$$Z \sim N(0,1) := \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}}_{f(z)} \qquad \mathrm{Supp}[Z] = \mathbb{R}$$

the "standard normal" or "standard Gaussian" or "standard bell curve"

Is f(z) a density?

(a) $f(z) \geq 0 \ \forall z \in \mathbb{R}$? Yes

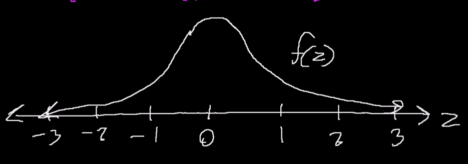
(b) $\int_{\mathrm{Supp}[Z]} f(z)\,dz = 1$ $\qquad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1$

Assume the Gaussian Integral: $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$

Proof is in Math 201 and you can prove it via a double integral and change to polar coordinates.

$$= \frac{1}{\sqrt{\cancel{2}\pi}} \int_{-\infty}^{\infty} e^{-u^2} \cancel{\sqrt{2}}\, du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1 \ \checkmark$$

let $u^2 = \frac{z^2}{2} \Rightarrow u = \frac{z}{\sqrt{2}} \Rightarrow du = \frac{1}{\sqrt{2}} dz \Rightarrow dz = \sqrt{2}\, du$



$$E[Z] := \int_{\mathbb{R}} z f(z)\,dz = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} z e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \cancel{z} e^{-u} \cancel{\tfrac{1}{z}}\, du$$

let $u = \frac{z^2}{2} \Rightarrow \frac{du}{dz} = z \Rightarrow dz = \frac{1}{z} du$

$$= \frac{1}{\sqrt{2\pi}} \left(- \left[e^{-u} \right]_{z=-\infty}^{z=\infty} \right) = -\frac{1}{2\pi} \left[e^{-z^2/2} \right]_{-\infty}^{\infty} = -\frac{1}{2\pi}(0 - 0) = 0 \ \checkmark$$

$$Var[Z] = E[Z^2] - \cancel{\mu^2}^{\,0} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} z^2 e^{-z^2/2} dz = \ldots\ldots = 1 = \sigma^2 \Rightarrow \sigma = 1$$

integration by parts

symmetry in f(z)

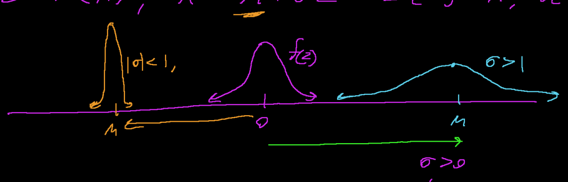
$$Med[Z] = 0$$

$$F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du =$$ this is an impossible integral to express in closed form... so it can only be approximated in a computer

$\Rightarrow F^{-1}(q)$ is also not available in closed form

Famous probabilities of the standard normal
P(Z ∈ [-1, 1]) = P(Z ∈ [±1σ]) = 68%
P(Z ∈ [-2, 2]) = P(Z ∈ [±2σ]) = 95%
P(Z ∈ [-3, 3]) = P(Z ∈ [±3σ]) = 99.7%
These three numbers are famous. They are called the "3σ rule", "empirical rule" and "68-95-99.7 rule"

$$Z \sim N(0,1), \quad X = \mu + \sigma Z \Rightarrow E[X] = \mu, \ SD[X] = \sigma$$



$$F_X(x) = P(X \leq x) = P(\mu + \sigma Z \leq x) = P\left(Z \leq \frac{x-\mu}{\sigma}\right) = F_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$f_X(x) = \frac{d}{dx}\left[F_Z\left(\frac{x-\mu}{\sigma}\right)\right] = \frac{d}{dv}\frac{dv}{dx} F_Z(v) = \frac{1}{\sigma}\frac{d}{dv}\left[F_Z(v)\right]$$

let $v = \frac{x-\mu}{\sigma} \Rightarrow \frac{dv}{dx} = \frac{1}{\sigma}$

$$= \frac{1}{\sigma} f_Z(v) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-v^2/2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = N(\mu, \sigma^2)$$

the general normal density

Supp[X] = ℝ. This rv has two parameters whose parameter space is μ ∈ ℝ, σ > 0.

Why is the normal rv so important?? We're going to answer this now but it will take awhile... We first talk about a totally different topic...

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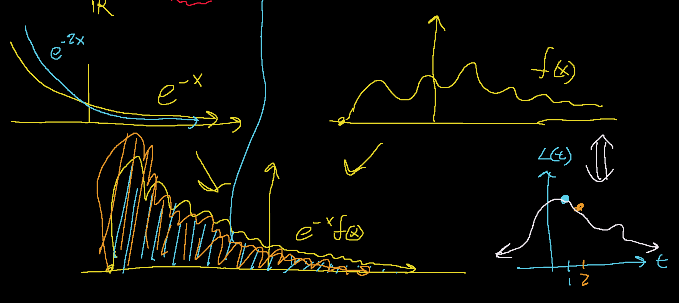
let $L(t) := \int_{\mathbb{R}} e^{-tx} f(x)\,dx$

This is called the "bilateral Laplace transform" of the function f.

$$L(2) = \int_{\mathbb{R}} e^{-2x} f(x)\,dx = 16.3$$



I'm transforming f into L if I do the integral for general values of t.

Why do we care about this transform? Well, if L(t) exists, then it is proven than it is 1:1 with f(x). So L(t) is kind of like another way to express f(x). Maybe like DNA is the same as someone's face. This relates to what we've been doing all semester with rv's:

$$M_X(t) := E\left[e^{tX}\right] \ \overset{\text{continuous}}{=} \ \int_{\mathbb{R}} e^{tx} f_X(x)\,dx \qquad \overset{\text{discrete}}{=} \ \sum e^{tx} p_X(x)$$

moment generating function (MGF) which is another way to look at the PDF or PMF of a rv

The MGF has many useful properties (0) $M_X(0) = E[e^{t(0)}] = 1$

(I) $M_X(t) = M_Y(t) \Longrightarrow X \overset{d}{=} Y$

this is due to the thm that says L(t) and f(x) are 1:1 which we won't prove since we need advanced math to do it

$M_X(t) = E[e^{tX}]$ generally speaking...

$$M_X'(t) = \frac{d}{dt}\left[E[e^{tX}]\right] = E\left[\frac{d}{dt}\left[e^{tX}\right]\right] = E\left[X e^{tX}\right]$$

$$M_X'(0) = E[X]$$

$$M_X''(t) = \qquad = E\left[\frac{d}{dt}\left[X e^{tX}\right]\right] = E\left[X^2 e^{tX}\right]$$

$$M_X''(0) = E[X^2]$$

$\vdots$

(II) $M_X^{(k)}(0) = E[X^k]$ the "moment generating property"

(III) $Y = aX + c$ where $a, c \in \mathbb{R}$

$$M_Y(t) = E[e^{tY}] = E\left[e^{t(aX+c)}\right] = E\left[e^{taX + tc}\right]$$

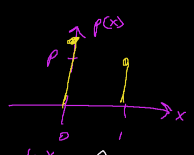
$$= E\left[e^{taX} e^{tc}\right] = e^{tc} E\left[e^{\overset{t'}{ta}X}\right] = e^{tc} E\left[e^{t'X}\right]$$

$$= e^{tc} M_X(t') = e^{tc} M_X(at)$$ shift and scale property

(IV) $Y = X_1 + X_2$ where $X_1, X_2 \overset{ind}{\sim}$

$$M_Y(t) := E\left[e^{t(X_1+X_2)}\right] = E\left[e^{tX_1} e^{tX_2}\right] = E[e^{tX_1}]E[e^{tX_2}] = M_{X_1}(t) M_{X_2}(t)$$

Let's find the MGF of the Bernoulli

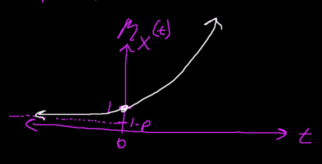


$$X \sim Bern(p) = p^x (1-p)^{1-x} = p(x)$$

$$M_X(t) = \sum_{x \in \mathrm{Supp}[X]} e^{tx} p(x) = \sum_{x \in \{0,1\}} e^{tx} p^x (1-p)^{1-x}$$

⇕ 1:1

$$= 1 - p + pe^t$$



$$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$M_Z(t) = E[e^{tZ}] = \int_{\mathbb{R}} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{z^2}{2} + tz} dz = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(z-t)^2} e^{\frac{t^2}{2}} dz$$

$$-\frac{1}{2}z^2 + tz = -\frac{1}{2}(z^2 - 2tz) = -\frac{1}{2}\left((z-t)^2 - t^2\right) = \frac{1}{2}(z-t)^2 + \frac{t^2}{2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{t^2/2} \int_{\mathbb{R}} e^{-\frac{1}{2}(z-t)^2} dz = e^{t^2/2} \int_{\mathbb{R}} \underbrace{\frac{1}{\sqrt{2\pi(1^2)}} e^{-\frac{1}{2(1^2)}(z-t)^2}}_{f_X(x) \text{ for } X \sim N(t,1)} dz = e^{t^2/2}$$

$$X \sim N(\mu, \sigma^2) \Rightarrow X = \mu + \sigma Z \overset{(II)}{\Rightarrow} M_X(t) = e^{t\mu} M_Z(\sigma t)$$

$$= e^{t\mu} e^{\sigma^2 t^2/2}$$

$$= e^{t\mu + \sigma^2 t^2/2}$$