MATH 341/641 Lec 9

Two populations:

DGP_1: $X_{1,1}, \ldots X_{1,n_1} \overset{iid}{\sim} f_1$

DGP_2: $X_{2,1}, \ldots, X_{2,n_2} \overset{iid}{\sim} f_2$

$H_0: DGP_1 = DGP_2$ i.e. $F_1(x) = F_2(x)$

$H_a: DGP_1 \neq DGP_2$ i.e. $F_1(x) \neq F_2(x)$

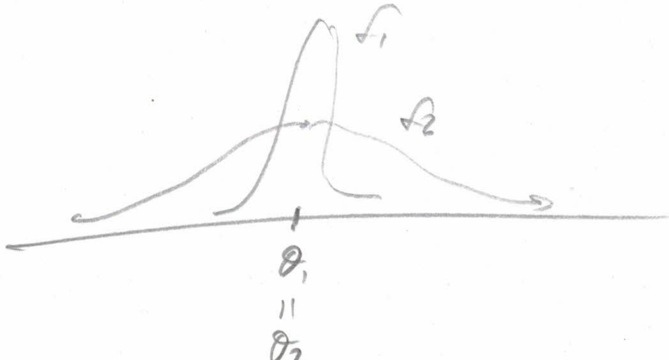
Recall

$f_1 = N(\theta_1, \sigma^2)$ and $f_2 = N(\theta_2, \sigma^2)$, σ^2 known

Can we use this test to test $\theta_1 = \theta_2$? Yes... But

if you know normality, you will get higher power there

This test can be used for $\mu_1 = \mu_2$ but $\sigma_1^2 \neq \sigma_2^2$ e.g.



Test statistic: $D_{n_1,n_2} := \sup_x \left\{ \left| \hat{F}_1(x) - \hat{F}_2(x) \right| \right\}$

$$\sqrt{\frac{n_1 n_2}{n_1+n_2}}\, D_{n_1,n_2} \xrightarrow{d} K$$

Let's put CMT / Slutsky's thm to work immediately by deriving the two proportion z-test.

Def: an estimator $\hat{\theta}$ is "consistent" if $\hat{\theta} \xrightarrow{p} \theta$ i.e. the prob bunches up more and more around what the estimator wishes to estimate.

Pop 1 has DGP: $X_1, \ldots X_{n_1} \overset{iid}{\sim} \text{Bern}(\theta_1)$

Pop 2 has DGP: $X_1, \ldots X_{n_2} \overset{iid}{\sim} \text{Bern}(\theta_2)$

$H_a: \theta_1 \neq \theta_2 \iff H_0: \theta_1 = \theta_2$

To estimate θ_1, θ_2 we use $\hat{\theta}_1 = \bar{X}_1$, $\hat{\theta}_2 = \bar{X}_2$, the UMVUE

also $\bar{X}_1 \sim N\left(\theta_1, \sqrt{\frac{\theta_1(1-\theta_1)}{n_1}}^2\right)$ by CLT,

$\bar{X}_2 \sim N\left(\theta_2, \sqrt{\frac{\theta_2(1-\theta_2)}{n_2}}^2\right)$ by CLT

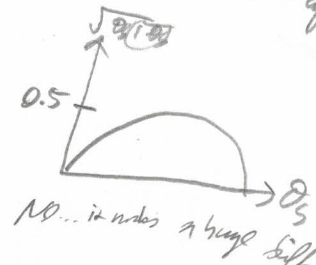
$$\Rightarrow \bar{X}_1 - \bar{X}_2 \sim N\left(\theta_1 - \theta_2, \sqrt{\frac{\theta_1(1-\theta_1)}{n_1} + \frac{\theta_2(1-\theta_2)}{n_2}}^2\right)$$

$$\hat{Z} := \frac{(\bar{X}_1 - \bar{X}_2) - (\theta_1 - \theta_2)}{\sqrt{\frac{\theta_1(1-\theta_1)}{n_1} + \frac{\theta_2(1-\theta_2)}{n}}} \sim N(0,1)$$

Under H_0, $\theta_1 - \theta_2 = 0$ and $\theta_1 - \theta_2 = \theta_s$ ← s for "shared"

$$\hat{Z} \mid H_0 = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{\theta_s(1-\theta_s)}{n_1} + \frac{\theta_s(1-\theta_s)}{n_2}}} = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\sqrt{\theta_s(1-\theta_s)}} \sim N(0,1)$$

Can we make it up?



NO... it makes a huge diff!

But we have a big problem... θ_s is unknown!!

Let's say we find a consistent estimator $\hat{\theta}_s$ for θ_s

i.e. $\hat{\theta}_s \xrightarrow{p} \theta_s$. Can we use it? How?

Consider:

$$\frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\sqrt{\hat{\theta}_s(1-\hat{\theta}_s)}} = \underbrace{\frac{\sqrt{\theta_s(1-\theta_s)}}{\sqrt{\hat{\theta}_s(1-\hat{\theta}_s)}}}_{A} \underbrace{\frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\theta_s(1-\theta_s)}}}_{B}$$

We know $B \xrightarrow{d} N(0,1)$. If $A \xrightarrow{p} 1 \Rightarrow AB \xrightarrow{d} N(0,1)$ by Slutsky's

let $g(x) = \frac{\sqrt{\theta_s(1-\theta_s)}}{\sqrt{x(1-x)}} \Rightarrow A = g(\hat{\theta}_s)$

since $\hat{\theta}_s \xrightarrow{p} \theta_s \overset{\text{by CMT}}{\Rightarrow} g(\hat{\theta}_s) \xrightarrow{p} g(\theta_s) = \frac{\sqrt{\theta_s(1-\theta_s)}}{\sqrt{\theta_s(1-\theta_s)}} = 1$ ✓

$\Rightarrow$

$$Z|H_0 = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\sqrt{\hat{\theta}_s(1-\hat{\theta}_s)}} \sim N(0,1)$$

What is $\hat{\theta}_s$?

Consider $\hat{\theta}_s = \frac{X_{1,1} + \dots X_{1,n_1} + X_{2,1} + \dots + X_{2,n_2}}{n_1 + n_2} \xrightarrow{p} \theta_s$ by WLLN

under H_0
all draw is iid
regardless of population

hence is consistent!

$H_0: \theta_1 = \theta_2, \ \alpha = 5\%$

e.g. $n_1 = 81, \ \sum X_{1i} = 27$ $\qquad \hat{\theta}_1 - \hat{\theta}_2 = \frac{27}{81} - \frac{12}{79} = 0.181$

from a previous exam

$n_2 = 79, \ \sum X_{2i} = 12$ $\qquad \hat{\hat{\theta}}_0 = \frac{27+12}{81+79} = 0.244$

$$\hat{z} = \frac{0.181}{\sqrt{\frac{1}{81} + \frac{1}{79}}\sqrt{0.244(1-.244)}} = 2.66 \notin [-2, 2] \Rightarrow \text{Reject}$$

$$p_{val} = 2\Phi(-2.66) \approx 0.007$$

Note: some textbooks say the following is a 2-prop z-test.

$$\hat{z} \mid H_0 = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{\bar{X}_1(1-\bar{X}_1)}{n_1} + \frac{\bar{X}_2(1-\bar{X}_2)}{n_2}}}$$

This also works but is less powerful!!

HW: Find a case where our test rejects and textbook retains.

Let's also derive the CI for θ when DGP: $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(\theta)$

$$\frac{\bar{X} - \theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \overset{\cdot}{\sim} N(0,1) \text{ by CLT}$$

$\Downarrow$ invert the test

$$\hat{CI}_{\theta, 1-\alpha} = \left[\bar{X} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\theta(1-\theta)}{n}}\right]$$

problem!! we don't know θ. We can't calculate this..

Use same trick:

if $A \xrightarrow{p} 1$, by Slutsky's thm.

$$\frac{\sqrt{\frac{\theta(1-\theta)}{n}}}{\sqrt{\frac{\theta(1-\theta)}{n}}} \frac{\bar{X} - \theta}{\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}} = \underbrace{\frac{\sqrt{\frac{\theta(1-\theta)}{n}}}{\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}}}_{A} \underbrace{\frac{\bar{X} - \theta}{\sqrt{\frac{\theta(1-\theta)}{n}}}}_{B \xrightarrow{d} N(0,1)} \xrightarrow{d} N(0,1)$$

let $g(x) = \frac{\sqrt{\frac{\theta(1-\theta)}{n}}}{\sqrt{\frac{x(1-x)}{n}}}$, $A = g(\bar{X}) \xrightarrow{p} g(\theta) = 1$ ✓ (by CMT)

$$\Rightarrow \frac{\bar{X} - \theta}{\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}} \overset{\cdot}{\sim} N(0,1) \Longrightarrow \hat{CI}_{\theta, 1-\alpha} = \left[\bar{X} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\bar{X}(1-\bar{X})}{n}}\right]$$

One prop CI !!!

double approx: CLT + Slutsky's $\Rightarrow$ very approx.

e.g. $n = 81$, $\sum x_i = 27 \Rightarrow \bar{x} = \frac{27}{81} = .333$

$$\hat{CI}_{\theta, 95\%} = \left[.333 \pm 2\sqrt{\frac{.333 \cdot .667}{81}}\right] = [.205, .461]$$

Def: An estimator is "consistent" if $\hat{\theta}_n \xrightarrow{p} \theta$.
This means as n gets large the probability piles up around what you wish to estimate.

Core MLE and MM Thm

(1) all $\hat{\theta}^{MM}$ are consistent, all $\hat{\theta}^{MLE}$ are consistent

(2) all $\hat{\theta}^{MM}$ are asymptotically normal, all $\hat{\theta}^{MLE}$ are asymptotically normal

(3) $\hat{\theta}^{MM}$ is "asymptotically inefficient" i.e. its variance does not converge generally to the CRLB,
$\hat{\theta}^{MLE}$ is "asymptotically efficient" i.e. its variance does converge to the CRLB. Huge Bonus!!!

$$\Rightarrow \hat{\theta}^{MM} \sim N\left(\theta, \sqrt{\frac{\text{expression}}{n}}^{\,2}\right)$$ beyond scope of course

$$\Rightarrow \hat{\theta}^{MLE} \sim N\left(\theta, \sqrt{\frac{I(\theta)^{-1}}{n}}^{\,2}\right)$$

We will prove this last statement only

p472 of C&B...