let $X_1, X_2, \ldots \overset{iid}{\sim}$ s.t. $E[X] = \mu < \infty$, $Var[X] = \sigma^2 < \infty$

$$\bar{X}_n := \frac{X_1 + \ldots + X_n}{n}$$

$$Var[\bar{X}_n] = \frac{1}{n^2} Var[X_1 + \ldots + X_n] \overset{\text{ind.}}{=} \frac{1}{n^2} \sum Var[X_i] \overset{\text{by equal distr.}}{=} \frac{1}{n^2} n \, Var[X] = \frac{\sigma^2}{n}$$

let $Z_n := \dfrac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$, the "standardized" average.

Why standardized?

$$E[Z_n] = E\left[\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}\right] = \frac{1}{\frac{\sigma}{\sqrt{n}}} \left(E[\bar{X}_n] - \mu\right) = \frac{1}{\frac{\sigma}{\sqrt{n}}} (\mu - \mu) = 0$$

$$SD[Z_n] = \sqrt{Var[Z_n]} = \sqrt{\frac{1}{\frac{\sigma^2}{n}} Var[\bar{X}_n - \mu]} = \sqrt{\frac{1}{\frac{\sigma^2}{n}} Var[\bar{X}_n]} = \sqrt{\frac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}}} = 1$$

r.v.'s that have mean zero and standard deviation one are called "standardized"!

We showed before that

$$\phi_{\bar{X}_n}(t) = \left(\phi_X\left(\frac{t}{n}\right)\right)^n \qquad Z_n = \frac{\sqrt{n}}{\sigma} \bar{X}_n - \frac{\sqrt{n}}{\sigma} \mu$$

$$\phi_{Z_n}(t) \overset{P2}{=} e^{-\frac{it\sqrt{n}\mu}{\sigma} \frac{\sqrt{n}}{\sqrt{n}}} \left(\phi_X\left(\frac{\sqrt{n}}{\sigma} \cdot \frac{t}{n}\right)\right)^n$$

$$= e^{-\frac{it\mu n}{\sigma\sqrt{n}}} \left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n$$

$$= e^{\frac{-it\mu n}{\sigma\sqrt{n}}} e^{\ln\left(\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n\right)}$$

$$= e^{\frac{-it\mu n}{\sigma\sqrt{n}} + n \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}$$

$$= e^{\dfrac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{\frac{1}{n}} \cdot \dfrac{\frac{t^2}{\sigma^2}}{\frac{t^2}{\sigma^2}}}$$

$$= e^{\frac{t^2}{\sigma^2}\left(\dfrac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln \phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)}{\frac{t^2}{n\sigma^2}}\right)}$$

Now let's take limit as $n \to \infty$

$$\lim_{n\to\infty} \phi_{Z_n}(t) = e^{\frac{t^2}{\sigma^2}\lim_{n\to\infty}\left(\dfrac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{\frac{t^2}{n\sigma^2}}\right)}$$

let $V = \frac{t}{\sigma\sqrt{n}}$ if $n \to \infty$, $V \to 0$

$$\overset{*}{=} e^{\frac{t^2}{\sigma^2}\lim_{V\to 0} \frac{-i\mu V + \ln(\phi_X(V))}{V^2}}$$

Note: if I subst. $V=0$, and use (P0), we get $\frac{0}{0}$

L'Hôpital's

$$= e^{\frac{t^2}{2\sigma^2}\lim_{V\to 0}\frac{-i\mu + \frac{\phi_X'(V)}{\phi_X(V)}}{V}}$$

(P4)

$\phi_X'(0) = iE[X]$

$\phi_X(0) = 1$

we get $\frac{0}{0}$ again!!

L'Hôpital's

$$= e^{\frac{t^2}{2\sigma^2}\lim_{V\to 0}\frac{\phi_X(V)\phi_X''(V) - (\phi_X'(V))^2}{(\phi_X(V))^2}}$$

(P4)

$\phi_X''(0) = i^2 E[X^2] = -E[X^2]$

$$= e^{\frac{t^2}{2\sigma^2}\left(-E[X^2] - i^2 E[X]^2\right)} = e^{\frac{t^2}{2\sigma^2}\left(-\left(\underbrace{E[X^2] - E[X]^2}_{\sigma^2}\right)\right)} = e^{-t^2/2} = \phi_Z(t)$$

By (P8)

$$\lim_{n\to\infty}\phi_{Z_n}(t) = \phi_Z(t) \Rightarrow Z_n \xrightarrow{d} Z$$

How do we find $f_Z(z)$?

Let's use (P6). Is $\phi_Z(t) \in L^1$?

Gaussian Integral done in MATH 201

$$\int_{\mathbb{R}} e^{-t^2/2}\,dt = \sqrt{2\pi} < \infty \Rightarrow \text{Yes!}$$

$$f_Z(z) \overset{(P6)}{=} \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz}\,\phi_Z(t)\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz}e^{-t^2/2}\,dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-(itz+t^2/2)}\,dt$$

Note $\frac{t^2}{2}+itz = \left(\frac{t}{\sqrt{2}}+\frac{\sqrt{2}\,iz}{2}\right)^2 - \overbrace{\left(\frac{\sqrt{2}\,iz}{2}\right)^2}^{\frac{i^2z^2}{2} \;=\; -\frac{z^2}{2}}$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}}+\frac{\sqrt{2}\,iz}{2}\right)^2} e^{-\frac{z^2}{2}}\,dt$$

let $v = \frac{t}{\sqrt{2}}+\frac{\sqrt{2}\,iz}{2}$ $\quad \frac{dv}{dt}=\frac{1}{\sqrt{2}} \Rightarrow dt=\sqrt{2}\,dv$, $t\to\infty \Rightarrow v\to\infty$, $t\to-\infty \Rightarrow v\to-\infty$

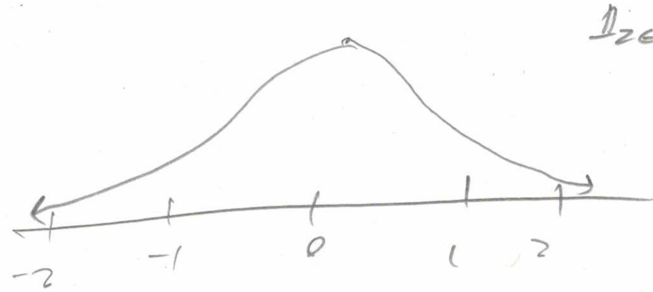
$$= \frac{1}{2\pi}e^{-\frac{z^2}{2}}\int_{\mathbb{R}} e^{-v^2}\sqrt{2}\,dt = \frac{1}{\pi\sqrt{2}}e^{-\frac{z^2}{2}}\underbrace{\int_{\mathbb{R}} e^{-v^2}\,dv}_{\text{Gaussian Integral } = \sqrt{\pi}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$$

no indicator! $S_Z = \mathbb{R}$!!!

$\mathbb{1}_{z\in\mathbb{R}} = 1$

$= N(0,1)$ the Standard normal!!!

$$\Rightarrow \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{\phi} N(0,1)$$



The CENTRAL LIMIT THM!

let $Z \sim N(0,1)$, $\phi_Z(t) = e^{-t^2/2}$

$E[Z] = ?$ $\qquad \phi_Z'(t) = -te^{-t^2/2}$, $\phi_Z'(0) = 0 \Rightarrow E[Z] \overset{P4}{=} 0$

$E[Z^2] = ?$ $\qquad \phi_Z''(t) = t^2 e^{-t^2/2} - e^{-t^2/2}$, $\phi_Z''(0) = -1$, $E[Z^2] \overset{P4}{=} \frac{\phi_Z''(0)}{i^2} = \frac{-1}{-1} = 1$

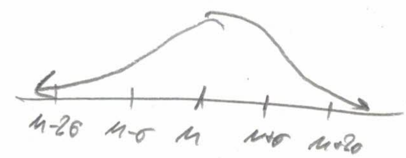
$Var[Z] = E[Z^2] - \cancel{E[Z]^2}^{\,0} = 1$

$SD[Z] = 1$

let $X = \underbrace{\mu + \sigma Z}_{}$ (shifted and scaled normal) $\Rightarrow \phi_X(t) \overset{P2}{=} e^{it\mu} e^{-\frac{(\sigma t)^2}{2}} = e^{it\mu - \frac{\sigma^2 t^2}{2}}$

$$f_X(x) \overset{P6}{=} \int_{\mathbb{R}} e^{-itx} \phi_X(t)\,dt = \dots = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = N(\mu, \sigma^2)$$

This is the general normal with mean μ, σ^2



$E(X) = \mu + \sigma \cancel{E(Z)}^{\,0} = \mu$, $Var(X) = Var(\mu + \sigma Z) = \sigma^2 Var(Z) = \sigma^2$

$X_1 \sim N(\mu_1, \sigma_1^2)$ indep of $X_2 \sim N(\mu_2, \sigma_2^2)$

$T = X_1 + X_2 \sim f_T(t) = ?$

$$\phi_T(t) \overset{P3}{=} \phi_{X_1}(t)\,\phi_{X_2}(t) = e^{it\mu_1 + \frac{\sigma_1^2 t^2}{2}} e^{it\mu_2 + \frac{\sigma_2^2 t^2}{2}} = e^{it(\mu_1+\mu_2) + \frac{(\sigma_1^2 + \sigma_2^2)t^2}{2}}$$

$$\overset{P1}{\Longrightarrow} T \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Convolution method:

difficult, but possible (M4 HW)

$$f_T(t) = \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\,dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}(t-x-\mu_2)^2} dx$$

$X \sim N(\mu, \sigma^2)$, $Y = aX+b \sim f_Y(y) = ?$

$$\phi_Y(t) \overset{(P2)}{=} e^{itb}\phi_X(at) = e^{itb} e^{iat\mu + \frac{\sigma^2 a^2 t^2}{2}} = e^{it(b+a\mu) + \frac{(a^2\sigma^2)t^2}{2}}$$

$$\overset{(P1)}{\Rightarrow} Y \sim N(b+a\mu, a^2\sigma^2)$$

$Z \sim N(0,1)$, $F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du$ does not have closed form!

Define $\Phi(z) :=$ ↑, the std. normal CDF function, need a computer!

However, there are some values to memorize to solve problems:

$\Phi(-3) = 0.1\%$ $\quad \Phi(3) = 99.9\%$

$\Phi(-2) = 2.5\%$ $\quad \Phi(2) = 97.5\%$ $\quad \Rightarrow P(Z \in [-3,3]) \approx 99.9\%$

$\Phi(-1) = 16\%$ $\quad \Phi(1) = 84\%$ $\quad P(Z \in [-2,2]) \approx 95\%$

$\Phi(0) = 50\%$ $\quad P(Z \in [-1,1]) = 68\%$

"68-95-99.9 Rule"

"Empirical Rule"

"3σ rule"

Def: if $X_n \xrightarrow{d} X \sim f_X(x)$ then for large n, $X_n \mathrel{\dot\sim} f_X(x)$ and is read "X_n is asymptotically distr as $f_X(x)$". It means "approx. distr."

If $X_1, X_2, \ldots$ iid distr with mean μ, var σ^2,

then $\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \mathrel{\dot\sim} N(0,1)$ for large n

$\Rightarrow \bar{X} \mathrel{\dot\sim} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

$\Rightarrow T \mathrel{\dot\sim} N(n\mu, (\sigma\sqrt{n})^2)$ where $T = X_1 + \ldots + X_n$ since $T = n\bar{X}$.

The CLT can be used to solve cool problems e.g.

$1 bet on lucky #7 pays $36 but its prob is 1 in 38.

$$X \sim \underbrace{\begin{cases} 36 & \text{w.p. } \frac{1}{38} \\ -1 & \text{w.p. } \frac{37}{38} \end{cases}}_{p(x)}$$

$$E(X) = 36\cdot\frac{1}{38} + -1\cdot\frac{37}{38} = -.026 = \mu$$

$$Var(X) = (36 - -.026)^2\frac{1}{38} + (-1 - -.026)^2\frac{37}{38} = 35.08$$

$$\Rightarrow \sigma = 5.92$$

What is $P(\text{you win any money after 50 bets?}) = P(T > 0)$

$X_1, X_2, \ldots, X_{50} \overset{iid}{\sim} p(x)$

$T = X_1 + \cdots + X_{50} \overset{\cdot}{\sim} N(50\mu, (\sqrt{n}\,\sigma)^2) = N(-1.31, 41.88^2)$

standardize the normal r.v.

$$P(T > 0) \approx P\left(\frac{T - -1.31}{41.88} > \frac{0 - -1.31}{41.88}\right) = P(Z > 0.03) = 1 - \Phi(.03) = 48.8\%$$

use computer

The alternative is to figure out the exact convolution of these PMF! HARD!!