EE1101 Signals and Systems JAN—MAY 2019
Tutorial 4 Solutions

1) For causal Discrete time LTI systems, h[n]=0 for n<0
For stable Discrete time LTI systems, $\sum\limits_{n=-\infty}^{\infty}|h[n]| < \infty$.

For Causal Continuous time LTI systems, h(t)=0 for t<0

For Stable Continuous time LTI systems,$\int_{\tau=-\infty}^{\infty}|h(\tau)|d\tau < \infty$.

(a) h[n] $= (-\frac{1}{2})^n u[n] + (1.01)^n u[1-n]$
Since u[1-n]$\neq$ 0 for n$<$ 0 , substituting $n < 0$ gives h[n] $\neq 0$
Therefore ,the system is NON CAUSAL
$\sum\limits_{n=-\infty}^{\infty}|h[n]| = \sum\limits_{n=-\infty}^{\infty}|(-\frac{1}{2})^n u[n] + (1.01)^n u[1-n]|$
$\sum\limits_{n=-\infty}^{\infty}|(-\frac{1}{2})^n u[n]| = \sum\limits_{n=0}^{\infty}|(-\frac{1}{2})^n| < \infty$
$\sum\limits_{n=-\infty}^{\infty}|(1.01)^n u[1-n]| = \sum\limits_{n=-\infty}^{1}|(1.01)^n| < \infty$
Therefore, $\sum\limits_{n=-\infty}^{\infty}|h[n]| < \infty$. So the system is STABLE

(b) h[n] $= n(\frac{1}{3})^n u[n-1]$
Since u[n-1]= 0 for n$<$ 1, h[n] = 0 for n<0
Therefore ,the system is CAUSAL

Intuitively, $n(\frac{1}{3})^n u[n-1]$ has a converging sum.Therefore, $\sum\limits_{n=-\infty}^{\infty}|h[n]| < \infty$.
To prove mathematically,

$$\begin{aligned}\sum_{n-\infty}^{\infty}|h[n]| &= \sum_{n=-\infty}^{\infty}|n(\frac{1}{3})^n u[n-1]| \\ &= \sum_{n=1}^{\infty}|n(\frac{1}{3})^n u[n-1]| \\ = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} &+ ....(sum\,of\,infinite\,AGP) \\ &= 0.75 < \infty\end{aligned}$$

Therefore,the system is STABLE

(c) h(t) $= e^{2t}u(-1-t)$
Since u(-1-t)$\neq$ 0 for t$<$ 0, h(t))$\neq$ 0 for t $<$0;
Therefore,system is NON CAUSAL

$$\begin{aligned}\int_{\tau=-\infty}^{\infty}|h(\tau)|d\tau = \int_{\tau=-\infty}^{\infty}|e^{2\tau}u(-1-\tau)|d\tau \\ = \int_{\tau=-\infty}^{-1}|e^{2\tau}|d\tau \\ = \frac{e^{-2}}{2} < \infty\end{aligned}$$

Therefore,the system is STABLE

(d) h(t) = $e^{-6|t|}$
Since h(t) has values at all instants of time, h(t))$\neq$ 0 for t <0;
Therefore,system is NON CAUSAL

$$\begin{aligned}\int_{\tau=-\infty}^{\infty}|h(\tau)|d\tau = \int_{\tau=-\infty}^{\infty}|e^{-6|\tau|}|d\tau \\ = \int_{\tau=-\infty}^{0}|e^{6\tau}|d\tau + \int_{\tau=0}^{\infty}|e^{-6\tau}|d\tau \\ = |\frac{1}{6}| + |\frac{1}{6}| = \frac{1}{3} < \infty\end{aligned}$$

Therefore,the system is STABLE

(e) h(t) = $(2e^{-t} - e^{(t-100)/100})u(t)$
Since h(t) is some function multiplied by u(t), h(t))= 0 for t <0;
Therefore,system is CAUSAL

$$\begin{aligned}\int_{\tau=-\infty}^{\infty}|h(\tau)|d\tau = \int_{\tau=-\infty}^{\infty}|(2e^{-\tau} - e^{(\tau-100)/100})u(\tau)|d\tau \\ = \int_{\tau=0}^{\infty}|(2e^{-\tau} - e^{(\tau-100)/100})|d\tau\end{aligned}$$

In this, We have an exponentially decaying part (($2e^{-\tau}$),and
Exponentially increasing part ($e^{(\tau-100)/100}$)
The exponentially increasing part makes the integral sum to infinte. Therefore ,system is UNSTABLE

2)

(a) (a) h[n] = $(-1/2)^n$u[n]

for n< 0

$$s[n] = 0$$

for $n \geq 0$

$$s[n] = \sum_{k=0}^{n}(\frac{-1}{2})^k$$

Using G.P. formula for finite series

$$s[n]= \frac{1}{3}(2+(\frac{-1}{2})^n)u[n]$$

(b) h[n] = nu[n]

for $n < 0$

$$s[n] = 0$$

for n$\geq$ 0

$$s[n] = \sum_{k=0}^{n} k$$

$$s[n]= \frac{n(n+1)}{2}u[n]$$

(c) h(t) = $e^{-|t|}$

for $t < 0$

$$s(t)= \int_{-\infty}^{0} e^{\tau} d\tau = e^t$$

for t$\geq$ 0

$$s(t) = \int_{-\infty}^{0} e^{\tau} d\tau + \int_{0}^{t} e^{-\tau} d\tau = 2- e^{-\tau}$$

$$s(t)=\begin{cases} e^t & t<0 \\ 2-e^{-t} & t \geq 0 \end{cases}$$

(d) h(t)=(1/4)(u(t)-u(t-4))

for $t < 0$

$$s(t) = 0$$

for $t < 4$

$$s(t) = \frac{1}{4}\int_{0}^{t} d\tau= \frac{1}{4}t$$

for t$\geq$ 4

$$s(t) =\frac{1}{4}\int_{0}^{4} d\tau= 1$$

$$s(t) = \begin{cases} 0 & t<0 \\ \frac{1}{4}t & 0 \leq t < 4 \\ 1 & t \geq 4 \end{cases}$$

(e) h(t) = u(t)

for $t < 0$

$$s(t) = 0$$

for t$\geq$ 0

$$s(t)= \int_{0}^{t} d\tau = t$$

$$s(t) = tu(t)$$

3)

(a) True. If $h(t)$ periodic and nonzero, then

$$\int_{-\infty}^{\infty} |h(t)|dt = \sum_{n=-\infty}^{+\infty} \int_{(n-1)T}^{nT} |h(t)|dt.$$

Since each summand is same , the infinite sum is unbounded. Thus $h(t)$ is unstable.

(b) False. For instance, suppose that the inverse of $h[n] = \delta[n - n_0]$ is $g[n]$. Then,

$$\Rightarrow h[n] * g[n] = \delta[n]$$
$$\Rightarrow \sum_{k=-\infty}^{\infty} \delta[k - n_0]g[n - k] = g[n - n_0] = \delta[n]$$
$$\Rightarrow g[n] = \delta[n + n_0]$$

which is noncausal.

(c) False. For example $h[n] = u[n]$ implies that

$$\sum_{-\infty}^{\infty} |h[n]| = \infty.$$

This is an unstable system.

(d) True. Assuming that $h[n]$ is bounded in the range $n_1 \le n \le n_2$,

$$\sum_{k=n_1}^{n_2} |h[k]| < \infty.$$

This implies that the system is stable.

(e) False. For example, $h(t) = tu(t)$ is causal but not stable.

(f) False. For example, the cascade of a causal system with impulse response $h_1[n] = \delta[n - 1]$ and a non-causal system with impulse response $h_2[n] = \delta[n + 1]$ leads to a system with overall impluse response given by $h[n] = h_1[n] * h_2[n] = \delta[n]$.

(g) False. For example, if $h(t) = e^{-t}u(t)$, then $s(t) = e^{-t}u(t) * u(t) = \int_{\tau=-\infty}^{\infty} e^{-\tau}u(\tau)u(t - \tau)d\tau$ $= \int_{\tau=0}^{t} e^{-\tau}u(\tau)d\tau = (1 - e^{-t})u(t)$ and

$$\int_0^{\infty} |1 - e^{-t}|dt = t + e^{-t}|_0^{\infty} = \infty.$$

Although the system is stable, the step response is not absolutely integrable.

(h) True. We may write $u[n] = \Sigma_{k=0}^{\infty}\delta[n - k]$. Therefore,

$$s[n] = \Sigma_{k=-\infty}^{\infty}u[k]h[n - k] = \Sigma_{k=0}^{\infty}h[n - k].$$

If $s[n] = 0$ for $n < 0$, then $h[n] = 0$ for $n < 0$ and the system is causal.

4) Given: System A is LTI and system B is inverse of A. Let $y_1(t)$ and $y_2(t)$ be outputs of system A for inputs $x_1(t)$ and $x_2(t)$ respectively. Combining these informations, we get,

$$x_1(t) \xrightarrow{\text{A}} y_1(t) \xrightarrow{\text{B}} x_1(t). \tag{1}$$

And,

$$x_2(t) \xrightarrow{\text{A}} y_2(t) \xrightarrow{\text{B}} x_2(t). \tag{2}$$

(a) **To prove system B is linear**. Assume that system B is not linear. From equations 1 and 2, we observe that an input $ax_1(t) + bx_2(t)$ to system A can generate $ay_1(t) + by_2(t)$, i.e.,

$$ax_1(t) + bx_2(t) \xrightarrow{\text{A}} ay_1(t) + by_2(t).$$

This is due to linearity property of the system A. Our assumption that system B is not linear implies that the output of B for the input $ay_1(t) + by_2(t)$ is not $ax_1(t) + bx_2(t)$, i.e. the outputs of system B does not add up linearly even if the inputs combine linearly. Therefore, we arrive at a situation which is as follows:

$$ax_1(t) + bx_2(t) \xrightarrow{\text{A}} ay_1(t) + by_2(t) \overset{\text{B}}{\not\to} ax_1(t) + bx_2(t).$$

This contradicts the fact that B is inverse of A. Hence, our assumption is incorrect, and so, system B is linear.

(b) **To prove system B is time-invariant**. Assume system B is time variant. By time-invariant property of system A, we have,

$$x_1(t - \tau) \xrightarrow{\text{A}} y_1(t - \tau).$$

When this output of system A, is fed to system B, we must not expect its response to be $x_1(t-\tau)$ because of our assumption. So, we land up in a situation where,

$$x_1(t - \tau) \xrightarrow{\text{A}} y_1(t - \tau) \overset{\text{B}}{\not\to} x_1(t - \tau).$$

This contradicts the fact that B is the inverse of A. So, the assumption about system B is incorrect. Therefore, system B is also time-invariant.

5)

$$s[n] = h[n] * u[n] = \begin{cases} \sum_{k=0}^{n} (k+1)\alpha^k & n \geq 0 \\ \\ 0 & otherwise \end{cases}$$

Given that

$$\sum_{k=0}^{n} (k+1)\alpha^k = \frac{d}{d\alpha} \sum_{k=0}^{n+1} \alpha^k = \frac{d}{d\alpha} \left[\frac{1 - \alpha^{n+2}}{1 - \alpha} \right]$$

$$s[n] = \left[\frac{1 - (n+2)\alpha^{n+1}}{1 - \alpha} + \frac{1 - \alpha^{n+2}}{1 - \alpha} \right] u[n]$$

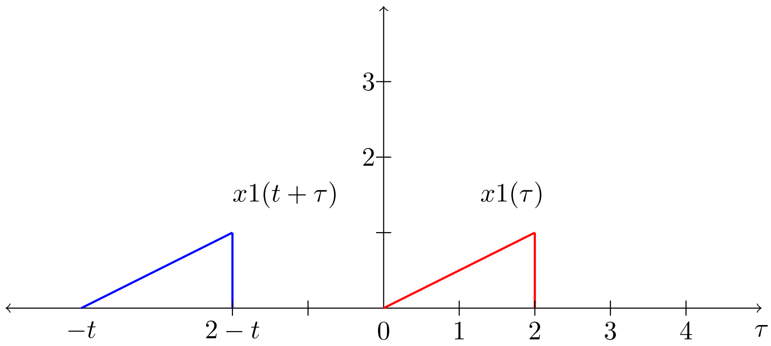
$$s[n] = \left[\frac{1}{(\alpha - 1)^2} - \frac{\alpha^2}{(\alpha - 1)^2}\alpha^n + \frac{\alpha}{(\alpha - 1)}(n+2)\alpha^n \right] u[n]$$

$$= \left[\frac{1}{(\alpha - 1)^2} - \frac{\alpha^2}{(\alpha - 1)^2}\alpha^n + \frac{\alpha(n+1)}{(\alpha - 1)}\alpha^n + \frac{\alpha}{(\alpha - 1)}\alpha^n \right] u[n]$$

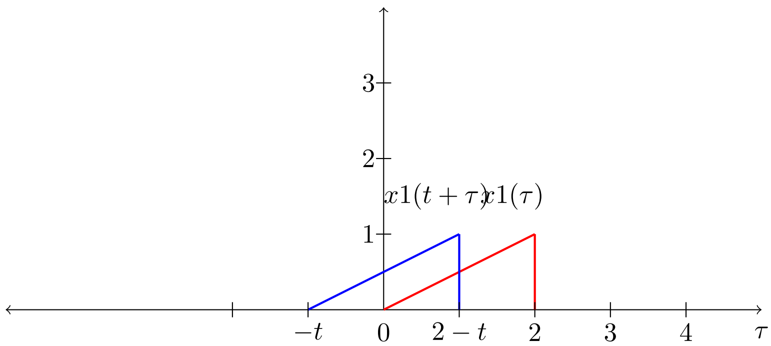
$$= \left[\frac{1}{(\alpha - 1)^2} - \frac{\alpha^2}{(\alpha - 1)^2}\alpha^n + \frac{\alpha(n+1)}{(\alpha - 1)}\alpha^n + \frac{\alpha(\alpha - 1)}{(\alpha - 1)^2}\alpha^n \right] u[n]$$

$$= \left[\frac{1}{(\alpha - 1)^2} - \frac{\alpha}{(\alpha - 1)^2}\alpha^n + \frac{\alpha}{(\alpha - 1)}(n+1)\alpha^n \right] u[n]$$

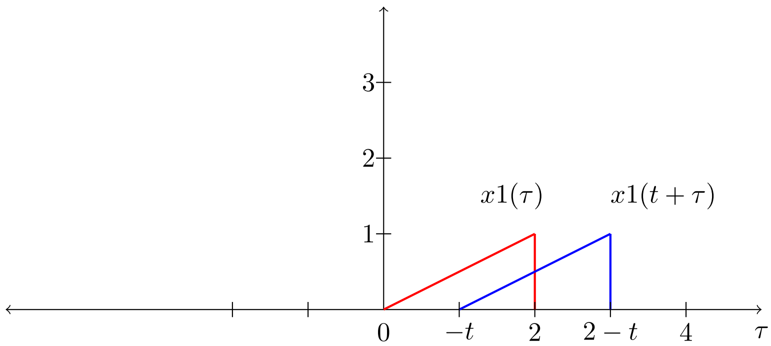
6) (a) The autocorrelation function for x1(t):
case 1:$t > 2$ $\phi_{x_1x_1}(t) = 0$



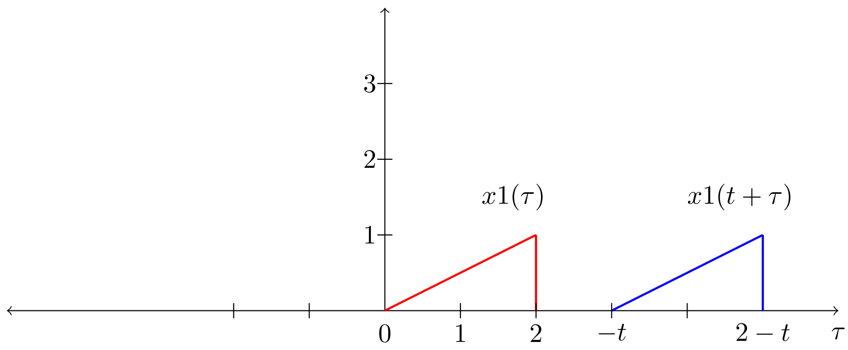
case 2:$0 \leq t \leq 2$ $\phi_{x_1x_1}(t) = \int_0^{2-t} (\tau/2) * ((t+\tau)/2) d\tau = (t^3/24) - (t/2) + (2/3)$



case 3:$-2 \leq t \leq 0$ $\phi_{x_1x_1}(t) = \int_{-t}^{2} (\tau/2) * ((t+\tau)/2) d\tau = (-t^3/24) + (t/2) + (2/3)$



case 4:$t \leq -2$ $\phi_{x_1x_1}(t) = 0$

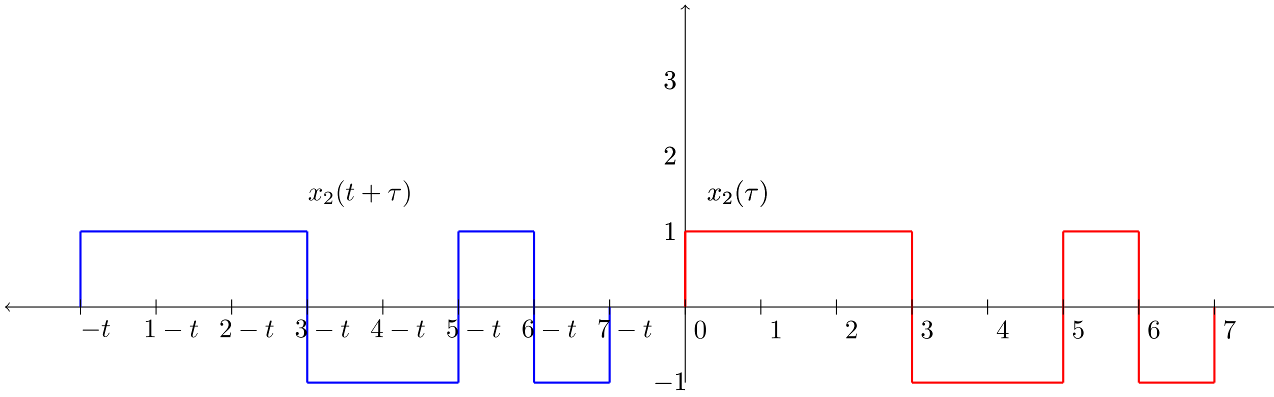


Consolidating all the cases,

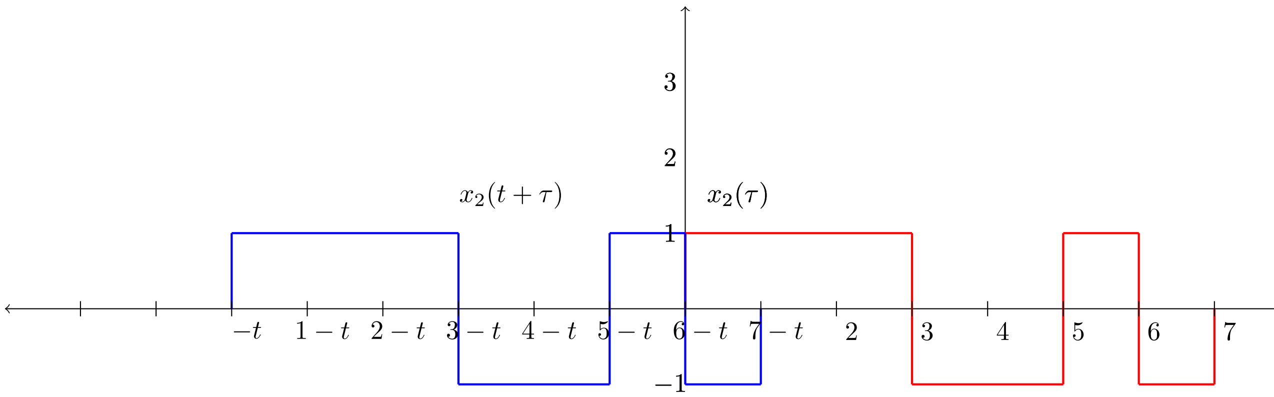
$$\phi_{x_1x_1}(t) = \begin{cases} (|t|^3/24) - (|t|/2) + (2/3), & 0 \leq |t| \leq 2 \\ 0, & |t| > 2 \end{cases}$$

and $\phi_{x_1x_1}(t) = \phi_{x_1x_1}(-t)$

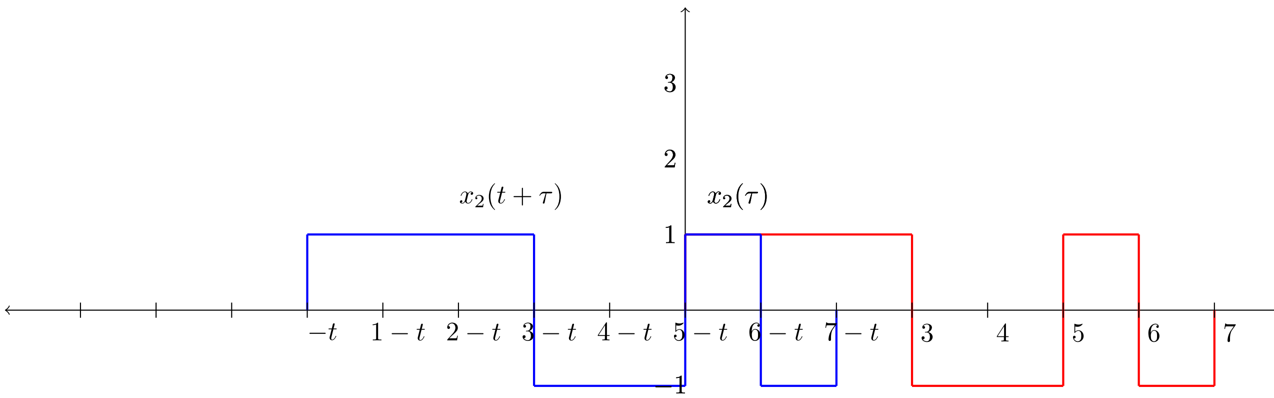
The autocorrelation function for x2(t):
case 1:$t > 7$ $\phi_{x_2x_2}(t) = 0$



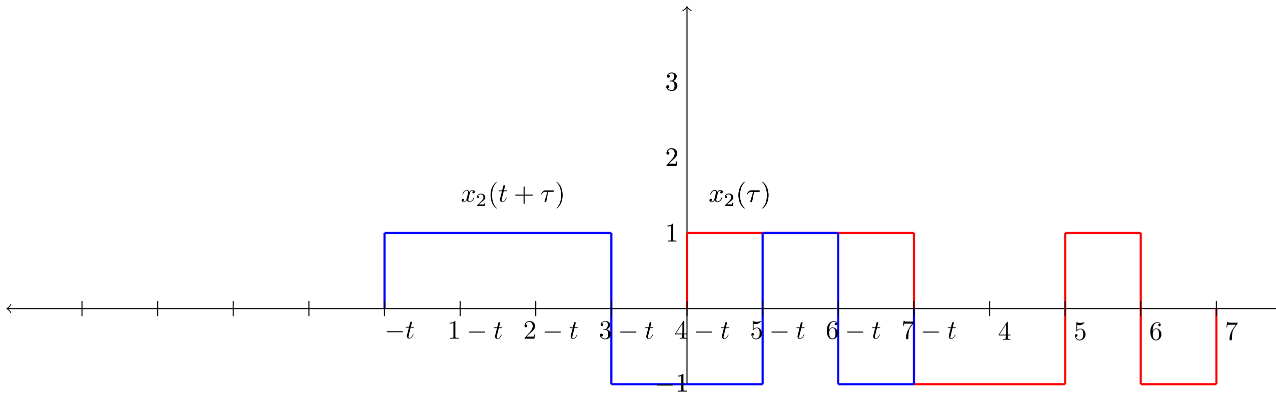
case 2:$6 \leq t \leq 7$ $\phi_{x_2x_2}(t) = \int_0^{7-t} -1 d\tau = t - 7$



case 3:$5 \leq t \leq 6$ $\phi_{x_2x_2}(t) = \int_0^{6-t} 1 d\tau + \int_{6-t}^{7-t} -1 d\tau = 5 - t$



case 4:$4 \leq t \leq 5$ $\phi_{x_2x_2}(t) = t - 5$

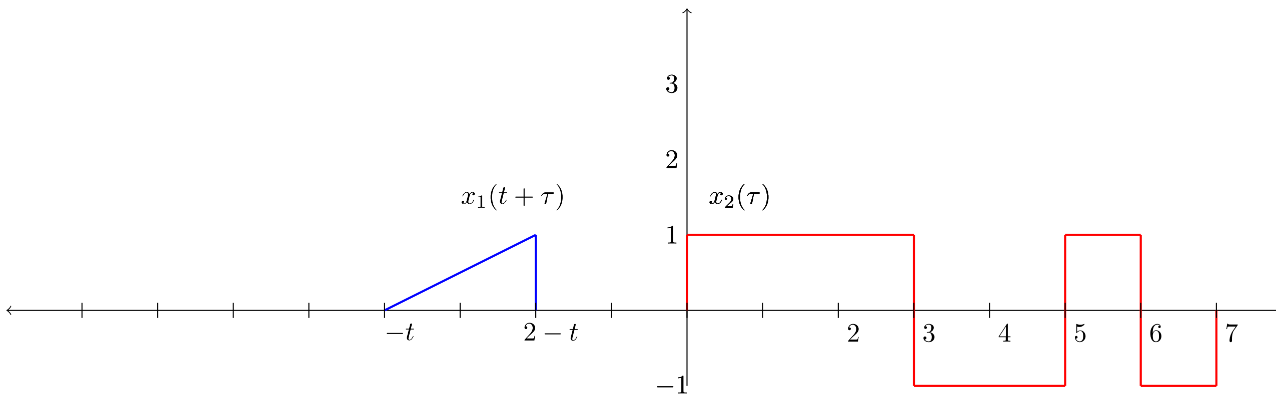


By shifting the signal further we get the following cases
case 5:$3 \leq t \leq 4$ $\phi_{x_2x_2}(t) = 3 - t$
case 6:$2 \leq t \leq 3$ $\phi_{x_2x_2}(t) = t - 3$
case 7:$1 \leq t \leq 2$ $\phi_{x_2x_2}(t) = 1 - t$
case 8:$0 \leq t \leq 1$ $\phi_{x_2x_2}(t) = 7(1 - t)$
case 9:$t < -7$ $\phi_{x_2x_2}(t) = 0$
case 10:$-7 \leq t \leq -6$ $\phi_{x_2x_2}(t) = -t - 7$
case 11:$-6 \leq t \leq -5$ $\phi_{x_2x_2}(t) = 5 + t$
case 12:$-5 \leq t \leq -4$ $\phi_{x_2x_2}(t) = -t - 5$
case 13:$-4 \leq t \leq -3$ $\phi_{x_2x_2}(t) = 3 + t$
case 14:$-3 \leq t \leq -2$ $\phi_{x_2x_2}(t) = -t - 3$
case 15:$-2 \leq t \leq -11$ $\phi_{x_2x_2}(t) = 1 + t$
case 16:$-1 \leq t \leq 0$ $\phi_{x_2x_2}(t) = 7(1 + t)$
Consolidating all the cases,

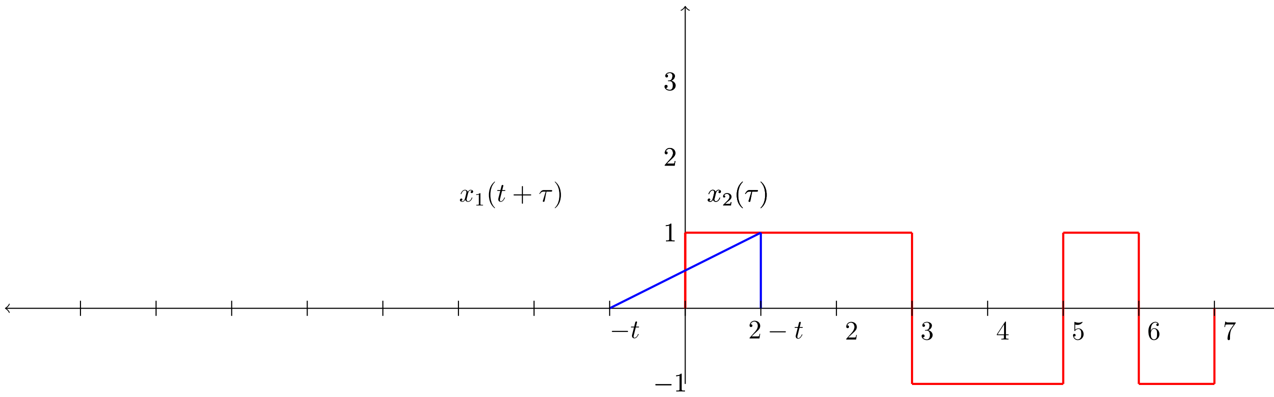
$$\phi_{x_2x_2}(t) = \begin{cases} 7(1-|t|), & 0 \leq |t| \leq 1 \\ 1-|t|, & 1 \leq |t| \leq 2 \\ |t|-3, & 2 \leq |t| \leq 3 \\ 3-|t|, & 3 \leq |t| \leq 4 \\ |t|-5, & 4 \leq |t| \leq 5 \\ 5-|t|, & 5 \leq |t| \leq 6 \\ |t|-7, & 6 \leq |t| \leq 7 \\ 0, & |t| > 7 \end{cases}$$

and $\phi_{x_2x_2}(t) = \phi_{x_2x_2}(-t)$

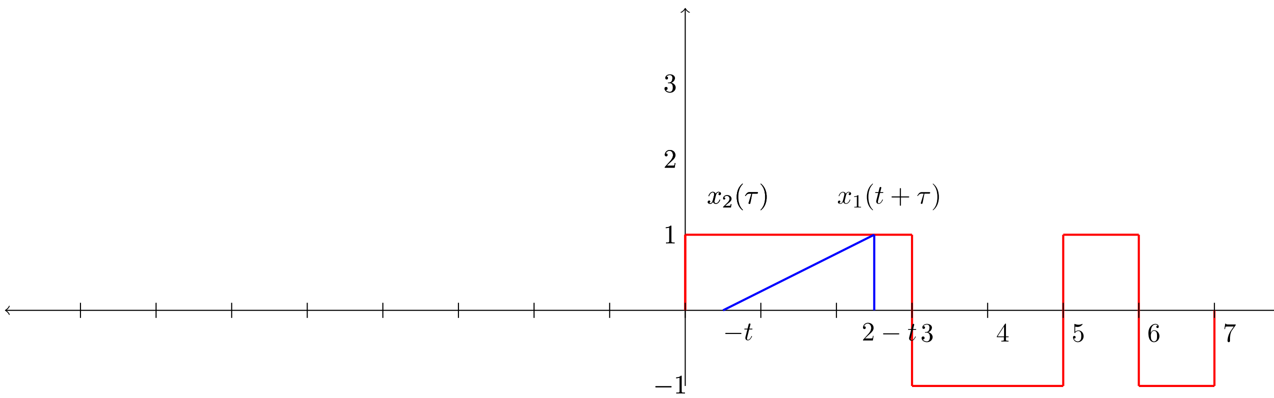
(b) The cross correlation function of x2(t) and x1(t):
$\phi_{x_1x_2}(t) = \int_{-\infty}^{+\infty} x1(t+\tau)x2(\tau)d\tau$
case 1:$t > 2$ $\phi_{x_1x_2}(t) = 0$



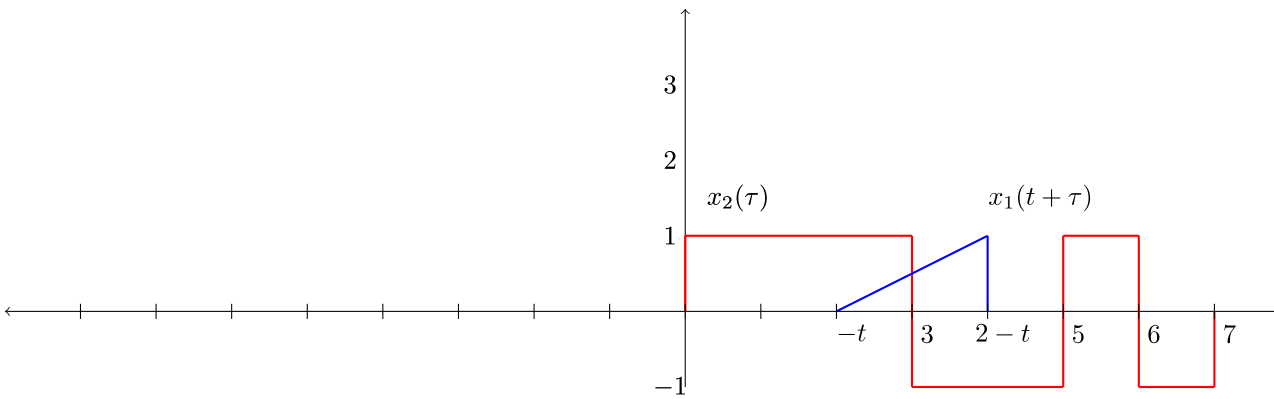
case 2:$0 \leq t \leq 2$ $\phi_{x_1x_2}(t) = \int_0^{2-t}(t+\tau)/2d\tau = 1 - (t^2/4)$



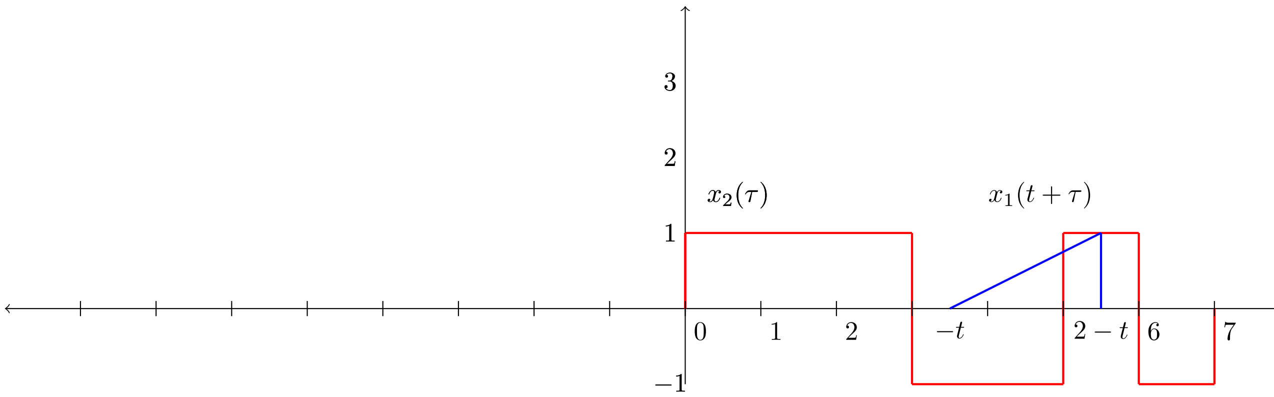
case 3:$-1 \leq t \leq 0$ $\phi_{x_1x_2}(t) = \int_{-t}^{2-t} (t+\tau)/2 d\tau = 1$



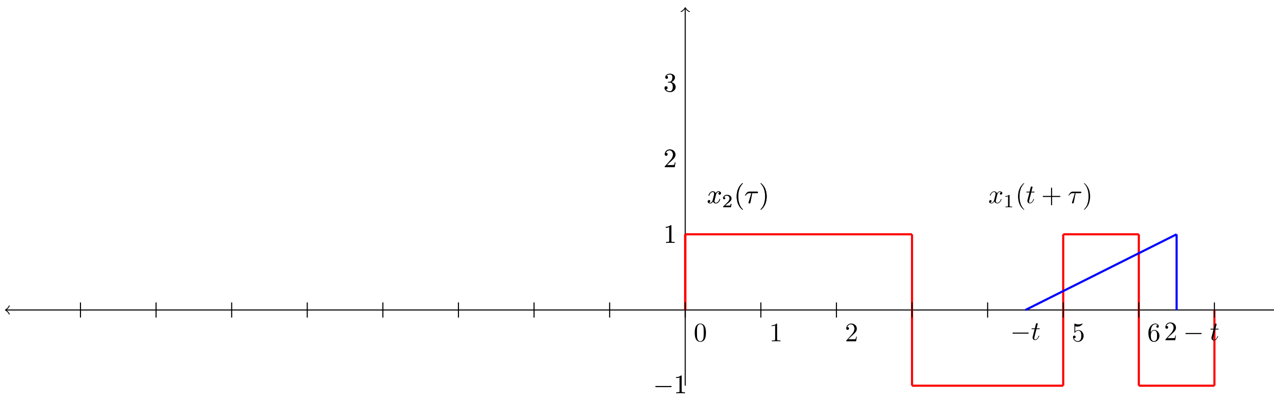
case 4:$-3 \leq t \leq -1$ $\phi_{x_1x_2}(t) = \int_{-t}^{3} (t+\tau)/2 d\tau + \int_{3}^{2-t} -(t+\tau)/2 d\tau = 3t + 3.5$



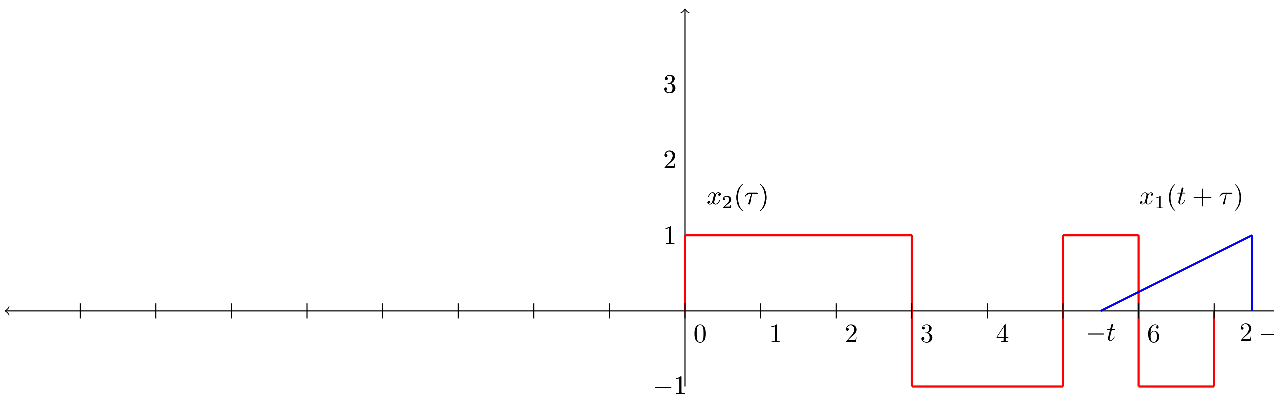
case 5:$-4 \leq t \leq -3$ $\phi_{x_1x_2}(t) = \int_{-t}^{5} -(t+\tau)/2 d\tau + \int_{5}^{2-t} (t+\tau)/2 d\tau = -(t^2/2) - 5t - (23/2)$



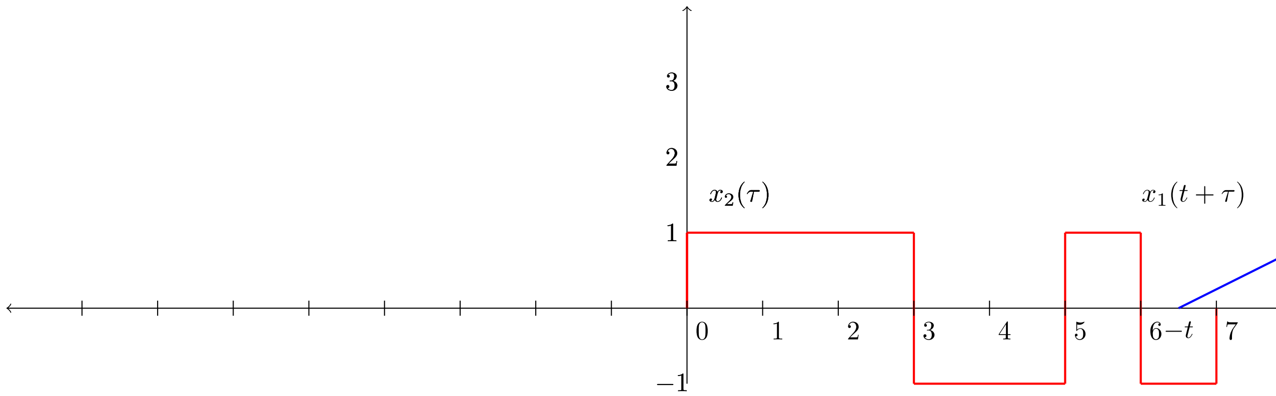
case 6:$-5 \leq t \leq -4$ $\phi_{x_1x_2}(t) = \int_{-t}^{5} -(t+\tau)/2d\tau + \int_{5}^{6}(t+\tau)/2d\tau \int_{6}^{2-t} -(t+\tau)/2d\tau = (3t/2) + (29/4)$



case 7:$-6 \leq t \leq -5$ $\phi_{x_1x_2}(t) = \int_{-t}^{6}(t+\tau)/2d\tau + \int_{6}^{7} -(t+\tau)/2d\tau = (t^2/4) + (5t/2) + (49/4)$



case 8:$-7 \leq t \leq -6$ $\phi_{x_1x_2}(t) = \int_{-t}^{7} -(t+\tau)/2d\tau = -(t^2/4) - (7t/2) - (49/4)$



case 9:$-t < -7$ $\phi_{x_1x_2}(t) = 0$

(c) The crosscorrelation function:

$$\phi_{xy}(t) = \int_{-\infty}^{+\infty} x(t+\tau)y(\tau)d\tau$$

Let τ=-u then

$$\phi_{xy}(t) = \int_{-\infty}^{+\infty} x(t-u)y(-u)du$$

So,

$$\phi_{xy}(t) = x(t) * y(-t)$$

7) Impulse response of the system is

$$\text{h[n]}=\delta[n]+ \text{a}\delta[n-k]$$

$h^{inv}[n]$ is causal as mentioned in the question

$$\text{h[n]} = \begin{cases} 1 & n=0 \\ a & n=k \end{cases}$$

$$h[n]*h^{inv}[n] = \sum_{l=-\infty}^{l=\infty} h[l]h^{inv}[n-l]$$

$$= h[0]h^{inv}[n-0] + h[k]h^{inv}[n-k]$$

We know that $h[n]*h^{inv}[n] = \delta[n]$

$$\delta[n] = h^{inv}[n] + ah^{inv}[n-k]$$

since $h^{inv}[n]$ is causal so $ah^{inv}[n-k]$ exists only for positive values of k

for $n < 0$

$$h^{inv}[n] = 0$$

for $n = 0$

$$1 = h^{inv}[0] + 0$$

for $n > 0$

$$h^{inv}[n] = -ah^{inv}[n-k]$$

which means $h^{inv}[n]$is nonzero only for postive multiples of k, this statement is verified by susbstituting some values of n

Substituting $n = 1$

$$h^{inv}[1] = -ah^{inv}[1-k]$$

$$h^{inv}[1] = \begin{cases} -a & k = 1 \\ 0 & k > 1 \end{cases}$$

Substituting $n = 2$

$$h^{inv}[2] = -ah^{inv}[2-k]$$

$$h^{inv}[2] = \begin{cases} (-a)^2 & k = 1 \\ (-a) & k = 2 \\ 0 & k > 2 \end{cases}$$

Substituting $n = 3$

$$h^{inv}[3] = -ah^{inv}[3-k]$$

$$h^{inv}[3] = \begin{cases} (-a)^3 & k = 1 \\ 0 & k = 2 \\ (-a) & k = 3 \\ 0 & k > 3 \end{cases}$$

Hence $h^{inv}[n]$can be written as

$$h^{inv}[n] = \sum_{p=0}^{p=\infty} (-a)^p \delta[n - pk]$$