MATH341/641 Lec11 Last class we proved $\frac{\hat{\theta}^{MLE} - \theta}{\sqrt{I(\theta)^{-1}/n}} \xrightarrow{d} N(0,1)$

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This result can be used immediately to run one sample hypothesis tests where $H_0: \theta = \theta_0$ as the denominator (std. err.) can be computed. But to generate CI's using the $\hat{\theta}^{MLE}$, one more step is required...

$$\frac{\hat{\theta}^{MLE} - \theta}{\sqrt{\frac{I(\hat{\theta}^{MLE})^{-1}}{n}}} = \underbrace{\frac{\sqrt{\frac{I(\theta)^{-1}}{n}}}{\sqrt{\frac{I(\hat{\theta}^{MLE})^{-1}}{n}}}}_{A} \cdot \underbrace{\frac{\hat{\theta}^{MLE} - \theta}{\sqrt{\frac{I(\theta)^{-1}}{n}}}}_{B} \xrightarrow{d} N(0,1)$$

since $A \xrightarrow{p} 1$ by Slutsky's (A)

$B \xrightarrow{d} N(0,1)$ by MLE thm

let $g(x) = \frac{\sqrt{\frac{I(\theta)^{-1}}{n}}}{\sqrt{\frac{I(x)^{-1}}{n}}}$ $A = g(\hat{\theta}^{MLE}) \xrightarrow{p} g(\theta) = 1$ by CMT

$\overset{\text{inverting the test}}{\Longrightarrow}$ $\hat{CI}_{\theta, 1-\alpha} = \left[\hat{\theta}^{MLE} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{I(\hat{\theta}^{MLE})^{-1}}{n}}\right]$

due to double approx MLE thm + Slutsky's, this is very approx.

When is this useful?

DGP: $X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(\theta) \Rightarrow \hat{\theta}^{mle} = \bar{X} \,\dot\sim\, N\left(\theta, \sqrt{\frac{\theta(1-\theta)}{n}}^2\right)$ by CLT

DGP: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2) \Rightarrow \hat{\theta}^{mle} = \bar{X} \,\dot\sim\, N\left(\theta, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$ by CLT

DGP: $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\theta) \Rightarrow \hat{\theta}^{mle} = \bar{X} \,\dot\sim\, N\left(\theta, \sqrt{\frac{\theta}{n}}^2\right)$ by CLT

$\Rightarrow$ This thm only is useful if $\hat{\theta}^{mle} \neq \bar{X}$.

Let's see an example of this.

$$X_1, \ldots, X_n \overset{iid}{\sim} \text{Gumbel}(\theta, 1) := e^{-\left((x-\theta) + e^{-(x-\theta)}\right)}$$

Let's find the MLE for this DGP:

$$\mathcal{L}(\theta; x_1, \ldots, x_n) = \prod_{i=1}^{n} e^{-\left((x_i-\theta) + e^{-(x_i-\theta)}\right)}$$

$$= e^{-\sum (x_i-\theta) + e^{-(x_i-\theta)}}$$

$$\ell(\theta; x_1, \ldots, x_n) = -\sum (x_i-\theta) + e^{-(x_i-\theta)}$$

$$= -\sum x_i + \sum \theta - \sum e^{-x_i} e^{\theta}$$

$$= -n\bar{x} + n\theta - e^{\theta} \sum e^{-x_i}$$

$$\ell'(\theta; x_1, \ldots, x_n) = n - e^{\theta} \sum e^{-x_i} \overset{set}{=} 0$$

$$\Rightarrow n = e^{\theta} \sum e^{-x_i} \Rightarrow e^{\theta} = \frac{n}{\sum e^{-x_i}} \Rightarrow \hat{\theta}^{mle} = \ln\left(\frac{n}{\sum e^{-x_i}}\right) \neq \bar{X}$$

To use the thm to run tests and construct CI's, we need to compute $I(\theta)$

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$$l''(\theta;x) = -e^{\theta}\sum e^{-x_i}$$

$$I_n(\theta) = E(-l''\theta;x_i) = e^{\theta}E\left[\sum e^{-x_i}\right] = n e^{\theta}E\left[e^{-x_i}\right]$$ How do we do this?

$$E\left[e^{-x_i}\right] = \int_{\mathbb{R}} e^{-x}e^{-((x-\theta)+e^{-(x_i-\theta)})}dx = \int_{\mathbb{R}} e^{-u-\theta}e^{-u+e^{-u}}du = e^{-\theta}\int_{\mathbb{R}} e^{-2u+e^{-u}}du$$

let $u = x-\theta \Rightarrow x = u+\theta \Rightarrow dx = du$

$$= e^{-\theta}\int_{\mathbb{R}} e^{-2u}e^{-e^{-u}}du = e^{-\theta}\int_{\mathbb{R}}(e^{-u})^2 e^{-e^{-u}}du = e^{-\theta}\int_{\infty}^{0} v^2 e^{-v}\left(-\frac{1}{v}\right)dv$$

let $v = e^{-u} \Rightarrow \frac{dv}{du} = -e^{-u} \Rightarrow du = -\frac{1}{e^{-u}}dv$, $u=-\infty \Rightarrow v=\infty$
$= -\frac{1}{v}dv$ $u=\infty \Rightarrow v=0$

$$= e^{-\theta}\int_0^{\infty} \underbrace{v}_{r}\underbrace{e^{-v}dv}_{ds} = e^{-\theta}\left([rs]_0^{\infty} - \int_0^{\infty} s\,dr\right) = e^{-\theta}\left([-ve^{-v}]_0^{\infty} - \int_0^{\infty} -e^{-v}dv\right)$$

$$= e^{-\theta}\left(0 + [-e^{-v}]_0^{\infty}\right) = e^{-\theta}(1-0) = e^{-\theta}$$

$$\Rightarrow I_n(\theta) = n e^{\theta}e^{-\theta} = n \Rightarrow I(\theta) = 1$$

$$\Rightarrow \frac{\hat{\theta}^{mle}-\theta}{\sqrt{\frac{1}{n}}} \xrightarrow{d} N(0,1) \Rightarrow \hat{\theta}^{mle} \sim N\left(\theta, \left(\tfrac{1}{\sqrt{n}}\right)^2\right)$$

$H_0: \theta = \theta_0, \alpha = 5\%$

$$\Rightarrow RET = \left[\theta_0 \pm 2\tfrac{1}{\sqrt{n}}\right]$$

$$\Rightarrow \hat{CI}_{\theta,95\%} = \left[\hat{\theta}^{mle} \pm 2\tfrac{1}{\sqrt{n}}\right]$$

let $x = (2.15, 1.91, 3.66, 4.85, 3.03, 1.03, 3.58)$, $n=7 \Rightarrow \hat{\theta}^{mle} = \ln\left(\frac{7}{e^{-2.15}+\dots+e^{-3.58}}\right) = 2.93$

$H_0: \theta = 2$ $\quad RET = \left[2 \pm 2\tfrac{1}{\sqrt{7}}\right] = [1.24, 2.76] \Rightarrow$ Reject H_0. $\quad \hat{z} = \frac{2.93-2}{\frac{1}{\sqrt{7}}} = 2.46 \notin [-2,2]$

$$\hat{CI}_{\theta,95\%} = \left[2.93 \pm 2\tfrac{1}{\sqrt{7}}\right] = [2.17, 3.68]$$

Let's construct a completely new type of test...

Recall the score function: let $W_i := \ell'(\theta; X_i)$

$$S(\theta; X_1 \ldots X_n) := \ell'(\theta; X_1 \ldots X_n) = \sum_{i=1}^{n} \ell'(\theta; X_i) = \sum_{i=1}^{n} W_i$$

$$\Rightarrow \frac{1}{n} S(\theta; X_1 \ldots X_n) = \bar{W}$$

we know

$E[W_i] = 0$ (see last class)

$Var[W_i] = I(\theta)$ (see last class)

$\Rightarrow E[\bar{W}] = 0$

$Var[\bar{W}] = \frac{I(\theta)}{n}$

$$\Rightarrow \frac{\frac{1}{n} S(\theta; X_1 \ldots X_n)}{\sqrt{\frac{I(\theta)}{n}}} \xrightarrow{d} N(0,1) \text{ by CLT}$$

$$\Rightarrow \frac{S(\theta; X_1 \ldots X_n)}{\sqrt{n I(\theta)}} = \frac{S(\theta; X_1 \ldots X_n)}{\sqrt{I_n(\theta)}} \xrightarrow{d} N(0,1) \text{ by CLT}$$

Who cares? Well if you wish to test $H_0: \theta = \theta_0$, then under the null hypothesis,

$$\hat{Z} | H_0 := \frac{S(\theta_0; X_1 \ldots X_n)}{\sqrt{I_n(\theta)}} \sim N(0,1) \underset{\text{for } \alpha = 5\%}{\Rightarrow} RET = [-2, 2]$$

Now, we just calculate $\hat{\hat{Z}}$ and see if it lands outside of RET

"Score Test" or "LaGrange Multiplier Test" (Rao, 1949).