# Math 340 / 640 Fall 2025
# Midterm Examination One

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Full Name ______________________________

## Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

**Cheating** Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

______________________________ signature ______________ date

## Instructions

This exam is 110 minutes (variable time per question) and closed-book. You are allowed **one** page (front and back) of a "cheat sheet", blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

**Problem 1** This question is about the uniform discrete rv and the Bernoulli rv.

(a) [5 pt / 5 pts] Let $Y \sim U(\{a, b, c, d\})$ where $a, b, c, d$ are unique real numbers. Write $p(y)$, the PMF of $Y$ below.

(b) [5 pt / 10 pts] Let $X \sim \text{Bernoulli}\,(p)$. Write $F(x)$, the CDF which is valid for all $x \in \mathbb{R}$. Hint: use indicator functions.

(c) [7 pt / 17 pts] Let $X_1, X_2, \ldots \stackrel{iid}{\sim} \text{Bernoulli}\,(0.5)$. Let $Y \sim U(\{a, b, c\})$ where $a \neq b \neq c$ and $a, b, c$ are unique real numbers. Write $Y$ as a function of $a, b, c, X_1, X_2, \ldots$. Write as many terms as necessary to illustrate your answer unambiguously. Hint: you have to use all $X_1, X_2, \ldots$. Note: this question demonstrates that any uniform discrete rv can be generated via binary Bernoulli rv's. Advice: leave this question for last.

(d) [3 pt / 20 pts] Let $X$ be a rv and $S(x)$ be its survival function valid for all $x \in \mathbb{R}$ (i.e., this is not the "old" form). Provide three properties that $S(x)$ must satisfy.

(i)

(ii)

(iii)

**Problem 2** This problem is about characteristic functions.

(a) [5 pt / 25 pts] In class we spoke about betting \$1 on lucky number 7 in American roulette. In this game, you win \$36 with probability 1/38 and lose your dollar otherwise. If $X$ models the result of this bet, find $\phi_X(t)$.

(b) [3 pt / 28 pts] Let $X \sim \text{Exp}(\lambda)$. It can be shown that $\phi_X(t) = \frac{\lambda}{\lambda - it}$. Find the characteristic function of $Y := \frac{1}{a}X$ where $a$ is a nonzero real number.

(c) [6 pt / 34 pts] Let $X \sim \text{Exp}(\lambda)$. Let $Y := \frac{1}{a}X$ where $a$ is a nonzero real number. Find the distribution of $Y$. Hint: it will be one of the brand name rv's we studied.

**Problem 3** Let $X \sim \text{Geometric}\,(p)$ independent of $Y \sim \text{NegBin}\,(2, p)$.

(a) [3 pt / 37 pts] Which is true? Circle one of the following.

(i) $\mathbb{P}\,(X > Y) < \frac{1}{2}$

(ii) $\mathbb{P}\,(X > Y) = \frac{1}{2}$

(iii) $\mathbb{P}\,(X > Y) > \frac{1}{2}$

(b) [8 pt / 45 pts] Compute $\mathbb{P}\,(X > Y)$. Get as far as possible given the tools we studied. But simplify as much as possible.

**Problem 4** Consider a rv $X$ with $S_X \geq 0$. We realize $n$ (assumed to be large) iid realizations from this rv and calculate the following quantities rounded to two decimals:

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} x_i &= 2.51, & \frac{1}{n}\sum_{i=1}^{n} x_i^6 &= 3071.00 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^2 &= 7.89, & \frac{1}{n}\sum_{i=1}^{n} x_i^7 &= 16475.31 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^3 &= 29.62, & \frac{1}{n}\sum_{i=1}^{n} x_i^8 &= 91664.09 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^4 &= 127.21, & \frac{1}{n}\sum_{i=1}^{n} x_i^9 &= 523306.42 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^5 &= 602.66, & \frac{1}{n}\sum_{i=1}^{n} x_i^{10} &= 3042898.00
\end{aligned}
$$

(a) [4 pt / 49 pts] Provide an approximation for this rv's expectation.

(b) [3 pt / 52 pts] Which theorem did you make use of to answer the previous question?

(c) [4 pt / 56 pts] Estimate the quantity $\phi_X''''(0)$.

**Problem 5** Consider the following rv

$$X \sim \text{arcsin} := \frac{1}{\pi\sqrt{x(1-x)}}\mathbb{1}_{x\in(0,1)}$$

which has this name because its CDF can be written as

$$F(x) = \frac{2}{\pi}\text{arcsin}(\sqrt{x}).$$

(a) [2 pt / 58 pts] Is this rv discrete or continuous?

(b) [2 pt / 60 pts] How many parameter(s) does this rv have?

(c) [2 pt / 62 pts] What is the support of this rv?

(d) [2 pt / 64 pts] What is the name of the probability principle we used many times in class which demands that $\int_0^1 \frac{1}{\pi\sqrt{x(1-x)}}dx = 1$?

**Problem 6** Consider the following rv

$$X \sim \text{Frechet}(c, s) := \frac{s}{c}\left(\frac{x}{c}\right)^{-1-s} e^{\left(\frac{x}{c}\right)^{-s}} \mathbb{1}_{x>0}$$

(a) [2 pt / 66 pts] Is this rv discrete or continuous?

(b) [2 pt / 68 pts] How many parameter(s) does this rv have?

(c) [2 pt / 70 pts] What is the support of this rv?

(d) [5 pt / 75 pts] Assume $c$ is positive. Why can't $s$ be negative?

(e) [8 pt / 83 pts] Let $X_1, X_2 \overset{iid}{\sim} \text{Frechet}(c, s)$ and $T = X_1 + X_2$. Get as far as possible calculating $f_T(t)$. But simplify as much as possible.

**Problem 7** Consider the following two-dimensional vector rv

$$\boldsymbol{X} \sim \begin{cases} 0.1 & \text{if } \ x_1 = 0 \ \text{ and } \ x_2 = 0 \\ 0.2 & \text{if } \ x_1 = 1 \ \text{ and } \ x_2 = 0 \\ 0.3 & \text{if } \ x_1 = 0 \ \text{ and } \ x_2 = 1 \\ 0.4 & \text{if } \ x_1 = 1 \ \text{ and } \ x_2 = 1 \end{cases}$$

(a) [3 pt / 86 pts] The expression above after the "$\sim$" symbol is the ... of $\boldsymbol{X}$. Circle one:

(i) PMF (ii) JMF (iii) PDF (iv) JDF

(b) [5 pt / 91 pts] It can be shown that $X_2 \sim \text{Bernoulli}\,(0.7)$. Find the distribution of $X_1$.

(c) [4 pt / 95 pts] Use the Cauchy-Schwartz corollary to find an upper bound for the absolute value of $\mathbb{C}\text{ov}\,[X_1, X_2]$. Hint: the variance of a rv distributed as a $\text{Bernoulli}\,(p)$ is $p(1-p)$. Round to two significant digits.

(d) [5 pt / 100 pts] Compute $\mathbb{C}\text{ov}\,[X_1, X_2]$. Round to two significant digits.