MATH 241 Fall 2021 Homework #2

Professor Adam Kapelner

Due by email 11:59PM October 2, 2021

(this document last updated Wednesday 15$^{\text{th}}$ September, 2021 at 11:35am)

Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out". Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, read the section about sample spaces in Chapter 2 and relevant parts of Chapter 1 in Ross. Chapter references are from the 7th edition.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to [see syllabus] points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks not on this printout. Keep this first page printed for your records. Write your name below.

NAME: ______________________________________

Problem 1

Examine the following words and tell me how many *permutations* there are of the letters. We do not care about keeping track of the individual common letters. For example, in the word *dad*, there are two $d's$ and we want to treat the permutation d_1d_2a the *same* as d_2d_1a.

(a) [easy] town

(b) [easy] tsktsk (yes, this is a real word!)

(c) [harder] mississippi

(d) [difficult] supercalifragilisticexpialidocious

Problem 2

Below is a standard chessboard. Rows one and eight have the following pieces: two rooks, two knights, two bishops, a king and a queen. Rows two and seven have 8 pawns. Rows one and two have all black pieces and rows seven and eight have all white pieces.

(a) [easy] How many ways are there to place the black queen on a white square?

(b) [harder] How many ways are there to set up the pieces in the back ranks of both white and black *i.e.* arrange the two rooks, two knights, two bishops, king and queen on the first row of 8 squares. Note that this game is called "Fischer Random Chess" after the famous grandmaster Bobby Fischer who proposed the idea to make standard chess more exciting.

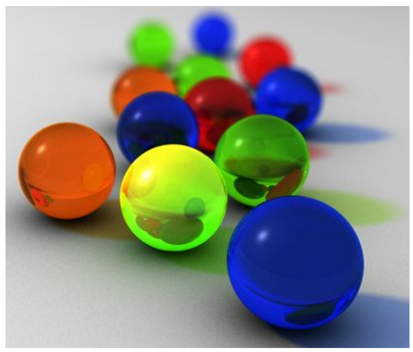
(c) [difficult] The game progresses and white takes two black pawns and black takes two white pawns. How many ways are there to arrange the pieces on the board? We don't care about pieces of a type being unique (*i.e.* all white pawns are the same, all black rooks are the same, etc).

(d) [difficult] Are all arrangements "equally likely" during an actual chess game? Explain why or why not.

More counting These counting questions will give you more practice in computing probabilities. Due to computations involving large factorials, we will also review Stirling's Approximation.

Problem 3

We have 4 blue marbles, 4 green marbles, 2 orange marbles, and 2 red marbles. For the following questions, if you are using "choose notation", please write your choose notation, then write the formula using factorials, then write the actual number after you compute it.



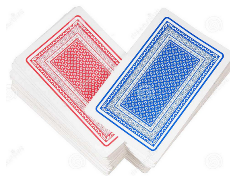
(a) [easy] Viewing all the marbles as *unique*, how many ways are there to order the marbles? Note that "order" is another way of saying "permute."

(b) [harder] Viewing all marbles of the same color as *interchangeable*, how many ways is there to order the marbles?

(c) [E.C.] If I pick 4 marbles at random from the collection, how many ways are there to get two-of-a-kind *i.e.* two marbles of one color and two marbles of a different color.

Problem 4

Imagine you have a bag of 10 cards where 6 are blue and 4 are red. A "draw" means one card is taken out of the bag at random and the color is revealed. If the problem asks "what is the probability," this means an explicit computation is required unless otherwise stated.



(a) [easy] What is the probability of getting a blue card when drawing one card?

(b) [easy] What is the probability of drawing 3 red cards in a row *without replacement*?

(c) [harder] Five cards are drawn. What is the probability of having 3 reds and 2 blues without regards to any order of the cards?

(d) [difficult] Five cards are drawn. What is the probability of having 3 reds and 2 blues in that order? Think carefully about the numerator and denominator in this probability computation.

Problem 5

Imagine you are putting together musical performances and you are employing musicians at random. There are many available for hire: 23 guitarists, 15 vocalists, 6 drummers, 14 bassists, 8 violinists, 9 violas, 6 cellists.

(a) [easy] If we hire 4 musicians at random, what is the probability we get a rock band (a vocalist, a guitarist, a bassist and a drummer)?

(b) [easy] If we hire 4 musicians at random, what is the probability we get a string quartet (two violinists, 1 cellist and one violist)?

(c) [easy] If we hire 4 musicians at random, what is the probability we get a doowop group (four vocalists)?

(d) [easy] We now move to a different city and the musicians for hire are different. Here, we have 10 guitarists, 10 vocalists, 10 drummers, 10 bassists. What is the probability we form a rock band when hiring four musicians at random?

(e) [difficult] Given the same situation in part (d), what is the probability we get two pairs of musicians (e.g. two guitarists and two bassists or two drummers and two bassists)?

(f) [difficult] Given the same situation in part (d), what is the probability we get all four musicians be the same type?

Problem 6

Combinations are not only useful in probability problems. They come up all over mathematics.

(a) [easy] Below is "Pascal's Triangle" up to $n = 4$.

$$
\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Explain why the 6 in the middle of the $n = 4$ row is equivalent to $\binom{4}{2}$ by using the fact proved in class.

(b) [easy] In the expansion $(a+b)^{100}$, how many terms are of the form a^2b^{98}?

(c) [E.C.] Prove the binomial theorem for arbitrary $n \in \mathbb{N}$.

Philosophy of Probability We covered many definitions of probability and discussed some philosophy of probability.

Problem 7

Answer the following questions by writing a paragraph or two *in English*.

(a) [easy] Previously we defined probability as $\mathbb{P}(A) := \frac{|A|}{|\Omega|}$. Describe a situtation where this fails to produce the correct probability that is not the spinner used in lecture.

(b) [easy] Which definition of probability does the book use and why do you think the authors chose this definition?

(c) [easy] Give an example of an event whose probability cannot be approximated by the limiting frequency.

(d) [E.C.] Who picks $\omega \in \Omega$ *i.e.* the outcome from the set of possible outcomes in the universe? This is an issue we ignored. Discuss your thoughts.

(e) [easy] What are some problems with the long run frequency definition of probability?

(f) [harder] How did Chevalier de Mere in 1654 know that the $\mathbb{P}\left(\text{one or more double sixes in 24 rolls of two dice}\right) < \frac{1}{2}$?

(g) [easy] What are some problems with the propensity definition of probability?

(h) [E.C.] What idea(s) inspired Karl Popper to invent the propensity definition?

(i) [easy] What is the main problem with the subjective definition of probability?

(j) [easy] According to Laplace (and his interpretation of Newton), if all information was known about physical systems including all laws and all initial conditions, would there be randomness? Yes/no and discuss.

(k) [difficult] According to Laplace, what is randomness? I've uploaded Laplace's quote in lecture 5 on the course homepage. You can answer this in a few words.

(l) [difficult] Knowing what we known in the 21st century, if all information was known about physical systems including all laws and all initial conditions, would there be randomness? If so, what theory has demonstrated evidence for this?

(m) [difficult] What is the prevailing historical theory about why probability wasn't formalized using mathematics prior to the 1600's?

Problem 8

We will get our feet wet with basic "axioms" and theorems. Assume all capital letters are sets. If the problem asks you to prove a fact, you may only use your knowledge of set theory and the definition of $\mathbb{P}(\cdot)$ given in the book / lecture. Most of the answers are in the book or in my lecture notes. Try to do them yourself and only use the book if you are having trouble. This section is strongly recommended for math majors but most of it is optional.

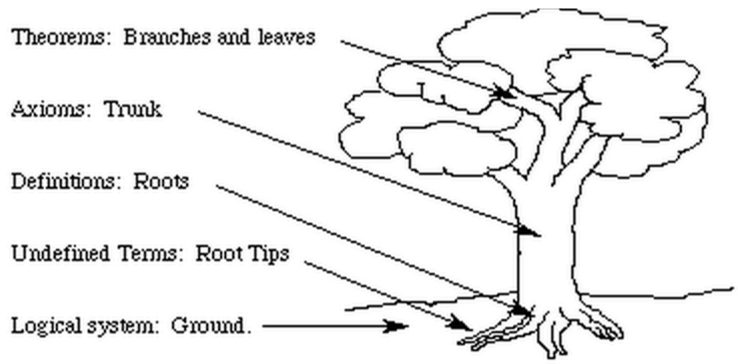


Figure 1: An axiomatic system is like a tree.

(a) [easy] List (1) all assumptions prior to and (2) the three conditions that make $\mathbb{P}(\cdot)$, the set function that returns a probability. These latter three conditions are also known as the "axioms of probability."

(b) [E.C.] [OPTIONAL] Explain why the three conditions are not technically axioms.

(c) [easy] [OPTIONAL] Prove that $\mathbb{P}(\varnothing) = 0$.

(d) [harder] [OPTIONAL] Prove that $\mathbb{P}(A) \leq 1$.

(e) [difficult] [OPTIONAL] Assuming the previous theorem that $\mathbb{P}(A) \leq 1$, prove that $\mathbb{P}(A) \in [0, 1]$.

(f) [difficult] [OPTIONAL] Prove that if $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

(g) [difficult] [OPTIONAL] Prove the law of inclusion-exclusion for two arbitrary sets: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A, B)$ (in the notes).

(h) [E.C.] [OPTIONAL] On a separate sheet of paper, prove the general law of inclusion-exclusion i.e. $\mathbb{P}\left(\cup_{i=1}^{n} A_i\right) =$

(i) [difficult] [OPTIONAL] Some authors write $\mathbb{P}(A) \in [0, 1]$ insetad of the condition hich we did in class, $\mathbb{P}(A) \geq 0 \;\forall A$. Why is this addition detail (of being probability being ≤ 1) unncessary? Hint: see (d).

(j) [difficult] [OPTIONAL] Prove that if $A = \{\omega_1, \omega_2, \ldots\}$ possibly infinite in cardinality, then

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(\{\omega_i\})$$

Hint: look at the proof of why $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$ in the case of equally likely outcomes.

Problem 9

Computations of combinations and permutations is impossible for a computer due to the factorial computations. In this problem, we investigate Stirling's formula. For CS majors, doing these problems is strongly encouraged. But they are optional.

(a) [easy] [OPTIONAL] Use the natural log to derive an expression for $\binom{1500}{300}$ using sums by recalling that $\ln(n!) = \sum_{i=1}^{n} \ln(i)$. Do not compute. Leave in terms of sums and natural logs.

(b) [easy] [OPTIONAL] Show that Stirling's approximation is equivalent to:

$$\ln(n!) \approx \frac{1}{2}\ln(2\pi) + \left(n + \frac{1}{2}\right)\ln(n) - n$$

(c) [difficult] [OPTIONAL] Use the expression in (c) to approximate the probability of getting 300 Heads in 1500 coin flips.