MATH 341/641 Fall 2024 Homework #2

Professor Adam Kapelner

Due by email 11:59PM September 23, 2024

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Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out." Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, review MATH 340 concepts: random variables, PMF's, PDF's, CDF's, binomial, and review MATH 241 concepts: the normal distribution.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems. "[MA]" are for those registered for 621 and extra credit otherwise.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to 5 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. The easiest way to use overleaf is to copy the raw text from hwxx.tex and preamble.tex into two new overleaf tex files with the same name. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks which are *not* on this printout. Keep this first page printed for your records.

NAME: ____

Problem 1

Here we will do a binomial exact test using the survey data from our class. We want to demonstrate that the iphone users in our class is *greater* than the national average (which is 52.4%). Recall that our data was as follows: for $n = 28$, the $\hat{\hat{\theta}} = 0.893$ where the estimator we chose was the sample proportion (or equivalently, the sample average).

(a) [easy] Write down H_a then H_0.

(b) [easy] Declare your α level desired for this test. You do not need to justify it. It is what you are comfortable with.

(c) [harder] Because we want to show something is greater than a point value, it is called a right-tailed test. In any test, we need to find the distribution (or approximate the distribution of) the estimator under the null hypothesis. Because we will reject on the right, why is the most conservative value of θ to choose when deriving the null sampling distribution to be largest value in the null hypothesis region (in this case $\theta = \theta_0 = 0.524$)?

(d) [easy] Regardless of if you understood the previous question or not, what is the exact null sampling distribution for $\hat{T} := n\hat{\theta}$? Your answer should be a PMF.

(e) [easy] Draw the PMF of the null sampling distribution. Label all axes carefully and provide sufficient tick marks. Marked easy because you can copy from class. Probabilities below are rounded to the nearest 3 digits. Other values in the support have probability < 0.001. Leave space below your plot under the horizontal axis as we will use the space for later assignment problems.

6	7	8	9	10	11	12	13	14	15	16	17	18
0.001	0.002	0.006	0.015	0.032	0.058	0.091	0.123	0.145	0.149	0.133	0.103	0.070

19	20	21	22	23
0.040	0.020	0.008	0.003	0.001

(f) [harder] What are all the ten smallest possible sizes of this test?

(g) [easy] Given your choice of α, which size will you use for the test and why?

(h) [easy] What is $\mathbb{P}$ (Type I error) in this test?

(i) [easy] Indicate the RET and the rejection region in the above illustration. People may have different answers based on α, which was your choice

(j) [harder] Were you able to create a rejection region at your exact level of α? Yes / no and why?

(k) [easy] What is $\mathbb{P}\left(\hat{\theta} \notin RET\right)$ in this test?

(l) [easy] Declare δ, your margin of equivalence for this test and explain why you chose it.

(m) [easy] Run the test using the evidence $\hat{\hat{t}} := n\hat{\hat{\theta}}$. Write your conclusion in English. Comment in your conclusion on the estimate's statistical significance and the estimate's practical significance.

(n) [in the notes] What is the definition of the pval for this test?

(o) [harder] Calculuate the p-value of this test to the nearest three digits (use the PMF table on the previous page for the terms in the sum).

(p) [easy] Assuming $\theta = 0.7$, what is the true sampling distribution? Write its PMF below.

(q) [difficult] Assuming $\theta = 0.7$, calculate the $\mathbb{P}$ (Type II error) of this test.

(r) [easy] Easy only if you have the answer to the previous question: calculate the power in this test.

Problem 2

Here we will review theory testing from a conceptual point of view. For each question, state whether the theory under consideration should become a null hypothesis or alternative hypothesis. If null, also write the alternative; if alternative also write the null.

(a) [easy] A new grand unified theory of physics.

(b) [easy] The latest conspiracy theory about the president.

(c) [easy] You are a shareholder in a pharmaceutical company. Your new drug cures cancer.

Problem 3

We will revisit the concept of a degenerate point estimator as in the first problem but this time let the DGP be $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\theta, \sigma^2)$ where σ^2 is considered known. We are focused on point estimation for θ and the range of possible values is all numbers i.e. $\Theta = \mathbb{R}$. Of course we use $\hat{\theta} = \bar{X}$. But we also consider the "bad" point estimator $\hat{\theta}_{\text{bad}} = 1$.

(a) [harder] Graph the bias of $\hat{\theta}_{\text{bad}}$ over all θ. Label your axes.

(b) [harder] Graph the risk of $\hat{\theta}$ and $\hat{\theta}_{\text{bad}}$ under squared error loss. Label your axes and provide appropriate tick marks on the axes.

(c) [difficult] Compare $\hat{\theta}_{\text{bad}}$ to $\hat{\theta}$ using the sup risk under squared error loss. How much better is $\bar{X}$?

Problem 4

We will now do a *one-sided one-sample exact Z test* using the following sample of heights: $n = 13$ and $\bar{x} = 68.85''$. We want to test at $\alpha = 1\%$ if the population that this sample was drawn from has a *greater* mean than the American female height mean of 65". According to this article, female American height is $\stackrel{iid}{\sim} \mathcal{N}(65, 3.5^2)$.

(a) [easy] Write the alternative and null hypotheses.

(b) [easy] As we discussed in class, which value of θ_0 do we use to generate the null sampling distribution?

(c) [difficult] Why do we use that value of θ_0 when there are potentially infinite values to choose from?

(d) [easy] Write the null sampling distribution on both the original scale (inches) and the standardized scale.

(e) [easy] Write the RET region as a set on both the original scale (inches) and the standardized scale.

(f) [easy] What is the $\mathbb{P}$ (Type I error) in this test?

(g) [easy] Is it possible to calculate the $\mathbb{P}$ (Type II error) of this test given the information you have? Yes / no

(h) [easy] Is it possible to calculate the power of this test given the information you have? Yes / no

(i) [easy] What is the test statistic on the original scale (in inches)?

(j) [easy] Calculate the test statistic on the standardized scale (unitless).

(k) [easy] Run this test using both RET regions (the original scale and standardized scale). Show both answers are the same.

(l) [easy] Calculate the p-val for this test. Use a z-calculator on your graphing calculator or find one on the Internet.

(m) [easy] Was this an exact test? Yes / no

(n) [easy] Write a conclusion of this test in English.

(o) [easy] Was this conclusion expected? Did you expect such a low/high p-val? Discuss.

(p) [harder] Find the power function for this test.

(q) [easy] Regardless of the other inputs to the power function you found in the previous question, if $n \to \infty$, what does power converge to?

Problem 5

Assume the DGP for male height is $\overset{iid}{\sim}$ normal with standard deviation 4". and the DGP for female height is $\overset{iid}{\sim}$ normal with standard deviation 3.5". Using the male and female height data from class, we will now prove that the QC STEM 300-level population mean male height is greater than the QC STEM 300-level population mean female height.

(a) [harder] Write the hypotheses, declare an α of your choosing, find the sampling distribution under H_0, find the RET region, run the test, calculate the p-val and provide a conclusion sentence.

(b) [easy] Was this an exact test? Yes / no

Problem 6

These questions will be about the Method of Moments (MM) procedure for generating estimators/estimates directly from the assumption of the DGP.

(a) [easy] From MATH 340: define the kth moment of a rv.

(b) [easy] For sample size n, define the of the kth sample moment *estimator* of a rv.

(c) [easy] For sample size n, define the of the kth sample moment *estimate* of a rv.

(d) [easy] Give an example of a DGP with two parameters.

(e) [harder] For a DGP with $K = 3$ parameters, write the system of equations that relates each moment to parameters. There should be 3 equations with 3 unknowns. Then write the system of equations that relates each parameter to moments. There should be another 3 equations with 3 unknowns.

(f) [in the notes] For any iid DGP with finite mean, find an MM estimator for the mean.

(g) [in the notes] For any iid DGP with finite variance, find an MM estimator for the variance.

(h) [in the notes] Consider the $X_1, \ldots, X_n \overset{iid}{\sim} \text{Binomial}\,(\theta_1,\, \theta_2)$ DGP and derive the MM estimators $\hat{\theta}_1^{\text{MM}}$ and $\hat{\theta}_2^{\text{MM}}$ for θ_1 and θ_2 and express them in terms of $\bar{X}$ and $\hat{\sigma}^2$.

(i) [easy] Provide an example dataset (different from the one in class) where the MM estimates $\hat{\hat{\theta}}_1^{\text{MM}}$ and $\hat{\hat{\theta}}_2^{\text{MM}}$ in (h) are illegal and explain why they're illegal.

(j) [harder] Imagine you are a NYPD officer at precinct 100 in Queens. You want to estimates of the number of crimes in your precinct and people's propensity to phone in crimes. The number of daily phone reports for two weeks are: 13, 21, 25, 21, 15, 19, 15, 15, 17, 23, 16, 15, 19, 15. Estimate the true mean number of daily total crimes in the precinct and probability of the crime being phoned in.

(k) [difficult] What exactly what you need to know if you wanted to test if the true mean number of crimes daily exceeds 20? This is conceptual and should be answered with a sentence or two.

(l) [harder] Derive the MM estimator $\hat{\theta}^{\text{MM}}$ for θ in the $X_1, \ldots, X_n \stackrel{iid}{\sim} \text{U}(\theta,\, 17)$ DGP.

(m) [easy] Provide an example dataset where the MM estimate $\hat{\hat{\theta}}^{\text{MM}}$ in the previous question is illegal and explain why it is illegal.

(n) [difficult] [MA] Derive the MM estimators $\hat{\theta}_1^{\text{MM}}$ and $\hat{\theta}_2^{\text{MM}}$ for the $\overset{iid}{\sim} \text{U}(\theta_1, \theta_2)$ DGP. Then estimate θ_1 and θ_2 given the dataset 9.8, 3.1, 1.2, -0.1, 12.1, 15.9, -9, 3.4, 14.9, -3.6, 16.5, -9.6, 11.2, 11.6, -3.9, -9.3, -1 using these new MM estimators. Remember that θ_1 is defined to be $< \theta_2$!