A = {1,2,3}

Cardinality operator |.|: simply the size of the set (number of elements in the set).

|A| = 3
B = {1, 2, ..., n}
|B| = n
|ℕ| = |{1, 2, 3, ...}|| = countable infinity
|ℤ| = countable infinity
|ℚ| = countable infinity (FYI)
|ℝ| = uncountable infinity > |ℕ|

Proof that the real numbers are not countable. Consider [0, 1] ⊂ ℝ and write them in binary (base 2). Enumerate all of them:

[1] 0.100000...
[2] 0.101000...
[3] 0.011000..
[4] .
[5] .
[6] .

With all these binary changes, you change every single number on the list but the number created with the switches 010... is not any number of the list to begin with. Cantor proved that in late 1800's. This proves that the reals are a "bigger infinity" than the counting numbers.

Back to set theory... We will now define the "powerset" of a set which is set function:

$$2^A := \{B : B \subseteq A\}$$

A = {1,2,3}, $2^A = \{\underset{\{\}}{\phi}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \underset{\{1,2,3\}}{A}\}$

$\{1\} \subset A$ $\quad \phi \subset A$

$1 \in A$ $\quad \{\} \subset A$,

FFF 123 | TFF 123 | FTF 123 | FFT 123 | TTF 123 | TFT 123 | FTT 123 | TTT 123

$|2^A| = 8$

There's a simple "heuristic" (a rule of thumb) to count the elements of the subset:

$$\underset{1}{\underbrace{T/F}_{2}} \cdot \underset{2}{\underbrace{T/F}_{2}} \cdot \underset{3}{\underbrace{T/F}_{2}} = 8 = 2^3$$

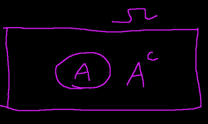
Thm: for any set A, $|2^A| = 2^{|A|}$.

In probability theory, there is a special set called the "universe", "scope", "sample space" or "space of discourse" which is all elements we are considering and it's denoted Ω (the greek letter Omega)

e.g. Ω = F ∪ M = {Mary, Susan, Jane, Dana, Bob, Joe, Max}
F ⊆ Ω, M ⊆ Ω
F ∩ Ω = F, F ∪ Ω = Ω
∅ ∩ Ω = ∅, ∅ ∪ Ω = Ω
F \ Ω = ∅
Ω \ F = {Bob, Joe, Max} ≠ M

We now define "set complement" as the last example:

$$A^c := \Omega \setminus A$$



$(A^c)^c = A$,

$A \cap A^c = \phi$, $A \cup A^c = \Omega$, $\Omega^c = \phi$, $\phi^c = \Omega$

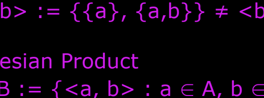
The sets A, A-complement are "mutually exclusive" meaning if an element is in one, it is not in the other. Also, the sets A, A-complement are "collectively exhaustive" meaning between the two of them, all elements in the universe are present.

$|A| + |A^c| = |\Omega|$ for a finitely sized universe

let $A_1, A_2, \ldots, A_n$ be collectively exhaustive $\bigcup_{i=1}^{n} A_i = \Omega$

$= A_1 \cup A_2 \cup \ldots \cup A_n$

let $A_1, A_2, \ldots, A_n$ be mutually exclusive $A_i \cap A_j = \phi,\ i \neq j$



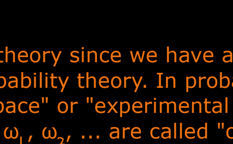
Ordered Pair
<a, b> := {{a}, {a,b}} ≠ <b, a> := {{b}, {a,b}}

"product operator"

Cartesian Product
A x B := {<a, b> : a ∈ A, b ∈ B}
e.g. A = {1,2}, B = {3,4}
A x B = {<1,3>, <1,4>, <2,3>, <2,4>}
|A x B| = |A| |B| = (2)(2) = 4 in general... $|A_1 \times A_2 \times \ldots \times A_n| = \prod_{i=1}^{n} |A_i|$

A^2 = A x A

$\mathbb{R}^2 =$ ℝ x ℝ



$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

We're done with set theory since we have all the tools we need to begin discussing probability theory. In probability theory, the Ω is called the "sample space" or "experimental space" or "outcome space". Its elements ω_1, ω_2, ... are called "outcomes" (lowercase omega in the Greek alphabet). Our first experiment is the "fair coin flipping experiment".

$$\Omega = \{\underset{\omega_1}{H}, \underset{\omega_2}{T}\}$$

Is {H} and {T} mutually exclusive? Yes
Is {H} and {T} collectively exhaustive? Yes
2^Ω = {∅, {H}, {T}, {H,T}} for a total cardinality of 4

Let an event "A" be defined as a subset of the universe, A ⊆ Ω e.g. A = {H}. What is the probability of the event A?

$\boxed{P(A) = \frac{|A|}{|\Omega|}} = \frac{1}{2}$, $P(\phi) = 0 = \frac{|\phi|}{|\Omega|} = \frac{0}{2}$, $P(\Omega) = \frac{|\Omega|}{|\Omega|} = \frac{2}{2} = 1$

this is our working definition of probability (we'll see that it's busted in a few lectures from now).

$P(H) = \frac{|H|}{|\Omega|} = \frac{?}{2} =$ undefined, $P(\{H\}) = \frac{1}{2}$

$$P : 2^\Omega \to [0, 1]$$

Probability is a set function taking in an element of the powerset of Ω (i.e. a subset of Ω) and returning a number between 0 and 1 including 0 (the probability of trivial event ∅) and 1 (the probability of the trivial event Ω).