MATH 340/640 Fall 2023 Homework #4

Professor Adam Kapelner

Due by email 11:59PM October 29, 2023

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Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out." Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, review Math 241 concerning the exponential.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems. "[MA]" are for those registered for 621 and extra credit otherwise.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to 5 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. The easiest way to use overleaf is to copy the raw text from hwxx.tex and preamble.tex into two new overleaf tex files with the same name. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks which are *not* on this printout. Keep this first page printed for your records.

NAME: ______________________________

Problem 1

These exercises will deepen the understanding of the Multinomial distribution. Consider $\boldsymbol{X} \sim \text{Multinomial}\left(17,\ [0.1\ 0.2\ 0.3\ 0.4]^\top\right)$.

(a) [easy] What is $\dim[\boldsymbol{X}]$?

(b) [harder] Find $\mathbb{V}\text{ar}[\boldsymbol{X}]$.

(c) [harder] If $x_1 = 1$, what is the JDF of the remaining rv's?

(d) [harder] If $x_1 = 1$ and $x_2 = 6$, what is the JDF of the remaining rv's?

(e) [easy] If $x_1 = 1$, $x_2 = 6$ and $x_3 = 3$, how is the remaining rv distributed?

Problem 2

These exercises will introduce the famous inequalities.

(a) [easy] Prove Markov's Inequality. State the assumptions clearly.

(b) [harder] Prove a Markov's-like Inequality for the bound on the probability of the left tail for a negative r.v. X.

(c) [easy] Prove Chebyshev's Inequality. State the assumptions clearly.

(d) [easy] Prove the lemma of Chebyshev's Inequality for the tail which looks like $\mathbb{P}\left(X \geq b\right) \leq$ .... State the assumptions clearly.

(e) [harder] Let X be a non-negative rv. Prove $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X^3]}{a^3}$ where $a > 0$.

(f) [harder] Prove that if $\mathbb{E}[|X|]$ is finite then $\mathbb{E}[X]$ is finite.

(g) [difficult] [MA] Prove that if $\mathbb{E}[X]$ is finite then $\mathbb{E}[|X|]$ is finite.

(h) [difficult] Let $X_n \sim \text{Exp}(n)$. Compute upper bounds for $\mathbb{P}(X \geq 3)$ via Markov and Chebyshev. Does one go to zero "faster" than the other? Explain.

Problem 3

These questions are about convergence.

(a) [easy] Given

$$X_n \sim \begin{cases} 0 & \text{w.p.} \;\; 1 - \frac{1}{(n+1)^2} \\ 1 & \text{w.p.} \;\; \frac{1}{(n+1)^2} \end{cases}$$

Show that $X_n \xrightarrow{d} 0$. You can use the theorem we never proved that $\lim \mathbb{1}_{a_n} = \mathbb{1}_{\lim a_n}$.

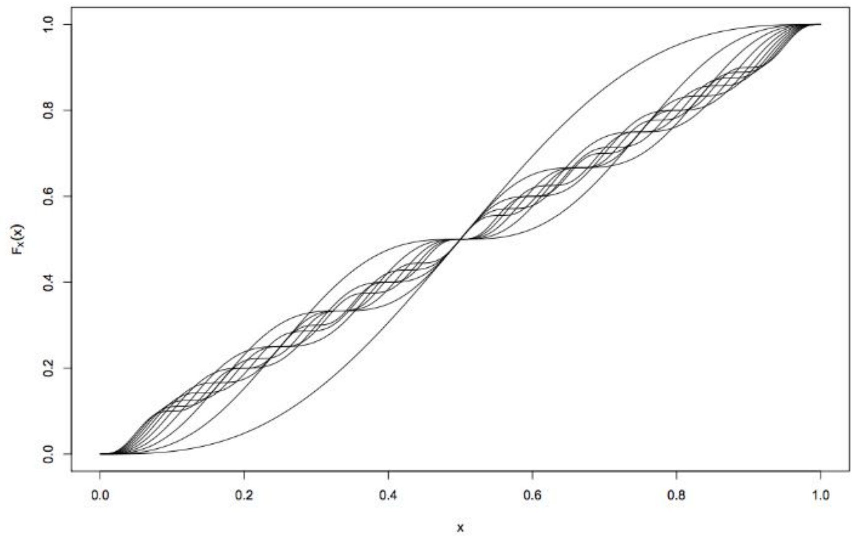
(b) [harder] Prove that $X_n \xrightarrow{p} 0$. For a full proof of $\forall \epsilon$, you need to show it for $\epsilon < 1$ and $\epsilon \geq 1$ separately but since we only care about small epsilon, you can just demonstrate it for $\epsilon < 1$.

(c) [harder] If $X_n \sim \text{Exp}(n)$, prove that $X_n \xrightarrow{d} 0$.

(d) [harder] If $X_n \sim \text{Exp}(n)$, prove that $X_n \xrightarrow{p} 0$. You can prove this using its CDF or using Markov's inequality.

(e) [difficult] [MA] Prove that PDF convergence implies CDF convergence (i.e., that $X_n \xrightarrow{d} X$). You will need to use the dominated convergence theorem (DCT). Justifying the use of the DCT is slightly harder.

(f) [harder] Let $X_n \sim f_{X_n}(x) = (1 - \cos(2\pi n x))\mathbb{1}_{x \in [0,1]}$. Show that $X_n \xrightarrow{d} \text{U}(0, 1)$. Hint: use the fact that $\lim_{n\to\infty} \frac{\sin(nx)}{n} = 0$ which should have been proven using the "squeeze theorem" in your calculus class. The CDF is pictured below for $n = 1, 2, 3, 4, \ldots$.



(g) [harder] [MA] Show that $\lim_{n\to\infty} f_{X_n}$ does not exist. This is a counterexample to the conjecture that CDF convergence implies PDF convergence for finite continuous distributions with a limiting continuous distribution. Hint: it is just a calculus exercise.

Problem 4

These exercises will give you practice with the continuous mapping theorem (CMT) and Slutsky's theorems.

(a) [easy] State the CMT.

(b) [easy] State Slutsky's theorem A.

(c) [easy] State Slutsky's theorem B.

(d) [easy] If $X_n \overset{d}{\to} X$ and $\lim_{n\to\infty} a_n = a$, prove $a_n X_n \overset{d}{\to} aX$.

(e) [easy] Write the rv S_n^2 as a function of $X_1, \ldots, X_n$.

(f) [easy] Prove $S_n^2 \xrightarrow{p} \sigma^2$.

(g) [easy] Prove $S_n \xrightarrow{p} \sigma$.

(h) [easy] Prove $\sqrt{n}(\bar{X}_n - \mu)/S \xrightarrow{d} \mathcal{N}(0, 1)$.

Problem 5

These exercises will give you practice with transformations of discrete r.v.'s.

(a) [easy] Let $X \sim \text{Binomial}(n, p)$. Find the PMF of $Y = \ln(X + 1)$.

(b) [harder] Show that for any r.v. X (discrete or continuous), if $Y = aX + b$, then $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$.

(c) [harder] Let $X \sim \text{NegBin}\,(k,\, p)$. Find the PMF of $Y = X^2$. Is $g(X)$ monotonic? Does that matter for this r.v.?

(d) [difficult] Let $X \sim \text{Binomial}\,(n,\, p)$ where n is an even number. Find an expression for the PMF of $Y = \text{mod}(X, 2)$.

Problem 6

These exercises will give you practice with transformations of continuous r.v.'s.

(a) [easy] Let g be a strictly decreasing function and X be a continuous rv and $Y = g(X)$. Find a formula for the PDF of Y. Justify each step.

(b) [easy] Let $g(x) = ax + b$, X be a continuous rv and $Y = g(X)$. Find a formula for the PDF of Y.

(c) [easy] Let $g(x) = ax + b$, X be a continuous rv and $Y = g(X)$. Find a formula for the PDF of Y.

(d) [harder] Let $X \sim \text{Exp}\,(\lambda)$. Show that $Y = aX$ is an exponential rv and find its paramter. Use the transformation formula (not ch.f.'s).

(e) [difficult] Let $X \sim \text{Logistic}(0, 1)$. Find the PDF of $Y = g(X) = \frac{1}{1+e^{-X}}$. If this is a brand name r.v., mark it so and include its parameter values.

(f) [harder] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = ke^X$ where $k > 0$. This will be a brand name r.v., so mark it so and include its parameter values.

(g) [harder] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = \ln(X)$.

(h) [difficult] If $X \sim \text{Exp}(\lambda)$ then show that $Y = X^\beta \sim \text{Weibull}$ where $\beta > 0$. Find the resulting Weibull's parameters in terms of the parameterization we learned in class.

(i) [E.C.] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = \sin(X)$. Don't attempt this unless you have extra time.

(j) [easy] Rederive the $X \sim \text{Laplace}(0, 1)$ r.v. model by taking the difference of two standard exponential r.v.'s.

(k) [easy] Show that $\mathcal{E} \sim \text{Laplace}(0, \sigma)$ satisfies the three conditions of the definition of an "error distribution".

Problem 7

These exercises will give you practice with the gamma function.

(a) [easy] Write the definition of $\Gamma(x)$.

(b) [difficult] Prove $\Gamma(k+1) = k\Gamma(k)$ for $k > 0$.

(c) [harder] Write the definition of $Q(x, a)$ without using the gamma function.

(d) [harder] If $0 < a < b < \infty$, find an integral expression for $\Gamma(x, b) - \gamma(x, a)$.

(e) [harder] Let $X \sim \text{Gamma}(\alpha, \beta)$. Prove the Humpty Dumpty identity.

(f) [easy] Write the PMF's and parameter spaces of both the extended negative binomial rv and the negative binomial rv model. Explain how the latter "upgrades" the former.

(g) [easy] Write the PDF's and parameter spaces of both the gamma rv and the Erlang rv model. Explain how the latter "upgrades" the former.