

Math 340 / 640 Fall 2025

Final Examination

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Full Name _____

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

signature

date

Instructions

This exam is 120 minutes (variable time per question) and closed-book. You are allowed **three** pages (front and back) of a “cheat sheet”, blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 These questions are theoretical.

- (a) [8 pt / 8 pts] Let $X \sim U(0, 1)$ and $Y | X = x \sim \text{Poisson}(x)$. Find the marginal distribution of Y expressed as a regularized incomplete gamma function (P or Q) times the support term $\mathbb{1}_{y \in \mathbb{S}_Y}$.

- (b) [5 pt / 13 pts] Let $X \sim U(0, 1)$ and $Y | X = x \sim \text{Poisson}(x)$. Find $\mathbb{E}[Y]$ using the law of iterated expectation.

- (c) [8 pt / 21 pts] Let $X \sim \text{Exp}(1)$ and $Y | X = x \sim \mathcal{N}(0, x)$. Find the marginal distribution of Y . Show that Y is distributed as a brand name rv we've studied and find that name brand rv's parameter values. To do this problem, you'll need to make use of the following integral identity:

$$\int_0^\infty u^{-\frac{1}{2}} e^{-\frac{\alpha}{u}-u} du = \sqrt{\pi} e^{-2\sqrt{\alpha}}$$

- (d) [5 pt / 26 pts] Let $X \sim \text{Exp}(1)$ and $Y | X = x \sim \mathcal{N}(0, x)$. Find $\text{Var}[Y]$ using the law of total variance.

- (e) [8 pt / 34 pts] Let X and Y be independent rv's with chf's ϕ_X and ϕ_Y respectively. Find the joint chf for the vector rv $\mathbf{U} := \begin{bmatrix} X - Y \\ X + Y \end{bmatrix}$ as a function of ϕ_X and ϕ_Y . Simplify as much as you can.
- (f) [5 pt / 39 pts] Let $X_1, \dots, X_{37} \stackrel{iid}{\sim} \text{Weibull}(k, \lambda)$. Find the PDF of $X_{(17)}$. No need to simplify.

- (g) [8 pt / 47 pts] Let X_1 and X_2 be independent continuous rv's. Find the most simple formula which computes the PDF of $M = \ln(X_1)X_2$ as a function of the densities of X_1 and X_2 .

(h) [8 pt / 55 pts] Let $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Exp}(\lambda)$. Show that the minimum is distributed as a brand name rv we've studied and find that brand name rv's parameter values.

(i) [5 pt / 60 pts] Compute $\int_{\mathbb{R}} e^{(2\sqrt{\pi})x - \pi x^2} dx$ and simplify as much as possible.

(j) [6 pt / 66 pts] Let $\mathbf{Z} \sim \mathcal{N}_n(\mathbf{0}_n, \mathbf{I}_n)$. What is $\mathbb{P}(\mathbf{Z} > \mathbf{0}_n)$? No need to show work.

(k) [6 pt / 72 pts] Let $X \sim \text{ParetoI}(17, \lambda)$. Let $Y = X \mid X > 37$. Show that Y is distributed as a brand name rv we've studied and find that name brand rv's parameter values.

(l) [6 pt / 78 pts] Let $u = .1234$ be a sample from $U \sim U(0, 1)$. Using the algorithm we discussed in class, draw a realization x from $X \sim \text{Logistic}(0, 1)$ using this value of u . Round to two decimals.

(m) [8 pt / 86 pts] Let $\mathbf{X} \sim \mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Find $k_{\mathbf{X}}(\mathbf{x})$, the kernel of its JDF.

(n) [6 pt / 92 pts] Let $\mathbf{X}_1 \sim \mathcal{N}_n(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ independent of $\mathbf{X}_2 \sim \mathcal{N}_n(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$. Find the JDF of $\mathbf{X}_1 + \mathbf{X}_2$ by using joint chf's. Make sure you state the properties you are using at the steps that use the properties.

(o) [8 pt / 100 pts] Prove the following integral identity:

$$B(a, b) = \int_0^{\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt$$

- (p) [10 pt / 110 pts] Extra Credit. Do not attempt this problem until you are finished with the rest of the test. Let $X \sim \text{Logistic}(0, 1)$. Show that $\phi_X(t) = \Gamma(1 - c(t))\Gamma(1 + c(t))$ and find $c(t) \in \mathbb{C}$.