MA2020 - Differential Equations-2017
Assignment - 1

1. Use variables separable method to solve the following problems.

   (i) $(y+\sqrt{x^2+y^2})dx - xdy = 0, y(1) = 0.$

   Ans: $y+\sqrt{x^2+y^2} = x^2.$

   (ii) $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin\left(\dfrac{y}{x}\right)$

   Ans: $y = 2x\text{Arc}\tan cx.$

   (iii) $2y\exp(x/y)dx + (y - 2x\exp(x/y))dy = 0$

   Ans: $2\exp(x/y) + \ln y = c.$

   (iv) $(x\exp(y/x) - y\sin(y/x))dx + x\sin(y/x)dy = 0.$

   Ans: $2\ln x - \exp(-y/x)(\sin(y/x) + \cos(y/x)) = c.$

   (v) $\dfrac{dy}{dx} = \dfrac{2x+3y+4}{4x+6y+5}$

   Ans: $14(2x+3y) - 9\ln(14x+21y+22) = 49x + c.$

   (vi) $\dfrac{dy}{dx} = \dfrac{4x-6y-1}{2x-3y+2}$

   Ans: $24x - 12y + 15\ln(8x-12y-7) = c.$

   (vii) $\dfrac{dy}{dx} = \dfrac{x+2y-3}{2x+y-3}$

   Ans: $(x+y-2) = c(x-y)^3.$

   (viii) $\dfrac{dy}{dx} = \dfrac{y+2}{x+y+1}$

   Ans: $x - 1 = (y+2)(\ln(y+2) + c).$

2. Find the general solution of each of the following equations.

   (i) $xy' + y = x^3$

   Ans: $4xy = x^4 + c.$

   (ii) $xy' + y = y^2\ln x$

   Ans: $y\ln x + y + cxy = 1.$

   (iii) $\frac{dx}{dy} + 2xy = \exp(-y^2)$

   Ans: $x = \exp(-y^2)(y + c).$

   (iv) $\frac{dr}{d\theta} = (r + \exp(-\theta))\tan\theta$

   Ans: $2r = c\sec\theta - \exp(-\theta)(\tan\theta + 1).$

   (v) $(1-x^3)y' = 2(1+x)y + y^{5/2}$

Ans: $y^{-3/2} = -\frac{3}{4(1+x+x^2)} + c\frac{(1-x)^2}{1+x+x^2}$.

3. Consider the differential equation [a generalization of the Bernoulli equation]

$$\frac{dy}{dx} + P(x)h(y) = f(x)g(y), h(y) = g(y)\int \frac{dy}{g(y)}.$$

Assume that the functions P, f, g, h are continuous on $\Re$. Show that the general solution of the differential equation is given by

$$e^{\int P(x)dx} \int \frac{dy}{g(y)} - \int f(x)e^{\int P(x)dx}dx = c,$$

where c is an arbitrary constant.

Hint: Use the transformation $u(x) = h(y)/g(y)$.

4. The number of cells $y = y(x)$ growing within a tumour is governed by the Gompertz equation $\frac{dy}{dx} - ay\ln(b/y) = 0$, where a and b are positive constants. Obtain y explicitly.

Ans: $y = b\exp(c\exp(-ax))$, where c is an arbitrary constant.

For problems 5(i)-(iii), you may use the idea of probelm 3.

5. Solve

(i) $\frac{dy}{dx} - \frac{\tan y}{(1+x)} = (1+x)\exp(x)\sec y$.

Ans: $\sin y = (\exp(x) + c)(1+x)$.

(ii) $\tan y\frac{dy}{dx} + \tan x = \cos y\cos^2 x$.

Ans: $\sec y\sec x = \sin x + c$.

(iii) $\frac{dz}{dx} + \left(\frac{z}{x}\right)\ln z = \frac{z}{x}\ln^2 z$.

Ans: $(x\ln z)^{-1} = x^{-1} + c$.

(iv) $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$.

Hint: View x as the dependent variable and the equation turns out to be a Bernoulli equation.

Ans: $\sqrt{x/y} = (\ln y)/2 + c$.

6. Show that each of the following equations is exact and find a one -parameter family of solutions.

(i)$(3x^2y + 8xy^2)dx + (x^3 + 8x^2y + 12y^2)dy = 0$

(ii) $2xydx + (x^2 + y^2)dy = 0$

(iii) $\cos y dx - (x \sin y - y^2)dy = 0$

7. Test each of the following equations for exactness. If it is not exact, find an integrating factor and hence solve.

   (i)$(x^2 + y^2 + x)dx + xydy = 0$

   Ans: $3x^4 + 4x^3 + 6x^2y^2 = c$; integrating factor x.

   (ii)$y(2x + y^3)dx - x(2x - y^3)dy = 0$

   Ans:$x^2 + xy^3 = cy^2$; integrating factor y^{-3}.

   (iii)$\exp(x)(x + 1)dx + (y \exp(y) - x \exp(x))dy = 0$

   Ans: $2x\exp(x - y) + y^2 = c$; integrating factor $\exp(-y)$.

   (iv)$(y^2 - 3xy - 2x^2)dx + (xy - x^2)dy = 0$

   Ans: $x^2y^2 - 2x^3y - x^4 = c$; integrating factor $2x$.

8. Consider the differential equation

$$\frac{dy}{dx} + P(x)h(y) = f(x)g(y), h(y) = g(y) \int \frac{dy}{g(y)}.$$

   Assume that the functions P, f, g, h are continuously differentiable on $\Re$.

   (i) Test for the exactness of the differential equation.

   (ii)If the differential equation is not exact, find an integrating factor.

   (iii) Obtain the solution.

   Hint: see Problem 3.

9. Plot the direction field and approximate solution curves for the following problems:

   (i) $y' = x^2$ (ii) $y' = -x/y$ (iii) $y' = x + y$ (iv) $y' = y^2$.

THE END