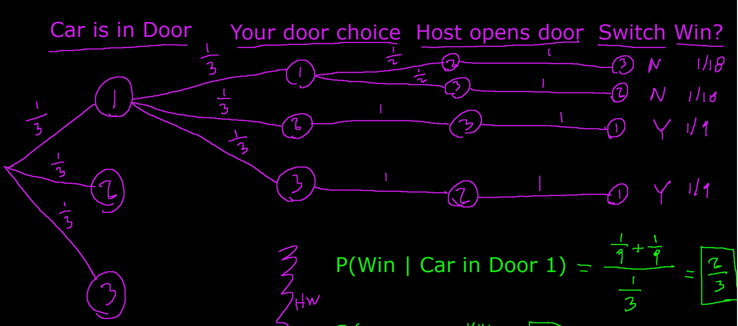


HW

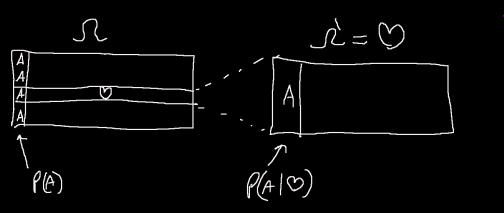
$$P(\text{Win} \mid \text{Car in Door 1}) = \frac{\frac{1}{9}+\frac{1}{9}}{\frac{1}{3}} = \boxed{\frac{2}{3}}$$

$$P(\text{Win}) = \overset{HW}{\dots\dots} = \boxed{\frac{2}{3}}$$

Return to the universe of the 52-card deck and draw one card.

$$P(A \mid \Omega) = \frac{4}{52} = \frac{1}{13}, \quad P(A \mid \heartsuit) = \frac{1}{13} = \frac{P(A, \heartsuit)}{P(\heartsuit)} = \frac{\frac{1}{\cancel{52}}}{\frac{13}{\cancel{52}}}$$

Did providing the information that the card was a heart change the probability? NO. Knowing it was a heart was "informationally irrelevant" or "probabilistic independence".



The heart event and the ace event are "independent" events.

P(IBM stock goes up today | raining in DC) = P(IBM stock goes up today)

If two events A, B are independent then: P(A | B) = P(A), P(B | A) = P(B)

$$\Rightarrow P(A \mid B) = \frac{P(A,B)}{P(B)} = P(A) \Rightarrow P(A,B) = P(A)P(B)$$

the "multiplication rule"

If you flip 5 coins. What is the probability of all heads?

$$P(H_1, H_2, H_3, H_4, H_5) = P(H_1) \cdot \ldots \cdot P(H_5) = \left(\tfrac{1}{2}\right)^5 \overset{\text{old}}{=} \frac{1}{|\Omega|} = \frac{1}{|\Omega_1|^5} \quad \frac{1}{2^5}$$

We will now see if Chevalier de Mere's gambling conjecture is correct. He claimed that P(one or more double 6 in 24 dice rolls) < 1/2.

= P(1 6-6) + P(2 6-6's) + ... + P(24 6-6's)   due to [A3]
= 1 - P(no 6-6's)   complement rule
= 1 - P(roll 1 is not 6-6, roll 2 is not 6-6, ..., roll 24 is not 6-6)
= 1 - P(roll 1 is not 6-6) P(roll 2 is not 6-6) * ... * P(roll 24 is not 6-6) mult rule
= 1 - (1 - P(6,6))^24   complement rule, same event
= 1 - (1 - P(6)P(6))^24   mult rule
= 1 - (1 - 1/6 * 1/6)^24 = 1 - (1 - 1/36)^24 = 1 - (35/36)^24 = 0.4914 < 0.5

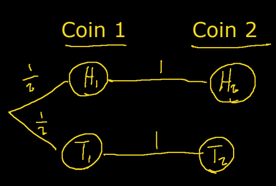
If P(A | B) ≠ P(A) then A, B and "not independent" i.e. "dependent".

P(dying of SARS-Cov-2 | vaccine) < P(dying of SARS-Cov-2)

If you can prove this... then the two events are dependent AKA "informationally relevant".

P(Q64 is late | snowstorm) > P(Q64 is late)
P(lung cancer | smoking) > P(lung cancer)

Consider two magic coins. You flip them together and they always come up the same.



Prove that heads in coin 1 and heads in coin 2 are dependent.

$$1 = P(H_2 \mid H_1) \neq P(H_2) = \frac{1}{2}$$

Let's do some famous fun problems. First is the "birthday problem". What is the probability that at least one pair of people share a birthday in this class? Intuitively, there are 365 days and only 22 people so it's probably low.

= P(at least one birthday pair in 22)
= P(1 pair) + P(2 pairs) + ... + P(22 pairs)   [A3]
= 1 - P(no pairs)
= 1 - 365/365 * 364/365 * 363 / 365 * ... * 344/365

$$= 1 - \frac{{}_{365}P_{22}}{365^{22}} = 1 - \frac{\frac{365!}{343!}}{365^{22}} = 1 - .524 = \boxed{.476}$$

$$\text{P(at least one birthday pair in 60)} = 1 - \frac{{}_{365}P_{60}}{365^{60}} = 99.5\%$$

Another famous problem is the "hat problem". This is difficult. n people walk into a room and toss their hats in a pile. They each then grab a random hat. What is the probability at least one person gets their own hat?

= P(1 person gets hat) + P(2 people get hat) + ... + P(n people get hat)

Let A_i := event i'th person gets their hat

= P(A_1 ∪ A_2 ∪ ... ∪ A_n)
Then using the inclusion-exclusion formula,

$$= \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \ldots + \ldots - \ldots + \ldots$$

$$P(A_1) = \frac{1}{1}\,\frac{n-1}{2}\,\frac{n-2}{3}\cdots\frac{1}{n} \Rightarrow \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$P(A_2) = \frac{n-1}{1}\,\frac{1}{2}\,\frac{n-2}{3}\cdots\frac{1}{n} \Rightarrow \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$\Rightarrow P(A_i) = \frac{1}{n}$$

$$P(A_1 \cap A_2) = \frac{1}{1}\,\frac{1}{2}\,\frac{n-2}{3}\cdots\frac{1}{n} \Rightarrow \frac{(n-2)!}{n!} = P(A_i \cap A_j)$$

$$P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!}$$

$$= \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k)$$

$$\sum P(A_i) = \sum \frac{1}{n} = n \cdot \frac{1}{n} = 1 = \frac{1}{1!}$$

$$\sum P(A_i \cap A_j) = \sum \frac{(n-2)!}{n!} = \binom{n}{2}\frac{(n-2)!}{n!} = \frac{\cancel{n!}}{\cancel{(n-2)!}\,2!}\,\frac{\cancel{(n-2)!}}{\cancel{n!}} = \frac{1}{2!}$$

$$\sum P(A_i \cap A_j \cap A_k) = \sum \frac{(n-3)!}{n!} = \binom{n}{3}\frac{(n-3)!}{n!} = \frac{1}{3!}$$

$$= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + - + - \ldots = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!} \overset{n\ \text{large}}{\approx} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!}$$

Taylor series

$$= 1 - \frac{1}{e} \approx \boxed{67\%}$$