Math 341 / 641 Fall 2025
Midterm Examination One

Professor Adam Kapelner

Sept 30, 2025

Full Name ______________________________

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

______________________________ ______________
signature date

Instructions

This exam is 110 minutes (variable time per question) and closed-book. You are allowed **one** page (front and back) of a "cheat sheet", blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 Consider the following rv and some of its properties:

$$X \sim \text{Gamma}(\theta_1, \theta_2) := \frac{\theta_2^{\theta_1}}{\Gamma(\theta_1)} x^{\theta_1 - 1} e^{-\theta_2 x} \mathbb{1}_{x>0}, \quad \theta_1 > 0, \quad \theta_2 > 0, \quad \mathbb{E}[X] = \frac{\theta_1}{\theta_2}, \quad \mathbb{V}\text{ar}[X] = \frac{\theta_1}{\theta_2^2}$$

We realize $n = 100$ iid realizations from this DGP and calculate the following quantities rounded to two decimals:

$$\begin{aligned}
\bar{x} := \frac{1}{n}\sum_{i=1}^{n} x_i &= 3.51 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^2 &= 7.89 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^3 &= 29.62 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^4 &= 127.21 \\
\frac{1}{n}\sum_{i=1}^{n} x_i^5 &= 602.66
\end{aligned}$$

(a) [3 pt / 3 pts] Observing $n = 100$ iid realizations from this DGP is equivalent to sampling an SRS of $n = 100$ from a population with which propert(ies)?

(b) [3 pt / 6 pts] Find the method of moments (MM) estimator for $\mathbb{E}[X]$.

(c) [3 pt / 9 pts] What is the MM estimate for $\mathbb{E}[X]$? Round to the nearest two decimals.

(d) [6 pt / 15 pts] Find the MM estimator for θ_2.

(e) [3 pt / 18 pts] What is the MM estimate for θ_2? Round to the nearest two decimals.

(f) [4 pt / 22 pts] Does the MM estimate for θ_2 make sense? Why or why not?

(g) [5 pt / 27 pts] Find a MM estimator for θ_1.

Now consider the scenario where we know that the value of $\theta_1 = 4$. The constant in the denominator can be calculated to be 6. Now, there is only one parameter, so we simplify the expression and just denote it θ. Thus,

$$X \sim \text{Gamma}(4, \theta) := \frac{\theta^4}{6} x^3 e^{-\theta x} \mathbb{1}_{x>0}$$

(h) [7 pt / 34 pts] Show that $\hat{\theta}^{\text{MLE}} = \frac{4}{\bar{X}}$ for general sample of size n. All data is properly realized and are positive values and thus you can drop the indicator term during your calculation.

(i) [3 pt / 37 pts] Show that this MLE is indeed a maximum of the log-likelihood by using a second derivative check. Explain your reasoning.

(j) [6 pt / 43 pts] Derive the CRLB for θ in the DGP $X \sim \text{Gamma}(4, \theta)$.

(k) [1 pt / 44 pts] It can be shown that

$$\mathbb{E}\left[\hat{\theta}^{\text{MLE}}\right] = \frac{4n}{4n-1}\theta.$$

Is $\hat{\theta}^{\text{MLE}}$ unbiased? Yes or no.

(l) [6 pt / 50 pts] It can be shown that

$$\mathbb{V}\text{ar}\left[\hat{\theta}^{\text{MLE}}\right] = \frac{(4n)^2}{(4n-1)^2(4n-2)}\theta^2.$$

Calculate $\mathbb{M}\text{SE}\left[\hat{\theta}^{\text{MLE}}\right]$ as a function of n and θ. Simplify as much as you can.

(m) [5 pt / 55 pts] Under squared error loss, compute $\sup_{\theta\in\Theta}\left\{R(\hat{\theta}^{\text{MLE}},\theta)\right\}$.

Problem 2 Consider a survey of iPhoneness. $n = 27$ students were chosen outside of Kiely Hall on Tuesday afternoon, Sept 30, 2025 were asked if they had an iPhone ($x_i = 1$) or not ($x_i = 0$). In total, 23 students had iPhones. We wish to infer about the population of all QC undergraduate students.

(a) [4 pt / 59 pts] Is this sample an SRS from the population of interest? Why or why not?

Regardless of what you wrote in (a), consider this sample to be an SRS from the population of interest for the rest of this question.

(b) [9 pt / 68 pts] The US mean countrywide proportion of iPhones is estimated to be 62%. Prove via a hypothesis test at $\alpha = 5\%$ using the one-proportion Z-test that the QC population has higher iPhone proportion than the countrywide mean. Your answer must include a clear setup of the two hypotheses, a retainment region and a decision. Do not interpret the decision nor find Fisher's p_{val}. Those will be asked in subsequent questions.

(c) [4 pt / 72 pts] Interpret your decision from the previous question.

(d) [2 pt / 74 pts] If your decision was an error, what type of error was it?

(e) [6 pt / 80 pts] Represent Fisher's p_{val} by using the Φ notation (i.e. the CDF of the standard normal).

(f) [3 pt / 83 pts] Why is the p_{val} from the previous question approximate?

(g) [1 pt / 84 pts] If you wanted an exact test, what is the name of another test (besides the one we are considering) you would use?

(h) [3 pt / 87 pts] In order to calculate the power of this test we are considering, what information would you need?

(i) [3 pt / 90 pts] If you ran the test we are considering with $\alpha = 6\%$ instead of the original 5%, would power increase or decrease?

(j) [5 pt / 95 pts] If the null hypothesis were true, as the sample size approaches infinity, what would the decision of the test we are considering be?

(k) [5 pt / 100 pts] In order to calculate a 95% CI for θ by inverting the test we are considering, what specific information do you need? Note: this is a difficult question.