1. An infinitely long cylinder of radius a has its axis along the z-axis. Its magnetization is given in cylindrical polar coordinates by $\mathbf{M} = M_0\,(\rho/a)^2\,\hat{e}_\phi$, where M_0 is a constant. Find $\mathbf{J}_b$ and $\mathbf{K}_b$ as well as $\mathbf{B}$ and $\mathbf{H}$ both inside and outside the cylinder.

We are given $\mathbf{M} = M_0\frac{\varrho^2}{a^2}\ \hat{e}_\varphi$ inside the cylinder. Inside the cylinder, we have

$$\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} = \frac{3M_0\varrho}{a^2}\ \hat{e}_z\ .$$

Outside the cylinder, $\mathbf{J}_b = 0$. The bound surface current is given by

$$\mathbf{K}_b = \mathbf{M}\times\hat{e}_\varrho\Big|_{\varrho=a} = -M_0\ \hat{e}_z\ .$$

Since $\mathbf{J}_f = 0$ in this problem, we can determine $\mathbf{B}$ using the bound currents that we just determined. Symmetry considerations imply that we can choose $\mathbf{B} = B(\varrho)\ \hat{e}_\varphi$. To determine, $B(\varrho)$, we choose as an Amperian loop a circle of radius R centered about the z-axis and lying in a plane given by $z =$ constant. Using Ampère's law, for $R < a$, we get

$$\begin{aligned}\int_{C_R}\mathbf{B}\cdot\mathbf{dl} = (2\pi R)\ B(R) &= \mu_0\int_0^R\left(\mathbf{J}_b(\varrho)\cdot\hat{e}_z\right)\ 2\pi\varrho d\varrho\ ,\\ &= \frac{2\pi\mu_0 M_0 R^3}{a^2}\ .\end{aligned}$$

This implies that $B(\varrho) = \frac{\mu_0 M_0\varrho^2}{a^2}$ for $\varrho < a$. For $R > a$, a similar computation gives

$$\begin{aligned}\int_{C_R}\mathbf{B}\cdot\mathbf{dl} = (2\pi R)\ B(R) &= \mu_0\int_0^a\mathbf{J}_b(\varrho)\cdot\hat{e}_z\ 2\pi\varrho d\varrho + \mu_0(2\pi a)(-M_0),\\ &= \frac{2\pi\mu_0 M_0 a^3}{a^2} + \mu_0(2\pi a)(-M_0)\\ &= 0 \implies B(R) = 0\ .\end{aligned}$$

We thus obtain

$$\boxed{\mathbf{B} = \begin{cases}\frac{\mu_0 M_0\varrho^2}{a^2}\ \hat{e}_\varphi & \text{for } \varrho < a\ ,\\ 0 & \text{for } \varrho > a\ .\end{cases}} \tag{1}$$

The discontinuity in $\mathbf{B}$ at $\varrho = a$ is expected due to the presence of a non-zero surface current. The student is asked to check that the discontinuity is as expected i.e. $\hat{n}\times(\mathbf{B}_2 - \mathbf{B}_1) = \mu_0\ \mathbf{K}_b$. We determine $\mathbf{H}$ using the relation $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$ to obtain

$$\boxed{\mathbf{H} = 0 \text{ everywhere.}}$$

The vanishing of $\mathbf{H}$ is related to the absence of free currents.

2. Consider a toroid in which a wedge-shaped region of small angle ψ is absent, as shown in the figure. A steady current I flows in it. The inner radius of the toroid is R, and the total number of turns in it is N. Assume that the magnetic field $\mathbf{B}$ in the air gap is still along $\hat{e}_\phi$. Find $\mathbf{H}$ in the toroid given that the core of the toroid is a LIH magnetic material with magnetic susceptibility χ_m.

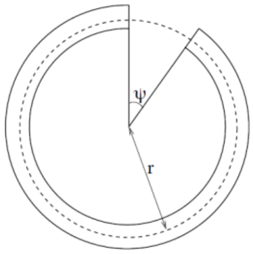


Figure 1: Top view of the toroid

Recall that for a toroidal coil, there is a constant auxiliary field $\mathbf{H}$ (and $\mathbf{B}$) directed along $\hat{e}_\varphi$ in its interior. The magnitude is determined in terms of the current and the total number of coils. We assume that the magnetic field $\mathbf{H}$ (and $\mathbf{B}$) is along $\hat{e}_\varphi$ – this is a good approximation if the wedge angle ψ is small. At the boundary of the wedge, the normal component of $\mathbf{B}$ must be continuous. Since the normal to the wedge is along $\hat{e}_\varphi$, we have $\mathbf{B}_{\text{air}} = \mathbf{B}_{\text{toroid}}$. Thus, the auxiliary field is given by

$$\mathbf{B}_{\text{toroid}} = \mu_0(1+\chi_m)\ \mathbf{H}_{\text{toroid}} \quad , \quad \mathbf{B}_{\text{air}} = \mu_0 \mathbf{H}_{\text{air}} \ . \tag{2}$$

The continuity of the normal component of $\mathbf{B}$ implies that

$$\mathbf{H}_{\text{air}} = (1+\chi_m)\ \mathbf{H}_{\text{toroid}} \ .$$

In order to determine $\mathbf{H}$, we choose an Amperian loop of radius r (indicated by a dashed line in Figure 1) passing through the interior of the toroid. Carrying out the line integral over $\mathbf{H}$, we get

$$\begin{aligned} NI &= H_{\text{toroid}}(2\pi-\psi)\ r + H_{\text{air}}\psi\ r \ , \\ &= H_{\text{toroid}}\ r\,(2\pi - \psi + (1+\chi_m)\psi) \\ &= H_{\text{toroid}}\ r\,(2\pi + \chi_m\psi) \end{aligned}$$

$$\boxed{\implies \mathbf{H}_{\text{toroid}} = \frac{NI}{(2\pi+\chi_m\psi)r}\ \hat{e}_\varphi \ .}$$

3. An infinite planar magnetic sheet of thickness d having a nonuniform permeability given by $\mu(z) = \mu_0\,[1+(z/d)]^2$ occupies the region $0 \le z \le d$. There is vacuum on either side of the sheet. A magnetic field $\mathbf{B} = B_0\,\hat{e}_y$ (where B_0 is a constant) is applied in the entire space. The sheet has no free current on it. Find the magnetization surface current densities at $z=0$ and $z=d$, and also the magnetization volume current density as a function of z.

   Translation invariance in the x and y directions implies that

$$\mathbf{H} = \mathbf{H}(z) = H_1(z)\ \hat{e}_x + H_2(z)\ \hat{e}_y + H_3(z)\ \hat{e}_z \ .$$

   The boundary conditions on $H(z)$ is

$$\lim_{|z|\to\infty} H(z) = \frac{B_0}{\mu_0} \ .$$

But $(\nabla \times \mathbf{H}) = 0$ implies that $dH_i(z)/dz = 0$ or $H_i(z) =$ constant for $i = 1, 2$ with no condition on $H_3(z)$. We thus obtain

$$\mathbf{H} = \frac{B_0}{\mu_0}\ \hat{e}_y + H_3(z)\ \hat{e}_z \quad \text{everywhere} .$$

To fix $H_3(z)$, we need to impose $\nabla \cdot \mathbf{B} = 0$. Outside the strip it implies $H_3(z)$ is a a constant which vanishes due to the boundary conditions as $|z| \to \infty$. This implies that

$$\mathbf{H} = \frac{B_0}{\mu_0}\ \hat{e}_y \quad \text{outside the strip.}$$

Inside the strip, $\nabla \cdot \mathbf{B} = 0$ implies that $\mu(z)H_3(z)$ is a constant. Continuity of the z component of $\mathbf{B}$, at $z = 0$, forces this constant to vanish. Thus, we obtain that $H_3(z) = 0$ everywhere.

$$\boxed{\mathbf{H} = \frac{B_0}{\mu_0}\ \hat{e}_y \quad \text{everywhere} .}$$

From this we obtain the magnetic field to be

$$\mathbf{B} = \begin{cases} B_0\left(1 + \frac{z}{d}\right)^2\ \hat{e}_y\ , & \text{for } 0 < z < d\ , \\ B_0\ \hat{e}_y\ , & \text{outside the magnetic sheet}\ . \end{cases}$$

The magnetization is determined using $\mathbf{M} = \frac{\mathbf{B}}{\mu_0} - \mathbf{H}$ and we get

$$\mathbf{M} = \begin{cases} \frac{B_0}{\mu_0}\left(2\frac{z}{d} + \frac{z^2}{d^2}\right)\ \hat{e}_y\ , & \text{for } 0 < z < d\ , \\ 0\ , & \text{outside the magnetic sheet}\ . \end{cases}$$

We can determine the current densities using the formulae $\mathbf{J}_b = \nabla \times \mathbf{M}$ and $\mathbf{K}_b = (\mathbf{M} \times \hat{n})$. We get that volume charge density for $0 < z < d$ is given by

$$\boxed{\mathbf{J}_b = -\frac{2B_0}{\mu_0 d}\left(1 + \frac{z}{d}\right)\ \hat{e}_x\ .}$$

Using ($\mathbf{M} = 0$, $\hat{n} = -\hat{e}_z$) at $z = 0$ and ($\mathbf{M} = (3B_0/\mu_0)\ \hat{e}_y$, $\hat{n} = \hat{e}_z$) at $z = d$, the surface current densities are

$$\boxed{\mathbf{K}_b = \begin{cases} 0\ , & \text{at } z = 0\ , \\ \frac{3B_0}{\mu_0}\hat{e}_x\ , & \text{at } z = d\ . \end{cases}}$$

Again, the student is asked to check the consistency of the expressions for $\mathbf{B}$ and $\mathbf{K}_b$ by checking that the discontinuity at the two interfaces are consistent with $\hat{n} \times (\mathbf{B}_2 - \mathbf{B}_1) = \mu_0\ \mathbf{K}_b$.

4. Suppose the field inside a large piece of magnetic material is $\mathbf{B_0}$ so that $\mathbf{H_0} = (1/\mu_0)\,\mathbf{B_0} - \mathbf{M}$. Where $\mathbf{M}$ is 'frozen-in'magnetization. Find the field, Find magnetic field in terms of $\mathbf{B_0}$ and $\mathbf{M}$, and $\mathbf{H}$ in terms $\mathbf{H_0}$ and $\mathbf{M}$ at the centre of a

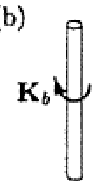
(a) at the centre of a small spherical cavity hollowed out of the material, in terms of $\mathbf{B_0}$ and $\mathbf{M}$. Also find $\mathbf{H}$ at the centre of the cavity, in terms $\mathbf{H_0}$ and $\mathbf{M}$.

(b) a long needle shaped cavity running parallel to $\mathbf{M}$

(c) a thin wafer shaped cavity perpendicular to $\mathbf{M}$

(a) The field of a magnetized sphere is $\frac{2}{3}\mu_0\mathbf{M}$, so $\boxed{\mathbf{B} = \mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M},}$ with the sphere removed.

In the cavity, $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$, so $\mathbf{H} = \frac{1}{\mu_0}\left(\mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M}\right) = \mathbf{H}_0 + \mathbf{M} - \frac{2}{3}\mathbf{M} \Rightarrow \boxed{\mathbf{H} = \mathbf{H}_0 + \frac{1}{3}\mathbf{M}.}$

(b) 

The field inside a long solenoid is $\mu_0 K$. Here $K = M$, so the field of the bound current on the inside surface of the cavity is $\mu_0 M$, pointing *down*. Therefore

$$\boxed{\mathbf{B} = \mathbf{B}_0 - \mu_0\mathbf{M};}$$

$$\mathbf{H} = \frac{1}{\mu_0}(\mathbf{B}_0 - \mu_0\mathbf{M}) = \frac{1}{\mu_0}\mathbf{B}_0 - \mathbf{M} \Rightarrow \boxed{\mathbf{H} = \mathbf{H}_0.}$$

(c) 

This time the bound currents are small, and far away from the center, so $\boxed{\mathbf{B} = \mathbf{B}_0,}$ while $\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}_0 = \mathbf{H}_0 + \mathbf{M} \Rightarrow \boxed{\mathbf{H} = \mathbf{H}_0 + \mathbf{M}.}$

[*Comment*: In the wafer, $\mathbf{B}$ is the field in the medium; in the needle, $\mathbf{H}$ is the $\mathbf{H}$ in the medium; in the sphere (intermediate case) both $\mathbf{B}$ and $\mathbf{H}$ are modified.]

5. A coaxial cable consists of two very long thin cylindrical tubes of radius a and b ($a < b$) separated by linear insulating materials of magnetic susceptibility χ_m. A current I, uniformly distributed over the cylinder, flows down the inner cylinder and returns along the outer one. Find the magnetic field in the region between the tubes. As a check, calculate magnetization and bound current, and confirm that together with free currents they generate the correct field.

$\oint \mathbf{H}\cdot d\mathbf{l} = I_{f_{enc}} = I$, so $\mathbf{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\phi}}$. $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} = \boxed{\mu_0(1+\chi_m)\frac{I}{2\pi s}\hat{\boldsymbol{\phi}}.}$ $\mathbf{M} = \chi_m\mathbf{H} = \boxed{\frac{\chi_m I}{2\pi s}\hat{\boldsymbol{\phi}}.}$

$$\mathbf{J}_b = \nabla\times\mathbf{M} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\chi_m I}{2\pi s}\right)\hat{\mathbf{z}} = \boxed{0.} \quad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = \boxed{\begin{cases} \frac{\chi_m I}{2\pi a}\hat{\mathbf{z}}, & \text{at } s = a; \\ -\frac{\chi_m I}{2\pi b}\hat{\mathbf{z}}, & \text{at } r = b. \end{cases}}$$

Total enclosed current, for an amperian loop between the cylinders:

$$I + \frac{\chi_m I}{2\pi a}2\pi a = (1+\chi_m)I, \quad \text{so} \quad \oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}} = \mu_0(1+\chi_m)I \Rightarrow \mathbf{B} = \frac{\mu_0(1+\chi_m)I}{2\pi s}\hat{\boldsymbol{\phi}}. \checkmark$$