$X, Y \overset{iid}{\sim} \text{Exp}(1) = e^{-x}\mathbb{1}_{x\in(0,\infty)}$

$D := X - Y \sim f_D(d) = ?$

let $Z = -Y \sim e^{--y}\mathbb{1}_{-y\in(0,\infty)} = e^{y}\mathbb{1}_{y\in(-\infty,0)}$, $S_Z = (-\infty,0)$

$$D = X + Z \sim \int_{x\in S_X} f_X(x)\, f_Z(d-x)\,\mathbb{1}_{d-x\in S_Z}\, dx$$

$$= \int_0^\infty e^{-x} e^{d-x} \underbrace{\underbrace{\mathbb{1}_{d-x\in(-\infty,0)}}_{x-d\in(0,\infty)}}_{x\in(d,\infty)} dx$$

$$= e^d \begin{cases} \int_0^\infty e^{-2x}dx & \text{if } d\le 0 \\ \int_d^\infty e^{-2x}dx & \text{if } d>0 \end{cases} = -\frac{e^d}{2}\begin{cases} \left[e^{-2x}\right]_0^\infty & \text{if } d\le 0 \\ \left[e^{-2x}\right]_d^\infty & \text{if } d>0 \end{cases}$$

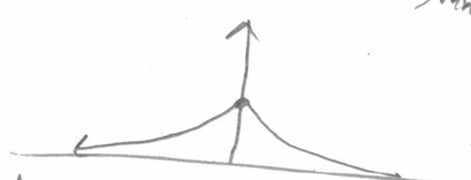
$$= -\frac{e^d}{2}\begin{cases} -1 & \text{if } d\le 0 \\ -e^{-2d} & \text{if } d>0 \end{cases} = \frac{1}{2}\begin{cases} e^{d} & \text{if } d\le 0 \\ e^{-d} & \text{if } d>0 \end{cases} = \frac{1}{2}e^{-|d|} = \text{Laplace}(0,1)$$

"or Double Exponential"

Standard Laplace

$E[D] = E[X] - E[Y] = 1 - 1 = 0$

$Var[D] = Var[X] + Var[Y] = 1 + 1 = 2$



let $X \sim \text{Laplace}(0,1)$, $Y = \mu + \sigma X \sim \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}$

where $\mu\in\mathbb{R}$, $\sigma>0$

CDF &

Hw: gen'l CDF

Laplace came up with this distr in 1774 and called it the "first law of errors". What was he trying to solve?

Imagine you want to measure a value μ but you measure it with error ε. So you observe $m = \mu + \varepsilon$.
What is the distr of errors? $M = \mu + \varepsilon$, $\varepsilon \sim ?$

He made a few assumptions...

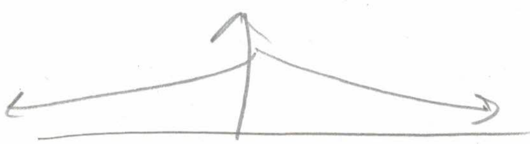
$E[M] = \mu$

① $E[\varepsilon] = 0$. So on average you measure μ ("unbiased")

② $f_\varepsilon(\varepsilon) = f_\varepsilon(-\varepsilon)$. A size $+\varepsilon$ error is equally likely as a size $-\varepsilon$ error (symmetric around zero)

$\Rightarrow$ 50% errors > 0, 50% errors < 0.

③ $f'(\varepsilon) < 0$ for $\varepsilon > 0$ and $f'(\varepsilon) > 0$ for $\varepsilon < 0$.



Higher magnitude errors are less likely than lower magnitude errors

There are many distr's that satisfy ①,②,③... Normal, Logistic

Laplace then considered if $f''(\varepsilon) = f'(\varepsilon) \Rightarrow f(\varepsilon) = c e^{-d\varepsilon}$

Solving for constants for Humpty Dumpty $\Longrightarrow$ $\varepsilon \sim \text{Laplace}(0,1)$.

Second Law of Errors (1778)

$f(\varepsilon) = c e^{-d\varepsilon^2} \Rightarrow f(\varepsilon) = N(0,1)$

Solving for constants for Humpty-Dumpty

Another one...

$$X \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad Y = e^X \iff X = \ln(Y) = g^{-1}(Y)$$

$$\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right| = \frac{1}{|y|}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2} \frac{1}{|y|} \underbrace{\mathbb{1}_{\ln(y) \in \mathbb{R}}}_{\mathbb{1}_{y \in (0,\infty)}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 y^2}} e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2}$$

$$= \text{LogN}(\mu, \sigma^2) \text{ "the log-normal"}$$

Since if you transform $X = \ln(Y) \Rightarrow X \sim N(\mu, \sigma^2)$

It can easily have been called "exponentiated-normal"

$X_1, \dots, X_k \overset{iid}{\sim} \text{Exp}(\lambda)$, $\lambda > 0$ another waiting time distr. wait for k Exp(λ)'s to realize..

$X = X_1 + \dots + X_k \sim \text{Erlang}(k, \lambda)$, $k \in \mathbb{N}$, $\lambda > 0$

$$= \frac{1}{(k-1)!} \lambda^k e^{-\lambda x} x^{k-1} \mathbb{1}_{x \in (0,\infty)}$$

Is this a valid PDF? Yes, it came from the convolution formula. Let's investigate its Humpty-Dumpty identity below:

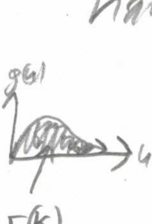
$$1 = \int_0^\infty \frac{1}{(k-1)!} \lambda^k e^{-\lambda x} x^{k-1} \, dx$$

$$= \frac{\lambda^k}{(k-1)!} \int_0^\infty e^{-\lambda x} x^{k-1} \, dx = \frac{\lambda^k}{(k-1)!} \int_0^\infty e^{-u} \left(\frac{u}{\lambda}\right)^{k-1} \frac{1}{\lambda} \, du = \frac{\cancel{\lambda^k}}{(k-1)!} \frac{1}{\cancel{\lambda^k}} \int_0^\infty e^{-u} u^{k-1} \, du$$

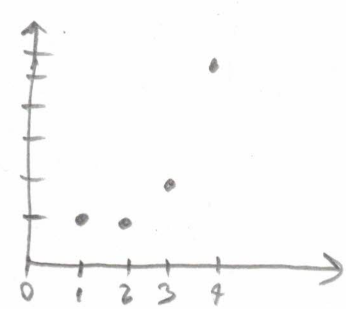
let $u = \lambda x \Rightarrow x = \frac{u}{\lambda} \Rightarrow \frac{du}{dx} = \lambda \Rightarrow dx = \frac{1}{\lambda} du$, $x = 0 \Rightarrow u = 0$, $x = \infty \Rightarrow u = \infty$

$$\Longrightarrow (k-1)! = \int_0^\infty e^{-u} u^{k-1} \, du \quad \text{very interesting identity!}$$

The integral on the rhs is very common, so it gets its own name and symbol, the gamma function:



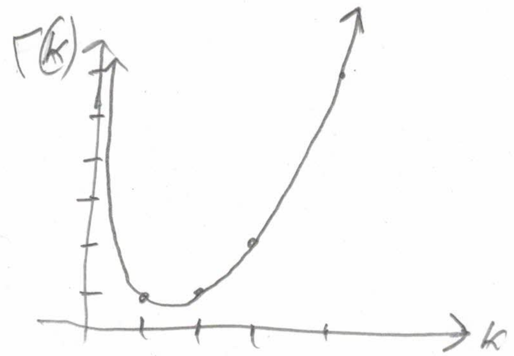
$$\Gamma(k) := \int_0^\infty \overbrace{e^{-u} u^{k-1}}^{g(u)} \, du$$



Very cool function! Note: $\int e^{-u} u^{k-1} du$ has no closed form antiderivative

For the purposes of this class,
The Γ function exists for $k \in (0,\infty)$. At $k=0$, the integral absolutely diverges. (It also exists for $k \in (-\infty,0) \setminus \mathbb{Z}$ but we are not interested in negative k values in this class.)

Using numerical integration, you can show the function looks like:



And you can prove it's strictly increasing from $(\approx 1.46, \infty)$

Why not use this function to extend the factorial function to all $k \in (0,\infty)$? We can!

The major property of the gamma function.

Just like $k! = k(k-1)!$ $\forall k \in \mathbb{N}$ (undefined elsewhere)

So too $\Gamma(k+1) = k\,\Gamma(k)$ $\forall k > 1$ Proof:

$$\Gamma(k) := \int_0^\infty \underbrace{e^{-u}}_{u}\, \underbrace{u^{k-1}\,du}_{dv} \overset{\text{integration by parts}}{=} [uv]_0^\infty - \int_0^\infty v\,du = \left[\frac{e^{-u}u^k}{k}\right]_0^\infty - \int_0^\infty \frac{u^k}{k}\left(-e^{-u}\right)du$$

$$= (0-0) - -\frac{1}{k}\int_0^\infty e^{-u}u^{(k+1)-1}\,du = \frac{1}{k}\Gamma(k+1) \Rightarrow \Gamma(k+1) = k\,\Gamma(k)$$

Rewriting the PDF of the Erlang using the gamma function,

$$X \sim \text{Erlang}(k,\lambda) = \frac{1}{\Gamma(k)}\lambda^k e^{-\lambda x} x^{k-1}\,\mathbb{1}_{x \in (0,\infty)}$$

this PDF integrates to 1 $\forall k > 0$ not only $k \in \mathbb{N}$.

We've now augmented the parameter space of the Erlang. This new rv historically gets a new name: the Gamma Distr!

$$X \sim \text{Gamma}(\alpha, \beta) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} e^{-\beta x} x^{\alpha - 1} \mathbb{1}_{x \in (0,\infty)}$$

where $\alpha > 0$, $\beta > 0$

Why is it called the gamma? (Verifying the Hypsy-Dupsy property uses the gamma function)

Some more definitions. Let $a > 0$

$$\Gamma(k) = \int_0^\infty e^{-u} u^{k-1}\, du = \underbrace{\int_0^a e^{-u} u^{k-1}\, du}_{\gamma(k,a)} + \underbrace{\int_a^\infty e^{-u} u^{k-1}\, du}_{\Gamma(k,a)}$$

$\gamma(k,a)$ lower incomplete gamma function

$\Gamma(k,a)$ upper incomplete gamma funct

$$\Rightarrow \Gamma(k) = \gamma(k,a) + \Gamma(k,a)$$

$$\Rightarrow 1 = \underbrace{\frac{\gamma(k,a)}{\Gamma(k)}}_{P(k,a)} + \underbrace{\frac{\Gamma(k,a)}{\Gamma(k)}}_{Q(k,a)} \Rightarrow 1 = P(k,a) + Q(k,a)$$

$P(k,a)$ lower regularized gamma function

$Q(k,a)$ upper regularized gamma function

$\lim_{a\to\infty} \gamma(k,a) = \Gamma(k)$ $\quad \lim_{a\to\infty} \Gamma(k,a) = 0$

$\lim_{a\to 0} \gamma(k,a) = 0$ $\quad \lim_{a\to 0} \Gamma(k,a) = \Gamma(k)$

$\lim_{a\to\infty} P(k,a) = 1$ $\quad \lim_{a\to\infty} Q(k,a) = 0$

$\lim P(k,a) = 0$ $\quad \lim Q(k,a) = 1$

Useful gamma-family integrals:

$u=\frac{v}{c}$, $u=q \Rightarrow v=qc$

let $v=cu \Rightarrow \frac{dv}{du}=c \Rightarrow du=\frac{1}{c}dv$, $u=0\Rightarrow v=0$, $u=\infty \Rightarrow v=\infty$

$$\int_0^\infty u^{k-1}e^{-cu}\,du = \int_0^\infty \left(\frac{v}{c}\right)^{k-1} e^{-v}\frac{1}{c}\,dv = \frac{1}{c^k}\int_0^\infty v^{k-1}e^{-v}\,du = \frac{\Gamma(k)}{c^k}$$

$$\int_0^q u^{k-1}e^{-cu}\,du = \int_0^{qc}\left(\frac{v}{c}\right)^{k-1}e^{-v}\frac{1}{c}\,dv = \frac{1}{c^k}\int_0^{qc} v^{k-1}e^{-v}\,du = \frac{\gamma(k,qc)}{c^k}$$

$$\int_q^\infty u^{k-1}e^{-cu}\,du = \int_0^\infty u^{k-1}e^{-cu}\,du - \int_0^q u^{k-1}e^{-cu}\,du = \frac{\Gamma(k)-\gamma(k,qc)}{c^k} = \frac{\Gamma(k,qc)}{c^k}$$

$X\sim \mathrm{gamma}(\alpha,\beta)$

$$F_X(x) = \int_0^x \frac{1}{\Gamma(\alpha)}\beta^\alpha e^{-\beta y}y^{\alpha-1}\,dy = \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}e^{-\beta y}\,dy = \frac{\cancel{\beta^\alpha}}{\Gamma(\alpha)}\frac{\gamma(\alpha,\beta x)}{\cancel{\beta^\alpha}} = P(\alpha,\beta x)$$

$S_X(x) = 1-F_X(x) = Q(\alpha,\beta x)$

Let $n\in\mathbb{N}$

$$\Gamma(n,q) = \int_q^\infty \underbrace{u^{n-1}}_{q}\,\underbrace{e^{-u}\,du}_{dr} = [qr]_q^\infty - \int_q^\infty r\,dq$$

$$= \left[u^{n-1}(-e^{-u})\right]_q^\infty - \int_q^\infty (-e^{-u})(n-1)u^{n-2}\,du$$

$$= q^{n-1}e^{-q} + (n-1)\int_q^\infty e^{-u}u^{n-2}\,du$$

$$= q^{n-1}e^{-q} + (n-1)\Gamma(n-1,q)$$

$$= q^{n-1}e^{-q} + (n-1)\left(q^{n-2}e^{-q} + (n-2)\Gamma(n-2,q)\right)$$

$$= q^{n-1}e^{-q} + (n-1)q^{n-2}e^{-q} + (n-1)(n-2)\Gamma(n-2,q)$$

$$= e^{-q}\left(q^{n-1} + (n-1)q^{n-2} + (n-1)(n-2)q^{n-3} + (n-1)(n-2)(n-3)q^{n-4} + \dots + (n-1)!\,q^0\right)$$

$$\Gamma(1,q) = \int_q^\infty e^{-u}\,du = [-e^{-u}]_q^\infty = e^{-q}$$

$$= e^{-q}\,\frac{(n-1)!}{(n-1)!}\left(q^{n-1} + (n-1)q^{n-2} + \dots + (n-1)!\,q^{0}\right)$$

$$= e^{-q}(n-1)!\left(\frac{q^{n-1}}{(n-1)!} + \frac{q^{n-2}}{(n-2)!} + \dots + \frac{q^{0}}{0!}\right)$$

$$= e^{-q}(n-1)! \sum_{i=0}^{n-1} \frac{q^{i}}{i!} = \Gamma(n,q)$$

$$\Rightarrow \Gamma(n+1,q) = e^{-q}\, n! \sum_{i=0}^{n} \frac{q^{i}}{i!}$$

Recall the Negative Binomial $\quad \frac{(k+t-1)!}{(k-1)!\,t!}$ $\quad \parallel$

$X_1, \dots, X_{k} \stackrel{iid}{\sim} \text{Geom}(p)$

$$X = X_1 + \dots + X_k \sim \text{NegBin}(k,p) = \binom{k+t-1}{k-1}(1-p)^{t}p^{k}\,\mathbb{1}_{t\in\mathbb{N}_0}$$

We are waiting for k geometric rv's to realize. Can $k=3.5$? No... but conceptually it makes sense... Can we extend $k\in\mathbb{N} \to k\in(0,\infty)$? Yes:

$$X \sim \text{ExtNegBin}(k,p) = \frac{\Gamma(k+t)}{\Gamma(k)\,t!}(1-p)^{t}p^{k}\,\mathbb{1}_{t\in\mathbb{N}_0}$$

extended negative binomial

$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \mathbb{1}_{z \in \mathbb{R}}$, $Y = Z^2 = g(Z)$ not 1:1! We cannot use our formula!!

How to solve... let's go back to CDF's...

$$F_Y(y) = P(Y \le y) = P(Z^2 \le y) = P\left(Z \in (-\sqrt{y}, \sqrt{y})\right)$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

by symmetry around zero of $f_Z(z)$

$$= 2\left(F_Z(\sqrt{y}) - \underbrace{F_Z(0)}_{\frac{1}{2}}\right) = 2F_Z(\sqrt{y}) - 1$$

$$f_Y(y) = \frac{d}{dy}\left[2F_Z(\sqrt{y}) - 1\right] = 2 \cdot \frac{1}{2} y^{-\frac{1}{2}} f_Z(\sqrt{y}) = y^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{y})^2} \mathbb{1}_{\underbrace{\sqrt{y} \in \mathbb{R}}_{y \in (0,\infty)}}$$

$$= \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \mathbb{1}_{y \in (0,\infty)} = \chi^2_1$$

"Chi-Squared with 1 degree of freedom"

the parameter is called degrees of freedom and it's denoted as a subscript

$= \underbrace{\text{ChiSq}(1)}$

the notation you're used to