# MATH 340/640 Fall 2024 Homework #4

Professor Adam Kapelner

Due by email 11:59PM October 27, 2024

(this document last updated Wednesday 16$^{\text{th}}$ October, 2024 at 12:14am)

**Instructions and Philosophy**

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out." Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, review Math 241 concerning the exponential.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems. "[MA]" are for those registered for 621 and extra credit otherwise.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to 5 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. The easiest way to use overleaf is to copy the raw text from hwxx.tex and preamble.tex into two new overleaf tex files with the same name. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks which are *not* on this printout. Keep this first page printed for your records.

NAME: ______________________________________________

## Problem 1

These exercises will introduce the Multinomial distribution.

(a) [in the notes] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim[\boldsymbol{X}] = k$, what is the parameter space for both $n$ and $\boldsymbol{p}$?

(b) [in the notes] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim[\boldsymbol{X}] = k$, what is $\text{Supp}\,[\boldsymbol{X}]$?

(c) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim[\boldsymbol{X}] = k$, what is $\dim[\boldsymbol{p}]$?

(d) [in the notes] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim[\boldsymbol{X}] = 2$, express $p_2$ as a function of $p_1$.

(e) [in the notes] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim[\boldsymbol{X}] = 2$, prove that $X_2 \sim \text{Binomial}\,(n,\, p_2)$.

(f) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ and $n = 1$ and $\dim[\boldsymbol{X}] = 7$ as a column vector, give an example value of $\boldsymbol{x}$, a realization of the r.v. $\boldsymbol{X}$. Use the notation $[\ldots]^\top$ to write it as a row vector transposed.

(g) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ and $n = 10$ and $\dim[\boldsymbol{X}] = 7$ as a column vector, give an example value of $\boldsymbol{x}$, a realization of the r.v. $\boldsymbol{X}$. Use the notation $[\ldots]^\top$ to write it as a row vector transposed.

(h) [harder] Is a binomial rv a multinomial rv? Yes or no and explain. This is subtle.

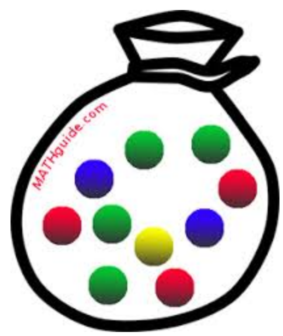
(i) [harder] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim[\boldsymbol{X}] = k$, prove that the sum over the JMF is 1. To do this, use the multinomial theorem.

(j) [difficult] [MA] Let $\boldsymbol{X}_1, \boldsymbol{X}_2 \overset{iid}{\sim} \text{Multinomial}\,(n,\, \boldsymbol{p})$ with $\dim[\boldsymbol{X}_1] = \dim[\boldsymbol{X}_2] = k$. Find the JMF of $\boldsymbol{T}_2 = \boldsymbol{X}_1 + \boldsymbol{X}_2$ from the definition of convolution. This looks harder than it is! First, use the definition of convolution and factor out the terms that are not a function of $x_1, \ldots, x_K$. Finally, use Theorem 1 in this paper: [click here] for the summation.

(k) [harder] Explain in English why the following should be true. Remember how the sampling from the bag works.

$$\binom{n}{x_1, x_2, \ldots, x_K} = \binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-(x_1+x_2)}{x_3}\cdot \ldots \cdot \binom{n-(x_1+x_2+\ldots x_{K-1})}{x_K}$$

(l) [easy] Consider the following bag of 4 green, 3 red, 2 blue and 1 yellow marbles:




Draw one marble with replacement 37 times. What is the probability of getting 10 red, 17 green, 6 blue and 4 yellow? Compute explicitly to the nearest two significant digits.

(m) [E.C.] [MA] If $\boldsymbol{X} \sim \text{Multinomial}\,(n, \boldsymbol{p})$, find the JMF of any subset of $X_1, \ldots, X_k$. Is it technically multinomial? This is not much harder than the previous problem if formulated carefully.

(n) [easy] Consider the following rv for the remaining exercises in this problem: $\boldsymbol{X} \sim \text{Multinomial}\left(17,\, [0.1\ 0.2\ 0.3\ 0.4]^\top\right)$. What is $\dim[\boldsymbol{X}]$?

(o) [harder] Find $\mathbb{V}\text{ar}[\boldsymbol{X}]$.

(p) [harder] If $x_1 = 1$, what is the JDF of the remaining rv's?

(q) [harder] If $x_1 = 1$ and $x_2 = 6$, what is the JDF of the remaining rv's?

(r) [easy] If $x_1 = 1$, $x_2 = 6$ and $x_3 = 3$, how is the remaining rv distributed?

## Problem 2

These exercises will give you practice with transformations of discrete r.v.'s.

(a) [easy] Let $X \sim \text{Binomial}\,(n,\, p)$. Find the PMF of $Y = \ln{(X+1)}$.

(b) [easy] Let $X \sim \text{Binomial}\,(n,\, p)$. Find the PMF of $Y = X^3$.

(c) [harder] Show that for any r.v. $X$ (discrete or continuous), if $Y = aX + b$, then $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$.

(d) [harder] Let $X \sim \text{NegBin}\,(k,\, p)$. Find the PMF of $Y = X^2$. Is $g(X)$ monotonic? Does that matter for this r.v.?

(e) [difficult] Let $X \sim \text{Binomial}\,(n,\, p)$ where $n$ is an even number. Find an expression for the PMF of $Y = \text{mod}(X, 2)$.

## Problem 3

These exercises will give you practice with transformations of continuous r.v.'s.

(a) [in the notes] Let $g$ be a strictly decreasing function and $X$ be a continuous rv and $Y = g(X)$. Find a formula for the PDF of $Y$. Justify each step.

(b) [in the notes] Let $g(x) = ax + b$, $X$ be a continuous rv and $Y = g(X)$. Find a formula for the PDF of $Y$. Do this step-by-step (i.e., first find the inverse function).

(c) [harder] Let $X \sim \text{Exp}(\lambda)$. Show that $Y = aX$ is an exponential rv and find its paramter. Use the transformation formula (not ch.f.'s).

(d) [difficult] Let $X \sim \text{Logistic}(0, 1)$. Find the PDF of $Y = g(X) = \frac{1}{1+e^{-X}}$. If this is a brand name r.v., mark it so and include its parameter values.

(e) [harder] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = ke^X$ where $k > 0$. This will be a brand name r.v., so mark it so and include its parameter values.

(f) [harder] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = \ln(X)$.

(g) [difficult] If $X \sim \text{Exp}(\lambda)$ then show that $Y = X^\beta \sim$ Weibull where $\beta > 0$. Find the resulting Weibull's parameters in terms of the parameterization we learned in class.

(h) [E.C.] Let $X \sim \text{Exp}(\lambda)$. Find the PDF of $Y = g(X) = \sin(X)$. Don't attempt this unless you have extra time.

(i) [easy] Rederive the $X \sim \text{Laplace}(0, 1)$ r.v. model by taking the difference of two standard exponential r.v.'s.

(j) [easy] Show that $\mathcal{E} \sim \text{Laplace}(0, \sigma)$ satisfies the three conditions of the definition of an "error distribution".

## Problem 4

These exercises will give you practice with the gamma function.

(a) [in the notes] Write the definition of $\Gamma(x)$.

(b) [difficult] Prove $\Gamma(k+1) = k\Gamma(k)$ for $k > 0$.

(c) [harder] Write the definition of $Q(x, a)$ without using the gamma function.

(d) [harder] If $0 < a < b < \infty$, find an integral expression for $\Gamma(x, b) - \gamma(x, a)$.

(e) [harder] Let $X \sim \text{Gamma}(\alpha, \beta)$. Prove the Humpty Dumpty identity. How do you think the Gamma rv got its name?

(f) [in the notes] Write the PMF's and parameter spaces of both the extended negative binomial rv and the negative binomial rv model. Explain how the latter "upgrades" the former.

(g) [in the notes] Write the PDF's and parameter spaces of both the gamma rv and the Erlang rv model. Explain how the latter "upgrades" the former.