

Math 341 / 641 Fall 2025 Final Examination **Solutions**

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Full Name _____

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Instructions

This exam is 120 minutes (variable time per question) and closed-book. You are allowed **three** pages (front and back) of a “cheat sheet”, blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 Consider the iid Poisson DGP where instead of being parameterized by the usual $\lambda > 0$, we parameterize it by $\theta = g(\lambda) = \ln(\lambda)$ so that $\lambda = e^\theta$ hence,

$$X_1, \dots, X_n \stackrel{iid}{\sim} \text{Poisson}(\theta) := \frac{e^{-e^\theta} (e^\theta)^x}{x!} \mathbb{1}_{x \in \mathbb{N}_0}.$$

(a) [2 pt / 2 pts] What is Θ ?

$$\lambda \in (0, \infty) \Rightarrow \ln(\lambda) = \theta \in \mathbb{R} = \Theta$$

(b) [4 pt / 6 pts] Show the following and justify each step:

$$\mathcal{L}(\theta; \mathbf{X}) = e^{-ne^\theta} e^{n\bar{X}\theta} \prod_{i=1}^n \frac{\mathbb{1}_{X_i \in \mathbb{N}_0}}{X_i!}$$

$$\mathcal{L}(\theta; \mathbf{X}) = \prod_{i=1}^n \frac{e^{-e^\theta} (e^\theta)^{X_i}}{X_i!} \mathbb{1}_{X_i \in \mathbb{N}_0} = e^{-ne^\theta} (e^\theta)^{\sum_{i=1}^n X_i} \prod_{i=1}^n \frac{\mathbb{1}_{X_i \in \mathbb{N}_0}}{X_i!} = e^{-ne^\theta} e^{n\bar{X}\theta} \prod_{i=1}^n \frac{\mathbb{1}_{X_i \in \mathbb{N}_0}}{X_i!}$$

(c) [3 pt / 9 pts] Find $\ell(\theta; \mathbf{X})$.

$$\ell(\theta; \mathbf{X}) = -ne^\theta + n\bar{X}\theta + \sum_{i=1}^n \ln \left(\frac{\mathbb{1}_{X_i \in \mathbb{N}_0}}{X_i!} \right)$$

(d) [2 pt / 11 pts] Find $\ell'(\theta; \mathbf{X})$.

$$\ell'(\theta; \mathbf{X}) = -ne^\theta + n\bar{X}$$

(e) [4 pt / 15 pts] Find $\hat{\theta}^{\text{MLE}}$. Simplify.

$$\ell'(\theta; \mathbf{X}) \stackrel{\text{set}}{=} 0 \quad \Rightarrow \quad -ne^\theta + n\bar{X} = 0 \quad \Rightarrow \quad e^\theta = \bar{X} \quad \Rightarrow \quad \hat{\theta}^{\text{MLE}} = \ln(\bar{X})$$

(f) [2 pt / 17 pts] Find $I_n(\theta)$.

$$I_n(\theta) = \mathbb{E}[-\ell''(\theta; \mathbf{X})] = \mathbb{E}[-(-ne^\theta)] = ne^\theta$$

(g) [2 pt / 19 pts] Find $I(\theta)$.

$$I_n(\theta) = nI(\theta) \quad \Rightarrow \quad ne^\theta = nI(\theta) \quad \Rightarrow \quad I(\theta) = e^\theta$$

(h) [3 pt / 22 pts] Given your answer in (a), what is the Laplace prior on θ ?

Since the parameter space is all \mathbb{R} , $f(\theta) \propto 1$.

(i) [1 pt / 23 pts] Is Laplace's prior proper? Yes / no.

No

- (j) [3 pt / 26 pts] Show that the kernel of Jeffrey's prior on θ is $e^{\theta/2}$.

$$f(\theta) \propto \sqrt{I(\theta)} = \sqrt{e^\theta} = (e^\theta)^{\frac{1}{2}} = e^{\theta/2}$$

- (k) [3 pt / 29 pts] Is Jeffrey's prior proper? Yes / no and justify your answer.

No, because $\int_{\Theta} e^{\theta/2} d\theta = \infty$ and thus Humpty Dumpty cannot be satisfied.

For the rest of this question, you will need to know about the following new rv:

$$Y \sim \text{LogGamma}(a, b) := \frac{b^a}{\Gamma(a)} e^{ay - be^y}, \quad a, b > 0, \quad \mathbb{E}[Y] = \psi(a) - \ln(b), \quad \text{Mode}[Y] = \ln\left(\frac{a}{b}\right)$$

where ψ is called the "digamma" function.

- (l) [6 pt / 35 pts] Using Laplace's prior, derive the posterior. If you did not figure out Laplace's prior, let $f(\theta) \propto 1$ going forward. Show the posterior is a LogGamma distribution and find the posterior parameters.

$$\begin{aligned} f(\theta \mid \mathbf{x}) &= \frac{f(\mathbf{x} \mid \theta)f(\theta)}{f(\mathbf{x})} \propto f(\mathbf{x} \mid \theta)f(\theta) \propto \mathcal{L}(\theta; \mathbf{x}) = \frac{e^{-ne^\theta} (e^\theta)^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} \prod_{i=1}^n \mathbb{1}_{x_i \in \mathbb{N}_0} \\ &\propto e^{-ne^\theta} (e^\theta)^{\sum_{i=1}^n x_i} \\ &= e^{n\bar{x}\theta - ne^\theta} \\ &\propto \text{LogGamma}(n\bar{x}, n) \end{aligned}$$

(m) [3 pt / 38 pts] Is this posterior always proper? Yes / no and justify your answer.

No. If $\bar{x} = 0$ (which is possible for a Poisson rv), then $b = 0$ and the posterior is improper.

(n) [2 pt / 40 pts] Write Jeffrey's prior as a LogGamma (whether proper or improper).

$$e^{\theta/2} \propto \text{LogGamma}(1/2, 0)$$

(o) [6 pt / 46 pts] Now let $f(\theta) = \text{LogGamma}(a, b)$, the general rv where $a, b > 0$, and rederive the posterior.

Note $f(\theta) \propto e^{a\theta - be^\theta}$. We can now piggyback from part (l) and start here:

$$\begin{aligned} f(\theta \mid \mathbf{x}) &\propto e^{n\bar{x}\theta - ne^\theta} f(\theta) \\ &\propto e^{n\bar{x}\theta - ne^\theta} \left(e^{a\theta - be^\theta} \right) \\ &= e^{(n\bar{x}+a)\theta - (n+b)e^\theta} \\ &\propto \text{LogGamma}(n\bar{x} + a, n + b) \end{aligned}$$

(p) [3 pt / 49 pts] Is the LogGamma the conjugate prior for the iid Poisson with our parameterization of $\theta = \ln(\lambda)$? Yes / no and justify your answer.

Yes. The prior is LogGamma and the posterior is LogGamma hence we satisfied the definition of conjugacy.

- (q) [3 pt / 52 pts] Interpret the likely significance of the hyperparameter b in the context of the pseudoobservations.

Given that it adds to n , we can surmise that $b = n_0$, the number of pseudoobservations.

- (r) [4 pt / 56 pts] Interpret the likely significance of the hyperparameter a in the context of the pseudoobservations.

Given that it adds to $n\bar{x} = \sum_{i=1}^n x_i$, we can surmise that $a = n_0\mu_0$, where μ_0 is the average of the n_0 pseudoobservations. Or alternatively, a is the sum of the n_0 pseudoobservations which in the context of the Poisson DGP is the number of pseudosuccesses.

- (s) [1 pt / 57 pts] Are the interpretations of a, b the same as the interpretation of α, β in the Gamma(α, β) prior when we parameterize the Poisson with λ , the canonical parameterization we used in class? Yes / no.

Yes.

- (t) [3 pt / 60 pts] Is Haldane's prior the same as Laplace's prior in this model? Yes / no and justify your answer.

Yes. Haldane's prior is setting $n_0 = 0$. When doing so, we get the same posterior as Laplace's prior, therefore the prior must be identical.

- (u) [4 pt / 64 pts] Consider the setting with a sample size of 17 observations. Provide an example of an informative conjugate prior.

All we need to do is let $n_0 \gg 17$ and μ_0 doesn't matter. So the following qualifies: $\text{LogGamma}(\cdot, 1000)$.

For the remainder of this problem, let

$$f(\theta \mid \mathbf{x}) = \text{LogGamma}(n\bar{x} + n_0\mu_0, n + n_0)$$

where we have hyperparameters $n_0 \geq 0, \mu_0 \geq 0$.

- (v) [2 pt / 66 pts] Find $\hat{\theta}^{\text{MMSE}}$ explicitly as a function of n, \bar{x}, n_0, μ_0 or write a mathematical expression that will compute it.

$$\hat{\theta}^{\text{MMSE}} := \mathbb{E}[\theta \mid \mathbf{x}] = \psi(n\bar{x} + n_0\mu_0) - \ln(n + n_0)$$

- (w) [4 pt / 70 pts] Find $\hat{\theta}^{\text{MMAE}}$ explicitly as a function of n, \bar{x}, n_0, μ_0 or write a mathematical expression that will compute it.

$$\hat{\theta}^{\text{MMAE}} := \text{Med}[\theta \mid \mathbf{x}] = \left\{ \theta : \int_{-\infty}^{\theta} \frac{(n + n_0)^{n\bar{x} + n_0\mu_0}}{\Gamma(n\bar{x} + n_0\mu_0)} e^{(n\bar{x} + n_0\mu_0)u - (n + n_0)e^u} du = 0.5 \right\}$$

- (x) [2 pt / 72 pts] Find $\hat{\theta}^{\text{MAP}}$ explicitly as a function of n, \bar{x}, n_0, μ_0 .

$$\hat{\theta}^{\text{MAP}} := \text{Mode}[\theta \mid \mathbf{x}] = \ln \left(\frac{n\bar{x} + n_0\mu_0}{n + n_0} \right)$$

For the remainder of this problem, let $\text{qlgamma}(q, a, b)$ be the function that returns the q th quantile of the rv $\text{LogGamma}(a, b)$ and $\text{plgamma}(x, a, b)$ be the CDF $F(x)$ of the rv $\text{LogGamma}(a, b)$.

- (y) [5 pt / 77 pts] Write an expression that will return a 95% credible region for θ after seeing the data \mathbf{x} .

$$CR_{\theta,95\%} = [\text{qlgamma}(.025, n\bar{x} + n_0\mu_0, n + n_0), \text{qlgamma}(.975, n\bar{x} + n_0\mu_0, n + n_0)]$$

- (z) [7 pt / 84 pts] Note: we are switching to Frequentist inference for this one question. Find an approximate 95% confidence interval for θ after seeing the data \mathbf{x} . You will need to use the fact that if $Y \sim \text{Poisson}(\lambda)$ then $\text{Var}[Y] = \lambda$.

For this we need the delta method. In a previous question we found that $\hat{\theta}^{\text{MLE}} = \ln(\bar{x})$. We have $\theta = g(\lambda) = \ln(\lambda)$ and thus $g'(\lambda) = \lambda^{-1}$. You either have proved or can prove quickly that $\hat{\lambda}^{\text{MLE}} = \bar{x}$. Hence $\text{SD}[\hat{\lambda}^{\text{MLE}}] = \frac{\sigma^2}{n} = \frac{\lambda}{n}$ and thus $\hat{\text{SD}}[\hat{\lambda}^{\text{MLE}}] = \frac{\bar{x}}{n}$. We now have everything we need for the formula:

$$CI_{\theta,95\%} \approx \left[\hat{\theta}^{\text{MLE}} \pm 1.96 g'(\hat{\lambda}^{\text{MLE}}) \hat{\text{SD}}[\hat{\lambda}^{\text{MLE}}] \right] = \left[\ln(\bar{x}) \pm 1.96 \left(\frac{1}{\bar{x}} \right) \left(\frac{\bar{x}}{n} \right) \right] = \left[\ln(\bar{x}) \pm \frac{1.96}{n} \right]$$

- (aa) [1 pt / 85 pts] If n is large, do you expect $CR_{\theta,95\%} \approx CI_{\theta,95\%}$? Yes / no.

Yes.

- (ab) [5 pt / 90 pts] Write an expression that will return a the Bayesian p-value for the test $H_0 : \theta = \theta_0$ with a margin of equivalence δ after seeing the data \mathbf{x} .

$$p_{\text{val}} = \text{plgamma}(\theta_0 + \delta, n\bar{x} + n_0\mu_0, n + n_0) - \text{plgamma}(\theta_0 - \delta, n\bar{x} + n_0\mu_0, n + n_0)$$

- (ac) [10 pt / 100 pts] Derive the distribution of X_* , the next future observation, given the data \mathbf{x} . Don't forget the $\mathbb{1}_{x_* \in S_{X_*}}$ term. If the answer is a brand name rv, indicate it as so and find its parameters.

$$\begin{aligned}
p_{X_*|\mathbf{x}}(X_* | \mathbf{x}) &= \int_{\mathbb{R}} p_{X_*|\theta}(X_*|\theta) f_{\theta|\mathbf{x}}(\theta | \mathbf{x}) d\theta \\
&= \int_{\mathbb{R}} \left(\frac{e^{-e^\theta} (e^\theta)^{x_*}}{x_*!} \mathbb{1}_{x_* \in \mathbb{N}_0} \right) \left(\frac{(n + n_0)^{n\bar{x} + n_0\mu_0}}{\Gamma(n\bar{x} + n_0\mu_0)} e^{(n\bar{x} + n_0\mu_0)\theta - (n + n_0)e^\theta} \right) d\theta \\
&\propto \frac{\mathbb{1}_{x_* \in \mathbb{N}_0}}{x_*!} \int_{\mathbb{R}} e^{(n\bar{x} + n_0\mu_0 + x_*)\theta - (n + n_0 + 1)e^\theta} d\theta \\
&= \frac{\Gamma(n\bar{x} + n_0\mu_0 + x_*)}{(n + n_0 + 1)^{n\bar{x} + n_0\mu_0 + x_*} x_*!} \mathbb{1}_{x_* \in \mathbb{N}_0} \\
&\propto \frac{\Gamma(n\bar{x} + n_0\mu_0 + x_*)}{x_*!} \left(\frac{1}{n + n_0 + 1} \right)^{x_*} \mathbb{1}_{x_* \in \mathbb{N}_0} \\
&\propto \text{ExtNegBin} \left(n\bar{x} + n_0\mu_0, \frac{n + n_0}{n + n_0 + 1} \right)
\end{aligned}$$

Note: this is the same posterior predictive distribution we derived in class using the hyperparameters specified by pseudodata, i.e., $f(\theta) = \text{Gamma}(\alpha = n_0\mu_0, \beta = n_0)$. It makes sense that the posterior predictive distribution will be the same regardless of the parameterization as the parameter itself has been margined out when computing $p_{X_*|\mathbf{x}}(X_* | \mathbf{x})$. (It shouldn't matter which scale the parameter is measured).