$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right), \quad T = \sum_{i=1}^{n} X_i \sim p(t) = ? \qquad \text{Supp}[T] = \{0, 1, \ldots, n\}$

$$T \sim \begin{cases} 0 \text{ w.p. } \binom{n}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^n \\ 1 \text{ wp } \binom{n}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{n-1} \\ 2 \text{ w.p. } \binom{n}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{n-2} \\ \vdots \\ n-1 \text{ wp } \binom{n}{n-1}\left(\frac{1}{3}\right)^{n-1}\left(\frac{2}{3}\right)^1 \\ n \text{ wp } \binom{n}{n}\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^0 \end{cases}$$

$$X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(p). \quad T = \sum_{i=1}^{n} X_i \sim p(t) = \begin{cases} 0 \text{ wp } \binom{n}{0} p^0 (1-p)^n \\ 1 \text{ wp } \binom{n}{1} p^1 (1-p)^{n-1} \\ \vdots \\ n-1 \text{ wp } \binom{n}{n-1} p^{n-1}(1-p)^1 \\ n \text{ wp } \binom{n}{n} p^n (1-p)^0 \end{cases} = \binom{n}{t} p^t (1-p)^{n-t} = \text{Bin}(n,p)$$

$\Downarrow$

$T \sim \text{Bin}(n, p)$

---

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$ A possibly infinite sequence of iid Bernoullis

let $T := \min\{t : X_t = 1\} \qquad \text{Supp}[T] = \{1, 2, \ldots\} = \mathbb{N}$

a "stopping time"

$p(1) = P(T=1) = P(X_1 = 1) = p$

$p(2) = P(T=2) = P(X_1 = 0, X_2 = 1) = P(X_1 = 0)P(X_2 = 1) = (1-p)p$

$p(3) = (1-p)(1-p)p = (1-p)^2 p$

$\vdots$

$p(t) = (1-p)^{t-1} p =$ Geometric(p)

$X \sim \text{Geom}(p) := (1-p)^{x-1} p, \quad \text{Supp}[X] = \mathbb{N}, \quad p \in (0,1)$

$X \sim \text{Geom}(1) = 0^{x-1}(1) = \text{Deg}(1).$

$X \sim \text{Geom}(0) = 1^{x-1}(0) = 0$ Not even degenerate, it is not a rv!

$$1 \overset{?}{=} \sum_{x \in \text{Supp}[X]} p(x) = \sum_{x=1}^{\infty} (1-p)^{x-1} p = p \sum_{x=1}^{\infty} (1-p)^{x-1} = p \sum_{y=0}^{\infty} (1-p)^y = p \frac{1}{1-(1-p)} = \frac{p}{p} = 1$$

let $y = x - 1 \Leftrightarrow x = y + 1$

The geometric series formula

$q \in (0,1)$

$$S = \sum_{i=1}^{\infty} q^i = q^0 + q^1 + q^2 + q^3 + \ldots = 1 + q + q^2 + q^3 + \ldots$$

$$S = 1 + q(\underbrace{1 + q + q^2 + \ldots}_{S})$$

$$\Rightarrow S = 1 + qS \Rightarrow S - qS = 1 \Rightarrow (1-q)S = 1$$

$$\Rightarrow \boxed{S = \frac{1}{1-q}}$$

$F(x) := P(X \le x), \quad \underbrace{1 - F(x) = P(X > x)}_{\text{easier}} = (1-p)^x \Rightarrow \boxed{F(x) = 1 - (1-p)^x}$

$P(X > x)$ $\quad \underset{1}{0} \; \underset{2}{0} \; \underset{3}{0} \; 0 \; \ldots \; \underset{x}{0} \; | \; \underset{x+1}{\_} \; \underset{x+2}{\_} \; \ldots \; 1 \; \ldots$

$= P(X_1 = 0, X_2 = 0, \ldots, X_x = 0) = (1-p)^x$

$$P(X > x) = \sum_{i=x+1}^{\infty} (1-p)^{i-1} p = \sum_{j=0}^{\infty} (1-p)^{j+x+1-1} = (1-p)^x \boxed{\sum_{j=0}^{\infty} (1-p)^j}$$

let $j = i - x - 1 \Leftrightarrow i = j + x + 1$

---

What is the probability you win the NYS 6-card lottery in the next 30yr if you play every day? The lottery odds are 1 in 45,057,474 which means p = 2.22e-8. 30yr = 10,958 days.

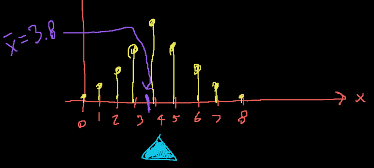
$X \sim \text{Geom}(2.22 \times 10^{-8})$

$$P(X \le 10958) = F(10958) = 1 - (1 - 2.22 \times 10^{-8})^{10958}$$

$=$ 0.00024 = 1 in 4,111

---

Consider $X \sim \text{Bin}(8, \frac{1}{2}) = p(x)$




4 is the "pivot" of the seesaw which balances both sides. How do we calculate this pivot?

$X_1, X_2, \ldots, X_{10} \overset{iid}{\sim} \text{Bin}(8, \frac{1}{2})$

$x_1 = 5 \quad x_6 = 1$

$x_2 = 5 \quad x_7 = 3$

$x_3 = 7 \quad x_8 = 2$

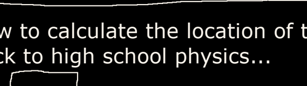
$x_4 = 5 \quad x_9 = 2$

$x_5 = 3 \quad x_{10} = 5$

$\bar{x} := \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{38}{10} = 3.8$

The "sample average"

How to calculate the location of the pivot point? Let's go back to high school physics...




$$\sum_i w_i (d_i - d^*) = 0 \Rightarrow \sum w_i d_i - \sum w_i d^* = 0$$

$$\Rightarrow \sum w_i d_i = d^* \sum w_i \Rightarrow d^* = \frac{\sum w_i d_i}{\sum w_i}$$ "weighted distance"

Let's call the pivot the "mean" of rv or the "expectation" of the rv and denote it E[X] or μ. We can calculate using this HS physics example where the weights are now the probabilities at each value x ∈ Supp[X] i.e. p(x) and the distances are the x's:

$$\mu := E[X] = \frac{\sum_i p(x_i) x_i}{\sum_i p(x_i)} = \frac{\sum_{x \in \text{Supp}[X]} p(x) x}{\underbrace{\sum_{x \in \text{Supp}[X]} p(x)}_{1}} = \sum_{x \in \text{Supp}[X]} x \, p(x) \in \mathbb{R}$$

scalar

It is the probability-weighted average of the realization values. The square brackets E[] mean "function of a function" or "operator". Since the rv X is technically a function, we write E[X] instead of E(X) to indicate E is a function of a function. Same for Supp[X].

It appears the sample average, $\bar{x} \approx E[X]$. This is called the "Law of Large Numbers". "Expect" means you expect the average to be close to this value.