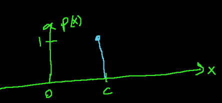
$X \sim Deg(c) := \{ c \text{ w.p. } 1$

$M_X(t) := E[e^{tX}] = \sum_{x \in \{c\}} e^{tx} p(x) = e^{tc}$



There is one more property of the MGF which we need:

(V) Continuity

Let $X_1, X_2, \dots, X_n$ be a sequence of rv's

$$\lim_{n\to\infty} M_{X_n}(t) = M_Y(t) \Rightarrow X_n \xrightarrow{d} Y$$

The expression on the rhs means "converges in distribution" which means as n → ∞, then X_n becomes more and more equal in distribution to Y.

$X_1, \dots, X_n$ sequence of r.v's iid $\quad \bar{X}_n \xrightarrow{d} \mu \quad$ Pf:

$Y \sim Deg(\mu)$ so $\bar{X}_n \xrightarrow{d} Y$. Let $T_n = X_1 + \dots + X_n$

$\Rightarrow m_Y(t) = e^{t\mu} \qquad \bar{X}_n = \frac{1}{n} T_n$

$$M_{\bar{X}_n}(t) \overset{III}{=} M_{T_n}\left(\frac{t}{n}\right) \overset{IV}{=} M_{X_1}\left(\frac{t}{n}\right) M_{X_2}\left(\frac{t}{n}\right) \cdots M_{X_n}\left(\frac{t}{n}\right) \overset{iid,\ I}{=} M_X\left(\frac{t}{n}\right)^n$$

$a^b = e^{\ln(a^b)} = e^{b \ln(a)}$

$$= e^{n \ln\left(M_X\left(\frac{t}{n}\right)\right)} = e^{\frac{\ln\left(M_X\left(\frac{t}{n}\right)\right)}{\frac{1}{n}}}$$

$\frac{0}{0}$ $\ln(1)$ (0)

Now let's take the limit as n → ∞ in hopes of finding something cool

$$\lim_{n\to\infty} M_{\bar{X}_n}(t) = e^{\lim_{n\to\infty} \frac{\ln\left(M_X\left(\frac{t}{n}\right)\right)}{\frac{1}{n}}} = e^{\lim_{u\to 0} \frac{\ln(M_X(ut))}{u}} \overset{?}{=} e^{\frac{\ln(M_X(0))}{0}}$$

let $u = \frac{1}{n} \quad n \to \infty \Rightarrow u \to 0$

L'Hopital's Rule

$$= e^{\lim_{u\to 0} \frac{t M_X'(ut) / M_X(ut)}{1}} = e^{t \frac{M_X'(0)}{M_X(0)}} \overset{0, II}{=} e^{t \frac{E[X]}{1}}$$

$$= e^{t\mu} \overset{(V)}{\Longrightarrow} \bar{X}_n \xrightarrow{d} \mu$$

Law of Large Numbers! (LLN)

But still... why is the normal distribution special?

let $X_1, \dots, X_n \overset{iid}{\sim}$ with mean μ and variance σ²

$E[\bar{X}_n] = \mu, \quad Var[\bar{X}] = \frac{\sigma^2}{n}$

$$Z_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \qquad E[Z_n] = E\left[\frac{\bar{X}_n}{\frac{\sigma}{\sqrt{n}}}\right] - E\left[\frac{\mu}{\frac{\sigma}{\sqrt{n}}}\right] = \frac{\mu}{\frac{\sigma}{\sqrt{n}}} - \frac{\mu}{\frac{\sigma}{\sqrt{n}}} = 0$$

$$Var[Z_n] = Var\left[\frac{\bar{X}}{\frac{\sigma}{\sqrt{n}}}\right] = \frac{1}{\frac{\sigma^2}{n}} Var[\bar{X}] = \frac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}} = 1$$

standardizing a rv
i.e. subtract the mean and divid by the standard deviation yields a rv with mean zero and Var & SD = 1

$$Z_n = \frac{\sqrt{n}\left(\frac{X_1 + \dots + X_n}{n} - \mu\right)}{\sigma} \overset{\mu = \frac{n\mu}{n}}{=} \frac{\sqrt{n}\left(\frac{X_1 + \dots + X_n}{n} - \frac{\mu + \dots + \mu}{n}\right)}{\sigma}$$

$$= \frac{\cancel{\sqrt{n}}\left((X_1 - \mu) + (X_2 - \mu) + \dots + (X_n - \mu)\right)}{\sqrt{n}\ \cancel{n}\sigma}$$

let $Z_i := \frac{X_i - \mu}{\sigma}$ stdize $E[Z_i] = 0$, $Var[Z_i] = 1$

$$= \frac{1}{\sqrt{n}}\left(\frac{X_1 - \mu}{\sigma} + \frac{X_2 - \mu}{\sigma} + \dots + \frac{X_n - \mu}{\sigma}\right) = \frac{Z_1 + \dots + Z_n}{\sqrt{n}}$$

$$M_{Z_n}(t) \overset{III}{=} M_{Z_1 + \dots + Z_n}\left(\frac{t}{\sqrt{n}}\right) \overset{IV}{=} M_{Z_1}\left(\frac{t}{\sqrt{n}}\right) \cdot \ldots \cdot M_{Z_n}\left(\frac{t}{\sqrt{n}}\right) \overset{iid,\ I}{=} M_Z\left(\frac{t}{\sqrt{n}}\right)^n$$

$$= e^{n \ln\left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)} = e^{\frac{\ln\left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)}{\frac{1}{n}}}$$

Now we take the limit of the MGF for Z_n as n → ∞ in the hope of finding something cool

$$\lim_{n\to\infty} M_{Z_n}(t) = e^{\lim_{n\to\infty} \frac{\ln\left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)}{\frac{1}{n}}} = e^{\lim_{u\to 0} \frac{\ln(M_Z(tu))}{u^2}}$$

let $u = \frac{1}{\sqrt{n}} \Rightarrow n \to \infty \Rightarrow u \to 0$

L'Hopital's Rule

$$= e^{\lim_{u\to 0} \frac{t \frac{M_Z'(tu)}{M_Z(tu)}}{2u}} \overset{?}{=} e^{\frac{t \frac{0}{1}}{0}} = e^{\frac{0}{0}} \quad BAD$$

L'Hopital's Rule

$$= e^{\frac{t}{2} \lim_{u\to 0} \frac{t M_Z(tu) M_Z''(tu) - t M_Z'(tu)^2}{1 \cdot 0^2}}$$

$$= e^{\frac{t^2}{2}\left(M_Z(0) M_Z''(0) - M_Z'(0)^2\right)} = e^{t^2/2}$$

$E[Z^2] = Var[Z] - E[Z]^2 = 1$

so the result is the MGF for the rv N(0, 1) hence.....

$$Z_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

This is called the "Central Limit Theorem" (CLT) which is probably the most famous thm in all of probability. This is unbelievable since we made no assumptions about the X's except that they were iid and they had mean μ and variance σ². They could've been any rv's!! But their standardized average no matter what becomes more and more normal.

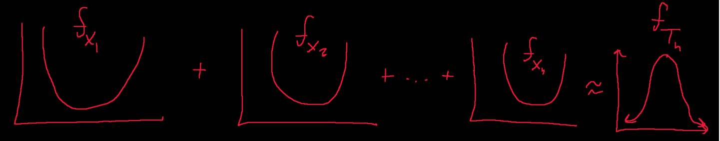
Implications of the CLT. Recall, limits don't exist so, it's never exactly normally distribution, only "approximately distributed" so:

$$Z_n = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \overset{\cdot}{\sim} N(0,1)$$

$$\Rightarrow \bar{X}_n \overset{\cdot}{\sim} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) \quad *$$

useful

$$\Rightarrow T_n \overset{\cdot}{\sim} N\left(n\mu, (\sigma\sqrt{n})^2\right) \quad *$$



Imagine you're standing on a line and with 50% prob you take a step forward and with 50% prob you take a step backward (indep). Let's say you take 100 such random steps.
(1) Where do you finish on average?
(2) What is the approx probability you are more than 10 steps away from where you started?

(1) $X_1, \dots, X_{100} \overset{iid}{\sim} \begin{cases} +1 & \text{wp } 50\% \\ -1 & \text{wp } 50\% \end{cases}$

$\mu = (+1)(.5) + (-1)(.5) = 0$

$\sigma^2 = (+1-0)^2(.5) + (-1-0)^2 \cdot .5 = .5 + .5 = 1 \Rightarrow \sigma = 1$

$E[\bar{X}_n] = \mu = 0$

$Y \sim N(\mu, \sigma^2)$

$Z = \frac{Y - \mu}{\sigma} \sim N(0,1)$

(2) P(|T| > 10) = P(T > 10) + P(T < -10)

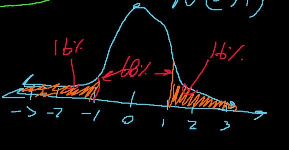
CLT: $T_n \overset{\cdot}{\sim} N\left(n\mu, (\sigma\sqrt{n})^2\right)$

$\Rightarrow T_{100} \overset{\cdot}{\sim} N\left(100\mu, (\sigma\sqrt{100})^2\right) = N(0, 10^2)$

$$P(T > 10) = P\left(\frac{T - 0}{10} > \frac{10 - 0}{10}\right)$$

$$= P(Z > 1) = 16\%$$

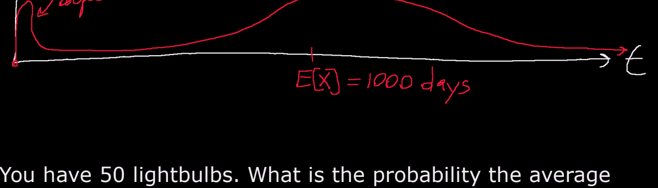
Std normal



$$P(T < -10) = P\left(\frac{T - 0}{10} < \frac{-10 - 0}{10}\right) = P(Z < -1) = 16\%$$

$$\Rightarrow P(|T| > 10) \approx 32\%$$

Here is the distribution of light bulb failure time, rv X:



You have 50 lightbulbs. What is the probability the average burn out is more than 1300 hr?