$X, Y \overset{iid}{\sim} \text{Geom}(p)$

$$P(X>Y) = \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} p_{X,Y}(x,y) \mathbb{1}_{x>y}$$

$$= \sum_{x\in\mathbb{R}} \sum_{y\in\mathbb{R}} p_X(x)\, p_Y(y) \mathbb{1}_{x>y}$$

$$= \sum_{x\in S_X} \sum_{y\in S_Y} p_X^{old}(x)\, p_Y^{old}(y) \mathbb{1}_{x>y}$$

$$= \sum_{x\in\mathbb{N}_0} \sum_{y\in\mathbb{N}_0} (1-p)^x p\, (1-p)^y p\, \mathbb{1}_{x>y}$$

$$= p^2 \sum_{y\in\mathbb{N}_0} (1-p)^y \underbrace{\sum_{x\in\{y+1, y+2, \ldots\}} (1-p)^x}_{\text{looks "close" to geometric series}}$$

let $i := x-(y+1)$ then $i \in \{0, 1, \ldots\} = \mathbb{N}_0$

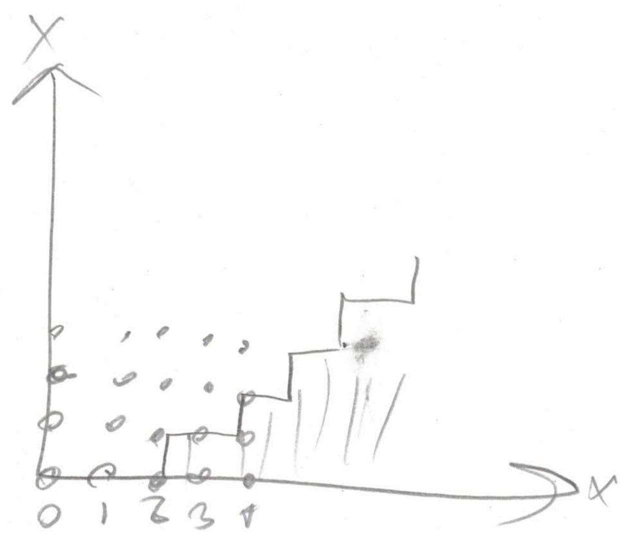
$\Rightarrow x = i+y+1$

$$= p^2 \sum_{y\in\mathbb{N}_0} (1-p)^y \sum_{i\in\mathbb{N}_0} (1-p)^{i+y+1}$$

$$= \cancel{p^2} \sum_{y\in\mathbb{N}_0} (1-p)^{2y+1} \underbrace{\sum_{i\in\mathbb{N}_0} (1-p)^i}_{\frac{1}{1-(1-p)} = \frac{1}{\cancel{p}}}$$

$$= p(1-p) \underbrace{\sum_{y\in\mathbb{N}_0} \left((1-p)^2\right)^y}_{\frac{1}{1-(1-p)^2} = \frac{1}{\cancel{1}-\cancel{1}+2p-p^2}} = \frac{\cancel{p}(1-p)}{\cancel{p}(2-p)} = \frac{1-p}{2-p}$$

as $p \to 0$, $P(X>Y) \to \frac{1}{2}$



CDF of Geometric (corrected)

$B_1, B_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$ # of zeroes

$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \ldots\ \underline{\quad}\ \underline{\quad}$

$1 \quad 2 \quad 3 \quad \ldots \quad x \quad x+1$

$$S(x) = P(X > x) = P(X \geq x+1)$$

x+1 zeroes or more before the first one

$$= \underbrace{P(X \geq x+1, X = x+1) + P(X \geq x+1, X = x+2) + P(X \geq x+1, X = x+3) + \ldots}$$

$$P(B_1 = 0, B_2 = 0, \ldots, B_{x+1} = 0) = (1-p)^{x+1}$$

$$\Rightarrow F(x) = 1 - (1-p)^{x+1}$$

By the book:

let $j := i-(x+1) \Rightarrow i = j+x+1$

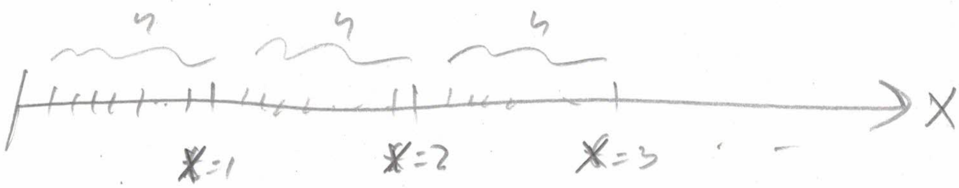
$$S(x) = P(X \geq x+1) = \sum_{i \in \{x+1, x+2, \ldots\}} p(1-p)^x = p \sum_{j \in \mathbb{N}_0} (1-p)^{j+x+1} = \cancel{p}(1-p)^{x+1} \underbrace{\sum_{j \in \mathbb{N}_0} (1-p)^j}_{\frac{1}{\cancel{1-(1-p)}}}$$

$$= (1-p)^{x+1}$$

$X \sim \text{Geom}(p)$

Imagine X is time on a clock, maybe every "second". Measure "survival in seconds".

What if instead, we had $n \overset{iid}{\sim} \text{Bern}(p)$ per second:



$\Rightarrow F_{X_n}(x) = 1-(1-p)^{nx+1}$

Let X_n denote stopping time $X_n \sim p_{X_n}(x) = (1-p)^{nx}\, p\, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1, 1+\frac{1}{n}, 1+\frac{2}{n}, \dots\}}$

The process "stops" almost immediately if n is large and p is modest

But what if p was very small? $p \to 0,\ n \to \infty$ s.t. $\lambda = pn > 0$

$\Leftrightarrow p = \frac{\lambda}{n}$

And we substitute...

$$p_{X_n}(x) = \left(1-\frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n}\, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \dots\}}$$

Now we let $n \to \infty$ effectively running infinite Bernoullis in each second:

$$(x) = \lim_{n\to\infty} p_{X_n}(x) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n}\, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \dots\}}$$

Recall from calculus... $\lim_{n\to\infty} f(n)g(n) = \lim_{n\to\infty} f(n) \lim_{n\to\infty} g(n)$

$\lim_{n\to\infty} g(f(n)) = g\left(\lim_{n\to\infty} f(n)\right)$

$$= \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nx} \lim_{n\to\infty} \frac{\lambda}{n} \lim_{n\to\infty} \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \dots\}}$$

$\lim f(n) + g(n)$
$= \lim f(n) + \lim g(n)$

$$= \Bigg(\underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}}_{e^{-\lambda}}\Bigg)^{x}\left(\lim_{n\to\infty}\frac{\lambda}{n}\right)\lim_{n\to\infty}\mathbb{1}_{x\in\{0,\frac{1}{n},\frac{2}{n},\dots\}}$$

continuous support

$$= \left(e^{-\lambda x}\right)(0)\,\mathbb{1}_{x\in[0,\infty)}$$

$$|S_{X_\infty}| \neq |\mathbb{N}|$$

ctbl ∞

$= 0$ Not a PMF!!!

No longer a discrete rv

$$|S_{X_\infty}| = |\mathbb{R}|$$

unctbl ∞

$\Rightarrow$ We're in trouble!

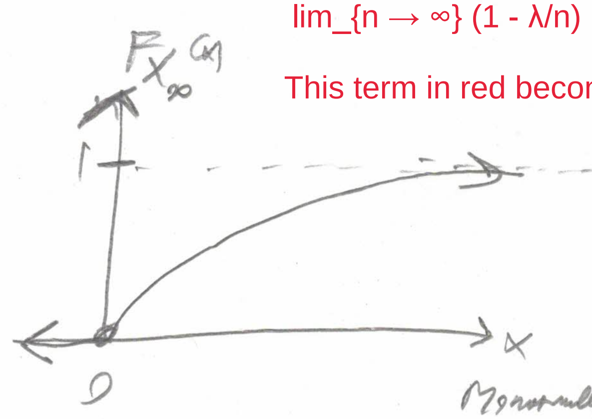
Let's take a look at the CDF...

$$F_{X_\infty}(x) = \lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nx+1} = 1-\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}\right)^{x} \quad *$$

$\lim_{n\to\infty}(1-\lambda/n)$

This term in red becomes 1

$$= 1-e^{-\lambda x}$$



Is this a valid CDF?

~~1) $F(x)\in[0,1]$ since it's a prob~~

~~If $x<0 \Rightarrow F(x)=0$~~

~~If $x>0 \Rightarrow F(x)\in[0,1]$ since $1-e^{-\lambda x}\in[0,1]$~~

Monotonically incr.

③ $\frac{d}{dx}F(x) \geq 0$

$\lambda e^{-\lambda x} > 0$ for all $x \geq 0$ and $=0$ for all $x<0$ ✓

1) $\lim_{x\to-\infty} F(x) = 0$ since $x<0 \Rightarrow F(x)=0$

2) $\lim_{x\to\infty} F(x) = 1$ $\quad \lim_{x\to\infty} 1-e^{-\lambda x} = 1-\lim_{x\to\infty} e^{-\lambda x} = 1-0 = 1$ ✓

$\Rightarrow$ It's a valid CDF

Def: a cont rv X has $|S_X| = |\mathbb{R}|$ and no PMF,

the PMF is $p(x) = 0 \ \forall x$ $\quad P(X = x) = 0$!!

They have CDF's. The deriv. of the CDF

(how fast the rv collects prob at any x) is useful:

$f_X(x) := \frac{d}{dx}[F_X(x)]$ is called the rv's prob "density" function (PDF)

Why density? It's the density of prob in any given region:

$$a < b, \ P(X \in (a, b)) = F(b) - F(a) \overset{FTC}{=} \int_a^b f_X(x)\,dx \Rightarrow$$

$$P(X \in (-\infty, b]) = F(b) - \cancel{F(-\infty)}^{0} = \int^b f_X(x)\,dx$$

Also, many times the PDF is avail. in closed form ~~but~~ the CDF is not! Properties:

$$1 = P(X \in (-\infty, \infty)) = \int_{\mathbb{R}} f_X(x)\,dx$$ the Humpty-Dumpty ~~formula~~ for cont. rv's.

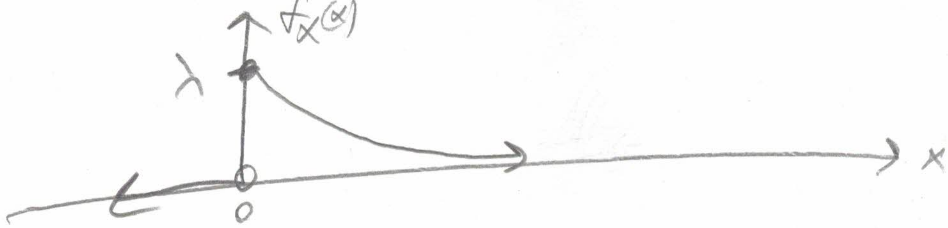
$f_X(x) \geq 0$ since $F_X(x)$ is monotonically

Note: f_X can be > 0. It's not a prob like the PMF!!!

$$S_X = \{x : f(x) > 0\}$$

The X_{90} we discussed is a famous brand name rv., the exponential

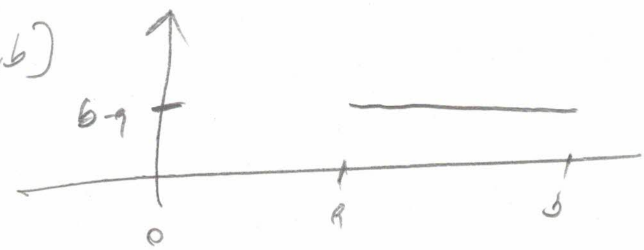
$$X \sim \text{Exp}(\lambda) := \underbrace{\overbrace{\lambda e^{-\lambda x}}^{f_X(x)}}_{[illegible]} \mathbb{1}_{x \geq 0} \qquad \lambda \in (0, \infty) \quad \text{same } p \in (0,1),\ n \in \mathbb{N}$$



Cont.

Another famous cont rv is the uniform r.v. or just "uniform"

$X \sim U(a,b) := \frac{1}{b-a} \mathbb{1}_{x \in [a,b]}$



$S_X = [a,b]$, $a \in \mathbb{R}, b \in \mathbb{R}, a < b$

Valid density? $\int_{\mathbb{R}} \frac{1}{b-a} \mathbb{1}_{x \in [a,b]} = \int_a^b \frac{1}{b-a} dx = \frac{1}{b-a} (x)_a^b = \frac{b-a}{b-a} = 1$ ✓

$\forall x \; f(x) \geq 0$ If $x \in (-\infty, a) \Rightarrow f(x) = 0$, $x \in [a,b] \Rightarrow f(x) = b-a > 0$
$x \in (b, \infty) \Rightarrow f(x) = 0$ ✓

If $a = 0, b = 1 \Rightarrow X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$ the standard uniform

Very important r.v. especially in comp. science!

if $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$ $\quad f_{\vec{X}}(\vec{x}) = f_{X_1,\dots,X_n}(x_1,\dots,x_n)$ joint density function (JDF)

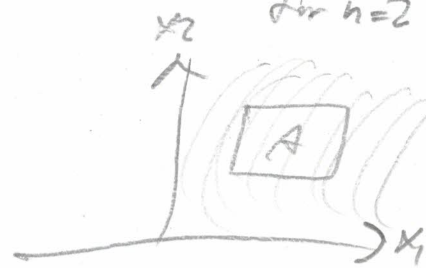
if $X_1 \dots X_n$ ind $\Rightarrow$ $= f_{X_1}(x_1)\, f_{X_2}(x_2) \cdots f_{X_n}(x_n)$

if $X_1,\dots,X_n \overset{iid}{\sim}$ $\Rightarrow$ $= f(x_1)\, f(x_2) \cdots f(x_n)$

$$\int_{\mathbb{R}^n} f_{\vec{X}}(\vec{x})\, d\vec{x} = 1$$ Humpty Dumpty

To get prob's from a JDF, you need to integrate over a region $P(\vec{X} \in A) = \int_A f_{\vec{X}}(\vec{x})\, d\vec{x}$

for $n=2$



Volume under blanket in A

A calc. fact: Leibnitz's Rule

$$\frac{d}{dt}\int_{a(t)}^{b(t)} g(x,y)\,dy = g(x,b(t))\,b'(t) + g(x,a(t))\,a'(t)$$

If $b(t) = t$, $a(t) = -\infty$

$$\frac{d}{dt}\int_{-\infty}^{t} g(x,y)\,dy = g(x,t)$$ since $a(t)$ is a const $\Rightarrow a'(t) = 0$

let X_1, X_2 be two cont. rv's

$$T = X_1 + X_2 \sim f_T(t) = ?$$

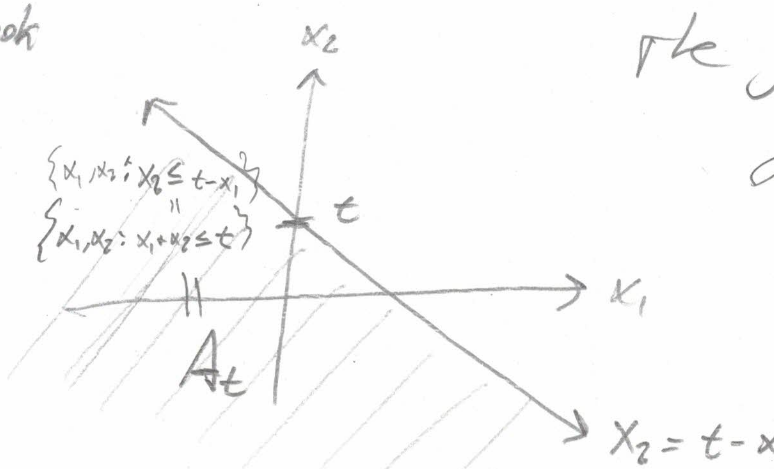
Method 1

Note: if $F_T(t)$ is known $\Rightarrow f_T(t) = \frac{d}{dt}\left[F_T(t)\right]$ what if not known?

Method 2

We want to derive a general formula like before as a function of the jdf or densities (if indep.)

p145 of the textbook



$$F_T(t) = P(T \le t) = P(\vec{X} \in A_t) = \int_A f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2 = \int_{x_1 \in \mathbb{R}} \int_{x_2=-\infty}^{x_2 = t - x_1} f_{X_1,X_2}(x_1,x_2)\,dx_2\,dx_1$$

~~We need to make this look like Leibniz's rule so~~ $\cancel{\int_{-\infty}^{t} \ldots}$

~~To do so we need to reindex similar to what we did with the sums for the geometric problem with~~ $\cancel{P(X > k)}$

~~let~~ $\cancel{X_2 = v - X_1 \Rightarrow v = x_2 + x_1 \Rightarrow dv = dx_2}$

$V = X_2 + X_1$

$\Uparrow$

let $x_2 = V - X_1$ which is a reindexing trick like we did with the $P(X > Y)$ geometric iid problem

$\Rightarrow X_2 = -\infty \Rightarrow V = -\infty$

$\Rightarrow X_2 = t - X_1 \Rightarrow V = t$

$\frac{dx_2}{dv} = 1 \Rightarrow dv = dx_2$

drop subscript on $x_1 \Rightarrow X$

$$= \int_{X \in \mathbb{R}} \int_{v=-\infty}^{v=t} f_{X_1, X_2}(x, v-x)\, dv\, dx$$

free variable of integration

$$F_T(t) = \int_{-\infty}^{t} \underbrace{\left(\int_{X \in \mathbb{R}} f_{X_1, X_2}(x, v-x)\, dx \right)}_{\text{density}} dv$$

Recall

$$P(X \in (-\infty, b]) = F(b) - \cancel{F(-\infty)}^{0}$$

$$= \int_{-\infty}^{b} f_X(x)\, dx$$

$$\Rightarrow f_T(t) = \int_{x \in \mathbb{R}} f_{X_1, X_2}(x, t-x)\, dx$$

general conv. formula

Any integral that looks like this means density is within!

$$F(x) = \int_{-\infty}^{x} f_X(v)\, dv$$

this is the density

free variable of integration