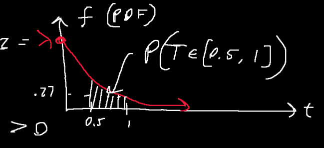
$F(t) = 1 - e^{-\lambda t} \Rightarrow f(t) = \lambda e^{-\lambda t}$

let $\lambda = 2$ $\quad f(1) = 0.27,\ p(1) = 0$

$f(0.1) = 2e^{-2(.1)} = 1.63 > 0$



PDF's are not probabilities. They are probability "densities"
i.e. how fast the CDF is changing. Its value doesn't matter, only
(1) its integrals matter
(2) its ratios matter

$$\frac{f(0.1)}{f(1)} = \frac{1.63}{0.27} \approx 6$$

This ratio means that realizations "near" 0.1 are 6x more likely than realizations near 1.

$$\lim_{\varepsilon \to 0} \frac{P(T \in [0.1, 0.1+\varepsilon])}{P(T \in [1, 1+\varepsilon])} = \frac{F(0.1+\varepsilon) - F(0.1)}{F(1+\varepsilon) - F(1)}$$

$$= \lim_{\varepsilon \to 0} \frac{\frac{F(0.1+\varepsilon) - F(.1)}{\varepsilon}}{\frac{F(1+\varepsilon) - F(1)}{\varepsilon}} = \frac{f(0.1)}{f(1)}$$

$$P(T \in (-\infty, \infty)) = 1 \Rightarrow \int_{-\infty}^{\infty} f(t)\,dt = 1$$

equiv of $\sum_{x \in Supp(X)} p(x) = 1$ for discrete rv

Properties of continuous rv X

① $|Supp[X]| = |\mathbb{R}|$

② $f(x) \geq 0$

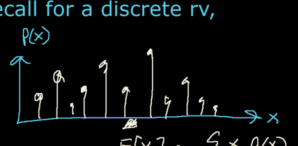
③ $\int_{\mathbb{R}} f(x)\,dx = 1$

④ $Supp[X] = \{x : f(x) > 0\}$

⑤ $Q[X, q] = F^{-1}(q) = \underset{x}{\mathrm{argmin}}\{F(x) \geq q\}$

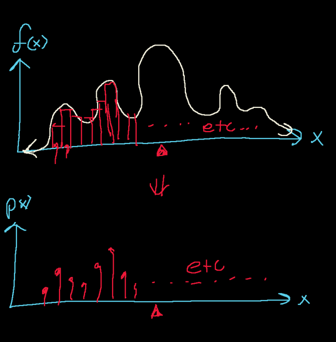
⑥ Definition of equals in distribution $X_1 \overset{d}{=} X_2 \Rightarrow f_{X_1}(x) = f_{X_2}(x)$ or $F_{X_1}(x) = F_{X_2}(x)$

How do we define expectation of a continuous rv, E[X]?
Recall for a discrete rv,



$$E[X] = \sum_{x \in Supp[X]} x\,p(x)$$

For a continous rv, there are no "probability masses", only density:



We can approximate any continous rv as a discrete rv by defining a "bin width" and taking integrals over the grid via a left/right/mid rectangle approx.

This is called a Riemann sum approximation. If the width of the rectangle goes to 0 in its limit, then the sum becomes the definition of the integral.

$$E[X] = \sum_{x \in Supp[X]} x\,p(x) \longrightarrow \int_{Supp[X]} x\,f(x)\,dx$$

Thus, the definition of expectation for a continuous rv is

$$\mu := E[X] := \int_{Supp[X]} x\,f(x)\,dx$$

You can prove then for any transformation g(X) that

$$E[g(X)] = \int_{Supp[X]} g(x)\,f(x)\,dx$$

Thus, we can define variance which is nothing but an expectation of a transformation of X, the squared error loss,

$$\sigma^2 := Var[X] := E[(X-\mu)^2] = \int_{Supp(X)} (x-\mu)^2 f(x)\,dx$$

We proved lots of rules for expectation and variance of discrete rv's. You can show all those rules are the same for continuous rv's. Here's a list:

$$E[aX + c] = a\,E[X] + c = a\mu + c$$

$$Var[aX + c] = a^2\,Var[X] = a^2\sigma^2$$

$$\Downarrow$$

$$SD[aX+c] = |a|\sqrt{Var[X]} = |a|\,\sigma$$

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = n\mu$$

if $X_1, \ldots, X_n$ equally distributed

$$Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i] = n\sigma^2$$

if $X_1, \ldots, X_n$ ind $\quad$ if $X_1, \ldots, X_n$ iid

When we "scrunched up" the geometric last class, we invented a new rv, the "exponential rv"

$$P(X > x) = e^{-\lambda x}$$

$$X \sim Exp(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{f(x)}, \quad \underbrace{F(x)}_{P(X \leq x)} = 1 - e^{-\lambda x}, \quad \lambda > 0$$

from its def. with n,p last class

$$Supp[X] = [0, \infty)$$

$$E[X] = \int_0^\infty x\,\lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty x e^{-\lambda x}\,dx$$

Recall integration by parts: $\int u\,dv = uv - \int v\,du$

let $u = x \Rightarrow du = dx$, $dv = e^{-\lambda x}dx \Rightarrow v = \int e^{-\lambda x}dx = -\frac{1}{\lambda}e^{-\lambda x}$

$$= \lambda\left(\left[x\left(-\frac{1}{\lambda}e^{-\lambda x}\right)\right]_0^\infty - \int_0^\infty -\frac{1}{\lambda}e^{-\lambda x}\,dx\right)$$

$$= \lambda\left(-\frac{1}{\lambda}\overbrace{\left[\frac{x}{e^{\lambda x}}\right]_0^\infty}^{0-0} - \frac{1}{\lambda^2}\overbrace{\left[e^{-\lambda x}\right]_0^\infty}^{0-1}\right) = \frac{1}{\lambda}$$

We said the geometric rv was "memoryless". Is the exponential rv memoryless?

$$P(X > a+b \mid X > b) = \frac{P(X > a+b \text{ and } X > b)}{P(X > b)} = \frac{P(X > a+b)}{P(X > b)}$$

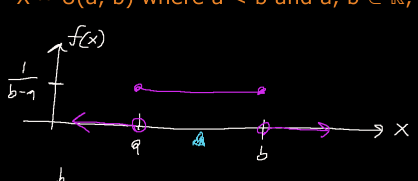
$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda b}} = \frac{e^{-\lambda a}\cancel{e^{-\lambda b}}}{\cancel{e^{-\lambda b}}}$$

$$= e^{-\lambda a} = P(X > a)$$

Recall the uniform discrete rv. e.g. X ~ U({1, 7, 19}) $= \begin{cases} 1 & wp\ \frac{1}{3} \\ 7 & wp\ \frac{1}{3} \\ 19 & wp\ \frac{1}{3} \end{cases}$

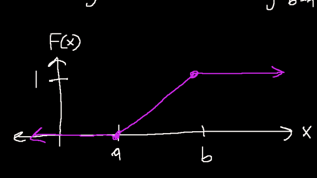
There is a uniform continuous rv called the "uniform rv"

X ~ U(a, b) where a < b and a, b ∈ ℝ, Supp[X] = [a, b]



$$1 = \int_a^b c\,dx = c[x]_a^b = c(b-a) \Rightarrow c = \frac{1}{b-a}$$

$$F(x) = \int f(x)\,dx + c = \int \frac{1}{b-a}\,dx + c = \frac{x}{b-a} + c, \quad c = ?$$



$F(a) = 0,\ F(b) = 1$ $\Rightarrow \frac{a}{b-a} + c = 0 \Rightarrow c = -\frac{a}{b-a}$

$\Rightarrow F(x) = \frac{x}{b-a} - \frac{a}{b-a}$

$= \frac{x-a}{b-a}$

$$E[X] = \int_a^b x\,\frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \boxed{\frac{a+b}{2}}$$

$$Var[X] = \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx$$

let's use a different formula...

$$= E[X^2] - \mu^2$$

$$= \int_a^b x^2 \frac{1}{b-a}\,dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{b-a}\int_a^b x^2\,dx - \frac{a^2+2ab+b^2}{4} = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{a^2+2ab+b^2}{4} = \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$\begin{array}{r} b^2 + ab + a^2 \\ b-a \,\overline{)\, b^3 - a^3} \\ -(b^3 - ab^2) \\ \hline ab^2 - a^3 \\ -(ab^2 - a^2b) \\ \hline a^2b - a^3 \\ a^2b - a^3 \\ \hline 0 \end{array}$$

$$= \frac{4b^2+4ab+4a^2}{12} - \frac{3a^2+6ab+3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \boxed{\frac{(b-a)^2}{12}}$$

$$\Rightarrow SD[X] = \frac{b-a}{\sqrt{12}}$$

Let's compute the median of X, Med[X] := Q[X, 0.5] = $F^{-1}(0.5)$

$F^{-1}(q) = ?$ $\quad F(x) = q \Rightarrow \frac{x-a}{b-a} = q \Rightarrow x - a = q(b-a)$

$\Rightarrow x = q(b-a) + a$

$= qb - qa + a$

$= qb + a - qa$

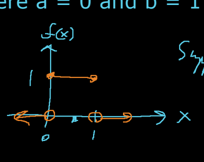
also known as the "quantile function" $\boxed{F^{-1}(q) = qb + (1-q)a}$

$$Med[X] = F^{-1}(0.5) = 0.5b + (1-0.5)a = \frac{a+b}{2} = E[X]$$

in this case

There is a "standard uniform" rv where a = 0 and b = 1

$$X \sim U(0,1) = \frac{1}{1-0} = 1 = f(x)$$



$Supp[X] = [0,1]$

$$F(x) = x$$

$$E[X] = \frac{1}{2} = Med[X]$$

This is one of the most widely-used rv's. It's used in every programming language e.g. Math.random() in Java realized a standard uniform rv.