Math 340 / 640 Fall 2023
Final Examination Solutions

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Full Name ____

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

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signature date

Instructions

This exam is 110 minutes (variable time per question) and closed-book. You are allowed **three** 8.5" $\times$ 11" page (front and back) "cheat sheets", blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. Show as much partial work as you can and justify each step. No food is allowed, only drinks.

Problem 1 Below are mostly unrelated problems.

(a) [8 pt / 8 pts] Let $X \sim \chi^2_k$. Find Mode$[X]$ as a function of k and indicate which values of k are valid for the expression to be the mode.

$$
\begin{aligned}
X &\sim \chi^2_k \propto k(x) = x^{k/2-1} e^{-x/2} \mathbb{1}_{x \in (0,\infty)} \\
h(x) &:= \ln\left(k(x)\right) = (k/2-1)\ln(x) - \frac{x}{2} \\
\text{Mode}[X] &= \underset{x \in (0,\infty)}{\arg\max} \;\{h(x)\} \\
h'(x) &= \frac{k/2-1}{x} - \frac{1}{2} \overset{set}{=} 0 \Rightarrow \frac{k/2-1}{x} = \frac{1}{2} \Rightarrow x_\star = \boxed{\text{k - 2}} \\
h''(x) &= -\frac{k/2-1}{x^2} < 0 \;\text{ for all }\; x \in (0,\infty) \;\text{ and for all }\; \boxed{k > 2}
\end{aligned}
$$

(b) [10 pt / 18 pts] Let $X \sim \text{BetaBinomial}(n, \alpha, \beta)$. Find $k(x)$.

$$
\begin{aligned}
X \sim p_X(x) &= \binom{n}{x} \frac{B(x+\alpha, n-x+\beta)}{B(\alpha,\beta)} \\
&\propto \frac{n!}{x!(n-x)!} B(x+\alpha, n-x+\beta) \mathbb{1}_{x \in \{0,1,\ldots,n\}} \\
&\propto \frac{1}{x!(n-x)!} \Gamma\left(x+\alpha\right) \Gamma\left(n-x+\beta\right) \mathbb{1}_{x \in \{0,1,\ldots,n\}} = k(x)
\end{aligned}
$$

(c) [8 pt / 26 pts] Let $X \sim \text{Poisson}(\lambda)$ and $Y = X\mathbb{1}_{X>0}$. Find $p_Y(y)$. Your answer must be only a function of λ and y.

$$
\begin{aligned}
X &\sim \frac{e^{-\lambda}\lambda^x}{x!} \mathbb{1}_{x \in \mathbb{N}_0} \\
p_Y(y) &= \frac{p_X(y)\mathbb{1}_{y>0}}{\sum_{u>0} p_X(u)} \\
&= \frac{p_X^{old}(y)\mathbb{1}_{y \in \mathbb{N}_0}\mathbb{1}_{y>0}}{1 - p_X(0)} \\
&= \frac{e^{-\lambda}\lambda^y}{(1-e^{-\lambda})y!} \mathbb{1}_{y \in \{1,2,\ldots\}}
\end{aligned}
$$

(d) [5 pt / 31 pts] Let $X \sim \text{Weibull}(0.5, 0.5)$. Let $a = \mathbb{P}(X > 17)$ and let $b = \mathbb{P}(X > 37 \mid X > 20)$. Circle the larger quantity: a or $\boxed{\text{b}}$

$$\begin{aligned} k \in (0,1) &\Rightarrow \forall\, x, c\; \mathbb{P}(X > x + c \mid X > c) > \mathbb{P}(X > x) \\ &\Rightarrow \mathbb{P}(X > 37 \mid X > 20) > \mathbb{P}(X > 17) \;\Rightarrow\; b > a \end{aligned}$$

(e) [13 pt / 44 pts] Let $Y \mid X = x \sim \text{Gamma}(x+1,\, \beta)$ and $X \sim \text{Geometric}(p)$. Find $f_Y(y)$ and identify it as one of the brand name rv's we studied and identify its parameter(s). Hint: $e^a = \sum_{i=0}^{\infty} \frac{a^i}{i!}$. Advice: leave this problem for last.

$$\begin{aligned} f_Y(y) &= \sum_{x \in \mathbb{R}} f_{Y|X}(y, x) p_X(x) \\ &= \sum_{x \in \mathbb{R}} \left(\frac{\beta^{x+1}}{\Gamma(x+1)} y^{x+1-1} e^{-\beta y} \mathbb{1}_{y \in (0,\infty)} \right) \left((1-p)^x p \mathbb{1}_{x \in \mathbb{N}_0} \right) \\ &= p\beta e^{-\beta y} \mathbb{1}_{y \in (0,\infty)} \sum_{x \in \mathbb{N}_0} \frac{1}{\Gamma(x+1)} (\beta y (1-p))^x \\ &= p\beta e^{-\beta y} \mathbb{1}_{y \in (0,\infty)} \sum_{x \in \mathbb{N}_0} \frac{(\beta y(1-p))^x}{x!} \\ &= p\beta e^{-\beta y} \mathbb{1}_{y \in (0,\infty)} e^{\beta y (1-p)} \\ &= p\beta e^{\beta y(1-p) - \beta y} \mathbb{1}_{y \in (0,\infty)} \\ &= p\beta e^{\beta y - \beta p y - \beta y} \mathbb{1}_{y \in (0,\infty)} \\ &= p\beta e^{-\beta p y} \mathbb{1}_{y \in (0,\infty)} \\ &= \text{Exp}(p\beta) = \text{Weibull}(1, \beta p) = \text{Gamma}(1,\, \beta p) \end{aligned}$$

(f) [8 pt / 52 pts] Let $Y \mid X = x \sim \text{Exp}\,(x)$ and $X \sim \text{InvGamma}(\alpha, \beta)$. Find $\mathbb{E}\,[Y]$.

By the law of iterated expectation,

$$\mathbb{E}\,[Y] = \mathbb{E}_X\,[\mathbb{E}_Y\,[Y \mid X]] = \mathbb{E}_X\left[\frac{1}{X}\right] = \frac{\alpha}{\beta}$$

The last equality follows from letting $U = \frac{1}{X}$. Thus $U \sim \text{Gamma}\,(\alpha,\, \beta)$ and $\mathbb{E}_X\left[\frac{1}{X}\right] = \frac{\alpha}{\beta}$.

You can also compute the expectation above manually:

$$\begin{aligned}
\mathbb{E}_X\left[\frac{1}{X}\right] &= \int_{\mathbb{R}} \frac{1}{x} f_X(x)\,dx = \int_{\mathbb{R}} \frac{1}{x}\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}\cdot x^{-\alpha-1}\cdot e^{-\frac{\beta}{x}}\cdot \mathbb{1}_{x\in(0,\infty)}\,dx \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty \left(\frac{1}{x}\right)^{\alpha+2} e^{-\beta\frac{1}{x}}\,dx \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_\infty^0 u^{\alpha+2}e^{-\beta u}\,\frac{-1}{u^2}\,du \qquad \text{Now let:} \quad \boxed{u = \frac{1}{x},\ dx = \frac{-1}{u^2}\,du} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^\infty u^{\alpha+1-1}e^{-\beta u}\,du \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\cancel{\beta^\alpha}}{\cancel{\Gamma(\alpha)}}\cdot\frac{\alpha\cdot\cancel{\Gamma(\alpha)}}{\cancel{\beta^\alpha}\cdot\beta} = \frac{\alpha}{\beta}
\end{aligned}$$

(g) [8 pt / 60 pts] Let $X_1, \ldots, X_{37} \overset{iid}{\sim} \text{ParetoI}(1, 53)$. Let $X_{(k)}$ denote the kth order statistic. Find $f_{X_{(17)}}(x)$ as a function of x only.

$$\begin{aligned}
f_{X_{(j)}}(x) &= \frac{n!}{(j-1)!(n-j)!} f(x)F(x)^{j-1}(1-F(x))^{n-j} \\
f_{X_{(j)}}(x) &= \frac{n!}{(j-1)!(n-j)!}\left(\frac{\lambda k^\lambda}{x^{\lambda+1}}\mathbb{1}_{x\in(0,\infty)}\right)\left(1-\left(\frac{k}{x}\right)^\lambda\right)^{j-1}\left(\left(\frac{k}{x}\right)^\lambda\right)^{n-j} \\
f_{X_{(17)}}(x) &= \frac{37!}{16!\,20!}\left(\frac{53}{x^{54}}\mathbb{1}_{x\in(0,\infty)}\right)\left(1-\left(\frac{1}{x}\right)^{53}\right)^{16}\frac{1}{x^{(53)(20)}} \\
f_{X_{(17)}}(x) &= \frac{(53)37!}{16!\,20!}\frac{1}{x^{1114}}\left(1-\left(\frac{1}{x}\right)^{53}\right)^{16}\mathbb{1}_{x\in(0,\infty)}
\end{aligned}$$

(h) [7 pt / 67 pts] Let $Y = aX + b$ where $a, b \in \mathbb{R}$. From the definition of the ch.f., prove $\phi_Y(t) = e^{iub/a}\phi_X(u)$ where $u = at$.

$$\phi_Y(t) = \phi_Y\left(\frac{u}{a}\right) = \mathbb{E}\left[e^{i\left(\frac{u}{a}\right)Y}\right] = \mathbb{E}\left[e^{i\left(\frac{u}{a}\right)(aX+b)}\right] = \mathbb{E}\left[e^{iuX}e^{iub/a}\right] = e^{iub/a}\,\mathbb{E}\left[e^{iuX}\right] = e^{iub/a}\phi_X(u)$$

Let $\boldsymbol{Z} \sim \mathcal{N}_2(\mathbf{0}_2, \boldsymbol{I}_2)$. Let $\boldsymbol{X} = \boldsymbol{\mu} + A\boldsymbol{Z}$ where $\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$. Use these definitions for all of the following questions.

(i) [3 pt / 70 pts] What is S_{X_2}?

Since X_2 is normally distributed, $S_{X_2} = \mathbb{R}$

(j) [6 pt / 76 pts] Find $f_{\boldsymbol{X}}(\boldsymbol{x})$ as a function of x_1, x_2 only.

$$\boldsymbol{\Sigma} = AA^\top = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow \det[\boldsymbol{\Sigma}] = 1 \Rightarrow \boldsymbol{\Sigma}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^n \det[\boldsymbol{\Sigma}]}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$$

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{4\pi^2}} e^{-\frac{1}{2}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)^\top \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)}$$

(k) [8 pt / 84 pts] Find $f_{X_2}(x)$ as a function of x only.

Any subset of a multivariate normal is itself multivariate normal. A subset of dimension one is thus normal. It's mean corresponds to the index component of $\boldsymbol{\mu}$ and its variance corresponds to the index component of the diagonal of $\boldsymbol{\Sigma}$, i.e.

$$X_2 \sim f_{X_2}(x) = \mathcal{N}(\mu_2 = 2, \boldsymbol{\Sigma}_{2,2} = 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2}$$

An alternative solution is to via the joint ch.f. from (l). By P9, we can marginalize,

$$\phi_{X_2}(t) = \phi_{\boldsymbol{X}}\left(\begin{bmatrix} 0 \\ t \end{bmatrix}\right) = e^{i2t_2 - \frac{1}{2}t^2} \implies X_2 \sim \mathcal{N}(\mu = 2, \sigma^2 = 1) = f_{X_2}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2}$$

Where the implication above is via P1.

(l) [7 pt / 91 pts] Find $\phi_{\boldsymbol{X}}(\boldsymbol{t})$ as a function of t_1, t_2 only.

$$\begin{aligned}
\phi_{\boldsymbol{X}}(\boldsymbol{t}) &= e^{i\boldsymbol{t}^\top \boldsymbol{\mu} - \frac{1}{2}\boldsymbol{t}^\top \boldsymbol{\Sigma} \boldsymbol{t}} \\
&= e^{i[t_1 \;\; t_2]\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{2}[t_1 \;\; t_2]\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}} \\
&= e^{i(t_1+2t_2) - \frac{1}{2}[t_1 \;\; t_2]\begin{bmatrix} 2t_1 - t_2 \\ t_2 - t_1 \end{bmatrix}} \\
&= e^{i(t_1+2t_2) - \frac{1}{2}(2t_1^2 - 2t_2 t_1 + t_2^2)}
\end{aligned}$$

(m) [9 pt / 100 pts] Compute $\mathbb{E}[X_1 X_2]$ numerically.

$$\begin{aligned}
\mathbb{C}\text{ov}[X_1, X_2] &:= \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2] \\
\Rightarrow\; \mathbb{E}[X_1 X_2] &= \mathbb{C}\text{ov}[X_1, X_2] + \mathbb{E}[X_1]\,\mathbb{E}[X_2] \\
&= \boldsymbol{\Sigma}_{1,2} + \mu_1 \mu_2 \\
&= (-1) + (1)(2) = 1
\end{aligned}$$

An alternative solution uses the joint ch.f. from (l). First, we find $h_{t_1,t_2}(\boldsymbol{t})$:

$$\begin{aligned}
h_{t_1,t_2}(\boldsymbol{t}) &= \frac{\partial^2}{\partial t_1\, \partial t_2}\left[\phi_{\boldsymbol{X}}(\boldsymbol{t})\right] \\
&= \frac{\partial^2}{\partial t_1\, \partial t_2}\left[e^{i(t_1+2t_2) - \frac{1}{2}\left(2t_1^2 - 2t_2 t_1 + t_2^2\right)}\right] \\
&= \frac{\partial}{\partial t_1}\left[(2i + t_1 - t_2) \cdot \phi_{\boldsymbol{X}}(\boldsymbol{t})\right] \\
&= \phi_{\boldsymbol{X}}(\boldsymbol{t}) + (2i + t_1 - t_2) \cdot (i - 2t_1 + t_2) \cdot \phi_{\boldsymbol{X}}(\boldsymbol{t})
\end{aligned}$$

By property P0,

$$h_{t_1,t_2}(\boldsymbol{0}_2) = \phi_{\boldsymbol{X}}(\boldsymbol{0}_2) + (2i)(i) \cdot \phi_{\boldsymbol{X}}(\boldsymbol{0}_2) = 1 + 2i^2 = 1 - 2 = -1$$

Using the moment generation property P4,

$$\mathbb{E}[X_1 X_2] = \frac{h_{t_1,t_2}(\boldsymbol{0}_2)}{i^2} = \frac{-1}{-1} = 1$$