MATH 340/640

Lec 11

Let a_n be a sequence of constants s.t. $\lim_{n\to\infty} a_n = a$

Let $X_n \xrightarrow{d} X$, Prove that $a_n X_n \xrightarrow{d} aX$

Let $Y_n = Deg(a_n)$

Then she justifies this but its obvious

$\lim F_{Y_n}(y) = \lim \mathbb{1}_{y \geq a_n} = \mathbb{1}_{y \geq \lim a_n} = \underbrace{\mathbb{1}_{y\geq a}}_{\text{CDF of } Deg(a)} \Rightarrow Y_n \xrightarrow{d} a$

$\Rightarrow Y_n \xrightarrow{p} a$

$\Rightarrow a_n \xrightarrow{p} a$

$\Rightarrow Y_n X_n \xrightarrow{d} aX$ by Slutsky's (A)

If $X_n \xrightarrow{d} c \Rightarrow Y_n X_n \xrightarrow{d} ac \Rightarrow Y_n X_n \xrightarrow{p} ac$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

Why is there a $\frac{1}{n-1}$ and not $\frac{1}{n}$ like in $\bar{X}$? Answer is given in MATH 341.

$$= \frac{1}{n-1} \sum X_i^2 - 2X_i\bar{X} + \bar{X}^2$$

$$= \frac{1}{n-1}\left(\sum X_i^2 - 2\bar{X}\sum \bar{X} + \sum \bar{X}^2\right)$$

$$= \frac{1}{n-1}\left(\sum X_i^2 - 2n\bar{X}^2 + n\bar{X}^2\right)$$

$$= \frac{1}{n-1}\left(\sum X_i^2 - n\bar{X}^2\right) \cdot \frac{n}{n}$$

$$= \frac{n}{n-1}\left(\frac{1}{n}\sum X_i^2 - \bar{X}_n^2\right)$$

a function of n

Let $A_n := \frac{1}{n}\sum X_i^2$, $B_n = \bar{X}_n^2$, let $a = 1$, $b = -1$

$C_n := aA_n + bB_n$

$\bar{X}_n \xrightarrow{p} \mu$

$A_n \xrightarrow{p} E[X^2]$ by WLLN, $\Rightarrow B_n \xrightarrow{p} \mu^2$ by CMT

$$\Rightarrow \frac{1}{n}\sum X_i^2 - \bar{X}^2 \xrightarrow{p} E[X^2] - \mu^2 = Var[X] = \sigma^2$$

$$= \underbrace{\frac{n}{n-1}}_{a_n} C_n \xrightarrow{p} \sigma^2 \Rightarrow S_n^2 \xrightarrow{p} \sigma^2 \checkmark$$

$\lim a_n = a = 1$

let $g(x) = \sqrt{x}$, a cont. function

$$\Rightarrow g(S_n^2) = S_n$$
$$g(\sigma^2) = \sigma$$

$$S_n^2 \xrightarrow{p} \sigma^2 \underset{\text{CMT}}{\Rightarrow} S_n \xrightarrow{p} \sigma$$

$$\bar{X} = \frac{\sum X_i}{n} \xrightarrow{p} \mu$$

Consider $W = \frac{\sum X_i + 1}{n+2} \xrightarrow{p} ?$

We will see why we want to use this in 341

$$W = \frac{n}{n} \frac{\sum X_i + 1}{n+2} = \frac{n}{n+2} \frac{\sum X_i + 1}{n} = \frac{n}{n+2}\left(\frac{\sum X_i}{n} + \frac{1}{n}\right) = \underbrace{\frac{n}{n+2}}_{a_n} \bar{X} + \underbrace{\frac{1}{n+2}}_{b_n}$$

Since $\lim a_n = 1$, $W \xrightarrow{p} \mu$
and $\lim b_n = 0$, $\Rightarrow$
and $\bar{X} \xrightarrow{p} \mu$ by WLLN

Back to CLT... $X_1, \dots X_n \overset{iid}{\sim}$ s.t. $\mu < \infty$, $\sigma^2 < \infty$

$$\Rightarrow \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1) \Rightarrow \bar{X} \,\dot\sim\, N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right), \; T \,\dot\sim\, N\left(n\mu, (\sqrt{n}\,\sigma)^2\right)$$

Need knowledge of X's mean & variance

Is it true $\frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \xrightarrow{d} N(0,1) \Rightarrow \bar{X} \,\dot\sim\, N\left(\mu, \left(\frac{S}{\sqrt{n}}\right)^2\right), \; T \,\dot\sim\, N\left(n\mu, (\sqrt{n}\, S)^2\right)$

only need knowledge of mean

Yes ... proof ...

$$\frac{\overline{X}-\mu}{\frac{S}{\sqrt{n}}} = \underbrace{\frac{\frac{\sigma}{\sqrt{n}}}{\frac{S}{\sqrt{n}}}}_{A_n \xrightarrow{P} 1} \cdot \underbrace{\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}_{B_n \xrightarrow{d} N(0,1)} \xrightarrow{d} N(0,1)$$

by CLT

by Slutsky's (A)

let $g(x) = \frac{\frac{\sigma}{\sqrt{n}}}{\frac{x}{\sqrt{n}}} \Rightarrow g(S_n) = \frac{\frac{\sigma}{\sqrt{n}}}{\frac{S_n}{\sqrt{n}}}$, $g(\sigma) = \frac{\frac{\sigma}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}} = 1 \Rightarrow S_n \xrightarrow{P} \sigma \Rightarrow \frac{\frac{\sigma}{\sqrt{n}}}{\frac{S}{\sqrt{n}}} \xrightarrow{P} 1$

by CMT

This is a more powerful CLT but not so practical. Very important in statistics though! We will also use this fact later to prove Student's T-distr converges to the Std Normal Distr.

Let's return to transformations to derive new r.v.'s.

$Y = g(X) \Longleftrightarrow X = g^{-1}(Y)$

$Y \sim f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}[g^{-1}(y)] \right|$

$= \lambda e^{-\lambda x} \mathbb{1}_{x \in (0,\infty)}, \ \lambda > 0$ s.t. $k > 0$ — another waiting distribution, a generalization of the exponential (4)

$$X \sim Exp(\lambda), \quad Y = ke^X \Rightarrow \frac{Y}{k} = e^X \Rightarrow X = \ln(Y) - \ln(k) = g^{-1}(Y)$$

$$\left|\frac{d}{dy}[g^{-1}(y)]\right| = \left|\frac{1}{y}\right|, \quad f_Y(y) = \lambda \underbrace{e^{-\lambda(\ln(y) - \ln(k))}}_{\underbrace{e^{-\lambda \ln(y)} e^{+\lambda \ln(k)}}_{e^{\ln(y^{-\lambda})} e^{\ln(k^\lambda)}}} \mathbb{1}_{\underbrace{\ln(y) - \ln(k) \in (0,\infty)}_{\underbrace{\ln(y) \in (\ln(k), \infty)}_{y \in (k,\infty)}}} \left|\frac{1}{y}\right|$$

no need for abs value sign as $y > k > 0$

$$= \lambda k^\lambda y^{-\lambda} \mathbb{1}_{y \in (k,\infty)} \frac{1}{y}$$

$$= \frac{\lambda k^\lambda}{y^{\lambda+1}} \mathbb{1}_{y \in (k,\infty)} = \text{Pareto I}(k, \lambda)$$

$$S_Y = (k,\infty), \quad \underbrace{k > 0}_{\text{by assumption}}, \underbrace{\lambda > 0}_{\text{from } Exp(\lambda)}$$

$$F_Y(y) = \int_k^y \frac{\lambda k^\lambda}{x^{\lambda+1}} dx = \lambda k^\lambda \left[-\frac{x^{-\lambda}}{\lambda}\right]_k^y = k^\lambda (k^{-\lambda} - y^{-\lambda}) = 1 - \left(\frac{k}{y}\right)^\lambda$$

another waiting time

$X \sim Exp(1) = e^{-x} \mathbb{1}_{x \in (0,\infty)}$, $Y = \frac{1}{\lambda} X^{\frac{1}{k}}$ where $k, \lambda > 0$

$\Rightarrow \lambda Y = X^{\frac{1}{k}} \Rightarrow X = (\lambda Y)^k = \lambda^k Y^k = g^{-1}(Y)$

$$\frac{d}{dy}\left[g^{-1}(y)\right] = k\lambda^k y^{k-1}$$

$$f_Y(y) = e^{-\lambda^k y^k} \underbrace{\mathbb{1}_{\lambda^k y^k \in (0,\infty)}}_{\substack{y^k \in (0,\infty) \\ k\ln(y) \in \mathbb{R} \\ \ln(y) \in \mathbb{R} \\ y \in (0,\infty)}} |k\lambda^k y^{k-1}| = k\lambda(\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \in (0,\infty)}$$

no need for abs val as $y > 0$

$= Weibull(k, \lambda)$

$$F_Y(y) = \int_0^y k\lambda(\lambda x)^{k-1} e^{-(\lambda x)^k} dx = \int_0^{(\lambda y)^k} \cancel{k\lambda} \cancel{(\lambda x)^{k-1}} e^{-u} \frac{1}{\cancel{k\lambda(\lambda x)^{k-1}}} du$$

let $u = (\lambda x)^k = \lambda^k x^k$ $\quad \frac{du}{dx} = k\lambda^k x^{k-1} \Rightarrow dx = \frac{1}{k\lambda^k x^{k-1}} du$
$= \frac{1}{k\lambda(\lambda x)^{k-1}} du$

$x = 0 \Rightarrow u = 0$
$x = y \Rightarrow u = (\lambda y)^k$

$$= \left[-e^{-u}\right]_0^{(\lambda y)^k} = 1 - e^{-(\lambda y)^k}$$

$X \sim Exp(1) = e^{-x} \mathbb{1}_{x \in (0,\infty)}$, $\quad Y = g(X) = \ln\left(\frac{1}{X}\right) \Rightarrow e^Y = \frac{1}{X} \Rightarrow X = e^{-Y} = g^{-1}(Y)$

$\left|\frac{d}{dy}\left[e^{-y}\right]\right| = \left|-e^{-y}\right| = e^{-y}$

$$f_Y(y) = e^{-e^{-y}} \underbrace{\underbrace{\mathbb{1}_{e^{-y} \in (0,\infty)}}_{-y \in \mathbb{R}}}_{y \in \mathbb{R}} e^{-y} = e^{-(y + e^{-y})} = \text{Gumbel}(0,1)$$

"Standard Gumbel"

$$F_Y(y) = \int_{-\infty}^{y} e^{-x} e^{-e^{-x}} dx = \int_{\infty}^{e^{-y}} u e^{-u} \left(-\frac{1}{u} du\right) = \int_{e^{-y}}^{\infty} e^{-u} du = \left[-e^{-u}\right]_{e^{-y}}^{\infty} = e^{-e^{-y}}$$

let $u = e^{-x} \Rightarrow \frac{du}{dx} = -e^{-x} = -u \Rightarrow dx = -\frac{1}{u} du$, $\quad x = -\infty \Rightarrow u = \infty$; $x = y \Rightarrow u = e^{-y}$

let $X \sim \text{Gumbel}(0,1)$, $Y = \mu + \beta X \sim \frac{1}{\beta} e^{-\left(\frac{y-\mu}{\beta} + e^{-\left(\frac{y-\mu}{\beta}\right)}\right)} = \text{Gumbel}(\mu, \beta)$

where $\mu \in \mathbb{R}, \beta > 0$

"General ~~Gumbel~~"

Hw: general CDF