Math 340/640 Lec 4

Convergence in distribution

Consider a sequence of r.v.'s $X_1, X_2, \ldots$ which we denote X_n

We say X_n converges in distribution to X denoted $X_n \xrightarrow{d} X$

if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$. We've seen this before...

Let $X_n \sim \text{Geom}\left(\frac{\lambda}{n}\right)$ where $\lambda \in (0, \infty)$. We proved $X_n \xrightarrow{d} X$

where $X \sim \text{Exp}(\lambda)$. Note X_n are discrete and X is continuous.

Any permutation of discrete / cont. are acceptable.

We will do more examples on $\xrightarrow{d}$ soon. Back to the arc...

X_1, X_2 are cont r.v.'s

$T = X_1 + X_2 \sim f_T(t) = ?$

Method #1: $f_T(t) = F'(t)$ if CDF is known

Method #2: Long proof from last class...

$$f_T(t) = \int_{\mathbb{R}} f_{X_1, X_2}(x, t-x)\, dx$$

general conv. formula

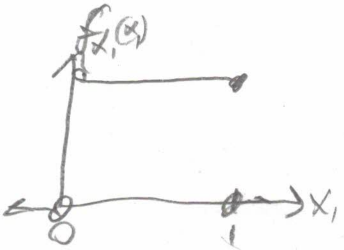
$$= \int_{x\in\mathbb{R}} f_{X_1}(x)\, f_{X_2}(t-x)\,dx = \int_{x\in S_{X_1}} f^{old}_{X_1}(x)\, f^{old}_{X_2}(t-x)\,dx$$

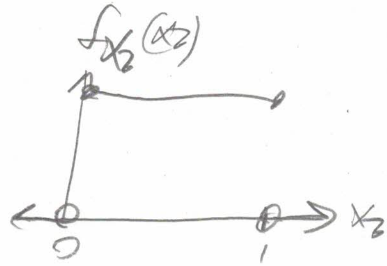
if $X_1, X_2 \overset{ind}{\sim}$

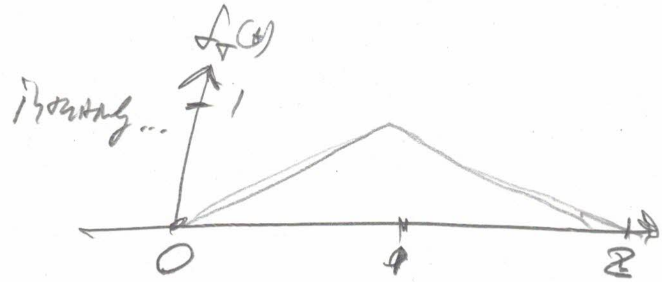
$$= \int_{x\in\mathbb{R}} f(x)\, f(t-x)\,dx = \int_{x\in S_X} f^{old}(x)\, f^{old}(t-x)\,dx$$

if $X_1, X_2 \overset{iid}{\sim}$

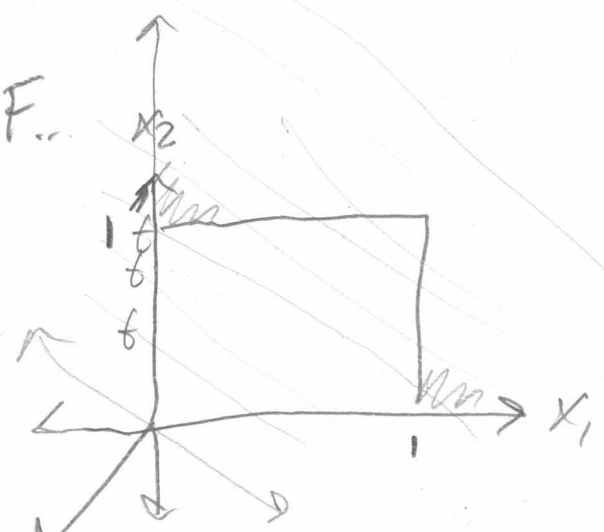
$X_1, X_2 \overset{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f(t) = ?$ $S_T = [0,2]$







JDF...



$f_{X_1,X_2}(x_1,x_2)$

$F_T(t) = P(T \le t)$

$$= \begin{cases} 0 & \text{if } t < 0 \\ t^2/2 & \text{if } t \in [0,1] \\ t^2/2 - \cancel{2}\frac{(t-1)^2}{\cancel{2}} & \text{if } t \in (1,2] \\ 1 & \text{if } t > 2 \end{cases}$$

$$\Rightarrow f_T(t) = \frac{d}{dt} F(t) = \begin{cases} 0 & \text{if } t<0 \\ t & \text{if } t\in[0,1] \\ t-(2t-2) = 2-t & \text{if } t\in(1,2] \\ 0 & \text{if } t>2 \end{cases}$$

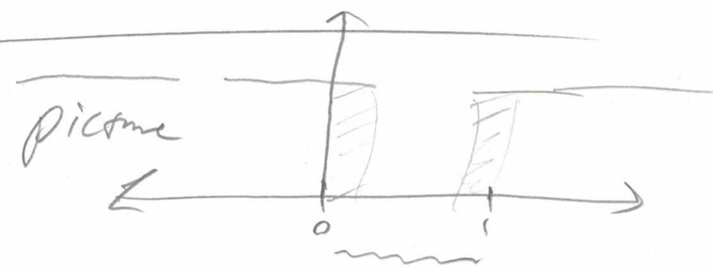
Why is the CDF method hard??

Because those volumes are double integrals under difficult blankets! This blanket $= 1$ always

$\Rightarrow$ EASY!!

Method #2: conv. formula

picture



Use iid dd-style formula

$$f_T(t) = \int_{x\in S_x} f^{dd}(x)\, f^{dd}(t-x)\, \mathbb{1}_{t-x\in S_x}\, dx = \int_{x\in[0,1]} (1)(1)\, \mathbb{1}_{\underbrace{t-x\in[0,1]}_{\underbrace{x-t\in[-1,0]}_{x\in[t-1,t]}}}\, dx = \int_{x\in[0,1]\cap[t-1,t]} dx$$

this intersection is different based on values of t

$[0,1]\cap[-2,-1]$

$[0,1]\cap[-0.7,0.3]$

$[0,1]\cap[0.7,1.7]$

$[0,1]\cap[2,3]$

$$= \begin{cases} 0 & \text{if } t<0 \\ t & \text{if } t\in[0,1] \\ 1-(t-1) = 2-t & \text{if } t\in(1,2] \\ 0 & \text{if } t>2 \end{cases}$$

easier and always works!

$X_1, X_2 \overset{iid}{\sim} Exp(\lambda) = \lambda e^{-\lambda x} \mathbb{1}_{x \in (0,\infty)}$, $T_2 = X_1 + X_2 \sim f_{T_2}(t) = ?$

$$f_{T_2}(t) = \int_{x \in (0,\infty)} \left(\lambda e^{-\lambda x}\right)\left(\lambda e^{-\lambda(t-x)}\right) \mathbb{1}_{\underbrace{\underbrace{t-x \in (0,\infty)}_{x-t \in (-\infty,0)}}_{x \in (-\infty,t)}} \, dx$$

Wait for two processes

$$= \lambda^2 e^{-\lambda t} \int_{x \in (0,\infty)} \mathbb{1}_{x \in (-\infty,t)}$$

$$= \lambda^2 e^{-\lambda t} \mathbb{1}_{t \in (0,\infty)} \int_{x \in (0,t)} dx$$

$$= t \lambda^2 e^{-\lambda t} \mathbb{1}_{t \in (0,\infty)}$$

$$\overset{\cdot\cdot}{=}$$

$Erlang(2,\lambda)$ $\quad S_{T_2} = (0,\infty)$, $\lambda \in (0,\infty)$

[illegible]

$X_1, X_2, X_3 \overset{iid}{\sim} Exp(\lambda)$ $\quad T_3 = X_1 + X_2 + X_3 = T_2 + X_3$

$$f_{T_3}(t) = \int_{x \in (0,\infty)} \left(x \lambda^2 e^{-\lambda x}\right) \lambda e^{-\lambda(t-x)} \mathbb{1}_{\underbrace{t-x \in (0,\infty)}_{x \in (-\infty,t)}} \, dx$$

$$= \lambda^3 e^{-\lambda t} \int_{x \in (0,\infty)} x \, \mathbb{1}_{x \in (-\infty,t)} \, dx$$

$$= \lambda^3 e^{-\lambda t} \mathbb{1}_{t \in (0,\infty)} \int_{x \in (0,t)} x \, dx$$

$$= \frac{t^2 \lambda^3 e^{-\lambda t}}{2} \mathbb{1}_{t \in (0,\infty)} = Erlang(3,\lambda)$$

$X_1, X_2, X_3, X_4 \overset{iid}{\sim} Exp(\lambda)$ $\quad T_4 = T_3 + X \sim f_T(t) = ?$

$$f_{T_4}(t) = \int_{x \in (0,\infty)} \frac{x^2 \lambda^3 e^{-\lambda x}}{2} \lambda e^{-\lambda(t-x)} \underbrace{\mathbb{1}_{t-x \in (0,\infty)}}_{x \in (0,t)} \, dx$$

$$= \frac{\lambda^4 e^{-\lambda t}}{2} \mathbb{1}_{t \in (0,\infty)} \int_{x \in (0,t)} x^2 \, dx$$

$$= \frac{t^3 \lambda^4 e^{-\lambda t}}{2 \cdot 3} \mathbb{1}_{t \in (0,\infty)} = Erlang(4, \lambda)$$

$\vdots$ Induction $\quad T_k = X_1 + \dots + X_k, \quad f_{T_k}(t) = ?$

$$f_{T_k}(t) = \frac{t^{k-1} \lambda^k e^{-\lambda t}}{(k-1)!} \mathbb{1}_{t \in (0,\infty)} = Erlang(k, \lambda)$$

$k \in \mathbb{N}, \lambda \in (0,\infty)$

$S_{T_k} = (0,\infty)$

discovered by AK Erlang sometime between 1900-1920. Used to "approximate" model # of phone calls in a given time period. Very useful. We'll see this example when we do the "Poisson-Process" later.

We've now done a review of most rv's from 241, PMF's, PDF's, CDF's, Convolutions (new), Erlang (new)

New Unit:

Moment Generating Functions (mgf's) and Characteristic Functions (ch.f's)

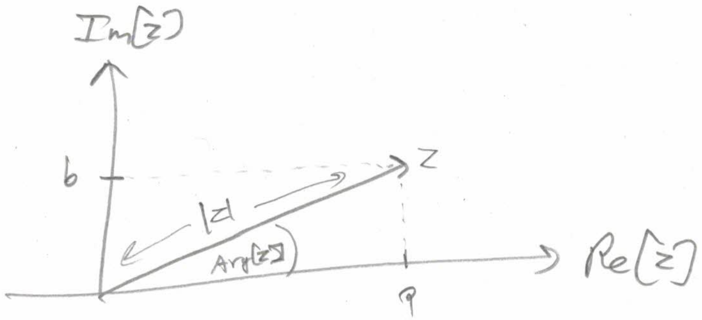
Review of imaginary #'s

$i := \sqrt{-1}$

complex #'s

let $a, b \in \mathbb{R}$ and $z := a + bi \in \mathbb{C}$,

$Re[z] = a$, $Im[z] = b$, $|z| = \sqrt{a^2+b^2}$



exponents of i

$i^2 = (\sqrt{-1})^2 = -1$

$i^3 = i^2 i = -\sqrt{-1} = -i$

$i^4 = (i^2)^2 = 1$

$i^5 = i^4 i = i$

$\vdots$

$i^n = i^{n \bmod 4}$

Recall from Calculus:

$$e^x = \sum_{k \in \mathbb{N}} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$e^{itx} = 1 + itx + \underbrace{\frac{(itx)^2}{2!}}_{= -t^2x^2} + \underbrace{\frac{(itx)^3}{3!}}_{= -it^3x^3} + \underbrace{\frac{(itx)^4}{4!}}_{= t^4x^4} + \underbrace{\frac{(itx)^5}{5!}}_{= it^5x^5} + \dots$$

Recall

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots$$

$$i\sin(tx) = itx - i\frac{t^3x^3}{3!} + i\frac{t^5x^5}{5!} - + \dots$$

$$\cos(tx) = -\frac{t^2x^2}{2!} + \frac{t^4x^4}{4!}$$

$$\Rightarrow e^{itx} = i\sin(tx) + \cos(tx)$$

if $tx = \pi$

$$e^{i\pi} = i\,\cancel{\sin(\pi)}^{\,0} + \underbrace{\cos(\pi)}_{-1} \Rightarrow e^{i\pi}+1=0 \quad \text{(Euler's Identity)}$$

Define $L^1 := \left\{ g : \int_{\mathbb{R}} |g(x)|\, dx < \infty \right\}$ "L^1 integrable" functions or "absolutely integrable" functions

Are all PDF's $\in L^1$? Yes! HW...

If $g \in L^1$, $\exists\, \hat{g}$, the "Fourier transform" of g:

$$\hat{g}(\omega) := \int_{\mathbb{R}} e^{-i2\pi\omega t} g(t)\, dx$$

$g(t)$ is called the "time domain", $\hat{g}(\omega)$ is called the "frequency domain"!

Also called "Forward Fourier Transform" or "Fourier Analysis".

Not so interesting.. but if $\hat{g} \in L^1$ (not guaranteed) then

$$g(t) = \int_{\mathbb{R}} e^{i2\pi\omega x} \hat{g}(\omega)\, d\omega$$

which is called "Inverse Fourier Transform" or "Reverse Fourier Transform" or "Fourier Synthesis"!

the Fourier transform is essentially a decomposition of $g(t)$ into possible infinite superimposed sines and cosines.

Fourier Inversion Thm: if g cont, $g, \hat{g} \in L^1 \Rightarrow g, \hat{g}$ are 1:1.

DEMO

Back to prob. Recall from 241

$$E[g(X)] \overset{\text{discrete}}{=} \sum_{x\in\mathbb{R}} g(x)p(x) \leftarrow \text{PMF}$$

$$\overset{\text{cts}}{=} \int_{x\in\mathbb{R}} g(x) f(x)\,dx \leftarrow \text{PDF}$$

$X_1, X_2 \overset{\text{ind}}{\sim}$

$$E[X_1 X_2] = E[X_1]E[X_2]$$

$$\Rightarrow E[g(X_1)h(X_2)] = E[g(X_1)]E[h(X_2)]$$

Define the ch.f. of X

$$\phi_X(\underset{\text{free variable}}{t}) := E[e^{itX}] = \sum_{x\in\mathbb{R}} e^{itx}p(x)$$

$$= \int_{x\in\mathbb{R}} e^{itx} f(x)\,dx$$

$X_1, \dots, X_n \overset{\text{ind}}{\sim}$

(P3) $\phi_T(t) = \prod_{i=1}^{n} \phi_{X_i}(t) \overset{\text{if iid}}{=} (\phi_X(t))^n$

Useful properties of ch.f.'s

(P0) $\phi_X(0) = E[e^{it(0)}] = E[1] = 1$

(P1) If $\phi_X(t) = \phi_Y(t) \iff X \overset{d}{=} Y$ "uniqueness"

(P2) If $Y = aX+b \Rightarrow \phi_Y(t) = e^{itb}\phi_X(at)$ Proof:

$$\phi_Y(t) = E[e^{itY}] = E[e^{it(aX+b)}] = E[e^{itaX}e^{itb}] = e^{itb}\phi_X(at)$$

(P3) If $X_1, X_2 \overset{\text{ind.}}{\sim}$ and $T = X_1 + X_2$ then $\phi_T(t) = \phi_{X_1}(t)\phi_{X_2}(t) \overset{\text{if iid}}{=} (\phi_X(t))^2$ Proof:

$$\phi_T(t) = E[e^{it(X_1+X_2)}] = E[e^{itX_1}e^{itX_2}] = E[e^{itX_1}]E[e^{itX_2}] = \phi_{X_1}(t)\phi_{X_2}(t)$$

(P4) "Moment Generation"

$$\phi_X'(t) = \frac{d}{dt}\left[E[e^{itX}]\right] = E\left[\frac{d}{dt}[e^{itX}]\right]$$

Conditions satisfied to exchange deriv. and integr. and deriv and summing