Transformations for Vector rv's

Let $\vec{X}$ be a continuous vector rv with dimension n and known jdf $f_{\vec{X}}(\vec{x})$.

Let $g : \mathbb{R}^n \to \mathbb{R}^n$, a vector-valued function and 1:1 and let $\vec{Y} = g(\vec{X})$. Find $f_{\vec{Y}}(\vec{y})$.

Recall what a vector function does. It is really n different $g_i : \mathbb{R}^n \to \mathbb{R}$ functions:

$$g_1(X_1, \ldots X_n) = Y_1,$$
$$g_2(X_1, \ldots X_n) = Y_2,$$
$$\vdots$$
$$g_n(X_1, \ldots X_n) = Y_n$$

Because g is 1:1 $\exists\, h$ which inverts the function $\vec{X} = h(g(\vec{X})) = h(\vec{Y})$, which is also n different $h_i : \mathbb{R}^n \to \mathbb{R}$ functions:

$$h_1(Y_1, \ldots Y_n) = X_1$$
$$h_2(Y_1, \ldots Y_n) = X_2$$
$$\vdots$$
$$h_n(Y_1, \ldots, Y_n) = X_n$$

The multivariate change of variable formula is for $X_1 \ldots X_n$ continuous is:

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h(\vec{y}))\, |J_h(\vec{y})|$$

where $J_h := \det \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial h_n}{\partial y_1} & & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$

I can't find a proof of this that doesn't involve heavy multivariable calculus

For the purpose of this class, we are only interested in finding the density of $Y = g_1(X_1,\dots,X_n)$, i.e. the first rv in $\vec{Y}$. So then there is an extra step:

$$f_Y(y) = \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} f_{\vec{Y}}(y, u_1,\dots,u_{n-1})\, du_1 \cdots du_{n-1}$$

i.e. margin out everything else. For the case with $n=2$, $f_{X_1,X_2}(x_1,x_2)$ known, $Y = g(X_1,X_2)$, $f_{Y,U}(y,u) = f_{X_1,X_2}(h_1(y,u), h_2(y,u)) \left| \frac{\partial h_1}{\partial y_1}\frac{\partial h_2}{\partial y_2} - \frac{\partial h_1}{\partial y_2}\frac{\partial h_2}{\partial y_1} \right|$

and then.. $f_Y(y) = \int_{\mathbb{R}} f_{Y,U}(y,u)\, du$

The first thing we will do is recover the convolution formula as a special case of an arbitrary transformation of two rv's.

$T = X_1 + X_2 = g_1(X_1,X_2)$ Now we need to find a function g_2 s.t. $U = g_2(X_1,X_2)$, so that we can find functions h_1, h_2 s.t. $X_1 = h_1(T,U)$, $X_2 = h_2(T,U)$

i.e. h is the inverse function.

let $U = X_1 = g_2(X_1,X_2) \Rightarrow_{X_1 = U} = h_1(T,U) \Rightarrow \frac{\partial h_1}{\partial t} = 0,\ \frac{\partial h_1}{\partial u} = 1$

$\Rightarrow T = U + X_2$

$\Rightarrow X_2 = T - U = h_2(T,U) \Rightarrow \frac{\partial h_2}{\partial t} = 1,\ \frac{\partial h_2}{\partial u} = -1 \Rightarrow J_h = \det\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = 0\cdot -1 + 1\cdot 1 = 1$

$\Rightarrow f_{T,U}(t,u) = f_{X_1,X_2}(h_1(t,u), h_2(t,u))\, |J_h| = f_{X_1,X_2}(u, t-u)\,|1| = f_{X_1,X_2}(u,t-u)$

convolution formula!

$\Rightarrow f_T(t) = \int_{\mathbb{R}} f_{X_1,X_2}(u,t-u)\, du \overset{\text{if } X_1,X_2 \text{ ind}}{=} \int_{\mathbb{R}} f_{X_1}(u) f_{X_2}(t-u)\, du \overset{\text{if } X_1,X_2 \text{ iid}}{=} \int_{\mathbb{R}} f(u) f(t-u)\, du$

Step-by-step procedure

① Find g_2, s.t. you can find h_1, h_2. This requires some playing around!

② Calculate $|J_h| = \left| \frac{\partial h_1}{\partial y} \cdot \frac{\partial h_2}{\partial u} - \frac{\partial h_1}{\partial u} \frac{\partial h_2}{\partial y} \right|$

③ Plug in values $f_{X_1,X_2}(h_1(y,u), h_2(y,u)) \, |J_h|$

④ Integrate out nuisance variable, u from step 3 ie $f_Y(y) = \int_{\mathbb{R}} \cdot \, du$

⑤ Simplify for $X_1, X_2 \overset{ind}{\sim}$ and $X_1, X_2 \overset{iid}{\sim}$ and for old densities it you know

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Another example: let $R = \frac{X_1}{X_2}$ (ratio). Find $f_R(r)$.

① let $U = X_2 = g_2(X_1, X_2) \implies X_2 = U = h_2(R, U)$

$\implies R = \frac{X_1}{U} \implies X_1 = RU = h_1(R, U)$

② $\frac{\partial h_1}{\partial r} = u, \; \frac{\partial h_1}{\partial u} = r, \; \frac{\partial h_2}{\partial r} = 0, \; \frac{\partial h_2}{\partial r} = 1$

$\implies |J_h| = |(u)(1) - (r)(0)| = |u|$

③ $f_{X_1,X_2}(ru, u) \, |u|$

④ $f_R(r) = \int_{\mathbb{R}} f_{X_1,X_2}(ru, u) \, |u| \, du \underset{X_1,X_2 \; ind}{\overset{⑤}{=}} \int_{\mathbb{R}} f_{X_1}(ru) f_{X_2}(u) \, |u| \, du = \int_{\mathbb{R}} f(ru) f(u) \, |u| \, du$

$= \int_{\mathcal{X}} f_{X_1}^{old}(ru) f_{X_2}^{old}(u) \mathbb{1}_{u \in \mathcal{S}_{X_2}} |u| \, du$

$= \int_{\mathcal{X}} f^{old}(ru) f^{old}(u) \mathbb{1}_{u \in \mathcal{S}_X} |u| \, du$

4

Back to derivation of Student's T distribution...

$$R = \frac{\frac{\bar{X}-\mu}{S}}{\sqrt{n}} = \frac{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\frac{n-1}{\sigma^2} S_n^2}{n-1}}}$$

$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ ← independent of each other ↓ proved due to Cochran's Thm.

$\frac{n-1}{\sigma^2} S_n^2 \sim \chi^2_{n-1} = \text{Gamma}\left(\frac{n-1}{2}, \frac{1}{2}\right)$

$\frac{\frac{n-1}{\sigma^2} S_n^2}{n-1} \sim \text{Gamma}\left(\frac{n-1}{2}, \frac{n-1}{2}\right)$

let $X \sim \text{Gamma}(\alpha, \beta)$, $Y = aX \sim f_Y(y) = ?$ did this last class!

$$\phi_X(t) = \left(\frac{\beta}{it-\beta}\right)^\alpha$$

should be "β - it" in the denominator

$$\phi_Y(t) = \phi_X(at) = \left(\frac{\beta}{iat-\beta}\right)^\alpha = \left(\frac{\frac{\beta}{a}}{it-\frac{\beta}{a}}\right)^\alpha \Rightarrow Y \sim \text{Gamma}\left(\alpha, \frac{\beta}{a}\right)$$

$$\Rightarrow \frac{\frac{n-1}{\sigma^2} S_n^2}{n-1} \sim \text{Gamma}\left(\frac{n-1}{2}, \frac{n-1}{2}\right) = \frac{\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} t^{\frac{n-1}{2}-1} e^{-\frac{n-1}{2}t} \mathbb{1}_{t>0}$$

let $X \sim \text{Gamma}(\alpha, \beta)$, $Y = \sqrt{X} \Leftrightarrow X = Y^2 = g^{-1}(Y)$ which is 1:1 on S_X

$$\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = 2|y|$$

$$f_Y(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} (y^2)^{\alpha-1} e^{-\beta y^2} \mathbb{1}_{\underbrace{y^2 \in (0,\infty)}_{y \in (0,\infty)}} 2y$$

$$= 2\frac{\beta^\alpha}{\Gamma(\alpha)} y^{2\alpha-1} e^{-\beta y^2} \mathbb{1}_{y \in (0,\infty)}$$

$$\Rightarrow \sqrt{\frac{\frac{n-1}{\sigma^2} S^2}{n-1}} \sim 2\frac{\left(\frac{n-1}{2}\right)^{\left(\frac{n-1}{2}\right)}}{\Gamma\left(\frac{n-1}{2}\right)} \underbrace{y^{2\left(\frac{n-1}{2}\right)-1}}_{y^{n-2}} e^{-\frac{n-1}{2}y^2} \mathbb{1}_{y \in (0,\infty)}$$

$R = \frac{X_1}{X_2}$ where $X_1 \sim N(0,1)$, $X_2 \sim$ sqrt gamma from previous page

$$f_R(r) = \int_{\mathbb{R}} f_{X_1}(ru)\, f_{X_2}(u)\, |u|\, du$$

$$= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(ru)^2} \right) \left(2 \frac{\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} u^{n-2} e^{-\frac{n-1}{2}u^2} \mathbb{1}_{u\in(0,\infty)} \right) |u|\, du$$

$$= \frac{2}{\sqrt{2\pi}} \frac{\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty u^{n-1} \underbrace{e^{-\frac{r^2u^2}{2} - \frac{n-1}{2}u^2}}_{e^{-\left(\frac{r^2}{2}+\frac{n-1}{2}\right)u^2}} du =$$

let $v = u^2 \Rightarrow \frac{dv}{du} = 2u = 2v^{\frac{1}{2}} \Rightarrow du = \frac{1}{2} v^{-\frac{1}{2}} dv$, $u=0 \Rightarrow v=0$, $u=\infty \Rightarrow v=\infty$

$\Rightarrow u = v^{\frac{1}{2}}$

$$\int_0^\infty \left(v^{\frac{1}{2}}\right)^{n-1} e^{-\left(\frac{r^2+n-1}{2}\right)v} \frac{1}{2} v^{-\frac{1}{2}} dv$$

$$= \frac{1}{2} \int_0^\infty v^{\frac{n-1}{2} - \frac{1}{2}} e^{-\frac{r^2+n-1}{2}v} dv$$

$$= \frac{1}{2} \int_0^\infty v^{\frac{n-2}{2} + \frac{2}{2} - 1} e^{-\frac{r^2+n-1}{2}v} dv$$

$$= \frac{1}{2} \int_0^\infty v^{\frac{n}{2} - 1} e^{-\frac{r^2+n-1}{2}v} dv$$

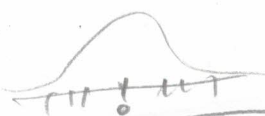
$$= \frac{\Gamma\left(\frac{n}{2}\right)}{2\left(\frac{r^2+n-1}{2}\right)^{n/2}}$$

$$\Rightarrow f_R(r) = \frac{\cancel{2}}{\sqrt{2\pi}} \frac{\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\cancel{2}} \left(\frac{r^2}{2} + \frac{n-1}{2}\right)^{-\frac{n}{2}} = \frac{2^{-\frac{1}{2}}}{\sqrt{\pi}} \frac{(n-1)^{\frac{n-1}{2}}\, 2^{-\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \Gamma\left(\frac{n}{2}\right) \frac{\left(\frac{n-1}{2}\right)^{-\frac{n}{2}}}{\left(\frac{n-1}{2}\right)^{-\frac{n}{2}}} \left(\frac{r^2}{2} + \frac{n-1}{2}\right)^{-\frac{n}{2}}$$

$$= \frac{2^{-\frac{1}{2}}}{\sqrt{\pi}} \frac{\cancel{(n-1)^{\frac{1}{2}}} (n-1)^{-\frac{1}{2}} \cancel{2^{-\frac{n}{2}}} \cancel{2^{\frac{1}{2}}}}{\Gamma\left(\frac{n-1}{2}\right)} \Gamma\left(\frac{n}{2}\right) \frac{(n-1)^{-\frac{n}{2}}}{\cancel{2^{-n/2}}} \left(\frac{\frac{r^2}{\cancel{2}}}{\frac{n-1}{\cancel{2}}} + 1 \right)^{-\frac{n-1+1}{2}}$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi(n-1)}\,\Gamma\left(\frac{n-1}{2}\right)} \left(1 + \frac{r^2}{n-1}\right)^{-\frac{n-1+1}{2}} = T_{n-1} := \text{Students } T(n-1)$$

Generally, $T_K := \dfrac{\Gamma\left(\frac{K+1}{2}\right)}{\sqrt{K\pi}\,\Gamma\left(\frac{K}{2}\right)} \left(1 + \dfrac{r^2}{K}\right)^{-\frac{K+1}{2}}$



$X_K \sim T_K$

$X_K \xrightarrow{d} ?$

$\dfrac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \xrightarrow{d} N(0,1)$ for any iid rv's seq $X_1, X_2, \ldots$

proved via Slutsky's

Hence... the T-distr. becomes more and more like $N(0,1)$. This is where the whole "if $n>30$, use Z-table" comes from

Let $V \sim \text{Gamma}(\alpha_1, \beta)$ indep. of $U \sim \text{Gamma}(\alpha_2, \beta)$

$$R = \frac{V}{U} \sim f_R(r) = \int_{\mathbb{R}} f_V(ru)\, f_U(u)\, |u|\, du$$

$$= \int_{\mathbb{R}} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (ru)^{\alpha_1 - 1} e^{-\beta r u} \underbrace{\mathbb{1}_{ru \in (0,\infty)}}_{r \in (0,\infty)} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} u^{\alpha_2 - 1} e^{-\beta u} |u| \underbrace{\mathbb{1}_{u \in (0,\infty)}}_{u>0} du$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1 - 1} \mathbb{1}_{r \in (0,\infty)} \int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-\beta(1+r)u}\, du$$

$$= \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} r^{\alpha_1-1} \mathbb{1}_{r\in(0,\infty)} \frac{\Gamma(\alpha_1+\alpha_2)}{\underbrace{(\beta(1+r))^{\alpha_1+\alpha_2}}_{\cancel{\beta^{\alpha_1+\alpha_2}}(1+r)^{\alpha_1+\alpha_2}}}$$

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{r^{\alpha_1-1}}{(1+r)^{\alpha_1+\alpha_2}} \mathbb{1}_{r\in(0,\infty)} = \text{BetaPrime}(\alpha_1, \alpha_2)$$