3(\hat{\mathbf{r}}\cdot\mathbf{r}')^2 - r'^2}{2}\right]dq + \cdots \qquad (2.61)\end{aligned}$$

It will be observed that successive terms of this expansion differ by a factor of the order of R'/r, where R' is a linear dimension characteristic of the charge distribution. Therefore, the dominant term of the distribution, when $r \gg R'$, will be the first nonvanishing term. If R' were of atomic dimensions and r was a macroscopic distance, then $R'/r \leq 10^{-4}$.

In certain cases, the point of observation may be enclosed by the charge distribution. If $r \ll R'$, where R' is the smallest dimension of the distribution, then the potential in the region $r < R'$ is represented by a series expansion in terms of r/r' rather than r'/r as in the previous case. The result is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int\frac{dq}{r'} + \mathbf{r}\cdot\int\frac{\mathbf{r}'}{r'^3}\,dq + \frac{1}{2}\int\left(\frac{3(\mathbf{r}\cdot\mathbf{r}')^2}{r'^5} - \frac{r^2}{r'^3}\right)dq\right] \qquad (2.62)$$

We will not discuss the interior problem [given by Eq. (2.62)] any further except in Example 2.17 and Problem 2.2. The exterior problem, however, will now be discussed in detail. Let us now consider the terms $\Phi^{(0)}$, $\Phi^{(1)}$, and $\Phi^{(2)}$ in Eq. (2.61) separately.

Monopole Term: $\Phi^{(0)}$. The potential $\Phi^{(0)}$ is called the monopole potential; it can be written as follows:

$$\Phi^{(0)} = \frac{1}{4\pi\varepsilon_0 r}\int dq = \frac{Q}{4\pi\varepsilon_0 r}$$

where Q is the total net charge of the charge distribution. The presence of this term simply indicates that far enough away the charge distribution in the lowest-order approximation looks like a point charge, Q.

Dipole Term: $\Phi^{(1)}$***—Dipoles in External Fields.*** The contribution $\Phi^{(1)}$ is called the dipole term; it can be written as follows:

where **p** is the vector called the *dipole moment* of the charge distribution, defined as

$$\mathbf{p} = \int \mathbf{r}'\, dq = \int \rho(\mathbf{r}')\mathbf{r}'\, dv' \tag{2.63}$$

and ρ is the charge density. The importance of the dipole term is that when $Q = 0$, it is the term that dominates the expansion. Because of the importance of "dipoles" in the discussion of electrical properties of matter, we catalog several properties of the dipole field, and of the dipole moment **p**.

1. In general, the dipole moment, **p**, depends upon one's choice of origin. In fact, it can always be set equal to zero for some origin if the total, net, charge of the distribution, Q, is not zero. If, however, $Q = 0$, then **p** has a value independent of origin.

2. The prototype dipole consists of two point charges of equal magnitude but opposite sign with a relative displacement δ. We see that the dipole moment can be expressed as

$$\mathbf{p} = \int \mathbf{r}'\, dq = \mathbf{r}_-(-q) + \mathbf{r}_+(+q) = q(\mathbf{r}_+ - \mathbf{r}_-)$$

where $\mathbf{r}_+$ and $\mathbf{r}_-$ are, respectively, the vector positions of q and $-q$ charges from an origin O. One often considers such dipoles as giving rise to dipole fields. However, it must be emphasized that the dipole potential $\Phi^{(1)}$ is the potential of this dipole only in the limit as $\delta/r \to 0$. The potential of two such point charges is not $\Phi^{(1)}$ when $\delta/r \approx 1$. Note that we separate any charge distribution into its positive and negative charge components, such the $dq = dq_+ + dq_-$; that is,

$$\mathbf{p} = \int \mathbf{r}'\, dq = \int \mathbf{r}'\, dq_+ + \int \mathbf{r}'\, dq_- = \langle \mathbf{r}'_+ \rangle Q_+ + \langle \mathbf{r}'_- \rangle Q_-$$

where $\langle \mathbf{r}'_+ \rangle$ and $\langle \mathbf{r}'_- \rangle$ represent the average displacements of the total positive charge Q_+ and the total negative charge Q_- from the origin. If the net charge of the distribution is zero, then $Q_- = -Q_+$, and

$$\mathbf{p} = Q_+[\langle \mathbf{r}'_+ \rangle - \langle \mathbf{r}'_- \rangle] \equiv Q_+ \delta$$

3. The energy required to place a dipole in an external electrostatic field of potential function Φ is

$$U^{(1)} = -q\Phi(\mathbf{r}_-) + q\Phi(\mathbf{r}_+)$$

where $\mathbf{r}_+ = \mathbf{r}_- + \delta$. If δ is small enough, we can approximate $\Phi(\mathbf{r}_- + \delta)$ by the first two terms of the Taylor expansion $\Phi(\mathbf{r}_- + \delta) = \Phi(\mathbf{r}_-) + (\delta \cdot \nabla)\Phi(\mathbf{r}_-)$. Therefore $U^{(1)} = -q\Phi(\mathbf{r}_-) + q\Phi(\mathbf{r}_-) + q\delta \cdot \nabla\Phi(\mathbf{r}_-)$. Taking $\mathbf{E} = -\nabla\Phi(\mathbf{r})$ gives

$$U^{(1)} = -\mathbf{p} \cdot \mathbf{E}(\mathbf{r}) \tag{2.64}$$

In asserting that δ is small enough, we imply that **E** is in fact constant "over the dipole." Then $U^{(1)}$ has the simple interpretation as the work necessary to displace a positive charge q by δ in the field **E**. However, it does not include the energy required to form the dipole in the absence of **E**.

4. The force **F** on a dipole immersed in an external field, **E**, is $\mathbf{F} = \mathbf{F}_- + \mathbf{F}_+$ where $\mathbf{F}_-$ and $\mathbf{F}_+$ are the forces acting on the $-q$ and q charges as shown in Fig. 2.25. Writing $\mathbf{F}_\pm = \pm q\mathbf{E}(\mathbf{r}_\pm)$, **F** becomes

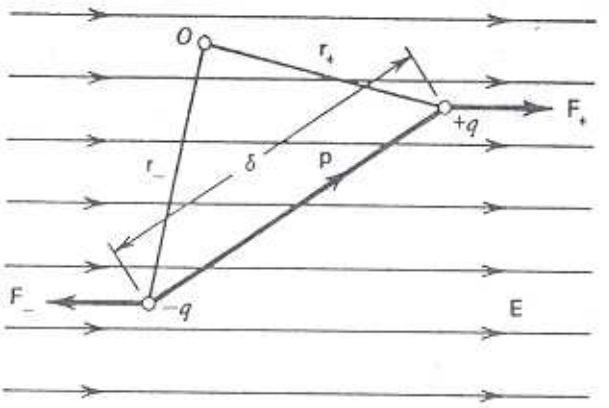


Figure 2.25 Schematic diagram of an electric dipole interacting with an external electric field by representing the dipole by two separated charges.

Keeping the first two terms in the Taylor expansion gives

$$\mathbf{F} \approx -qE(\mathbf{r}_-) + qE(\mathbf{r}_-) + q(\boldsymbol{\delta}\cdot\nabla)E(\mathbf{r}_-)$$

or

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \tag{2.65}$$

It is to be noted from Eq. (2.65) that the force on a dipole is zero if the field in which it is immersed is uniform.

5. The torque $\boldsymbol{\tau}$ on the dipole when placed in a uniform field $\mathbf{E}$ is just

$$\boldsymbol{\tau} = \mathbf{r}_- \times \mathbf{F}_- + \mathbf{r}_+ \times \mathbf{F}_+$$

where $\mathbf{F}_-$ and $\mathbf{F}_+$ are the forces exerted by the field on the $-q$ and q charges, as shown in Fig. 2.25. Substituting for $\mathbf{F}_\pm = \pm q\mathbf{E}$ gives

$$\boldsymbol{\tau} = (\mathbf{r}_+ - \mathbf{r}_-) \times q\mathbf{E} = \boldsymbol{\delta} \times q\mathbf{E} = \mathbf{p} \times \mathbf{E} \tag{2.66}$$

The torque is in such a direction as to align the dipole moment along the field E. If the field E is not uniform, one can directly show that

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} + \mathbf{r} \times \mathbf{F} \tag{2.67}$$

where F is the force acting on the dipole, and r is its displacement from the origin—about which the torque is computed.

6. Since a dipole may be conceived of two equal and opposite "monopoles" separated by the displacement δ, then referring to Fig. 2.25 we get

$$\Phi^{(1)} = \Phi^{(0)}(\mathbf{r}) - \Phi^{(0)}(\mathbf{r} + \boldsymbol{\delta})$$

Using the Taylor expansion gives

$$\Phi^{(1)} = \Phi^{(0)}(\mathbf{r}) - \Phi^{(0)}(\mathbf{r}) - (\boldsymbol{\delta}\cdot\nabla)\Phi^{(0)}(\mathbf{r}) = -\boldsymbol{\delta}\cdot\nabla\Phi^{(0)}(\mathbf{r}) \tag{2.68}$$

This expression was originally encountered in Example 2.9. In a similar fashion, it is easy to show that we may construct a multipole of order n from a multipole of order $(n-1)$. In fact,

$$\Phi^{(n)} = \Phi^{(n-1)}(\mathbf{r}) - \Phi^{(n-1)}(\mathbf{r} + \boldsymbol{\delta}^{(n)}) = -\boldsymbol{\delta}^{(n)}\cdot\nabla\Phi^{(n-1)} \tag{2.69}$$

where $\boldsymbol{\delta}^{(n)}$ is the

5

Example 2.15 Some Dipole Moments

We determine in this example the dipole moments of a number of charge distributi (shown in Fig. 2.26).

(1) *A Point Charge.* The dipole moment of the point charge q located a displacemen from the origin is $\mathbf{p} = q\mathbf{r}'$. If we had chosen the origin at the location of the charge, the dip moment would have been zero.

(2) *N Point Charges.* The dipole moment of N point charges $q_1, q_2, \ldots, q_N$ is given by

$$\mathbf{p} = \sum_{j=1}^{N} \mathbf{r}'_j q_j$$

where $\mathbf{r}'_j$ is the displacement of charge q_j from the origin. Only if

$$\sum_{j=1}^{N} q_j = 0$$

is $\mathbf{p}$ independent of the choice of origin.

(3) *A Circular Ring of Charge.* The dipole moment of a circular ring uniformly charged wi respect to an origin at the center of the ring is given by

$$p = \int \mathbf{r}' \, dq = \int_{\phi=0}^{2\pi} R\hat{\boldsymbol{\rho}}\lambda R \, d\phi = \lambda R^2 \int_0^{2\pi} \hat{\boldsymbol{\rho}} \, d\phi = 0$$

where R is the radius of the ring and λ is the linear charge density.

(4) *Rod of Charge.* The rod shown has the charge density

$$\rho = \alpha\left(z - \frac{l}{2}\right) \qquad \text{for } 0 \le z \le l$$

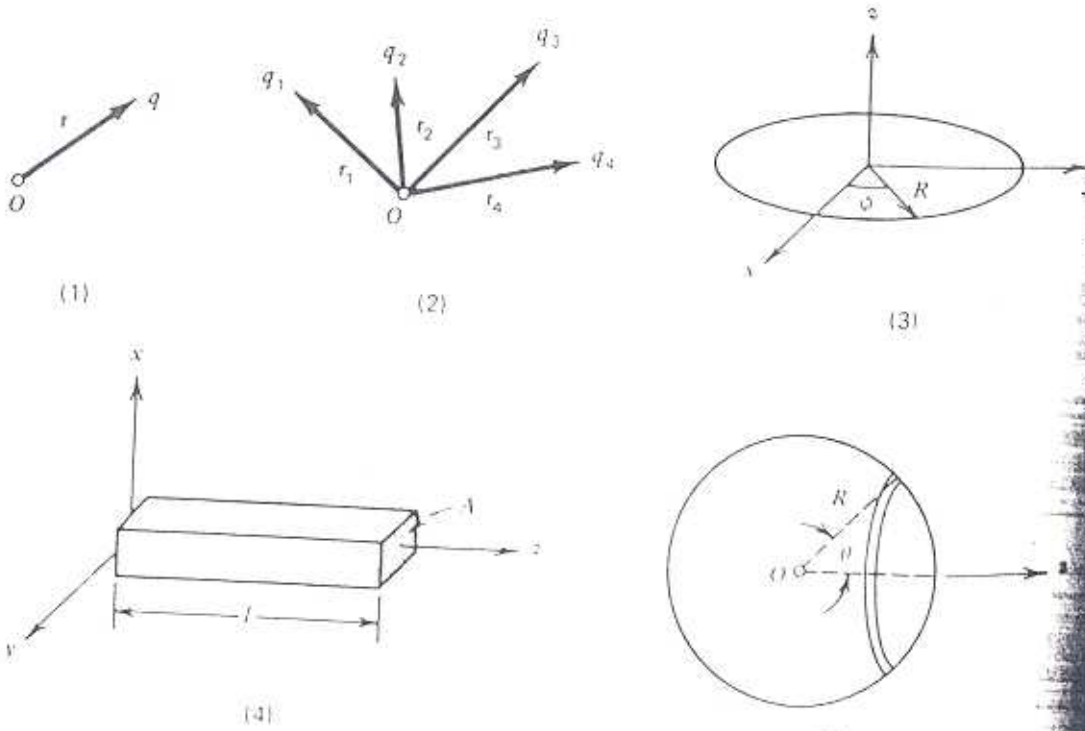


Figure 2.26 Some charge distributions

Its dipole moment is given by

$$\mathbf{p} = \int \mathbf{r}'\, dq = \hat{\mathbf{z}} \int_{z=0}^{l} z(\rho A\, dz) = \hat{\mathbf{z}}\alpha A \int_{z=0}^{l} z\left(z - \frac{l}{2}\right) dz = \frac{1}{12}\hat{\mathbf{z}} l^3 \alpha A \tag{2.70}$$

Note that the total charge in the rod $\int dq$ can be easily shown to be zero, and therefore $\mathbf{p}$ is independent of our choice of origin.

(5) *Sphere with an Angular Charge Distribution.* We consider the dipole moment of a sphere with a surface charge density given by $\sigma = \sigma_0 \cos\theta$. A point on the surface is located by $\mathbf{r}' = R\hat{\mathbf{r}}$, which is equal to $z'\hat{\mathbf{z}} + \rho'\hat{\boldsymbol{\rho}}$ in cylindrical coordinates, where $\rho'^2 + z'^2 = R^2$. The charge dq on a ring at an angle θ' is $dq = 2\pi R^2 \sigma \sin\theta'\, d\theta'$. Therefore $\mathbf{p} = \int \mathbf{r}'\, dq = \int (\hat{\mathbf{z}}z' + \hat{\boldsymbol{\rho}}\rho')dq$. The integral $\int \hat{\boldsymbol{\rho}}\rho'\, dq$ vanishes; thus

$$\mathbf{p} = \int \hat{\mathbf{z}} z'\, dq = \hat{\mathbf{z}} 2\pi R^3 \sigma_0 \int_{\theta=0}^{\pi} \cos^2\theta' \sin\theta'\, d\theta' = \hat{\mathbf{z}}\frac{4\pi R^3 \sigma_0}{3} = \hat{\mathbf{z}}\sigma_0 V \tag{2.71}$$

where V is the volume of the sphere. This dipole moment is independent of the choice of the origin because the total charge on the sphere is zero.

Quadrupole Term; $\Phi^{(2)}$. The contribution to the potential $\Phi^{(2)}$ is called the quadrupole term;

$$\Phi^{(2)} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{2r^5}\int [3(\mathbf{r}\cdot\mathbf{r}')^2 - r^2 r'^2]dq \tag{2.72}$$

We have written this term so that the integrand is completely symmetric with respect to r and r'. Expanding the integrand in cartesian coordinates, and noting that the integration over the charge distribution depends on the primed coordinates (x', y', z'), not on the coordinates of the point of observation, (x, y, z)] give

$$\begin{aligned}\int [3(\mathbf{r}\cdot\mathbf{r}')^2 - (rr')^2]dq = {} & (3x^2 - r^2)\int x'^2\, dq + 3xy\int x'y'\, dq + 3xz\int x'z'\, dq \\ & + 3xy\int x'y'\, dq + (3y^2 - r^2)\int y'^2\, dq + 3yz\int y'z'\, dq \\ & + 3zx\int z'x'\, dq + 3zy\int z'y'\, dq + (3z^2 - r^2)\int z'^2\, dq\end{aligned}$$

The "off-diagonal" elements of the array of terms are equal. The array of integrals is a matrix called the *quadrupole matrix*. Its elements will be denoted by Q_{xx}, Q_{xy}, Q_{xz}, etc. A knowledge of these elements completely specifies $\Phi^{(2)}$, just as a knowledge of the components of the dipole moment (p_x, p_y, p_z) specifies $\Phi^{(1)}$. Equation (2.72) can now be written as

$$\Phi^{(2)} = \frac{1}{4\pi\varepsilon_0}\sum_{i,j=1}^{3}\frac{Q_{ij}}{2r^5}(3x_i x_j - \delta_{ij}r^2) \tag{2.73}$$

where

$$Q_{ij} = \int x'_i x'_j\, dq \tag{2.74}$$

Occasionally the terms above are regrouped, so that instead of the diagonal terms

$$(3x^2 - r^2)\int x'^2\,dq + (3y^2 - r^2)\int y'^2\,dq + (3z - r^2)\int z'^2\,dq$$

one writes the equivalent expression (show this)

$$3x^2\int\left(x'^2 - \frac{1}{3}r'^2\right)dq + 3y^2\int\left(y'^2 - \frac{1}{3}r'^2\right)dq + 3z^2\int\left(z'^2 - \frac{1}{3}r'^2\right)dq$$

In terms of these integrals (denoted Q'_{xx}, Q'_{yy}, Q'_{zz}) the quadrupole matrix is said to be "reduced," a word chosen because $Q'_{xx} + Q'_{yy} + Q'_{zz} = 0$, whereas $Q_{xx} + Q_{yy} + Q_{zz} = \int r'^2\,dq$.

Thus the potential in (2.72) can alternatively be written as follows:

$$\Phi^{(2)} = \frac{1}{4\pi\varepsilon_0}\frac{1}{2r^5}\sum_{i,j=1}^{3} 3x_i x_j Q'_{ij} \tag{2.75}$$

where 1, 2, and 3, denote x, y, and z, and

$$Q'_{ij} = \int\left(x'_i x'_j - \frac{1}{3}\delta_{ij}r'^2\right)dq \tag{2.76}$$

It is however possible to simplify the description of this matrix yet further by properly choosing the coordinate axes. If they are chosen judiciously, the (off-diagonal) terms Q_{xy}, Q_{xz}, Q_{yz}, can be made to equal zero. It turns out that this can always be accomplished by choosing the axes to be perpendicular to planes of symmetry. A case of special importance is one where there is *rotational symmetry* about an axis, which we call the z axis. Rotational symmetry here implies that one cannot distinguish one orientation of the distribution about the z axis from any other. In such a case $Q_{xy} = 0$, $Q_{xz} = 0$, and $Q_{yz} = 0$. Moreover, $Q_{xx} = Q_{yy}$, since the rotational symmetry renders the x and y coordinates indistinguishable. In terms of the reduced quadrupole matrix, one sees in this case that $Q'_{xx} + Q'_{yy} = -Q'_{zz}$, implying that

$$Q'_{xx} = Q'_{yy} = -\frac{1}{2}Q'_{zz} \tag{2.77}$$

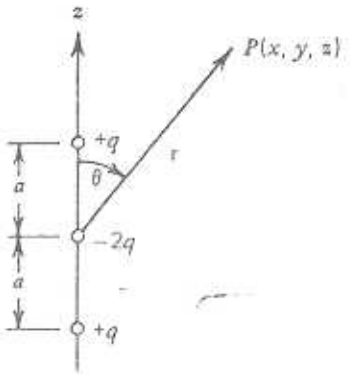
Thus, all of the nonzero quadrupole matrix elements are expressible in terms of Q'_{zz}, sometimes called the *quadrupole moment* of the distribution, as follows:

$$Q'_{zz} = \int(z'^2 - \frac{1}{3}r'^2)dq = \frac{1}{3}\int r'^2(3\cos^2\theta' - 1)dq \tag{2.78}$$

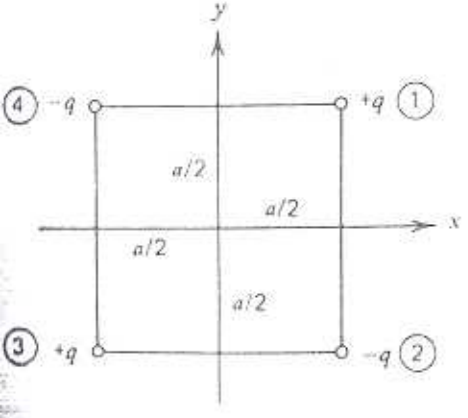
where θ' denotes the angle between the z axis and the charge element dq. If $Q'_{zz} > 0$, one has a cigar-shaped ellipsoid, whereas if $Q'_{zz} < 0$, one has a saucer-shaped (oblate) ellipsoid of charge. In this sense, the quadrupole moment is a measure of the deformation of the ellipsoid from spherical symmetry.

Example 2.16 Quadrupole Distributions of Point Charges

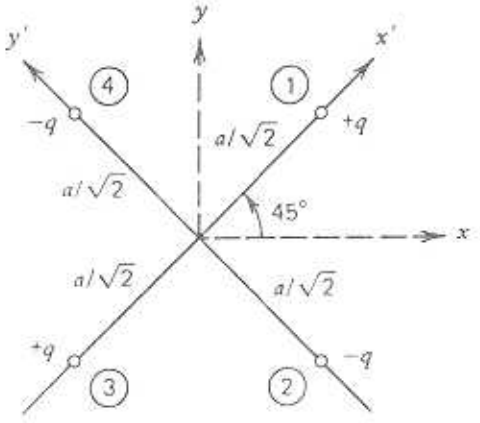
In this example we determine the quadrupole matrix of the point charge distribution shown in Figs. 2.27*a* and 2.27*b*. In the case where one is dealing with point charges alone, the quadrupole potential $\Phi^{(2)}$ reduces to the sum



(a)



(b)



(c)

Figure 2.27 Quadrupole charge distribution. (*a*) Linear quadrupole. (*b*) Two-dimensional quadrupole. (*c*) Same as (*b*) but with axes rotated.

and the reduced quadrupole matrix has elements of the form

$$Q'_{ij} = \sum_{m=1}^{N} \left(x'_{im} x'_{jm} - \frac{1}{3} \delta_{ij} r'^2_m \right) q_m \tag{2.79}$$

where the sum is over the number of charges, and q_m and x_{im} are the magnitude and the ith coordinate of the mth charge.

The set of charges shown in Fig. 2.27*a* has zero monopole and dipole moments. Its reduced quadrupole matrix elements are easily calculated by using Eq. (2.79):

$$Q'_{xy} = Q'_{xz} = Q'_{yz} = 0$$

$$Q'_{yy} = Q'_{xx} = q\left(-\frac{1}{3}a^2\right) + 0 + q\left(-\frac{1}{3}a^2\right) = -\frac{2}{3}qa^2$$

$$Q'_{zz} = 2q\left[a^2 - \frac{1}{3}a^2\right] = \frac{4}{3}qa^2$$

$Q'_{xx} + Q'_{yy} + Q'_{zz} = 0$. The potential $\Phi^{(2)}$ is therefore (for $r > a$)

$$\Phi^{(2)} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{2r^5} \left[3x^2\left(-\frac{2}{3}qa^2\right) + 3y^2\left(-\frac{2}{3}qa^2\right) + 3z^2\left(\frac{4}{3}qa^2\right) \right]$$

This array of charges, called a "linear quadrupole," has rotational symmetry about the z axis, and its "quadrupole moment" is $Q'_{zz} = 4qa^2/3$.

2. Another point quadrupole charge distribution is shown in Fig. 2.27*b*. The monopole and dipole moments are here zero. The reduced quadrupole matrix has matrix elements:

$$Q'_{xx} = \sum q_i\left(x_i^2 - \frac{1}{3}r_i^2\right) = 0$$

$$Q'_{yy} = Q'_{zz} = 0$$

$$Q_{xy} = \sum q_i x_i y_i = qa^2$$

$$Q_{xz} = Q_{yz} = 0$$

that is, only $Q_{xy} = Q_{yx}$ are not zero. The potential $\Phi^{(2)}$ is thus given by

$$\Phi^{(2)} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{2r^5}\cdot[6xyQ_{xy}] = \frac{1}{4\pi\varepsilon_0}\frac{3xy}{r^5}qa^2$$

If this array of charges were referred to the new set of axes shown—that is, x' and y' in Fig. 2.27*c*—we would have had $Q'_{xx} = 2q[\frac{1}{2}a^2] = qa^2$, $Q'_{yy} = -qa^2$, $Q'_{zz} = 0$, and $Q'_{xy} = Q'_{xz} = Q'_{yz} = 0$. Therefore we would have

$$\Phi^{(2)} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{2r^5}[3x'^2\cdot qa^2 - 3y'^2\cdot qa^2] = \frac{1}{4\pi\varepsilon_0}\cdot\frac{3qa^2}{2r^5}[x'^2 - y'^2]$$

The expressions for the potentials have different forms. However, if the transformations between the different coordinate systems are made, or

$$x' \to \frac{1}{\sqrt{2}}(x + y) \qquad \text{and} \qquad y' \to \frac{1}{\sqrt{2}}(-x + y)$$

one finds that the latter expression for $\Phi^{(2)}$ becomes the former.

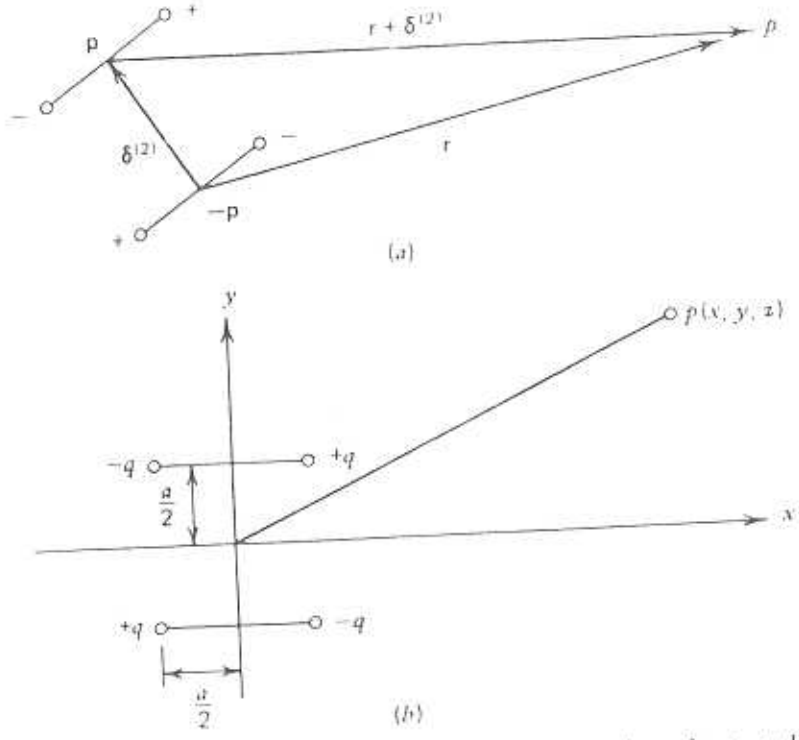


Figure 2.28 Determination of the fields of a quadrupole as a differential of a dipole field. (*a*) General quadrupole distribution. (*b*)

3. It is interesting to point out that the "pure" quadrupole distributions shown are simply pairs of dipoles, one displaced from the other with the signs of the charges reversed. If $\delta^{(2)}$ is the displacement of the one dipole from the other (see Fig. 2.28*a*), we have, from Eq. (2.69),

$$\Phi^{(2)} = \Phi^{(1)}(\mathbf{r}) - \Phi^{(1)}(\mathbf{r} + \delta^{(2)}) = (-\delta^{(2)} \cdot \nabla)\Phi^{(1)}(\mathbf{r}) \tag{2.80}$$

This result is analogous to the derivation of the dipole field from the field of a simple monopole [see Eq. 2.68)]. For the quadrupole of Fig. 2.28*b*,

$$\Phi^{(0)}(r) \equiv \frac{q}{4\pi\varepsilon_0 r} \qquad r = \sqrt{x^2 + y^2 + z^2} \qquad \delta^{(1)} = \delta^{(1)}\hat{\mathbf{x}} = a\hat{\mathbf{x}} \qquad \delta^{(2)} = \delta^{(2)}\hat{\mathbf{y}} = a\hat{\mathbf{y}}$$

Substituting these expressions in Eq. (2.80) gives

$$\Phi^{(2)} = -\frac{\partial}{\partial y}\Phi^{(1)}(\mathbf{r}) = -a\frac{\partial}{\partial y}\left[-a\frac{\partial}{\partial x}\left(\frac{q}{4\pi\varepsilon_0 r}\right)\right]$$

or

$$\Phi^{(2)} = a^2\frac{\partial^2}{\partial x\,\partial y}\left\{\frac{q}{4\pi\varepsilon_0\sqrt{x^2 + y^2 + z^2}}\right\} = \frac{a^2 q}{4\pi\varepsilon_0}\frac{3xy}{r^5}$$

just as previously obtained.

Example 2.17 Potential by Multipole Expansion

This example explains the use of the multipole expansion in calculating the potential due to an angular charge distribution. Consider a spherical shell of radius R, carrying a surface charge distribution $\sigma = \sigma_0 \cos\theta$, where θ is the angle with respect to the z axis. From Eq. (2.61), the potential outside the sphere ($r > R$) is

$$\Phi(\mathbf{r}) = \Phi^{(0)}(\mathbf{r}) + \Phi^{(1)}(\mathbf{r}) + \Phi^{(2)}(\mathbf{r}) + \cdots$$

where

$$\Phi^{(0)}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0 r}\int_{s'} \sigma_0 \cos\theta'\, da'$$

$$\Phi^{(1)}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}\cdot\int_{s'} \mathbf{r}'\,\sigma_0 \cos\theta'\, da'$$

$$\Phi^{(2)}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\int_{s'}\left[\frac{3(\mathbf{r}\cdot\mathbf{r}')^2}{r^5} - \frac{r'^2}{r^3}\right]\sigma_0 \cos\theta'\, da'$$

where θ' is the angle between $\mathbf{r}'$ and the z axis, and s' is the surface of the shell. The potential $\Phi^{(0)}(\mathbf{r}) = 0$ since the total charge on the sphere is zero.

The potential $\Phi^{(1)}$ can be evaluated as follows: One first writes $\mathbf{r}'$ in terms of θ' and ϕ', as follows:

$$\mathbf{r}' = (R\sin\theta'\cos\phi')\hat{\mathbf{x}} + (R\sin\theta'\sin\phi')\hat{\mathbf{y}} + R\cos\theta'\,\hat{\mathbf{z}}$$

Thus the integral in $\Phi^{(1)}$ becomes

$$\sigma_0\int \mathbf{r}'\cos\theta'\, da' = R^3\sigma_0\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}[(\sin^2\theta'\cos\theta'\cos\phi')\hat{\mathbf{x}} + (\sin^2\theta'\cos\theta'\sin\phi')\hat{\mathbf{y}} + \cos^2\theta'\sin\theta'\,\hat{\mathbf{z}}]\,d\theta'\,d\phi'$$

The first two terms vanish because

$$\int_0^{2\pi}\sin\phi'\,d\phi' \qquad \int_0^{2\pi}\cos\phi'\,d\phi' = 0$$

and the last term gives $\mathbf{p} = (4\pi/3)\sigma_0 R^3 \hat{\mathbf{z}}$. Therefore

$$\Phi^{(1)}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

The potential $\Phi^{(1)}(\mathbf{r})$ is the first nonvanishing contribution; it is a dipole potential with a dipole moment $\mathbf{p}$ along the z axis and moment equal to σ_0 times the volume bounded by the shell. This potential agrees with the direct calculations of the dipole moment of the shell [see Eq. (2.71)].

Using a similar procedure one can show that $\Phi^{(2)}(\mathbf{r})$ as well as the other higher multipoles vanish, indicating that the field of this charge distribution is a dipole field; that is,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \qquad r > R \tag{2.81}$$

Inside the sphere—that is, for $r < R$—we use Eq. (2.62). Thus

$$\Phi(\mathbf{r}) = \Phi^{(0)}(\mathbf{r}) + \Phi^{(1)}(\mathbf{r}) + \Phi^{(2)}(\mathbf{r}) + \cdots$$

where

$$\Phi^{(0)}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{s'}\frac{\sigma_0\cos\theta'}{r'}\,da'$$

$$\Phi^{(1)}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\mathbf{r}\cdot\int_{s'}\frac{\mathbf{r}'}{r'^3}\sigma_0\cos\theta'\,da'$$

$$\Phi^{(2)}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{2}\int_{s'}\left[\frac{3(\mathbf{r}\cdot\mathbf{r}')^2}{r'^5} - \frac{r^2}{r'^3}\right]\sigma_0\cos\theta'\,da'$$

Substituting R for the magnitude of $\mathbf{r}'$, and taking it outside the integral gives

$$\Phi^{(0)} = \frac{1}{4\pi\varepsilon_0 R}\int\sigma_0\cos\theta'\,da' = 0$$

since the integral is just the total charge on the sphere. Doing the same thing in the expression for $\Phi^{(1)}$ gives

$$\Phi^{(1)} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}}{R^3}\cdot\int\mathbf{r}'\sigma_0\cos\theta'\,da' = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{R^3}\cdot\mathbf{r}$$

where $\mathbf{p}$, the dipole moment, is as defined above. Again, as in the region exterior to the sphere, $\Phi^{(2)}$ and the other higher-order terms vanish. Therefore

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{R^3}\cdot\mathbf{r} = \frac{\sigma_0}{3\varepsilon_0}z \qquad r < R \tag{2.82}$$

It is apparent that the potential inside the shell depends only on z and is independent of the size of the shell, unlike the potential outside the shell. The corresponding electric field is $(\sigma_0/3\varepsilon_0)\hat{\mathbf{z}}$, which is uniform and directed along the negative z axis.

The previous discussion indicates that the electric multipoles can be used to approximate the electric field or potential of an arbitrary charge distribution. This process, in fact, is equivalent to the approximation of the charge distribution itself by a combination of a point charge, a point dipole, a point quadrupole, and so forth, as shown schematically in Fig. 2.29. The applications of such approximation

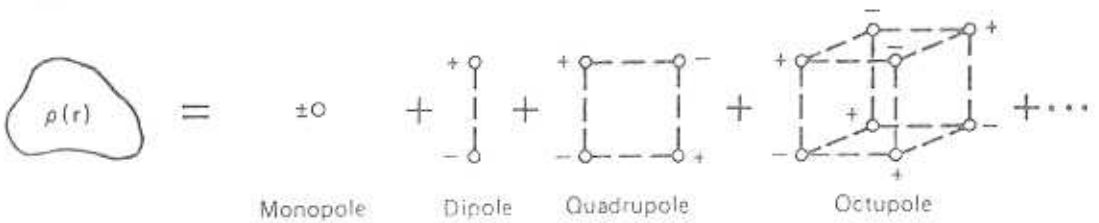


Figure 2.29 Schematic diagram of the representation of a localized charge distribution by its various multipoles.

2.9 Summary

Electrostatics is the subject matter that deals with electric charges that are at rest. Coulomb's law defines the electrostatic force law between a point charge q_0 at the origin and a point charge q located at $\mathbf{r}$; that is,

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_0 q\hat{\mathbf{r}}}{r^2}$$

where

$$\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9\ \mathrm{N \cdot m^2/C^2}$$

in MKS units. Writing

$$\mathbf{F} = q\mathbf{E}$$

defines the electrostatic field $\mathbf{E}$ associated with the charge q_0

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_0\hat{\mathbf{r}}}{r^2}$$

Coulomb's law and the electric field can be generalized to many point charges or continuous charge distributions that may residue in volumes, on surfaces, or along lines such that the element of charge dq is given by

$$dq = \rho\, d\tau,\ \sigma\, da,\ \text{or}\ \lambda\, dl$$

where ρ, σ, and λ are the volume, surface, and line charge densities, respectively. For a point charge q_i located at $\mathbf{r}_i$,

$$\rho(\mathbf{r}) = q_i\delta(\mathbf{r} - \mathbf{r}_i)$$

where δ is the Dirac delta function. Since forces add vectorially, then

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \tag{2.15}$$

This electrostatic field is said to be conservative. In other words,

$$\nabla \times \mathbf{E} = 0 \tag{2.37}$$

Its divergence, on the other hand, depends linearly on the charge density

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{2.33}$$

This is often called the differential form of Gauss's law, and it is one of four fundamental laws of electromagnetism as we understand them today (Maxwell's equations). It is even satisfied by time-dependent fields. The curl property, however, is true only for electrostatics and will be modified later when time-varying sources are considered.

The integral law of Gauss' law follows from the differential law by integrating both sides over an arbitrary volume V bounded by the surface S, and applying the divergence theorem to the left side; that is,

$$\oint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{1}{\varepsilon_0}\int \sigma\,dv = \frac{Q_{\text{int}}}{\varepsilon_0} \tag{2.27}$$

where Q is the total charge enclosed by S. In cases of symmetry where the electric field is of constant magnitude over all elements of the surface, and of constant direction relative to the surface, Gauss' law simplifies the calculation of the electric field at the surface. A powerful implication of Gauss' law, along with the fact that the electric field inside a conductor is zero, is the fact that charge on a conductor must reside on its outer surface, with the fields just outside the conductor being $\mathbf{E} = (\sigma/\varepsilon_0)\hat{\mathbf{n}}$. The law also shows that the fields just below and just above a surface charge distribution, $\mathbf{E}_2$ and $\mathbf{E}_1$, are discontinuous, with the discontinuity being

$$(\mathbf{E}_2 - \mathbf{E}_1)\cdot\hat{\mathbf{n}} = \frac{\sigma}{\varepsilon_0} \tag{2.35}$$

At distances from the charge distribution that are large compared to the largest dimension of the distribution, the charge distribution can be approximated by a combination of a point charge Q, a point dipole $\mathbf{p}$, a point quadrupole Q_{ij}, etc., where

$$Q = \int \rho(\mathbf{r}')dr' \qquad \mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')dr' \qquad Q'_{ij} = \int (x'_i x'_j - \frac{1}{3}\delta_{ij}r'^2)dq \tag{2.63)–(2.76}$$

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{\mathbf{r}^2} + \frac{1}{2\mathbf{r}^5}\sum_{i,j=1}^{3} 3x_i x_j Q'_{ij}\right)$$

where 1, 2, and 3 denote x, y, and z, respectively. The applications of such approximation in fact goes beyond just the electric or the potential of the distribution.

When a point charge is placed in an electrostatic potential Φ of corresponding field $\mathbf{E}$, the charge experiences a force $\mathbf{F} = q\mathbf{E}$, and the potential energy U of the charge is

$$U = q\Phi$$

If a dipole $\mathbf{p}$ is placed in such a field, we have for the force and energy

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad U = -\mathbf{p}\cdot\mathbf{E} \tag{2.64),(2.65}$$

The dipole will also experience a torque τ as follows:

$$\tau = \mathbf{p}\times\mathbf{E} + \mathbf{r}\times\mathbf{F} \tag{2.67}$$

Problems

2.1 Four point charges, $q = 2\times10^{-5}$ C each, are on the corners of a square of length 4 m. Find the force on a point charge $q_0 = 10^{-4}$ C located 3 m just above the center of the square.

2.2 Determine the electric field at the center of curvature of a uniformly charged semicircular rod with a total charge q.

2.3 A filamentary charge is distributed along the z axis with a charge density $\lambda = \lambda_0$ for $|z| > 5$ and $\lambda = 0$ for $|z| < 5$. Find $\mathbf{E}$ on the x axis, 2 m from the origin.

2.4 A filamentary charge is distributed along the z axis with a charge density $\lambda = \lambda_0$ for $|z| < d$ and $\lambda = 0$ for $|z| > d$. Determine the $\mathbf{E}$ field in the $x - y$ plane a distance R from the z axis.

2.5 Determine, by direct integration, the E field on the axis of a uniformly charged disk of radius a and charge density σ.

2.6 A circular disk of radius a lies in the $z = 0$ plane with its center at the origin. The disk has a surface charge density $\sigma = \sigma_0/\rho$. Determine the electric field at $z = h$ along the z axis. Discuss the nature of the field when $h \gg a$.

2.7 A sheet of charge lies in the $z = 0$ plane and occupies the area $0 \leq x < 2$ m and $0 \leq y < 2$ m. The surface charge density is $\sigma = 2x(x^2 + y^2 + 4)^{3/2}$ C/m^2. Determine the electric field 2 m above the sheet on the z axis.

2.8 A circular disk of radius a has a nonuniform charge density $\sigma = \sigma_0 \sin^2 \phi$. Determine E on its axis at $z = h$.

2.9 Two infinite, uniform sheets of charge in the $y - z$ plane of charge densities σ and σ' are located at $x = 1$ and $x = -1$, respectively. Determine the E in all regions for both cases $\sigma' = \sigma$, and $\sigma' = -\sigma$.

2.10 A sheet of uniform charge density $\sigma = -10^{-7}$ C/m^2 occupies the $y = 2$ m plane. Next to it a line of uniform charge density $\lambda = 0.4$ μC/m lies parallel to the x axis at $y = -1$ m and $z = 2$ m. Determine the region in which E will be zero.

2.11 Use Gauss' law to determine the E field produced by a very long cyclinder of charge of volume density $\rho = 5re^{-2r}$ C/m^3, where r is the distance from the axis of the cyclinder.

2.12 Use Gauss' law to determine the E field produced by a spherical charge distribution of density $\rho = \alpha/r^2$, where α is a constant.

2.13 Consider a closed surface S. Calculate the net flux that crosses it due to the following charge distributions enclosed by it. (a) Three point charges: $q_1 = 3 \times 10^{-8}$ C, $q_2 = 1.5 \times 10^{-7}$ C, and $q_3 = -7 \times 10^{-8}$ C. (b) A circular disk of charge of radius 2 m and $\sigma = (\sin^2 \phi)/\rho$ C/m^2. (c) A circular disk of charge of radius 2 m and $\sigma = \sin \phi$ C/m^2. (d) Two point charges $q_1 = 20 \times 10^{-7}$ C and $q_2 = -20 \times 10^{-7}$ C.

2.14 Determine the electric flux density 5 m from a point charge, $q = 3 \times 10^{-8}$ C.

2.15 Find the charge densities that produce the following fields in V/m.

(a) $\mathbf{E} = 10 \sin \theta \, \hat{\mathbf{r}} + 2 \cos \theta \, \hat{\boldsymbol{\theta}}$.

(b) $\mathbf{E} = \frac{1}{2}\alpha\left(\rho - \frac{a^2}{\rho}\right)\hat{\boldsymbol{\rho}}$ for $a \leq \rho \leq b$ $\qquad$ $\mathbf{E} = \frac{1}{2}\frac{\alpha}{\rho}(b^2 - a^2)\hat{\boldsymbol{\rho}}$ for $\rho > b$.

where α, a, and b are constants.

2.16 A thin ring has its inner and outer diameters equal to ρ_0 and $\rho_0 + W$, respectively. The ring has a uniform surface charge density σ. Determine the potential at the center of the ring. Does it depend on ρ_0?

2.17 A charge is distributed uniformly along a straight line of finite length $2l$. (a) Determine the potential at distance r_1 just above the midpoint. (b) What is the potential in the limit $l \gg r_1$? (c) Determine for two points, r_1 and r_2, just above the midpoint the potential difference Φ_{12} if $l \gg r_1 > r_2$. How does this potential difference compare with the potential of an infinite line charge?

2.18 (a) Determine the E field associated with the potential

$$\Phi = \frac{a \cos \theta}{r^2} + \frac{b}{r}$$

(b) What is the charge distribution responsible for this potential? (c) Find the charge distribution that gives rise to the potential $\Phi(r) = -qe^{-\alpha r}/r$, where q and α are constants.

2.19 This problem is given to illustrate the power of linear superposition as applied to calculations of electric fields and potentials. A spherical cavity of radius a is hollowed from the interior of a uniformly charged sphere of radius R and charge density ρ_0. The distance between the centers of the sphere and the cavity is d. (a) Determine the E field at a distance $r < R$ from the center of the sphere, assuming that there is no