$$H = \gamma^0 (\gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z) + \gamma^0 m$$

but $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$ and $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$

$$\therefore \gamma^0 \gamma^i = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

(note can treat these 4x4 matrices as blocks of 2x2)

$$\therefore H = \gamma^0 (\gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z) + \gamma^0 m$$

$$= \gamma^0 \vec{\gamma} \cdot \vec{p} + \gamma^0 m = \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & 0 \end{pmatrix} + m \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$\therefore [\hat{H}, \hat{\vec{\Sigma}}] = \hat{H} \hat{\vec{\Sigma}} - \hat{\vec{\Sigma}} \hat{H}$$

$$= \left[\begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & 0 \end{pmatrix} + m \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \right] \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

$$- \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \left[\begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & 0 \end{pmatrix} + m \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \right]$$

$$= \left[\frac{1}{2} \begin{pmatrix} 0 & (\vec{\sigma}\cdot\vec{p})\vec{\sigma} \\ (\vec{\sigma}\cdot\vec{p})\vec{\sigma} & 0 \end{pmatrix} + \cancel{\frac{\omega}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}} \right]$$

$$- \left[\frac{1}{2} \begin{pmatrix} 0 & \vec{\sigma}(\vec{\sigma}\cdot\vec{p}) \\ \vec{\sigma}(\vec{\sigma}\cdot\vec{p}) & 0 \end{pmatrix} + \cancel{\frac{\omega}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}} \right]$$

$$= \frac{1}{2} \left[\begin{pmatrix} 0 & [\vec{\sigma}\cdot\vec{p}, \vec{\sigma}] \\ [\vec{\sigma}\cdot\vec{p}, \vec{\sigma}] & 0 \end{pmatrix} \right]$$

$$= \frac{1}{2} \left[\begin{pmatrix} 0 & [\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z, \vec{\sigma}] \\ [\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_y, \vec{\sigma}] & 0 \end{pmatrix} \right]$$

but remembering that $[\sigma_1, \sigma_2] = 2i\sigma_3$

$$[\sigma_2, \sigma_3] = 2i\sigma_1$$

$$[\sigma_3, \sigma_1] = 2i\sigma_2$$

$$[\sigma_1, \sigma_1] = [\sigma_2, \sigma_2] = [\sigma_3, \sigma_3] = 0$$

then we have that:

$$[\hat{H}, \hat{\vec{\Sigma}}_x] = \frac{1}{2}\left[\begin{pmatrix} 0 & [\sigma_1 \hat{p}_x + \sigma_2 \hat{p}_y + \sigma_3 \hat{p}_z, \sigma_1] \\ [\sigma_1 \hat{p}_x + \sigma_1 \hat{p}_y + \sigma_3 \hat{p}_z, \sigma_1] & 0 \end{pmatrix}\right.$$

Now keep in mind that $p_{x,y,z}$ are just numbers so

$$[\sigma_1 \hat{p}_x, \sigma_1] = \hat{p}_x [\sigma_1, \sigma_1]$$

$$[\sigma_2 \hat{p}_y, \sigma_1] = \hat{p}_y [\sigma_2, \sigma_1]$$

etc

thus $[\hat{H}, \hat{\Sigma}_x] =$

$$= \frac{1}{2}\begin{pmatrix} 0 & \hat{p}_y[\sigma_2, \sigma_1] + \hat{p}_z[\sigma_3, \sigma_1] \\ \hat{p}_y[\sigma_3, \sigma_1] + \hat{p}_z[\sigma_3, \sigma_1] & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -2i\hat{p}_y\sigma_3 + 2i\hat{p}_z\sigma_2 \\ -2i\hat{p}_y\sigma_3 + 2i\hat{p}_z\sigma_2 & 0 \end{pmatrix}$$

Similarly $[\hat{H}, \hat{\Sigma}_y] =$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2i\sigma_3 p_x - 2i\sigma_1 p_z \\ 2i\sigma_3 p_x - 2i\sigma_1 p_z & 0 \end{pmatrix}$$

and $[\hat{H}, \hat{\Sigma}_2] =$

$$= \frac{1}{2}\begin{pmatrix} 0 & -2i\sigma_2 p_x + 2i\sigma_1 p_y \\ -2i\sigma_2 p_x + 2i\sigma_1 p_y & 0 \end{pmatrix}$$

Now we showed in the first part of this question that:

$$[\hat{H}, \hat{\vec{L}}] = -i\gamma^0 \begin{pmatrix} \gamma^2 p_z - \gamma^3 p_y \\ \gamma^3 p_x - \gamma^1 p_z \\ \gamma^1 p_y - \gamma^2 p_x \end{pmatrix}$$

earlier we also showed

$$\boxed{\gamma^0\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}}$$

$$\therefore [\hat{H}, \hat{L}_x] = -i\begin{pmatrix} 0 & \sigma_2 p_z - \sigma_3 p_y \\ \sigma_2 p_z - \sigma_3 p_y & 0 \end{pmatrix}$$

$$= -[\hat{H}, \hat{\Sigma}_x]$$

$$\therefore [\hat{H}, \hat{L}_x] + [\hat{H}, \hat{\Sigma}_x] = 0$$

$$\therefore \ [\hat{H}, L_x + \Sigma_x] = 0$$

Similarly

$$[\hat{H}, \hat{L}_y] = -i \begin{pmatrix} 0 & \sigma^3 p_x - \sigma^1 p_z \\ \sigma^3 p_x - \sigma^1 p_z & 0 \end{pmatrix}$$

$$= -[\hat{H}, \hat{\Sigma}_y]$$

$$\therefore \ [\hat{H}, L_y + \hat{\Sigma}_y] = 0$$

and $[\hat{H}, \hat{L}_z] = -i \begin{pmatrix} 0 & \sigma_1 \hat{p}_y - \sigma_2 \hat{p}_x \\ \sigma_1 \hat{p}_y - \sigma_2 \hat{p}_x & 0 \end{pmatrix}$

$$= -[\hat{H}, \hat{\Sigma}_z]$$

$$\therefore \ [\hat{H}, \hat{L}_z + \hat{\Sigma}_z] = 0$$

$$\boxed{\text{So have just proven that } \hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{\Sigma}} \ \underline{\underline{\text{is}}} \text{ conserved}}$$