Four cases from last class:
(a) X ~ Hyper(n = 2, K = 4, N = 10), N-K = 6, Supp[X] = {0, 1, 2}
(b) X ~ Hyper(n = 5, K = 4, N = 10), N-K = 6, Supp[X] = {0, 1, 2, 3, 4}
(c) X ~ Hyper(n = 8, K = 4, N = 10), N-K = 6, Supp[X] = {2, 3, 4}
(d) X ~ Hyper(n = 5, K = 7, N = 10), N-K = 3, Supp[X] = {2, 3, 4, 5}

(a) n < K and n < N - K => Supp[X] = {0, ..., n}
(b) n ≥ K and n < N - K => Supp[X] = {0, ..., K}
(c) n ≥ K and n ≥ N - K => Supp[X] = {n - (N - K), ..., K}
(d) n < K and n ≥ N - K => Supp[X] = {n - (N - K), ..., n}

Supp[X] = {max(0, n - (N - K)), ..., min(n, K)}

$\sum_{x \in Supp[X]} p(x) \overset{?}{=} 1$ True... difficult to prove... see HW

Hypergeometric has three params: n, K, N. We can "reparameterize" it using another set of three params which is 1:1 with the original params (i.e. an "equivalent parameterization"). Here is one such equivalent parameterization:

N = N
K = pN where p represents the proportion of the bag that is special ($p := \frac{K}{N}$)
n = n
{n, K, N} equivalent to {n, p, N}

X ~ Hyper(n, p, N) := $\frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$

$N \in \mathbb{N} \setminus \{1\}$
$n \in \{1, 2, \ldots, N-1\}$
$p \in \{\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\}$

X ~ Hyper(6, 0.5, 100)
P(X = 3) = $\frac{\binom{50}{3}\binom{50}{3}}{\binom{100}{6}} = 0.3223$

X ~ Hyper(6, 0.5, 1000)
P(X = 3) = $\frac{\binom{500}{3}\binom{500}{3}}{\binom{1000}{6}} = 0.3134$

X ~ Hyper(6, 0.5, 10000)
P(X = 3) = $\frac{\binom{5000}{3}\binom{5000}{3}}{\binom{10000}{3}} = 0.3126$

As N → ∞, is this probability converging to a specific value?
Is there a limiting rv model? Let's see...

X ~ Hyper(n, p, N) := p(x). What is the PMF as N → ∞?

$$\lim_{N\to\infty} \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}} = \lim_{N\to\infty} \frac{\frac{(pN)!}{x!(pN-x)!}\frac{((1-p)N)!}{(n-x)!((1-p)N-(n-x))!}}{\frac{N!}{n!(N-n)!}}$$

$$= \underbrace{\frac{\frac{1}{x!}\frac{1}{(n-x)!}}{\frac{1}{n!}}} \lim_{N\to\infty} \frac{\frac{(pN)!}{(pN-x)!}\frac{((1-p)N)!}{((1-p)N-(n-x))!}}{\frac{N!}{(N-n)!}}$$

e.g. $\frac{10!}{(10-4)!} = \frac{10!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7$

$\binom{n}{x} = \frac{n!}{x!(n-x)!}$

$$= \binom{n}{x} \lim_{N\to\infty} \frac{\overbrace{\overbrace{(pN)(pN-1)\cdot\ldots\cdot(pN-x+1)}^{x \text{ terms}}\overbrace{((1-p)N)((1-p)N-1)\cdot\ldots\cdot((1-p)N-(n-x)+1)}^{n-x \text{ terms}}}^{n \text{ terms}}}{\underbrace{(N)(N-1)(N-2)\cdot\ldots\ldots\cdot(N-n+1)}_{n \text{ terms}}}$$

$\lim_{x\to\infty} \frac{ax+b}{cx+d} = \frac{a}{c}$

$\lim f(x)g(x) = \lim f(x) \lim g(x)$

$$= \binom{n}{x} \underbrace{\underbrace{\lim \frac{pN}{N}}_{p} \underbrace{\lim \frac{pN-1}{N-1}}_{p} \cdot\ldots\cdot \underbrace{\lim \frac{pN-x+1}{N-x+1}}_{p}}_{x \text{ terms}} \underbrace{\underbrace{\lim \frac{(1-p)N}{N-x}}_{1-p} \underbrace{\lim \frac{(1-p)N-1}{N-x-1}}_{1-p} \cdot\ldots\cdot \underbrace{\lim \frac{(1-p)N-(n-x)+1}{N-n+1}}_{1-p}}_{n-x \text{ terms}}$$

$= \binom{n}{x} p^x (1-p)^{n-x}$ = Binomial(n, p)

As N → ∞, sampling without replacement becomes sampling with replacement.

Param space: n ∈ ℕ, p ∈ (0, 1)
Supp[X] = {0, 1, ..., n}

Bin(n, 0) := $\binom{n}{x} 0^x (1-0)^{n-x} = \begin{cases} 1 \cdot 0^0\, 1^n = 1 & \text{if } x = 0 \\ n \cdot 0^1\, 1^{n-1} = 0 & \text{if } x = 1 \\ \vdots & \vdots \\ 0 & \text{if } x = n \end{cases}$ = Deg(0)

$p = 0$
$p = 1$

Bin(n, 1) := . . . - - - - - - - - - = Deg(n)

X ~ Bin(1, p) := $\binom{1}{x} p^x (1-p)^{1-x} = p^x (1-p)^{1-x}$ = Bernoulli(p)

Supp[X] = {0, 1}, $\binom{1}{0} = 1$, $\binom{1}{1} = 1$

$1 \overset{?}{=} \sum_{x \in Supp[X]} p(x) = \sum_{x=0}^{n} \binom{n}{x} \underbrace{p^x}_{a} \underbrace{(1-p)^{n-x}}_{b} = (p + (1-p))^n = 1^n = 1$ ✓ by the binomial thm

X_1 and X_2 are independent ($X_1, X_2 \overset{ind}{\sim}$) if ...

$P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$ or

$P(X_2 = x_2 \mid X_1 = x_1) = P(X_2 = x_2)$ or

$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1) P(X_2 = x_2)$ joint mass function (JMF)

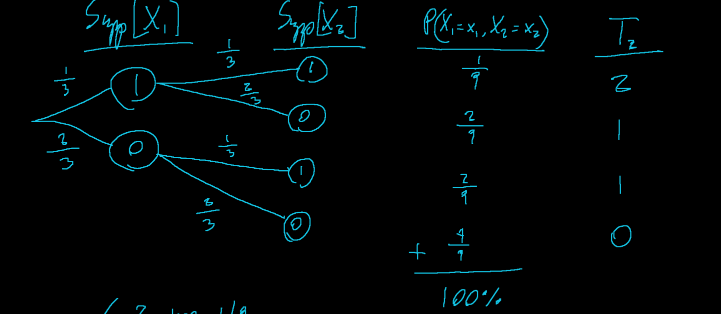
$\forall x_1 \in Supp[X_1]$ and $\forall x_2 \in Supp[X_2]$

If X_1, X_2 are independent and identically distributed rv's then they are independent and $X_1 \overset{d}{=} X_2$ i.e. their PMFs are the identical. We denote this $X_1, X_2 \overset{iid}{\sim}$

let $X_1, X_2 \overset{iid}{\sim} Bern(\frac{1}{3})$

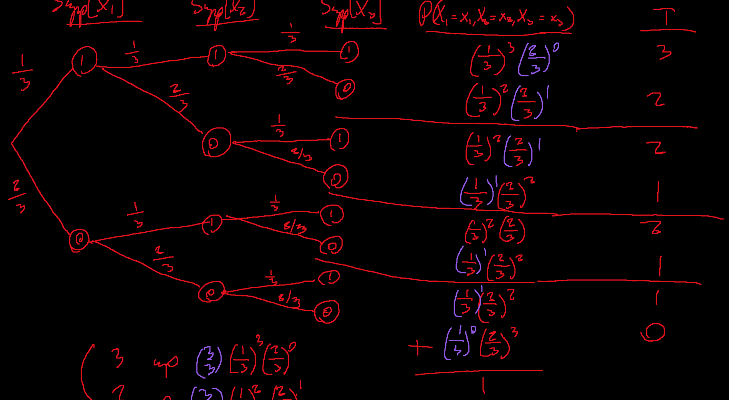
let $T_2 := X_1 + X_2 = g(X_1, X_2) \sim p(x) = ?$

I have a new rv defined as a function of other rv's. What is its PMF? Let's use a tree to figure it out:



$T_2 \sim \begin{cases} 2 & \text{wp } 1/9 \\ 1 & \text{wp } 4/9 = 2/9 + 2/9 \\ 0 & \text{wp } 4/9 \end{cases}$

$X_1, X_2, X_3 \overset{iid}{\sim} Bern(\frac{1}{3})$, $T_3 = X_1 + X_2 + X_3 = p(t) = ?$



$T_3 \sim \begin{cases} 3 & \text{wp } \binom{3}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 \\ 2 & \text{wp } \binom{3}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 \\ 1 & \text{wp } \binom{3}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 \\ 0 & \text{wp } \binom{3}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 \end{cases}$