Math 341 / 641 Fall 2023
Final Examination Solutions

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Full Name ____

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

______________________________________ ________________

signature date

Instructions

This exam is 120 minutes (variable time per question) and closed-book. You are allowed **three** 8.5" $\times$ 11" pages (front and back) "cheat sheets", blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. Show as much partial work as you can and justify each step. No food is allowed, only drinks.

Distribution of r.v.	Quantile Function	PMF / PDF function	CDF function	Sampling Function
beta	`qbeta`(p, α, β)	`d-`(x, α, β)	`p-`(x, α, β)	`r-`(α, β)
betabinomial	`qbetabinom`(p, n, α, β)	`d-`(x, n, α, β)	`p-`(x, n, α, β)	`r-`(n, α, β)
gamma	`qgamma`(p, α, β)	`d-`(x, α, β)	`p-`(x, α, β)	`r-`(α, β)
ext negative binomial	`qnbinom`(p, r, θ)	`d-`(x, r, θ)	`p-`(x, r, θ)	`r-`(r, θ)
normal	`qnorm`(p, θ, σ)	`d-`(x, θ, σ)	`p-`(x, θ, σ)	`r-`(θ, σ)
inversegamma	`qinvgamma`(p, α, β)	`d-`(x, α, β)	`p-`(x, α, β)	`r-`(α, β)
Student's T	`qt.scaled`(p, k, μ, σ)	`d-`(x, k, μ, σ)	`p-`(x, k, μ, σ)	`r-`(k, μ, σ)

Table 1: Functions from `R` (in alphabetical order) that can be used on this assignment and exams. The hyphen in colums 3, 4 and 5 is shorthand notation for the full text of the r.v. which can be found in column 2.

Problem 1 An uber driver is rated on a 1-5 ★ scale. Most ratings are 5 ★. Anything less means something went wrong thus we are interested in 5 ★ vs < 5★. So let 1 indicate 5 ★ and 0 indicate < 5★. We model a driver's n ratings with the DGP: $X \sim \text{Binomial}\,(n,\, \theta)$. The number of rides n is fixed and we seek inference on the parameter θ.

(a) [3 pt / 3 pts] What is the parameter space of θ?

$(0, 1)$

(b) [3 pt / 6 pts] Using $f(\theta) = \text{Beta}\,(\alpha,\, \beta)$, how is the posterior $f(\theta \,|\, X)$ distributed?

$f(\theta \,|\, X) = \text{Beta}\,(\alpha + x,\, \beta + n - x)$

(c) [6 pt / 12 pts] Show that $\hat{\theta}^{\text{MMSE}}$ is asymptotically normal. Find its mean and variance.

By the CLT or MLE monster theorem, $\bar{X} \mathrel{\dot\sim} \mathcal{N}\left(\theta,\, \frac{\theta(1-\theta)}{n}\right)$. Using this fact,

$$\begin{aligned}
\hat{\theta}^{\text{MMSE}} &= \frac{\alpha + X}{\alpha + \beta + n} = \frac{\alpha}{\alpha + \beta + n} + \frac{n}{\alpha + \beta + n}\bar{X} \\
&\mathrel{\dot\sim} \mathcal{N}\left(\frac{\alpha}{\alpha + \beta + n} + \frac{n}{\alpha + \beta + n}\theta,\, \frac{n\theta(1-\theta)}{(\alpha + \beta + n)^2}\right)
\end{aligned}$$

(d) [6 pt / 18 pts] Compute $\mathbb{B}\text{ias}\left[\hat{\theta}^{\text{MAP}}\right]$.

$$\begin{aligned}
\hat{\theta}^{\text{MAP}} &= \frac{\alpha + X - 1}{\alpha + \beta + n - 2} = \frac{\alpha - 1}{\alpha + \beta + n - 2} + \frac{n}{\alpha + \beta + n - 2}\bar{X} \\
\mathbb{B}\text{ias}\left[\hat{\theta}^{\text{MAP}}\right] &:= \mathbb{E}\left[\hat{\theta}^{\text{MAP}}\right] - \theta = \mathbb{E}\left[\frac{\alpha - 1}{\alpha + \beta + n - 2} + \frac{n}{\alpha + \beta + n - 2}\bar{X}\right] - \theta \\
&= \frac{\alpha - 1}{\alpha + \beta + n - 2} + \frac{n}{\alpha + \beta + n - 2}\theta - \theta \\
&= \frac{\alpha - 1}{\alpha + \beta + n - 2} + \left(\frac{n}{\alpha + \beta + n - 2} - 1\right)\theta = \frac{\alpha - 1 + (\alpha + \beta - 2)\,\theta}{\alpha + \beta + n - 2}
\end{aligned}$$

A new driver in the United States takes 5 rides in the first day and get only one 5 ★ rating and 4 < 5★ ratings. Use this data for the rest of the questions in this problem.

(e) [4 pt / 22 pts] Using the Jeffrey's prior, calculate $\hat{\hat{\theta}}^{\text{MMSE}}$ to two significant digits.

$$\hat{\hat{\theta}}^{\text{MMSE}} = \frac{\alpha + x}{\alpha + \beta + n} = \frac{1 + \frac{1}{2}}{\frac{1}{2} + \frac{1}{2} + 5} = 0.25$$

(f) [4 pt / 26 pts] Using Haldane's prior, write an expression for a 95% credible region. Use numerical values for parameters.

$$\begin{aligned} f(\theta \,|\, \boldsymbol{X}) &= \text{Beta}\,(\alpha + x,\, \beta + n - x) = \text{Beta}\,(x,\, n - x) = \text{Beta}\,(1,\, 4) \\ CR_{\theta,95\%} &= [\text{qbeta}(0.025, 1, 4), \text{qbeta}(0.975, 1, 4)] \end{aligned}$$

(g) [5 pt / 31 pts] Interpret the interval from the previous question.

The probability that θ is contained within the set defined by $CR_{\theta,95\%}$ is 95%.

(h) [5 pt / 36 pts] Using Laplace's prior, write an expression for the Bayesian p-value for the test whether this driver has a worse-than-average rating. The average average rating is 0.96 in the United States. Use numerical values for parameters.

$$\begin{aligned} & H_a : \theta < 0.96 \;\Rightarrow\; H_0 : \theta \geq 0.96, \\ p_{val} &:= \mathbb{P}\,(H_0 \mid X) = \mathbb{P}\,(\theta \geq 0.96 \mid X) = 1 - \text{pbeta}(0.96, \alpha + x, \beta + n - x) \\ &= 1 - \text{pbeta}(0.96, 2, 5) \end{aligned}$$

Five rides is too small to make a meaningful estimate. Instead we rely on previous data. Fitting a beta distribution to previous data yields $\hat{\alpha}^{\text{MLE}} = 961.24$ and $\hat{\hat{\beta}}^{\text{MLE}} = 40.58$. We now use these maximum likelihood estimates as α and β in $f(\theta)$ for the remainder of the problem.

(i) [2 pt / 38 pts] Would this prior be considered objective? Yes / $\boxed{\text{no}}$

(j) [4 pt / 42 pts] Compute the shrinkage metric ρ for $\hat{\hat{\theta}}^{\text{MMSE}}$ to three significant digits.

$$\rho = \frac{\alpha + \beta}{\alpha + \beta + n} = \frac{961.24 + 40.58}{961.24 + 40.58 + 5} = 0.995$$

(k) [4 pt / 46 pts] Write an expression for $\hat{\hat{\theta}}^{\text{MMAE}}$. Use numerical values for parameters.

$$\hat{\hat{\theta}}^{\text{MMAE}} = \text{qbeta}(0.5, \alpha + x, \beta + n - x) = \text{qbeta}(0.5, 962.24, 44.58)$$

(l) [5 pt / 51 pts] For the driver's next $1,000$ rides, what is the distribution of the count of future 5 ★ ratings, i.e. $\mathbb{P}(X_* \mid X = 1)$? Use numerical values for parameters.

$$\mathbb{P}(X_* \mid X = 1) = \text{BetaBinomial}(n_*, \alpha + x, \beta + n - x) = \text{BetaBinomial}(1000, 962.24, 44.58)$$

Problem 2 You are the data scientist at T-Mobile trying to understand the number of customer support calls on a weekday at primetime at the East Coast switchboard per minute. For two weeks Monday-Friday from 6:00:00PM - 6:00:59PM, here are the number of calls:

583 565 580 524 644 598 564 594 564 538

You model the number of calls per minute as the DGP: $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\theta)$. The average number of calls is $\bar{x} = 575.4$.

(a) [3 pt / 54 pts] Using the Laplace prior, what is $f(\theta)$? Use numerical values for parameters.

$$f(\theta) = \text{Gamma}(1,\ 0)$$

(b) [4 pt / 58 pts] Using the Laplace prior, what is the posterior distribution of $\theta \mid \boldsymbol{X}$? Use numerical values for parameters.

$$f(\theta \mid \boldsymbol{X}) = \text{Gamma}\left(1 + \sum x_i,\ n\right) = \text{Gamma}(5755,\ 10)$$

(c) [5 pt / 63 pts] Using the Haldane prior, create a 90% credible region for $\theta \mid \boldsymbol{X}$. Use numerical values for parameters.

$$\begin{aligned} f(\theta \mid \boldsymbol{X}) &= \text{Gamma}\left(\sum x_i,\ n\right) = \text{Gamma}(5754,\ 10) \\ CR_{\theta,95\%} &= [\text{qgamma}(0.05, 5754, 10), \text{qgamma}(0.95, 5754, 10)] \end{aligned}$$

(d) [6 pt / 69 pts] Using the Jeffrey's prior, create a 95% posterior predictive interval (PI) for the number of calls from 6:00:00PM - 6:00:59PM on the next weekday. Use numerical values for parameters.

$$\begin{aligned} \mathbb{P}(X_* \mid \boldsymbol{X}) &= \text{ExtNegBinom}\left(\alpha + \sum x_i, \frac{\beta + n}{\beta + n + 1}\right) = \text{ExtNegBinom}\left(0.5 + 5754, \frac{10}{10 + 1}\right) \\ &= \text{ExtNegBinom}(5754.5, 0.909) \\ PI_{X_*,95\%} &= [\text{qnbinom}(0.025, 5754.5, 0.909), \text{qnbinom}(0.975, 5754.5, 0.909)] \end{aligned}$$

Problem 3 You are a data scientist studying Chevron's dividends. The following are the $n = 27$ quarterly dividend returns of Chevron stock (as a %) dating back to February, 2016:

```
3.6 3.6 3.4 4.2 4.6 5.5 5.0 5.5 5.9 5.9 3.6 4.7 3.9 4.1
3.9 4.0 3.8 3.5 4.0 3.8 4.0 4.1 3.8 4.0 4.2 4.2 5.0
```

You model the quarterly returns as the DGP: $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\theta, \sigma^2)$ where both θ and σ^2 are unknown parameters. The average dividend return is $\bar{x} = 4.289$ and the sample variance of the returns is $s^2 = 0.526$.

(a) [6 pt / 75 pts] Using the Jeffrey's prior, what is the posterior distribution, $f(\theta, \sigma^2 \mid \boldsymbol{X})$? Round the parameters to three decimal places.

$$\begin{aligned} f(\theta, \sigma^2 \mid \boldsymbol{X}) &= \text{NormalInverseGamma}\left(\bar{x}, n, \frac{n}{2}, \frac{(n-1)s^2}{2}\right) \\ &= \text{NormalInverseGamma}\left(4.289, 27, 13.5, 6.838\right) \end{aligned}$$

(b) [6 pt / 81 pts] Using the Jeffrey's prior, write an expression for the Bayesian p-value for the test for $H_a : \theta < 4$. Use numerical values for parameters.

$$\begin{aligned} & H_a : \theta < 4 \;\Rightarrow\; H_0 : \theta \geq 4, \\ f(\theta \mid \boldsymbol{X}) &= T_{n-1}\left(\bar{x}, \frac{s^2}{n}\right) = T_{26}\left(4.289, \frac{0.526}{27}\right) \\ p_{val} &:= \mathbb{P}\left(H_0 \mid X\right) = \mathbb{P}\left(\theta \geq 4 \mid X\right) = 1 - \text{pt.scaled}(4, 26, 4.289, 0.0195) \end{aligned}$$

(c) [6 pt / 87 pts] Find an expression for the probability that Chevron's next quarter's dividend will exceed 5%. Use numerical values for parameters.

$$\begin{aligned} f(X_* \mid \boldsymbol{X}) &= T_{n-1}\left(\bar{x}, \frac{n+1}{n} s^2\right) = T_{26}\left(4.289, 0.545\right) \\ \mathbb{P}\left(X_* \geq 5 \mid \boldsymbol{X}\right) &= 1 - \text{pt.scaled}(5, 26, 4.289, 0.545) \end{aligned}$$

Problem 4 Consider the DGP:

$$X_1, \dots, X_n \overset{iid}{\sim} \text{Geometric}\,(\theta) := \theta(1-\theta)^x \mathbb{1}_{x \in \mathbb{N}_0} \quad \text{where} \quad \mathbb{E}\,[X] = \frac{1-\theta}{\theta}, \; \mathbb{V}\text{ar}\,[X] = \frac{1-\theta}{\theta^2}$$

(a) [6 pt / 93 pts] Find the posterior distribution of θ and the parameters of this distribution. Assume the Laplace prior.

$$\begin{aligned} f(\theta \,|\, \boldsymbol{X}) &\propto \left(\prod_{i=1}^{n} \theta(1-\theta)^{x_i}\right) f(\theta) \propto \theta^n (1-\theta)^{\sum x_i} = \theta^{(n+1)-1}(1-\theta)^{(\sum x_i + 1)-1} \\ &\propto \text{Beta}\left(1+n,\, 1+\sum_{i=1}^{n} x_i\right) \end{aligned}$$

(b) [7 pt / 100 pts] Find the Jeffrey's prior and its parameters for this DGP.

$$\begin{aligned} \mathcal{L}(\theta; \boldsymbol{X}) &= \prod_{i=1}^{n} \theta(1-\theta)^{x_i} = \theta^n (1-\theta)^{\sum x_i} \\ \ell(\theta; \boldsymbol{X}) &= n \ln(\theta) + \left(\sum_{i=1}^{n} x_i\right) \ln(1-\theta) \\ \ell'(\theta; \boldsymbol{X}) &= \frac{n}{\theta} - \frac{\sum_{i=1}^{n} x_i}{1-\theta} \\ \ell''(\theta; \boldsymbol{X}) &= -\frac{n}{\theta^2} - \frac{\sum_{i=1}^{n} x_i}{(1-\theta)^2} \\ I_n(\theta) &= \mathbb{E}\left[-\ell''(\theta; \boldsymbol{X})\right] = \mathbb{E}\left[\frac{n}{\theta^2} + \frac{\sum_{i=1}^{n} X_i}{(1-\theta)^2}\right] = \frac{n}{\theta^2} + \frac{\sum_{i=1}^{n} \mathbb{E}\,[X_i]}{(1-\theta)^2} = \frac{n}{\theta^2} + \frac{\sum_{i=1}^{n} \frac{1-\theta}{\theta}}{(1-\theta)^2} \\ &= \frac{n}{\theta^2} + \frac{n}{\theta(1-\theta)} = n\left(\frac{1-\theta}{\theta^2(1-\theta)} + \frac{\theta}{\theta^2(1-\theta)}\right) = n\theta^{-2}(1-\theta)^{-1} \\ f_J(\theta) &\propto \sqrt{I_n(\theta)} \propto \sqrt{\theta^{-2}(1-\theta)^{-1}} = \theta^{-1}(1-\theta)^{-1/2} = \theta^{(0)-1}(1-\theta)^{(1/2)-1} \\ &\propto \text{Beta}\,(0,\, 1/2) \end{aligned}$$