Math 241 Fall 2021
Midterm Examination One

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Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

By taking this exam, you acknowledge and agree to uphold this Code of Academic Integrity.

Instructions

This exam is 70 minutes (variable time per question) and closed-book. You are allowed **one** page (front and back) of a "cheat sheet", blank scrap paper and a graphing calculator. Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 [10min] (and 10min will have elapsed) These questions are about the philosophy of probability and the mathematization of probability.

- [15 pt / 15 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

(a) The theory of probability is as old as geometry, dating back to the ancient Greeks.

(b) Notions of equally likely outcomes, conditional probability and independence began around the 1600's.

(c) Some famous people are saying the probability the stock market will crash this month is high; they are employing a subjective defintion of probability.

(d) The long run frequency definition of probability can help us find the value of the probability the stock market will crash this month.

(e) Karl Poppoer's propensity definition of probability can help us find the value of the probability the stock market will crash this month.

(f) Chevalier de Mere likely employed the long run frequency definition of probability to conjecture that the chance of getting one or more double six in 24 rolls of a pair of dice is less than a half.

(g) The problem with the long run frequency definition is that we cannot actually observe infinite experiments even if events were repeatable.

(h) According to Laplace, randomness is an illusion due to our own ignorance.

(i) The development of the physical theory of quantum mechanics seems to indicate the world is not deterministic (at least when talking about very small objects like electrons).

(j) The definition $\mathbb{P}(A) = |A|/|\Omega|$ is always valid.

(k) The definition $\mathbb{P}(A) = |A|/|\Omega|$ is an "objective" definition of probability.

(l) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ if A, B are disjoint is one of Kolmogorov's axiom properties of the $\mathbb{P}()$ function.

(m) $\mathbb{P}(A) = 1 - \mathbb{P}\left(A^C\right)$ is one of Kolmogorov's axiom properties of the $\mathbb{P}()$ function.

(n) Kolmogorov's definitions have been proven absolutely true.

(o) Kolmogorov's definitions solved the philosophical problem of defining probability in the physical world.

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 2 [12min] (and 22min will have elapsed) We flip a coin three times. The first flip yields either H_1 or T_1. The second flip yields either H_2 or T_2. The third flip yields either H_3 or T_3. Let Ω be the universe of outcomes for all three flips.

- [19 pt / 34 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

  (a) The total number of unique outcomes is 3

  (b) The total number of unique outcomes is 2×3

  (c) The total number of unique outcomes is 2^3

  (d) The total number of unique outcomes is 2^{2^3}

  (e) The total number of "events" are the total number of probability questions one can ask

  (f) The total number of probability questions one can ask is 3

  (g) The total number of probability questions one can ask is 2×3

  (h) The total number of probability questions one can ask is 2^3

  (i) The total number of probability questions one can ask is 2^{2^3}

  (j) $\mathbb{P}(H_3) = 1/2$

  (k) $\mathbb{P}(H_3) = 1/8$

  (l) $\mathbb{P}(H_1, H_2, H_3) = 1/8$

  (m) $\mathbb{P}(H_1, T_2, H_3) = 1/8$

  (n) $\mathbb{P}(\text{2 heads and 1 tail in that order}) = 1/8$

  (o) $\mathbb{P}(\text{2 heads and 1 tail in any order}) = 1/8$

  (p) $\mathbb{P}(\text{2 heads and 1 tail in any order}) = \binom{3}{2}/2^3$

  (q) $\mathbb{P}(\text{2 heads and 1 tail in any order}) = \binom{3}{1}/2^3$

  (r) $\mathbb{P}(\text{2 heads and 0 tail in any order}) = \binom{3}{2}\binom{3}{0}/2^3$

  (s) $\mathbb{P}(\text{1 head and 1 tail in any order in } \textit{only} \text{ two flips}) = 1/2$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 3 [12min] (and 34min will have elapsed) 10 people go to a party. Five are men: James, Robert, Michael, William, David (J, R, M, W, D) and five are women: Patricia, Linda, Elizabeth, Barbara, Susan (P, L, E, B, S).

- [13 pt / 47 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

  (a) If there are 10 chairs, there are 10 unique orderings in which the 10 people could be seated.

  (b) If there are 10 chairs, there are 10! unique orderings in which the 10 people could be seated.

  (c) If there are 10 chairs, there are ${}_{10}P_{10}$ unique orderings in which the 10 people could be seated.

  (d) If there are 10 chairs, there are $\binom{10}{10}$ unique orderings in which the 10 people could be seated.

  (e) If there are 6 chairs, there are 6 unique orderings in which the 10 people could be seated.

  (f) If there are 6 chairs, there are 6! unique orderings in which the 10 people could be seated.

  (g) If there are 6 chairs, there are ${}_{10}P_6$ unique orderings in which the 10 people could be seated.

  (h) If there are 6 chairs, there are $\binom{10}{6}$ unique orderings in which the 10 people could be seated.

  (i) If 4 people are chosen at random, the probability they are all men is 4/10.

  (j) If 4 people are chosen at random, the probability they are all men is ${}_{10}P_4$ / ${}_{10}P_{10}$.

  (k) If 4 people are chosen at random, the probability they are all men is 5 / $\binom{10}{4}$.

  (l) If 4 people are chosen at random, the probability they are all men is $\binom{5}{4}$ / $\binom{10}{4}$.

  (m) If 4 people are chosen at random, the probability they are all men is $\binom{5}{4}$ / $\binom{10}{6}$.

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 4 [9min] (and 43min will have elapsed) Let $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$. Let $a, b \in \mathbb{R}$.

- [10 pt / 57 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

  (a) $(a+b)^n = \binom{n}{0}a^0b^n + \binom{n}{1}a^1b^{n-1} + \binom{n}{2}a^2b^{n-2} + \ldots + \binom{n}{n}a^nb^0$

  (b) $(y+1)^5 = 1 + 5y + 10y^2 + 10y^3 + 5y^4 + y^5$

  (c) $(y+1)^5 = \sum_{i=1}^{5} \binom{5}{i} y^i$

  (d) When expanding $(a+b)^4$, there will be 4 terms after combining like terms

  (e) When expanding $(a+b)^4$, there will be 5 terms after combining like terms

  (f) When expanding $(a+b)^4$, there will be 16 terms after combining like terms

  (g) When expanding $(a+b)^4$, there will be a term $2a^2b^2$ after combining like terms

  (h) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = n!$

  (i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = n^2$

  (j) The generation of Pascal's triangle reveals a cominatorial identity

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 5 [13min] (and 56min will have elapsed) Let A and B be sets and Ω be the universe where $|\Omega| > 0$ and finite.

- [26 pt / 83 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

(a) $A \subseteq \Omega$

(b) $A \subset \Omega$

(c) $A = \{x : x \in \Omega\}$

(d) $|A| \leq |\Omega|$

(e) $2^A \in \Omega$

(f) $2^A \subseteq \Omega$

(g) $A \times A \subseteq \Omega$

(h) $A \cup B \subseteq \Omega$

(i) $A \cup \varnothing = \Omega$

(j) $A \cap \varnothing = \Omega$

(k) $A \cup B \subseteq \Omega$

(l) $\varnothing \neq \Omega$

(m) $\Omega \cup \varnothing = \Omega$

(n) $|A \cup B| = |A| + |B|$ if A and B are mutually exclusive

(o) $|A \cup B| = |A| + |B|$ if A and B are collectively exhaustive

(p) $A \cup B^C = \Omega$

(q) $B \cup B^C = \Omega$

(r) $A \cap B^C = \varnothing$

(s) $B \cap B^C = \varnothing$

(t) $A \backslash B = (A \backslash B) \backslash B$

(u) $A \backslash B = A \backslash (B \backslash B)$

(v) $\{2x : x \in \mathbb{Z}\} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$

(w) $\mathbb{N} \backslash \mathbb{Z} = \{\ldots, -2, -1, 0\}$

(x) $\mathbb{R} \backslash \mathbb{Z} = \varnothing$

(y) $\mathbb{R} \cap \mathbb{Z} = \mathbb{Z}$

(z) $\{2x : x \in \mathbb{R}\} = \mathbb{R}$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 6 [14min] (and 70min will have elapsed) In basketball there are three positions: guard, forward and center. Some players can play more than one position well. We are assuming for this problem that each player only can play one position. In the NY Knicks there are 19 players where 9 are guards, 3 are centers and 7 are forwards.

- [13 pt / 96 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

(a) The probability of a randomly sampled player on the NY Knicks being a center is $3/19$.

(b) The probability of two randomly sampled players on the NY Knicks being centers is $3/19 \times 2/18$.

(c) The probability of two randomly sampled players on the NY Knicks being centers is ${}_3P_2 / {}_{19}P_2$.

(d) The probability of two randomly sampled players on the NY Knicks being centers is $\binom{3}{2}/\binom{19}{2}$.

(e) The probability of sampling 7 players and all 7 being centers is 0.

(f) The probability of sampling 7 players and all 7 being forwards is 1.

(g) The probability of sampling 7 players and all 7 being forwards is 0.

(h) The number of ways of sampling 4 players and getting two pairs of types of positions players is $\binom{9}{2}\binom{3}{2} + \binom{9}{2}\binom{7}{2} + \binom{3}{2}\binom{7}{2}$ where order doesn't matter.

(i) The number of ways of sampling 4 players and getting two pairs of types of positions players is $\binom{3}{2}$ where order doesn't matter.

(j) The number of ways of sampling 4 players and getting two pairs of types of positions players is $\binom{4}{2}$ where order doesn't matter.

(k) The number of ways of choosing five players without replacement where order doesn't matter $\binom{19}{5}$.

Assume that during gameplay, only five players are on the court for each time: the 2-1-2 lineup which is 2 guards, 1 center and 2 forwards. These five players are called the "team" and the order of the players in the team does not matter.

(l) The probability of five randomly sampled players on the NY Knicks forming a team is $\binom{9}{2}\binom{3}{1}\binom{7}{2}/\binom{19}{5}$.

(m) The probability of five randomly sampled players being 2 guards, 1 center and 2 forwards *in that order* is $\binom{9}{2}\binom{3}{1}\binom{7}{2}/\binom{19}{5}$.

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.