P(3 of a kind) = $\dfrac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$

the rank of the 3 of a kind

the 3 suited cards

the other two cards that are not a pair (their ranks)

(their suits)

$AA66 = 66AA$

P(2 pair) = $\dfrac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$

the ranks of the two pairs

the suits of the two pairs

the 5th card's rank

the 5th card's suit

$\times \quad \binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}$

$\frac{13 \cdot 12}{2} = \binom{13}{2} \neq \binom{13}{1}\binom{12}{1} = 13 \cdot 12$

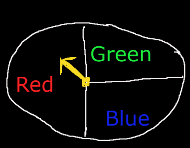
$AAA66 \neq AA666$

This term was in the full house count but not in the 2-pair

Let's return to our "working definition" of probability (also called the "classic definition"):

$$P(A) = \frac{|A|}{|\Omega|}$$

Is this definition always correct? Consider the following random experiment of a spinner:



$$P(\text{Red}) \stackrel{?}{=} \frac{|\{R\}|}{|\{R,G,B\}|} = \frac{1}{3}$$

This is clearly incorrect! It should be 1/2 not 1/3. So our definition it wrong. Our definition is only correct when... all outcomes are equally likely i.e. ∀ω ∈ Ω P({ω}) = 1 / |Ω|.

Do we have a definition that *always* works?? NO! Let's explore the "philosophy of probability". There are a number of contending definitions.

(I) Long run frequency definition. Define the indicator function:

$$\mathbb{1}_{\omega \in A} := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

$$P(A) := \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \mathbb{1}_{\omega_i \in A}}{n}$$

Story: John Kerrich was a statistician and was POW caught by the Nazis in WW2 and he flipped a coin in prison 10,000x and got 5,067 heads ⇒ 0.5067 was his probability estimate of heads (1946).

Story: Chevalier de Mere in 1654 he claimed that

P({≥1 double 6 in 24 rolls of two dice}) < 0.5

The exact probability is 0.4914. He used the long run freq. def. and kept track in his head. He was a genius who devoted his brains to gambling.

Problems with the long run freq def. of prob:
(1) Limts aren't real. It is impossible to do an experiment infinite times.
 (i) We never know the true values of the probabilities
 (ii) We can be quite wrong if n is low

(2) Some experiments are one-off
 P(OJ Simpson is guilty)
 P(hurrican Ida will hit New Orleans)
 P(North Korea gets nuclear weapons next year)

You cannot repeat these experiments! Thus you can't even estimate the probabilities using this definition.

(II) Propensity Theory (Karl Popper, 1957)
Objects have inherent disposition towards their ω's. He got the idea from quantum mechanics. You can prove using theory of quantum mechanics that the half life of U238 is 4.5 billion years which means:

P(one U238 atom exploding in < 4.5 billion years) = 0.5

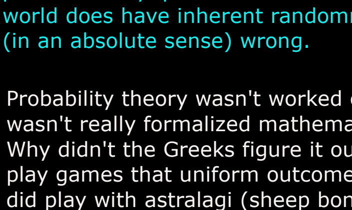
Problem: you can't calculate anything "real"

I and II are called "objective" theories of probability. They assume probability exists with or without our belief in it.

(III) Subjective definition of probability: people use their own evidence to come up with their own probability values. Thus probability is a "degree of belief" which follows certain rational rules (Ramsey, 1926 and de Finetti, 1928).

Here's another question: what is "randomness"? Is randomness real? Laplace said "no" in 1814 since everything is deterministic i.e. calculable using laws of physics and knowledge of anterior states. Thus, randomness is an illusion due to our own ignorance.

In the 1920's there was an experiment that shot electrons from a gun through two slits and detected them on a screen behind the slits:



It seems to land in a "random" place with the pattern being predictable by quantum mechanics. Thus.... it seems.... the world does have inherent randomness and thus Laplace was (in an absolute sense) wrong.

Probability theory wasn't worked out until late 1800's and wasn't really formalized mathematically until 1930's. WHY? Why didn't the Greeks figure it out?? Maybe because they didn't play games that uniform outcomes (e.g. coins, dice, etc) but they did play with astralagi (sheep bones which are all unique) which had completely different probabilities of their sides.

We have no definition of probability but we have a mathematical theory (and math isn't physically real). For the test, you need to know the three "axioms".

Consider a set Ω ≠ ∅ and a function P() which satisfies Kolmogorov's (1933) three axioms:
(1) P(Ω) = 1
(2) For all A ⊆ Ω, P(A) ≥ 0
(3) If A_1, A_2, ... are disjoint (mutually exclusive) then
 P(A_1 ∪ A_2 ∪ ....) = P(A_1) + P(A_2) + ...

From these three axioms and basic set theory, you can derive all the rules we did so far in class. The following proofs are NOT COVERED on this class's exams! But the consequences are:

Thm I Complement Rule

$\Omega = A \cup A^c$

$P(\Omega) = P(A \cup A^c)$

$P(\Omega) = P(A) + P(A^c) \quad [A3]$

$1 = P(A) + P(A^c) \quad [A1]$

$P(A) = 1 - P(A^c)$

Thm 2: something must happen

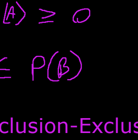
$P(\Omega) = 1 - P(\Omega^c) \quad [T1]$

$P(\Omega) = 1 - P(\emptyset)$

$1 = 1 - P(\emptyset) \quad [A1]$

$\Rightarrow P(\emptyset) = 0.$

Thm 3



$A \subseteq B$ $\Rightarrow C \cup A = B$

let $C : B \setminus A$ $\Rightarrow C \cap A = \emptyset$

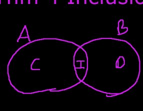
$\overset{P(B)}{\overbrace{P(C \cap A)}} = P(C) + P(A) \quad [A3]$

$P(B) - P(A) = P(C)$

$P(B) - P(A) \geq 0 \quad [A2]$

$P(A) \leq P(B)$

Thm 4 Inclusion-Exclusion Law



$C = A \setminus B$
$D = B \setminus A$
$I = A \cap B$
$\Rightarrow C \cup I = A,\ D \cup I = B,\ C \cap I = \emptyset,\ D \cap I = \emptyset$

$P(A) = P(C) + P(I) \quad [A3] \Rightarrow P(C) = P(A) - P(I)$

$P(B) = P(D) + P(I) \quad [A3] \Rightarrow P(D) = P(B) - P(I)$

$P(A \cup B) = P(C) + P(D) + P(I) \quad [A3]$

$\Downarrow$

$P(A \cup B) = P(A) - \cancel{P(I)} + P(B) - P(I) + \cancel{P(I)}$

$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$