MATH 340/640 Lec 17

$$X \sim \text{Weibull}(k, \lambda) := (k\lambda)(\lambda x)^{k-1} e^{-(\lambda x)^k} \mathbb{1}_{x \in (0,\infty)}$$

$$= \underbrace{k\lambda^k}_{c} \underbrace{x^{k-1} e^{-(\lambda x)^k} \mathbb{1}_{x \in (0,\infty)}}_{k(x)}$$

$$X \sim \text{Gamma}(\alpha, \beta) := \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{c} \underbrace{x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \in (0,\infty)}}_{k(x)}$$

$$X \sim \text{Logistic}(0,1) := \underbrace{\frac{e^{-x}}{(1+e^{-x})^2}}_{k(x)}, \quad c = 1$$

$$X \sim \text{Laplace}(\mu, \sigma) := \underbrace{\frac{1}{2\sigma}}_{c} \underbrace{e^{-\frac{|x-\mu|}{\sigma}}}_{k(x)}$$

$$x^a e^{-bx} \mathbb{1}_{x \in (0,\infty)} \overset{?}{\propto} \text{Gamma}(a+1, b), \quad e^{-d|x|} \overset{?}{\propto} \text{Laplace}\left(0, \frac{1}{d}\right)$$

This is how we use it to solve problems... we eliminate constants to find kernels then hope we can recognize the kernels.

E.g. let $X_1 \sim \text{Gamma}(\alpha_1, \beta)$ indep. of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$
$\alpha_1, \alpha_2, \beta > 0$
we know from using ch.f.'s that $T = X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$.

Let's use the convolution formula. We'll discover something interesting

$$f_T(t) = \int_0^\infty f_{X_1}^{\text{old}}(x)\, f_{X_2}^{\text{old}}(t-x)\, \mathbb{1}_{t-x\in(0,\infty)}\, dx$$

$$= \int_0^\infty \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1-1} \cancel{e^{-\beta x}}\, \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (t-x)^{\alpha_2-1} \underbrace{e^{-\beta(t-x)}}_{e^{-\beta t}\cancel{e^{\beta x}}}\, \mathbb{1}_{\underbrace{\underbrace{t-x\in(0,\infty)}_{x-t\in(-\infty,0)}}_{x\in(-\infty,t)}}\, dx$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta t}\, \mathbb{1}_{t\in(0,\infty)} \int_0^t x^{\alpha_1-1}(t-x)^{\alpha_2-1}\, dx$$

$$\propto e^{-\beta t}\, \mathbb{1}_{t\in(0,\infty)} \int_0^1 x^{\alpha_1-1}(t-x)^{\alpha_2-1}\, dx$$

let $u = \frac{x}{t} \Rightarrow x = ut \Rightarrow \frac{dx}{du} = t \Rightarrow dx = t\,du \Rightarrow x = 0 \Rightarrow u = 0,\ x = t \Rightarrow u = 1$

$$= e^{-\beta t}\, \mathbb{1}_{t\in(0,\infty)} \int_0^\infty (ut)^{\alpha_1-1}(t-ut)^{\alpha_2-1}\, t\, du$$

$$= e^{-\beta t}\, t^{\alpha_1-1}\, t^{\alpha_2-1}\, t\, \mathbb{1}_{t\in(0,\infty)} \underbrace{\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\, du}_{g(\alpha_1,\alpha_2)}$$

killer u substitution!!

$$\propto e^{-\beta t}\, t^{\alpha_1+\alpha_2-1}\, \mathbb{1}_{t\in(0,\infty)}$$

$g(\alpha_1,\alpha_2)$ i.e. not a function of t and $g(\alpha_1,\alpha_2) < \infty \ \forall \alpha_1,\alpha_2 > 0$

$$\propto \text{Gamma}(\alpha_1+\alpha_2, \beta)$$

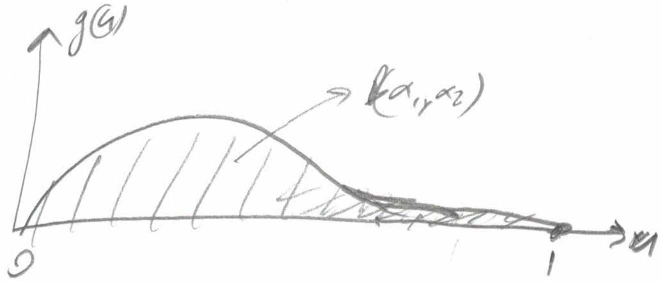
Easier than without using kernels!!

Now let's examine this integral; it turns out it's famous... It's called the "Beta Function"!

$$B(\alpha_1,\alpha_2) := \int_0^1 \underbrace{u^{\alpha_1-1}(1-u)^{\alpha_2-1}}_{g(u)}\,du$$

No closed form expression. Need computer to approximate

finite for all $\alpha_1 > 0,\ \alpha_2 > 0$



Rehydrating the constants, we know the rhs. The lhs is by definition

$$\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2-1} e^{-\beta t} \mathbb{1}_{t\in(0,\infty)} = \text{Gamma}(\alpha_1+\alpha_2,\beta) = \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} B(\alpha_1,\alpha_2)\, t^{\alpha_1+\alpha_2-1} e^{-\beta t} \mathbb{1}_{t\in(0,\infty)}$$

$$\Rightarrow B(\alpha_1,\alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$$, the beta-gamma function identity

$$\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\,du = \frac{\int_0^\infty u^{\alpha_1-1}e^{-u}\,du \int_0^\infty u^{\alpha_2-1}e^{-u}\,du}{\int_0^\infty u^{\alpha_1+\alpha_2-1}e^{-u}\,du}$$

Really? That isn't obvious at all!

$$\Rightarrow \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} = \frac{1}{B(\alpha_1,\alpha_2)}$$

Using this identity, we can now provide the usual PDF expressions for BetaPrime and FisherSnedecor distr's:

$$R \sim \text{BetaPrime}(\alpha_1, \alpha_2) = \frac{1}{B(\alpha_1, \alpha_2)} \frac{r^{\alpha_1 - 1}}{(1+r)^{\alpha_1+\alpha_2}} \mathbb{1}_{r \in (0,\infty)}$$

$$R \sim F_{k_1, k_2} = \frac{1}{B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} \left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}} r^{\frac{k_1}{2}-1} \left(1 + r\frac{k_1}{k_2}\right)^{-\frac{k_1+k_2}{2}} \mathbb{1}_{r \in (0,\infty)}$$

Also...

Consider $k(x) = x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x \in [0,1]}$ with $\alpha, \beta > 0$

What is $f(x)$? Solve for $c = \frac{1}{\int_{\mathbb{R}} x^{\alpha-1}(1-x)^{\beta-1} \mathbb{1}_{x\in(0,1)}\, dx} = \frac{1}{B(\alpha,\beta)}$

$$\Rightarrow f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx = \text{Beta}(\alpha, \beta) \text{ distribution}$$

It gets its name from the Beta function see prove the Hungry-Dupey!

Recall last class $X_1, \ldots X_n \overset{iid}{\sim} U(0,1)$

$$X_{(k)} \sim f_{X_{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!} x^{k-1}(1-x)^{n-k} \mathbb{1}_{x\in(0,1)}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\,\Gamma(n-k+1)} x^{k-1}(1-x)^{n-k} \mathbb{1}_{x\in(0,1)}$$

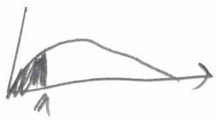
$$= \frac{1}{B(k, n-k+1)} x^{k-1}(1-x)^{n-k} \mathbb{1}_{x\in(0,1)} = \text{Beta}(k, n-k+1)$$

The Beta distr is the distr of the order statistics of standard uniforms!!

Just like the gamma function, the beta function has some related functions:

let $a \in (0,1)$, $B(a, \alpha, \beta) = \int_0^a u^{\alpha-1}(1-u)^{\beta-1}\,du$ the "incomplete beta function"

there's also "lower" and "upper"



$a \in (0,1)$ $I_a(\alpha, \beta) := \dfrac{B(a, \alpha, \beta)}{B(\alpha, \beta)} \in [0,1]$, the prop. of the area $\leq a$.

Regularized
Incomplete Beta Function

$$X \sim \text{Beta}(\alpha, \beta), \quad F(x) = P(X \leq x) = \int_0^x \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}\,dy$$

$$= \frac{B(x, \alpha, \beta)}{B(\alpha, \beta)} = I_x(\alpha, \beta)$$

$$X \sim \text{Beta}(1,1) = \underbrace{\frac{1}{B(1,1)}}_{\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} = \frac{1!}{0!\,0!} = 1} x^{1-1}(1-x)^{1-1}\,\mathbb{1}_{x \in (0,1)} = \mathbb{1}_{x \in (0,1)} = U(0,1)$$

$\Rightarrow$ Uniform is special case of the Beta!

$$X \sim \text{Beta}(\alpha, \beta) \quad E(X) = \int_0^1 x \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{1}{B(\alpha,\beta)} \int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1}\,dx = \frac{B(\alpha+1, \beta)}{B(\alpha,\beta)} = \frac{\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}} = \frac{\frac{\alpha \cancel{\Gamma(\alpha)}}{(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}}}{\frac{\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha+\beta)}}} = \boxed{\frac{\alpha}{\alpha+\beta}}$$

Where else does the Beta distr come up?

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ indep. of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$

let $P = \frac{X_1}{X_1 + X_2} \sim ?$ The proportion of the first waiting time over the sum of the waiting times

On HW... $f_P(p) = \int_{\mathbb{R}} f_{X_1, X_2}(pu, u(1-p)) |u| \, du \underset{X_1, X_2 \text{ ind}}{=} \int_{\mathbb{R}} f_{X_1}(pu) f_{X_2}(u(1-p)) |u| \, du \underset{X_1, X_2 \text{ iid}}{=} \int_{\mathbb{R}} f(pu) f(u(1-p)) |u| \, du$

$$f_P(p) = \int_{\mathbb{R}} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} (pu)^{\alpha_1 - 1} \cancel{e^{-\beta p u}} \mathbb{1}_{pu \in (0,\infty)} \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)} (u(1-p))^{\alpha_2 - 1} \underbrace{e^{-\beta u(1-p)}}_{e^{-\beta u} \cancel{e^{\beta u p}}} \mathbb{1}_{u(1-p) \in (0,\infty)} u \, du$$

$$\propto p^{\alpha_1 - 1}(1-p)^{\alpha_2 - 1} \int_{\mathbb{R}} u^{\alpha_1 + \alpha_2 - 1} e^{-\beta u} \mathbb{1}_{pu \in (0,\infty)} \,\&\, (1-p)u \in (0,\infty) \, du$$

$\mathbb{1}_{pu \in (0,\infty)} \mathbb{1}_{u(1-p) \in (0,\infty)} = \mathbb{1}_{p \in (0,1)} \mathbb{1}_{u \in (0,\infty)}$

$pu > 0 \;\&\; (1-p)u > 0$

If $p > 1$ this is false for $\forall u \in \mathbb{R}$

If $p < 0$ " " " " " " " "

$\Rightarrow p \in (0,1) \Rightarrow u > 0$

$$= p^{\alpha_1 - 1}(1-p)^{\alpha_2 - 1} \mathbb{1}_{p \in (0,1)} \int_0^\infty u^{\alpha_1 + \alpha_2 - 1} e^{-\beta u} \, du$$

I know this is gamma but idc! It's not a function of p!!

$$\propto p^{\alpha_1 - 1}(1-p)^{\alpha_2 - 1} \mathbb{1}_{p \in (0,1)}$$

$\propto \text{Beta}(\alpha_1, \alpha_2)$

Sampling!

the "quantile operator"

Define the "q^{th} quantile" of a rv X denoted $Q[X, q]$

$:= \min\{x : q \le F(x)\} = \min\{x : F(x) \ge q\}$. Also $100q$ is called the $100q^{th}$ percentile

Suppose value s.t. x has $100q$ % of the mass/density below it.

e.g. $X \sim U(\{2, 4, 6, \ldots, 18, 20\})$

Since all rv's have CDF's, quantiles $q \in (0,1)$ always exist.

Let $MED[X] := Q[X, 0.5]$, the median of X, another measure of central tendency

x	$p(x)$	$F(x)$
2	0.1	0.1
4	0.1	0.2
6	″	0.3
8	″	0.4
10	″	0.5
12	″	0.6
14	″	0.7
16	″	0.8
18	″	0.9
20	0.1	1.0

$Q[X, 0.3] = 6$

$Q[X, 0.85] = 18$ $\quad \nexists\, x$ s.t. $F(x) = 0.85$ but $F(18) = 0.9$, the smallest x s.t. $F(x) \ge .85$

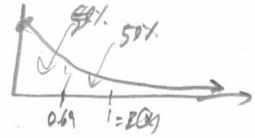
For continuous rv, $Q[X,q] := \min\{x : q \le F(x)\} \overset{\text{since } \exists x\, F(x) = q\ \forall q \in (0,1)}{=} \min\{x : q = F(x)\}$

$= F^{-1}(q)$, the inverse CDF

e.g. $X \sim Exp(1) = e^{-x} \mathbb{1}_{x \in (0,\infty)} \Rightarrow F(x) = 1 - e^{-x} = q \Rightarrow 1 - q = e^{-x} \Rightarrow -x = \ln(1-q)$

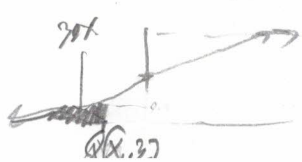
$\Rightarrow x = \ln\left(\frac{1}{1-q}\right)$

$Q[X, q] = \ln\left(\frac{1}{1-q}\right)$



$MED[X] = \ln\left(\frac{1}{1-0.5}\right) = \ln(2) = 0.69$

$X \sim N(0,1)$ $\quad Q[X, .3] = F^{-1}(0.3) = \Phi^{-1}(0.3)$ $\quad$ inverse std norm CDF, not available in closed form



$= \left\{x : \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du\right\}$ use numerical methods