Math 341 / 641 Fall 2025
Midterm Examination Two

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Full Name ____

Code of Academic Integrity

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Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

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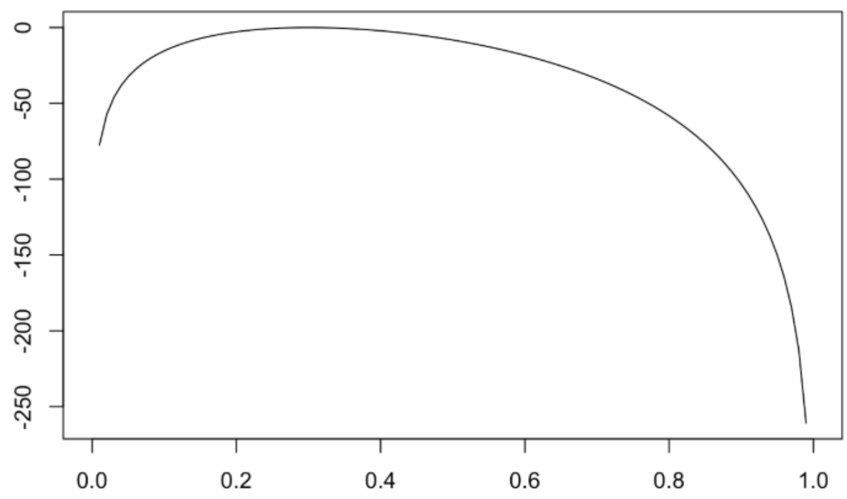
signature date

Instructions

This exam is 110 minutes (variable time per question) and closed-book. You are allowed **two** pages (front and back) of a "cheat sheet", blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 Consider the following plot where the x-axis is θ and the y-axis is $\ell(\theta; \boldsymbol{x})$. We wish to test $H_0 : \theta = \theta_0$ at significance level $\alpha = 5\%$.



(a) [2 pt / 2 pts] Estimate $\hat{\hat{\theta}}^{\text{MLE}}$ from the plot above.

≈ 0.3

(b) [2 pt / 4 pts] What is the name of the test we learned about that rejects H_0 if the distance between $\hat{\hat{\theta}}^{\text{MLE}}$ and θ_0 is large?

Wald Test

(c) [2 pt / 6 pts] What is the name of the test we learned about that rejects H_0 if the absolute value of the slope of the tangent line to $\ell(\theta_0; \boldsymbol{x})$ is large?

Score Test

(d) [2 pt / 8 pts] What is the name of the test we learned about that rejects H_0 if the distance between $\ell(\theta_0; \boldsymbol{x})$ and $\ell(\hat{\hat{\theta}}^{\text{MLE}}; \boldsymbol{x})$ is large?

Likelihood Ratio Test

(e) [5 pt / 13 pts] Test the null hypothesis when $\theta_0 = 0.8$.

We only have enough information to run the Likelihood Ratio Test. We approximate $\ell(\hat{\hat{\theta}}^{\text{MLE}} = 0.3; \boldsymbol{x}) = -5$ and $\ell(\theta_0 = 0.8; \boldsymbol{x}) = -50$. Thus, $\hat{\hat{\Lambda}} = 2((-5) - (-50)) = 90 > 3.84$, the cutoff for the χ^2_1 distribution at $\alpha = 5\%$. Thus, we reject H_0.

Problem 2 Consider the following rv and some of its properties:

$$X \sim \text{Lomax}(\theta, 3) := \frac{\theta}{3}\left(1+\frac{x}{3}\right)^{-(\theta+1)} \mathbb{1}_{x>0}, \quad \theta > 0, \quad \mathbb{E}[X] = \frac{3}{\theta - 1}, \quad \mathbb{V}\text{ar}[X] = \frac{9\theta}{(\theta-1)^2(\theta-2)}$$

Assume $X_1, \ldots, X_n \overset{iid}{\sim} \text{Lomax}(\theta, 3)$ and that the following is correct:

$$\begin{aligned}
\mathcal{L}(\theta; X_1, \ldots, X_n) &= \prod_{i=1}^{n} \frac{\theta}{3}\left(1+\frac{X_i}{3}\right)^{-(\theta+1)} = \frac{\theta^n}{3^n}\left(\prod_{i=1}^{n}\left(1+\frac{X_i}{3}\right)\right)^{-(\theta+1)} \\
\ell(\theta; X_1, \ldots, X_n) &= n\ln(\theta) - n\ln(3) - (\theta+1)\sum_{i=1}^{n}\ln\left(1+\frac{X_i}{3}\right) \\
\ell'(\theta; X_1, \ldots, X_n) &= \frac{n}{\theta} - \sum_{i=1}^{n}\ln\left(1+\frac{X_i}{3}\right) \quad \Rightarrow \quad \hat{\theta}^{\text{MLE}} = \frac{n}{\sum_{i=1}^{n}\ln\left(1+\frac{X_i}{3}\right)} \\
\ell''(\theta; X_1, \ldots, X_n) &= -\frac{n}{\theta^2}
\end{aligned}$$

Then, assume we are testing $H_0 : \theta = \theta_0$.

(a) [3 pt / 16 pts] If we were to estimate $\mathbb{V}\text{ar}[X]$ from data, why do we employ Bessel's correction?

To have an unbiased estimator.

(b) [3 pt / 19 pts] Find $\sqrt{I_n(\theta_0)}$.

$$I_n(\theta_0) = \mathbb{E}\left[-\ell''(\theta_0; X_1, \ldots, X_n)\right] = \mathbb{E}\left[-\left(-\frac{n}{\theta_0^2}\right)\right] = \frac{n}{\theta_0^2} \quad \Rightarrow \quad \sqrt{I_n(\theta_0)} = \frac{\sqrt{n}}{\theta_0}$$

(c) [4 pt / 23 pts] Compute the asymptotically normal Wald statistic estimator $\hat{Z} \mid H_0$ (the test which comes directly from the monster theorem) as a function of $X_1, \ldots, X_n$, n and θ_0. Simplify.

$$\hat{Z} \mid H_0 = \frac{\hat{\theta}^{\text{MLE}} - \theta_0}{\frac{1}{\sqrt{I_n(\theta_0)}}} = \frac{\sqrt{n}}{\theta_0}\left(\frac{n}{\sum_{i=1}^{n}\ln\left(1+\frac{X_i}{3}\right)} - \theta_0\right) = \frac{n^{3/2}}{\theta_0\sum_{i=1}^{n}\ln\left(1+\frac{X_i}{3}\right)} - \sqrt{n}$$

(d) [4 pt / 27 pts] Compute the asymptotically normal score statistic estimator $\hat{Z} \mid H_0$ as a function of $X_1, \ldots, X_n$, n and θ_0. Simplify.

$$\hat{Z} \mid H_0 = \frac{s(\theta_0; \boldsymbol{X})}{\sqrt{I_n(\theta)}} = \frac{\frac{n}{\theta_0} - \sum_{i=1}^n \ln\left(1 + \frac{X_i}{3}\right)}{\frac{\sqrt{n}}{\theta_0}} = \sqrt{n} - \frac{\theta_0}{\sqrt{n}} \sum_{i=1}^n \ln\left(1 + \frac{X_i}{3}\right)$$

(e) [4 pt / 31 pts] Compute the asymptotically χ_1^2 likelihood ratio statistic estimator $\hat{\Lambda} \mid H_0$ as a function of $X_1, \ldots, X_n$, n and θ_0. Simplify.

$$\begin{aligned}
\hat{\Lambda} \mid H_0 &= 2\ln\left(\frac{\mathcal{L}(\hat{\theta}^{\mathrm{MLE}}; \boldsymbol{X})}{\mathcal{L}(\theta_0; \boldsymbol{X})}\right) = 2\ln\left(\frac{\frac{(\hat{\theta}^{\mathrm{MLE}})^n}{3^n}\left(\prod_{i=1}^n \left(1 + \frac{X_i}{3}\right)\right)^{-(\hat{\theta}^{\mathrm{MLE}}+1)}}{\frac{\theta_0^n}{3^n}\left(\prod_{i=1}^n \left(1 + \frac{X_i}{3}\right)\right)^{-(\theta_0+1)}}\right) \\
&= 2\ln\left(\left(\frac{\hat{\theta}^{\mathrm{MLE}}}{\theta_0}\right)^n \left(\prod_{i=1}^n \left(1 + \frac{X_i}{3}\right)\right)^{(\theta_0+1)-(\hat{\theta}^{\mathrm{MLE}}+1)}\right) \\
&= 2\left(n\ln\left(\hat{\theta}^{\mathrm{MLE}}\right) - n\ln(\theta_0) + \left(\theta_0 - \hat{\theta}^{\mathrm{MLE}}\right)\sum_{i=1}^n \ln\left(1 + \frac{X_i}{3}\right)\right)
\end{aligned}$$

For the rest of this question let $n = 100$ and $\sum_{i=1}^n \ln\left(1 + \frac{x_i}{3}\right) = 4.567$. We now wish to test $H_0 : \theta = 17$ at significance level $\alpha = 5\%$.

(f) [3 pt / 34 pts] Compute the Wald statistic estimate to four decimal places. Then write the result of the hypothesis test decision.

$$\hat{\hat{z}} = \frac{100^{3/2}}{17 \cdot 4.567} - \sqrt{100} = 2.880 \notin [\pm 1.96] \quad \Rightarrow \quad \text{Reject } H_0$$

(g) [3 pt / 37 pts] Compute the score statistic estimate to four decimal places. Then write the result of the hypothesis test decision.

$$\hat{\hat{z}} = \sqrt{100} - \frac{17}{\sqrt{100}} \cdot 4.567 = 2.2361 \notin [\pm 1.96] \quad \Rightarrow \quad \text{Reject } H_0$$

(h) [3 pt / 40 pts] Compute the likelihood ratio statistic estimate to four decimal places. Then write the result of the hypothesis test decision.

$$\begin{aligned} \hat{\hat{\theta}}^{\text{MLE}} &= 100/4.567 = 21.89621 \\ \hat{\hat{\Lambda}} &= 2\left(100 \ln(21.89621) - 100 \ln(17) + (17 - 21.89621) \cdot 4.567\right) \\ &= 5.8981 \notin [0, 3.84] \quad \Rightarrow \quad \text{Reject } H_0 \end{aligned}$$

(i) [2 pt / 42 pts] Is it guaranteed that the outcomes of this hypothesis test would be the same for all three of these tests above? Yes/no.

No

(j) [2 pt / 44 pts] Given the information you have, is it possible to determine which test has the highest power? Yes/no.

No

(k) [2 pt / 46 pts] Which of these three tests were approximate?

all of them

(l) [4 pt / 50 pts] Compute a 95% confidence interval for θ to two decimals and denote it properly.

$$\begin{aligned} \hat{\hat{CI}}_{\theta, 95\%} &\approx \left[\hat{\hat{\theta}}^{\text{MLE}} \pm 1.96 \frac{1}{\sqrt{I_n(\hat{\hat{\theta}}^{\text{MLE}})}}\right] = \left[\hat{\hat{\theta}}^{\text{MLE}} \pm 1.96 \frac{\hat{\hat{\theta}}^{\text{MLE}}}{\sqrt{n}}\right] = \left[21.89621 \pm 1.96 \frac{21.89621}{\sqrt{100}}\right] \\ &= [17.61, 26.19] \end{aligned}$$

(m) [3 pt / 53 pts] Provide three interpretations of this interval.

- Before you collect data, the probability $\hat{CI}_{\theta,95\%}$ will contain θ is $\approx 95\%$.
- If you collect many different datasets and compute a confidence interval for each dataset, $\approx 95\%$ of $\hat{\hat{CI}}_{\theta,95\%}$'s will contain θ.
- The probability that θ is in the computed $\hat{\hat{CI}}_{\theta,95\%}$ is either zero or one.

(n) [5 pt / 58 pts] Compute a 95% confidence interval for $\phi := g(\theta) = \ln(\theta)$ to two decimals and denote it properly.

$$\hat{\hat{CI}}_{\phi,95\%} \approx \left[g(\hat{\hat{\theta}}^{\text{MLE}}) \pm 1.96|g'(\hat{\hat{\theta}}^{\text{MLE}})|\hat{\text{SE}}(\hat{\hat{\theta}}^{\text{MLE}})\right] = \left[\ln\left(\hat{\hat{\theta}}^{\text{MLE}}\right) \pm 1.96\frac{1}{\left|\hat{\hat{\theta}}^{\text{MLE}}\right|}\frac{\hat{\hat{\theta}}^{\text{MLE}}}{\sqrt{n}}\right]$$

$$= \left[\ln(21.89621) \pm 1.96\frac{1}{\sqrt{100}}\right] = [2.89, 3.28]$$

Problem 3 In a standard double-slit interference experiment, electrons are fired one at a time toward a pair of slits separated by distance d. The detection screen is at distance L, and the position X of each detected electron (measured in cm from the center line) follows a probability density function proportional to

$$I(x) = I_0 \cos^2\left(\frac{\pi d\, x}{\lambda L}\right),$$

where λ is the de Broglie wavelength, d is slit separation, and L is slit-to-screen distance.

For our particular setting of d, L, the system is calibrated so that the theoretical probability density for the impact location x in the interval $(-1, 1)$ is

$$f(x) = \frac{1}{2}(1 + \cos(3\pi x))$$

and the CDF on the interval $(-1, 1)$ can then be found by calculus to be:

$$F(x) = \frac{1}{2}\left(1 + x + \frac{\sin(3\pi x)}{3\pi}\right).$$

The experimenter collects $n = 20$ independent electron impact position observations which we denote $x_1, \dots, x_n$ and then computes their theoretical CDF values and the difference with the estimated CDF values. Below are these values sorted by the value of x:

i	x_i	$F(x_i)$	$\lvert F(x_i) - \hat{\hat{F}}(x_i)\rvert$
1	-0.96	0.038	0.012
2	-0.77	0.128	0.022
3	-0.74	0.143	0.007
4	-0.60	0.204	0.004
5	-0.51	0.248	0.002
6	-0.40	0.304	0.004
7	-0.32	0.343	0.007
8	-0.19	0.401	0.001
9	-0.10	0.445	0.005
10	-0.04	0.474	0.006
11	0.02	0.505	0.005
12	0.11	0.549	0.001
13	0.17	0.580	0.010
14	0.24	0.613	0.013
15	0.36	0.665	0.011
16	0.48	0.715	0.015
17	0.55	0.744	0.016
18	0.63	0.775	0.005
19	0.79	0.830	0.030
20	0.92	0.873	0.027

(a) [2 pt / 60 pts] Find a formula for $\hat{\hat{F}}_i := \hat{\hat{F}}(x_i)$ given that the x_i's are sorted smallest to largest.

$$\hat{\hat{F}}_i = \frac{i}{n} = \frac{i}{20}$$

Henceforth, our goal is to test whether these observations match the theoretical double-slit density derived from physics.

(b) [2 pt / 62 pts] What is the null hypothesis H_0?

$$H_0 : X_1, \dots, X_n \overset{iid}{\sim} f(x)$$

(c) [2 pt / 64 pts] What is the alternative hypothesis H_a?

$$H_0 : X_1, \dots, X_n \text{ are not distributed } \overset{iid}{\sim} f(x)$$

(d) [5 pt / 69 pts] Test H_a from the previous question at $\alpha = 0.05$. Indicate the decision. Interpret the decision.

We use the one-sample Kolmogorov-Smirnov test. At $\alpha = 0.05$, the critical value of the Kolmogorov distribution is 1.359. From the table above, we find the supremum difference between theoretical CDF and empirical CDF to be $\hat{\hat{D}}_n = 0.03$. We then calculate the test statistic using the formula below.

$$\hat{\hat{K}} = \sqrt{n}\hat{\hat{D}}_n = \sqrt{20} \cdot 0.03 = 0.134 \in [0, 1.359] \quad \Rightarrow \quad \text{Retain } H_0$$

There is insufficient evidence to suggest that these observed electron impact positions are not distributed according to the theoretical impact location in the double-slit experiment.

(e) [1 pt / 70 pts] Is the test in the previous question exact or approximate?

approximate

Problem 4 A researcher wants to know whether daily exercise frequency is associated with (dependent on) a person's stress level. A random sample of 240 adults is surveyed, and the results are summarized in the table below.

Stress Level $\rightarrow$ **Exercise Frequency** $\downarrow$	**Low**	**Medium**	**High**	**Total**
None	18	42	30	90
1–2 days/week	32	48	20	100
3–5 days/week	28	16	6	50
Total	78	106	56	240

Let $\theta_{i,j}$ denote the joint probability of having exercise frequency of row i and stress level of column j. Let $\theta_{i\cdot}$ denote the marginal probability of having exercise frequency of row i. Let $\theta_{\cdot j}$ denote the marginal probability of having stress level of column j.

Below are some 95%iles of chi-squared distributions by degrees of freedom.

d degrees of freedom	1	2	3	4	5	6	7	8	9	10
x s.t. $F_{\chi^2_d}(x) = .95$	3.84	5.99	7.81	9.49	11.07	12.59	14.07	15.51	16.92	18.31

Using the θ notation above, the null and alternative hypotheses are

$$H_a : \exists i, j \quad \theta_{i,j} \neq \theta_{i\cdot}\theta_{\cdot j}$$
$$H_0 : \forall i, j \quad \theta_{i,j} = \theta_{i\cdot}\theta_{\cdot j}$$

(a) [7 pt / 77 pts] Run the test for H_a from part (a) at $\alpha = 5\%$. Indicate the decision. Interpret the decision.

We compute the marginal probabilities of each row and column under H_0 (in the margins to three decimal places). Then in the cells, we multiply these row and column marginal probabilities times n to get the expected counts (to two decimal places):

Exercise Frequency	**Low Stress**	**Medium Stress**	**High Stress**	**Marginal Proportion**
None	29.25	39.78	20.07	.375
1–2 days/week	32.53	44.23	22.32	.417
3–5 days/week	16.22	22.06	11.13	.208
Marginal Proportion	.325	.442	.233	240

The chi-squared test statistic has $(r-1)(c-1) = (3-1)(3-1) = 4$ degrees of freedom. Hence the cutoff at $\alpha = 5\%$ is given in the table in the problem header as 9.49. Now we compute the chi-squared test statistic:

$$\begin{aligned}\hat{\hat{\phi}} &= \frac{(18-29.25)^2}{29.25} + \frac{(42-39.78)^2}{39.78} + \frac{(30-20.07)^2}{20.07} + \\ &\quad \frac{(32-32.53)^2}{32.53} + \frac{(48-44.23)^2}{44.23} + \frac{(20-22.32)^2}{22.32} + \\ &\quad \frac{(28-16.22)^2}{16.22} + \frac{(16-22.06)^2}{22.06} + \frac{(6-11.13)^2}{11.13} \approx 22 \notin [0, 9.49] \quad \Rightarrow \quad \text{Reject } H_0\end{aligned}$$

There is sufficient evidence to suggest that exercise level and stress are dependent.

(b) [1 pt / 78 pts] Is the test in the previous question exact or approximate?

approximate

Problem 5 You observe the following dataset from measurements on a metallurgic fabrication device:

```
1.59  0.92  0.30  1.45 -0.55
0.46 -0.73  0.01  1.48  1.07
```

And we compute $\bar{x} = 0.600$ and $s = 0.840$. It is important to prove $H_a : \theta > 0$ where θ is the expectation of the DGP.

Below are some 95%iles of Student's T distributions by degrees of freedom.

d degrees of freedom	1	2	3	4	5	6	7	8	9	10
t s.t. $F_{T_d}(t) = .95$	6.31	2.92	2.35	2.13	2.02	1.94	1.89	1.86	1.83	1.81

(a) [4 pt / 82 pts] Assuming the DGP is normal, run the appropriate test based on what is important to prove about θ at $\alpha = 5\%$.

$$H_0 : \theta \leq 0, \quad \hat{\hat{t}} = \frac{\bar{x} - \theta_0}{\frac{s}{\sqrt{n}}} = \frac{0.6 - 0}{\frac{0.840}{\sqrt{10}}} = 2.259 \notin (-\infty, 1.83] \quad \Rightarrow \quad \text{Reject } H_0$$

(b) [2 pt / 84 pts] Create an expression for the p_{val} using the CDF function $F_{T_d}(t)$ where you specify the values of d and t numerically and compare it to α using either $>, <, =$.

$$p_{val} = 1 - F_{T_9}(2.259) < \alpha$$

(c) [1 pt / 85 pts] Is the test in (a) exact or approximate?

exact

(d) [4 pt / 89 pts] Assuming the DGP is *not* normal, run the appropriate test based on what is important to prove about θ at $\alpha = 5\%$.

$$H_0 : \theta \leq 0, \quad \hat{\hat{z}} = \frac{\bar{x} - \theta_0}{\frac{s}{\sqrt{n}}} = \frac{0.6 - 0}{\frac{0.840}{\sqrt{10}}} = 2.259 \notin (-\infty, 1.96] \quad \Rightarrow \quad \text{Reject } H_0$$

(e) [2 pt / 91 pts] Create an expression for the p_{val} based on notation we used in class and compare it to α using either $>, <, =$.

$$p_{val} = 1 - \Phi(2.259) < \alpha$$

(f) [1 pt / 92 pts] Is the test in (d) exact or approximate?

approximate

You now observe the following dataset from measurements on a second metallurgic fabrication device:

```
0.68  2.51  1.35  1.81  0.44
2.69  1.97  1.36  0.85 -1.35
```

And we compute $\bar{x} = 1.231$ and $s = 1.174$. We wish to test $H_0 : \theta_1 = \theta_2$ where θ_1 is the mean of the first metallurgic fabrication device and θ_2 is the mean of the second metallurgic fabrication device.

(g) [2 pt / 94 pts] Assuming the DGP of the first device and the DGP of the second device are both normal with different variances, what is the exact distribution of the test statistic used to test H_0? Specify the numeric value(s) of this distribution or discuss how they are computed.

The test statistic is exactly Fisher-Behrens-distributed.

(h) [2 pt / 96 pts] Assuming the DGP of the first device and the DGP of the second device are both normal with different variances, what is the approximate distribution of the test statistic used to test H_0? This approximate distribution is the most popoular means of testing this null hypothesis. Specify the numeric value(s) of this distribution or discuss how they are computed.

The test statistic is approximately T_d-distributed with d equal to the Welch-Satterthwaite formula.

(i) [2 pt / 98 pts] Assuming the DGP of the first device and the DGP of the second device are both normal with equal variances, what is the exact distribution of the test statistic used to test H_0? Specify the numeric value(s) of this distribution or discuss how they are computed.

the test statistic is exactly T_{18}-distributed.

(j) [2 pt / 100 pts] Assuming the DGP of the first device and the DGP of the second device are both non-normal with equal variances, what is the approximate distribution of the test statistic used to test H_0? Specify the numeric value(s) of this distribution or discuss how they are computed.

The test statistic is approximately Z- or standard normal-distributed.