DEPARTMENT OF PHYSICS
INDIAN INSTITUTE OF TECHNOLOGY, MADRAS

PH1020 Physics II

Tutorial 4 (19.2.2018)

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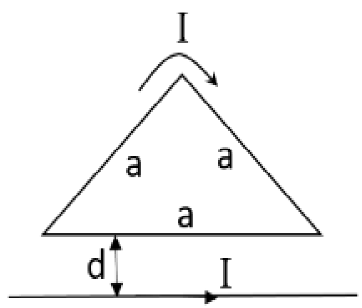
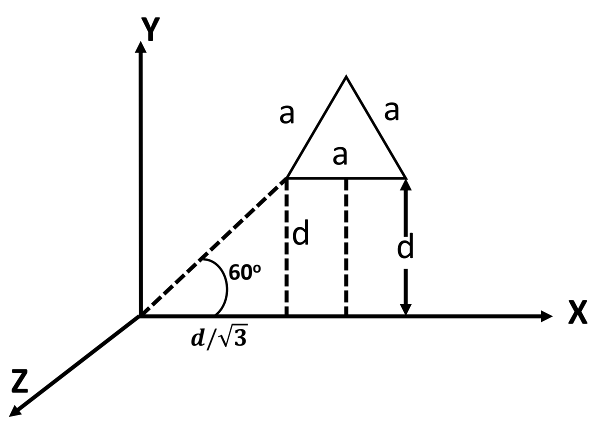
1. Find the force on a triangular loop due to a current carrying wire (see figure). Both the loop and the infinite wire carry a steady current I.



Solution:



Consider the bottom side. **B** due to a current carrying wire $= \frac{\mu_0 \mathrm{I}}{2\pi \mathrm{y}}\ \hat{z}$.

$$\therefore \text{ The force on the bottom side } = \mathrm{I} \int \vec{\mathrm{dl}} \times \vec{\mathrm{B}}$$

$$= \frac{\mathrm{I}\mu_0 \mathrm{I}}{2\pi \mathrm{d}} \int \mathrm{dl}$$

$$= \frac{\mu_0\ \mathrm{I}^2\ \mathrm{a}}{2\pi \mathrm{d}} \qquad \text{(upward deflection } \hat{y}\text{)}$$

On the left side, $\mathbf{B} = \frac{\mu_0\ \mathrm{I}}{2\pi \mathrm{y}}\ \hat{e}_z$

$$d\mathbf{F} = \mathrm{I}\ (\ d\mathbf{l} \times \mathbf{B}\ )$$

$$= \mathrm{I}\ (\ dx\ \hat{\mathrm{e}}_{\mathrm{x}} +\ dy\ \hat{\mathrm{e}}_{\mathrm{y}} +\ dz\ \hat{\mathrm{e}}_{\mathrm{z}}) \ \times \left(\frac{\mu_0\ \mathrm{I}}{2\pi \mathrm{y}}\ \hat{\mathrm{e}}_z \right)$$

$$= \frac{\mu_0\ \mathrm{I}^2}{2\pi \mathrm{y}}\ (\ -\ dx\ \hat{\mathrm{e}}_{\mathrm{y}} +\ dy\ \hat{\mathrm{e}}_{\mathrm{x}})$$

But the x component cancels the corresponding term from the right hand side.
So,

$$\mathrm{F}_{\mathrm{y}} = -\ \frac{\mu_0\ \mathrm{I}^2}{2\pi} \int_{\frac{\mathrm{d}}{\sqrt{3}}}^{\frac{d}{\sqrt{3}}+\frac{\mathrm{a}}{2}} \frac{1}{\mathrm{y}}\ dx$$

Here $\mathrm{y} = \sqrt{3}\ \mathrm{x}$.

$$\mathrm{F}_{\mathrm{y}} = -\ \frac{\mu_0\ \mathrm{I}^2}{2\pi} \int_{\frac{\mathrm{d}}{\sqrt{3}}}^{\frac{d}{\sqrt{3}}+\frac{\mathrm{a}}{2}} \frac{1}{\sqrt{3}\ \mathrm{x}}\ dx$$

$$= -\ \frac{\mu_0\ \mathrm{I}^2}{2\sqrt{3}\pi}\ \ln \left(\frac{\frac{d}{\sqrt{3}} + \frac{\mathrm{a}}{2}}{\frac{\mathrm{d}}{\sqrt{3}}} \right)$$

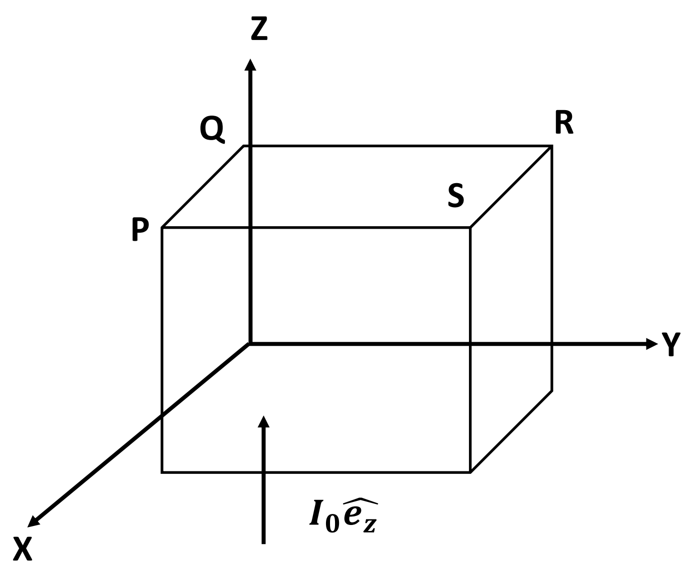
$$= -\ \frac{\mu_0\ \mathrm{I}^2}{2\sqrt{3}\pi}\ \ln \left(1 + \ \frac{\sqrt{3}\ \mathrm{a}}{2\ \mathrm{d}} \right)$$

The force on the right side is same, so the net force on the triangle is the sum of forces on 3 sides

$$= \frac{\mu_0 \, \mathrm{I}^2}{2\pi} - \frac{\mu_0 \, \mathrm{I}^2}{\sqrt{3}\pi} \ln\left(1 + \frac{\sqrt{3}\,\mathrm{a}}{2\,\mathrm{d}}\right)$$

$$= \frac{\mu_0 \, \mathrm{I}^2}{2\pi}\left[1 - \frac{2}{\sqrt{3}} \ln\left(1 + \frac{\sqrt{3}\,\mathrm{a}}{2\,\mathrm{d}}\right)\right]$$

2. A conducting material with rectangular cross section PQRS is placed with the sides PQ along the x-axis and QR along y-axis. A uniform current $I_0 \hat{e_z}$ flows across the cross section. Conduction electrons therefore move with a drift velocity $\mathbf{v} = -v_0 \hat{e}_z$ and the conductor is placed in a magnetic field $\mathbf{B} = B_0 \hat{e}_y$. (a) How are the electrons deflected? (b) Find the resulting potential difference between the opposite faces containing QR and PS.
   Solution: Given $\mathbf{B} = \mathrm{B}_0 \, \hat{\mathrm{e}}_\mathrm{y}$ and $\mathbf{v} = -\mathrm{v}_0 \, \hat{\mathrm{e}}_\mathrm{z}$ We have,



a)

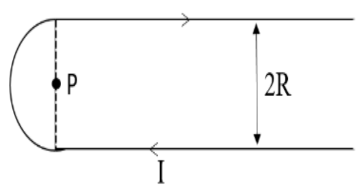
$$
\begin{aligned}
\mathbf{F} &= q\ (\mathbf{v} \times \mathbf{B}) \\
&= -e\ (\mathbf{v} \times \mathbf{B}) \\
&= -e\ (-v_0\ \hat{e}_z \times B_0\ \hat{e}_y) \\
&= -e\ (v_0\ B_0\ \hat{e}_x) \\
\mathbf{F} &= -e\ v_0\ B_0\ \hat{e}_x
\end{aligned}
$$

The electrons get deflected along $\hat{e}_x$.

b) Electric field due to accumulation of charges balance the magnetic field in steady state i.e.

$$
\begin{aligned}
E_x &= ev_0\ B_0 \\
\text{but } E_x &= \frac{V}{d} \\
\therefore\quad V &= \text{ Hall voltage} = E_x\ d
\end{aligned}
$$

3. A thin conducting wire in the configuration shown in the figure carries a steady current I. Find the magnetic field **B** at the point P, the center of the semicircle.



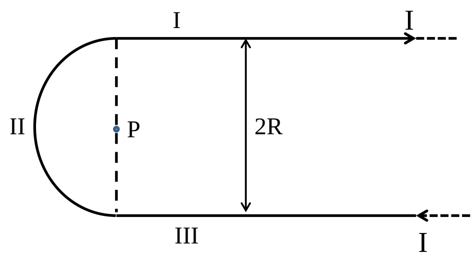
Solution:

Magnetic field at point P due to part I (lower semi-infinite wire) is

$$\mathbf{B} = \frac{1}{2}\frac{\mu_0 I}{2\pi R} \qquad \textit{(into the plane)}$$

Similarly for part III (upper semi-infinite wire)

$$\mathbf{B} = \frac{1}{2}\frac{\mu_0 I}{2\pi R} \qquad \textit{(into the plane)}$$



Magnetic field due to the semicircle is

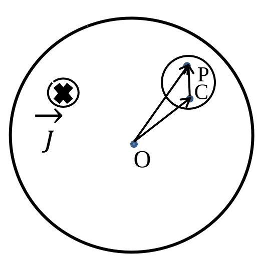
$$\mathbf{B} = \frac{1}{2}\frac{\mu_0 I}{2R} \qquad \textit{(into the plane)}$$

Therefore the net field at point P is

$$\mathbf{B} = \frac{\mu_0 I}{4R} + \frac{\mu_0 I}{4\pi R} + \frac{\mu_0 I}{4\pi R} \qquad \textit{(into the plane)}$$

$$\therefore \ \mathbf{B} = \frac{\mu_0 I}{4R}(1 + \frac{2}{\pi}) \qquad \textit{(into the plane)}$$

4. A long cylindrical conductor of radius R has a cylindrical hole of radius $b(b < R)$. The axis of the hole is parallel to the axis of the conductor. The remaining portion of the conductor has a uniform volume current density $\mathbf{J}$ parallel to the axis. Show that the magnetic field in the hole is uniform.
   Solution:



The current density in the hole is zero. This can be considered as due to the superposition of $\mathbf{J}$ and $-\mathbf{J}$, where $-\mathbf{J}$ flows through the cylinder of radius b. Thus, the magnetic field at a point P inside the hole can be written as a superposition of fields due to $\mathbf{J}$ and $-\mathbf{J}$.

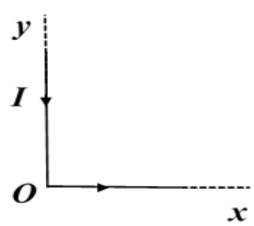
The direction of $\mathbf{B}(P)$ due to $\mathbf{J}$ is given by $\mathbf{J} \times \mathbf{r}$, where $\mathbf{r}$ is the position vector of P and the direction of $\mathbf{B}(P)$ due to $-\mathbf{J}$ is given by $-\mathbf{J} \times \overrightarrow{\mathrm{CP}}$.

So, the net field at P is

$$\begin{aligned}
\mathbf{B}(P) &= \frac{\mu_0}{2}\mathbf{J} \times \mathbf{r} - \frac{\mu_0}{2}\mathbf{J} \times \overrightarrow{\mathrm{CP}} \\
&= \frac{\mu_0}{2}\mathbf{J} \times (\mathbf{r} - \overrightarrow{\mathrm{CP}}) \\
&= \frac{\mu_0}{2}\mathbf{J} \times \overrightarrow{\mathrm{OC}}
\end{aligned}$$

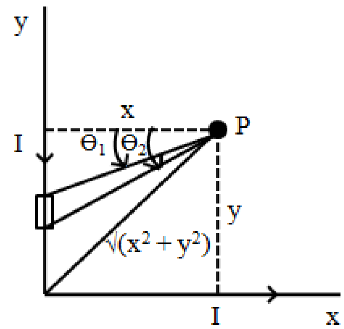
So, the magnetic field in the hole is uniform.

5. A steady current I flows through an L-shaped wire as shown in the figure. Calculate the magnetic field in the xy-plane over the domain $x > 0$ and $y > 0$.



Solution:

$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$



Magnetic field at any point at a distance d from a straight current carrying conductor is given by

$$\mathbf{B}_1 = \frac{\mu_0 I}{4\pi d}\int_{\theta_1}^{\theta_2} cos\theta d\theta = \frac{\mu_0 I}{4\pi d}(sin\theta_2 - sin\theta_1)$$

in this case $\theta_1 \to -\pi/2$

Therefore,

$$\mathbf{B}_1 = \frac{\mu_0 I}{4\pi d}(sin\theta_2 - sin(-\pi/2))$$

$$\mathbf{B}_1 = \frac{\mu_0 I}{4\pi x}(\frac{y}{\sqrt{x^2+y^2}} + 1)(\text{out of plane})$$

Similarly

$$\mathbf{B}_2 = \frac{\mu_0 I}{4\pi y}(\frac{x}{\sqrt{x^2+y^2}} + 1)(\text{out of plane})$$

6. A steady current density in a medium is given by $\mathbf{J}(\rho, \phi, z) = J_0 e^{-\lambda\rho^2}\hat{e}_z$ where J_0 and λ are positive constant of appropriate dimensions. Assume $\mu = \mu_0$ for the medium. (a) Find the magnetic field arising out of this current. (b) Sketch the magnitude of the field as a function ρ.

   $\mathbf{J} = J_0 e^{-\lambda\rho^2}\hat{e}_z$

   (a) Ampere's Law

$$\begin{aligned} |\mathbf{B}(\rho)|(2\pi\rho) &= \mu_0 \int_0^{\rho} \rho' d\rho' \int_0^{2\pi} d\phi' J_0 e^{-\lambda\rho'^2} \\ &= 2\pi J_0 \mu_0 \int_0^{\rho} \rho' e^{-\lambda\rho'^2} d\rho' \end{aligned}$$

put $\lambda\rho'^2 = t$
$2\lambda\rho' d\rho' = dt \Rightarrow \rho' d\rho' = \frac{1}{2\lambda} dt$
Therefore,

$$\begin{aligned}
\mathbf{B}(\rho) &= \frac{\mu_0 J_0}{\rho} \int_0^{\lambda\rho^2} e^{-t} dt/2\lambda \\
&= \frac{\mu_0 J_0}{2\lambda\rho} [-e^{-t}]_0^{\lambda\rho^2} \\
&= \frac{-J_0 \mu_0}{2\lambda\rho} [e^{-\lambda\rho^2} - 1]
\end{aligned}$$

$$\mathbf{B}(\rho) = \frac{\mu_0 J_0}{2\lambda\rho} [1 - e^{-\lambda\rho^2}]$$

(b)

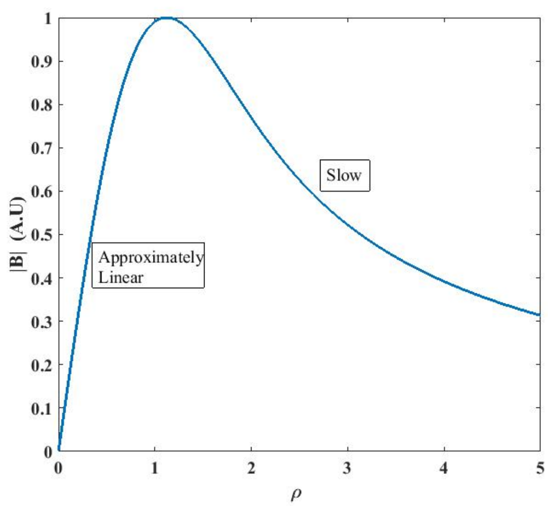


Figure 1: Magnitude of field as a function of ρ