Lec 10 MATH 340

First close the two loops from last class...

Let g be an invertible function on S_X. Let $Y = g(X)$, $X = g^{-1}(Y)$

Let $h(y) := g^{-1}(y)$ just for notational convenience.

$$g(h(y)) = y$$

$$\Rightarrow \frac{d}{dy}[g(h(y))] = \frac{d}{dy}[y]$$

$$\Rightarrow g'(h(y))\, h'(y) = 1$$

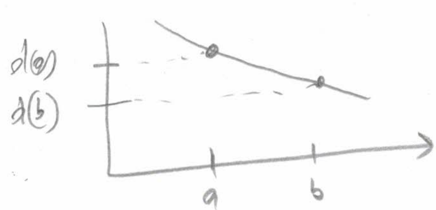
$$\Rightarrow h'(y) = \frac{1}{g'(h(y))}$$

If g is strictly increasing $g'(x) > 0 \ \forall x \Rightarrow h'(y) > 0 \ \forall y$

If g is strictly decreasing $g'(x) < 0 \ \forall x \Rightarrow h'(y) < 0 \ \forall y$

If $d(x)$ is a strictly decreasing function then for $a \le b \Rightarrow d(a) \ge d(b)$

"the inequality is reversed"



Let g be a strictly decreasing function $\Rightarrow g^{-1}(y)$ is also strictly decreasing

take g^{-1} of both sides

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}[1 - F_X(g^{-1}(y))] = -f_X(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)] = -f_X(g^{-1}(y))\left(-\left|\frac{d}{dy}[g^{-1}(y)]\right|\right) = f_X(g^{-1}(y))\left|\frac{d}{dy}[g^{-1}(y)]\right|$$

$$Y \sim \text{Logistic}(0,1) := \frac{e^y}{(e^y+1)^2} \cdot \frac{(e^{-y})^2}{(e^{-y})^2} = \frac{e^{-y}}{1+e^{-y}}$$

proof it is symmetric around zero

$$F_Y(y) = \int_{-\infty}^{y} \frac{e^x}{(e^x+1)^2}\,dx = \int_{1}^{1+e^y} \frac{\cancel{u-1}}{u^2}\frac{1}{\cancel{u-1}}\,du = \left[-u^{-1}\right]_1^{1+e^y} = 1 - \frac{1}{1+e^y} = \frac{e^y}{1+e^y} = \frac{1}{1+e^{-y}}$$

= standard logistic function $L=1$, $x_0=0$, $k=1$

let $u = 1+e^x \Rightarrow u-1 = e^x \Rightarrow x = \ln(u-1) \Rightarrow \frac{du}{dx} = e^x \Rightarrow dx = \frac{1}{e^x}du \Rightarrow dx = \frac{1}{u-1}du$

$\Rightarrow x=-\infty \Rightarrow u=1$, $x=y \Rightarrow u=1+e^y$

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Recall $X_n \xrightarrow{d} X$ which means $\lim_{n\to\infty} F_n(x) = F(x)$

and for a special case of $X \sim Deg(c)$, $X_n \xrightarrow{d} c$ " " " $\lim_{n\to\infty} F_n(x) = \begin{cases} 1 & \text{if } x \geq c \\ 0 & \text{if } x < c \end{cases}$ ← CDF of $Deg(c)$

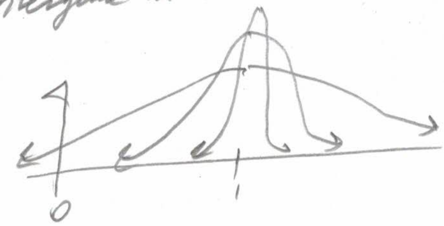
~~Would it be nice if for a continuous function g if~~

$$\cancel{X_n \xrightarrow{d} c \Rightarrow g(X_n) \xrightarrow{d} g(c)\ ?}$$

~~Yep. And this is true. But it's a long road to prove it.~~

We will now cover "convergence in probability", a different type of convergence than "convergence in distribution".

Consider $X_n \sim N\left(1, \frac{1}{n}\right)$



For $Y \sim N(\mu, \sigma^2)$, $\phi_Y(t) = e^{i\mu t - \frac{\sigma^2 t^2}{2}}$

It is clear that $X_n \xrightarrow{d} 1$. Why?

$$\phi_{X_n}(t) = e^{it - \frac{t^2}{2n}}$$

$$\phi_X(t) = \lim_{n\to\infty} \phi_{X_n}(t) = e^{it - \lim_{n\to\infty} \frac{t^2}{2n}} = e^{i(1)t} \overset{P1}{\Longrightarrow} X \sim Deg(1) \overset{P2}{\Longrightarrow} X_n \xrightarrow{d} 1$$

But note how all the probability is piling up near $x=1$.

Is it true that

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} P(|X_n - 1| \geq \varepsilon) = 0\ ?$$ Yes. Proof:

since $\mu = E[X_n] = 1$,
By Chebyshev's Inequality, $P(|X_n - 1| \geq \varepsilon) \leq \frac{Var[X_n]}{\varepsilon^2} = \frac{1}{n\varepsilon^2}$

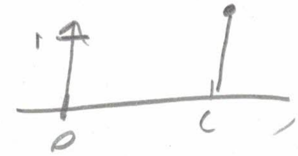
$$\Rightarrow \lim_{n\to\infty} P(|X_n - 1| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{1}{n\varepsilon^2} = 0 \ \checkmark$$

Define "convergence in probability to a constant" is:

$X_n \xrightarrow{P} c$ means by definition $\lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$.

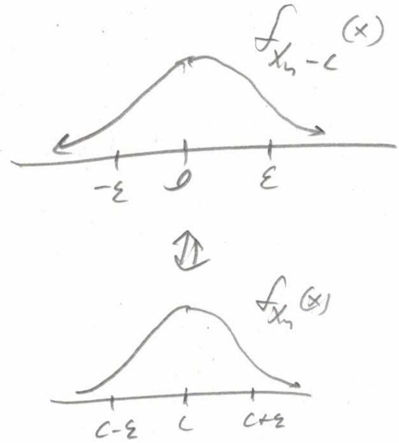
Colloquially, all the probability in X_n "piles up" near c eventually.

If $X_n \xrightarrow{d} c$ then eventually the PMF of X_n looks like



the Deg(c). So are these two definitions equivalent? Yes!

Proof: $X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{P} c$



Consider $\lim_{n\to\infty} P(|X_n - c| \geq \varepsilon)$ when $\varepsilon > 0$, WTS $= 0$

$= \lim_{n\to\infty} P(X_n - c \geq \varepsilon) + P(X_n - c \leq -\varepsilon)$

$= \lim_{n\to\infty} P(X_n \geq c+\varepsilon) + P(X_n \leq c-\varepsilon)$

$= \lim_{n\to\infty} 1 - F_{X_n}(c+\varepsilon) + F_{X_n}(c-\varepsilon)$

$= 1 - \lim_{n\to\infty} F_{X_n}(c+\varepsilon) + \lim_{n\to\infty} F_{X_n}(c-\varepsilon)$

since $X_n \xrightarrow{d} c \Rightarrow \lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 1 \text{ if } x \geq c \\ 0 \text{ if } x < c \end{cases}$

$= 1 - 1 + 0$

$= 0 \Rightarrow X_n \xrightarrow{P} c$

Recall $X_1, X_2, \ldots \overset{iid}{\sim}$ with $\mu < \infty$, $\bar{X}_n := \frac{X_1 + \ldots + X_n}{n}$ However, if $\sigma^2 < \infty \Rightarrow$ Use Chebyshev's

using chf's

$\Rightarrow \bar{X}_n \xrightarrow{d} \mu \Rightarrow \bar{X}_n \xrightarrow{P} \mu$, the Weak Law of Large Numbers (WLLN)

To prove equivalence, we need both directions.

Proof $X_n \xrightarrow{P} c \Rightarrow X_n \xrightarrow{d} c$

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 0$$

$$\Rightarrow \lim_{n\to\infty} P(X_n - c \geq \varepsilon) + P(X_n - c \leq -\varepsilon) = 0$$

$$\Rightarrow \lim_{n\to\infty} P(X_n \geq c+\varepsilon) + \lim_{n\to\infty} P(X_n \leq c-\varepsilon) = 0$$

Since Prob's $\in (0,1)$

$$\Rightarrow \lim_{n\to\infty} P(X_n \geq c+\varepsilon) = 0 \quad \text{AND} \quad \lim_{n\to\infty} P(X_n \leq c-\varepsilon) = 0$$

if $a+b=0$ and $a \geq 0, b \geq 0 \Rightarrow a=b=0$

$$\Rightarrow \lim_{n\to\infty} 1 - F_{X_n}(c+\varepsilon) = 0 \quad \text{AND} \quad \lim_{n\to\infty} F_{X_n}(c-\varepsilon) = 0$$

$$\Rightarrow \lim_{n\to\infty} F_{X_n}(c+\varepsilon) = 1 \quad \text{AND} \quad \lim_{n\to\infty} F_{X_n}(c-\varepsilon) = 0$$

$$\Rightarrow \lim_{n\to\infty} F_{X_n}(x) = F_X(x) = \begin{cases} 1 \text{ if } x \geq c+\varepsilon & \text{by def of CDF} \\ ? \text{ if } x \in (c-\varepsilon, c+\varepsilon) & \\ 0 \text{ if } x \leq c-\varepsilon & \text{by def of CDF} \end{cases}$$

Since this is valid $\forall \varepsilon > 0$, the middle set can be arbitrarily small

$$\Rightarrow \lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 1 \text{ if } x \geq c \\ 0 \text{ if } x < c \end{cases} \Rightarrow X \sim Deg(c) \checkmark$$

Now we want to prove the "Continuous mapping thm" (CMT)

if g is a continuous function and $X_n \xrightarrow{P} c \Rightarrow g(X_n) \xrightarrow{P} g(c)$

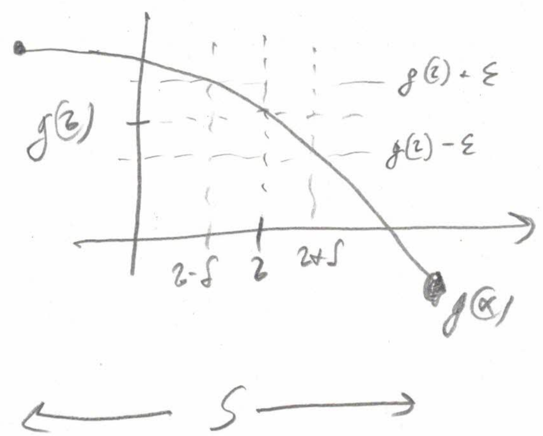
Def of continuity (Weierstrass) from real analysis:

The function $g(x)$ is continuous at x_0, $\forall x \in S$ if

$$\forall \varepsilon > 0 \;\exists \delta_\varepsilon > 0 \text{ if } |x - x_0| < \delta_\varepsilon \Rightarrow |g(x) - g(x_0)| < \varepsilon$$

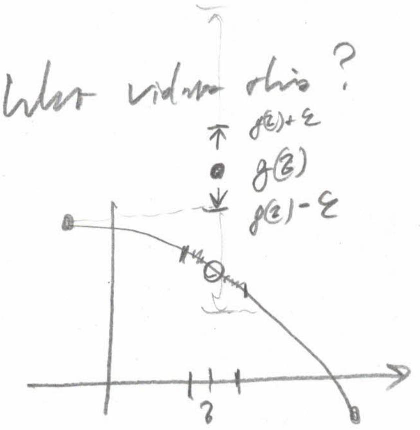
$$\Leftrightarrow |g(x) - g(x_0)| > \varepsilon \Rightarrow |x - x_0| > \delta_\varepsilon$$

let $x_0 = 2$



Regardless of the small window or neighborhood in the g or y-axis, that $g(x)$ remains in, we can make a window on the x-axis that contain all those $g(x)$'s.

What violates this?



there's no δ s.t. all the $g(x)$'s are close to $g(2)$.

Pf: Mostly rigorous...

$$P(|g(X_n) - g(c)| > \varepsilon) = P(|X_n - c| > \delta_\varepsilon)$$

$$\lim_{n\to\infty} P(|g(X_n) - g(c)| > \varepsilon) = \lim_{n\to\infty} P(|X_n - c| > \delta_\varepsilon) = 0 \Rightarrow g(X_n) \xrightarrow{P} g(c)$$

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Implications

$\overline{X}_n \xrightarrow{p} \mu$ by WLLN

$\frac{1}{\overline{X}_n} \xrightarrow{p} ?$ let $g(x) = \frac{1}{x}$

by CMT, $g(\overline{X}_n) \xrightarrow{p} g(\mu) \Rightarrow \frac{1}{\overline{X}_n} \xrightarrow{p} \frac{1}{\mu}$

Slutsky's Thm's

(A) $X_n \xrightarrow{d} X,\ Y_n \xrightarrow{p} c \Rightarrow X_n Y_n \xrightarrow{d} cX$

Can't find simple proof of this... maybe will find one later...

e.g.

$X_n \xrightarrow{d} N(0,1),\ U_n \sim N(2, \frac{1}{n}) \Rightarrow U_n \xrightarrow{p} 2$

$\frac{X_n}{U_n} \xrightarrow{d} ?$

let $g(x) = \frac{1}{x} \Rightarrow g(U_n) = \frac{1}{U_n} \to g(1) = \frac{1}{1} = 1$

$X_n g(U_n) \xrightarrow{d} (2)\, N(0,1) = N(0, 2^2).$

(B) $X_n \xrightarrow{p} c,\ Y_n \xrightarrow{p} d$

$\Rightarrow aX_n + bY_n \xrightarrow{p} ac + bd$

Also will try to find a proof for this later

Lemma $X_n \xrightarrow{d} X,\ \lim\limits_{n\to\infty} a_n = a \Rightarrow a_n X_n \xrightarrow{d} aX$

sequence

let $Y_n = \mathrm{Deg}(a_n) \Rightarrow F_{Y_n}(y) = \mathbb{1}_{y \geq a_n}$

$\lim F_{Y_n}(y) \doteq \lim \mathbb{1}_{y \geq a_n} = \mathbb{1}_{y \geq \lim a_n} = \mathbb{1}_{y \geq a} = F_Y(y) \Rightarrow Y_n \xrightarrow{d} Y \Rightarrow Y_n \xrightarrow{p} Y \sim \mathrm{Deg}(a)$