Lec 21 MATH 340/640

Let's do a general proof. Let $\vec{Y}$ be a r.v.ec with dim n. Let $\vec{\mu} = E[\vec{Y}]$

Let $A \in \mathbb{R}^{m \times n}$ Recall $\Sigma := Var[\vec{Y}] = E[\vec{Y}\vec{Y}^T] - E[\vec{Y}]E[\vec{Y}]^T$

$E[A\vec{Y}] = A\vec{\mu}$, $E[\vec{Y}^T A^T] = \vec{\mu}^T A^T$ ← from previous class

$$\begin{aligned}
Var[A\vec{Y}] &= E[A\vec{Y}(A\vec{Y})^T] - E[A\vec{Y}]E[A\vec{Y}]^T \\
&= E[A\vec{Y}\vec{Y}^T A^T] - (A\vec{\mu})(A\vec{\mu})^T \\
&= A\,E[\vec{Y}\vec{Y}^T]A^T - A\vec{\mu}\vec{\mu}^T A^T \\
&= A\left(E[\vec{Y}\vec{Y}^T]A^T - \vec{\mu}\vec{\mu}^T A^T\right) \\
&= A\left(E[\vec{Y}\vec{Y}^T] - \vec{\mu}\vec{\mu}^T\right)A^T \\
&= A\Sigma A^T
\end{aligned}$$

Let $\vec{Z} \sim N_n(\vec{0}_n, I_n)$, $\vec{X} = \vec{\mu} + A\vec{Z}$

$\Rightarrow Var[\vec{X}] = Var[\vec{\mu} + A\vec{Z}] = Var[A\vec{Z}] = AIA^T = AA^T = \Sigma$

Σ is symmetric as $(AA^T)^T = A^{T^T}A^T = AA^T$

Let $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$

$D := (\vec{X} - \vec{\mu})^T \Sigma^{-1} (\vec{X} - \vec{\mu}) \sim ?$ AKA the "Mahalanobis Distance"

Remember $\Sigma^{-1} = (A^{-1})^T A^{-1}$ i.e. $\exists A$ such that this is true!

$\Rightarrow D = (\vec{X} - \vec{\mu})^T (A^{-1})^T A^{-1} (\vec{X} - \vec{\mu}) = \left((\vec{X} - \vec{\mu})A^{-1}\right)^T \left(A^{-1}(\vec{X} - \vec{\mu})\right) = \vec{Z}^T\vec{Z} \sim \chi^2_n$

Recall $\vec{Z} = h(\vec{X}) = (\vec{X} - \vec{\mu})A^{-1}$

Beyond scope of course:

Given Σ, how to find A?

$\Sigma = P^T D P$ diagonalization

$= P^T D^{\frac{1}{2}} D^{\frac{1}{2}} P$

let $A = P^T D^{\frac{1}{2}}$

$$\vec{Z} \sim N_n(\vec{0}_n, I_n) \iff Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1)$$

$$\phi_{\vec{Z}}(\vec{t}) = E\left[e^{i\vec{t}^T\vec{Z}}\right] = \phi_Z(t_1)\cdot\ldots\cdot\phi_Z(t_n) = \prod_{i=1}^{n} e^{-\frac{t_i^2}{2}} = e^{-\frac{1}{2}\sum t_i^2} = e^{-\frac{1}{2}\vec{t}^T\vec{t}}$$

For $A \in \mathbb{R}^{n\times n}$ invertible, $\vec{\mu} \in \mathbb{R}^n$

$$\vec{X} = A\vec{Z} + \vec{\mu} \sim N_n(\vec{\mu}, \Sigma) \quad \text{where } \Sigma = AA^T$$

$$\phi_{\vec{X}}(\vec{t}) \overset{P2}{=} e^{i\vec{t}^T\vec{\mu}}\,\phi_{\vec{Z}}(A^T\vec{t}) = e^{i\vec{t}^T\vec{\mu}}\, e^{-\frac{1}{2}(A^T\vec{t})^T(A^T\vec{t})} = e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^TAA^T\vec{t}}$$
$$= e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}}$$

For any $A \in \mathbb{R}^{m\times n}$, $\vec{\mu} \in \mathbb{R}^m$. Is it possible to define..

$$\vec{X} = A\vec{Z} + \vec{\mu}$$

Seemingly... $\phi_{\vec{X}}(\vec{t}) = e^{i\vec{t}^T\vec{\mu} - \frac{1}{2}\vec{t}^T\Sigma\vec{t}} \Rightarrow \vec{X} \sim N_m(\vec{\mu}, \Sigma)$

Can this go wrong?? $\quad = \dfrac{1}{\sqrt{(2\pi)^m \det[\Sigma]}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}$

Yes!! $\det(\Sigma)$ must be > 0 ! Otherwise density doesn't exist

$\phi_{\vec{X}}(\vec{t}) \notin L^1$

You must be able to invert. Σ must be

$\Rightarrow \Sigma = AA^T$ and full rank $\iff$ Σ is "positive definite"

$\Rightarrow \Sigma$ is symmetric

In 343 we prove

$$\text{rank}(\Sigma) = \text{rank}(A) \Rightarrow \Sigma \in \mathbb{R}^{m\times m} \Rightarrow m \le n$$

since rank of any matrix $\le$ # columns..

Hw: $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$ dim n

$\vec{Y} = B\vec{X} + \vec{c}$ where $B \in \mathbb{R}^{m\times n}$, $\vec{c} \in \mathbb{R}^m$ $\quad \vec{Y} \sim$? Conditions on $B, \vec{c}$?

$\vec{X} \sim N_n(\vec{\mu}, \Sigma)$, $X_i \sim ?$ Use (P9)...

$$\phi_{\vec{X}}\left(\begin{bmatrix}0\\ \vdots\\ 0\\ t\\ 0\\ \vdots\\ 0\end{bmatrix}\right) = e^{i[0 \cdots 0\ t\ 0 \cdots 0]\vec{\mu} - \frac{1}{2}[0 \cdots 0\ t\ 0 \cdots 0]\,\Sigma \begin{bmatrix}0\\ \vdots\\ 0\\ t\\ 0\\ \vdots\\ 0\end{bmatrix}}$$

$$= e^{it\mu_i - \frac{1}{2}[0 \cdots 0\ t\ 0 \cdots 0]\,\vec{\Sigma}_{\cdot i}}$$ ← the ith col of Σ

$$= e^{it\mu_i - \frac{1}{2}\Sigma_{ii}}$$

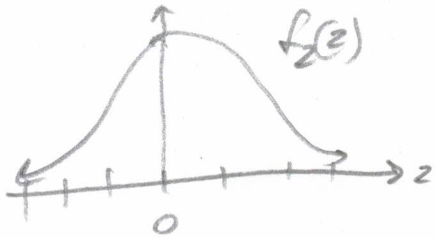
$$\overset{(P1)}{\Longrightarrow} X_i \sim N(\mu_i, \Sigma_{ii})$$

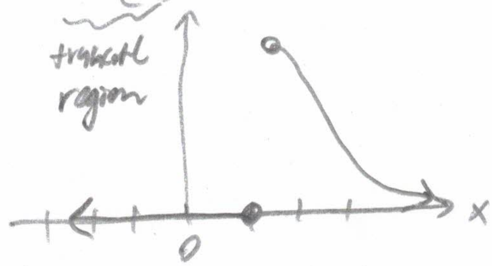
Hw: $\vec{X} \sim N_n(\vec{\mu}, \Sigma)$, $\begin{bmatrix}X_i\\ X_j\end{bmatrix} \sim ?$ Use (P9) then (P1)...

It will be MVN

Truncated distributions

Let $Z \sim N(0,1)$, $X = Z \mathbb{1}_{Z\in(1,\infty)} \sim f_X(x) = ?$ Draw a picture:





Same shape as $f_Z(z)$ in the truncated region so all we need to do is scale this truncated density so it integrates to 1.

$$\Rightarrow f_X(x) = \frac{f_Z(x)\,\mathbb{1}_{x\in(1,\infty)}}{\int_{(1,\infty)} f_Z(u)\,du} \approx \frac{1}{0.16}\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,\mathbb{1}_{x\in(1,\infty)}$$

In general, given density $Y \sim f_Y(y)$, $X = Y\mathbb{1}_{Y\in R} \sim \dfrac{f_Y(x)\,\mathbb{1}_{x\in R}}{\int_R f_Y(u)\,du}$

As promised, a discussion of the Weibull. What we've proved already

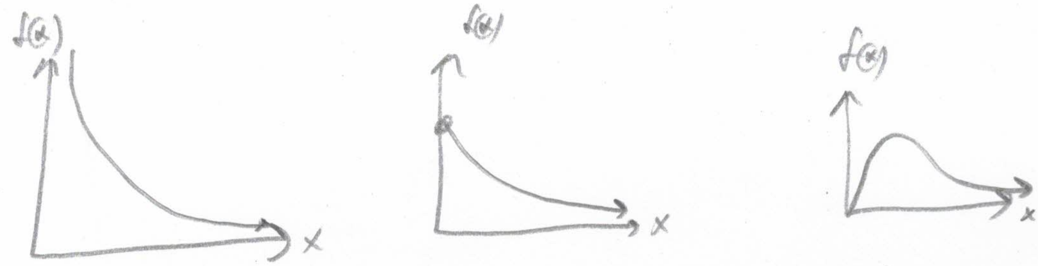
$X \sim \text{Weibull}(k,\lambda) = k\lambda^k x^{k-1} e^{-(\lambda x)^k} \mathbb{1}_{x \in (0,\infty)}$, $F(x) = 1 - e^{-(\lambda x)^k} \Rightarrow S(x) = P(X > x) = e^{-(\lambda x)^k}$

$\lambda > 0, k > 0$

if $k=1$ $\text{Weibull}(1,\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \in (0,\infty)} = \text{Exp}(\lambda)$

$\Rightarrow$ Weibull is a generalization of the exponential r.v.

Three types of Weibulls:



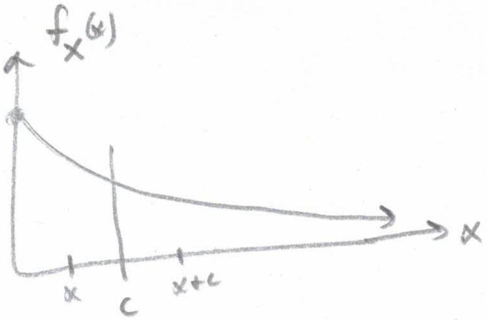
$K \in (0,1)$ $\quad$ $k=1$ (exponential) $\quad$ $K > 1$

Consider the conditional probability

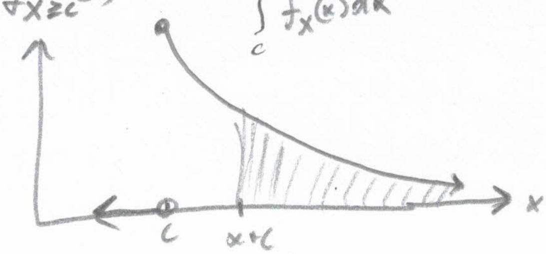
$P(X \geq x + c \mid X \geq c)$ where $c > 0$

A truncated distr

$$f_{X \geq c}(x) = \frac{f_X(x)\mathbb{1}_{x \geq c}}{\int_c^\infty f_X(x)\,dx}$$



$\Rightarrow$



$$P(X \geq x+c \mid X \geq c) = \frac{P(X \geq x+c \cap X \geq c)}{P(X \geq c)} = \frac{P(X \geq x+c)}{P(X \geq c)} = \frac{e^{-(\lambda(x+c))^k}}{e^{-(\lambda c)^k}}$$

$$= e^{-(\lambda(x+c))^k + (\lambda c)^k} = e^{\lambda^k(c^k - (x+c)^k)}$$

K is called the "Weibull modulus"

If $k=1$, $P(X \geq x+c \mid X>c) = e^{\lambda^1(c^1-(x+c)^1)} = e^{-\lambda x} = P(X>x) = e^{-\lambda x}$ $\forall x,c>0$

This is known as the "memorylessness" property of the exponential.

No matter how long you wait, the rv "resets" itself.

If $k>1$, $P(X \geq x+c \mid X>c) = e^{\lambda^k(c^k-(x+c)^k)} < P(X>x) = e^{-(\lambda x)^k}$ $\forall x,c>0$

Examples?
human lifespan

Proof let $r := \dfrac{e^{\lambda^k(c^k-(x+c)^k)}}{e^{-\lambda^k x^k}}$ WTS $r<1 \iff \ln(r)<0$

$r = e^{\lambda^k(c^k+x^k-(x+c)^k)} \Rightarrow \ln(r) = \underbrace{\lambda^k}_{>0}(c^k+x^k-(x+c)^k)$

WTS $c^k+x^k-(x+c)^k<0 \xRightarrow{\text{div by } x^k} \left(\frac{c}{x}\right)^k+1-\left(1+\frac{c}{x}\right)^k<0 \Rightarrow d^k+1-(1+d)^k<0$ let $d=\frac{c}{x}>0$

$d^k+1<(1+d)^k$: let $k=1+\beta \Rightarrow \beta>0$ since $k>1$

$\Rightarrow dd^\beta+1<(1+d)(1+d)^\beta=(1+d)^\beta+d(1+d)^\beta$

true since $1<(1+d)^\beta \xRightarrow{\ln} 0<\underbrace{\beta}_{>0}\underbrace{\ln(1+d)}_{>0}$

$\cancel{d}d^\beta < \cancel{d}(1+d)^\beta \Rightarrow \cancel{\beta}\ln(d) < \cancel{\beta}\ln(1+d)$ since $\ln$ is a strictly increasing function
div by pos #

If $k\in(0,1)$ $P(X \geq x+c \mid X>c) = e^{\lambda^k(c^k-(x+c)^k)} > P(X>x) = e^{-(\lambda x)^k}$ $\forall x,c>0$

Examples?
infant mortality,
age-hardening
alloys

WTS $r>1 \iff \ln(r)>0 \Rightarrow d^k+1>(1+d)^k$

let $k=1+\beta \Rightarrow \beta<0$

$\Rightarrow dd^\beta+1>(1+d)^\beta+d(1+d)^\beta$

true since $1>(1+d)^\beta \xRightarrow{\ln} 0>\underbrace{\beta}_{<0}\underbrace{\ln(1+d)}_{>0}$

$dd^\beta > d(1+d)^\beta \xRightarrow{\ln} \beta\ln(d) > \beta\ln(1+d) \Rightarrow \ln(d)<\ln(1+d)$
div by neg #