Math 340 / 640 Fall 2023
Midterm Examination One

Solutions

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Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

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Instructions

This exam is 110 minutes (variable time per question) and closed-book. You are allowed **one** page (front and back) of a "cheat sheet", blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 This problem is about a new rv,

$$X \sim \text{Poisson}(\lambda) := \frac{\lambda^x e^{-\lambda}}{x!} \mathbb{1}_{x \in \mathbb{N}_0}$$

whose parameter space for its sole parameter is $\lambda > 0$.

(a) [3 pt / 3 pts] What is the support of X? $\mathbb{S}_X =$ $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$

(b) [4 pt / 7 pts] Circle one: this rv is discrete / continuous.

(c) [4 pt / 11 pts] $p^{old}(x) =$ $\dfrac{\lambda^x e^{-\lambda}}{x!}$

(d) [5 pt / 16 pts] $\mathbb{P}(X \in [-2, 2]) =$ $e^{-\lambda} + \lambda e^{-\lambda} + \dfrac{\lambda^2 e^{-\lambda}}{2}$

(e) [7 pt / 23 pts] Recall the Taylor series for the exponential function of a where $a \in \mathbb{R}$:

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \frac{a^4}{4!} + \ldots = \sum_{x=0}^{\infty} \frac{a^x}{x!}$$

Using this fact, prove the Humpty-dumpty rule for the PMF of X.

by the Taylor series above for e^a

$$\sum_{x \in \mathbb{R}} p(x) = \sum_{x \in \mathbb{N}_0} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x \in \mathbb{N}_0} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1$$

(f) [6 pt / 29 pts] Write an expression for the CDF of X below denoted $F_X(x)$. For simplicity, assume only $x \in \mathbb{S}_X$ will be evaluated by this function. Note: the CDF is not available in closed form. Simplify as much as possible.

$$F_X(x) := P(X \le x) = \sum_{y \in (-\infty, x)} p(y) = \sum_{y=0}^{x} \frac{e^{-\lambda} \lambda^y}{y!} = e^{-\lambda} \sum_{\cancel{x}=0}^{x} \frac{\lambda^y}{y!}$$

(g) [7 pt / 36 pts] Recall the following combinatorial identity from Math 241:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Let $X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$ and $T = X_1 + X_2$. Prove the convolution of $p_T(t) = p_{X_1}(x) \star p_{X_2}(x)$ is Poisson and find its parameter value. You may not use characteristic functions to solve this problem. For maximum partial credit, provide the appropriate formula for the convolution and justify each intermediate step. This question is difficult. You may want to do parts (h) and (i) before doing this problem.

$$p_T(t) = \sum_{x \in S_x} p^{old}(x)\, p^{old}(t-x)\, \mathbb{1}_{t-x \in S_x} = \sum_{x \in \mathbb{N}_0} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{t-x} e^{-\lambda}}{(t-x)!} \underbrace{\underbrace{\mathbb{1}_{t-x \in \mathbb{N}_0}}_{x-t \in \{0,-1,-2,\dots\}}}_{x \in \{t, t-1, t-2, \dots\}}$$

$$= e^{-2\lambda} \lambda^t \mathbb{1}_{t \in \mathbb{N}_0} \sum_{x \in \{0,1,\dots,t\}} \frac{1}{x!} \frac{1}{(t-x)!} \cdot \frac{t!}{t!}$$

by combinatorial identity above

$$= \frac{e^{-2\lambda}\lambda^t}{t!} \mathbb{1}_{t \in \mathbb{N}_0} \sum_{x=0}^{t} \binom{t}{x} = \frac{e^{-2\lambda}\lambda^t}{t!} \mathbb{1}_{t \in \mathbb{N}_0}\, 2^t$$

$$= \frac{e^{-2\lambda}(2\lambda)^t}{t!} \mathbb{1}_{t \in \mathbb{N}_0} = \text{Poisson}(2\lambda)$$

(h) [6 pt / 42 pts] Prove $\phi_X(t) = e^{\lambda(e^{it}-1)}$. For maximum partial credit, provide the definition of the ch.f. and justify each intermediate step.

$$\phi_X(t) := \mathbb{E}[e^{itX}] = \sum_{x \in \mathbb{R}} e^{itx} p(x) = \sum_{x \in \mathbb{N}_0} e^{itx} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x \in \mathbb{N}_0} \frac{(\lambda e^{it})^x}{x!}$$

by Taylor Series of e^q

$$= e^{-\lambda} e^{\lambda e^{it}} = e^{\lambda e^{it} - \lambda} = e^{\lambda(e^{it}-1)}$$

(i) [5 pt / 47 pts] Let $X_1, X_2 \overset{iid}{\sim}$ Poisson (λ) and $T = X_1 + X_2$. Using the characteristic function of the Poisson, prove that T is a Poisson rv and find its parameter value. Justify each step.

$$\phi_T(t) = (\phi_X(t))^2 = \left(e^{\lambda(e^{it}-1)}\right)^2 = e^{2\lambda(e^{it}-1)} \Longrightarrow T \sim \text{Poisson}(2\lambda)$$

by (P3) for iid rv's

(P1)

(j) [6 pt / 53 pts] Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ Poisson (17) and let $\bar{X}_n = \frac{1}{n}(X_1 + \ldots + X_n)$. If n is very large, what real-number value will $\bar{x}_n$ be approximately equal to and why? Hint: for $X \sim$ Poisson (λ), then $\phi'_X(t) = i\lambda e^{it} e^{\lambda e^{it}} e^{-\lambda}$.

By LLN, $\bar{X}_n \xrightarrow{d} \mu \Rightarrow \bar{X}_n \approx \mu$ for large n

$$\mu = E[X] = \frac{\phi_X'(0)}{i} = \frac{\cancel{i}\lambda e^{i(0)} e^{\lambda e^{i(0)}} e^{-\lambda}}{\cancel{i}} = \lambda(1) e^{\lambda(1)} e^{-\lambda} = \lambda$$

$\Rightarrow \bar{X}_n \approx 17$

(k) [6 pt / 59 pts] Let $X_n \sim$ Poisson (λ/n), $n \in \mathbb{N}$. Prove that $X_n \xrightarrow{d} 0$. Justify each step.

$$\phi_{X_n}(t) = e^{\frac{\lambda}{n}(e^{it}-1)}$$

$$\lim_{n\to\infty} \phi_{X_n}(t) = e^{\lim_{n\to\infty} \frac{\lambda}{n}(e^{it}-1)} = e^0 = e^{it(0)} \overset{(P7)}{\Longrightarrow} X_n \xrightarrow{d} \text{Deg}(0) \quad \text{which is the def. of } X_n \xrightarrow{d} 0$$

Problem 2 Consider rolling a fair die 21 times. The die has 6 sides marked 1, 2, 3, 4, 5, 6 where the side that faces up upon the rolling of the die is uniform discrete.

(a) [6 pt / 65 pts] Find an expression for the probability of getting one 1, two 2's, three 3's, four 4's, five 5's and six 6's where the order of those rolls does not matter.

$X \sim \text{Multinom}\left(21, \frac{1}{6}\vec{1}_6\right)$

$P\left(X = [1\ 2\ 3\ 4\ 5\ 6]^T\right) = \binom{21}{1,2,3,4,5,6}\left(\frac{1}{6}\right)^{21}$

(b) [6 pt / 71 pts] If X_1 is the rv that represents the number of 1's rolled in the 21 rolls, find an expression for $\mathbb{P}(X_1 = x)$ where x can be any real number.

$p_{X_1}(x) = \text{Binom}\left(21, \frac{1}{6}\right) = \binom{21}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{21-x}$

Problem 3 This problem has disconnected theory questions.

(a) [6 pt / 77 pts] Prove that $\mathbb{C}\text{ov}[aX, X] = a\mathbb{V}\text{ar}[X]$ from the definition of covariance.

$Cov[aX,X] = E[(aX - E[aX])(X - E[X])] = E[a(X-\mu)(X-\mu)] = a\,E[(X-\mu)^2] = a\,Var[X]$

(b) [6 pt / 83 pts] Let X be a discrete non-negative non-degenerate rv. Prove that $\mathbb{E}[X] > 0$.

$E[X] = \sum_{x \in S_X} \underbrace{x}^{\geq 0} \underbrace{p^{old}(x)}_{\substack{>0 \\ \text{for all } x}} > 0$

$\exists x$ s.t. $x > 0$ since X cannot be $Deg(0)$

(c) [5 pt / 88 pts] Under what condition(s) is the following identity true?

$$g(t) = \int_{\mathbb{R}} e^{2\pi i\omega t} \int_{\mathbb{R}} e^{-2\pi i\omega t} g(t)dtd\omega.$$

(1) $g(t) \in L^1$

(2) $\hat{g}(\omega) \in L^1$ where $\hat{g}(\omega) := \int_{\mathbb{R}} e^{-2\pi i \omega t} g(t) dt$

(d) [6 pt / 94 pts] Let $X_1, \ldots, X_n \overset{iid}{\sim}$ with mean 1 and variance 2. Let $T = X_1 + \ldots + X_n$. Find the approximate probability $T > 3$. Leave your answer in terms of the Φ function.

$T \dot{\sim} N(n\mu, (\sigma\sqrt{n})^2) = N(4, (\sqrt{24})^2)$

$P(T>3) = P\left(\frac{T-4}{\sqrt{24}} > \frac{3-4}{\sqrt{24}}\right) \approx P\left(Z > \frac{3-4}{\sqrt{24}}\right) = 1 - \Phi\left(\frac{3-4}{\sqrt{24}}\right)$

(e) [6 pt / 100 pts] Let $X \sim$ Erlang (4, 2). Find an integral expression for $\mathbb{P}(X > 3)$. Simplify as much as possible.

$X \sim \frac{x^3 2^4 e^{-2x}}{3!} \mathbb{1}_{x>0} = \frac{16}{6} x^3 e^{-2x} \mathbb{1}_{x>0}$

$F_X(x) = \frac{8}{3} \int_0^x y^3 e^{-2y} dy$

$P(X>3) = 1 - F_X(3) = 1 - \frac{8}{3} \int_0^3 y^3 e^{-2y} dy$