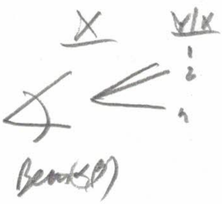
# Lec 19 MATH 340/640

Let $Y|X=x \sim \text{Bin}(n,x)$, $X \sim \text{Beta}(\alpha,\beta)$. Find $p(y)$. $Y$ is a "compound" discr distr.
$S_X = \Theta_p$

$$p(y) = \int_{\mathbb{R}} p_{Y|X}(x,y)\, f(x)\, dx = \int_{\mathbb{R}} \binom{n}{y} x^y (1-x)^{n-y} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x\in(0,1)}\, dx$$

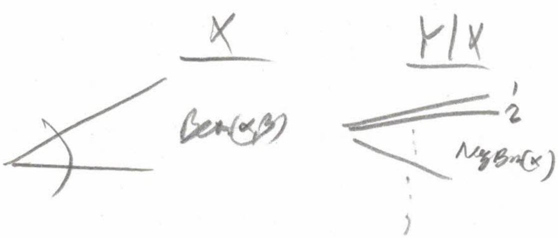



$$= \frac{\binom{n}{y}}{B(\alpha,\beta)} \int_0^1 x^{y+\alpha-1}(1-x)^{n-y+\beta-1}\, dx = \frac{\binom{n}{y} B(y+\alpha, n-y+\beta)}{B(\alpha,\beta)} = \text{BetaBinom}(n,\alpha,\beta)$$

take the place of $p$

Let $Y|X=x \sim \text{ExtNegBin}(k,x)$, $X \sim \text{Beta}(\alpha,\beta)$. $Y$ is a compound discr. distr. Find $p(y)$
$S_X = \Theta_p$

$$p(y) = \int_{\mathbb{R}} p_{Y|X}(x,y)\, f(x)\, dx = \int_{\mathbb{R}} \frac{\Gamma(k+y)}{\Gamma(k)\, y!} (1-x)^y x^k \mathbb{1}_{y\in\mathbb{N}_0} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x\in(0,1)}\, dx$$



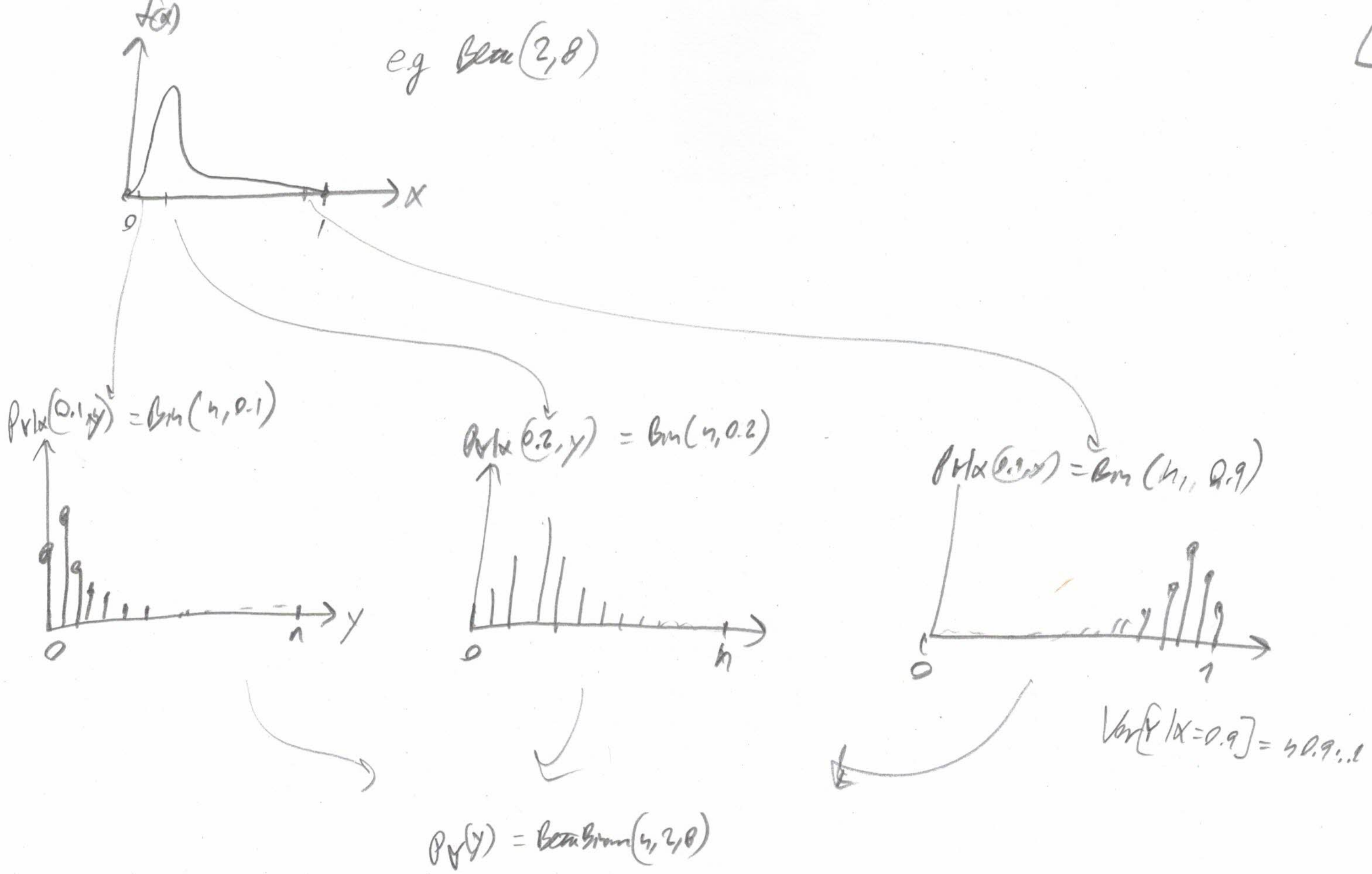

$$= \frac{\Gamma(k+y)}{\Gamma(k)\, y!\, B(\alpha,\beta)} \mathbb{1}_{y\in\mathbb{N}_0} \int_0^1 x^{k+\alpha-1}(1-x)^{y+\beta-1}\, dx = \frac{\Gamma(k+y)\, B(k+\alpha, y+\beta)}{\Gamma(k)\, y!\, B(\alpha,\beta)} \mathbb{1}_{y\in\mathbb{N}_0}$$

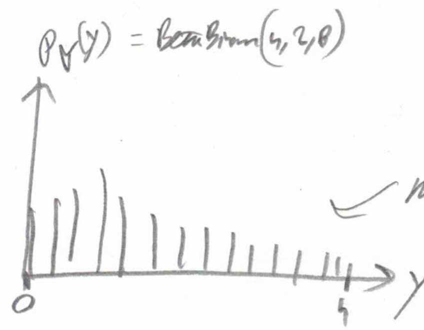
$$= \text{BetaNegativeBinomial}(k,\alpha,\beta)$$

take the place of $p$

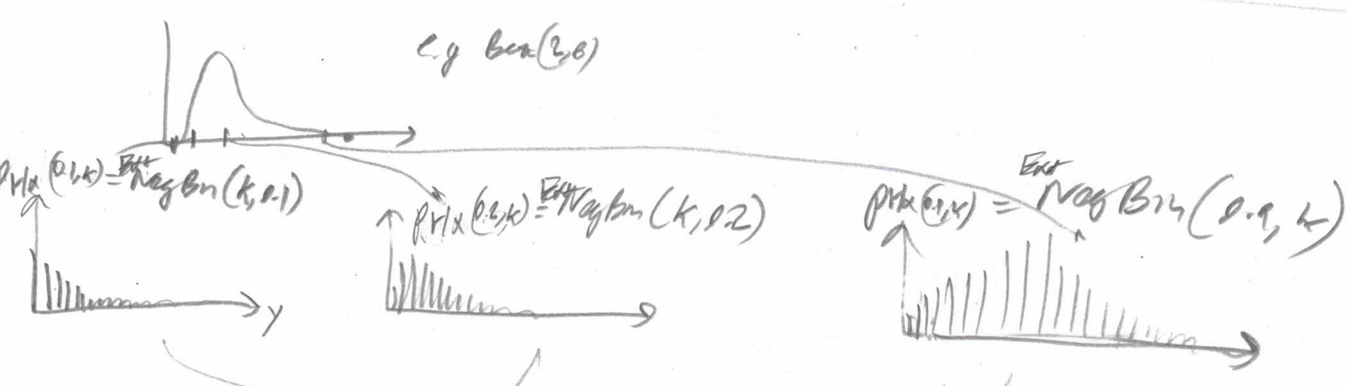
Let $Y|X=x \sim \text{Exp}(x)$, $X \sim \text{Gamma}(\alpha,\beta)$. $Y$ is a compound cont. distr.
$S_X = \Theta_\lambda$

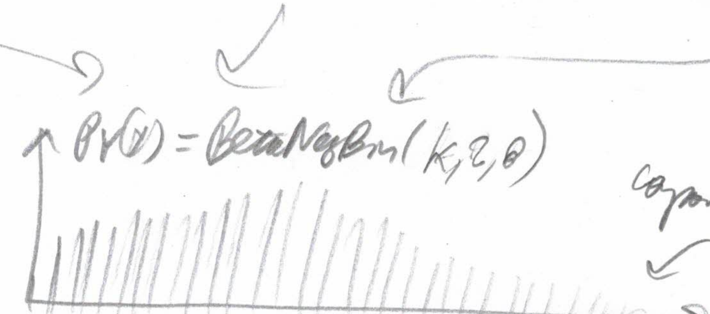
Then $Y \sim \text{Lomax}(\beta,\alpha)$, another heavy tail distr.




compound distr
more dispersed variance larger than
a Binomial rv




compound distr
more dispersed, variance
larger than Ext Neg Binomial

let's practice a very useful ~~kernel~~... let $a \in \mathbb{R}, b > 0$

$$X \sim N\left(\frac{a}{2b}, \frac{1}{2b}\right) = \frac{1}{\sqrt{2\pi\left(\frac{1}{2b}\right)}} e^{-\frac{1}{2\left(\frac{1}{2b}\right)}\left(x - \frac{a}{2b}\right)^2}$$

$$= \sqrt{\frac{b}{\pi}}\, e^{-b\left(x - \frac{a}{2b}\right)^2}$$

$$= \sqrt{\frac{b}{\pi}}\, e^{-b\left(x^2 - \frac{a}{b}x + \frac{a^2}{4b^2}\right)}$$

$$= \sqrt{\frac{b}{\pi}}\, e^{-bx^2 + ax - \frac{a^2}{4b}}$$

$$= \sqrt{\frac{b}{\pi}}\, e^{-\frac{a^2}{4b}} e^{ax - bx^2} \propto e^{ax-bx^2} \propto N\left(\frac{a}{2b}, \frac{1}{2b}\right)$$

What is

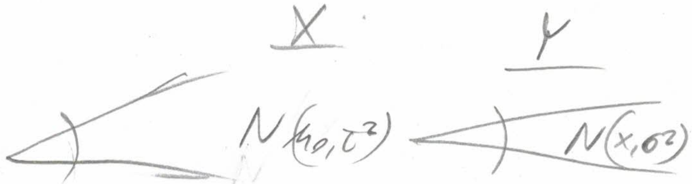
$$\int_{\mathbb{R}} e^{ax-bx^2}\,dx = \sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}} \int_{\mathbb{R}} \sqrt{\frac{b}{\pi}}\, e^{-\frac{a^2}{4b}} e^{ax-bx^2}\,dx = \sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}}$$

cool result!

A

let $Y|X=x \sim N(x,\sigma^2)$, $X \sim N(\mu_0,\tau^2)$ where $\mu_0 \in \mathbb{R}$, $\tau^2>0$, $\sigma^2>0$

Y is a conjugate cont. distr




$$f(y) = \int_{\mathbb{R}} f_{Y|X}(x,y)\, f_X(x)\, dx$$

$$= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-x)^2}\right)\left(\frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{1}{2\tau^2}(x-\mu_0)^2}\right) dx$$

$$\propto \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}(y-x)^2 - \frac{1}{2\tau^2}(x-\mu_0)^2} dx$$

$$= \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}y^2 + \frac{y}{\sigma^2}x - \frac{1}{2\sigma^2}x^2 - \frac{1}{2\tau^2}x^2 + \frac{\mu_0}{\tau^2}x - \frac{\mu_0^2}{2\tau^2}} dx$$

$$\propto e^{-\frac{1}{2\sigma^2}y^2} \int_{\mathbb{R}} e^{\underbrace{\left(\frac{y}{\sigma^2}+\frac{\mu_0}{\tau^2}\right)}_{a} x - \underbrace{\left(\frac{1}{2\sigma^2}+\frac{1}{2\tau^2}\right)}_{b} x^2} dx \qquad b = \frac{1}{2}\left(\frac{1}{\sigma^2}+\frac{1}{\tau^2}\right) = \frac{1}{2}\,\frac{\sigma^2+\tau^2}{\sigma^2\tau^2}$$

$$= e^{-\frac{1}{2\sigma^2}y^2} \sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}} \qquad \frac{a^2}{4b} = \frac{\frac{y^2}{\sigma^4} + \frac{2\mu_0 y}{\sigma^2\tau^2} + \frac{\mu_0^2}{\tau^4}}{\frac{2(\sigma^2+\tau^2)}{\sigma^2\tau^2}} = \frac{y^2\tau^2}{2\sigma^2(\sigma^2+\tau^2)} + \frac{\mu_0 y}{\sigma^2+\tau^2} + C$$

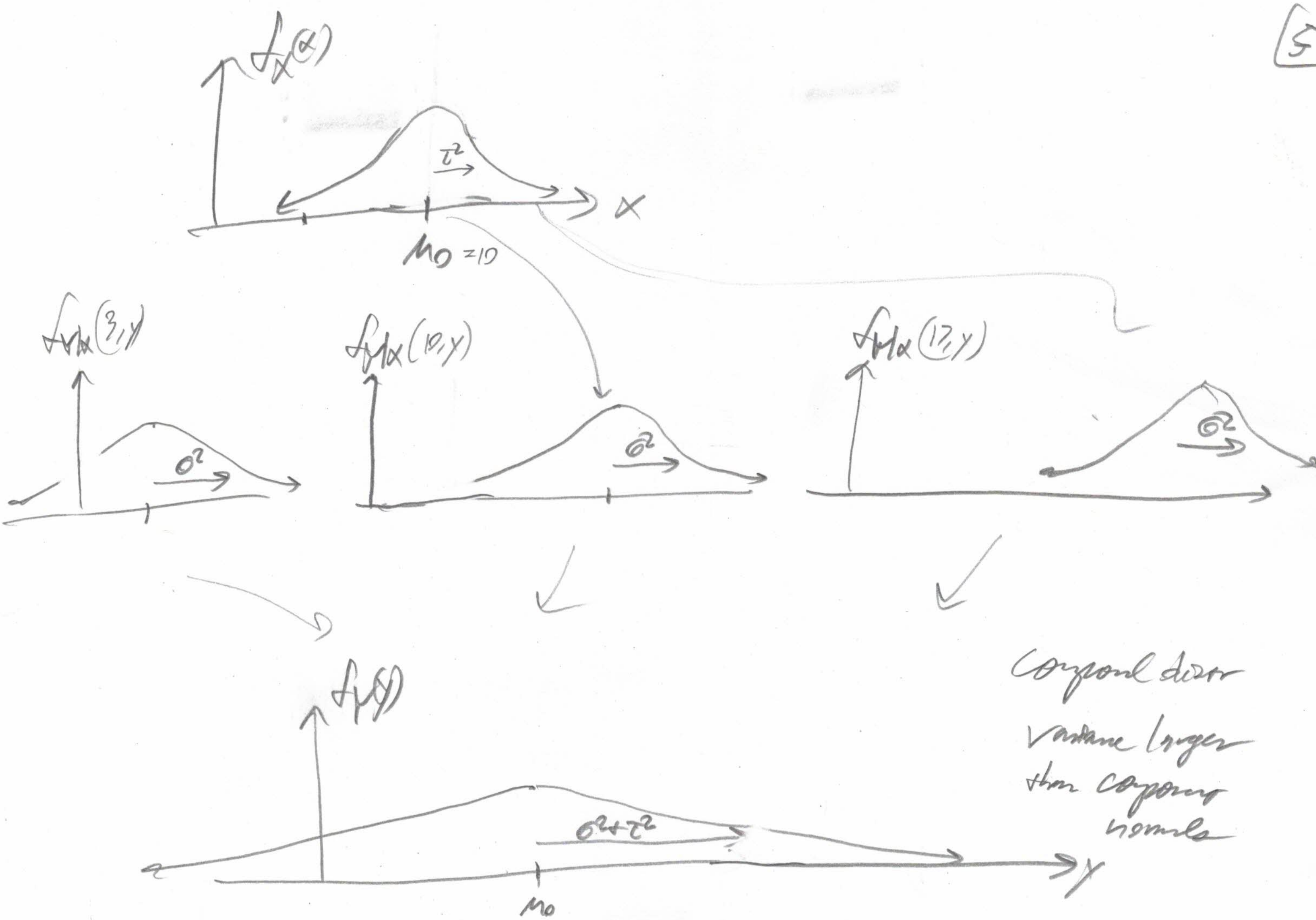
$$\propto e^{-\frac{1}{2\sigma^2}y^2} e^{\frac{a^2}{4b}}$$

$$\propto e^{-\frac{1}{2\sigma^2}y^2 + \frac{\tau^2}{2\sigma^2(\sigma^2+\tau^2)}y^2 + \frac{\mu_0}{\sigma^2+\tau^2}y}$$

$$= e^{\overbrace{\frac{\mu_0}{\sigma^2+\tau^2}}^{c} y - \overbrace{\left(\frac{-\tau^2}{2\sigma^2(\sigma^2+\tau^2)} + \frac{1}{2\sigma^2}\right)}^{d} y^2} \qquad 2d = \frac{-\tau^2}{\sigma^2(\sigma^2+\tau^2)} + \frac{1}{\sigma^2} = \frac{\sigma^2+\tau^2-\tau^2}{\sigma^2(\sigma^2+\tau^2)} = \frac{\sigma^2}{\sigma^2(\sigma^2+\tau^2)} = \frac{1}{\sigma^2+\tau^2}$$

$$\Rightarrow \frac{1}{2d} = \sigma^2+\tau^2$$

$$\propto N\left(\frac{c}{2d}, \frac{1}{2d}\right) = \boxed{N(\mu_0, \sigma^2+\tau^2)} \qquad \Rightarrow \frac{c}{2d} = \frac{\mu_0}{\sigma^2+\tau^2}(\sigma^2+\tau^2) = \mu_0$$




Compound distr
variance larger
than component
normals

Let's do the Poisson rv!

Thm if $\lim\limits_{n\to\infty} p_{X_n}(x) = p_X(x) \Longrightarrow X_n \xrightarrow{d} X$. Proof:

interchange sum/limits

$$\sum_{x\in(-\infty,y]} \lim_{n\to\infty} p_{X_n}(y) = \sum_{x\in(-\infty,y]} p_X(y) \Longrightarrow \lim_{n\to\infty} \sum_{x\in(-\infty,y]} p_{X_n}(y) = \sum_{x\in(-\infty,y]} p_X(y)$$

$\Rightarrow \lim\limits_{n\to\infty} F_{X_n}(x) = F_X(x)$, i.e. the def of convergence in dist.

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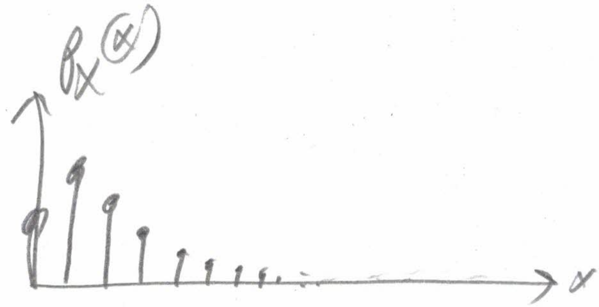
Let $X_n \sim Bin\left(n, \frac{\lambda}{n}\right)$ $= \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$ where $\lambda := np$. Consider $n$ large, $p$ small just like when we derived the exponential rv from the geometric. E.g. call center with tons of customers and prob they call is very small.

$$\lim_{n\to\infty} p_{X_n}(x) = \lim_{n\to\infty} \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x}{x!} \lim \frac{n!}{(n-x)!}\frac{1}{n^x} \underbrace{\lim\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \underbrace{\left(\lim\left(1-\frac{\lambda}{n}\right)\right)^{-x}}_{1} \mathbb{1}_{x\in\{0,\dots,n\}}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \lim \frac{\overbrace{n(n-1)(n-2)\cdots\cdots(1)}^{n \text{ terms}}}{\underbrace{n\cdot n\cdots n}_{x}\underbrace{(n-x)(n-x-1)\cdots 1}_{n-x \text{ terms}}} \lim \mathbb{1}_{x\in\{0,\dots,n\}} = \frac{\lambda^x e^{-\lambda}}{x!} \lim \frac{n}{n} \lim\frac{n-1}{n}\cdots \lim\frac{n-x+1}{n} \lim\frac{n-x}{n-x}\frac{n-x-1}{n-x-1}\cdots\frac{1}{1} \mathbb{1}_{x\in\mathbb{N}_0}$$

$$\Rightarrow \text{Poisson}(\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}\mathbb{1}_{x\in\mathbb{N}_0}$$

Param Space: $n\in\mathbb{N}$, $\lambda\in(0,\infty)$




Looks like Binomial except no max $\{S_x\}$

Let $Y|X=x \sim \text{Poisson}(x)$, $X \sim \text{Gamma}(\alpha,\beta)$

$$p_Y(y) = \int_{\mathbb{R}} p_{Y|X}(x,y)\, f_X(x)\, dx = \int_{\mathbb{R}} \frac{x^y e^{-x}}{y!} \mathbb{1}_{y\in\mathbb{N}_0} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x\in(0,\infty)}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y\in\mathbb{N}_0} \int_0^\infty x^{y+\alpha-1} e^{-(\beta+1)x}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{y!} \mathbb{1}_{y\in\mathbb{N}_0} \frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}}$$

$1-p = 1-\frac{\beta}{\beta+1} = \frac{\beta+1-\beta}{\beta+1} = \frac{1}{\beta+1}$

let $p=\frac{\beta}{\beta+1}$, let $k=\alpha$

$$= \frac{\Gamma(y+\alpha)}{\Gamma(\alpha)\, y!} \left(\frac{\beta}{\beta+1}\right)^\alpha \left(\frac{1}{\beta+1}\right)^y \mathbb{1}_{y\in\mathbb{N}_0} = \frac{\Gamma(y+k)}{\Gamma(k)\, y!} (1-p)^y p^k \mathbb{1}_{y\in\mathbb{N}_0} = \text{ExtNegBin}(k,p)$$