Two coin flip experiment:

$\Omega' = \Omega^2 = \{\langle H,H\rangle, \langle H,T\rangle, \langle T,H\rangle, \langle T,T\rangle\}$ (outcome space)

$P(\underbrace{\{\langle H,H\rangle\}}_{A}) = \frac{|A|}{|\Omega|} = \frac{|\{\langle H,H\rangle\}|}{|\Omega|} = \frac{1}{4}$

$2^{\Omega'} = \{\phi, \{HH\}, \{HT\}, \{TH\}, \{T,T\}, \{HH,HT\}, \{HH,TH\}, \ldots, \{HH,HT,TH\}, \ldots \Omega\}$

(event space) How large is this event space? $|2^{\Omega'}| = 2^{|\Omega'|} = 2^4 = 16$
This means there are 16 possible probability questions to ask on this experiment. That's it.

What is the probability of at least one tail in two coin flips?
B = {at least one tail} = {HT, TH, TT}

$P(B) = \frac{|B|}{|\Omega'|} = \frac{3}{4}$

What's the probability of exactly one head in two coin flips?
D = {exactly one head} = {HT, TH}

$P(D) = \frac{|D|}{|\Omega'|} = \frac{2}{4}$

A new experiment: a fair die roll. $\Omega = \{1,2,3,4,5,6\}$

$|2^{\Omega}| = 2^{|\Omega|} = 2^6 = 64$

What is the probability of rolling an even number?
A = {even numbers} = {2,4,6}

$P(A) = \frac{|A|}{|\Omega|} = \frac{3}{6}$

The formula to solve probability problems:
Step 1: translate sample space from English into Ω
Step 2: compute |Ω|
Step 3: translate event from English into A
Step 4: compute |A|
Step 5: divide step 4 by step 2

Step 3 and 4 are the hardest...

Now we flip four coins. What is the probability of getting exactly 2 heads?

① $\Omega' = \{HHHH, HHHT, HHTH, \ldots, TTTT\}$
$" \Omega^4$

② $|\Omega'| = |\Omega^4| = |\Omega|^4 = 2^4 = 16$

③ $A \in 2^{\Omega'}$, $A = \{HTHT, HHTT, TTHH, THTH, HTTH, THHT\}$

④ $|A| = 6$

⑤ $P(A) = \frac{6}{16}$

$P(\{HHHH\}) = P(\{HHTT\}) = \frac{1}{16} \neq P(2H) = \frac{6}{16}$

What is the probability of getting at least one head?

③ $A = \{HTTT, TTTH, \ldots\}$ A lot!

Is there an easier way to do this problem?

Recall: Ω = A ∪ A^C and A ∩ A^C = ∅ => |Ω| = |A| + |A^C|
This fact is true for finite sample sizes.

$\Rightarrow |A| = |\Omega| - |A^c|$

An alternative if Step 3 was too difficult:
Step 3': translate event complement from English into A^C
Step 4': compute |A^C|
Step 5': divide step (|Ω| - step 4) by step 2

③' $A^c = \{TTTT\}$ ④' $|A^c| = 1$, ⑤' $P(A) = \frac{|\Omega| - |A^c|}{|\Omega|} = \frac{16-1}{16} = \frac{15}{16} = \frac{|A|}{|\Omega|}$

We now flip 10 coins. What is the probability of getting exactly 4 H?

① $\Omega' = \Omega^{10} = \{HHHHHHHHHH, \ldots, TTTTTTTTTT\}$

② $|\Omega'| = |\Omega^{10}| = |\Omega|^{10} = 2^{10} = 1{,}024$. $|2^{\Omega'}| = 2^{1024}$

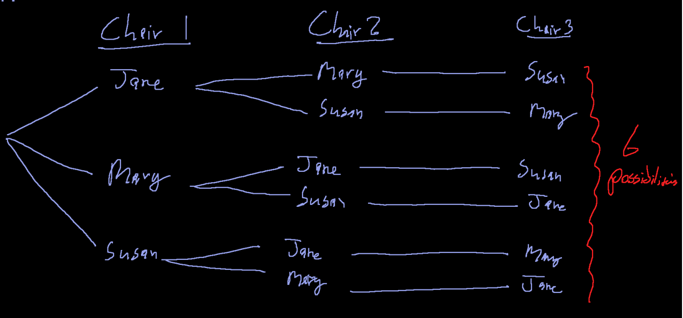
③ $A = \{\ldots\}$ or ③' $A^c = \{\ldots\}$
④ $|A| = ?$ ④' $|A^c| = ?$

both of these are too difficult!

⑤ $P(A) = \frac{|A|}{1024}$

In order to develop a method to count the size of A, let's develop some tools and return to this problem later.

Recall B = {Jane, Mary, Susan}. We want to sit them in 3 chairs. How many ways to do this? Let's draw a "tree diagram" chair-by-chair:



Now that we saw the tree (and its pattern), we can do this quicker using the following strategy:

possibilities: $\underset{\text{Chair 1}}{\underline{3}} \cdot \underset{\text{Chair 2}}{\underline{2}} \cdot \underset{\text{Chair 3}}{\underline{1}} = 3! = 6$

If the seating arrangements is Ω then |Ω| = 6.
Note that Ω ≠ B^3 = {<Jane, Jane, Jane>, ...} which has 27 elements.

Our Ω is the set of samples "without replacement". Each time you seat someone, they cannot be seated again. Another example: a bag with 17 balls. If you sample 3 without replacement, they are all unique.

Our B^3 is the set of samples "with replacement". Each time you seat someone, they unseat themselves and can be seated again.

The number of ways to sample n elements *without* replacement is: n! = (n)(n-1)(n-2) x ... x (2)(1).

The number of ways to sample n elements *with* replacement is: n^n = (n)(n) x ... x (n)

How many ways to seat 10 people in 10 chairs? 10! ≈ 3.6 million

How many ways to seat 10 people in 3 chairs?

possibilities: $\underset{\text{Chair 1}}{\underline{10}} \cdot \underset{\text{Chair 2}}{\underline{9}} \cdot \underset{\text{Chair 3}}{\underline{8}} = \frac{10!}{7!}$

The number of ways to sample k elements *without replacement* from a set of n total elements is:

#: $\underset{1}{\underline{n}} \cdot \underset{2}{\underline{n-1}} \cdot \ldots \cdot \underset{k}{\underline{n-k+1}} = \frac{n!}{(n-k)!} =: {}_nP_k$ this is called the number of "permutations"
Chair:

${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$ Thus, 0! := 1. This is a definition.

Let's seat Jane, Mary, Susan in three seats. What is the probability that Jane and Susan sit next to each other?

1) Ω is set B sampled without replacement
2) |Ω| = 6
3) A = {MJS, JSM, SJM, MSJ}
4) |A| = 4
6) P(A) = 4/6

Now you have 6 people made up of 3 couples: Bob-Jane, Richard-Susan, Charles-Mary. You seat all six people in six chairs. What is the probability the couples all sit with each other?

1) Ω is just all six people sampled without replacement
2) |Ω| = 6! = 720t
3) A = {...} This set is too hard.

Let's try to "skip" step 3 since we only need step 4's results to answer P(A). Let's try to get |A| using our heuristic:

possibilities: $\underset{\text{Chair 1}}{\underline{6}} \; \underset{2}{\underline{1}} \; \underset{3}{\underline{4}} \; \underset{4}{\underline{1}} \; \underset{5}{\underline{2}} \; \underset{6}{\underline{1}} = 6 \cdot 4 \cdot 2$

4) |A| = 48
5) P(A) = 48/720 = 6.66%

Another heuristic to solve this problem: consider couples instead of individuals. And imagine 3 loveseats and not 6 chairs.

possibilities: $\underbrace{\underset{\text{loveseat 1}}{\underline{3}}}_{\underset{\text{Chair 1}}{\underline{2}} \cdot \underset{\text{Chair 2}}{\underline{1}}} \cdot \underset{\text{loveseat 2}}{\underline{2}} \cdot \underset{\text{loveseat 3}}{\underline{1}} = 3!\,2^3 = 3 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 2 = 48$