Lec 15 MATH 340/640

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$X \sim T_K$ in our context, $K \in \mathbb{N}$ but defined on $k > 0$ due to $\Gamma$'s.

Let $V \sim \text{Gamma}(a,a)$ indep of $U \sim \text{Gamma}(b,b)$

$R = \frac{V}{U} \sim f_R(r) = \int_{\mathbb{R}} f_V(ru) f_U(u) |u| du$ This should be similar to the previous exercise

$$= \int_{\mathbb{R}} \frac{a^a}{\Gamma(a)} (ru)^{a-1} e^{-aru} \mathbb{1}_{\underbrace{ru \in (0,\infty)}_{r \in (0,\infty)}} \frac{b^b}{\Gamma(b)} u^{b-1} e^{-bu} |u| \mathbb{1}_{u \in (0,\infty)} du$$

$$= \frac{a^a}{\Gamma(a)} \frac{b^b}{\Gamma(b)} r^{a-1} \mathbb{1}_{r \in (0,\infty)} \int_0^\infty u^{a+b-1} e^{-(ar+b)u} du$$

$$= \frac{a^a}{\Gamma(a)} \frac{\cancel{b^b}}{\Gamma(b)} r^{a-1} \mathbb{1}_{r \in (0,\infty)} \frac{\Gamma(a+b)}{(ar+b)^{a+b} \frac{b^a b^b}{b^{a+b}}}$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left(\frac{a}{b}\right)^a r^{a-1} \left(1 + \frac{a}{b} r\right)^{-(a+b)} \mathbb{1}_{r \in (0,\infty)}$$

Let $V \sim \chi^2_{k_1}$ independent of $U \sim \chi^2_{k_2}$, $\frac{V}{k_1} \sim \text{Gamma}\left(\frac{k_1}{2}, \frac{k_1}{2}\right)$, $\frac{U}{k_2} \sim \text{Gamma}\left(\frac{k_2}{2}, \frac{k_2}{2}\right)$

$$R = \frac{V/k_1}{U/k_2} \sim \frac{\Gamma\left(\frac{k_1+k_2}{2}\right)}{\Gamma\left(\frac{k_1}{2}\right)\Gamma\left(\frac{k_2}{2}\right)} \left(\frac{k_1}{k_2}\right)^{\frac{k_1}{2}} r^{\frac{k_1}{2}-1} \left(1 + \frac{k_1}{k_2} r\right)^{-\frac{k_1+k_2}{2}} \mathbb{1}_{r \in (0,\infty)} = F_{k_1,k_2}$$

$= \text{Snedecor } F(k_1, k_2)$

$k_1, k_2 \in \mathbb{N}$

but defined for all $k_1, k_2 > 0$

Let $R \sim T_K \Rightarrow R = \frac{Z}{\sqrt{\frac{U}{K}}}$, $Z \sim N(0,1)$, $U \sim \chi^2_K$, $\frac{U}{K} \sim \text{Gamma}\left(\frac{k}{2}, \frac{k}{2}\right)$

$R^2 = \frac{Z^2/1}{\frac{U}{K}} \sim F_{1,K}$. The square of a Student's T-distr is an F-distr!

let $Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

$R = \frac{Z_1}{Z_2}$ $\quad P(R>0) = P(Z_1>0)\,P(Z_2<0) + P(Z_1<0)\,P(Z_2<0) = \frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{1}{2} = \frac{1}{4}$

another $R = \frac{Z_1}{\sqrt{Z_2^2}} = \frac{Z_1}{|Z_2|}$ $\quad P(R>0) = P(Z_1>0) = \frac{1}{2}$

$$\Rightarrow \frac{Z_1}{Z_2} \overset{d}{=} \frac{Z_1}{|Z_2|} \overset{d}{=} \frac{Z_1 \sim N(0,1)}{\sqrt{\frac{Z_2^2 \sim \chi^2_1}{1}}} \sim T_1 = \frac{\Gamma\left(\frac{1+1}{2}\right)}{\sqrt{(1)\pi}\,\Gamma\left(\frac{1}{2}\right)}\left(1+\frac{r^2}{1}\right)^{-\frac{1+1}{2}}$$

$$= \frac{\Gamma(1)}{\sqrt{\pi}\sqrt{\pi}}\,\frac{1}{(1+r^2)} = \frac{1}{\pi(1+r^2)} = \text{Cauchy}(0,1)$$

Maybe you don't trust this calculation? Let's do it from scratch...

$$f_R(r) = \int_{\mathbb{R}} f(ru)\,f(u)\,|u|\,du = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}r^2u^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\,|u|\,du = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{(1+r^2)}{2}u^2}\,|u|\,du$$

$$= \frac{1}{2\pi}\left(\int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} u\,du + \int_{-\infty}^0 e^{-\left(\frac{1+r^2}{2}\right)u^2}(-u)\,du\right)$$

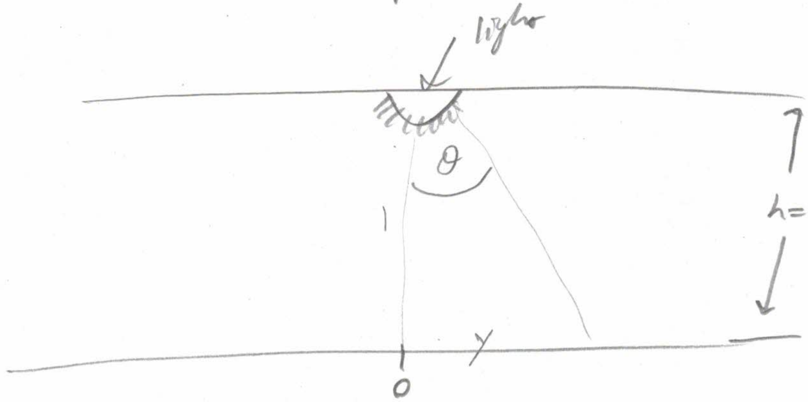
let $t=-u \Rightarrow u=-t$, $\frac{du}{dt} = -1 \Rightarrow dt = -du$, $u=0 \Rightarrow t=0$, $u=\infty \Rightarrow t=\infty$

$$= \frac{1}{2\pi}\left(\int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} u\,du + \int_\infty^0 e^{-\left(\frac{1+r^2}{2}\right)t^2} t\,(-dt)\right) = \frac{1}{2\pi}\left(\int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} u\,du + \int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)t^2} t\,dt\right)$$

$$= \frac{1}{\pi}\int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} u\,du = \frac{1}{\pi}\int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)v}\,\cancel{u}\,\frac{1}{2\cancel{u}}\,dv = \frac{1}{\cancel{2}\pi}\,\frac{1}{\frac{1+r^2}{\cancel{2}}}\left[-e^{-\left(\frac{1+r^2}{2}\right)v}\right]_0^\infty = \frac{1}{\pi}\,\frac{1}{1+r^2}$$

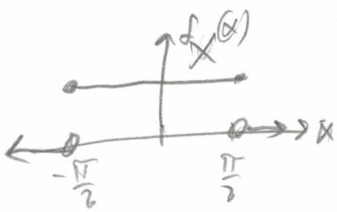
let $v = u^2 \Rightarrow \frac{dv}{du} = 2u \Rightarrow du = \frac{1}{2u}\,dv \Rightarrow u=0 \Rightarrow v=0$, $u=\infty \Rightarrow v=\infty$

There is a nice physical interpretation of the Cauchy Distr:




Imagine a light uniformly shining over all $180° = \pi$ radians. What is the distribution on the floor?

let $X \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{1}{\pi} \mathbb{1}_{x \in [\frac{\pi}{2}, \frac{\pi}{2}]}$




let $Y = \tan(X) \iff X = \arctan(Y) = g^{-1}(Y)$ $\quad \left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{1}{1+y^2}\right| = \frac{1}{1+y^2}$

$$f_Y(y) = f_X(\arctan(y)) \frac{1}{1+y^2} = \frac{1}{\pi} \underbrace{\mathbb{1}_{\arctan(y) \in [\frac{\pi}{2}, \frac{\pi}{2}]}}_{y \in \mathbb{R}} \frac{1}{1+y^2} = \frac{1}{\pi}\frac{1}{1+y^2} = \text{Cauchy}(0,1)$$

let $X \sim \text{Cauchy}(0,1)$

$$Y = \mu + \sigma X \sim \frac{1}{\pi\sigma} \frac{1}{1+\left(\frac{y-\mu}{\sigma}\right)^2} = \text{Cauchy}(\mu, \sigma)$$

where $\sigma > 0$, $\mu \in \mathbb{R}$

$X \sim \text{Cauchy}(0,1)$ $\qquad$ let $u = 1+x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{1}{2x} du$, $x = \pm\infty$

$$E(X) = \int_{\mathbb{R}} x \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi} \int_{\mathbb{R}} \frac{x}{1+x^2} dx \quad \underset{\text{Try}}{\frac{1}{\pi}} \left[\frac{1}{2} \ln(1+x^2)\right]_c^{\infty} = \infty \Rightarrow E(X) \text{ does not exist!}$$

Recall the improper integral $\int_{-\infty}^{\infty} g(x)\,dx$ exists if $\int_{-\infty}^{c} g(x)\,dx$ and $\int_{c}^{\infty} g(x)\,dx$ exist for all $c \in \mathbb{R}$ otherwise it does not exist!

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What is the ch.f. of $X \sim Cauchy(0,1)$? $\phi_X(t) = \int_{\mathbb{R}} e^{itx} \frac{1}{\pi(1+x^2)} dx$ ... difficult!

Here's an amazing proof. Guess the answer to be $\phi_X(t) = e^{-|t|} \in L^1$

If we invert this ch.f. and return the density of X, it must be the answer as a function and its Fourier transform are 1:1.

$$\frac{1}{\pi(1+x^2)} \stackrel{?}{=} f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} e^{-|t|} dt$$

Note $\cos(tx) = \cos(-tx)$ which is even $\int_{-a}^{a} \text{even}\, dx = 2\int_0^a \text{even}\, dx$

$$\Rightarrow \frac{2}{1+x^2} \stackrel{?}{=} \int_{\mathbb{R}} (i\sin(tx) + \cos(tx)) e^{-|t|} dt$$

$$= \cancel{i\int_{\mathbb{R}} \frac{\sin(tx)}{e^{|t|}} dt} + \int_{\mathbb{R}} \frac{\cos(tx)}{e^{|t|}} dt$$

Note $\sin(tx) = -\sin(-tx)$ sin is odd

$e^{|t|} = e^{|-t|}$ $e^{|t|}$ is even

$\Rightarrow \frac{\sin(tx)}{e^{|t|}}$ is odd. $\int_{-a}^{a} \text{odd}\, dx = 0$

$$\frac{2}{1+x^2} \stackrel{?}{=} 2\int_0^\infty \frac{\cos(tx)}{e^t}$$

Let $I(x) := \int_0^\infty \underbrace{\cos(tx)}_{u} \underbrace{e^{-t} dt}_{dv}$.

$$I(x) = [uv]_0^\infty - \int_0^\infty v\,du = \underbrace{[\cos(tx)(-e^{-t})]_0^\infty}_{-(0-1)} - \int_0^\infty (-e^{-t})(-x\sin(tx))\,dt$$

$$= 1 - x\int_0^\infty \underbrace{\sin(tx)}_{u} \underbrace{e^{-t} dt}_{dv}$$

$$= 1 - x\left(\underbrace{[\sin(tx)(-e^{-t})]_0^\infty}_{-(0-0)} - \int_0^\infty (-e^{-t})(x\cos(tx))\,dt\right)$$

$$= 1 - x^2\int_0^\infty \cos(tx) e^{-t} dt = 1 - x^2 I(x)$$

$$\Rightarrow I(x) - 1 = -x^2 I(x)$$

$$\Rightarrow \frac{I(x)-1}{I(x)} = -x^2 \Rightarrow \frac{1}{I(x)} - 1 = x^2 \Rightarrow \frac{1}{I(x)} = x^2+1$$

$$\Rightarrow I(x) = \frac{1}{x^2+1}$$

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$\phi_X(t) = e^{-|t|}$, $\phi_X'(t) = \begin{cases} -e^{-|t|} & \text{if } t \neq 0 \\ \text{undefined} & \text{if } t = 0 \end{cases}$ $E(X) = \frac{\phi_X'(0)}{i}$ ... undefined! In fact... no moments exist!

Recall WLLN $\bar{X}_n \xrightarrow{P} \mu$ This is only if $\mu$ exists!

Let $X_1 \ldots X_n \overset{iid}{\sim}$ Cauchy$(0,1)$

How is $\bar{X}_n \sim$ ?

$$\phi_{\bar{X}_n}(t) = \phi_{X_1+\cdots+X_n}\left(\frac{t}{n}\right) = \left(\phi_{X_i}\left(\frac{t}{n}\right)\right)^n = \left(e^{-\left|\frac{t}{n}\right|}\right)^n = e^{-\frac{|t|}{n}n} = e^{-|t|}$$

$\Rightarrow \bar{X}_n \sim$ Cauchy$(0,1)$. Weird!!

Make DEMO!!

Midterm II ↑

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Final ↓