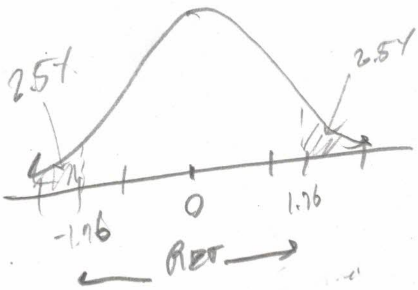
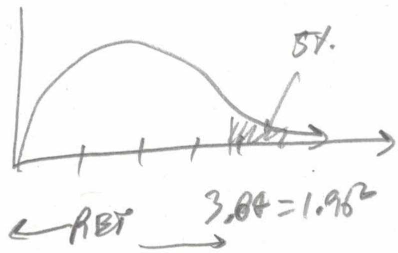
Lec 19 MATH 341/641

Facts from 340..

$Z \sim N(0,1)$

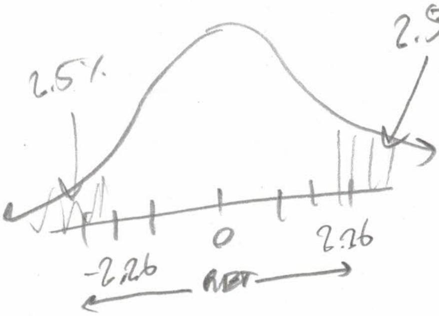


$Y = Z^2 \sim \chi^2_1$

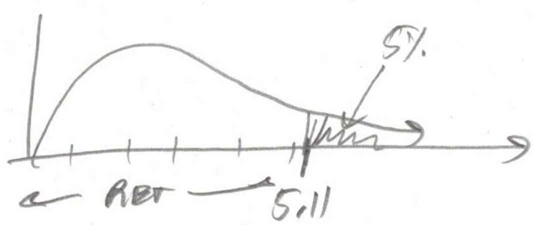


$\Rightarrow$ every 2-sided Z test is also a χ^2 test

$T \sim T_d$



$Y = T^2 \sim F_{1,d}$



$\Rightarrow$ every 2-sided T-test is an F-test

2 populations DGP: $X_{1,1},\ldots,X_{1,n_1} \overset{iid}{\sim} N(\theta_1,\sigma_1^2)$ indep of

$X_{2,1},\ldots X_{2,n_2} \overset{iid}{\sim} N(\theta_2,\sigma_2^2)$ (no params known)

What if I wanted to prove $H_a: \sigma_1^2 \neq \sigma_2^2 \Rightarrow H_0: \sigma_1^2=\sigma_2^2=\sigma^2$

Test statistic to gauge departure from H_0:

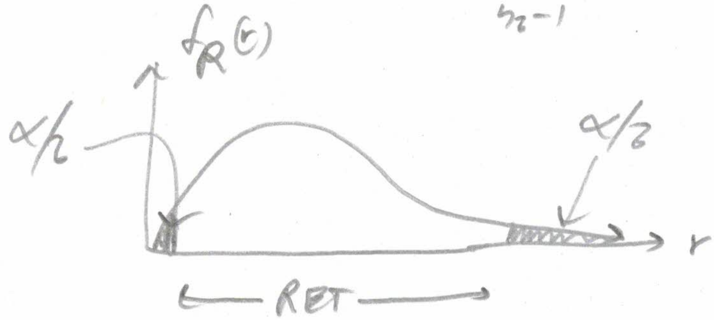
$$R = \frac{S_1^2}{S_2^2}$$ if $R<1$ by "enough" or $R>1$ by "enough" $\Rightarrow$ Reject H_0

What is distr of R?

"F-test for homogeneity of variance"

$$R = \frac{S_1^2}{S_2^2} \cdot \frac{\frac{n_1-1}{n_1-1}}{\frac{n_2-1}{n_2-1}} \frac{\sigma^2}{\sigma^2} = \frac{\dfrac{\frac{n_1-1}{\sigma^2}S_1^2}{n_1-1}}{\dfrac{\frac{n_2-1}{\sigma^2}S_2^2}{n_2-1}} \sim F_{n_1-1,\,n_2-1}$$

$\frac{n_1-1}{\sigma^2}S_1^2 \sim \chi^2_{n_1-1}$, $\frac{n_2-1}{\sigma^2}S_2^2 \sim \chi^2_{n_2-1}$



You can also test $H_a: \sigma_1^2 > \sigma_2^2$ and $H_a: \sigma_1^2 < \sigma_2^2$ by doing just the appropriate one-sided test.

Do test on male/female heights drawn from class let $\theta_j := P(\text{die rolls face } j)$, $j \in \{1, \dots 6\}$ B

Let's say we roll a 6-sided die. We wish to test if

H_a: die is unfair i.e. $\exists j$ s.t. $\vec{\theta}_j \neq \frac{1}{6}\vec{1}_6$

H_0: die is fair i.e. $\vec{\theta} = \frac{1}{6}\vec{1}_6$

Consider data for $n=15$ rolls $\vec{x} = \langle 4,3,6,1,6,5,1,1,3,1,3,2,5,6,6 \rangle$

To run the test, we need a test statistic that measures departure from H_0. Consider the frequency table:

Category	1	2	3	4	5	6	~~tot~~
#obs $O_j := X_j$	4	1	3	2	1	4	15
#exp $E_j := n\theta_{0j}$	2.5	2.5	2.5	2.5	2.5	2.5	15

What is a good test statistic?

Maybe $\hat{\phi} = \sum_{j=1}^{6} \frac{O_j}{E_j}$? The problem is they can balance each other out

$\hat{\phi} = \sum_{j=1}^{6} |O_j - E_j|$ or $\hat{\phi} = \sum_{j=1}^{6} (O_j - E_j)^2$ These work but we don't know the distr.

How about?

$\hat{\phi} = \sum \frac{(O_j - E_j)^2}{E_j} \xrightarrow{d} \chi^2_5 \Rightarrow \hat{\phi} \mathrel{\dot\sim} \chi^2_5$ and we have an asymptotically valid test

In general if Obs: $\vec{X} \sim \text{Multinom}(n, \vec{\theta})$ where $\dim(\vec{X}) = K$

then $\hat{\phi} = \sum_{j=1}^{K} \frac{(O_j - E_j)^2}{E_j} \xrightarrow{d} \chi^2_{K-1}$

"Chi-Sq Goodness of Fit Test" (Karl Pearson, 1900).

I hope to prove this result in 340/640. We need a multinomial CLT result.

$$\hat{\phi} = \frac{(4-2.5)^2}{2.5} + \frac{(1-2.5)^2}{2.5} + \frac{(3-2.5)^2}{2.5} + \frac{(2-2.5)^2}{2.5} + \frac{(1-2.5)^2}{2.5} + \frac{(4-2.5)^2}{2.5} = 3.8 \in RET$$

$\Rightarrow$ Retain H_0

No evidence die is unfair.

$\exists$ Possible Type II error due to not having enough power

$f_{\hat{\phi}}(\hat{\phi}) = \chi^2_5$

$F_{\hat{\phi}}(11.07) = 95\%$

Why are there 5 "degrees of freedom" in this null distr?

Different setting. Observe $n=279$ ~~men~~ and record hair and eye color.
Here is the raw data as a "cross-tabulation" or "contingency table".

DGP: ~~HW~~

Hair color	Brown	Blue	Hazel	Green	
Black	$32 = O_{1,1}$	$11 = O_{1,2}$	10	3	$= O_{1\cdot} = 56$
Brown	$53 = O_{2,1}$	50	25	15	$O_{2\cdot} = 143$
Red	10	10	7	7	$= O_{3\cdot} = 34$
Blond	3	30	5	8	$= O_{4\cdot} = 46$
	$= O_{\cdot 1} = 98$	$O_{\cdot 2} = 101$	$= O_{\cdot 3} = 47$	$O_{\cdot 4} = 33$	279

\# rows $r = 4$

\# cols $c = 4$

Maybe we are interested in testing:

H_a: Hair Color and Eye Color are dependent

H_0: Hair Color and Eye Color are independent

How do we write these hypotheses mathematically?

Let $\theta_{ij} := P$(each hair/eye combination)

Let $\theta_{i\cdot} := P$ (each hair color)

Let $\theta_{\cdot j} := P$(each eye color)

$H_1: \exists i j \;\; \theta_{ij} \neq \theta_{i\cdot}\theta_{\cdot j}$

Mult. Rule for indep events

$H_0: \forall i j \;\; \theta_{ij} = \theta_{i\cdot}\theta_{\cdot j} \Rightarrow P(\text{hair color } i, \text{ eye color } j) = P(\text{hair color } i)\, P(\text{eye color } j)$

We need a statistic that measures departure from H_0

Let's try the one from before:

$$\hat{\phi} = \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

What is the value of E_{jk}? This is the expected count under the null.

$$E_{jk} := n\theta_{jk} \overset{\text{under } H_0}{=} n\,\theta_{i\cdot}\,\theta_{\cdot j}$$ but we can't calculate it! We don't know the true $\theta_{i\cdot}$'s nor $\theta_{\cdot j}$'s

What do we do? Use $\hat{E}_{jk} = n\,\hat{\theta}_{i\cdot}\,\hat{\theta}_{\cdot j} = n \frac{O_{i\cdot}}{n}\frac{O_{\cdot j}}{n} = \frac{O_{i\cdot}O_{\cdot j}}{n}$

Then we use

$$\hat{\phi} = \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\left(O_{ij} - \frac{O_{i\cdot}O_{\cdot j}}{n}\right)^2}{\frac{O_{i\cdot}O_{\cdot j}}{n}}, \quad \hat{\phi} = \frac{\left(32 - \frac{56\cdot 98}{279}\right)^2}{\frac{56\cdot 98}{279}} + \frac{\left(53 - \frac{143\cdot 98}{279}\right)^2}{\frac{143\cdot 98}{279}} + \ldots = 41.28$$

as the test statistic

How is $\hat{\phi}$ distr? In general...

$$\hat{\phi} = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(O_{ij} - \hat{E}_{jk})^2}{\hat{E}_{jk}} \xrightarrow{d} \chi^2_{(r-1)(c-1)}$$

So in our case $\hat{\phi} \sim \chi^2_{(4-1)(4-1)} = \chi^2_9$

Why only 9 degrees of freedom? Previously, there would be $16-1=15$

It's because we don't know the $\theta_{i\cdot}$'s nor $\theta_{\cdot j}$'s. We estimate them from the data. For the first row, there are only 3 θ_{ij}'s since the last is forced to be a value given $\hat{\theta}_{1\cdot}$

lets run the test. at $\alpha = 5\%$.

$F_{\chi^2_9}(16.99) = 95\% \Rightarrow RET = [0, 16.99]$

Since $\hat{\phi} \notin RET \Rightarrow$ Reject H_0. There is compelling evidence hair color and eye color are associated/dependent.

This test can be thought of differently

Let $\vec{X}_{1\cdot} \sim \text{Multinom}(n_{1\cdot}, \vec{\theta}_{1\cdot})$ sampled from all pop of Black-haired people

Obs: $\vec{X}_{2\cdot} \sim \text{Multinom}(n_{2\cdot}, \vec{\theta}_{2\cdot})$ " " " " " " " " Brown-haired " " "

$\vec{X}_{3\cdot} \sim \text{Multinom}(n_{3\cdot}, \vec{\theta}_{3\cdot})$ " " " " " Red-haired " " "

$\vec{X}_{4\cdot} \sim \text{Multinom}(n_{4\cdot}, \vec{\theta}_{4\cdot})$ all independent of each other Blond-haired " " "

$\vec{\theta}_{1\cdot} = \begin{bmatrix} \theta_{11} \\ \theta_{12} \\ \theta_{13} \\ \theta_{14} \end{bmatrix} \rightarrow$ prob of brown eye in black-haired pop.. etc..

$H_0: \vec{\theta}_{1\cdot} = \vec{\theta}_{2\cdot} = \vec{\theta}_{3\cdot} = \vec{\theta}_{4\cdot} \Rightarrow \theta_{i,j} = \theta_{k,j} \;\; \forall j \; \forall i \neq k$

H_a: At least one $\vec{\theta}_{i\cdot} \neq \vec{\theta}_{k\cdot}$ for $i \neq k$

We need a test statistic to measure departure from H_0. Let's use same as before!

$$\hat{\phi} = \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

What is E_{ij} here?

\# sampled in pop i

under H_0, all θ_j are the same

$E_{ij} := n\,\theta_{ij} = n_{i\cdot}\,\theta_j$

But we don't know θ_j so we used $\hat{\theta}_{\cdot j} = \bar{\theta}$ stand for the column

$$\hat{\theta}_{\cdot j} = \frac{O_{\cdot j}}{n} \Rightarrow \hat{E}_{ij} = n_{i\cdot} \frac{O_{\cdot j}}{n} = \frac{O_{i\cdot} O_{\cdot j}}{n}$$

which is the same as χ^2 test of independence!

Since the test statistic is the same, the test decision is the same. But it came from a different setting ⇒ conclusion is different. Here, we reject H_0 and conclude the distr of eye color is different across the populations of Brown, Black, Red and Blonde haired men.