Math 340 / 640 Fall 2025
Midterm Examination Two

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Full Name ______________________________

Code of Academic Integrity

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Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

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signature date

Instructions

This exam is 110 minutes (variable time per question) and closed-book. You are allowed **two** pages (front and back) of a "cheat sheet", blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 These questions are about inequalities.

(a) [5 pt / 5 pts] Let $X_1, \ldots, X_n \overset{iid}{\sim}$ with mean $\mu < \infty$ and $\mathbb{S}_X > 0$. Find a bound for $\mathbb{P}\left(\bar{X} \geq 2\mu\right)$.

$$\mathbb{P}\left(\bar{X} \geq 2\mu\right) \leq \frac{\mathbb{E}\left[\bar{X}\right]}{2\mu} = \frac{\mu}{2\mu} = \frac{1}{2}$$

(b) [5 pt / 10 pts] Let $Z \sim \mathcal{N}(0,\, 1)$. What is the Chebyshev's bound on $\mathbb{P}\left(Z \notin (-2, 2)\right)$?

$$\mathbb{P}\left(|X-\mu| > a\right) \leq \frac{\sigma^2}{a^2} \quad \Rightarrow \quad \mathbb{P}\left(|Z| > 2\right) \;\leq\; \frac{\sigma^2}{2^2} = \frac{1}{4} = .25$$

Problem 2 These questions are about transformations of rv's.

(a) [5 pt / 15 pts] Let $X \sim \text{Binomial}\,(n,\, p) := \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,\ldots,n\}}$ and let $Y = \sqrt{X}$. Find $p_Y(y)$. Make sure you have a term in your answer that looks like $\mathbb{1}_{y \in \mathbb{S}_Y}$.

g is a 1:1: function on $\mathbb{S}_X$ so we can use our formula. We first find the inverse function $X = g^{-1}(Y) = Y^2$ and then substitute:

$$p_Y(y) = p_X(g^{-1}(y)) = p_X(y^2) = \binom{n}{y^2} p^{y^2} (1-p)^{n-y^2} \mathbb{1}_{y \in \left\{0,1,\sqrt{2},\ldots,\sqrt{n}\right\}}$$

(b) [5 pt / 20 pts] Consider a new brand name rv you will see later on the homework:

$$U \sim \text{Lomax}(\lambda, k) := \frac{\lambda}{k}\left(1 + \frac{u}{k}\right)^{-(\lambda+1)} \mathbb{1}_{u>0}$$

and let $X \sim \text{ParetoI}(k, \lambda) := \frac{\lambda k^\lambda}{x^{\lambda+1}} \mathbb{1}_{x>k}$. Let $Y = X - k$. Show that $Y \sim \text{Lomax}(\lambda, k)$.

We use the linear shift formula and then use algebra to massage it into the right format.

$$\begin{aligned}
f_Y(y) = f_X(y - (-k)) = f_X(y+k) &= \frac{\lambda k^\lambda}{(y+k)^{\lambda+1}} \mathbb{1}_{y+k>k} \\
&= \lambda k^\lambda (y+k)^{-(\lambda+1)} \mathbb{1}_{y>0} \\
&= \lambda k^\lambda k^{-(\lambda+1)} \left(1 + \frac{y}{k}\right)^{-(\lambda+1)} \mathbb{1}_{y>0} \\
&= \lambda k^{-1} \left(\frac{y}{k} + 1\right)^{-(\lambda+1)} \mathbb{1}_{y>0} \\
&= \frac{\lambda}{k}\left(1 + \frac{y}{k}\right)^{-(\lambda+1)} \mathbb{1}_{y>0} =: \text{Lomax}(\lambda, k)
\end{aligned}$$

(c) [6 pt / 26 pts] Let $X \sim \text{U}(0, 1)$. Let $Y = \ln\left(\frac{X}{1-X}\right)$. Show that Y is distributed as a brand name rv we've studied and find that name brand rv's parameter values. Derive the result using transformation of variables.

$$\begin{aligned}
Y = \ln\left(\frac{X}{1-X}\right) \quad \Rightarrow \quad e^Y &= \frac{X}{1-X} \quad \Rightarrow \quad e^{-Y} = \frac{1}{X} - 1 \quad \Rightarrow \quad e^{-Y} + 1 = \frac{1}{X} \\
\Rightarrow \quad g^{-1}(Y) &= \frac{1}{1+e^{-Y}} \quad \Rightarrow \quad \frac{d}{dy}\left[g^{-1}(Y)\right] = \frac{e^{-Y}}{(1+e^{-Y})^2} \\
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}\left[g^{-1}(y)\right] \right| \\
&= \mathbb{1}_{\frac{1}{1+e^{-y}} \in (0,1)} \frac{e^{-y}}{(1+e^{-y})^2} \\
&= \frac{e^{-y}}{(1+e^{-y})^2} \mathbb{1}_{1+e^{-y} \in (1,\infty)} \\
&= \frac{e^{-y}}{(1+e^{-y})^2} \mathbb{1}_{e^{-y} \in (0,\infty)} \\
&= \frac{e^{-y}}{(1+e^{-y})^2} \mathbb{1}_{-y \in (-\infty,\infty)} \\
&= \frac{e^{-y}}{(1+e^{-y})^2} = \text{Logistic}(0, 1)
\end{aligned}$$

(d) [8 pt / 34 pts] Let $X_1, X_2 \overset{iid}{\sim} \text{U}(0, 1)$. Let $R = X_1/X_2$. Find the PDF of R. Get as far as you can. Partial credit will be given at each step. Hint: the answer is a piecewise function. Hint: don't forget the support term(s).

$$
\begin{aligned}
f_R(r) = \int_{\mathbb{R}} f(ru)f(u)|u|du &= \int_{\mathbb{R}} \mathbb{1}_{ru\in(0,1)}\mathbb{1}_{u\in(0,1)}|u|du \quad \text{(formula for iid rv's)} \\
&= \int_0^1 \mathbb{1}_{ru\in(0,1)} udu \quad \text{(by IIP and the fact that } u > 0\text{)} \\
&= \int_0^1 \mathbb{1}_{u\in(0,1/r)} udu \\
&= \begin{cases} 0 & \text{if} \quad r < 0 \\ \int_0^1 udu & \text{if} \quad r \in [0,1] \\ \int_0^{1/r} udu & \text{if} \quad r > 1 \end{cases} \\
&= \mathbb{1}_{r>0} \begin{cases} \left[\frac{u^2}{2}\right]_0^1 & \text{if} \quad r \in [0,1] \\ \left[\frac{u^2}{2}\right]_0^{1/r} & \text{if} \quad r > 1 \end{cases} \\
&= \mathbb{1}_{r>0} \begin{cases} \frac{1}{2} & \text{if} \quad r \in [0,1] \\ \frac{1}{2r^2} & \text{if} \quad r > 1 \end{cases} \\
&= \frac{1}{2}\mathbb{1}_{r>0} \begin{cases} 1 & \text{if} \quad r \in [0,1] \\ r^{-2} & \text{if} \quad r > 1 \end{cases} \\
&= \frac{1}{2}\left(\mathbb{1}_{r\in[0,1]} + r^{-2}\mathbb{1}_{r>1}\right)
\end{aligned}
$$

Problem 3 Let $\boldsymbol{X} \sim \text{Multi}_k(n, \boldsymbol{p})$ where $k > 1$ and $n > 17$.

(a) [2 pt / 36 pts] How many entries are in the vector $\boldsymbol{p}$?

$$k$$

(b) [3 pt / 39 pts] What is the probability that $X_1 = 17$ as a function of n, p_1?

$$X_1 \sim \text{Binomial}(n, p_1) \quad \Rightarrow \quad \mathbb{P}(X_1 = 17) = \binom{n}{17} p_1^{17}(1-p_1)^{n-17}$$

(c) [4 pt / 43 pts] Find $\mathbb{V}\text{ar}[X_1 + X_2]$ as a function of n, p_1, p_2.

$$\begin{aligned}\mathbb{V}\text{ar}[X_1 + X_2] &= \mathbb{V}\text{ar}[X_1] + \mathbb{V}\text{ar}[X_2] + 2\mathbb{C}\text{ov}[X_1, X_2] \\ &= np_1(1-p_1) + np_2(1-p_2) + 2(-np_1p_2) \\ &= n(p_1(1-p_1) + p_2(1-p_2) - 2p_1p_2)\end{aligned}$$

(d) [4 pt / 47 pts] If $x_1 = 17$, find the JMF of the vector $\boldsymbol{X}_{-1}$, i.e., $\boldsymbol{X}$ without X_1. If the answer is a brand name rv, you can use the brand name notation from class to represent the JMF.

$$\boldsymbol{X}_{-1} \sim \text{Multi}_{k-1}\left(n - 17,\ \frac{1}{1-p_1}\boldsymbol{p}_{-1}\right)$$

(e) [4 pt / 51 pts] If $x_1 + x_2 + \ldots + x_{k-1} = 17$, find the PMF of X_k. If the answer is a brand name rv, you can use the brand name notation from class to represent the PMF.

$$X_k \sim \text{Deg}(n-17) := \mathbb{1}_{x=n-17}$$

Problem 4 Consider $X_1, \ldots, X_n \overset{iid}{\sim}$ with finite μ and finite σ^2.

(a) [5 pt / 56 pts] Consider

$$\frac{n}{X_1 + X_2 + \ldots + X_n} \xrightarrow{p} c$$

and find c as a function of μ, σ and justify the steps you took to prove it.

$$\frac{n}{X_1 + X_2 + \ldots + X_n} = \frac{1}{\bar{X}} \xrightarrow{p} \frac{1}{\mu} \quad \text{(by WLLN and CMT with } g(t) = 1/t\text{)}$$

(b) [5 pt / 61 pts] Consider

$$\frac{X_1^2 + X_2^2 + \ldots + X_n^2}{X_1 + X_2 + \ldots + X_n} \xrightarrow{p} c$$

and find c as a function of μ, σ and justify the steps you took to prove it.

$$\frac{X_1^2 + X_2^2 + \ldots + X_n^2}{X_1 + X_2 + \ldots + X_n} = \frac{\frac{1}{n}(X_1^2 + X_2^2 + \ldots + X_n^2)}{\frac{1}{n}(X_1 + X_2 + \ldots + X_n)} \xrightarrow{p} \frac{\mathbb{E}[X^2]}{\mathbb{E}[X]} = \frac{\sigma^2 + \mu^2}{\mu}$$

(by 2x WLLN and CMT with $g(t) = 1/t$)

(c) [5 pt / 66 pts] Does the following expression converge in distribution?

$$\frac{\bar{X}^2 - \mu \bar{X}}{\frac{\sigma}{\sqrt{n}}}$$

If so, find that limiting distribution's PDF and its parameters as a function of μ, σ and justify the steps you took to prove it. If the answer is a brand name rv, you can use the brand name notation from class to represent the PDF.

$$\frac{\bar{X}^2 - \bar{X}\mu}{\frac{\sigma}{\sqrt{n}}} = \bar{X}\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} \mu\, \mathcal{N}(0,\, 1) = \mathcal{N}\left(0,\, \mu^2\right) \quad \text{(by WLLN, CLT and Slutsky's A)}$$

(d) [5 pt / 71 pts] Prove that $\frac{1}{n-2}\sum_{i=1}^{n}(X_i - \bar{X})^2 \xrightarrow{p} \sigma^2$ and justify the steps you took to prove it. Hint: use the convergence that we proved in class for S_n^2.

$$\frac{1}{n-2}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{n-1}{n-1}\frac{1}{n-2}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{n-1}{n-2}S_n^2 \xrightarrow{p} \sigma^2$$

The sequence $\frac{n-1}{n-2}$ has the limit as $n \to \infty$ of 1 and $S_n^2 \xrightarrow{p} \sigma^2$ from the class result. We then use Slutsky's A corollary to complete the proof.

Problem 5 These are some distribution questions.

(a) [5 pt / 76 pts] Let $X \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1} \mathbb{1}_{x>0}$. Verify the Humpty-Dumpty identity for $f_X(x)$. Show each step. Hint: use the gamma-like integral.

$$\int_{\mathbb{R}} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1} \mathbb{1}_{x>0} dx \;=\; \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-\beta x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{\beta^\alpha} = 1$$

For the remainder of this problem, let $Z_1, Z_2, \ldots, Z_n \overset{iid}{\sim} \mathcal{N}(0, 1)$.

(b) [7 pt / 83 pts] Let $\boldsymbol{Z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ and consider the following function:

$$\boldsymbol{Z}^\top \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix} \boldsymbol{Z}$$

Find a function of Z_1, Z_2 that is independent of this function. No need to simplify. Justify your steps.

We are given above a quadratic form where the determining matrix $\boldsymbol{B}_1$ is

$$\begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix}$$

This matrix is symmetric and has rank 1 (the first column equals $\sqrt{3}$ times the second column. We now find $\boldsymbol{B}_2$ so that $\boldsymbol{B}_1 + \boldsymbol{B}_2 = \boldsymbol{I}_2$:

$$\boldsymbol{B}_2 = \begin{bmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{bmatrix}$$

This matrix is symmetric and has rank 1 (the second column equals $-\sqrt{3}$ times the first column.

The ranks of $\boldsymbol{B}_1, \boldsymbol{B}_2$ sum to two. Thus we can use Cochran's theorem to conclude that $\boldsymbol{Z}^\top \boldsymbol{B}_1 \boldsymbol{Z}$ and $\boldsymbol{Z}^\top \boldsymbol{B}_2 \boldsymbol{Z}$ are independent. The function given is $\boldsymbol{Z}^\top \boldsymbol{B}_1 \boldsymbol{Z}$. So we the answer is just

$$\boldsymbol{Z}^\top \boldsymbol{B}_2 \boldsymbol{Z} = \boldsymbol{Z}^\top \begin{bmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{bmatrix} \boldsymbol{Z}$$

and we do not need to simplify further.

(c) [2 pt / 85 pts] Would $Z_1 + Z_2$ be an error distribution? Yes/no.

Yes as $Z_1 + Z_2 \sim \mathcal{N}(0,\, 2)$ which has all three properties required.

(d) [5 pt / 90 pts] Let $U = \dfrac{1}{4}\left(\dfrac{Z_1}{Z_2} + \dfrac{Z_3}{Z_4} + \dfrac{Z_5}{Z_6} + \dfrac{Z_7}{Z_8}\right)$. How is U distributed? If the answer is a brand name rv, you can use the brand name notation from class to represent the PDF.

First note that $\frac{Z_i}{Z_j} \sim \text{Cauchy}(0,\, 1)$. Then U is just the average of four $\overset{iid}{\sim}$ Cauchy $(0,\, 1)$. We learned in class that the average of any number of $\overset{iid}{\sim}$ Cauchy $(0,\, 1)$'s is *itself* Cauchy $(0,\, 1)$ and thus $U \sim \text{Cauchy}(0,\, 1)$.

(e) [5 pt / 95 pts] Let $Y = Z_1^2 + \ldots + Z_n^2$ and let n be even. Circle all the following that are true:

- $Y \sim \chi^2_1$
- $Y \sim \chi^2_n$
- $Y \sim \chi^2_{n/2}$
- $Y \sim \text{Gamma}(n,\, 1/2)$
- $Y \sim \text{Gamma}(n/2,\, 1/2)$
- $Y \sim \text{Erlang}(n, 1/2)$
- $Y \sim \text{Erlang}(n/2, 1/2)$
- $Y \sim T_n$
- $Y \sim T_{n/2}$
- $Y \sim F_{n,1/2}$
- $Y \sim F_{n/2,1/2}$
- $Y \sim \text{Cauchy}(0, n)$
- $Y^2 \sim T_{n/2}$
- $Y^2 \sim F_{n/2,1/2}$

(f) [5 pt / 100 pts] Let $R \sim F_{1,2}$. Write the PDF of R below. Simplify as much as possible. Hint: your answer should not even include the gamma function but only feature fundamental constants. Don't forget the support term with the indicator function.

The general F distribution's PDF is below:

$$R \sim F_{a,b} \quad := \quad \frac{\Gamma((a+b)/2)}{\Gamma(a/2)\Gamma(b/2)} \left(\frac{a}{b}\right)^{a/2} r^{a/2-1} \left(1+\frac{a}{b}r\right)^{-(a+b)/2} \mathbb{1}_{r>0}$$

We now substitute $a = 1$ and $b = 2$ and simplify using the following identities we learned in class: $\Gamma(1) = 1$ and $\Gamma(x+1) = x\Gamma(x)$.

$$\begin{aligned}
R \sim F_{1,2} \quad &:= \quad \frac{\Gamma((1+2)/2)}{\Gamma(1/2)\Gamma(2/2)} \left(\frac{1}{2}\right)^{1/2} r^{1/2-1} \left(1+\frac{1}{2}r\right)^{-(1+2)/2} \mathbb{1}_{r>0} \\
&= \quad \frac{\Gamma(3/2)}{\Gamma(1/2)\Gamma(1)} \left(\frac{1}{2}\right)^{1/2} r^{1/2-1} \left(1+\frac{1}{2}r\right)^{-3/2} \mathbb{1}_{r>0} \\
&= \quad \frac{\frac{1}{2}\Gamma(1/2)}{\Gamma(1/2)\Gamma(1)} \left(\frac{1}{2}\right)^{1/2} r^{1/2-1} \left(1+\frac{1}{2}r\right)^{-3/2} \mathbb{1}_{r>0} \\
&= \quad \left(\frac{1}{2}\right)^{3/2} r^{1/2-1} \left(1+\frac{1}{2}r\right)^{-3/2} \mathbb{1}_{r>0} \\
&= \quad \frac{1}{\sqrt{r(2+r)^3}} \mathbb{1}_{r>0}
\end{aligned}$$