Lec 6 MATH 343

Survival Analysis / Reliability Analysis

X is a survival / reliability model if $S_X \geq 0$ and no maximum.

Some Brand Name
Discrete survival models: Geometric, ExtNegBinomial, Poisson

Some Brand Name
Continuous survival models: Gamma, LogNormal, Weibull, Pareto, F, BetaPrime

The Weibull is the most famous of the cont. models. Review...

Param space $k, \lambda > 0$

Let $Y_1, \ldots, Y_n \stackrel{iid}{\sim} \text{Weibull}(\lambda, k) = \lambda k (\lambda y)^{k-1} e^{-(\lambda y)^k} \mathbb{1}_{y \geq 0}$

$F_Y(y) = 1 - e^{-(\lambda y)^k} \Leftrightarrow$ "$\overset{1-F(y)}{S(y)} = e^{-(\lambda y)^k}$ survival function"

$\Rightarrow y^k = \frac{u}{\lambda^k} \Rightarrow y = \frac{u^{\frac{1}{k}}}{\lambda}$

let $u = \lambda^k y^k \Rightarrow \frac{du}{dy} = k \lambda^k y^{k-1}$

$\Rightarrow dy = \frac{1}{k\lambda^k y^{k-1}} du$, $y=0 \Rightarrow u=0$, $y=\infty \Rightarrow u=\infty$

$$\theta := E[Y] = \int_0^\infty y \lambda k (\lambda y)^{k-1} e^{-(\lambda y)^k} dy = k\lambda^k \int_0^\infty y^k e^{-\lambda^k y^k} dy = \cancel{k\lambda^k} \int_0^\infty y^k e^{-u} \frac{1}{\cancel{k\lambda^k} y^{k-1}} du$$

$$= \frac{1}{\lambda} \int_0^\infty u^{\frac{1}{k}+1-1} e^{-u} du = \frac{1}{\lambda}\Gamma\left(\frac{1}{k}+1\right), \quad \sigma^2 = Var(Y) = \cdots = \frac{1}{\lambda^2}\left(\Gamma\left(1+\frac{2}{k}\right) - \Gamma\left(1+\frac{1}{k}\right)^2\right) < \infty$$

MLE's (Hw) Page 2

$\Rightarrow$ By CLT $\bar{Y} \stackrel{\cdot}{\sim} N\left(\theta, \frac{\sigma^2}{n}\right)$ so the avg is also a good estimator if you're lazy

For non-extreme quantile estimation, bootstrap. MM works too.

But there is usually a big problem! Usually, not all the $Y_1, \ldots, Y_n$ are actually observed since observing large y_i's actually requires waiting for them to occur. The sampling frame usually has a maximum e.g. $t_f = 2\,yr$. So all y_i's after t_f...

Also ... we left out the following property of MLE's in 341.

Invariance of MLE Thm

with parameter space $\mathbb{R}^p$, $p \geq 1$.

let $\vec{\theta}$ be the parameters for a r.v., let $\vec{\tau} = g(\vec{\theta})$.

If $\hat{\vec{\theta}}^{MLE}$ is the MLE for $\vec{\theta}$ then $g(\hat{\vec{\theta}}^{MLE})$ is the MLE for $\vec{\tau}$.

Proof in the case of g being 1:1 (not 1:1 is a different proof).
and $X_1, \ldots, X_n$ iid.

$$\sup_{\vec{\tau} \in \mathbb{R}^p} l(\vec{\tau}; \vec{x}) = \sup_{\vec{\tau} \in \mathbb{R}^p} l\left(g^{-1}(\vec{\tau}); \vec{x}\right) = \sup_{\vec{\theta} \in \mathbb{R}^p} l(\vec{\theta}; \vec{x})$$

This sup is attained at both $\hat{\vec{\theta}}^{MLE}$ or $\hat{\vec{\tau}}^{MLE}$ hence they're equal

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$\Rightarrow$ the MLE of the mean is $\hat{\mu}^{MLE} = \dfrac{1}{\hat{\lambda}^{MLE}} \Gamma\left(\dfrac{1}{\hat{k}^{MLE}} + 1\right)$

realize by t_f are missing and termed "censored"
Let C be the binary censoring variable

Eg.

Y	C
0.85	0
0.71	0
? ($> t_f$)	1
0.13	0
0.60	0
? ($> t_f$)	1
⋮	⋮

So now the observed data is $\vec{y}, \vec{c}$.

How do we estimate $\mu = E[Y]$?

Let's write out the likelihood function:

$$\mathcal{L}(k, \lambda ; \vec{y}, \vec{c}, t_f) = \prod_{\{i : c_i = 0\}} f(y_i ; \lambda, k) \prod_{\{i : c_i = 0\}} P(Y_i > t_f)$$

$$= \prod_{\{i : c_i = 0\}} \lambda^k k y_i^{k-1} e^{-\lambda^k y_i^k} \prod_{\{i : c_i = 0\}} e^{-\lambda^k t_f^k}$$

let $n_0 := \sum \mathbb{1}_{c_i = 0}$
let $n_1 := n - n_0 = \sum \mathbb{1}_{c_i = 1}$

$$\ell(k, \lambda ; \vec{y}, \vec{c}, t_f) = n_0 k \ln(\lambda) + n_0 \ln(k) + (k-1) \sum_{\{i : c_i = 0\}} \ln(y_i) - \lambda^k \sum_{\{i : c_i = 0\}} y_i^k - n_1 \lambda^k t_f^k \overset{\text{set}}{=} 0$$

Solving this numerically yields $\hat{\lambda}^{MLE}, \hat{k}^{MLE} \Rightarrow \hat{\theta}^{MLE} \sim N(\theta, \text{var})$

What is the variance? Need Fisher Info Matrix (beyond scope of course) for λ, k, then use multivariate Delta Method. Packages in R do this for you.

Alternative: Use E-M algorithm

HW: t_f differs for each i
HW: data analysis ...
HW: E-M alg as alternative

⓪ Begin with guesses for y_i s.t. $c_i = 0$
① Compute $\hat{k}^{MLE}, \hat{\lambda}^{MLE}$ (M-step)
② Compute

$E[Y_i \mid Y_i > t_f]$ which will be the same for all i (E-step)

③ Repeat steps 1-2 until convergence