$$\vec{Y} = X\vec{\beta} + \vec{\varepsilon}$$

$$\downarrow$$

$$\vec{\varepsilon} \sim N_n(\vec{0}_n, \sigma^2 I_n)$$

$\Rightarrow \varepsilon_1, \ldots, \varepsilon_n \overset{iid}{\sim} N(0,1)$

D&P for misspecification error + ignorance error

Lec 10 MATH 343

$$\vec{Y} = X\vec{\beta} + \vec{\varepsilon} \leftarrow \text{Assume}$$

$\vec{Y}$: random through $\vec{\varepsilon}$; X: fixed constants; $\vec{\beta}$: unknown parameter

$$\Rightarrow \vec{B} = (X^T X)^{-1} X^T \vec{Y} \sim N_{p+1}\left(\vec{\beta}, \sigma^2 (X^T X)^{-1}\right) \Rightarrow B_j \sim N\left(\beta_j, \sigma^2 (X^T X)^{-1}_{jj}\right)$$

$$\Rightarrow \frac{B_j - \beta_j}{\sigma\sqrt{(X^T X)^{-1}_{jj}}} \sim N(0,1) \quad \forall j$$

But σ is unknown!

$\Rightarrow \hat{\vec{Y}} = X\vec{B}$

$$\Rightarrow \vec{Y} = X\vec{B} + \vec{E} = \hat{\vec{Y}} + \vec{E}$$

$\vec{E}$: D&P for $\vec{\varepsilon}$, the residuals in the regression

Left over from lec 9:

$$\Rightarrow \vec{E} = \vec{Y} - X\vec{B} \underset{\text{geometrically}}{=} (I-H)\vec{Y} = (I-H)(X\vec{\beta} + \vec{\varepsilon}) = \underbrace{(I-H)X\vec{\beta}}_{\vec{0}_{(p+1)\times(p+1)}} + (I-H)\vec{\varepsilon}$$

$$= (I-H)\vec{\varepsilon} \sim N_n\left(\vec{0}_n, (I-H)\sigma^2 I_n (I-H)^T\right) = N_n\left(\vec{0}, \sigma^2(I-H)\right)$$

$$\Rightarrow E_i \sim N\left(0, \sigma^2(1 - H_{ii})\right)$$

From MATH 340: $\vec{U} \sim N_n(\vec{0}_n, \Sigma)$, $\vec{\mu} \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n} \Rightarrow \vec{\mu} + A\vec{U} \sim N_m(\vec{\mu}, A\Sigma A^T)$

$$\hat{\vec{Y}} = X\vec{B} = H\vec{Y} = H(X\vec{\beta} + \vec{\varepsilon}) = HX\vec{\beta} + H\vec{\varepsilon} = X\vec{\beta} + H\vec{\varepsilon} \sim N_n(X\vec{\beta}, H\sigma^2 I_n H) = N_n(X\vec{\beta}, \sigma^2 H) \Rightarrow \hat{Y}_i \sim N(\vec{x}_{i\cdot}\vec{\beta}, \sigma^2 H_{ii})$$

In 340 we showed that if $\vec{Z} \sim N_n(\vec{0}, \sigma^2 I_n)$

$$\Rightarrow \vec{Z}^T \vec{Z} \sim \chi^2_n$$

From last class...

$$\vec{Z}^T \vec{Z} = \frac{1}{\sigma^2}\vec{\varepsilon}^T \underbrace{H}_{B_1} \vec{\varepsilon} + \frac{1}{\sigma^2}\vec{\varepsilon}^T \underbrace{(I-H)}_{B_1} \vec{\varepsilon}$$

Since $B_1 + B_2 = I_n$ and $\text{rank}(B_1) + \text{rank}(B_2) = (p+1) + (n-(p+1)) = n$, we can use Cochran's thm

$$\Rightarrow \frac{1}{\sigma^2}\vec{\varepsilon}^T H \vec{\varepsilon} \sim \chi^2_{p+1} \text{ indep of } \frac{1}{\sigma^2}\vec{\varepsilon}^T (I-H)\vec{\varepsilon} \sim \chi^2_{n-(p+1)}$$

last class $\quad$ measures all error

$$\frac{1}{\sigma^2}\vec{\varepsilon}^{\,T}(I-H)\vec{\varepsilon} = \underset{\downarrow}{\cdots\cdots} = \frac{1}{\sigma^2}\|\vec{E}\|^2 \sim \chi^2_{n-(p+1)}$$

$$\frac{1}{\sigma^2}\vec{\varepsilon}^{\,T}H\vec{\varepsilon} = \frac{1}{\sigma^2}\vec{\varepsilon}^{\,T}HH\vec{\varepsilon} = \frac{1}{\sigma^2}\vec{\varepsilon}^{\,T}H^T H\vec{\varepsilon} = \frac{1}{\sigma^2}(H\vec{\varepsilon})^T(H\vec{\varepsilon}) = \frac{1}{\sigma^2}\|H\vec{\varepsilon}\|^2$$

$$= \frac{1}{\sigma^2}\|H(\vec{Y}-X\vec{\beta})\|^2 = \frac{1}{\sigma^2}\|H\vec{Y}-HX\vec{\beta}\|^2 = \frac{1}{\sigma^2}\|\hat{\vec{Y}}-X\vec{\beta}\|^2$$

measures estimation error

$$= \frac{1}{\sigma^2}\|\underbrace{X\vec{B}}_{\hat{Y}} - \underbrace{X\vec{\beta}}_{E[Y]}\|^2 = \frac{1}{\sigma^2}\|X(\vec{B}-\vec{\beta})\|^2 \sim \chi^2_{p+1}$$

As $n\to\infty$ $\quad \vec{E}\to\vec{\varepsilon},\ \vec{B}\to\vec{\beta} \Rightarrow \|X(\vec{B}-\vec{\beta})\|^2 \to 0$

misspecification + ignorance error $\qquad$ estimation error disappears

$\Rightarrow \vec{E}$ and $\vec{B}$ are independent $\Rightarrow \|\vec{E}\|^2$ and B_j are independent

Consider the following estimator

from last class

$$\frac{\left.\dfrac{B_j-\beta_j}{\sigma\sqrt{(X^TX)^{-1}_{jj}}}\right\}\sim N(0,1)}{\sqrt{\left.\dfrac{\frac{1}{\sigma^2}\|\vec{E}\|^2}{n-(p+1)}\right\}\sim\chi^2_{n-(p+1)}}} \sim T_{n-(p+1)}$$

indep. $\qquad$ we just proved this $\qquad$ Student's T dist

$$= \frac{\dfrac{\cancel{1}}{\cancel{\sigma}}\dfrac{B_j-\beta_j}{\sqrt{(X^TX)^{-1}_{jj}}}}{\dfrac{\cancel{1}}{\cancel{\sigma}}\sqrt{\dfrac{\|\vec{E}\|^2}{n-(p+1)}}} \sim T_{n-(p+1)}$$

Hence, estimated test statistic is then

$$\frac{\dfrac{b_j-\beta_j}{\sqrt{(X^TX)^{-1}_{jj}}}}{\underbrace{\sqrt{\dfrac{SSE}{n-(p+1)}}}_{RMSE}} = \frac{b_j-\beta_j}{s_e\sqrt{(X^TX)^{-1}_{jj}}}$$

Assume level of test is α.

Usually the inference is for $H_1: \beta_j \neq 0 \Leftrightarrow H_0: \beta_j = 0$ why?

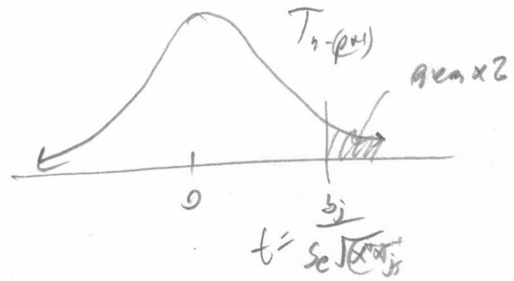
Because if $\beta_j = 0$, the j^{th} feature doesn't affect the response (linearly)

(which usually means the overall relationship is not too interesting).

In this case,

$$\frac{b_j}{s_e \sqrt{(X^TX)^{-1}_{jj}}} \in \left[\pm t_{1-\frac{\alpha}{2}, n-(p+1)}\right] \xRightarrow{\text{inverting the test}} CI_{\beta_j, 1-\alpha} = \left[b_j \pm t_{1-\frac{\alpha}{2}, n-(p+1)} \cdot s_e \sqrt{(X^TX)^{-1}_{jj}}\right]$$

How to get pval?



pval =

$$2P\left(|T_{n-(p+1)}| > \left|\frac{b_j}{s_e\sqrt{(X^TX)^{-1}_{jj}}}\right|\right)$$

Consider a new $\vec{x}_*$ and you want to predict its expected response $h^*(\vec{x}_*) = \vec{x}_* \vec{\beta}$. This is the average of many predictions of $\vec{x}_*$ since the noise mean is zero.

Let $\mu_* := \vec{x}_* \vec{\beta}$

$$\hat{Y}_* = \vec{x}_* \vec{B} \Rightarrow \hat{Y}_* \sim N\big(\underbrace{\vec{x}_* \vec{\beta}}_{\mu_*},\ \sigma^2\, \vec{x}_* (X^T X)^{-1} \vec{x}_*^T\big)$$

$$\vec{B} \sim N(\vec{\beta}, \sigma^2 (X^T X)^{-1}) \nearrow$$

$$\Downarrow$$

$$\frac{\hat{Y}_* - \mu_*}{\sigma \sqrt{\vec{x}_* (X^T X)^{-1} \vec{x}_*^T}} \sim N(0,1)$$

Again, σ is unknown so consider

$$\frac{\left.\dfrac{\hat{Y}_* - \mu_*}{\sigma \sqrt{\vec{x}_* (X^T X)^{-1} \vec{x}_*^T}}\right\} N(0,1)}{\sqrt{\left.\dfrac{\frac{1}{\sigma^2}\|\vec{E}\|^2}{n-(p+1)}\right\} \chi^2_{n-(p+1)}}} \sim T_{n-(p+1)}$$

indep, since $\hat{Y}_*$ is a function of $\vec{B}$

$$= \frac{\dfrac{\hat{Y}_* - \mu_*}{\sqrt{\vec{x}_* (X^T X)^{-1} \vec{x}_*^T}}}{\sqrt{\dfrac{\|\vec{E}\|^2}{n-(p+1)}}} \sim T_{n-(p+1)}$$

the test statistic for $H_a: \mu_* \neq \theta$ is

If $\dfrac{\hat{Y}_* - \theta}{s_e \sqrt{\vec{x}_* (X^T X)^{-1} \vec{x}_*^T}} \in \left[\pm t_{1-\frac{\alpha}{2},\, n-(p+1)}\right] \Rightarrow$ Retain H_0

which can be used to test. Inverting the test yields

$$CI_{\mu_*, 1-\alpha} = \left[\hat{y}_* \pm t_{1-\frac{\alpha}{2},\, n-(p+1)} \cdot s_e \sqrt{\vec{x}_* (X^T X)^{-1} \vec{x}_*^T}\right]$$