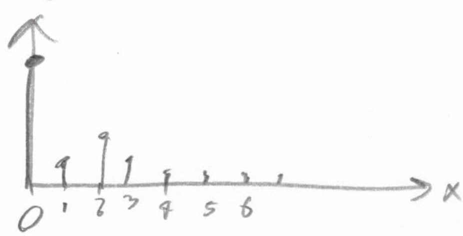
MATH 343 LEC 2

$$\langle \vec{\theta}_1, \vec{\theta}_2, \ldots \rangle \xRightarrow{\text{BURN \& THIN}} \{\vec{\theta}_B, \vec{\theta}_{B+l_0}, \vec{\theta}_{B+2l_0}, \ldots\}$$

Make sure # of samples remaining is a few thousand...

New Model... Poisson Count model with "hurdle".

What if data looks like this...



It appears the data > 0 has a Poisson distr. with expectation ≈ 2.5 but at zero ~~too~~ too many so for the Poisson.

Consider the model:

$$X_1, \ldots, X_n \overset{iid}{\sim} \begin{cases} 0 & \text{wp } \theta_1 \\ \text{Shifted Poisson}(\theta_2) & \text{wp } 1-\theta_1 \end{cases}$$

where we shift by +1 so support is $\{1, 2, \ldots\}$

$\theta_1 \in [0,1]$, $\theta_2 \in (0, \infty)$

Using Laplace priors, ...

$$f(\theta_1, \theta_2 \mid \vec{x}) \propto P(\vec{X} \mid \theta_1, \theta_2)$$

$$= \prod_{i=1}^{n} \theta_1^{\mathbb{1}_{x_i=0}} \left((1-\theta_1) \frac{\theta_2^{x_i-1} e^{-\theta_2}}{(x_i-1)!} \right)^{\mathbb{1}_{x_i>0}}$$

$$= \theta_1^{\sum \mathbb{1}_{x_i=0}} (1-\theta_1)^{\sum \mathbb{1}_{x_i>0}} \frac{\theta_2^{\sum (x_i-1)\mathbb{1}_{x_i>0}} \, e^{-\theta_2 \sum \mathbb{1}_{x_i>0}}}{\prod (x_i-1)!^{\mathbb{1}_{x_i>0}}}$$

let $n_0 := \sum \mathbb{1}_{x_i=0} \Rightarrow n-n_0 = \sum \mathbb{1}_{x_i>0}$

$$\propto \theta_1^{n_0}(1-\theta_1)^{n-n_0} \, \theta_2^{\sum (x_i-1)\mathbb{1}_{x_i>0}} \, e^{-(n-n_0)\theta_2}$$

not a function of θ_2

$$f(\theta_1 \mid \vec{x}, \theta_2) \propto \theta_1^{n_0}(1-\theta_1)^{n-n_0} \propto \text{Beta}(n_0+1, n-n_0+1)$$

$$f(\theta_2 \mid \vec{x}, \theta_1) \propto \theta_2^{\sum (x_i-1)\mathbb{1}_{x_i>0}} e^{-(n-n_0)\theta_2} \propto \text{Gamma}\left(1+\sum (x_i-1)\mathbb{1}_{x_i>0}, n-n_0\right)$$

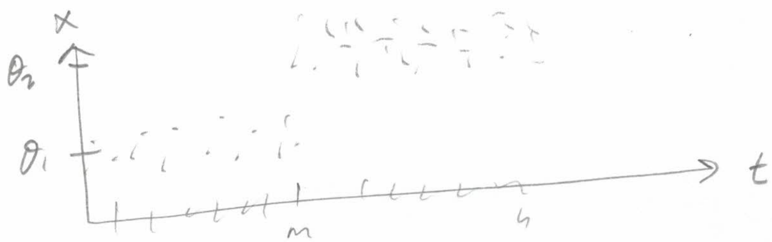
not a function of θ_1

Another example from lec 23, MATH 341

$X_1, \dots, X_m \overset{iid}{\sim} \text{Poisson}(\theta_1)$ $\qquad m, \theta_1, \theta_2$ unknown

$X_{m+1}, \dots, X_n \overset{iid}{\sim} \text{Poisson}(\theta_2)$

Goal: get $p(\theta_1, \theta_2, m \mid \vec{x})$. Assume $p(\theta_1, \theta_2, m) \propto 1$ Laplace Prior



$$p(\theta_1, \theta_2, m \mid \vec{X}) \propto p(\vec{X} \mid \theta_1, \theta_2, m) = \prod_{t=1}^{m} \frac{e^{-\theta_1} \theta_1^{X_t}}{X_t!} \prod_{t=1+m}^{n} \frac{e^{-\theta_2} \theta_2^{X_t}}{X_t!}$$

$$= \frac{e^{-m\theta_1} \theta_1^{\sum_{t=1}^{m} X_t}}{\prod_{t=1}^{m} X_t!} \frac{e^{-(n-m)\theta_2} \theta_2^{\sum_{t=m+1}^{n} X_t}}{\prod_{t=1+m}^{n} X_t!}$$

$$\propto e^{m\theta_2} e^{-m\theta_1} e^{-n\theta_2} \theta_1^{\sum_{t=1}^{m} X_t} \theta_2^{\sum_{t=m+1}^{n} X_t} \propto \; ?$$ Use Gibbs Sampling..

$$f(\theta_1 \mid \vec{X}, \theta_2, m) \propto e^{-m\theta_1} \theta_1^{\sum_{t=1}^{m} X_t} \mathbb{1}_{\theta_1 \in (0,\infty)} \propto \text{Gamma}\left(1 + \sum_{t=1}^{m} X_t, m\right)$$

$$f(\theta_2 \mid \vec{X}, \theta_1, m) \propto e^{-(n-m)\theta_2} \theta_2^{\sum_{t=m+1}^{n} X_t} \mathbb{1}_{\theta_2 \in (0,\infty)} \propto \text{Gamma}\left(1 + \sum_{t=m+1}^{n} X_t, n-m\right)$$

$$p(m \mid \vec{X}, \theta_1, \theta_2) \propto e^{m(\theta_2 - \theta_1)} \theta_1^{\sum_{t=1}^{m} X_t} \theta_2^{\sum_{t=m+1}^{n} X_t} \mathbb{1}_{m \in \{1, \dots, n\}}$$

$$\propto \frac{e^{m(\theta_2 - \theta_1)} \theta_1^{\sum X_t} \theta_2^{\sum X_t}}{\sum_{j=1}^{n} e^{j(\theta_2 - \theta_1)} \theta_1^{\sum_{t=1}^{j} X_t} \theta_2^{\sum_{t=1+j}^{n} X_t}}$$

Another Model

$$X_1,\ldots,X_n \overset{\text{ind}}{\sim} \text{Poisson}(\theta_0+\theta_1 t_i)$$

Assume

$$p(\theta_0,\theta_1)\propto 1$$

$$p(\theta_0,\theta_1|\vec{x}) \propto p(\vec{x}|\theta_0,\theta_1) = \prod_{i=1}^{n} \frac{e^{-(\theta_0+\theta_1 t_i)}(\theta_0+\theta_1 t_i)^{x_i}}{x_i!}$$

$$\propto e^{-n\theta_0-\theta_1\Sigma t_i}\prod_{i=1}^{n}(\theta_0+\theta_1 t_i)^{x_i}$$

$p(\theta_0|\vec{x},\theta_1) \propto e^{-n\theta_0}\prod_{i=1}^{n}(\theta_0+\theta_1 t_i)^{x_i} \not\propto$ a distr. we know

$p(\theta_1|\vec{x},\theta_0) \propto e^{-\theta_1\Sigma t_i}\prod_{i=1}^{n}(\theta_0+\theta_1 t_i)^{x_i} \not\propto$ a distr we know

Gibbs Sampler cannot be implemented unless we grid sample... very very slow!!!

Enter the Metropolis (1953) - Hastings (1970) Algorithm.