Lec 13 MATH 390/690

$Z \sim N(0,1)$, $Y = Z^2 \sim \chi^2_1 := \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \mathbb{1}_{y \in (0,\infty)}$

Investigate Hippy-Dippy...

$$1 = \int_0^\infty \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y}\, dy$$

by integral rule

$$\Rightarrow \sqrt{2\pi} = \int_0^\infty y^{\frac{1}{2}-1} e^{-\frac{1}{2}y}\, dy = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{2\pi}} \Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\tfrac{1}{2}}\sqrt{2\pi} = \sqrt{\pi}$$ nice value of gamma func

$$\Rightarrow Y \sim \chi^2_1 = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{1}{2}\right)^{\frac{1}{2}} y^{\frac{1}{2}-1} e^{-\frac{1}{2}y} \mathbb{1}_{y \in (0,\infty)} = \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$X \sim \text{Gamma}(\alpha, \beta)$ $\frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \in (0,\infty)}$

let $Z_1, \ldots, Z_K \overset{iid}{\sim} N(0,1)$

let $Y = Z_1^2 + \ldots + Z_K^2 \sim \chi^2_K$ "Chi-Squared with K degrees of freedom"

$f_Y(y) = ?$

$Z_1^2, \ldots, Z_K^2 \overset{iid}{\sim} \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$ sum of gammas?

Let's use ch.f's!

$X \sim \text{Gamma}(\alpha, \beta)$, $\phi_X(t) = E\left[e^{itX}\right] = \int_0^\infty e^{itx} \frac{1}{\Gamma(\alpha)} \beta^\alpha e^{-\beta x} x^{\alpha-1} \, dx$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta - it)x} \, dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta - it)^\alpha} = \left(\frac{\beta}{\beta - it}\right)^\alpha$$

$X_1, \ldots, X_n \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$

$T = X_1 + \ldots + X_n \sim f_T(t) = ?$

$\phi_T(t) = (\phi_X(t))^n = \left(\left(\frac{\beta}{\beta - it}\right)^\alpha\right)^n = \left(\frac{\beta}{\beta - it}\right)^{n\alpha} \Rightarrow T \sim \text{Gamma}(n\alpha, \beta)$

$\Rightarrow$ sums of gammas are gammas. This is totally expected!

$X_1 \ldots X_n \overset{iid}{\sim} \text{Erlang}(k, \lambda) \Rightarrow T = X_1 + \ldots + X_n \sim \text{Erlang}(nk, \lambda)$

since each Erlang is the sum of k $\text{Exp}(\lambda)$'s. So this is the grand sum of nk $\text{Exp}(\lambda)$'s!

$Z_1, \ldots, Z_k \overset{iid}{\sim} N(0,1)$

$Y = Z_1^2 + \ldots + Z_k^2 \sim \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right) = \chi^2_k = \frac{1}{\Gamma\left(\frac{k}{2}\right)} \left(\frac{1}{2}\right)^{\frac{k}{2}} x^{\frac{k}{2}-1} e^{-\frac{x}{2}} \mathbb{1}_{x \in (0,\infty)}$

$U \sim \chi^2_{k_1}$, $V \sim \chi^2_{k_2}$ $\quad U + V \sim \chi^2_{k_1 + k_2}$ just ~~adding~~ convolving two gammas

let $X \sim \chi^2_k$, $Y = \sqrt{X} \sim ?$ $\Rightarrow X = Y^2 = g^{-1}(Y)$ which is 1:1 on $\mathcal{X} = (0, \infty)$

$\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = 2|y|$

$$f_Y(y) = \frac{1}{\Gamma\left(\frac{k}{2}\right)} \left(\frac{1}{2}\right)^{\frac{k}{2}} (y^2)^{\frac{k}{2}-1} e^{-\frac{y^2}{2}} \underbrace{\mathbb{1}_{y^2 \in (0,\infty)}}_{y \in (0,\infty)} 2|y|$$

$$f_Y(y) = \frac{1}{\Gamma(\frac{k}{2})} \frac{1}{2^{k/2}} y^{k-2} e^{-\frac{y^2}{2}} 2y \, \mathbb{1}_{y \in (0,\infty)}$$

$$= \frac{1}{2^{\frac{k}{2}-1}} \frac{1}{\Gamma(\frac{k}{2})} y^{k-1} e^{-\frac{y^2}{2}} \mathbb{1}_{y\in(0,\infty)} = \chi_k$$

The chi distr. with k degrees of freedom

Not so many applications...

$\Leftrightarrow X_1 = \mu + \sigma Z_i \sim N(\mu, \sigma^2)$

$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2) \Rightarrow Z_1 := \frac{X_1 - \mu}{\sigma} \sim N(0,1),\ Z_2 := \frac{X_2 - \mu}{\sigma} \sim N(0,1)$

$Z_n := \frac{X_n - \mu}{\sigma} \sim N(0,1)$

$\Rightarrow Z_1, \ldots Z_n \overset{iid}{\sim} N(0, \sigma^2) \quad \Rightarrow \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \Rightarrow \frac{(\bar{X}-\mu)^2}{\frac{\sigma^2}{n}} = n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2 \sim \chi_1^2$

$\Rightarrow Z_1^2, \ldots, Z_n^2 \overset{iid}{\sim} \chi_1^2$

$\Rightarrow Z_1^2 + \ldots + Z_n^2 \sim \chi_n^2$, let $\vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \Rightarrow \vec{Z}^T \vec{Z} = Z_1^2 + \ldots + Z_n^2 \sim \chi_n^2$

Recall $Y = aX + b$, $X \sim N(\mu, \sigma^2) \Rightarrow f_Y(y) = \frac{1}{|a|} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\left(\frac{y-b}{a} - \mu\right)^2} = \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-\frac{1}{2a^2\sigma^2}(y - (b + a\mu))^2}$

$a > 0, b \in \mathbb{R}$ ch.f.'s work just as well for proving this $\Rightarrow Y \sim N(a\mu + b, a^2\sigma^2)$

$\vec{Z}^T \vec{1}_n = \vec{1}_n^T \vec{Z} = Z_1 + \ldots Z_n \sim N(0, \sqrt{n}^2)$, $\frac{1}{\sqrt{n}}(Z_1 + \ldots + Z_n) \sim N(0,1)$

$\Rightarrow \bar{Z} = \frac{1}{n}(Z_1 + \ldots + Z_n) \Rightarrow \bar{Z} = \frac{1}{\sqrt{n}}\left(\frac{Z_1 + \ldots + Z_n}{\sqrt{n}}\right) \Rightarrow \sqrt{n}\bar{Z} = \frac{Z_1 + \ldots + Z_n}{\sqrt{n}} \Rightarrow \sqrt{n}\bar{Z} \sim N(0,1)$

$\Rightarrow n\bar{Z}^2 \sim \chi_1^2$

$$\bar{Z} = \frac{\frac{X_1 - \mu}{\sigma} + \ldots + \frac{X_n - \mu}{\sigma}}{n} = \frac{\frac{\sum X_i - n\mu}{\sigma}}{n} = \frac{\bar{X} - \mu}{\sigma} \Rightarrow \bar{Z}^2 = \left(\frac{\bar{X}-\mu}{\sigma}\right)^2 \Rightarrow n\frac{(\bar{X}-\mu)^2}{\sigma^2} \sim \chi_1^2$$

$$\sum(Z_i^2 - \bar{Z} + \bar{Z})^2 = \sum\left((Z_i^2 - \bar{Z}) + \bar{Z}\right)^2 = \sum(Z_i^2 - \bar{Z})^2 + 2\sum(Z_i - \bar{Z})\bar{Z} + \sum \bar{Z}^2$$

$$= \sum(Z_i - \bar{Z})^2 + 2\sum(Z_i\bar{Z} - \bar{Z}^2) + \sum\bar{Z}^2$$

$$= \sum(Z_i - \bar{Z})^2 + 2\underbrace{\cancel{(n\bar{Z}^2 - n\bar{Z}^2)}}_{0} + n\bar{Z}^2$$

$$= \sum\left(\frac{X_i-\mu}{\sigma} - \frac{\bar{X}-\mu}{\sigma}\right)^2 + n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2$$

$$= \sum \frac{(X_i-\bar{X})^2}{\sigma^2} + n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2$$

$$= \frac{1}{\sigma^2}\sum(X_i-\bar{X})^2 + n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2$$

$$= \frac{n-1}{\sigma^2}\frac{1}{n-1}\sum(X_i-\bar{X})^2 + n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2$$

$$= \frac{n-1}{\sigma^2}S_n^2 + \underbrace{n\left(\frac{\bar{X}-\mu}{\sigma}\right)^2}_{\sim \chi^2_1} \sim \chi^2_n \qquad U \sim \chi^2_{n-1},\ V \sim \chi^2_1 \Rightarrow U+V \sim \chi^2_n$$

Conjecture: $\frac{n-1}{\sigma^2}S^2 \sim \chi^2_{n-1}$ and S^2 independent of $\bar{X}$.

This is true and published by Cochran in 1934 and the result he proved is called "Cochran's Thm". Involves more linear algebra than you have so far in MATH 231. So I won't prove it. Wikipedia has a good proof. Statement of thm:

If $Z_1, Z_2, \ldots, Z_n \overset{iid}{\sim} N(0,1)$, so $\vec{Z} := \begin{bmatrix} Z_1 \\ Z_n \end{bmatrix}$, $B_1, B_2, \ldots, B_k$ are symmetric $n \times n$ matrices where $B_1 + B_2 + \ldots + B_k = I_n$ then the following 3 statements are equivalent:

$\text{rank}[B_1] + \ldots + \text{rank}[B_k] = n \iff$ all $\vec{Z}^\top B_i \vec{Z}$'s are independent $\iff \vec{Z}^\top B_i \vec{Z} \sim \chi^2_{\text{rank}[B_i]}$

Let's see some examples... let $n=3$, $Z_1, Z_2, Z_3 \overset{iid}{\sim} N(0,1)$

$$B_1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad B_2=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix} \Rightarrow B_1+B_2=I_3, \text{ both } B_1, B_2 \text{ symmetric}$$

$\text{rank}[B_1]=1$, $\text{rank}[B_2]=2$ $\Rightarrow n=\text{rank}(B_1)+\text{rank}(B_2)$

$\Rightarrow$ we can use Cochran's thm!

$\Rightarrow \vec{Z}^T B_1 \vec{Z} \sim \chi^2_1$ let's see

$$[Z_1\ Z_2\ Z_3]\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}Z_1\\Z_2\\Z_3\end{bmatrix}=[Z_1\ Z_2\ Z_3]\begin{bmatrix}Z_1\\0\\0\end{bmatrix}=Z_1^2\sim\chi^2_1$$

$\Rightarrow \vec{Z}^T B_2 \vec{Z} \sim \chi^2_2$ let's see

$$[Z_1\ Z_2\ Z_3]\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}Z_1\\Z_2\\Z_3\end{bmatrix}=[Z_1\ Z_2\ Z_3]\begin{bmatrix}0\\Z_2\\Z_3\end{bmatrix}=Z_2^2+Z_3^2\sim\chi^2_2$$

$\Rightarrow \vec{Z}^T B_1 \vec{Z}$ is independent of $\vec{Z}^T B_2 \vec{Z}$. Let's see

Z_1^2 is definitely independent of $Z_2^2+Z_3^2$

Why do we care??

Recall
$$\vec{Z}^T\vec{Z}=\sum(Z_i-\bar{Z}+\bar{Z})^2=\underbrace{\sum(Z_i-\bar{Z})^2}+\underbrace{n\bar{Z}^2}$$
$$=\frac{n-1}{\sigma^2}S_n^2+n\frac{(\bar{X}-\mu)^2}{\sigma^2}$$

If we can show $\sum(Z_i-\bar{Z})^2=\vec{Z}^T B_1\vec{Z}$ and $n\bar{Z}^2=\vec{Z}^T B_2\vec{Z}$ where B_1, B_2 symmetric, $I=B_1+B_2$, and $\text{rank}(B_1)+\text{rank}(B_2)=n$, then we know $\frac{n-1}{\sigma^2}S_n^2\sim\chi^2_{\text{rank}(B_1)}$ and S_n^2 independent of $\bar{X}$.

Let's do it. First, B_2...

let $J_n := \begin{bmatrix}1&\cdots&1\\\vdots&&\vdots\\1&\cdots&1\end{bmatrix}$ then allows

$$n\bar{Z}^2=n\left(\frac{Z_1+\dots+Z_n}{n}\right)\left(\frac{Z_1+\dots+Z_n}{n}\right)=\frac{1}{n}(\vec{Z}^T\vec{1}_n)(\vec{1}_n^T\vec{Z})=\vec{Z}^T\left(\frac{1}{n}\vec{1}_n\vec{1}_n^T\right)\vec{Z}=\vec{Z}^T\underbrace{\left(\tfrac{1}{n}J_n\right)}_{B_2}\vec{Z}$$

$$B_2=\begin{bmatrix}\frac{1}{n}&\frac{1}{n}&\cdots&\frac{1}{n}\\\vdots&&&\vdots\\\frac{1}{n}&\cdots&&\frac{1}{n}\end{bmatrix}$$

symmetric? Yes!

$\text{rank}(B_2)=1$ only 1 linearly dep. column!

Let's do B_1 now...

$$\sum(Z_i-\bar{Z})^2 = \sum Z_i^2 - 2Z_i\bar{Z} + \bar{Z}^2 = \sum Z_i^2 - 2n\bar{Z}^2 + \bar{Z}^2 = \sum Z_i^2 - n\bar{Z}^2$$

$$= \vec{Z}^T\vec{Z} - \vec{Z}^T\left(\tfrac{1}{n}J_n\right)\vec{Z}$$

$$= \vec{Z}^T I_n \vec{Z} - \vec{Z}^T\left(\tfrac{1}{n}J_n\right)\vec{Z}$$

$$= \vec{Z}^T \underbrace{\left(I_n - \tfrac{1}{n}J_n\right)}_{B_1 = I_n - B_2}\vec{Z}$$

$$B_1 = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \\ -\frac{1}{n} & \cdots & & 1-\frac{1}{n} \end{bmatrix}$$

Obviously $B_1 + B_2 = I_n - B_2 + B_2 = I$

How do we prove the rank $[B_1] = n-1$? Not so clear from just looking at it..

Let's first note the following:

$$B_1B_1 = \left(I_n - \tfrac{1}{n}J_n\right)\left(I_n - \tfrac{1}{n}J_n\right) = I_nI_n - \tfrac{1}{n}I_nJ_n - \tfrac{1}{n}J_nI_n + \tfrac{1}{n^2}J_nJ_n$$

$$= I_n - \tfrac{2}{n}J_n + \tfrac{1}{n^2}nJ_n = I_n - \tfrac{2}{n}J_n + \tfrac{1}{n}J_n = I_n - \tfrac{1}{n}J_n = B_1$$

B_1 is called "idempotent".

Thm from 231: if A is symmetric and idempotent, then $\text{rank}(A) = \text{tr}(A) = \sum_{i=1}^{n} a_{ii}$

$$\Rightarrow \text{rank}[B_1] = \text{tr}[B_1] = \left(1-\tfrac{1}{n}\right) + \left(1-\tfrac{1}{n}\right) + \dots + \left(1-\tfrac{1}{n}\right) = n\left(1-\tfrac{1}{n}\right) = n-1$$

Cochran's Thm

$$\Rightarrow \frac{n-1}{\sigma^2}S_n^2 \sim \chi^2_{n-1} \quad \text{and} \quad S_n^2 \text{ independent of } \bar{X}_n$$

Fisher proved in 1925, Geary proved (1930) normal dist only distr. that has this feature

This is a cool result. We need one more core concept to derive the T, F distributions

$$\frac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}-\mu}{\frac{1}{\sqrt{n}}\sqrt{\frac{\sigma^2}{n-1}\frac{n-1}{\sigma^2}S^2}} = \frac{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\frac{n-1}{\sigma^2}S^2}{n-1}}}$$

$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$, $\frac{n-1}{\sigma^2}S^2 \sim \chi^2_{n-1}$

We need to derive PDFs of ratios of rv's!

$X \sim \text{Gamma}(\alpha, \beta)$

$$Y = aX \sim \frac{1}{a}\frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{x}{a}\right)^{\alpha-1} e^{-\beta\left(\frac{x}{a}\right)} \underbrace{\mathbb{1}_{\frac{x}{a}\in(0,\infty)}}_{x\in(0,\infty)} = \frac{\left(\frac{\beta}{a}\right)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\left(\frac{\beta}{a}\right)x} \mathbb{1}_{x\in(0,\infty)}$$

where $a > 0$

$$\Rightarrow Y \sim \text{Gamma}\left(\alpha, \frac{\beta}{a}\right)$$