MATH 340/640 Fall 2025 Homework #2

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Due by email 11:59PM by date on homepage

(this document last updated Thursday 4$^{\text{th}}$ September, 2025 at 10:52pm)

Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out." Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, review Math 241 concerning random variables, support, parameter space, PMF's, CDF's, bernoulli, binomial, geometric, negative binomial, exponential, Erlang, normal. Then read on your own about characteristic functions, the law of large numbers (LLN) and the central limit theorem (CLT).

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems. "[MA]" are for those registered for 621 and extra credit otherwise.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to 5 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. The easiest way to use overleaf is to copy the raw text from hwxx.tex and preamble.tex into two new overleaf tex files with the same name. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks which are *not* on this printout. Keep this first page printed for your records.

NAME: ____

Problem 1

These exercises introduce probabilities of conditional subsets of the supports of multiple r.v.'s.

(a) [difficult] Let $X \sim \text{Geometric}(p_x)$ independent of $Y \sim \text{Geometric}(p_y)$. Find $\mathbb{P}(X > Y)$ using the method we did in class.

(b) [harder] Let $p_x = p_y$. Find $\mathbb{P}(X > Y)$ in a different way by finding $\mathbb{P}(X = Y)$ and then using the law of total probability.

(c) [difficult] As both p_x and p_y are reduced to zero, but $r = \frac{p_x}{p_y}$, what is the asymptotic probability you found in (a)? This is similar exercise to what we did in class when we took the limit of n with $\lambda = np$ fixed.

Problem 2

These exercises will give you more practice with indicator functions.

(a) [easy] Resolve as best as possible: $\sum_{x \in \mathbb{Z}} \mathbb{1}_{x \in [0,c]}$ where $c \in \mathbb{N}_0$.

(b) [easy] Resolve as best as possible: $\sum_{x \in \{0,1,\ldots,d\}} \mathbb{1}_{x \in [0,c]}$ where $c, d \in \mathbb{N}_0$.

(c) [easy] Resolve as best as possible: $\int_{\mathbb{R}} \mathbb{1}_{x \in [0,c]} \mathrm{d}x$ where $c \in \mathbb{R}$.

(d) [easy] Resolve as best as possible: $\int_{-\infty}^{d} \mathbb{1}_{x \in [0,c]} \mathrm{d}x$ where $c, d \in \mathbb{R}$.

(e) [harder] Resolve as best as possible: $\int_d^{d+1} \mathbb{1}_{x \in [0,c]} \mathrm{d}x$ where $c, d \in \mathbb{R}$ and $c \geq 1$.

(f) [harder] Resolve as best as possible: $\int_d^{d+1} \mathbb{1}_{x \in [c,c+1]} \mathrm{d}x$ where $c, d \in \mathbb{R}$. and $c > 0$.

Problem 3

This question reviews the Exponential distribution and introduces the Erlang distribution.

(a) [harder] Let $X \sim \text{Geometric}(p)$. Assume $\mathbb{E}[X] = \frac{1-p}{p}$. Find $\mathbb{V}\text{ar}[X]$ from the definition of variance.

(b) [in the notes] Illustrate the PDF and CDF of $X \sim \text{Exp}(\lambda)$ one atop the other. Mark critical points on the axes.

(c) [in the notes] Derive the $\text{Exp}(\lambda)$ rv from the X_n Geometric r.v. with n experiments compacted per time period. In the process, write the CDF, PMF and PDF of the Exponential rv. Make sure the PMF and PDF are valid $\forall x \in \mathbb{R}$. The CDF you can leave in "old" style.

(d) [difficult] Let $X_1, \ldots, X_k \overset{iid}{\sim} \text{Exp}(\lambda)$ and let $T = X_1 + \ldots + X_k$. Using induction, show that $T \sim \text{Erlang}(k, \lambda)$ by finding its PDF.

(e) [harder] Find the expectation of $T \sim \text{Erlang}(k, \lambda)$. Use the expectation of the exponential rv and the linearity rules!

(f) [easy] If $X \sim \text{Exp}(\lambda)$, then $\mathbb{V}\text{ar}[X] = 1/\lambda^2$. Find the variance of $T \sim \text{Erlang}(k, \lambda)$. Use the variance of the exponential rv and and the linearity rules!

(g) [easy] Why is the geometric distribution analogous to the exponential distribution? Why is the negative binomial distribution analogous to the erlang distribution?

(h) [easy] If the length of time (in minutes) of a phone call is distributed $\text{Exp}\,(0.5)$ and all phone call lengths are $\overset{iid}{\sim}$, find an expression for the probability that the total sum time of 37 phone calls lasts longer than 17 minutes.

(i) [difficult] Show that for any exponential r.v. with rate parameter λ the distribution is "memoryless" meaning that for any c, a positive constant, $\mathbb{P}\,(X > x + c \mid X > c) = \mathbb{P}\,(X > x)$.

Problem 4

This problem introduces characteristic functions (ch.f.'s)!

(a) [in the notes] Prove that $\left|e^{i\theta}\right| = 1$ for all θ.

(b) [easy] Give one example function g where you show conclusively that $g \notin L^1$.

(c) [easy] Prove that all PDF's are $\in L^1$.

(d) [in the notes] List ch.f. properties P0-P7 below without proofs.

(e) [harder] Use P4 to compute the variance of rv X using only $\phi_X(t)$ and its derivative(s).

(f) [difficult] For continuous rv $X \sim f_X(x)$, given the Fourier inversion theorem, if $\phi_X(t) \in L^1$ then prove P6. Try to do this yourself first. Hint: translate the Fourier formulas (the ones with g and $\hat{g}$ into our vocabulary and then use u-substitutions.

SOLUTION: We begin with the statement of the forward Fourier transform. If $g \in L^1$,

$$\hat{g}(\omega) = \int_{\mathbb{R}} e^{-i2\pi\omega t} g(t) dt$$

and the statement of the inverse Fourier transform. If $\hat{g} \in L^1$,

$$g(t) = \int_{\mathbb{R}} e^{i2\pi\omega t} \hat{g}(\omega) d\omega.$$

We translate this to our vocabulary in the following way: let $f \leftarrow g, x \leftarrow t, v \leftarrow \omega$. The two statements then become:

$$\begin{aligned} \hat{g}(v) &= \int_{\mathbb{R}} e^{-i2\pi vx} f(x) dx \\ f(x) &= \int_{\mathbb{R}} e^{i2\pi vx} \hat{g}(v) dv. \end{aligned}$$

Now let $t = -2\pi v \Rightarrow v = -t/(2\pi), dv/dt = -1/(2\pi), dv = -\frac{1}{2\pi} dt, v = \infty \Rightarrow t = -\infty, v = -\infty \Rightarrow t = \infty$. Making these substitions yields:

$$\begin{aligned} \hat{g}(-t/(2\pi)) &= \int_{\mathbb{R}} e^{itx} f(x) dx \\ f(x) &= \int_{\infty}^{-\infty} e^{-itx} \hat{g}(-t/(2\pi)) \left(-\frac{1}{2\pi}\right) dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \hat{g}(-t/(2\pi)) dt. \end{aligned}$$

Letting $\phi_X(t) := \hat{g}(-t/(2\pi))$ completes the proof:

$$\begin{aligned} \phi_X(t) &= \int_{\mathbb{R}} e^{itx} f(x) dx \\ f(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \phi_X(t) dt. \end{aligned}$$

(g) [in the notes] Find the ch.f. of $X \sim \text{Bernoulli}\,(p)$.

(h) [easy] Find the ch.f. of $T \sim \text{Binomial}\,(n,\, p)$. Hint: use the binomial theorem.

(i) [easy] Using ch.f.'s, find $\mathbb{E}\,[T]$.

(j) [easy] Using ch.f.'s, find $\mathbb{V}\text{ar}\,[T]$.

(k) [in the notes] Using ch.f.'s, show that if $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}\,(p)$, then $T = X_1 + \ldots + X_n \sim \text{Binomial}\,(n,\, p)$.

(l) [harder] Using ch.f.'s, show that if $X_1, \ldots, X_n \overset{iid}{\sim} \text{Binomial}\,(m,\, p)$, then $T = X_1 + \ldots + X_n \sim \text{Binomial}\,(nm,\, p)$.

(m) [harder] Find the ch.f. of X_n, the Geometric rv whose probability of success is λ/n where there are n experiments per unit change in x (this is the rv we used to prove the exponential rv in class). Hint: use the reindexing trick.

(n) [harder] [MA] Prove the ch.f. from the previous problem limits to the ch.f. for $\text{Exp}\,(\lambda)$. This really is just an elaborate calculus exercise. The point being is that using ch.f's is not always easier to derive rv's than the direct method using PMF's and PDF's.

(o) [easy] Let $X \sim \mathrm{U}\,(a,\, b)$. Find $\phi_X(t)$.

Problem 5

The very weak law of large numbers (LLN) and the central limit theorem (CLT).

(a) [easy] State the setup / assumptions of the very weak LLN.

(b) [easy] Prove the very weak LLN (copy from the notes if you get stuck).

(c) [harder] What is the main implication of the LLN? Write in English.

(d) [easy] State the additonal assumption(s) of the CLT in addition to assumptions that were needed to prove the LLN.

(e) [easy] Prove the CLT i.e. prove the limiting ch.f. of the standardized average rv is $e^{-t^2/2}$, the ch.f. for $Z \sim \mathcal{N}(0, 1)$.

(f) [harder] What is the main implication of the CLT? Write in English.

(g) [difficult] [MA] Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ some distribution with mean μ_X and variance $\sigma_X^2 < \infty$ and let $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim}$ some distribution with mean μ_Y and variance $\sigma_Y^2 < \infty$ which are independent of the X_i's. Prove the following central limit theorem corollary:

$$\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{n}}} \overset{d}{\to} \mathcal{N}(0,\, 1)\,.$$

We need this fact to do two-sample testing in statistics. This looks harder than it is. Trace through the proof in (b), use algebra to simplify expressions and make substitutions in the appropriate places.

(h) [easy] Let $Z \sim \mathcal{N}(0, 1)$. Let $X = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$. Find $\phi_X(t)$.

(i) [easy] What is $f_X(x)$? Copy from class notes.

(j) [harder] [MA] Find the PDF of X using P6, inversion of its ch.f.