Math 340 / 640 Fall 2025
Midterm Examination Two Solutions

Professor Adam Kapelner

November 6, 2025

Full Name _____

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

_____   _____
signature   date

Instructions

This exam is 110 minutes (variable time per question) and closed-book. You are allowed **two** 8.5" × 11" page (front and back) "cheat sheets", blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. Show as much partial work as you can and justify each step. No food is allowed, only drinks.

Problem 1 Below are mostly unrelated problems.

(a) [6 pt / 6 pts]   Let $\mathcal{E} \sim \mathcal{N}(0, 1)$. Assume $\mathbb{E}[\mathcal{E}] = 0$ without proof. Show that $\mathcal{E}$ qualifies as an "error distribution".

The following is the density of the normal: $\mathcal{E} \sim f_{\mathcal{E}}(\epsilon) = \mathcal{N}(0, 1) = \frac{1}{\sqrt{2\pi}}e^{-\epsilon^2/2}$. Error distributions are defined by three conditions and these three are satisfied:

   (I) Mean center i.e. $\mathbb{E}[\mathcal{E}] = 0$. This is true as given.

   (II) Density is symmetric around zero i.e. $f_{\mathcal{E}}(\epsilon) = f_{\mathcal{E}}(-\epsilon)$. This is true as $\frac{1}{\sqrt{2\pi}}e^{-\epsilon^2/2} = \frac{1}{\sqrt{2\pi}}e^{-(-\epsilon)^2/2}$.

   (III) Density is strictly monotonic in both directions, i.e. $f_{\mathcal{E}}'(\epsilon) < 0$ for $\epsilon > 0$ and $f_{\mathcal{E}}'(\epsilon) > 0$ for $\epsilon < 0$. The derivative of the density is $f_{\mathcal{E}}'(\epsilon) = (-\epsilon)\frac{1}{\sqrt{2\pi}}e^{-\epsilon^2/2} = -\epsilon f_{\mathcal{E}}(\epsilon)$ which is negative for $\epsilon > 0$ and positive for $\epsilon < 0$.

(b) [10 pt / 16 pts]   Given the jdf of dependent rv's X_1 and X_2, $f_{X_1,X_2}(x_1, x_2)$, find an integral formula for the PDF of $M = X_1 X_2$.

   $\boxed{\text{Answer \#1}}$ Let $U = X_2$ thus $X_2 = h_2(M, U) = U$. Then we know $M = X_1 U$ thus $X_1 = h_1(M, U) = M/U$. We then have

$$\frac{\partial h_1}{\partial M} = \frac{1}{U}, \quad \frac{\partial h_1}{\partial U} = -M/U^2, \quad \frac{\partial h_2}{\partial M} = 0, \quad \frac{\partial h_2}{\partial U} = 1, \quad |J_h| = |(1/U)(1) - (-M/U^2)(0)| = 1/|U|$$

   Putting it all together, $f_M(m) = \int_{\mathbb{R}} f_{X_1,X_2}(h_1, h_2)|J_h|du = \int_{\mathbb{R}} f_{X_1,X_2}\left(\frac{m}{u}, u\right)\frac{1}{|u|}\,du$.

   $\boxed{\text{Answer \#2}}$ Let $U = X_1$ thus $X_1 = h_1(M, U) = U$. Then we know $M = UX_2$ thus $X_2 = h_2(M, U) = M/U$. We then have

$$\frac{\partial h_1}{\partial M} = 0, \quad \frac{\partial h_1}{\partial U} = 1, \quad \frac{\partial h_2}{\partial M} = 1/U, \quad \frac{\partial h_2}{\partial U} = -M/U^2, \quad |J_h| = |(0)(-M/U^2) - (1)(1/U)| = 1/|U|$$

   Putting it all together, $f_M(m) = \int_{\mathbb{R}} f_{X_1,X_2}(h_1, h_2)|J_h|du = \int_{\mathbb{R}} f_{X_1,X_2}\left(u, \frac{m}{u}\right)\frac{1}{|u|}\,du$.

(c) [7 pt / 23 pts]   Let $X \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\mathbb{1}_{x\in(0,\infty)}$. Prove $\mathbb{E}[X] = \frac{\alpha}{\beta}$ without using ch.f's.

$$\mathbb{E}[X] = \int_{\mathbb{R}} x\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\mathbb{1}_{x\in(0,\infty)}dx = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{(\alpha+1)-1}e^{-\beta x}dx = \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{\alpha\Gamma(\alpha)}{\beta\beta^\alpha} = \frac{\alpha}{\beta}$$

where the third equality follows from the gamma-like integral $\int_{\mathbb{R}} u^{k-1}e^{-cu}du = \frac{\Gamma(k)}{c^k}$.

(d) [5 pt / 28 pts]   Let $X \sim \text{Gamma}\,(17,\,37)$. Find an upper bound on the probability that $\mathbb{P}\,(X > 17)$. Hint: use the previous problem's result.

Since X is non-negative and by (c), $\mathbb{E}\,[X] = \frac{\alpha}{\beta} = \frac{17}{37}$, we can employ Markov's inequality:

$$\mathbb{P}\,(X > a) \leq \frac{\mathbb{E}\,[X]}{a} \quad \Rightarrow \quad \mathbb{P}\,(X > 17) \leq \frac{\frac{17}{37}}{17} \quad \Rightarrow \quad \mathbb{P}\,(X > 17) \leq \frac{1}{37}$$

(e) [6 pt / 34 pts]   Let $X \sim \chi_k^2$. Prove that $\mathbb{E}\,[X] = k$.

Remember that $X \sim \chi_k^2 = \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$ and by (c), $\mathbb{E}\,[X] = \frac{\alpha}{\beta} = \frac{k/2}{1/2} = k$.

(f) [6 pt / 40 pts]   Let $X \sim \text{Gamma}\,(\alpha,\,\beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \in (0,\infty)}$. Let $Y = \frac{1}{X}$. Find $f_Y(y)$. Simplify as much as you can. This is called the inverse gamma distribution.

$Y = g(X) = \frac{1}{X}$ implies $X = g^{-1}(Y) = \frac{1}{Y}$ and its absolute derivative is $\left| \frac{d}{dy}[g^{-1}(y)] \right| = y^{-2}$. Putting this all together we have:

$$
\begin{aligned}
f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right| &= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{y}\right)^{\alpha-1} e^{-\beta\left(\frac{1}{y}\right)} \mathbb{1}_{\left(\frac{1}{y}\right) \in (0,\infty)} y^{-2} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-\alpha+1} y^{-2} e^{-\frac{\beta}{y}} \mathbb{1}_{y \in (0,\infty)} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} e^{-\frac{\beta}{y}} \mathbb{1}_{y \in (0,\infty)}
\end{aligned}
$$

(g) [6 pt / 46 pts]   Let $X \sim T_k$. Let $Y = \mu + \sigma X$ where $\mu \in \mathbb{R}$ and $\sigma > 0$. Find $f_Y(y)$.

$$
\begin{aligned}
X \sim f_X(x) &= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi k}\,\Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{x^2}{k}\right)^{-\frac{k+1}{2}} \\
Y \sim f_Y(y) &= \frac{1}{|\sigma|} f_X\left(\frac{y - \mu}{\sigma}\right) \\
&= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi k \sigma^2}\,\Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{(y - \mu)^2}{k\sigma^2}\right)^{-\frac{k+1}{2}}
\end{aligned}
$$

3

(h) [5 pt / 51 pts]   If $\boldsymbol{X} \sim \text{Multinom}\left(17, \left[\frac{1}{2} \ \frac{1}{3} \ \frac{1}{6}\right]^\top\right)$, compute $\mathbb{Cov}[X_1, X_3]$.

$$\mathbb{Cov}[X_1, X_3] = -np_1 p_3 = -17 \cdot \frac{1}{2} \cdot \frac{1}{6} = -\frac{17}{12}$$

(i) [6 pt / 57 pts]   Let $X \sim \text{Binomial}(n, p)$. Let $Y = \ln(X+1)$. Find $p_Y(y)$.

$Y = g(X) = \ln(X+1)$ and thus $X = g^{-1}(Y) = e^Y - 1$ and thus

$$p_Y(y) = p_X(g^{-1}(y)) = p_X(e^y - 1) = \binom{n}{e^y - 1} p^{e^y - 1}(1-p)^{n - e^y + 1} \mathbb{1}_{e^y - 1 \in \{0, 1, \ldots, n\}}$$

$$= \binom{n}{e^y - 1} p^{e^y - 1}(1-p)^{n - e^y + 1} \mathbb{1}_{y \in \{\ln(1), \ln(2), \ldots, \ln(n+1)\}}$$

(j) [10 pt / 67 pts]   Prove the following and justify each step with theorems and results from class:

$$\frac{\bar{X} - \mu}{\frac{S_n}{\sqrt{n}}} \xrightarrow{d} \mathcal{N}(0, 1)$$

In class, we proved $S_n^2 \xrightarrow{p} \sigma^2$ and thus by the CMT we have $S_n \xrightarrow{p} \sigma$ where $g(x) = \sqrt{x}$. Again by the CMT we have

$$S_n \xrightarrow{p} \sigma \implies \frac{\frac{\sigma}{\sqrt{n}}}{\frac{S_n}{\sqrt{n}}} \xrightarrow{p} 1 \quad \text{where} \quad g(x) = \frac{\frac{\sigma}{\sqrt{n}}}{\frac{x}{\sqrt{n}}}$$

We also have the CLT result from class:

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Via the clever algebra trick from class,

$$\frac{\bar{X} - \mu}{\frac{S_n}{\sqrt{n}}} = \underbrace{\frac{\frac{\sigma}{\sqrt{n}}}{\frac{S_n}{\sqrt{n}}}}_{A_n} \underbrace{\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}}_{B_n} \xrightarrow{d} \mathcal{N}(0, 1)$$

where the $\xrightarrow{d} \mathcal{N}(0, 1)$ above follows from Slutsky's theorem Ⓐ as $A_n \xrightarrow{p} 1$ and $B_n \xrightarrow{d} \mathcal{N}(0, 1)$.

4

(k) [8 pt / 75 pts]   Let $X_n \sim \text{Weibull}(k, n)$ where $k > 0$. Prove $X_n \xrightarrow{p} 0$ from the definition of convergence in probability.

The $X_n \sim \text{Weibull}(k, n)$ rv is non-negative with CDF $F_{X_n}(x) = 1 - e^{-(nx)^k}$ and thus $\mathbb{P}\left(X_n > \epsilon\right) = 1 - F_{X_n}(x) = e^{-(nx)^k}$. By the definition, $X_n \xrightarrow{p} 0$ means:

$$\forall \epsilon > 0, \ \lim_{n \to \infty} \mathbb{P}\left(|X_n - 0| > \epsilon\right) = 0$$

We pick any positive ϵ and compute:

$$\lim_{n \to \infty} \mathbb{P}\left(|X_n - 0| > \epsilon\right) = \lim_{n \to \infty} \mathbb{P}\left(X_n > \epsilon\right) = \lim_{n \to \infty} e^{-(n\epsilon)^k} = \lim_{u \to \infty} \frac{1}{e^{cu}} = 0$$

where $u = n^k$, $c = \epsilon^k > 0$ and the first equality above follows from the non-negativity of X allowing us to drop the absolute value within the probability operator.

(l) [7 pt / 82 pts]   If $X \sim \text{ParetoI}(k = 0.17, \lambda = 2.37)$, then $\mathbb{E}\left[X\right] = 0.294$ and $\mathbb{V}\text{ar}\left[X\right] = 0.0986$. Let $X_1, X_2 \overset{iid}{\sim} \text{ParetoI}(k = 0.17, \lambda = 2.37)$, find an upper bound on $\mathbb{P}\left(X_1 + X_2 > 3.14\right)$ to the nearest three significant digits.

$\boxed{\text{Preferred Answer}}$ By Chebyshev's inequality corollary from class, if $b > 2\mu$ we have

$$\mathbb{P}\left(Y > b\right) \leq \frac{\sigma_Y^2}{(b - \mu_Y)^2}$$

Since $b = 3.14 > 2\mu = 2 \cdot 0.294 = 0.588$ we can employ this inequality and then substitute our numeric values to find:

$$\mathbb{P}\left(X_1 + X_2 > 3.14\right) \leq \frac{\mathbb{V}\text{ar}\left[X_1 + X_2\right]}{(3.14 - \mathbb{E}\left[X_1 + X_2\right])^2} = \frac{2\sigma^2}{(3.14 - 2\mu)^2} = \frac{2 \cdot 0.0986}{(3.14 - 2 \cdot 0.294)^2} = 0.0303$$

$\boxed{\text{Acceptable Answer}}$ By Markov's inequality and the fact that X_1, X_2 are both non-negative so their convolution is non-negative, we can employ this inequality and then substitute our numeric values to find:

$$\mathbb{P}\left(X_1 + X_2 > 3.14\right) \leq \frac{\mathbb{E}\left[X_1 + X_2\right]}{3.14} = \frac{2\mu}{3.14} = \frac{2 \cdot 0.294}{3.14} = 0.187$$

(m) [8 pt / 90 pts]   Let $Z_1, Z_2 \overset{iid}{\sim} \mathcal{N}\left(0, 1\right)$. Let $X = Z_1/Z_2$. What is the value of $\phi_X'(0)$?

The rv Z_1/Z_2 is Cauchy-distributed which means $\mathbb{E}\left[X\right]$ is undefined. Since $\phi_X'(0) = \frac{\mathbb{E}[X]}{i}$, the value of $\phi_X'(0)$ is undefined.

(n) [10 pt / 100 pts]   Let $Z_1, Z_2 \overset{iid}{\sim} \mathcal{N}(0, 1)$. Let $X = \frac{1}{2}Z_1^2 + Z_1 Z_2 + \frac{1}{2}Z_2^2$. How is X distributed?

Simplest way thanks to Bryan We can write $X = \frac{1}{2}(Z_1 + Z_2)^2$ Since $Z_1, Z_2 \overset{iid}{\sim} \mathcal{N}(0, 1)$, we know that $U := Z_1 + Z_2 \sim \mathcal{N}(0, 2)$. We can then write

$$X = \frac{1}{2}U^2 = \left(\frac{1}{\sqrt{2}}U\right)^2.$$

We know based on our transformation equations for the normal that $\frac{1}{\sqrt{2}}U \sim \mathcal{N}(0, 1)$ which means $X \sim \chi_1^2$.

Via Cochran's theorem Let $\boldsymbol{Z} = [Z_1\ Z_2]^\top$. We now want a quadratic form with vector $\boldsymbol{Z}$, so we need to conjure a determining matrix which we call B_1 and show this is an equivalent expression for our target rv X:

$$X = \boldsymbol{Z}^\top B_1 \boldsymbol{Z} = [Z_1\ Z_2]\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{B_1}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = [Z_1\ Z_2]\begin{bmatrix} aZ_1 + bZ_2 \\ cZ_1 + dZ_2 \end{bmatrix}$$

$$= aZ_1^2 + bZ_1 Z_2 + cZ_1 Z_2 + dZ_2^2$$
$$= aZ_1^2 + (b+c)Z_1 Z_2 + dZ_2^2$$

To match our target rv X: $a = \frac{1}{2}$, $(b+c) = 1$, and $d = \frac{1}{2}$. We add the condition that $b = c$ as B_1 must be symmetric, implying $b = c = 1/2$. Putting it together,

$$B_1 = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

This matrix B_1 has $\text{rank}[B_1] = 1$. To use Cochran's theorem, we conjure a matrix B_2 such that $B_1 + B_2 = I_2$:

$$B_2 = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

which has $\text{rank}[B_2] = 1$ which can be seen by taking the negative of column 2 and noticing it is the same as column 1.

Putting the facts above together, $B_1 + B_2 = I_2$ and $\text{rank}[B_1] + \text{rank}[B_2] = 2$. We now satisfied the conditions for Cochran's theorem and one of the three equivalent results. Thus, Cochran's Thm implies the additional result that the quadratic forms are chi-squared distributed. This result gives us our answer for the distribution of X:

$$X = \boldsymbol{Z}^\top B_1 \boldsymbol{Z} \sim \chi^2_{\text{rank}[B_1]} = \chi_1^2$$

Via Multivariate Transformation I do not think this is a possible route as the function $X = \frac{1}{2}(Z_1 - Z_2)^2$ is not invertible with a dummy dimension.