MATH 54[illegible] LEC 3

$$X, Y \overset{iid}{\sim} \text{Geom}(p)$$

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$$P(X > Y) = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} p_{X,Y}(x,y) \mathbb{1}_{x>y}$$

$$= \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} p_X(x)\, p_Y(y) \mathbb{1}_{x>y}$$

$$= \sum_{x \in S_X} \sum_{y \in S_Y} p_X^{dd}(x)\, p_Y^{dd}(y) \mathbb{1}_{x>y}$$

$$= \sum_{x \in \mathbb{N}_0} \sum_{y \in \mathbb{N}_0} (1-p)^x p\, (1-p)^y p\, \mathbb{1}_{x>y}$$

$$= p^2 \sum_{y \in \mathbb{N}_0} (1-p)^y \underbrace{\sum_{x \in \{y+1, y+2, \dots\}} (1-p)^x}_{\text{looks "close" to geometric series}}$$

let $i := x-(y+1)$ then $i \in \{0, 1, \dots\} = \mathbb{N}_0$

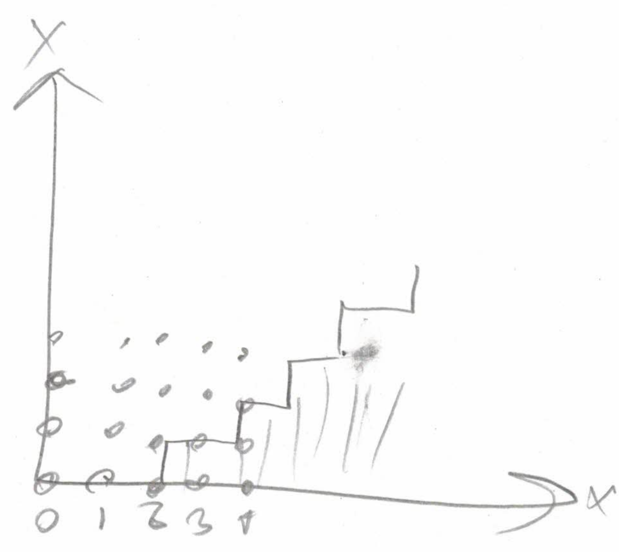
$\Rightarrow x = i + y + 1$

$$= p^2 \sum_{y \in \mathbb{N}_0} (1-p)^y \sum_{i \in \mathbb{N}_0} (1-p)^{i+y+1}$$

$$= p^{\cancel{2}} \sum_{y \in \mathbb{N}_0} (1-p)^{2y+1} \underbrace{\sum_{i \in \mathbb{N}_0} (1-p)}_{\frac{1}{1-(1-p)} = \frac{1}{\cancel{p}}}$$

$$= p(1-p) \underbrace{\sum_{y \in \mathbb{N}_0} \left((1-p)^2\right)^y}_{\frac{1}{1-(1-p)^2} = \frac{1}{\cancel{1-1}+2p-p^2}} = \frac{\cancel{p}(1-p)}{\cancel{p}(2-p)} = \frac{1-p}{2-p}$$

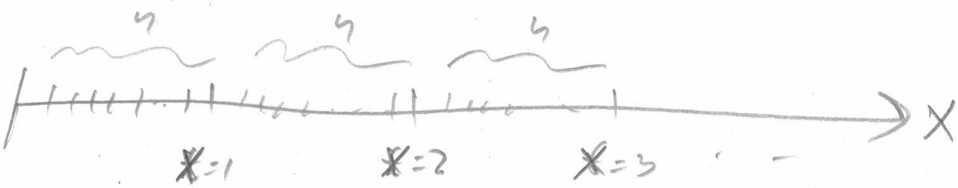
as $p \to 0$, $P(X > Y) \to \frac{1}{2}$



$X \sim \text{Geom}(p)$

Imagine X is time on a clock, + maybe every "second", measure "survival in seconds"!

What if instead, we had n iid Bern(p) per second:



$$\Rightarrow F_{X_n}(x) = 1-(1-p)^{nx}$$

Let X_n denote stopping time $X_n \sim p_{X_n}(x) = (1-p)^{nx}\, p\, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1, 1+\frac{1}{n}, 1+\frac{2}{n}, \ldots\}}$

The process "stops" almost immediately if n is large and p is modest

But what if p was very small? $p \to 0,\ n \to \infty$ s.t. $\lambda = pn > 0$

$\Leftrightarrow p = \frac{\lambda}{n}$

And we substitute...

$$p_{X_n}(x) = \left(1-\frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n}\, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots\}}$$

Now we let $n \to \infty$ effectively running infinite Bernoullis in each second:

$$(x) = \lim_{n\to\infty} p_{X_n}(x) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nx} \frac{\lambda}{n}\, \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots\}}$$

Recall from calculus... $\lim_{n\to\infty} f(n)g(n) = \lim_{n\to\infty} f(n) \lim_{n\to\infty} g(n)$

$\lim_{n\to\infty} g(f(n)) = g\left(\lim_{n\to\infty} f(n)\right)$

$$= \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nx} \lim_{n\to\infty} \frac{\lambda}{n} \lim_{n\to\infty} \mathbb{1}_{x \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots\}}$$

$\lim f(n) + g(n)$
$= \lim f(n) + \lim g(n)$

$$= \left(\overbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}}^{e^{-\lambda}}\right)^{x}\left(\lim_{n\to\infty}\frac{\lambda}{n}\right)\lim_{n\to\infty}\mathbb{1}_{x\in\{0,\frac{1}{n},\frac{2}{n},\ldots\}}$$

$$= \left(e^{-\lambda x}\right)(0)\,\mathbb{1}_{x\in[0,\infty)}$$ continuous support

$|S_{X_\infty}| \neq |\mathbb{N}|$ ctbl ∞

$= 0$ Not a PMF!!!

No longer a discrete rv

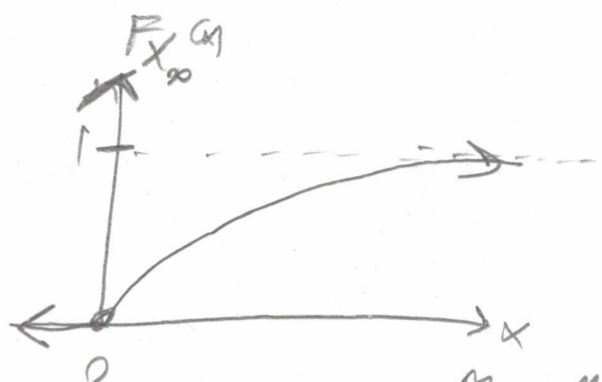
$\Rightarrow$ We're in trouble!

$|S_{X_\infty}| = |\mathbb{R}|$ unctbl ∞

Let's take a look at the CDF...

$$F_{X_\infty}(x) = \lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nx} = 1-\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)\right)^{x}$$

$$= 1 - e^{-\lambda x}$$



Is this a valid CDF?

1) ~~$F(x) \in [0,1]$ since it's a prob~~

~~If $x<0 \Rightarrow F(x)=0$~~

~~If $x>0 \Rightarrow F(x) \in [0,1]$ since $1-e^{-\lambda x} \in [0,1]$~~

Monotonically incr.

③ $\frac{d}{dx}F(x) \geq 0$

$\lambda e^{-\lambda x} > 0$ for all $x \geq 0$ and $= 0$ for all $x<0$ ✓

2) $\lim_{x\to-\infty} F(x) = 0$ since $x<0 \Rightarrow F(x)=0$

3) $\lim_{x\to\infty} F(x) = 1$ $\quad \lim 1-e^{-\lambda x} = 1-\lim_{x\to\infty} e^{-\lambda x} = 1-0 = 1$ ✓

$\Rightarrow$ It's a valid CDF

Def: a cont rv X has $|S_X| = |\mathbb{R}|$ and no PMF,

the PMF is $p(x) = 0 \;\forall x$ $\quad P(X = x) = 0$!!

They have CDF's. The deriv. of the CDF

(how fast the rv collects prob at any x) is useful:

$f_X(x) := \frac{d}{dx}[F_X(x)]$ is called the rv's prob "density" function (PDF)

Why density? It's the density of prob in any given region:

$$a < b,\; P(X \in (a, b)) = F(b) - F(a) \overset{FTC}{=} \int_a^b f_X(x)\,dx \Rightarrow \begin{array}{l} P(X \in (-\infty, b]) \\ = F(b) - \cancelto{0}{F(-\infty)} \\ = \int^b f_X(x)\,dx \end{array}$$

Also, many times the PDF is avail. in closed form but the CDF is not! Properties:

$$1 = P(X \in (-\infty, \infty)) = \int_{\mathbb{R}} f_X(x)\,dx$$ the Hungry-Dungry ~~formula~~ for cont. rv's.

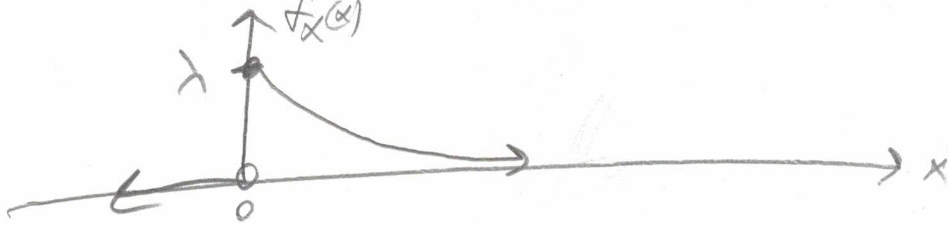
$f_X(x) \geq 0$ since $F_X(x)$ is monotonically

Note: f_X can be > 0. It's not a problem the PMF!!!

$S_X = \{x : f(x) > 0\}$

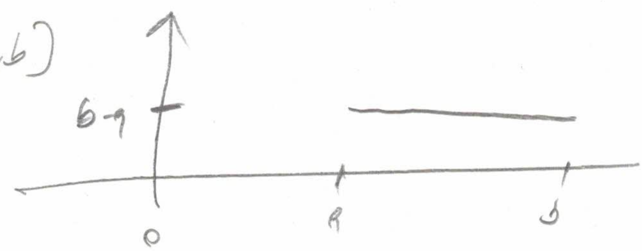
The X_{00} we discussed is a famous brand name rv., the exponential

$$X \sim \text{Exp}(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{f(x)} \mathbb{1}_{x \geq 0} \quad \lambda \in (0, \infty) \quad \text{since } p \in (0,1),\; n \in \mathbb{N}$$



Another famous cont. r.v. is the uniform r.v. or just "uniform"

$X \sim U(a,b) := \frac{1}{b-a} \mathbb{1}_{x \in [a,b]}$



$S_X = [a,b]$, $a \in \mathbb{R}, b \in \mathbb{R}, a < b$

Valid density? $\int_{\mathbb{R}} \frac{1}{b-a} \mathbb{1}_{x \in [a,b]} = \int_a^b \frac{1}{b-a} dx = \frac{1}{b-a} (x)_a^b = \frac{b-a}{b-a} = 1$ ✓

$\forall x \; f(x) \geq 0$ If $x \in (-\infty, a) \Rightarrow f(x) = 0$, $x \in [a,b] \Rightarrow f(x) = b-a > 0$

$x \in (b, \infty) \Rightarrow f(x) = 0$ ✓

If $a = 0, b = 1 \Rightarrow X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$ the standard uniform

Very important r.v. especially in comp. science!

if $\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$ $\quad f_{\vec{X}}(\vec{x}) = f_{X_1,\dots,X_n}(x_1,\dots,x_n)$ joint density function (JDF)

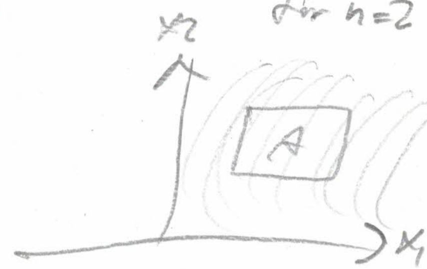
if $X_1,\dots,X_n$ ind $\Rightarrow \; = f_{X_1}(x_1)\, f_{X_2}(x_2) \cdot \ldots \cdot f_{X_n}(x_n)$

if $X_1,\dots,X_n \overset{iid}{\sim} \; \Rightarrow \; = f(x_1)\, f(x_2) \cdot \ldots \cdot f(x_n)$

$\int_{\mathbb{R}^n} f_{\vec{X}}(\vec{x})\, d\vec{x} = 1$ "Humpty Dumpty"

To get prob's from a JDF, you need to integrate over a region $P(\vec{X} \in A) = \int_A f_{\vec{X}}(\vec{x})\, d\vec{x}$

for $n=2$



Volume under blanket in A

A calc. fact: Leibnitz's Rule

$$\frac{d}{dt} \int_{a(t)}^{b(t)} g(x,y)\, dy = g(x, b(t))\, b'(t) + g(x, a(t))\, a'(t)$$

If $b(t) = t$, $a(t) = -\infty$

$$\frac{d}{dt} \int_{-\infty}^{t} g(x,y)\, dy = g(x,t)$$ since $a(t)$ is a const $\Rightarrow a'(t) = 0$

let X_1, X_2 be two cont. rv's

$$T = X_1 + X_2 \sim f_T(t) = ?$$

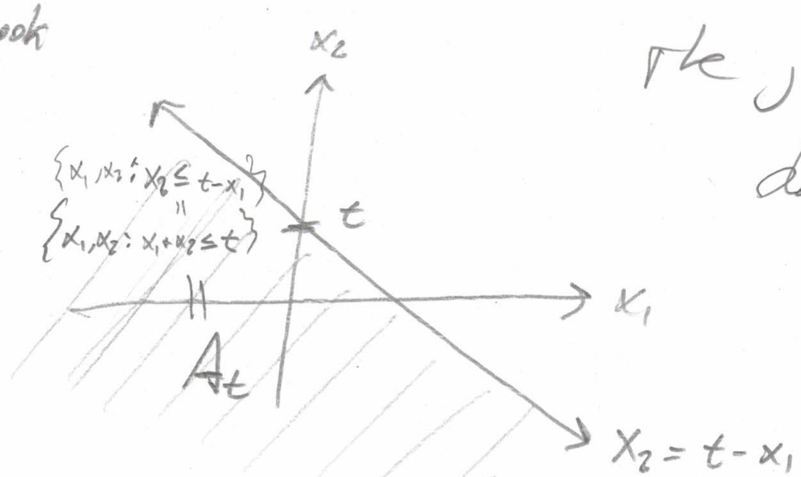
Method 1

Note: if $F_T(t)$ is known
$\Rightarrow f_T(t) = \frac{d}{dt}[F_T(t)]$ what if not known?

Method 2

We want to derive a general formula like before as a function of the jdf or densities (if indep.)

p145 of the textbook



$$F_T(t) = P(T \leq t) = P(\vec{X} \in A_t) = \int_A f_{X_1, X_2}(x_1, x_2)\, dx_1 dx_2 = \int_{x_1 \in \mathbb{R}} \int_{x_2 = -\infty}^{x_2 = t - x_1} f_{X_1, X_2}(x_1, x_2)\, dx_2\, dx_1$$

~~We need to make this look like Leibnitz's rule so $\int_{-\infty}^{t} \ldots$~~

~~To do so we need to reindex similar to what we did with the sums for the geometric problem with $P(X > Y)$~~

~~let $X_2 = V - X_1 \Rightarrow V = X_2 + X_1 \Rightarrow dv = dx_2$~~

$V = X_2 + X_1$

$\Updownarrow$

let $x_2 = V - X_1$ which is a reindexing trick like we did with the $P(X > Y)$ geometric iid problem

$\Rightarrow X_2 = -\infty \Rightarrow V = -\infty$

$\Rightarrow X_2 = t - X_1 \Rightarrow V = t$

$\frac{dx_2}{dv} = 1 \Rightarrow dv = dx_2$

drop subscript on $x_1 \Rightarrow x$

$$= \int_{x \in \mathbb{R}} \int_{v=-\infty}^{v=t} f_{X_1, X_2}(x, v-x)\, dv\, dx$$

free variable of integration

$$F_T(t) = \int_{-\infty}^{t} \underbrace{\left(\int_{x \in \mathbb{R}} f_{X_1, X_2}(x, v-x)\, dx \right)}_{\text{density}} dv$$

$$\Rightarrow f_T(t) = \int_{x \in \mathbb{R}} f_{X_1, X_2}(x, t-x)\, dx$$

general conv. formula

Recall

$$P(X \in (-\infty, b]) = F(b) - \cancel{F(-\infty)}^{0}$$

$$= \int_{-\infty}^{b} f_X(x)\, dx$$

Any integral that looks like this means density vs without!

$$F(x) = \int_{-\infty}^{x} f_X(v)\, dv$$

this is the density of integrand

free variable