Lec 23 MATH 341/641

$X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ Both unknown!

$$f(\theta, \sigma^2 | \vec{x}) = \frac{f(\theta, \sigma^2 | \vec{x})\, f(\theta, \sigma^2)}{f(\vec{x})}$$ ← prior for both!

$$\propto f(\theta, \sigma^2 | \vec{x})\, f(\theta, \sigma^2)$$

$$= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i - \theta)^2}\right) f(\theta, \sigma^2)$$

$$\propto (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2} k(\theta, \sigma^2)$$

Note: $\sum(x_i - \theta)^2 = \sum\left((x_i - \bar{x}) + (\bar{x} - \theta)\right)^2 = \sum(x_i - \bar{x})^2 + 2\sum(x_i - \bar{x})(\bar{x} - \theta) + \sum(\bar{x} - \theta)^2$

$$= (n-1)s^2 + 2\sum(x_i\bar{x} - \bar{x}^2 - x_i\theta + \bar{x}\theta) + n(\bar{x} - \theta)^2$$

$$= (n-1)s^2 + 2(\cancel{n\bar{x}^2} - \cancel{n\bar{x}^2} - \cancel{n\bar{x}\theta} + \cancel{n\bar{x}\theta}) + n(\bar{x}-\theta)^2$$

$$= (n-1)s^2 + n(\bar{x} - \theta)^2$$

$$= (\sigma^2)^{-n/2} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{1}{2\sigma^2}n(\bar{x}-\theta)^2} k(\theta, \sigma^2)$$

$$= \underbrace{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}_{N\left(\bar{x}, \frac{\sigma^2}{n}\right)} \underbrace{(\sigma^2)^{-\left(\frac{n}{2}+1\right)-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}}_{\text{InvGamma}\left(\frac{n+2}{2}, \frac{(n-1)s^2}{2}\right)} k(\theta, \sigma^2)$$

under Laplace

$$\text{NormalInvGamma}\left(\mu = \bar{x}, \lambda = n, \alpha = \frac{n+2}{2}, \beta = \frac{(n-1)s^2}{2}\right)$$

$$\Rightarrow k(\theta, \sigma^2) \propto e^{-\frac{1}{2\frac{\sigma^2}{\lambda_0}}(\theta - \mu_0)^2} (\sigma^2)^{-\alpha_0 - 1} e^{-\frac{\beta_0}{\sigma^2}} \propto \text{NormalInvGamma}(\mu_0, \lambda_0, \alpha_0, \beta_0)$$

We will not use general prior

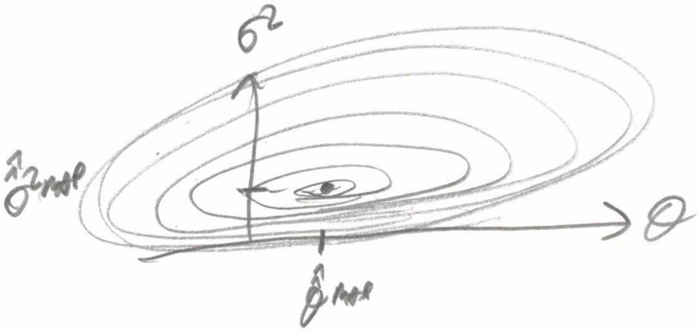
How to sample $\langle\theta, \sigma^2\rangle$ pairs from $f(\theta, \sigma^2 | \vec{x}) \overset{?}{=} f(\theta | \vec{x}, \sigma^2)\, f(\sigma^2 | \vec{x})$ ← Nice

① Sample σ^2 from $\text{InvGamma}\left(\frac{n+2}{2}, \frac{(n-1)s^2}{2}\right)$

② Sample θ from $N\left(\bar{x}, \frac{\sigma^2}{n}\right)$

How to sample only θ?
Sample $\langle\theta, \sigma^2\rangle$ pairs and return only θ's!

How to sample only σ^2
,, ,, ,, ,, ,, ,, σ^2!



Point Estimation:

$$\begin{bmatrix}\hat{\theta}\\ \hat{\sigma}^2\end{bmatrix}^{MAP}, \begin{bmatrix}\hat{\theta}\\ \hat{\sigma}^2\end{bmatrix}^{MMSE}, \begin{bmatrix}\hat{\theta}\\ \hat{\sigma}^2\end{bmatrix}^{MMAE}$$

$$= \begin{bmatrix}\hat{\theta}_{MAP}\\ \hat{\sigma}^2_{MAP}\end{bmatrix} \qquad = \begin{bmatrix}\hat{\theta}_{MMSE}\\ \hat{\sigma}^2_{MMSE}\end{bmatrix}$$

↑ we prove this soon

↑ Hard!

Credible region? 2-dimensional! We skip!

Hypothesis tests: $H_0: \begin{bmatrix}\theta\\ \sigma^2\end{bmatrix} \in A$, we skip!

Haldane = Jeffreys Prior? Most popular since Laplace doesn't make sense!

$$f_J(\theta, \sigma^2) = \underbrace{f_J(\theta \mid \sigma^2)}_{\underbrace{(1)}_{N(0,\infty)}} \underbrace{f_J(\sigma^2)}_{\underbrace{(\sigma^2)^{-1}}_{\text{InvGamma}(0,0)}}$$

Under Jeffreys prior

$$f(\theta, \sigma^2 \mid \vec{x}) \propto e^{-\frac{1}{2\sigma^2/n}(\theta - \bar{x})^2} (\sigma^2)^{-\frac{n}{2}} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \left((\sigma^2)^{-1}\right)$$

$$= e^{-\frac{1}{2\sigma^2/n}(\theta - \bar{x})^2} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}$$

$$\propto \text{NormalInverseGamma}\left(\bar{x}, n, \frac{n}{2}, \frac{(n-1)s^2}{2}\right)$$

← the only thing we will consider for this inference

What if you care about θ only? σ^2 is a nuisance parameter

$$f(\theta \mid \vec{x}) = \int_0^\infty f(\theta, \sigma^2 \mid \vec{x})\, d\sigma^2$$ Marginalize it out!!!

↑ "marginal posterior"

$$\propto \int_0^\infty e^{-\frac{1}{2\sigma^2/n}(\theta - \bar{x})^2} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}\, d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{n(\theta-\bar{x})^2/2 + (n-1)s^2/2}{\sigma^2}} d\sigma^2$$

$$= \frac{\Gamma(\frac{n}{2})}{\left(\frac{n(\theta-\bar{x})^2 + (n-1)s^2}{2}\right)^{n/2}}$$

$$\propto \left(n(\theta-\bar{x})^2 + (n-1)s^2\right)^{-\frac{n}{2}} \cdot \left(\frac{1}{(n-1)s^2}\right)^{-\frac{n}{2}}$$

$$= \left(1 + \frac{n(\theta-\bar{x})^2}{(n-1)s^2}\right)^{-\frac{n}{2}}$$

$$= \left(1 + \frac{1}{n-1}\frac{(\theta-\bar{x})^2}{\frac{s^2}{n}}\right)^{-\frac{(n-1)+1}{2}}$$

$$\propto T_{n-1}\left(\bar{x}, \frac{s^2}{n}\right)$$ makes sense!

$$CR_{\theta,1-\alpha} = \left[\text{qt.scaled}\left(\frac{\alpha}{2}, \bar{x}, \frac{s^2}{n}\right), \text{qt.scaled}\left(1-\frac{\alpha}{2}, \bar{x}, \frac{s^2}{n}\right)\right]$$

$H_a: \theta > \theta_0 \Rightarrow H_0: \theta \le \theta_0$

$$pval = P(H_0 \mid \vec{x}) = \text{pt.scaled}\left(\theta_0, \bar{x}, \frac{s^2}{n}\right)$$

What if you only care about σ^2? θ is a "nuisance parameter"

$$f(\sigma^2 \mid \vec{x}) = \int_{\mathbb{R}} f(\sigma^2, \theta \mid \vec{x})\, d\theta$$

$$\propto \int_{\mathbb{R}} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{1}{2\sigma^2/n}(\bar{x}-\theta)^2} e^{-\frac{(n-1)s^2}{2\sigma^2}} d\theta$$

$$= (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2}{2\sigma^2}} \int_{\mathbb{R}} \underbrace{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}_{\text{kernel of } N(\theta, \sigma^2)} d\theta$$

$$= (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)S^2/2}{\sigma^2}} \sqrt{2\pi \frac{\sigma^2}{n}} \propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)S^2/2}{\sigma^2}} (\sigma^2)^{\frac{1}{2}}$$

$$\propto (\sigma^2)^{-\frac{n-1}{2}-1} e^{-\frac{(n-1)S^2/2}{\sigma^2}}$$

$$\propto \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)S^2}{2}\right)$$

$$CR_{\sigma^2, 1-\alpha} = \left[\text{qinvgamma}\left(\frac{\alpha}{2}, \frac{n-1}{2}, \frac{(n-1)S^2}{2}\right), \text{qinvgamma}\left(1-\frac{\alpha}{2}, \frac{n-1}{2}, \frac{(n-1)S^2}{2}\right)\right]$$

$$H_a: \sigma^2 > c \Rightarrow H_0: \sigma^2 \leq c$$

$$\text{pval} = P(H_0 \mid \vec{x}) = \text{pinvgamma}\left(c, \frac{n-1}{2}, \frac{(n-1)S^2}{2}\right)$$

Formula Comparisons under Jeffrey's Prior:

$$P(\theta \mid \vec{x}, \sigma^2) = N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

$$P(\theta \mid \vec{x}) = T_{n-1}\left(\bar{x}, \frac{S^2}{n}\right)$$

$$P(\sigma^2 \mid \vec{x}, \theta) = \text{InvGamma}\left(\frac{n}{2}, \frac{n \overbrace{\hat{\sigma}^2_{MLE}}^{\sum(x_i-\theta)^2}}{2}\right)$$

$$P(\sigma^2 \mid \vec{x}) = \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)S^2}{2}\right)$$

Posterior Predictive Distr for $n_*=1$ under Jeffrey's Prior

$$P(x_* \mid \vec{x}) = \int_{\mathbb{R}}\int_{\mathbb{R}} \underbrace{P(x_* \mid \theta, \sigma^2)}_{N(\theta, \sigma^2)} \underbrace{\overbrace{P(\theta, \sigma^2 \mid \vec{x})}^{P(\theta \mid \vec{x}, \sigma^2)\, P(\sigma^2 \mid \vec{x})}}_{N\left(\bar{x}, \frac{\sigma^2}{n}\right) \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)S^2}{2}\right)} \, d\theta \, d\sigma^2$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_*-\theta)^2} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2} \frac{\left(\frac{(n-1)s^2}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} (\sigma^2)^{-\frac{n-1}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \, d\theta \, d\sigma^2 \, \mathbb{1}_{\sigma^2\in(0,\infty)}$$

$$\propto \int_{\sigma^2\in(0,\infty)} (\sigma^2)^{-\frac{1}{2}} (\sigma^2)^{-\frac{1}{2}} (\sigma^2)^{-\frac{n-1}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \underbrace{\int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}\left((x_*-\theta)^2+n(\bar{x}-\theta)^2\right)} \, d\theta}_{} \, d\sigma^2$$

$$\int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}\left(x_*^2-2x_*\theta+\theta^2+n\bar{x}^2-2n\bar{x}\theta+n\theta^2\right)} \, d\theta$$

$$= e^{-\frac{1}{2\sigma^2}(x_*^2+n\bar{x}^2)} \int_{\mathbb{R}} e^{\frac{x_*+n\bar{x}}{\sigma^2}\theta - \frac{1+n}{2\sigma^2}\theta^2} \, d\theta$$

$$= e^{-\frac{1}{2\sigma^2}(x_*^2+n\bar{x}^2)} \sqrt{\frac{\pi}{\frac{1+n}{2\sigma^2}}} \, e^{\frac{\left(\frac{x_*+n\bar{x}}{\sigma^2}\right)^2}{4\frac{1+n}{2\sigma^2}}}$$

$$\int_{\mathbb{R}} e^{a\theta-b\theta^2} \, d\theta = \sqrt{\frac{\pi}{b}} e^{a^2/4b}$$

$$\propto \int_0^\infty (\sigma^2)^{-\frac{1}{2}} \cancel{(\sigma^2)^{-\frac{1}{2}}} (\sigma^2)^{-\frac{n-1}{2}-1} \cancel{(\sigma^2)^{\frac{1}{2}}} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_*^2+n\bar{x}^2)} e^{\frac{x_*^2+2n\bar{x}x_*+n^2\bar{x}^2}{2(1+n)\sigma^2}} \, d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2 + x_*^2/2 + n\bar{x}^2/2 - (x_*^2+2n\bar{x}x_*+n^2\bar{x}^2)/(2(1+n))}{\sigma^2}} \, d\sigma^2$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\left(\frac{(n-1)s^2 + x_*^2 + n\bar{x}^2 - (x_*^2+2n\bar{x}x_*+n^2\bar{x}^2)/(1+n)}{2}\right)^{\frac{n}{2}}}$$

$$\propto \left(\underbrace{\left(1-\frac{1}{n+1}\right)}_{a} x_*^2 - \underbrace{\frac{2n\bar{x}}{n+1}}_{b} x_* + \underbrace{(n-1)s^2+n\bar{x}^2-\frac{n^2\bar{x}^2}{n+1}}_{c}\right)^{-n/2}$$

$$= \left(a x_*^2 - b x_* + c\right)^{-n/2} \cdot \left(\frac{1}{a}\right)^{-n/2} = \left(x_*^2 - \frac{b}{a}x_* + \frac{c}{a}\right)^{-n/2}$$

Note: $\left(x_* - \frac{b}{2a}\right)^2 = x_*^2 - \frac{b}{a}x_* + \frac{b^2}{2a^2}$

$$= \left(\left(x_* - \frac{b}{2a}\right)^2 - \frac{b^2}{2a^2} + \frac{c}{a}\right)^{-n/2} \cdot \left(\frac{1}{\frac{c}{a} - \frac{b^2}{2a^2}}\right)^{-n/2}$$

$$= \left(1 + \frac{\left(x_* - \frac{b}{2a}\right)^2}{\frac{c}{a} - \frac{b^2}{2a^2}}\right)^{-\frac{(n-1)+1}{2}} = \left(1 + \frac{1}{n-1}\,\frac{\left(x_* - \frac{b}{2a}\right)^2}{\frac{\frac{c}{a} - \frac{b^2}{2a^2}}{n-1}}\right)^{-\frac{(n-1)+1}{2}}$$

if n large

$$\propto T_{n-1}\left(\frac{b}{2a}, \frac{\frac{c}{a} - \frac{b^2}{4a^2}}{n-1}\right) = T_{n-1}\left(\bar{x}, \frac{n+1}{n}s^2\right) \approx N(\bar{x}, s^2) \approx N(\theta, \sigma^2)$$

makes sense

$= \frac{n}{n+1}$

$$\frac{b}{a} = \frac{\frac{2n\bar{x}}{n+1}}{1 - \frac{1}{n+1}} = \frac{\cancel{\frac{2n\bar{x}}{n+1}}}{\cancel{\frac{n}{n+1}}} = 2\bar{x} \Rightarrow \frac{b}{2a} = \frac{2\bar{x}}{2} = \bar{x}$$

$$\frac{b^2}{4a^2} = \left(\frac{b}{2a}\right)^2 = \bar{x}^2$$

$$\frac{c}{a} = \frac{(n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}}{\frac{n}{n+1}} = \frac{(n+1)(n-1)}{n}s^2 + (n+1)\bar{x}^2 - n\bar{x}^2 = \frac{(n+1)(n-1)}{n}s^2 + \bar{x}^2$$

$$\frac{\frac{c}{a} - \frac{b^2}{4a^2}}{n-1} = \frac{\frac{(n+1)(n-1)}{n}s^2 + \cancel{\bar{x}^2} - \cancel{\bar{x}^2}}{n-1} = \frac{n+1}{n}s^2$$