MATH 340/640 Fall 2024 Homework #3

Professor Adam Kapelner

Due by email 11:59PM October 6, 2024

(this document last updated Thursday 10th October, 2024 at 11:40am)

Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out." Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, review Math 241 concerning random variables, support, parameter space, PMF's, the central limit theorem (CLT), variance, covariance.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems. "[MA]" are for those registered for 621 and extra credit otherwise.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to 5 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. The easiest way to use overleaf is to copy the raw text from hwxx.tex and preamble.tex into two new overleaf tex files with the same name. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks which are *not* on this printout. Keep this first page printed for your records.

NAME: ____

Problem 1

These exercises will introduce implications of the CLT.

(a) [in the notes] State the CLT with all of its assumptions.

(b) [in the notes] State the asymptotic distribution of $\bar{X}_n$, the average of the rv's.

(c) [in the notes] State the asymptotic distribution of T_n, the total of the rv's.

(d) [harder] According to this site, the S&P500 delivers an average of 7.7% per year with a standard deviation of 19.1%. What is the probability the average yearly return over 20yr is positive? Note: this is a different question from "do you make money over 20yr given an initial investment"? We will learn how to answer that latter question later in the semester when we learn about the LogNormal rv.

Problem 2

These exercises will introduce the concept of covariance (the metric that gauges linear dependence between two rv's).

(a) [in the notes] Prove that $\mathbb{C}\text{ov}[X_1, X_2] = \mathbb{E}[(X_1 - \mu_1)(X_2 - \mu_2)]$.

(b) [in the notes] Prove that $\mathbb{C}\mathrm{ov}\left[X_1, X_2\right] = \mathbb{C}\mathrm{ov}\left[X_2, X_1\right]$.

(c) [in the notes] Prove that $\mathbb{C}\mathrm{ov}\left[X_1 + X_3, X_2\right] = \mathbb{C}\mathrm{ov}\left[X_1, X_2\right] + \mathbb{C}\mathrm{ov}\left[X_3, X_2\right]$.

(d) [difficult] Prove that

$$\mathbb{V}\mathrm{ar}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{C}\mathrm{ov}\left[X_i, X_j\right].$$

Hint: use induction and play with the sum notation.

(e) [harder] [MA] Prove that

$$\mathbb{C}\text{ov}\left[\sum_{i\in A} X_i, \sum_{j\in B} Y_j\right] = \sum_{i\in A}\sum_{j\in B} \mathbb{C}\text{ov}\left[X_i, Y_j\right]$$

.

(f) [in the notes] Prove the Cauchy-Schwartz Inequality.

(g) [in the notes] Prove the Covariance Inequality by invoking the Cauchy-Schwartz Inequality.

(h) [harder] Let Q be a non-negative, non-degenerate discrete rv. Prove $\mathbb{E}[Q] > 0$.

(i) [harder] Correlation ρ is a normalized unitless covariance metric. It is defined for for any two rv's X_1 and X_2 as:

$$\rho_{1,2} := \text{Corr}[X_1,\, X_2] := \frac{\sigma_{1,2}}{\sigma_1 \sigma_2} = \frac{\mathbb{C}\text{ov}[X_1, X_2]}{\sqrt{\mathbb{V}\text{ar}[X_1]\,\mathbb{V}\text{ar}[X_2]}}.$$

Prove that $\rho \in [-1, 1]$ for any two rv's X_1 and X_2.

Problem 3

These exercises will introduce the famous inequalities.

(a) [in the notes] Prove Markov's Inequality. State the assumptions clearly.

(b) [harder] Prove a Markov's-like Inequality for the bound on the probability of the left tail for a negative r.v. X.

(c) [in the notes] Prove Chebyshev's Inequality. State the assumptions clearly.

(d) [in the notes] Prove the lemma of Chebyshev's Inequality for the tail which looks like $\mathbb{P}(X \geq b) \leq ....$ State the assumptions clearly.

(e) [harder] Let X be a non-negative rv. Prove $\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X^3]}{a^3}$ where $a > 0$.

(f) [harder] Prove that if $\mathbb{E}[|X|]$ is finite then $\mathbb{E}[X]$ is finite.

(g) [difficult] [MA] Prove that if $\mathbb{E}[X]$ is finite then $\mathbb{E}[|X|]$ is finite.

(h) [difficult] Let $X_n \sim \text{Exp}(n)$. Compute upper bounds for $\mathbb{P}(X \geq 3)$ via Markov and Chebyshev. Does one go to zero "faster" than the other? Explain.

Problem 4

These questions are about convergence.

(a) [harder] Given

$$X_n \sim \begin{cases} 0 & \text{w.p. } 1 - \frac{1}{(n+1)^2} \\ 1 & \text{w.p. } \frac{1}{(n+1)^2} \end{cases}$$

Show that $X_n \xrightarrow{d} 0$. To do so, write out the PMF of X_n as a CDF using indicator functions in new format and then take the limit. You can also use the theorem we never proved that $\lim_{n\to\infty} \mathbb{1}_{a_n} = \mathbb{1}_{\lim_{n\to\infty} a_n}$.

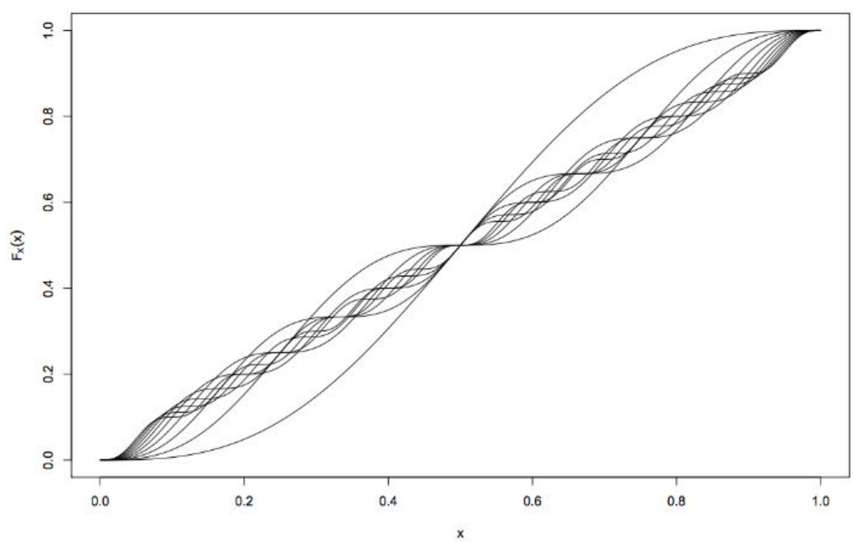
(b) [harder] Prove that $X_n \xrightarrow{p} 0$. For a full proof of $\forall \epsilon$, you need to show it for $\epsilon < 1$ and $\epsilon \geq 1$ separately but since we only care about small epsilon, you can just demonstrate it for $\epsilon < 1$.

(c) [harder] If $X_n \sim \text{Exp}(n)$, prove that $X_n \xrightarrow{d} 0$.

(d) [harder] If $X_n \sim \text{Exp}(n)$, prove that $X_n \xrightarrow{p} 0$. You can prove this using its CDF, using P7 of chf's or using Markov's inequality.

(e) [difficult] [MA] Prove that PDF convergence implies CDF convergence (i.e., that $X_n \overset{d}{\to} X$). You will need to use the dominated convergence theorem (DCT). Justifying the use of the DCT is slightly harder.

(f) [harder] Let $X_n \sim f_{X_n}(x) = (1 - \cos(2\pi n x))\mathbb{1}_{x \in [0,1]}$. Show that $X_n \overset{d}{\to} \text{U}(0,\, 1)$. Hint: use the fact that $\lim_{n\to\infty} \frac{\sin(nx)}{n} = 0$ which should have been proven using the "squeeze theorem" in your calculus class. The CDF is pictured below for $n = 1, 2, 3, 4, \ldots$.



(g) [harder] [MA] Show that $\lim_{n\to\infty} f_{X_n}$ does not exist. This is a counterexample to the conjecture that CDF convergence implies PDF convergence for finite continuous distributions with a limiting continuous distribution. Hint: it is just a calculus exercise.

Problem 5

These exercises will give you practice with the continuous mapping theorem (CMT) and Slutsky's theorems.

(a) [easy] State the CMT.

(b) [easy] State Slutsky's theorem A.

(c) [easy] State Slutsky's theorem B.

(d) [easy] If $X_n \overset{d}{\to} X$ and $\lim_{n\to\infty} a_n = a$, prove $a_n X_n \overset{d}{\to} aX$.

(e) [easy] Write the rv S_n^2 as a function of $X_1, \ldots, X_n$.

(f) [easy] Prove $S_n^2 \xrightarrow{p} \sigma^2$.

(g) [easy] Prove $S_n \xrightarrow{p} \sigma$.

(h) [easy] Prove $\sqrt{n}(\bar{X}_n - \mu)/S \xrightarrow{d} \mathcal{N}(0,\, 1)$.