MATH 380/680 Lec 5

AUDIO DEMO

Def of ch.f. of rv X:

$$\phi_X(t) := E[e^{itX}] = \sum_{x \in \mathbb{R}} e^{itx} p(x) \quad \text{(discrete rv's)}$$

$$= \int_{\mathbb{R}} e^{itx} f(x)\,dx \quad \text{(cont. rv's)}$$

Properties

(P0) $\phi_X(0) = 1 \quad \forall$ rv's

(P1) $\phi_X(t) = \phi_Y(t) \iff X \overset{d}{=} Y$ uniqueness

(P2) $Y = aX + b \implies \phi_Y(t) = e^{itb}\phi_X(at)$

(P3) $X_1, \ldots, X_n \overset{ind}{\sim}$, $T = X_1 + \cdots + X_n$ $\quad \phi_T(t) = \phi_{X_1}(t) \cdots \phi_{X_n}(t) = (\phi_X(t))^n$ if $X_1, \ldots, X_n \overset{iid}{\sim}$

(P4) "Moment Generation"

$$\phi_X'(t) = \frac{d}{dt}\left[E[e^{itX}]\right] = E\left[\frac{d}{dt}[e^{itX}]\right]$$

Cond. sufficient to exchange diff. and integr.

$= E\left[iXe^{itX}\right]$, $\phi_X'(0) = E[iX] \Rightarrow E[X] = \frac{\phi_X'(0)}{i}$

$$\phi_X''(t) = \frac{d^2}{dt^2}\left[E\left[e^{itX}\right]\right] = E\left[\frac{d^2}{dt^2}\left[e^{itX}\right]\right] = E\left[i^2X^2e^{itX}\right]$$

$$\phi_X''(0) = E\left[i^2X^2\right] \Rightarrow E(X^2) = \frac{\phi_X''(0)}{i^2}$$

$$\Rightarrow E\left(X^n\right) = \frac{\phi_X^{(n)}(0)}{i^n}$$ if the moment exists

There is no guarantee $\phi_X^{(n)}(0)$ is finite

(P5) Existence and Boundedness $\phi_X(t) \in [-1, 1]$ Proof:

$$\left|\phi_X(t)\right| = \left|E\left[e^{itX}\right]\right| \overset{cont}{=} \left|\int_{\mathbb{R}} e^{itx} f(x)\,dx\right| \le \int_{\mathbb{R}} \left|e^{itx} f(x)\right| dx = \int \left|e^{itx}\right| f(x)\,dx = \int f(x)\,dx = 1$$

$$\left|e^{itx}\right| = \sqrt{\sin^2(tx) + \cos^2(tx)} = 1$$

discr.

$$\left|\sum_{x\in\mathbb{R}} e^{itx} p(x)\right| \le \sum_{x\in\mathbb{R}} \left|e^{itx} p(x)\right| = \sum_{x\in\mathbb{R}} \left|e^{itx}\right| p(x) = \sum_{x\in\mathbb{R}} p(x) = 1$$

(P6) Inversion $\phi_X(t) \in L^1 \Rightarrow f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\phi_X(t)\,dt$ (on HW)

(P7) Levy's Continuity Thm $\lim \phi_{X_n}(t) = \phi_X(t) \Rightarrow X_n \xrightarrow{d} X$

(P8) Levy's CDF Thm. If $\phi_X(t) \notin L^1$ no inversion exists. This usually means that X is a discrete rv. For all ch.f's:

$$P(X \in [a,b]) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it}\phi_X(t)\,dt$$ we would use this

Let's find some ch.f's

$X \sim \mathrm{Deg}(c)$

$$\phi_X(t) = E[e^{itX}] = \sum_{x \in \mathbb{R}} e^{itx} \mathbb{1}_{x \in \{c\}} = e^{itc} \quad \text{Easy!}$$

$X \sim \mathrm{Bern}(p)$

$$\phi_X(t) = E[e^{itX}] = \sum_{x \in \mathbb{R}} e^{itx} p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} = \sum_{x \in \{0,1\}} e^{itx} p^x (1-p)^{1-x}$$

$$= \cancel{e^{it(0)}}^{1} p^{(0)} (1-p)^{1-(0)} + e^{it(1)} p^{(1)} (1-p)^{1-(1)}$$

$$= 1-p + e^{it} p$$

$X \sim \mathrm{Binom}(n,p)$

$$\phi_X(t) = E[e^{itX}] = \sum_{x \in \mathbb{R}} e^{itx} \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\ldots,n\}}$$

$$= \sum_{x \in \{0,1,\ldots,n\}} \binom{n}{x} (e^{it} p)^x (1-p)^{n-x}$$

Binom thm

$$= (e^{it} p + 1 - p)^n$$

Prove $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Bern}(p) \Rightarrow T = X_1 + \ldots + X_n \sim \mathrm{Binom}(n,p)$

By (P3), $\phi_T(t) = (\phi_X(t))^n = (1-p+e^{it}p)^n \Rightarrow T \sim \mathrm{Binom}(n,p)$ by (P1) !!

$X \sim \mathrm{Geom}(p)$

by geom series since $|e^{it}| = 1$

$$\phi_X(t) = E[e^{itX}] = \sum_{x \in \mathbb{R}} e^{itx} (1-p)^x p \mathbb{1}_{x \in \mathbb{N}_0} = p \sum_{x \in \mathbb{N}_0} (e^{it}(1-p))^x = \frac{p}{1 - e^{it}(1-p)}$$

$X \sim NegBin(r, p)$

$$\phi_X(t) = E[e^{itX}] = \sum_{x \in \mathbb{R}} e^{itx} \binom{r+x-1}{r-1} p^r (1-p)^x \mathbb{1}_{x \in \mathbb{N}_0} = \text{[illegible]}$$

$$= \left(\frac{p}{1 - e^{it}(1-p)} \right)^r$$

Since if $X_1, \dots, X_r \overset{iid}{\sim} Geom$ $T = X_1 + \dots + X_r$ $\phi_T(t) = \left(\frac{p}{1 - e^{it}(1-p)} \right)^r$ by P3,

by P1 $T \sim NegBin(r, p)$

$X \sim Exp(\lambda)$

$$\phi_X(t) = E[e^{itX}] = \int_{\mathbb{R}} e^{itx} \lambda e^{-\lambda x} \mathbb{1}_{x \in (0,\infty)} dx = \lambda \int_{x \in (0,\infty)} e^{(it-\lambda)x} dx$$

$$= \lambda \frac{1}{(it-\lambda)} \left[e^{(it-\lambda)x} \right]_0^\infty = \frac{\lambda}{it-\lambda} \left[e^{itx} e^{-\lambda x} \right]_0^\infty$$

$$= \frac{\lambda}{it-\lambda} \left(\overbrace{i\sin(tx) e^{-\lambda x}}^{\lim = 0} + \overbrace{\cos(tx) e^{-\lambda x}}^{\lim = 0} - \cancel{e^{it(0)}} e^{-\lambda(0)} \right)$$

$$= \frac{\lambda}{\lambda - it}$$

$\lim e^{itx}$ needs complex analysis for rigorous proof

but note $\forall x$ $|e^{itx}| \leq 1$ by P5

so it cannot grow to outweigh the $\lim e^{-\lambda x}$. So we can handwave..

$X \sim U(a,b)$ $\phi_X(t) = E[e^{itX}] = \int_{x \in [a,b]} e^{itx} \frac{1}{b-a} dx = \frac{1}{b-a} \frac{1}{it} \left(e^{itb} - e^{ita} \right)$

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let $X_1, X_2, \dots \overset{iid}{\sim}$ s.t. $E[X] = \mu < \infty$

let $\overline{X}_n := \frac{X_1 + \dots + X_n}{n}$ the "average r.v." $\xrightarrow{n \to \infty} \mu$ intuitively

$$\phi_{\overline{X}_n}(t) = \cancel{e^{it(0)}} \phi_{X_1 + \dots + X_n}\left(\frac{t}{n}\right) \text{ by P2}$$

$$= \left(\phi_X\left(\frac{t}{n}\right) \right)^n \text{ by P3}$$

What happens when $\lim_{n \to \infty} \phi_{\overline{X}_n}(t)$?

$$= \lim_{n\to\infty} \left(\phi_X\left(\tfrac{t}{n}\right)\right)^n = \lim_{n\to\infty} e^{\ln\left(\left(\phi_X\left(\frac{t}{n}\right)\right)^n\right)}$$ MAGIC!!!

$$= \lim_{n\to\infty} e^{n \ln\left(\phi_X\left(\frac{t}{n}\right)\right)} = e^{\lim_{n\to\infty} \frac{\ln\left(\phi_X\left(\frac{t}{n}\right)\right)}{\frac{1}{n}}}$$

let $v = \frac{t}{n} \Rightarrow n\to\infty \Rightarrow v \to 0 \Rightarrow \frac{1}{n} = \frac{v}{t}$

$$= e^{t \lim_{v\to 0} \frac{\ln(\phi_X(v))}{v}}$$ if I plug in $v=0$ we get $\frac{\ln(\overbrace{\phi(0)}^{1 \text{ by (P0)}})}{0} = \frac{0}{0}$

L'Hopital

$$= e^{t \lim_{v\to 0} \frac{\phi_X'(v)}{\phi_X(v)}} = e^{\frac{t\, \overbrace{\phi_X'(0)}^{i\mu \text{ by (P4)}}}{\underbrace{\phi_X(0)}_{1 \text{ by (P0)}}}} = e^{it\mu}$$

$$\Rightarrow \lim_{n\to\infty} \phi_{\overline{X}_n}(t) = \underbrace{e^{it\mu}}_{\text{ch.f for } \mathrm{Deg}(\mu)} \xRightarrow{(P8)} \overline{X}_n \xrightarrow{d} \mathrm{Deg}(\mu)$$

or just "$\overline{X}_n \xrightarrow{d} \mu$" colloquially

A "Very Weak Law of Large #'s". We will prove a stronger result later.
(LLN) The strongest result is proved in a measure theory class.