Combinations are the number of unique subsets of size k of a set A s.t. |A| = n ≥ k i.e.

|{B : B ⊆ A & |B| = k}|

This set theoretical definition leads to a cool identity:

2^A

$A = \{5, 9, 11\}$

$$|2^A| = |\{B : B \subseteq A\}|$$

$$|2^A| = |\{B : B \subseteq A \,\&\, |B| = 0\} \cup \{B : B \subseteq A \,\&\, |B| = 1\} \cup \ldots \cup \{B : B \subseteq A \,\&\, |B| = n\}|$$

these sets are mutually exclusive and collectively exhaustive

$$|2^A| = |\{B : B \subseteq A \,\&\, |B| = 0\}| + |\{B : B \subseteq A \,\&\, |B| = 1\}| + \ldots + |\{B : B \subseteq A \,\&\, |B| = n\}|$$

$$|2^A| = \sum_{k=0}^{n} |\{B : B \subseteq A \,\&\, |B| = k\}|$$

$$|2^A| = \boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

$$|2^A| = 2^{|A|}$$

Another nice and famous combinatorial identities

$$(a+b)^2 = \underbrace{a^2 + ab + ba + b^2}_{4 \text{ terms}} = 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 = (a+b)(a+b)(a+b) = \overbrace{aaa + aab + aba + baa + abb + bab + bba + bbb}^{8 \text{ terms}} = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$

$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b) = \binom{4}{4}a^4 + \binom{4}{3}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{1}ab^3 + \binom{4}{0}b^4 = \sum_{k=0}^{4}\binom{4}{k}a^k b^{4-k}$$

$\vdots$

$n \in \mathbb{N}$

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k}$$

This is the "binomial theorem"

$$(1+x)^n = (x+1)^n = \sum_{k=0}^{n}\binom{n}{k} x^k$$

Another famous combinatorial identity is from Pascal's triangle:

$$1 = \binom{0}{0}$$

$$\binom{1}{0} = 1 \quad 1 = \binom{1}{1}$$

$$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad 1 = \binom{2}{2}$$

$$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \quad \binom{4}{1} = 4 \quad \binom{4}{2} = 6 \quad \binom{4}{3} = 4 \quad \binom{4}{4} = 1$$

$\vdots$

$\binom{0}{k}$, $\binom{1}{k}$, $\binom{2}{k}$, $\binom{3}{k}$, $\binom{4}{k}$

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$$

This triangle motivates the following identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\frac{n!}{k!(n-k)!} = \left(\frac{(n-1)!}{((n-1)-(k-1))!\,(k-1)!} + \frac{(n-1)!}{(n-k-1)!\,k!}\right)\frac{n}{n}$$

$$= \frac{n!}{n}\left(\frac{1}{(n-k)!\,(k-1)!}\cdot\frac{k}{k} + \frac{1}{(n-k-1)!\,k!}\cdot\frac{n-k}{n-k}\right)$$

$$= \frac{n!}{n}\left(\frac{k}{(n-k)!\,k!} + \frac{n-k}{(n-k)!\,k!}\right) = \frac{n!}{\cancel{n}}\,\frac{\cancel{n}}{(n-k)!\,k!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

Consider the game "5 card draw poker" with no trade-in. You get 5 cards (called your "hand") and there are certain winning hands in order of payout: royal flush, straight flush, 4-of-a-kind, full house, flush, straight, 3-of-a-kind, 2 pair. We will now calculate the probability of each of these winning hands.

Flush: all cards are of the same suit
Straight: all cards are in order where A can be "low" or "high" i.e. A2345, ..., 56789, ..., 10JQKA
Straight flush: both flush and straight
Royal flush: the highest straight flush i.e. 10JQKA
4-of-a-kind: four of the same rank
full house: 3-of-a-kind and a 2-of-a-kind (pair)

$$P(\text{Royal Flush}) = \frac{|\{\text{Royal Flushes}\}|}{\binom{52}{5}} = \frac{\binom{4}{1}}{\binom{52}{5}}$$

$\binom{52}{5}$ ← the number of 5 card hands i.e. samples of 52 without replacement where order doesn't matter i.e 2,598,960

choose suit

$$P(\text{Straight Flush}) = \frac{\binom{4}{1}\binom{9}{1}}{\binom{52}{5}}$$

choose starting position

choose rank

$$P(\text{Four of a kind}) = \frac{\binom{13}{1}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}}$$

choose 5th card's rank and then 5th card's suit

rank of the 3-of-a-kind

the suit configuration of the 3 of a kind

$$P(\text{Full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$$

the rank of the pair

the suit configuration of the pair

suit

the five ranks

$$P(\text{flush}) = \frac{\binom{4}{1}\binom{13}{5} - \binom{4}{1}\binom{9}{1} - \binom{4}{1}}{\binom{52}{5}}$$

remove the straight flushes

suit selection of each card

starting positions

$$P(\text{straight}) = \frac{\binom{10}{1}\binom{4}{1}^5 - \binom{4}{1}\binom{9}{1} - \binom{4}{1}}{\binom{52}{5}}$$

remove straight flushes

remove royal flushes