Lec 20 MATH 280/680

Joint / Multivariate ch.f.'s! Let $\vec{X}$ be an n-dim vector rv

Def: $\phi_{\vec{X}}(\vec{t}) := E\left[e^{i\vec{t}^T\vec{X}}\right]$

$= E\left[e^{i(t_1X_1 + \ldots + t_nX_n)}\right]$

$= E\left[e^{it_1X_1 + \ldots + it_nX_n}\right]$

$= E\left[e^{it_1X_1} \cdot \ldots \cdot e^{it_nX_n}\right]$

if $X_1, \ldots, X_n \overset{ind}{\sim}$ $\to$ $= E\left[e^{it_1X_1}\right] \cdot \ldots \cdot E\left[e^{it_nX_n}\right]$

Properties analogous to lecture 5

$= \phi_{X_1}(t_1) \cdot \ldots \cdot \phi_{X_n}(t_n) \neq \phi_{X_1}(t) \cdot \ldots \cdot \phi_{X_n}(t) = \phi_{X_1 + \ldots + X_n}(t)$ — only one-dim t!

$X_1, \ldots, X_n \overset{iid}{\sim}$ $= \phi_X(t_1) \ldots \phi_X(t_n)$

(P0) $\phi_{\vec{X}}(\vec{0}_n) = E\left[e^{i(0)X_1} \cdot \ldots \cdot e^{i(0)X_n}\right] = E[1] = 1$

(P1) $\phi_{\vec{X}}(\vec{t}) = \phi_{\vec{Y}}(\vec{t}) \iff \vec{X} \overset{d}{=} \vec{Y}$ due to multivariate Fourier transform theorem (unproven)

(P2) Let $A \in \mathbb{R}^{m \times n}$, $\vec{b} \in \mathbb{R}^m$, $\underset{m\times 1}{\vec{Y}} = \underset{m\times n}{A}\,\underset{n\times 1}{\vec{X}} + \underset{m\times 1}{\vec{b}}$ where $\dim(\vec{Y}) = m$

$\phi_{\vec{Y}}(\underset{\text{dim } m}{\vec{t}}) = E\left[e^{i\vec{t}^T(A\vec{X}+\vec{b})}\right] = E\left[e^{i\underbrace{\vec{t}^TA}_{(A^T\vec{t})^T}\vec{X}}e^{i\vec{t}^T\vec{b}}\right] = e^{i\vec{t}^T\vec{b}}\,E\left[e^{i(A^T\vec{t})^T\vec{X}}\right] = e^{i\vec{t}^T\vec{b}}\,\phi_{\vec{X}}(\underbrace{A^T\vec{t}}_{\text{dim } n})$

(P3) $\vec{T} = \vec{X}_1 + \ldots + \vec{X}_K$ where $\vec{X}_1, \ldots, \vec{X}_K \overset{ind}{\sim}$

$\phi_{\vec{T}}(\vec{t}) = E\left[e^{i\vec{t}^T(\vec{X}_1 + \ldots + \vec{X}_K)}\right] = E\left[e^{i\vec{t}^T\vec{X}_1} \cdot \ldots \cdot e^{i\vec{t}^T\vec{X}_K}\right] = \phi_{\vec{X}_1}(\vec{t}) \cdot \ldots \cdot \phi_{\vec{X}_K}(\vec{t}) = \left(\phi_{\vec{X}}(\vec{t})\right)^K$ — $\vec{X}_1, \ldots, \vec{X}_K \overset{iid}{\sim}$

(4) Moment Generation

$$h_{i_1}(\vec{t}) := \frac{\partial}{\partial t_i}\left[\phi_{\vec{X}}(\vec{t})\right] = E\left[\frac{\partial}{\partial t_i}\left[e^{itX_1}\cdots e^{itX_i}\cdots e^{itX_n}\right]\right]$$

$$= E\left[iX_i\left(e^{itX_1}\cdots e^{itX_n}\right)\right]$$

$$\Rightarrow h_{i_1}(\vec{0}_n) = E[iX_i] \Rightarrow E(X_i) = \frac{h_{i_1}(\vec{0}_n)}{i}$$

Here: $h_{i_\ell}(\vec{t}) := \frac{\partial^\ell}{\partial t_i^\ell}\left(\phi_{\vec{X}}(\vec{t})\right) \Rightarrow E[X_i^\ell] = \frac{h_{i_\ell}(\vec{0}_n)}{i^\ell}$

Further... Consider

$$h_{i_1,j_1} := \frac{\partial^2}{\partial t_i \partial t_j}\left[\phi_{\vec{X}}(\vec{t})\right] = E\left[\frac{\partial^2}{\partial t_i \partial t_j}\left[e^{itX_1}\cdots e^{itX_i}\cdots e^{itX_j}\cdots e^{itX_n}\right]\right]$$

$$= E\left[iX_i \frac{\partial}{\partial t_j}\left[e^{itX_1}\cdots e^{itX_j}\cdots e^{itX_n}\right]\right]$$

$$= E\left[i^2 X_i X_j \left(e^{itX_1}\cdots e^{itX_n}\right)\right]$$

$$\Rightarrow h_{i_1,j_1}(\vec{0}_n) = E[i^2 X_i X_j] \Rightarrow E(X_i X_j) = \frac{h_{i_1,j_1}(\vec{0}_n)}{i^2}$$

This allows you to get any moment product you wish! For example...

$$E\left[X_{17}^3 X_{37}^5 X_{41}^2\right] = \frac{h_{17_3,37_5,41_2}(\vec{0}_n)}{i^{3+5+2}}$$

where $h_{17_3,37_5,41_2}(\vec{t}) = \frac{\partial^{3+5+2}}{\partial t_{17}^3\, \partial t_{37}^5\, \partial_{41}^2}\left[\phi_{\vec{X}}(\vec{t})\right]$

(P5) Existence and Boundedness

HW.. $|\phi_{\vec{X}}(\vec{t})| \in [-1,1]$ due to $|e^{i\vec{t}^T\vec{X}}| = |i\sin(\vec{t}^T\vec{X}) + \cos(\vec{t}^T\vec{X})| = \sin^2(\vec{t}^T\vec{X}) + \cos^2(\vec{t}^T\vec{X}) = 1$

(P6) Inversion

$$\phi_{\vec{X}}(\vec{t}) \in L^1 \Rightarrow f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\vec{t}^T\vec{x}} \phi_{\vec{X}}(\vec{t})\, d\vec{t}$$ difficult!!

(P7) Continuity (beyond scope of course)

(P8) Levy's CDF formula (")

\# NEW (P9) Marginalization

Consider $$\phi_{\vec{X}}\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ t \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = E\left[e^{i(0)X_1} \cdots e^{i(0)X_{i-1}} e^{itX_i} e^{i(0)X_{i+1}} \cdots e^{i(0)X_n}\right] = E\left[e^{itX_i}\right] = \phi_{X_i}(t)$$

i^{th} position

the ch.f. of the i^{th} component!!

Further...

$$\phi_{\vec{X}}\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ t_i \\ 0 \\ 0 \\ t_j \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = E\left[e^{it_iX_i} e^{it_jX_j}\right] = \phi_{\begin{bmatrix} X_i \\ X_j \end{bmatrix}}\left(\begin{bmatrix} t_i \\ t_j \end{bmatrix}\right)$$

You can get any subset joint ch.f. you want...

$$\underset{\text{dim } k}{\vec{X}} \sim \text{Multinom}(n, \vec{p}) := \binom{n}{x_1, \dots, x_k} p_1^{x_1} \cdots p_k^{x_k}$$

$$\phi_{\vec{X}}(\vec{t}) = E\left[e^{i\vec{t}^T\vec{X}}\right] = \sum_{\vec{x} \in S_X} e^{it_1x_1} \cdots e^{it_kx_k} \binom{n}{x_1, \dots, x_k} p_1^{x_1} \cdots p_k^{x_k}$$

$$= \sum_{\vec{x} \in S_X} \binom{n}{x_1 \cdots x_k} \left(e^{it_1}p_1\right)^{x_1} \cdots \left(e^{it_k}p_k\right)^{x_k}$$

by multinom thm

$$= \left(p_1 e^{it_1} + \dots + p_k e^{it_k}\right)^n$$

Lec 8

$$\vec{X}_1, \dots, \vec{X}_k \overset{iid}{\sim} \text{Multinom}(1, \vec{p}) \Rightarrow \phi_{\vec{X}}(\vec{t}) = p_1 e^{it_1} + \dots + p_k e^{it_k}$$

$$\vec{T} = \vec{X}_1 + \dots + \vec{X}_k \Rightarrow \phi_T(\vec{t}) \overset{P3}{=} \left(\phi_{\vec{X}}(\vec{t})\right)^n = \left(p_1 e^{it_1} + \dots + p_k e^{it_k}\right)^n \overset{P1}{\Rightarrow} T \sim \text{Multinom}(n, \vec{p})$$

$\vec{X} \sim \text{Multinom}(n, \vec{p})$

$$Cov[X_i, X_j] := \underbrace{E[X_iX_j]}_{\text{difficult ... 3D mat!}} - \overbrace{E[X_i]}^{np_i} \overbrace{E[X_j]}^{np_j}$$

By (P4), $E[X_iX_j] = \dfrac{h_{i,j}(\vec{0})}{i^2}$

$$h_{i,j}(\vec{t}) = \frac{\partial^2}{\partial t_i \partial t_j}\left[\left(p_1 e^{it_1} + \dots + p_k e^{it_k}\right)^n\right]$$

$= ip_1 e$

$\vec{X} \sim \text{Multinom}(n, p)$, $X_i \sim ?$

By (P9)

$$\phi_{\vec{X}}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \left(p_1 e^{i(0)} + \dots + p_{i-1} e^{i(0)} + p_i e^{it} + p_{i+1} e^{i(0)} + \dots + p_k e^{i(0)}\right)^n$$

$$= \left(p_1 + \dots + p_{i-1} + p_{i+1} + \dots + p_k + p_i e^{it}\right)^n$$

$$= \left(1 - p_i + p_i e^{it}\right)^n$$

$$\overset{P1}{\Rightarrow} X_i \sim \text{Bin}(n, p_i)$$

$\vec{X} \sim \text{Multnom}(1, \vec{p})$

When we were deriving $Cov[X_i, X_j] := \overbrace{E[X_i X_j]}^{(\text{difficult}) = 0} - \overbrace{E[X_i]}^{p_i}\overbrace{E[X_j]}^{p_j}$

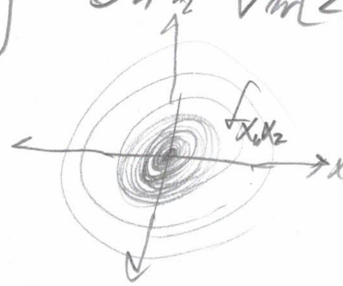
let's use cf now...

$$E[X_i X_j] = \frac{h_{i_1} h_{j_1}(\vec{0}_K)}{i^2}$$

$$h_{i_1, j_1}(\vec{t}) = \frac{\partial^2}{\partial t_i \partial t_j}\left[p_1 e^{it_1} + \ldots + p_K e^{it_K}\right] = \frac{\partial}{\partial t_j}\left[i p_i e^{it_i}\right] = 0 \Rightarrow E[X_i X_j] = 0 \checkmark$$

Let $Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$, $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$ n-dim vector rv

$$E[\vec{Z}] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}_n \qquad Var[\vec{Z}] = \begin{bmatrix} Var[Z_1] & Cov[Z_1,Z_2] & \cdots & \\ Cov[Z_1,Z_2] & Var[Z_2] & & \\ \vdots & & \ddots & \\ & & & Var[Z_n] \end{bmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} = I_n$$



$Z_1, \ldots Z_n$ iid

$$f_{\vec{Z}}(\vec{z}) = \prod_{i=1}^{n} f(z_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_i^2}{2}} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^{n} z_i^2} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\vec{z}^T\vec{z}} = N(\vec{0}_n, I_n)$$

Standard Multivariate normal (MVN) rv

$AA^{-1} = I \Rightarrow \det(AA^{-1}) = \det(I) = 1$

$\Rightarrow 1 = \det(A)\det(A^{-1}) \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

Consider $\vec{\mu} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ and full rank. Let $\underbrace{\vec{X} = \underset{n\times 1}{\vec{\mu}} + \underset{n\times n}{A}\underset{n\times 1}{\vec{Z}}}_{n\times 1}$

$f_{\vec{X}}(\vec{x}) = ?$ Let's use multivariate change of variables.

$$\vec{X} = g(\vec{Z}) = \vec{\mu} + A\vec{Z} \Rightarrow \vec{X} - \vec{\mu} = A\vec{Z} \Rightarrow \vec{Z} = h(\vec{X}) = \underbrace{A^{-1}}_{B}(\vec{X} - \vec{\mu})$$

$$h_1(\vec{x}) = \vec{b}_{1\cdot}(\vec{x} - \vec{\mu}) = \vec{b}_{1\cdot}\vec{x} - \vec{b}_{1\cdot}\vec{\mu}$$
$$h_2(\vec{x}) = \vec{b}_{2\cdot}(\vec{x} - \vec{\mu}) = \vec{b}_{2\cdot}\vec{x} - \vec{b}_{2\cdot}\vec{\mu}$$
$$\vdots$$
$$h_n(\vec{x}) = \vec{b}_{n\cdot}(\vec{x} - \vec{\mu}) = \vec{b}_{n\cdot}\vec{x} - \vec{b}_{n\cdot}\vec{\mu}$$

$$J_h = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial h_n}{\partial x_1} & \frac{\partial h_n}{\partial x_2} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = B$$

$$f_{\vec X}(\vec x) = f_{\vec Z}(h(\vec x))\,|J_h| = \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\left(A^{-1}(\vec x-\vec\mu)\right)^T\left(A^{-1}(\vec x-\vec\mu)\right)} \det(A^{-1})$$

$$= \frac{1}{(2\pi)^{n/2}\det[A]} e^{-\frac{1}{2}(\vec x-\vec\mu)^T (A^{-1})^T A^{-1}(\vec x-\vec\mu)}$$

let $\Sigma = AA^T \Rightarrow \Sigma^{-1} = (AA^T)^{-1} = (A^T)^{-1}A^{-1} = (A^{-1})^T A^{-1}$

$I = AA^{-1} \Rightarrow I^T = (AA^{-1})^T = (A^{-1})^T A^T = I \Rightarrow (A^T)^{-1} = (A^{-1})^T$

$\det[\Sigma] = \det[AA^T] = \det[A]\det[A^T] = \det[A]^2$

$\Rightarrow \det[A] = \sqrt{\det[\Sigma]}$

$$= \frac{1}{\sqrt{(2\pi)^n \det[\Sigma]}} e^{-\frac{1}{2}(\vec x-\vec\mu)^T\Sigma^{-1}(\vec x-\vec\mu)} = N_n(\vec\mu, \Sigma)$$

general MVN rv

Does Σ have any special meaning??

(If $\vec\mu = \vec 0_n$ and $\Sigma = I_n$ standard MVN)

$E[\vec X] = E[\vec\mu + A\vec Z] = \vec\mu + E[A\vec Z] = \vec\mu + A\vec 0_n = \vec\mu$

In general (not just for MVN). Let $\vec Y$ be a vector rv with dim n. Let $A \in \mathbb{R}^{m\times n}$.

$$E[A\vec Y] = E\begin{bmatrix}\vec a_1\cdot\vec Y\\ \vec a_2\cdot\vec Y\\ \vdots\\ \vec a_m\cdot\vec Y\end{bmatrix} = \begin{bmatrix}E[\vec a_1\cdot\vec Y]\\ E[\vec a_2\cdot\vec Y]\\ \vdots\\ E[\vec a_m\cdot\vec Y]\end{bmatrix} = \begin{bmatrix}\vec a_1\cdot\vec\mu\\ \vec a_2\cdot\vec\mu\\ \vdots\\ \vec a_m\cdot\vec\mu\end{bmatrix} = A\vec\mu = AE[\vec Y]$$

by same logic

$E[\vec Y^T A^T] = E[(A\vec Y)^T] = (E[A\vec Y])^T = (A\vec\mu)^T = \vec\mu^T A^T = E[\vec Y]^T A^T$

$Var[\vec X] = ?$

The above proof also works for matrix rv's Y.