Lec 10 MATH 390

First close the two loops from last class...

Let $g$ be an invertible function on $S_X$. Let $Y = g(X)$, $X = g^{-1}(Y)$

Let $h(y) := g^{-1}(y)$ just for notational convenience.

$$g(h(y)) = y$$

$$\Rightarrow \frac{d}{dy}\left[g(h(y))\right] = \frac{d}{dy}[y]$$

$$\Rightarrow g'(h(y))\, h'(y) = 1$$

$$\Rightarrow h'(y) = \frac{1}{g'(h(y))}$$

If $g$ is strictly increasing $g'(x) > 0 \ \forall x \Rightarrow h'(y) > 0 \ \forall y$

If $g$ is strictly decreasing $g'(x) < 0 \ \forall x \Rightarrow h'(y) < 0 \ \forall y$

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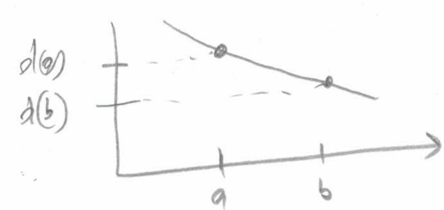
If $d(x)$ is a strictly decreasing function then for $a \le b \Rightarrow d(a) \ge d(b)$

"the inequality is reversed"

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Let $g$ be a strictly decreasing function $\Rightarrow g^{-1}(y)$ is also strictly decreasing

take $g^{-1}$ of both sides

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \ge g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

$$f_Y(y) = \frac{d}{dy}\left[1 - F_X(g^{-1}(y))\right] = -f_X(g^{-1}(y))\frac{d}{dy}\left[g^{-1}(y)\right] = -f_X(g^{-1}(y))\left(-\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right|\right) = f_X(g^{-1}(y))\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right|$$

$$Y \sim \text{Logistic}(0,1) := \frac{e^{y}}{(e^{y}+1)^2} \cdot \frac{(e^{-y})^2}{(e^{-y})^2} = \frac{e^{-y}}{1+e^{-y}}$$

proof it is symmetric around zero

denominator
should be squared above

$$F_Y(y) = \int_{-\infty}^{y} \frac{e^{x}}{(e^{x}+1)^2}\,dx = \int_{1}^{1+e^{y}} \frac{\cancel{u-1}}{u^2}\,\frac{1}{\cancel{u-1}}\,du = \left[-u^{-1}\right]_{1}^{1+e^{y}} = 1 - \frac{1}{1+e^{y}} = \frac{e^{y}}{1+e^{y}} = \frac{1}{1+e^{-y}}$$

= standard logistic function $L=1$, $n=0$, $k=1$

let $u = 1+e^{x} \Rightarrow u-1 = e^{x} \Rightarrow x = \ln(u-1) \Rightarrow \frac{du}{dx} = e^{x} \Rightarrow dx = \frac{1}{e^{x}}\,du \Rightarrow dx = \frac{1}{u-1}\,du$

$\Rightarrow x = -\infty \Rightarrow u = 1$, $x = y \Rightarrow u = 1+e^{y}$

Recall $X_n \xrightarrow{d} X$ which means $\lim_{n\to\infty} F_n(x) = F(x)$

and for a special case of $X \sim Deg(c)$, $X_n \xrightarrow{d} c$ " " " $\lim_{n\to\infty} F_n(x) = \begin{cases} 1 & \text{if } x \geq c \\ 0 & \text{if } x < c \end{cases}$ ← CDF of $Deg(c)$

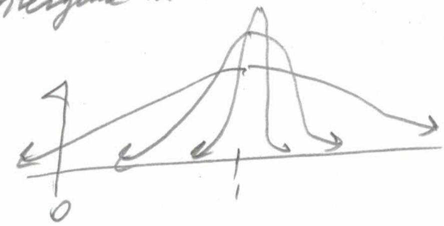
~~Would it be nice if for a continuous function g if~~

~~$X_n \xrightarrow{d} c \Rightarrow g(X_n) \xrightarrow{d} g(c)$ ?~~

~~Yes. And this is true. But it's a long road to prove it.~~

We will now cover "convergence in probability", a different type of convergence than "convergence in distribution".

Consider $X_n \sim N\left(1, \frac{1}{n}\right)$



For $Y \sim N(\mu, \sigma^2)$, $\phi_Y(t) = e^{i\mu t - \frac{\sigma^2 t^2}{2}}$

It is clear that $X_n \xrightarrow{d} 1$. Why?

$$\phi_{X_n}(t) = e^{it - \frac{t^2}{2n}}$$

$$\phi_X(t) = \lim_{n\to\infty} \phi_{X_n}(t) = e^{it - \lim_{n\to\infty} \frac{t^2}{2n}} = e^{i(1)t} \overset{(P1)}{\Longrightarrow} X \sim Deg(1) \overset{(P2)}{\Longrightarrow} X_n \xrightarrow{d} 1$$

But note how all the probability is piling up near $x = 1$.

Is it true that

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} P(|X_n - 1| \geq \varepsilon) = 0 \ ?$$ Yes. Proof:

since $\mu = E[X_n] = 1$,

By Chebyshev's Inequality, $P(|X_n - 1| \geq \varepsilon) \leq \frac{Var[X_n]}{\varepsilon^2} = \frac{1}{n\varepsilon^2}$

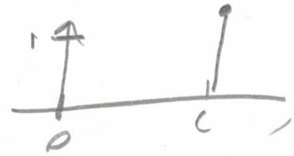
$$\Rightarrow \lim_{n\to\infty} P(|X_n - 1| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{1}{n\varepsilon^2} = 0 \quad ✓$$

Define "convergence in probability to a constant" is:

$X_n \xrightarrow{P} c$ means by definition $\lim\limits_{n\to\infty} P\left(|X_n - c| \geq \varepsilon\right) = 0 \quad \forall \varepsilon > 0.$

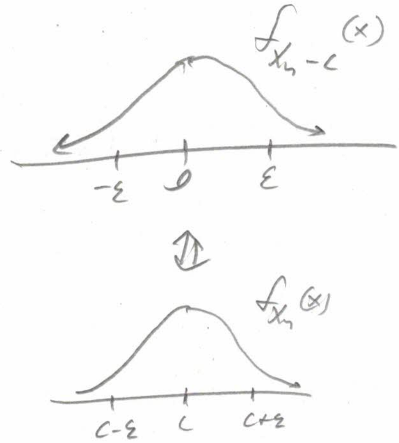
Colloquially, all the probability in $X_n$ "piles up" near $c$ eventually.

If $X_n \xrightarrow{d} c$ then eventually the PMF of $X_n$ looks like




the Deg(c). So are these two definitions equivalent? Yes!

Proof: $X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{P} c$




Consider $\lim\limits_{n\to\infty} P\left(|X_n - c| \geq \varepsilon\right)$ when $\varepsilon > 0$, WTS $= 0$

$= \lim\limits_{n\to\infty} P(X_n - c \geq \varepsilon) + P(X_n - c \leq -\varepsilon)$

$= \lim\limits_{n\to\infty} P(X_n \geq c+\varepsilon) + P(X_n \leq c-\varepsilon)$

$= \lim\limits_{n\to\infty} 1 - F_{X_n}(c+\varepsilon) + F_{X_n}(c-\varepsilon)$

$= 1 - \lim\limits_{n\to\infty} F_{X_n}(c+\varepsilon) + \lim\limits_{n\to\infty} F_{X_n}(c-\varepsilon)$

since $X_n \xrightarrow{d} c \Rightarrow \lim\limits_{n\to\infty} F_{X_n}(x) = \begin{cases} 1 & \text{if } x \geq c \\ 0 & \text{if } x < c \end{cases}$

$= 1 - 1 + 0$

$= 0 \Rightarrow X_n \xrightarrow{P} c$

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Recall $X_1, X_2, \ldots \overset{iid}{\sim}$ with $\mu < \infty$, $\bar{X}_n := \frac{X_1 + \ldots + X_n}{n}$ Hw, if $\sigma^2 < \infty \Rightarrow$ Use Chebyshev's

using ch.f.'s $\Rightarrow \bar{X}_n \xrightarrow{d} \mu \Longrightarrow \bar{X}_n \xrightarrow{P} \mu$, the Weak Law of Large Numbers (WLLN)

To prove equivalence, we need both directions.

Proof $X_n \xrightarrow{P} c \Rightarrow X_n \xrightarrow{d} c$

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 0$$

$$\Rightarrow \lim_{n\to\infty} P(X_n - c \geq \varepsilon) + P(X_n - c \leq -\varepsilon) = 0$$

$$\Rightarrow \lim_{n\to\infty} P(X_n \geq c+\varepsilon) + \lim_{n\to\infty} P(X_n \leq c-\varepsilon) = 0$$

Since Prob's $\in (0,1)$

$$\Rightarrow \lim_{n\to\infty} P(X_n \geq c+\varepsilon) = 0 \quad \text{AND} \quad \lim_{n\to\infty} P(X_n \leq c-\varepsilon) = 0$$

if $a+b=0$ and $a \geq 0, b \geq 0 \Rightarrow a=b=0$

$$\Rightarrow \lim_{n\to\infty} 1 - F_{X_n}(c+\varepsilon) = 0 \quad \text{AND} \quad \lim_{n\to\infty} F_{X_n}(c-\varepsilon) = 0$$

$$\Rightarrow \lim_{n\to\infty} F_{X_n}(c+\varepsilon) = 1 \quad \text{AND} \quad \lim_{n\to\infty} F_{X_n}(c-\varepsilon) = 0$$

$$\Rightarrow \lim_{n\to\infty} F_{X_n}(x) = F_X(x) = \begin{cases} 1 \text{ if } x \geq c+\varepsilon & \text{by def of CDF} \\ ? \text{ if } x \in (c-\varepsilon, c+\varepsilon) & \\ 0 \text{ if } x \leq c-\varepsilon & \text{by def of CDF} \end{cases}$$

Since this is valid $\forall \varepsilon > 0$, the middle set can be arbitrarily small

$$\Rightarrow \lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 1 \text{ if } x \geq c \\ 0 \text{ if } x < c \end{cases} \Rightarrow X \sim \text{Deg}(c) \checkmark$$

Now we want to prove the "continuous mapping thm" (CMT)

if $g$ is a continuous function and $X_n \xrightarrow{P} c \Rightarrow g(X_n) \xrightarrow{P} g(c)$

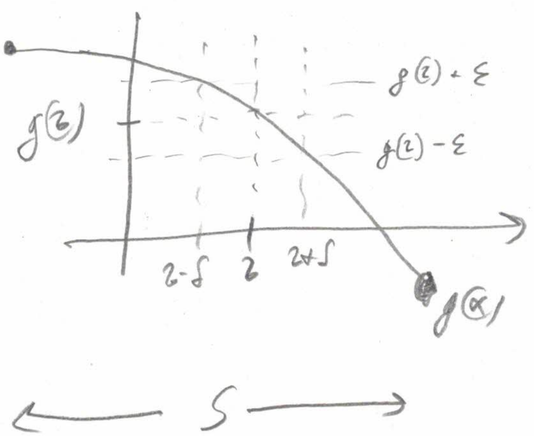
Def of continuity (Weierstrass) from real analysis:

The function $g(x)$ is continuous at $x_0$, $\forall x \in S$ if

$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 \text{ if } |x - x_0| < \delta_\varepsilon \Rightarrow |g(x) - g(x_0)| < \varepsilon$$

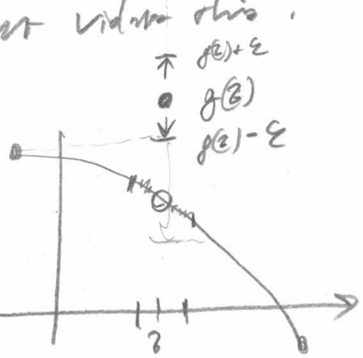
$$\Leftrightarrow |g(x) - g(x_0)| > \varepsilon \Rightarrow |x - x_0| > \delta_\varepsilon$$

let $x_0 = 2$




Regardless of the small window or neighborhood in the $g$ or $y$-axis, that $g(x)$ remains in, we can make a window on the $x$-axis that contains all those $g(x)$'s.

What violates this?




there's no $\delta$ s.t. all the $g(x)$'s are close to $g(2)$.

Pf: Mostly rigorous...

$$P(|g(X_n) - g(c)| > \varepsilon) = P(|X_n - c| > \delta_\varepsilon)$$

$$\lim_{n\to\infty} P(|g(X_n) - g(c)| > \varepsilon) = \lim_{n\to\infty} P(|X_n - c| > \delta_\varepsilon) = 0 \Rightarrow g(X_n) \xrightarrow{P} g(c)$$

Implications

$\overline{X}_n \xrightarrow{p} \mu$ by WLLN

$\frac{1}{\overline{X}_n} \xrightarrow{p} ?$ let $g(x) = \frac{1}{x}$

By CMT, $g(\overline{X}_n) \xrightarrow{p} g(\mu) \Rightarrow \frac{1}{\overline{X}_n} \xrightarrow{p} \frac{1}{\mu}$

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Slutsky's Thm's

(A) $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{p} c \Rightarrow X_n Y_n \xrightarrow{d} cX$

Can't find simple proof of this... maybe will find one later...

e.g.

$X_n \xrightarrow{d} N(0,1)$, $U_n \sim N(2, \frac{1}{n}) \Rightarrow U_n \xrightarrow{p} 2$

$\frac{X_n}{U_n} \xrightarrow{d} ?$

let $g(x) = \frac{1}{x} \Rightarrow g(U_n) = \frac{1}{U_n} \rightarrow g(1) = \frac{1}{1} = 1$

$X_n g(U_n) \xrightarrow{d} (2) N(0,1) = N(0, 2^2)$.

(B) $X_n \xrightarrow{p} c$, $Y_n \xrightarrow{p} d$

$\Rightarrow aX_n + bY_n \xrightarrow{p} ac + bd$

Also will try to find a proof for this later

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Lemma $X_n \xrightarrow{d} X$, $\lim_{n\to\infty} a_n = a \Rightarrow a_n X_n \xrightarrow{d} aX$

sequence

let $Y_n = \mathrm{Deg}(a_n) \Rightarrow F_{Y_n}(y) = \mathbb{1}_{y \geq a_n}$

$\lim F_{Y_n}(y) = \lim \mathbb{1}_{y \geq a_n} = \mathbb{1}_{y \geq \lim a_n} = \mathbb{1}_{y \geq a} = F_Y(y) \Rightarrow Y_n \xrightarrow{d} Y \Rightarrow Y_n \xrightarrow{p} Y \sim \mathrm{Deg}(a)$