In one of the studies of $n=81$ people, $\Sigma x = 27$ people got atrial fibrillation (AF) $\bar{X} = \frac{27}{81} = 0.333$. This is our point estimate of θ, the true prop. in the population who get AF.

We know $\bar{X} \xrightarrow{p} \mu$, it's unbiased and we can use it to create an asymptotically normal Z test. But what if we don't care about θ directly. Instead, we care about the "odds against" getting AF^c

i.e. $\phi(\theta) := \text{Odds}A(\theta) := \frac{1-\theta}{\theta}$. So if $\theta = \frac{1}{4}, 1-\theta = \frac{3}{4} \Rightarrow \text{Odds}A = 3:1$

So ϕ is the parameter of interest, not θ!

What pt. estimate do we use for ϕ?

Can we use $\hat{\phi} = \frac{1-\hat{\theta}}{\hat{\theta}}$? Why? By CMT $\hat{\phi} = \frac{1-\hat{\theta}}{\hat{\theta}} \xrightarrow{p} \frac{1-\theta}{\theta} = \phi$

But how can we do testing? CI's? We need the distr of $\hat{\phi}$!

Thm: **Delta Method**. If $\hat{\theta}$ is asymptotically normal and $\hat{\phi} = g(\hat{\theta})$ where g is differentiable, then,

$$\frac{g(\hat{\theta}) - g(\theta)}{|g'(\theta)| \, SE(\hat{\theta})} \xrightarrow{d} N(0,1)$$

Proof:

Ignore

$$g(\hat\theta) \approx g(\theta) + (\hat\theta - \theta)\, g'(\theta) \quad \text{by Taylor's Series}$$

$$\frac{\hat\phi - \phi}{g'(\theta)\, SE(\hat\theta)} = \frac{g(\hat\theta) - g(\theta)}{g'(\theta)\, SE(\hat\theta)} = \frac{\hat\theta - \theta}{SE(\hat\theta)} \xrightarrow{d} N(0,1)$$

If $g'(\theta) < 1$, then

$$= \frac{\hat\phi - \phi}{-|g'(\theta)|\, SE(\hat\theta)} \xrightarrow{d} N(0,1) \quad \text{due to symmetry}$$

$$\Rightarrow \frac{\hat\phi - \phi}{|g'(\theta)|\, SE(\hat\theta)} \xrightarrow{d} N(0,1) \quad \text{same regardless of sign of } g'(\theta)$$

$$\Rightarrow \frac{\hat\phi - \phi}{|g'(\hat\theta)|\, SE(\hat\theta)} \xrightarrow{d} N(0,1)$$

$$\Rightarrow \hat\phi = g(\hat\theta) \sim N(\phi, (g'(\theta)\, SE(\hat\theta))^2)$$

allows testing of $H_0 : \theta = \theta_0 \Leftrightarrow \phi = \phi_0$

$$\hat z_{H_0} = \frac{\hat\phi - \phi_0}{g'(\theta_0)\, SE(\hat\theta)}$$

by CMT + Slutsky's (A)

$$\hat\phi = g(\hat\theta) \sim N(\phi, g'(\hat\theta)\, SE[\hat\theta]^2)$$

$\Rightarrow$ allows testing when $SE[\hat\theta]$ must be estimated from the data as well as CI's constructed via test inversion

$$\hat{CI}_{\phi, 1-\alpha} \approx \left[\hat\phi \pm z_{1-\frac{\alpha}{2}}\, g'(\hat\theta)\, SE[\hat\theta]\right]$$

---

at $\alpha = 5\%$

Let's use this to test $H_a : \phi \neq 1 \Rightarrow H_0 : \phi = 1 \Leftrightarrow \theta = .5$

and to build a $CI_{95\%}$ for ϕ. $n = 81$, $\sum x_i = 27 \Rightarrow \hat\theta = \frac{1}{3}$

$\phi = g(\theta) = \frac{1-\theta}{\theta}$, $g'(\theta) = \frac{d}{d\theta}\left[\frac{1}{\theta} - 1\right] = -\theta^{-2} \Rightarrow g'(\theta_0) = -.5^{-2} = -\left(\frac{1}{2}\right)^{2} = -4$

$\hat\theta = .333 \Rightarrow \hat\phi = \frac{1-.333}{.333} = 2$

$$\Rightarrow SE(\hat\theta) = \sqrt{\frac{\theta_0(1-\theta_0)}{n}} = \sqrt{\frac{\frac{1}{2}\cdot\frac{1}{2}}{81}} = .0556$$

$$\hat z = \frac{2-1}{|-4|(.0556)} = \frac{1}{.222} = 4.5 \notin [-2, 2] \Rightarrow \text{Reject } H_0$$

$-.333^{-2} = -9$

$$CI_{\phi, 95\%} = \left[\hat\phi \pm z_{1-\frac{\alpha}{2}}\, |g'(\hat\theta)| \sqrt{\frac{\hat\theta(1-\hat\theta)}{n}}\right] = \left[2 \pm 2|-9|\sqrt{\frac{.333 \cdot .667}{81}}\right] = [2 \pm .47] = [1.53, 2.47]$$

Example from Problem 11, Midterm 2020. Root survival

$X_1, \ldots, X_n \overset{iid}{\sim}$ DGP unknown. $n = 30$, $\bar{x} = 2.57$, $s = 1.00$
but $\theta < \infty$, $\sigma^2 < \infty$.

Create $CI_{\phi, 95\%}$ where $\phi = \ln(\theta)$, the log mean survival. $\hat{\theta} = \bar{X}$

$g'(\theta) = \frac{1}{\theta}$, $SE[\hat{\theta}] = \frac{\sigma}{\sqrt{n}} \Rightarrow \hat{SE}[\hat{\theta}] = \frac{S}{\sqrt{n}} = \frac{1}{\sqrt{30}} =$

$g'(\hat{\theta}) = \frac{1}{2.57} = .389$, $\hat{\phi} = \ln(2.57) = .944$

$CI_{\phi, 95\%} = \left[\hat{\phi} \pm z_{1-\frac{\alpha}{2}} |g'(\hat{\theta})| \hat{SE}[\hat{\theta}]\right] = [.944 \pm 2 \cdot .389 \cdot .183]$

$= [.802, 1.086]$

Finally, the T-test...

$X_1, \dots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ neither θ, σ^2 known but inference desired for θ,

we know $\frac{\bar{X} - \theta}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ and $\frac{\bar{X} - \theta}{\frac{S}{\sqrt{n}}} \xrightarrow{d} N(0,1)$

Can we do better than using an approx. test here?

thicker tails than $N(0,1)$

$\frac{\bar{X} - \theta}{\frac{S}{\sqrt{n}}} \sim T_{n-1}$. Note: this statistic needs $n \geq 2$ otherwise $S \sim \text{Deg}(0)$

If n large $\Rightarrow$ just use Z test as the distr. T_{n-1} is virtually indistinguishable from $N(0,1)$.

Define $t_{n-1,\alpha} := \{ t : F_{T_{n-1}}(t) = \alpha \}$,

e.g. $\alpha = 5\%$ $t_{9, 97.5\%} = 2.26 > 1.96$ (16% difference)

$t_{19, 97.5\%} = 2.09 > 1.96$ (7% difference)

$t_{49, 97.5\%} = 2.01 > 1.96$ (2.5% difference)

$t_{99, 97.5\%} = 1.98 > 1.96$ (1% difference)

Testing is same as Z-test except you create a T-stat and compare to the slightly different RET. $H_0: \theta = \theta_0$

$$\hat{T} \mid H_0 = \frac{\bar{X} - \theta_0}{\frac{S}{\sqrt{n}}} \overset{?}{\in} RET = \left[t_{n-1, \frac{\alpha}{2}}, t_{n-1, 1-\frac{\alpha}{2}} \right] = [-2.09, 2.09]$$

e.g. $n = 20$

$$\hat{CI}_{\theta, 1-\alpha} = \left[\bar{X} \pm t_{n-1, 1-\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}} \right]$$

How about 2-populations testing $H_0: \theta_1 - \theta_2 = 0$ with sample sizes n_1, n_2

Two cases: (I) $\sigma_1^2 = \sigma_2^2 = \sigma^2$ (II) $\sigma_1^2 \neq \sigma_2^2$

$$\hat{Z} | H_0 = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \overset{\text{(II)}}{\sim} N(0,1) \quad \text{or} \quad \frac{\bar{X}_1 - \bar{X}_2}{\sigma \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \overset{\text{(I)}}{\sim} N(0,1)$$

σ_1^2, σ_2^2 unknown so we use

$$\text{(II)} \quad \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \xrightarrow{d} N(0,1) \quad \text{or} \quad \text{(I)} \quad \frac{\bar{X}_1 - \bar{X}_2}{S_{pooled} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \xrightarrow{d} N(0,1)$$

$$S^2_{pooled} = \frac{\sum (x_{i1} - \hat{\theta}_1)^2 + \sum (x_{i2} - \hat{\theta}_2)^2}{n_1 + n_2} = \frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2}$$

$\approx$ weighted avg

Can we get an exact test? Yes only under (I)

$$\hat{T} | H_0 = \frac{\bar{X}_1 - \bar{X}_2}{S_{pooled} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim T_{n_1 + n_2 - 2} \quad \text{exactly}$$

Prove extra credit

$$\Rightarrow RET = \left[t_{n_1+n_2-2, \frac{\alpha}{2}}, t_{n_1+n_2-2, 1-\frac{\alpha}{2}} \right]$$

$$\Rightarrow \hat{CI}_{\theta_1 - \theta_2, 1-\alpha} = \left[\bar{X}_1 - \bar{X}_2 \pm t_{n_1+n_2-2} \cdot \frac{S_{pooled}}{\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \right]$$

In case (II), the test statistic is NOT Student's T distributed.

So either use Z-approximate test or use $\frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \sim$ Fisher-Behrens distr (1929/1935)

the "Welch-Satterthwaite Approximation"

In 1966 proven exists in closed form

2018 it was [illegible]

(1946)

6

Welch-Satterthwaite Approx p314-315 C&B

$$\frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \sim T_w, \quad w := \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{S_1^4}{n_1^2(n_1-1)} + \frac{S_2^4}{n_2^2(n_2-1)}}$$

Still used today! Very classic formula!

do demo of class heights (male vs. female).