Math 241 Fall 2021
Midterm Examination Two

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Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating   Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

By taking this exam, you acknowledge and agree to uphold this Code of Academic Integrity.

Instructions

This exam is 70 minutes (variable time per question) and closed-book. You are allowed **one** page (front and back) of a "cheat sheet", blank scrap paper and a graphing calculator. Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 [12min] (and 12min will have elapsed) Consider an apartment building with three floors. Each apartment has a floor number and an apartment letter. The first floor has 5 apartments denoted 1A, 1B, 1C, 1D, 1E; the second floor has 3 apartments denoted 2A, 2B, 2C; the third floor has only a penthouse denoted 3A. Bob walks into this apartment building and selects a floor uniformly (i.e. each floor is equally likely). He then enters an apartment on that floor uniformly.

- [15 pt / 15 pts]   Record the letter(s) of all the following that are **true** in general. At least one will be true.

  (a) The probability Bob picks floor 3 is less than the probability Bob picks floor 1.

  (b) The probability Bob picks floor 3 is more than the probability Bob picks floor 1.

  (c) The probability of selecting each apartment in the entire building is uniform.

  (d) The probability of selecting each apartment on floor 2 is uniform if you are told Bob is on floor 2.

  (e) The probability Bob enters the penthouse is $1/3$.

  (f) The probability Bob enters a C apartment is $2/3$.

  (g) Given that Bob is in a C apartment, the probability he is on the first floor is zero.

  (h) Given that Bob is in a C apartment, the probability he is on the first floor is $1/3$.

  (i) Given that Bob is in a C apartment, the probability he is on the first floor is $9/24$.

  (j) Given that Bob is in a C apartment, the probability he is on the first floor is $15/24$.

  (k) Given that Bob is in a C apartment, the probability he is on the third floor is zero.

  (l) Given that Bob is in a C apartment, the probability he is on the third floor is $1/3$.

  (m) Given that Bob is in a C apartment, the probability he is on the third floor is $9/24$.

  (n) Given that Bob is in a C apartment, the probability he is on the third floor is $15/24$.

  (o) The floor selection is independent of the apartment letter selection

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

deik

Problem 2 [12min] (and 24min will have elapsed) Consider a game where someone spins the spinner pictured on the right. The numbers represent payouts in $. The spinner has four colors: red, blue, green, yellow with payouts 2,4, or 8; 5 or 7; 3 or 6; 1 respectively. Let X be the rv whose realization values are the payouts. Assume the spinner is fair.

- [18 pt / 33 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

  (a) X is a discrete rv

  (b) X is a uniform discrete rv

  (c) X could be a binomial rv where $n = 8$

  (d) X could be a hypergeometric rv where $N = 8$

  (e) $\mathbb{S}\text{upp}[X] = \{1, 2, ..., 8\}$

  (f) $\mathbb{S}\text{upp}[X] = [1, 8]$

  (g) $|\mathbb{S}\text{upp}[X]| = |\mathbb{N}|$

  (h) If we let $\Omega = \{\text{Red, blue, green, yellow}\}$ then we can construct a rv X that maps $\omega \in \Omega$ to the support values of X.

  (i) If we let $\Omega = [0, 2\pi]$ where ω represents the angle of the spinner from the right horizontal level between states 2 and 3, then we can construct a rv X that maps $\omega \in \Omega$ to the support values of X.

  (j) $\mathbb{E}[X] = 4$

  (k) $\mathbb{E}[X] = 4.5$

  (l) $\mathbb{E}[X] = 5$

  (m) $\text{Med}[X] = 4$

  (n) $\text{Med}[X] = 4.5$

  (o) $\text{Med}[X] = 5$

  (p) $Q[X, 0.1] = 1$

  (q) $Q[X, 0.1] = 2$

  (r) $\text{Range}[X] = 8$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

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abeikmp

Problem 3 [8min] (and 32min will have elapsed) Consider a game where someone spins the spinner pictured on the right. The numbers represent payouts in dollars. The spinner has four colors: red, blue, green, yellow. Red payouts are 2,4, or 8; blue payouts are 5 or 7; green payouts are 3 or 6; yellow only pays out 1. Let X be the rv whose realization values are the payouts. Assume the spinner is fair. In this rv, $\mu = 4.5$ and $\mathbb{Supp}[X] = \{1, 2, ..., 8\}$.

ETA 4013

- [14 pt / 47 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

    (a) $p(x)$ is monotonically increasing

    (b) $F(x)$ is monotonically increasing

    (c) $p(x) < 1$ for all $x \in \mathbb{Supp}[X]$

    (d) $F(x) < 1$ for all $x \in \mathbb{Supp}[X]$

    (e) $F(4) = 0$

    (f) $F(4) = 0.5$

    (g) $F(4) = 1$

    (h) $p(4) = 0.5$

    (i) $\sigma^2 := \mathbb{Var}[X] = (1/8)(1^2 + 2^2 + 3^2 + \ldots + 8^2)$

    (j) $\sigma^2 := \mathbb{Var}[X] = (1/4)(0.5^2 + 1.5^2 + 2.5^2 + 3.5^2)$

    (k) $\mathbb{E}[X^2] = 4.5^2$

    (l) $\sigma := \mathbb{SD}[X] = \mathbb{E}[X] - 4.5$

    (m) The unit of the value of the variance of X is dollars

    (n) The unit of the value of the standard deviation of X is dollars

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

bcfjn

Problem 4 [10min] (and 42min will have elapsed) Consider a game where someone spins the spinner pictured on the right. The numbers represent payouts in dollars. The spinner has four colors: red, blue, green, yellow. Red payouts are 2,4, or 8; blue payouts are 5 or 7; green payouts are 3 or 6; yellow only pays out 1. Let X be the rv whose realization values are the payouts. Assume the spinner is fair. In this rv, $\mu = 4.5$ and $\mathbb{Supp}[X] = \{1, 2, ..., 8\}$. The variance is $\sigma^2 := \mathbb{Var}[X] = 5.25$ and $\sigma := \mathbb{SD}[X] = 2.29$ rounded to the nearest two digits.

ETA 4013

- [13 pt / 60 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

(a) Another way to define variance (i.e. mean "distance" from the mean) could be $\mathbb{E}[|X - \mu|]$

(b) Another way to define variance (i.e. mean "distance" from the mean) could be $\mathbb{E}[X - \mu]$

(c) Another way to define variance (i.e. mean "distance" from the mean) could be $\mathbb{E}[(X - \mu)^{100}]$

(d) $\mathbb{E}[X^2] = 5.25 + 4.25^2 = 23.3125$

(e) $\mathbb{E}[X^2] = 5.25 - 4.25^2 = -12.8125$

(f) $\mathbb{E}[X^2] = 5.25^2 - 4.25^2 = 9.5$

For the rest of these questions let Y denote the rv for the following game. You play the game X but you have to pay a 20% of the winnings to the casino, then after the tax is taken, you have to pay \$4 to the casino.

(g) $X \overset{d}{=} Y$

(h) X, Y are independent rv's

(i) Y is a "fair game" for you (the player)

(j) $\mathbb{Var}[Y] = 0.8\sigma^2$

(k) $\mathbb{SD}[Y] = 0.8\sigma$

(l) The probability of winning money if you play the game Y is 3/8

(m) $\mathbb{Var}[X + Y]$ would have a non-zero covariance term in its expression

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

acdklm

Problem 5 [9min] (and 51min will have elapsed) Consider a game where someone spins the spinner pictured on the right. The numbers represent payouts in dollars. The spinner has four colors: red, blue, green, yellow. Red payouts are 2,4, or 8; blue payouts are 5 or 7; green payouts are 3 or 6; yellow only pays out 1. Let X be the rv whose realization values are the payouts. Assume the spinner is fair. In this rv, $\mu = 4.5$ and $\mathbb{Supp}[X] = \{1, 2, ..., 8\}$. The variance is $\sigma^2 := \mathbb{Var}[X] = 5.25$ and $\sigma := \mathbb{SD}[X] = 2.29$ rounded to the nearest two digits. Also consider playing this game $n = 1000$ times where the spinner is reset to the same position and you flick it with approximately the same force each time. The payouts for these n games are denoted by rv's $X_1, \ldots, X_n$.

- [11 pt / 71 pts]   Record the letter(s) of all the following that are **true** in general. At least one will be true.

  (a) $X_1, \ldots, X_n \overset{iid}{\sim}$

  (b) $X_1, \ldots, X_n \overset{ind}{\sim}$

  (c) The total winnings after all n games is expected to be 4500

  (d) The variance in the total winnings after all n games is 5250

  (e) The standard deviation in the total winnings after all n games is 72.46 rounded to the nearest two digits.

  (f) The standard deviation in the total winnings after all n games is 2290

  (g) The average winnings per game after all n games is expected to be 4.5

  (h) The variance in the average winnings per game after all n games is 0.00525

  (i) The standard deviation in the average winnings per game after all n games is 0.00229 rounded to the nearest 3 digits.

  (j) The standard deviation in the average winnings per game after all n games is 0.0724 rounded to the nearest 3 digits.

  (k) As n gets larger, the average winnings per game after all n games approaches a rv centered at 4.5 with variance shrinking to zero.

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

abcdghjk

6

Problem 6 [9min] (and 60min will have elapsed) In the previous game, the probability of winning is $p := 3/8$ and you played $n = 1000$ times. Each game was $\overset{iid}{\sim}$. Let $X_1, \ldots, X_n$ now denote whether each game was won or not ($1 = $ win and $0 = $ lose). These are different rv's than the previous problem! Let T denote the total number of wins out $\boxed{\text{of}}$ the n games.

- [17 pt / 88 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

  (a) $X_1, \ldots, X_n \overset{iid}{\sim}$

  (b) $X_{17} \sim \text{Bernoulli}\,(3/8)$

  (c) $X_{17} \sim \text{Bernoulli}\,(5/8)$

  (d) $T \sim \text{Binomial}\,(n, p)$

  (e) $T \sim \text{Hypergeometric}\,(n, np, n)$

  (f) T is independent of X_{17}

  (g) $\mathbb{P}\,(T = 17) = 3/8$

  (h) $\mathbb{P}\,(T = 17) = \binom{1000}{17}(3/8)^{17}(5/8)^{983}$

  (i) $\mathbb{P}\,(X_{17} = 17) = 3/8$

  (j) $\mathbb{P}\,(X_{17} = 17) = (3/8)^{17}$

  (k) $\mathbb{P}\,(X_{17} = 17) = \binom{1000}{17}(3/8)^{17}(5/8)^{983}$

  (l) $\mathbb{E}\,[T] = (3/8)^{17}$

  (m) $\mathbb{E}\,[T] = 1000 \times 3/8$

  (n) $\mathbb{V}\text{ar}\,[T] = 1000 \times 3/8 \times 5/8$

  (o) $\mathbb{V}\text{ar}\,[T] = \sum_{t=1}^{1000} \binom{1000}{t}(3/8)^t(5/8)^{1000-t}$

  (p) $\mathbb{V}\text{ar}\,[T] = \sum_{t=1}^{1000} t\binom{1000}{t}(3/8)^t(5/8)^{1000-t}$

  (q) If you kept playing this game until you won (i.e. no limit on n, the number of games), then the number of games total played would be a memoryless rv.

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

abdhimnq

Problem 7 [10min] (and 70min will have elapsed) Consider a cup with $N = 8$ coins inside and $K = 3$ coins are marked with a permanent marker. Let $p := 3/8$ denote the proportion of marked coins of the total number of coins. You shake the cup and then reach in and pull out $n = 6$ coins so 6 are in your hand and 2 are left in the cup. Let $X_1, X_2, \ldots, X_6$ denote the rv that models if each of the 6 coins in your hand are marked or unmarked ($X_i = 1$ if marked and $X_i = 0$ if unmarked). Let T denote the number of coins in your hand that are marked with the permanent marker. Thus, T is the sum of $X_1, X_2, \ldots, X_6$.

- [16 pt / 104 pts]   Record the letter(s) of all the following that are **true** in general. At least one will be true.

  (a) $X_1, X_2, \ldots, X_6 \overset{iid}{\sim}$

  (b) $X_3 \overset{d}{=} X_6$

  (c) $X_3 \sim \text{Bernoulli}(p)$

  (d) $X_3 \sim \text{Bernoulli}(K/N)$

  (e) $T \sim \text{Binomial}(n, p)$

  (f) $T \sim \text{Binomial}(N, p)$

  (g) $T \sim \text{Hypergeometric}(n, K, N)$

  (h) $T \sim \text{Hypergeometric}(n, np, N)$

  (i) T is independent of X_3

  (j) $\mathbb{Supp}[T] = \{1, 2, 3, 4, 5, 6, 7, 8\}$

  (k) $\mathbb{Supp}[T] = \{1, 2, 3, 4, 5, 6\}$

  (l) $\mathbb{Supp}[T] = \{1, 2, 3\}$

  (m) $\mathbb{P}(T = t) = \dfrac{\binom{3}{t}\binom{5}{6-t}}{\binom{8}{6}}$

  (n) $\mathbb{E}[T] = np$

  (o) $\mathbb{E}[T] = n\frac{K}{N}$

  (p) Given a calculator and enough time, $\mathbb{Var}[T]$ can be computed given the information on this page

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

bcdglmnop