Lecture 20 MATH 341/641

New DGP: $X_1, \ldots, X_n \stackrel{iid}{\sim} \text{Poisson}(\theta) := \frac{\theta^x e^{-\theta}}{x!} \mathbb{1}_{x \in \mathbb{N}_0}$, $\theta \in (0, \infty)$

prior & posterior continuous

$$f(\theta \mid \vec{x}) = \frac{p(\vec{x} \mid \theta) f(\theta)}{p(\vec{x})} \propto p(\vec{x} \mid \theta) f(\theta) = \left(\prod_{i=1}^{n} \frac{\theta^{x_i} e^{-\theta}}{x!} \right) f(\theta)$$

$$= \frac{\theta^{\Sigma x_i} e^{-n\theta}}{\prod x_i} f(\theta)$$

$$\propto \theta^{\Sigma x_i} e^{-n\theta} k(\theta)$$ What does the conjugate kernel look like?

$$\propto \left(\theta^{\Sigma x_i} e^{-n\theta} \right) \left(\theta^{\alpha - 1} e^{-\beta\theta} \right) \Rightarrow f(\theta) = \text{Gamma}(\alpha, \beta), \text{ s.t. } \alpha, \beta > 0$$

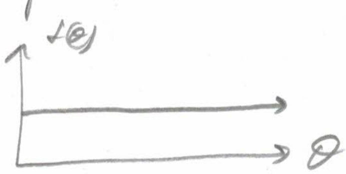
$$= \theta^{\alpha + \Sigma x_i - 1} e^{-(\beta + n)\theta} \propto \text{Gamma}(\alpha + \Sigma x_i, \beta + n)$$

Pseudodata interpretation? Look at units!!

$\alpha = \#$ pseudosuccesses

$\beta = n_0 = \#$ pseudotrials

Laplace Prior?



No way to have a flat prior on $\theta \in (0, \infty)$.

Trick: let $f(\theta) \propto 1$. If so...

$$f(\theta \mid \vec{x}) \propto \theta^{\Sigma x_i} e^{-n\theta} (1) \propto \text{Gamma}(1 + \Sigma x_i, n) \Rightarrow f(\theta) = \text{Gamma}(1, 0)$$

improper since $\beta \not> 0$!

Haldane Prior? Total ignorance

$\#$ pseudosuccesses $= 0 \Rightarrow \alpha = 0$

$n_0 = 0 \Rightarrow \beta = 0$

$\Rightarrow f(\theta) = \text{Gamma}(0, 0)$ improper!

Jeffreys Prior under 1, Q1 in Also Improper

$$f_J(\theta) \propto \sqrt{I(\theta)} \stackrel{\downarrow}{=} \sqrt{\frac{n}{\theta}} \propto \theta^{-\frac{1}{2}} = \theta^{\frac{1}{2}-1} \propto \text{Gamma}\left(\tfrac{1}{2}, 0\right)$$

Point Estimates?

$$\hat{\theta}^{\text{MMSE}} = \frac{\alpha + \sum x_i}{\beta + n}$$

$\hat{\theta}^{\text{MMAE}} = \text{qgamma}(0.5, \alpha + \sum x_i, \beta + n)$ Not available in closed form

$$X \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

$\text{Mode}[X] = ?$ $\text{argmax}\{f_X(x)\} = \text{argmax}\{x^{\alpha-1} e^{-\beta x}\}$

$$\Rightarrow (x^{\alpha-1})(-\beta e^{-\beta x}) + ((\alpha-1)x^{\alpha-2})e^{-\beta x} \stackrel{\text{set}}{=} 0$$

$$\Rightarrow \cancel{x^{\alpha-2} e^{-\beta x}}(-\beta x + (\alpha-1)) = 0$$

$$\Rightarrow (\alpha-1) = \beta x \qquad x = \frac{\alpha-1}{\beta}$$ only a max if $\alpha > 1$, otherwise, 0

$$\Rightarrow \hat{\theta}^{\text{MAP}} = \frac{\alpha + \sum x_i - 1}{\beta + n}$$ for $\alpha + \sum x_i - 1 > 1$, otherwise, 0

$$\hat{\theta}^{\text{MLE}} \stackrel{\text{under Laplace i.e. } \alpha=1, \beta=0}{=} \hat{\theta}^{\text{MAP}} = \frac{\sum x_i}{n} = \bar{x}$$ if $\sum x_i > 0$, otherwise 0

2-sided

Credible Regions

$$CR_{\theta,1-\alpha_0} = \left[qgamma\left(\frac{\alpha_0}{2}, \alpha+\Sigma x_i, \beta+n\right), qgamma\left(1-\frac{\alpha_0}{2}, \alpha+\Sigma x_i, \beta+n\right) \right]$$

Hypothesis Tests at level α_0

$H_a: \theta > \theta_0 \Rightarrow H_0: \theta \leq \theta_0$

$pval = pgamma(\theta_0, \alpha+\Sigma x_i, \beta+n)$ Retain if $pval \geq \alpha_0$

$H_a: \theta < \theta_0 \Rightarrow H_0: \theta \geq \theta_0$

$pval = 1 - pgamma(\theta_0, \alpha+\Sigma x_i, \beta+n)$ Retain if $pval \geq \alpha_0$

$H_a: \theta \neq \theta_0 \Rightarrow H_0: \theta = \theta_0$

Using margin of equivalence δ

$\Rightarrow H_a: \theta \notin [\theta_0 \pm \delta] \Rightarrow H_0: \theta \in [\theta_0 \pm \delta]$

$pval = pgamma(\theta_0+\delta, \alpha+\Sigma x_i, \beta+n) - pgamma(\theta_0-\delta, \alpha+\Sigma x_i, \beta+n)$

Retain if $pval \geq \alpha_0$

Using CR method

Retain H_0 if $\theta_0 \in CR_{\theta,1-\alpha_0}$

Posterior Predictive Distribution

Observe $X_1, \dots, X_n$. What is distr of X_*, the future observation

Note: we only do the case of $n_* = 1$ otherwise very complicated.

$$P(X_* | X) = \int_{\mathbb{R}} \underbrace{P(X_* | \theta)}_{\text{Poisson}(\theta)} \underbrace{f(\theta | \vec{x})}_{\text{Gamma}(\alpha+\sum x_i, \beta+n)} d\theta \overset{\text{by Math 340 Lec 19}}{=} \text{ExtNegBin}\left(\alpha + \sum x_i, \frac{\beta+n}{\beta+n+1}\right)$$

Shrinkage...

$$\hat{\theta}^{\text{mmse}} = \frac{\alpha + \sum x_i}{\beta + n} = \frac{\alpha}{\beta+n} \cdot \frac{\beta}{\beta} + \frac{\sum x_i}{\beta+n} \cdot \frac{n}{n}$$

$$= \underbrace{\frac{\beta}{\beta+n}}_{\rho} \underbrace{\frac{\alpha}{\beta}}_{E(\theta)} + \underbrace{\frac{n}{\beta+n}}_{(1-\rho)} \underbrace{\bar{x}}_{\hat{\theta}^{\text{MLE}}}$$

Done!

New model!

$$\text{DGP: } X_1, \ldots X_n \overset{iid}{\sim} \overbrace{N(\theta, \sigma^2)}^{\text{likelihood and prior pred. distr conv.}} \quad \text{where } \sigma^2 \text{ known}$$

$$\underbrace{\theta \in \mathbb{R}}_{\text{prior and posterior}}$$

$$f(\theta \mid \vec{x}, \sigma^2) = \frac{f(\vec{x} \mid \theta, \sigma^2)\, f(\theta \mid \sigma^2)}{f(\vec{x} \mid \sigma^2)}$$

$$\propto f(\vec{x} \mid \theta, \sigma^2)\, f(\theta \mid \sigma^2)$$

$$\propto \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i - \theta)^2} \right) k(\theta \mid \sigma^2)$$

$$\propto e^{-\frac{1}{2\sigma^2}\sum (x_i - \theta)^2}\, k(\theta \mid \sigma^2)$$

$$= e^{-\frac{1}{2\sigma^2}\left(\sum x_i^2 - 2\theta \sum x_i + n\theta^2\right)}\, k(\theta \mid \sigma^2)$$

$$= e^{-\frac{1}{2\sigma^2}\sum x_i^2}\, e^{\frac{\sum x_i}{\sigma^2}\theta - \frac{n}{2\sigma^2}\theta^2}\, k(\theta \mid \sigma^2)$$

$$\propto e^{\frac{n\bar{x}}{\sigma^2}\theta - \frac{n}{2\sigma^2}\theta^2}\, k(\theta \mid \sigma^2)$$

What's the conjugate prior? $e^{a_0\theta - b_0\theta^2}$

$$= e^{\left(a_0 + \frac{n\bar{x}}{\sigma^2}\right)\theta - \left(b_0 + \frac{n}{2\sigma^2}\right)\theta^2}$$

$$\propto N\left(\frac{a_0 + \frac{n\bar{x}}{\sigma^2}}{2b_0 + \frac{n}{\sigma^2}},\ \frac{1}{2b_0 + \frac{n}{\sigma^2}} \right)$$

$$f(\theta \mid \sigma^2) = N\left(\frac{a_0}{2b_0}, \frac{1}{2b_0}\right) = N(\mu_0, \tau^2)$$

let $b_0 = \frac{1}{2\tau^2}$, $a_0 = \frac{\mu_0}{\tau^2}$

$$= N\left(\overbrace{\frac{\frac{\mu_0}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}}^{\theta_p},\ \overbrace{\frac{1}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}}^{\sigma_p^2} \right)$$

Point Estimation

$$\hat{\theta}^{MMSE} = \hat{\theta}^{MMAE} = \hat{\theta}^{MAP} = \theta_p = \frac{\frac{\mu_0}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}$$

Credible Region

$$CR_{\theta,1-\alpha} = \left[qnorm\left(\frac{\alpha}{2}, \theta_p, \sigma_p^2\right), \; qnorm\left(1-\frac{\alpha}{2}, \theta_p, \sigma_p^2\right) \right]$$

Hypothesis Tests

$H_0: \theta \leq \theta_0, \; H_1: \theta > \theta_0$

$pval = P(H_0 | \vec{x}) = P(\theta \leq \theta_0 | \vec{x}) = pnorm(\theta_0, \theta_p, \sigma_p^2)$

$H_0: \theta \geq \theta_0, \; H_1: \theta < \theta_0$

$pval = P(H_0 | \vec{x}) = P(\theta \geq \theta_0 | \vec{x}) = 1 - pnorm(\theta_0, \theta_p, \sigma_p^2)$