Math 340 / 640 Fall 2025
Midterm Examination One

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Full Name ______________________________

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Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

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Instructions

This exam is 110 minutes (variable time per question) and closed-book. You are allowed **one** page (front and back) of a "cheat sheet", blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 This question is about the uniform discrete rv and the Bernoulli rv.

(a) [5 pt / 5 pts] Let $Y \sim U(\{a, b, c, d\})$ where a, b, c, d are unique real numbers. Write $p(y)$, the PMF of Y below.

$$p(y) = \frac{1}{4}\mathbb{1}_{y \in \{a,b,c,d\}}$$

(b) [5 pt / 10 pts] Let $X \sim \text{Bernoulli}(p)$. Write $F(x)$, the CDF which is valid for all $x \in \mathbb{R}$. Hint: use indicator functions.

$$F(x) = (1-p)\mathbb{1}_{x \geq 0} + p\mathbb{1}_{x \geq 1}$$

(c) [7 pt / 17 pts] Let $X_1, X_2, \ldots \overset{iid}{\sim} \text{Bernoulli}(0.5)$. Let $Y \sim U(\{a, b, c\})$ where $a \neq b \neq c$ and a, b, c are unique real numbers. Write Y as a function of $a, b, c, X_1, X_2, \ldots$. Write as many terms as necessary to illustrate your answer unambiguously. Hint: you have to use all $X_1, X_2, \ldots$. Note: this question demonstrates that any uniform discrete rv can be generated via binary Bernoulli rv's. Advice: leave this question for last.

$$\begin{aligned} Y \;=\; & a\mathbb{1}_{X_1=1,X_2=1} + b\mathbb{1}_{X_1=1,X_2=0} + c\mathbb{1}_{X_1=0,X_2=1} + \\ & \mathbb{1}_{X_1=0,X_2=0}(a\mathbb{1}_{X_3=1,X_4=0} + b\mathbb{1}_{X_3=1,X_4=1} + c\mathbb{1}_{X_3=0,X_4=1} + \\ & \qquad \mathbb{1}_{X_3=0,X_4=0}(a\mathbb{1}_{X_5=1,X_6=0} + b\mathbb{1}_{X_5=1,X_6=1} + c\mathbb{1}_{X_5=0,X_6=1} + \ldots)) \end{aligned}$$

(d) [3 pt / 20 pts] Let X be a rv and $S(x)$ be its survival function valid for all $x \in \mathbb{R}$ (i.e., this is not the "old" form). Provide three properties that $S(x)$ must satisfy.

(i) $\lim_{x \to \infty} S(x) = 0$

(ii) $\lim_{x \to -\infty} S(x) = 1$

(iii) $\frac{d}{dx}[S(x)] \leq 0$ for all $x \in \mathbb{R}$

Problem 2 This problem is about characteristic functions.

(a) [5 pt / 25 pts] In class we spoke about betting \$1 on lucky number 7 in American roulette. In this game, you win \$36 with probability 1/38 and lose your dollar otherwise. If X models the result of this bet, find $\phi_X(t)$.

We first find the old-style PMF. No need to use indicator functions as we're not going to be convolving it.

$$X \sim \begin{cases} 36 & w.p. \;\; 1/38 \\ -1 & w.p. \;\; 37/38 \end{cases}$$

We now use the definition of the ch.f. to compute:

$$\phi_X(t) = \mathbb{E}\left[e^{itX}\right] = \sum_{x \in \{-1, 36\}} e^{itx} p^{old}(x) = e^{it(-1)}\frac{37}{38} + e^{it(36)}\frac{1}{38} = \frac{37e^{-it} + e^{36it}}{38}$$

(b) [3 pt / 28 pts] Let $X \sim \text{Exp}(\lambda)$. It can be shown that $\phi_X(t) = \frac{\lambda}{\lambda - it}$. Find the characteristic function of $Y := \frac{1}{a}X$ where a is a nonzero real number.

$$\phi_Y(t) = \phi_X\left(\frac{1}{a}t\right) = \frac{\lambda}{\lambda - \frac{it}{a}}$$

(c) [6 pt / 34 pts] Let $X \sim \text{Exp}(\lambda)$. Let $Y := \frac{1}{a}X$ where a is a nonzero real number. Find the distribution of Y. Hint: it will be one of the brand name rv's we studied.

We use the ch.f from the previous question and massage it so we can use property 1:

$$\phi_Y(t) = \frac{\lambda}{\lambda - \frac{it}{a}} \frac{a}{a} = \frac{a\lambda}{a\lambda - it} \quad \Rightarrow \quad Y \sim \text{Exp}(a\lambda)$$

Problem 3 Let $X \sim \text{Geometric}(p)$ independent of $Y \sim \text{NegBin}(2, p)$.

(a) [3 pt / 37 pts] Which is true? Circle one of the following.

(i) $\mathbb{P}(X > Y) < \frac{1}{2}$

(ii) $\mathbb{P}(X > Y) = \frac{1}{2}$

(iii) $\mathbb{P}(X > Y) > \frac{1}{2}$

(b) [8 pt / 45 pts] Compute $\mathbb{P}(X > Y)$. Get as far as possible given the tools we studied. But simplify as much as possible.

$$
\begin{aligned}
\mathbb{P}(X > Y) &= \sum_{x\in\mathbb{R}}\sum_{y\in\mathbb{R}} p_{X,Y}(x,y)\mathbb{1}_{x>y} \\
&= \sum_{y\in\mathbb{R}}\sum_{x\in\mathbb{R}} p_X(x)p_Y(y)\mathbb{1}_{x\in\{y+1,y+2,\ldots\}} \\
&= \sum_{y\in\mathbb{R}} p_Y(y) \sum_{x\in\{y+1,y+2,\ldots\}} p_X(x) \\
&= \sum_{y\in\mathbb{R}} \binom{y+2-1}{1}(1-p)^y p^2 \mathbb{1}_{y\in\mathbb{N}_0} \sum_{x\in\{y+1,y+2,\ldots\}} (1-p)^x p \mathbb{1}_{x\in\mathbb{N}_0} \\
&= p^3 \sum_{y\in\mathbb{N}_0} (y+1)(1-p)^y \sum_{x\in\{y+1,y+2,\ldots\}} (1-p)^x \\
&= p^3 \sum_{y\in\mathbb{N}_0} \left(y(1-p)^y + (1-p)^y\right) \sum_{x\in\{y+1,y+2,\ldots\}} (1-p)^x \\
&= p^3 \sum_{y\in\mathbb{N}_0} \left(y(1-p)^y + (1-p)^y\right) \sum_{i\in\mathbb{N}_0} (1-p)^{i+y+1} \\
&= p^3 \sum_{y\in\mathbb{N}_0} \left(y(1-p)^y + (1-p)^y\right)(1-p)^{y+1} \sum_{i\in\mathbb{N}_0} (1-p)^i \\
&= p^3 \sum_{y\in\mathbb{N}_0} \left(y(1-p)^y + (1-p)^y\right)(1-p)^{y+1} \left(\frac{1}{p}\right) \\
&= p^2(1-p) \sum_{y\in\mathbb{N}_0} \left(y(1-p)^{2y} + (1-p)^{2y}\right) \\
&= p^2(1-p) \left(\sum_{y\in\mathbb{N}_0} y\left((1-p)^2\right)^y + \sum_{y\in\mathbb{N}_0} \left((1-p)^2\right)^y\right) \\
&= p^2(1-p) \left(\sum_{y\in\mathbb{N}_0} y\left((1-p)^2\right)^y + \frac{1}{1-(1-p)^2}\right)
\end{aligned}
$$

This is as far as you can get given tools in class. However, it can be solved in closed form using a series related to the geometric series but we didn't study it in class. If interested, comment out the following code:

Problem 4 Consider a rv X with $S_X \geq 0$. We realize n (assumed to be large) iid realizations from this rv and calculate the following quantities rounded to two decimals:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} x_i &= 2.51, & \frac{1}{n}\sum_{i=1}^{n} x_i^6 &= 3071.00 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^2 &= 7.89, & \frac{1}{n}\sum_{i=1}^{n} x_i^7 &= 16475.31 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^3 &= 29.62, & \frac{1}{n}\sum_{i=1}^{n} x_i^8 &= 91664.09 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^4 &= 127.21, & \frac{1}{n}\sum_{i=1}^{n} x_i^9 &= 523306.42 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^5 &= 602.66, & \frac{1}{n}\sum_{i=1}^{n} x_i^{10} &= 3042898.00
\end{aligned}$$

(a) [4 pt / 49 pts] Provide an approximation for this rv's expectation.

$$\mathbb{E}[X] \approx \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x} = 2.51$$

(b) [3 pt / 52 pts] Which theorem did you make use of to answer the previous question?

very weak law of large numbers or WLLN or LLN

(c) [4 pt / 56 pts] Estimate the quantity $\phi_X''''(0)$.

$$\phi_X''''(0) = i^4\mathbb{E}\left[X^4\right] = \mathbb{E}\left[X^4\right] \approx \frac{1}{n}\sum_{i=1}^{n} x_i^4 = 127.21 \quad \text{via very WLLN once again}$$

Problem 5 Consider the following rv

$$X \sim \text{arcsin} := \frac{1}{\pi\sqrt{x(1-x)}}\mathbb{1}_{x\in(0,1)}$$

which has this name because its CDF can be written as

$$F(x) = \frac{2}{\pi}\text{arcsin}(\sqrt{x}).$$

(a) [2 pt / 58 pts] Is this rv discrete or continuous?

continuous

(b) [2 pt / 60 pts] How many parameter(s) does this rv have?

0

(c) [2 pt / 62 pts] What is the support of this rv?

$(0, 1)$

(d) [2 pt / 64 pts] What is the name of the probability principle we used many times in class which demands that $\int_0^1 \frac{1}{\pi\sqrt{x(1-x)}}dx = 1$?

Humpty Dumpty

Problem 6 Consider the following rv

$$X \sim \text{Frechet}(c, s) := \frac{s}{c}\left(\frac{x}{c}\right)^{-1-s} e^{\left(\frac{x}{c}\right)^{-s}}\mathbb{1}_{x>0}$$

(a) [2 pt / 66 pts] Is this rv discrete or continuous?

continuous

(b) [2 pt / 68 pts] How many parameter(s) does this rv have?

2

(c) [2 pt / 70 pts] What is the support of this rv?

$(0, \infty)$

(d) [5 pt / 75 pts] Assume c is positive. Why can't s be negative?

Because the PDF must be positive for $x > 0$

(e) [8 pt / 83 pts] Let $X_1, X_2 \overset{iid}{\sim}$ Frechet(c, s) and $T = X_1 + X_2$. Get as far as possible calculating $f_T(t)$. But simplify as much as possible.

$$\begin{aligned}
f_T(t) &= \int_{\mathbb{S}_X} f^{old}(x) f^{old}(t-x) \mathbb{1}_{t-x \in \mathbb{S}_X} dx \\
&= \int_0^\infty \left(\frac{s}{c}\left(\frac{x}{c}\right)^{-1-s} e^{\left(\frac{x}{c}\right)^{-s}}\right)\left(\frac{s}{c}\left(\frac{t-x}{c}\right)^{-1-s} e^{\left(\frac{t-x}{c}\right)^{-s}}\right) \mathbb{1}_{t-x \in (0,\infty)} dx \\
&= \frac{s^2}{c^{-2s}} \int_0^\infty (x(t-x))^{-1-s} e^{c^s(x^{-s}+(t-x)^{-s}} \mathbb{1}_{x-t \in (-\infty,0)} dx \\
&= \frac{s^2}{c^{-2s}} \int_0^\infty (x(t-x))^{-1-s} e^{c^s(x^{-s}+(t-x)^{-s}} \mathbb{1}_{x \in (-\infty,t)} dx \\
&= \frac{s^2}{c^{-2s}} \mathbb{1}_{t>0} \int_0^t (x(t-x))^{-1-s} e^{c^s(x^{-s}+(t-x)^{-s}} dx
\end{aligned}$$

Problem 7 Consider the following two-dimensional vector rv

$$\boldsymbol{X} \sim \begin{cases} 0.1 & \text{if} \quad x_1 = 0 \quad \text{and} \quad x_2 = 0 \\ 0.2 & \text{if} \quad x_1 = 1 \quad \text{and} \quad x_2 = 0 \\ 0.3 & \text{if} \quad x_1 = 0 \quad \text{and} \quad x_2 = 1 \\ 0.4 & \text{if} \quad x_1 = 1 \quad \text{and} \quad x_2 = 1 \end{cases}$$

(a) [3 pt / 86 pts] The expression above after the "∼" symbol is the ... of $\boldsymbol{X}$. Circle one:

(i) PMF (ii) JMF (iii) PDF (iv) JDF

(b) [5 pt / 91 pts] It can be shown that $X_2 \sim \text{Bernoulli}(0.7)$. Find the distribution of X_1.

Just like X_2, $\mathbb{S}_{X_1} = \{0, 1\}$ which means its Bernoulli. We only need to find the value of p which is by definition the $\mathbb{P}(X_1 = 1)$. Using the addition rule for disjoint events,

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = 1, X_2 = 0) + \mathbb{P}(X_1 = 1, X_2 = 1) = 0.2 + 0.4 = 0.6$$

hence $X_1 \sim \text{Bernoulli}(0.6)$.

(c) [4 pt / 95 pts] Use the Cauchy-Schwartz corollary to find an upper bound for the absolute value of $\mathbb{C}\text{ov}[X_1, X_2]$. Hint: the variance of a rv distributed as a Bernoulli (p) is $p(1-p)$. Round to two significant digits.

$$|\mathbb{C}\text{ov}[X_1, X_2]| \leq \sqrt{\mathbb{V}\text{ar}[X_1]\,\mathbb{V}\text{ar}[X_2]} = \sqrt{0.7(1-0.7)0.6(1-0.6)} = 0.22$$

(d) [5 pt / 100 pts] Compute $\mathbb{C}\text{ov}[X_1, X_2]$. Round to two significant digits.

We use the definition and the facts that both rv's are Bernoullis,

$$\begin{aligned} \mathbb{C}\text{ov}[X_1, X_2] &:= \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2] \\ &= \sum_{\boldsymbol{x} \in \mathbb{R}^2} x_1 x_2\, p_{\boldsymbol{X}}(x_1, x_2) - 0.7 \cdot 0.6 \end{aligned}$$

Now the sum above has only one nonzero element which is when $x_1 = 1, x_2 = 1$ which has probability of occuring of 0.4. Hence,

$$\mathbb{C}\text{ov}[X_1, X_2] = 0.4 - 0.7 \cdot 0.6 = -0.02$$