Math 340 / 640 Fall 2023
Midterm Examination Two Solutions

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Full Name ______________________________

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Instructions

This exam is 110 minutes (variable time per question) and closed-book. You are allowed **two** 8.5" × 11" page (front and back) "cheat sheets", blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. Show as much partial work as you can and justify each step. No food is allowed, only drinks.

Problem 1 Below are mostly unrelated problems.

(a) [6 pt / 6 pts] Let $\mathcal{E} \sim \mathcal{N}(0,\, 1)$. Assume $\mathbb{E}[\mathcal{E}] = 0$ without proof. Show that $\mathcal{E}$ qualifies as an "error distribution".

The following is the density of the normal: $\mathcal{E} \sim \mathcal{N}(0,\, 1) = \frac{1}{\sqrt{2\pi}} e^{-\epsilon^2/2}$. Error distributions are defined by three conditions

(I) Mean center i.e. $\mathbb{E}[\mathcal{E}] = 0$. This is true as given.

(II) Density is symmetric around zero i.e. $f_{\mathcal{E}}(\epsilon) = f_{\mathcal{E}}(-\epsilon)$. This is true as $\frac{1}{\sqrt{2\pi}} e^{-\epsilon^2/2} = \frac{1}{\sqrt{2\pi}} e^{-(-\epsilon)^2/2}$.

(III) Density is strictly monotonically decreases in both directions i.e. $f'_{\mathcal{E}}(\epsilon) < 0$ for $\epsilon > 0$ and $f'_{\mathcal{E}}(\epsilon) > 0$ for $\epsilon < 0$. The derivative of the density is $f'_{\mathcal{E}}(\epsilon) = \frac{1}{\sqrt{2\pi}}(-2\epsilon) e^{-\epsilon^2/2}$ which is negative for $\epsilon > 0$ and positive for $\epsilon < 0$.

(b) [10 pt / 16 pts] Given the jdf of dependent rv's X_1 and X_2, $f_{X_1,X_2}(x_1, x_2)$, find an integral formula for the PDF of $M = X_1 X_2$.

$\boxed{\text{Answer \#1}}$ Let $U = X_2$ thus $X_2 = h_2(M, U) = U$. Then we know $M = X_1 U$ thus $X_1 = h_1(M, U) = M/U$. We then have

$$\frac{\partial h_1}{\partial M} = U, \quad \frac{\partial h_1}{\partial U} = -M/U^2, \quad \frac{\partial h_2}{\partial M} = 0, \quad \frac{\partial h_2}{\partial U} = 1, \quad |J_h| = |(U)(1) - (-M/U^2)(0)| = |U|$$

Putting it all together, $f_M(m) = \int_{\mathbb{R}} f_{X_1,X_2}(h_1, h_2)|J_h| du = \int_{\mathbb{R}} f_{X_1,X_2}\left(\frac{m}{u}, u\right) |u|\, du$.

$\boxed{\text{Answer \#2}}$ Let $U = X_1$ thus $X_1 = h_1(M, U) = U$. Then we know $M = U X_2$ thus $X_2 = h_2(M, U) = M/U$. We then have

$$\frac{\partial h_1}{\partial M} = 0, \quad \frac{\partial h_1}{\partial U} = 1, \quad \frac{\partial h_2}{\partial M} = U, \quad \frac{\partial h_2}{\partial U} = -M/U^2, \quad |J_h| = |(0)(-M/U^2) - (1)(U)| = |U|$$

Putting it all together, $f_M(m) = \int_{\mathbb{R}} f_{X_1,X_2}(h_1, h_2)|J_h| du = \int_{\mathbb{R}} f_{X_1,X_2}\left(u, \frac{m}{u}\right) |u|\, du$.

(c) [7 pt / 23 pts] Let $X \sim \text{Gamma}(\alpha,\, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \in (0,\infty)}$. Prove $\mathbb{E}[X] = \frac{\alpha}{\beta}$ without using ch.f's.

$$\mathbb{E}[X] = \int_{\mathbb{R}} x \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x\in(0,\infty)} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{(\alpha+1)-1} e^{-\beta x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\alpha\Gamma(\alpha)}{\beta\beta^\alpha} = \frac{\alpha}{\beta}$$

where the third equality follows from the gamma-like integral $\displaystyle\int_{\mathbb{R}} u^{k-1} e^{-cu} du = \frac{\Gamma(k)}{c^k}$.

(d) [5 pt / 28 pts] Let $X \sim \text{Gamma}(17, 37)$. Find an upper bound on the probability that $\mathbb{P}(X > 17)$. Hint: use the previous problem's result.

Since X is non-negative and by (c), $\mathbb{E}[X] = \frac{\alpha}{\beta} = \frac{17}{37}$, we can employ Markov's inequality:

$$\mathbb{P}(X > a) \leq \frac{\mathbb{E}[X]}{a} \Rightarrow \mathbb{P}(X > 17) \leq \frac{\frac{17}{37}}{17} \Rightarrow \mathbb{P}(X > 17) \leq \frac{1}{37}$$

(e) [6 pt / 34 pts] Let $X \sim \chi^2_k$. Prove that $\mathbb{E}[X] = k$.

Remember that $X \sim \chi^2_k = \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right)$ and by (c), $\mathbb{E}[X] = \frac{\alpha}{\beta} = \frac{k/2}{1/2} = k$.

(f) [6 pt / 40 pts] Let $X \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \in (0,\infty)}$. Let $Y = \frac{1}{X}$. Find $f_Y(y)$. Simplify as much as you can. This is called the inverse gamma distribution.

$Y = g(X) = \frac{1}{X}$ implies $X = g^{-1}(Y) = \frac{1}{Y}$ and its absolute derivative is $\left|\frac{d}{dy}[g^{-1}(y)]\right| = y^{-2}$. Putting this all together we have:

$$\begin{aligned}
f_Y y = f_X(g^{-1}(y)) \left|\frac{d}{dy}[g^{-1}(y)]\right| &= \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{y}\right)^{\alpha-1} e^{-\beta\left(\frac{1}{y}\right)} \mathbb{1}_{\left(\frac{1}{y}\right) \in (0,\infty)} y^{-2} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-\alpha+1} y^{-2} e^{-\frac{\beta}{y}} \mathbb{1}_{y \in (0,\infty)} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} e^{-\frac{\beta}{y}} \mathbb{1}_{y \in (0,\infty)}
\end{aligned}$$

(g) [6 pt / 46 pts] Let $X \sim T_k$. Let $Y = \mu + \sigma X$ where $\mu \in \mathbb{R}$ and $\sigma > 0$. Find $f_Y(y)$.

$$\begin{aligned}
X \sim f_X(x) &= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi k}\,\Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{x^2}{k}\right)^{-\frac{k+1}{2}} \\
Y \sim f_Y(y) &= \frac{1}{|\sigma|} f_X\left(\frac{y-\mu}{\sigma}\right) \\
&= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi k \sigma^2}\,\Gamma\left(\frac{k}{2}\right)} \left(1 + \frac{(y-\mu)^2}{k\sigma^2}\right)^{-\frac{k+1}{2}}
\end{aligned}$$

(h) [5 pt / 51 pts] If $\boldsymbol{X} \sim \text{Multinom}\left(17, \begin{bmatrix}\frac{1}{2} & \frac{1}{3} & \frac{1}{6}\end{bmatrix}^\top\right)$, compute $\mathbb{C}\text{ov}\left[X_1, X_3\right]$.

$$\mathbb{C}\text{ov}\left[X_1, X_3\right] = -np_1p_3 = -17 \cdot \frac{1}{2} \cdot \frac{1}{6} = -\frac{17}{12}$$

(i) [6 pt / 57 pts] Let $X \sim \text{Binomial}\left(n,\, p\right)$. Let $Y = \ln\left(X+1\right)$. Find $p_Y(y)$.

$Y = g(X) = \ln\left(X+1\right)$ and thus $X = g^{-1}(Y) = e^Y - 1$ and thus

$$p_Y(y) = p_X(g^{-1}(y)) = p_X(e^y - 1) = \binom{n}{e^y - 1} p^{e^y - 1}(1-p)^{n - e^y + 1}$$

(j) [10 pt / 67 pts] Prove the following and justify each step with theorems and results from class:

$$\frac{\bar{X} - \mu}{\frac{S_n}{\sqrt{n}}} \overset{d}{\to} \mathcal{N}\left(0,\, 1\right)$$

In class, we proved $S_n^2 \overset{p}{\to} \sigma^2$ and thus by the CMT we have $S_n \overset{p}{\to} \sigma$ where $g(x) = \sqrt{x}$. Again by the CMT we have

$$S_n \overset{p}{\to} \sigma \;\Rightarrow\; \frac{\frac{\sigma}{\sqrt{n}}}{\frac{S_n}{\sqrt{n}}} \overset{p}{\to} 1 \;\text{ where }\; g(x) = \frac{\frac{\sigma}{\sqrt{n}}}{\frac{x}{\sqrt{n}}}$$

We also have the CLT result from class:

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \overset{d}{\to} \mathcal{N}\left(0,\, 1\right)$$

By algebra we have that

$$\frac{\bar{X} - \mu}{\frac{S_n}{\sqrt{n}}} = \underbrace{\frac{\frac{\sigma}{\sqrt{n}}}{\frac{S_n}{\sqrt{n}}}}_{A_n} \underbrace{\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}}_{B_n} \overset{d}{\to} \mathcal{N}\left(0,\, 1\right)$$

where the $\overset{d}{\to} \mathcal{N}\left(0,\, 1\right)$ above follows from Slutsky's theorem A.

(k) [8 pt / 75 pts] Let $X_n \sim \text{Weibull}(k, n)$ where $k > 0$. Prove $X_n \xrightarrow{p} 0$ from the definition of convergence in probability.

The $X_n \sim \text{Weibull}(k, n)$ rv is non-negative with CDF $F_{X_n}(x) = 1 - e^{-(nx)^k}$ and thus $S_{X_n}(x) = e^{-(nx)^k}$. By the definition, $X_n \xrightarrow{p} 0$ means:

$$\forall \epsilon > 0, \quad \lim_{n \to \infty} \mathbb{P}\left(|X_n - 0| > \epsilon\right) = 0$$

We pick any positive ϵ and compute:

$$\lim_{n \to \infty} \mathbb{P}\left(|X_n - 0| > \epsilon\right) = \lim_{n \to \infty} \mathbb{P}\left(X_n > \epsilon\right) = \lim_{n \to \infty} S_{X_n}(\epsilon) = \lim_{n \to \infty} e^{-(n\epsilon)^k} = 0$$

where the first equality above follows from the non-negativity of X.

(l) [7 pt / 82 pts] If $X \sim \text{ParetoI}(k = 0.17, \lambda = 2.37)$, then $\mathbb{E}[X] = 0.294$ and $\mathbb{V}\text{ar}[X] = 0.0986$. Let $X_1, X_2 \overset{iid}{\sim} \text{ParetoI}(k = 0.17, \lambda = 2.37)$, find an upper bound on $\mathbb{P}(X_1 + X_2 > 3.14)$ to the nearest three significant digits.

$\boxed{\text{Preferred Answer}}$ By Chebyshev's inequality corollary from class, if $b > 2\mu$ we have

$$\mathbb{P}(Y > b) \leq \frac{\sigma_Y^2}{(b - \mu_Y)^2}$$

Since $3.14 > 2\mu = 2 \cdot 0.294 = 0.588$ we can employ this inequality and substitute to find:

$$\mathbb{P}(X_1 + X_2 > 3.14) \leq \frac{\mathbb{V}\text{ar}[X_1 + X_2]}{(3.14 - \mathbb{E}[X_1 + X_2])^2} = \frac{2\sigma^2}{(3.14 - 2\mu)^2} = \frac{2 \cdot 0.0986}{(3.14 - 2 \cdot 0.294)^2} = 0.0303$$

$\boxed{\text{Acceptable Answer}}$ By Markov's inequality and the fact that X_1, X_2 are both non-negative so their convolution is non-negative, we have

$$\mathbb{P}(X_1 + X_2 > 3.14) \leq \frac{\mathbb{E}[X_1 + X_2]}{3.14} = \frac{2\mu}{3.14} = \frac{2 \cdot 0.294}{3.14} = 0.187$$

(m) [8 pt / 90 pts] Let $Z_1, Z_2 \overset{iid}{\sim} \mathcal{N}(0,\, 1)$. Let $X = Z_1/Z_2$. What is the value of $\phi_X'(0)$?

The rv Z_1/Z_2 is Cauchy-distributed which means $\mathbb{E}[X]$ is undefined. Since $\phi_X'(0) = \frac{\mathbb{E}[X]}{i}$, the value of $\phi_X'(0)$ is undefined.

(n) [10 pt / 100 pts] Let $Z_1, Z_2 \overset{iid}{\sim} \mathcal{N}(0, 1)$. Let $X = \frac{1}{2}Z_1^2 + Z_1 Z_2 + \frac{1}{2}Z_2^2$. How is X distributed? Hint: use Cochran's thm.

Let $\boldsymbol{Z} = [Z_1 \; Z_2]^\top$. By playing around, we conjure a B_1 matrix so that X is expressed as a quadratic form in the standard normal vector $\boldsymbol{Z}$:

$$
\begin{aligned}
X = \boldsymbol{Z}^\top B_1 \boldsymbol{Z} &= [Z_1 \; Z_2] \underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}_{B_1} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \\
&= [Z_1 \; Z_2] \begin{bmatrix} \frac{1}{2}Z_1 + \frac{1}{2}Z_2 \\ \frac{1}{2}Z_1 + \frac{1}{2}Z_2 \end{bmatrix} \\
&= \frac{1}{2}Z_1^2 + \frac{1}{2}Z_1 Z_2 + \frac{1}{2}Z_2 Z_1 + \frac{1}{2}Z_2^2 \\
&= \frac{1}{2}Z_1^2 + 2 Z_1 Z_2 + \frac{1}{2}Z_2^2
\end{aligned}
$$

This matrix B_1 has $\text{rank}[B_1] = 1$. To use Cochran's theorem, we conjure a matrix B_2:

$$
B_2 = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}
$$

which has $\text{rank}[B_2] = 1$ which can be seen by taking the negative of column 2 and noticing it is the same as column 1. Since $B_1 + B_2 = I_2$ and $\text{rank}[B_1] + \text{rank}[B_2] = 2$, we can use Cochran's Thm's result that the quadratic forms are chi-squared distributed and thus:

$$
X = \boldsymbol{Z}^\top B_1 \boldsymbol{Z} \sim \chi^2_{\text{rank}[B_1]} = \chi^2_1
$$