$$\overbrace{\text{P(alternating gender)}}^{A} = P(BGBGBG) + P(GBGBGB) = \frac{2(3\cdot3\cdot2\cdot2)}{6!} = \frac{\cancel{2}\cdot\cancel{3}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{2}}{\cancel{6}\cdot5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}} = \boxed{\frac{1}{10}}$$

$$P(BGBGBG) = \frac{3\cdot3\cdot2\cdot2}{6!}$$

3 3 2 2 1 1
Chair 1 2 3 4 5 6

$$P(GBGBGB) = \frac{3\cdot3\cdot2\cdot2}{6!}$$

3 3 2 2 1 1
Chair 1 2 3 4 5 6

Ω

BG... GB...

$$\text{P(Richard and Susan sit together)} = \frac{2\cdot5\cdot4!}{6!} = \frac{2\cdot5}{6\cdot5} = \frac{1}{3}$$

(R) (S)
1 1 4 3 2 1
Chair 1 2 3 4 5 6

(R) (S)
4 1 1 3 2 1
1 2 3 4 5 6

⋮

(S) (R)
1 1 4 3 2 1

⋮

(S) (R)
4 3 2 1 1 1

How many ways to seat them in a circle of 6 seats such that the circle is rotationally invariant?

Chairs: 6 (1), 5 (2), 4 (3), 3 (4), 2 (5), 1 (6) , 1 (6), 6 (1), 5 (2), 4 (3), 3 (4), 2 (5) , ... , 5 (2), 4 (3), 3 (4), 2 (5), 1 (6), 6 (1)

<J,B,R,S,C,M> = <M,J,B,R,S,C>

$= \frac{6!}{6}$ ← the # of invariance settings

The principle of "dividing out the invariance". You divide out the number of ways that are considered equivalent in your setting.

You have five flowers: three orchids O_1, O_2, O_3 and two chysanthemums: X_1, X_2. How many ways to place them into five flower pots where each flower pot is "distinct" (i.e. unique or special).

5 4 3 2 1 $= 5! = 120$
Chair: 1 2 3 4 5

How many ways to place them into 5 pots where the orchids are indistinct from one another (they're all the same) and the chysanthemums are indistinct from one another (they're all the same)?

$O_1 O_2 O_3 X_1 X_2$
$O_1 O_3 O_2 X_1 X_2$
$O_2 O_3 O_1 X_1 X_2$
$O_2 O_1 O_3 X_1 X_2$
$O_3 O_2 O_1 X_1 X_2$
$O_3 O_1 O_2 X_1 X_2$

$6 = 3\cdot2\cdot1 = 3!$

$O_1 O_2 O_3 X_2 X_1$
$O_1 O_3 O_2 X_2 X_1$
$O_2 O_1 O_3 X_2 X_1$
$O_2 O_3 O_1 X_2 X_1$
$O_3 O_2 O_1 X_2 X_1$
$O_3 O_1 O_2 X_2 X_1$

$12 = 3!\cdot2 = 3!\,2!$ OOOXX

$$\Rightarrow \frac{5!}{3!\,2!} = \frac{5\cdot\cancel{4}^{2}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 10 \rightarrow$$

OOOXX
OOXOX
OXOOX
XOOOX
XOOXO
XOXOO
XXOOO
OXXOO
OOXXO
OXOXO

two examples of dividing out the invariance

$$\overbrace{\text{P(getting four heads in ten coin flips)}}^{A} = \frac{|A|}{|\Omega|} = \frac{\frac{10!}{4!\,6!}}{2^{10}} = \boxed{.205}$$

H_1 T_1 H_2 T_2 H_3 T_3 T_4 T_5 T_6 H_4
1 2 3 4 5 6 7 8 9 10

How many ways can we order 4H and 6T in 10 flips such that all heads are indistinct and all tails are indistinct?

$$|A| = \frac{10!}{4!\,6!}$$

Recall our 6 people {J,B,S,R,M,C}.
How many ways to seat them in 6 chairs? $_6P_6 = \frac{6!}{0!} = 6!$

How many ways to seat them in 3 chairs? $_6P_3 = \frac{6!}{3!}$

How many ways to seat them in 3 chairs such that their order doesn't matter? $\frac{_6P_3}{_3P_3}$

$$\frac{\cancel{6}^{3}\cdot5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}} = 15$$

$$\shortparallel$$

How many ways to seat them in 4 chairs such that their order doesn't matter? $\frac{_6P_4}{_4P_4} = \frac{\frac{6!}{(6-4)!}}{\frac{4!}{(4-4)!}} = \frac{6!}{2!\,4!}$

Here are the 15 arrangements:

{JBSR}, {JBSM}, {JBSC}, {JSRM}, {JSRC}, {JSMC}, {JRMC}
{BSRM}, {BSRC}, {BSMC}, {BRMC}
{SRMC}n

<SRMC>, <SRCM>, <SMRC>, <SMCR>,
<RSMC>, <RSCM>, <RMSC>, <RMCS>
<MSRC>, <MSCR>, <MCSR>, <MCRS>
<CSRM>, <CSMR>, <CMSR>, <CMRS>, ...

The number of ways to "choose" 4 objects out of a set of 6 objects without replacement such that order doesn't matter = $\frac{6!}{4!\,2!}$

The number of ways to "choose" k objects out of a set of n objects without replacement such that order doesn't matter is:

$$_nC_k = \binom{n}{k} = \frac{_nP_k}{_kP_k} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{k!\,(n-k)!}$$

This is called "combinations". It's a permutation with an invariance of the order of the chosen objects divided out.
n must be $\in \mathbb{N}_0$.
k must be $\in$ {0, 1, 2, ..., n}
Identities:

$\binom{n}{1} = n$ $\binom{n}{n} = 1$

$\binom{n}{n-1} = n$ $\binom{n}{0} = 1$

$\binom{n}{k} = \binom{n}{n-k}$

n
n−k   k

Recall our 6 people {J,B,S,R,M,C}. Four of them sit down randomly. What is the probability Jane is seated?

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\binom{5}{3}}{\binom{6}{4}} =$$

$$A: \{J, ?, ?, ?\} \Rightarrow \binom{5}{3}$$

$$\frac{4(_5P_3)}{_6P_4}\cdot\frac{1}{\frac{4!}{4!}} = \frac{\frac{_5P_3}{3!}}{\frac{_6P_4}{4!}} = \frac{\frac{5!}{2!\,3!}}{\frac{6!}{2!\,4!}}$$

$$= \boxed{2/3}$$

order doesn't matter   order does matter