

Math 343 / 643 Spring 2025

Midterm Examination Two

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Full Name _____

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Instructions

This exam is 75 minutes (variable time per question) and closed-book. You are allowed **one** page (front and back) of a “cheat sheet”, blank scrap paper (provided by the proctor) and a graphing calculator (which is not your smartphone). Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 Consider the following full-rank design matrix:

$$\mathbf{X} := [\mathbf{1}_n \mid \mathbf{x}_{\cdot 1} \mid \dots \mid \mathbf{x}_{\cdot p}] = \begin{bmatrix} \mathbf{x}_{1\cdot} \\ \vdots \\ \mathbf{x}_{n\cdot} \end{bmatrix}$$

with column indices $0, 1, \dots, p$ and row indices $1, 2, \dots, n$. And let \mathbf{H} be the orthogonal projection matrix onto the column space of \mathbf{X} . We assume also a continuous (real-valued) response model which is linear in these measurements, i.e. $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. For the error term, we assume the “core assumption”,

$$\boldsymbol{\varepsilon} \sim \mathcal{N}_n(\mathbf{0}_n, \sigma^2 \mathbf{I}_n) \quad \text{where } \sigma^2 > 0.$$

Consider the following estimator for $\boldsymbol{\beta}$: $\mathbf{B} := (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}$ and let $\hat{\mathbf{Y}} := \mathbf{X} \mathbf{B}$ and $\mathbf{E} := \mathbf{Y} - \hat{\mathbf{Y}}$.

(a) [5 pt / 5 pts] Circle all of the following which are non-degenerate random variables.

$$n, \quad p, \quad \mathbf{X}, \quad \mathbf{x}_{\cdot 1}, \quad \mathbf{x}_{n\cdot}, \quad \mathbf{H}, \quad \mathbf{Y}, \quad \boldsymbol{\beta}, \quad \boldsymbol{\varepsilon}, \quad \sigma^2, \quad \mathbf{I}_n, \quad \mathbf{B}, \quad \hat{\mathbf{Y}}, \quad \mathbf{E}$$

(b) [3 pt / 8 pts] Of the random variables in the previous question, which two are independent of each other? No need to prove this.

$$\mathbf{B}, \mathbf{E}$$

(c) [5 pt / 13 pts] Derive the distribution of \mathbf{B} with only what is in the problem header, the fact about multivariate normal distributions from 340 and linear algebra manipulations. Show each step.

$$\begin{aligned} \mathbf{B} &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T (\mathbf{X} \boldsymbol{\beta} + \boldsymbol{\varepsilon}) = \boldsymbol{\beta} + (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \boldsymbol{\varepsilon} \\ &\sim \mathcal{N}_{p+1} \left(\boldsymbol{\beta}, (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \sigma^2 \mathbf{I}_n \left((\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \right)^T \right) \\ &= \mathcal{N}_{p+1} \left(\boldsymbol{\beta}, \sigma^2 (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{X} \left((\mathbf{X}^T \mathbf{X})^{-1} \right)^T \right) \\ &= \mathcal{N}_{p+1} \left(\boldsymbol{\beta}, \sigma^2 \left((\mathbf{X}^T \mathbf{X})^{-1} \right)^T \right) \\ &= \mathcal{N}_{p+1} \left(\boldsymbol{\beta}, \sigma^2 \left((\mathbf{X}^T \mathbf{X})^T \right)^{-1} \right) \\ &= \mathcal{N}_{p+1} \left(\boldsymbol{\beta}, \sigma^2 (\mathbf{X}^T \mathbf{X})^{-1} \right) \end{aligned}$$

(d) [6 pt / 19 pts] Prove misspecification error vanishes as $n \rightarrow \infty$.

Misspecification error is $g(\mathbf{x}) - h^*(\mathbf{x})$ where $h^*(\mathbf{x}) = \mathbf{x}\boldsymbol{\beta}$ by assumption and $g(\mathbf{x}) = \mathbf{x}\mathbf{b}$ because we are using OLS thus misspecification error is $\mathbf{x}(\boldsymbol{\beta} - \mathbf{b})$. Over the whole dataset the misspecification errors are $\mathbf{x}_1(\boldsymbol{\beta} - \mathbf{b}), \dots, \mathbf{x}_n(\boldsymbol{\beta} - \mathbf{b})$. One way to holistically measure all misspecification errors is to sum their squares, i.e, $\|\mathbf{X}(\boldsymbol{\beta} - \mathbf{b})\|^2$ as seen on the homework.

Now we prove this holistic measure goes to zero as n increases. First note that \mathbf{b} is a realization of \mathbf{B} , the OLS estimator which is also the MLE for $\boldsymbol{\beta}$. From 341, the monster theorem stated that MLE's are consistent. Thus in our setting, $\mathbf{B} \xrightarrow{p} \boldsymbol{\beta}$ and thus by the multivariate CMT from 340, $\mathbf{B} - \boldsymbol{\beta} \xrightarrow{p} \mathbf{0}_{p+1}$. Now we apply this result to our holistic measure. By the multivariate CMT, $\|\mathbf{X}(\boldsymbol{\beta} - \mathbf{b})\|^2 \xrightarrow{p} \|\mathbf{X}\mathbf{0}_{p+1}\|^2 = 0$.

Problem 2 Consider the Boston Housing Data which has $n = 506$ and response `medv` with $\bar{y} = 22.53$ and $s_y = 9.20$. We consider modeling `medv` using OLS on `zn` + `rm` + `nox` + `dis` + `lstat`, all continuous (non-categorical) features. Below is the $(\mathbf{X}^T \mathbf{X})^{-1}$ where \mathbf{X} is the design matrix:

	(Intercept)	zn	rm	nox	dis	lstat
(Intercept)	0.58000	4.4e-04	-5.1e-02	-2.7e-01	-1.8e-02	-3.0e-03
zn	0.00044	6.9e-06	-4.5e-05	-7.1e-05	-5.1e-05	-2.0e-07
rm	-0.05100	-4.5e-05	6.9e-03	7.5e-04	6.3e-04	4.4e-04
nox	-0.27000	-7.1e-05	7.5e-04	4.2e-01	1.5e-02	-1.9e-03
dis	-0.01800	-5.1e-05	6.3e-04	1.5e-02	1.5e-03	4.3e-05
lstat	-0.00300	-2.0e-07	4.4e-04	-1.9e-03	4.3e-05	8.9e-05

The RMSE for this regression is 5.289 and here are the slope estimates:

(Intercept)	zn	rm	nox	dis	lstat
16.14	0.06	4.44	-15.20	-1.44	-0.66

Assume the “core assumption” (see Problem 1 for its definition) except in (e,f,l,m) which make explicit a new assumption.

(a) [2 pt / 21 pts] Consider creating a $\hat{CI}_{\beta_{\text{nox}}, 95\%}$, the confidence interval for the true slope parameter of the variable `nox`. Which degrees of freedom value would you use to lookup the appropriate t value's quantile?

$$df_{\text{error}} := n - (p + 1) = 506 - 6 = 500$$

- (b) [5 pt / 26 pts] Compute $\hat{CI}_{\beta_{\text{nox}}, 95\%}$ to the nearest two digits. Regardless of the truly appropriate t value, use 1.96 as the t value.

$$\begin{aligned}\hat{CI}_{\beta_{\text{nox}}, 1-\alpha} &= \left[b_{\text{nox}} \pm t_{df_{\text{error}}, 1-\alpha/2} \cdot s_e \sqrt{(\mathbf{X}^T \mathbf{X})_{\text{nox}, \text{nox}}^{-1}} \right] \\ \hat{CI}_{\beta_3, 95\%} &= \left[b_3 \pm 1.96 \cdot s_e \sqrt{(\mathbf{X}^T \mathbf{X})_{4,4}^{-1}} \right] \\ &= \left[-15.20 \pm 1.96 \cdot 5.289 \sqrt{0.42} \right] = [-21.92, -8.48]\end{aligned}$$

- (c) [1 pt / 27 pts] The confidence interval in the previous question is... circle one:
☒ exact / ☐ approximate
- (d) [1 pt / 28 pts] Based on your confidence interval from the previous question, the null hypothesis that $\beta_{\text{nox}} = 0$ would be ... circle one:
☒ rejected / ☐ retained
- (e) [5 pt / 33 pts] Assume the errors are independent, mean centered and homoskedastic but now assume they are *not* normally distributed. Create a $\hat{CI}_{\beta, 95\%}$ for the variable **nox** to the nearest two digits.

$$[-21.92, -8.48]$$

- (f) [1 pt / 34 pts] The confidence interval in the previous question is... circle one:
☐ exact / ☒ approximate
- (g) [5 pt / 39 pts] Justify and record your decision for the test of $H_0 : \beta_{\text{rm}} = 3$, a test on the slope parameter for the variable **rm**. Regardless of the truly appropriate t value, use 1.96 as the t value.

$$\hat{t} := \frac{b_{\text{rm}} - \beta_{\text{rm}}}{s_e \cdot \sqrt{(\mathbf{X}^T \mathbf{X})_{\text{rm}, \text{rm}}^{-1}}} = \frac{4.44 - 3}{5.289 \cdot \sqrt{0.0069}} = 3.277 > 1.96 \Rightarrow \text{Reject } H_0$$

- (h) [5 pt / 44 pts] Compute R_{adj}^2 to the nearest two digits using the following calculations:

$$s_e := \sqrt{\frac{SSE}{df_{\text{error}}}} \Rightarrow SSE = df_{\text{error}} \cdot s_e^2 = 500 \cdot 5.289^2 = 13986.76$$

$$SST := \sum_{i=1}^n (y_i - \bar{y})^2 = (n-1) \cdot s_y^2 = 505 \cdot 9.20^2 = 42743.2$$

$$R_{adj}^2 := 1 - \frac{n-1}{df_{\text{error}}} \frac{SSE}{SST} = \frac{505}{500} \frac{13986.76}{42743.2} = 0.33$$

Below is the first six rows and six columns of the \mathbf{H} matrix. There are rownames and colnames displayed to help with finding entries (e.g., $\mathbf{H}_{2,4} = 0.0076$).

	1	2	3	4	5	6
1	0.0053	0.0020	0.0035	0.0039	0.0024	0.0036
2	0.0020	0.0058	0.0065	0.0076	0.0076	0.0072
3	0.0035	0.0065	0.0100	0.0110	0.0110	0.0085
4	0.0039	0.0076	0.0110	0.0130	0.0130	0.0110
5	0.0024	0.0076	0.0110	0.0130	0.0140	0.0110
6	0.0036	0.0072	0.0085	0.0110	0.0110	0.0110

- (i) [5 pt / 49 pts] Estimate the probability the residual for the fourth observation in the boston housing dataset will be greater than 5 as best as you can.

$$E_4 \sim \mathcal{N}(0, \sigma^2(1 - h_{4,4})) \Rightarrow \frac{E_4}{\sigma \sqrt{1 - h_{4,4}}} \sim \mathcal{N}(0, 1) \Rightarrow \frac{E_4}{s_e \sqrt{1 - h_{4,4}}} \dot{\sim} \mathcal{N}(0, 1)$$

$$\mathbb{P}(E_4 > 5) = \mathbb{P}\left(\frac{E_4}{s_e \sqrt{1 - h_{4,4}}} > \frac{5}{s_e \sqrt{1 - h_{4,4}}}\right) \approx \mathbb{P}\left(Z > \frac{5}{5.289 \sqrt{1 - 0.0130}}\right)$$

$$= \mathbb{P}(Z > 0.95) \approx 16\%$$

- (j) [6 pt / 55 pts] The predicted value for the first observation is $\hat{y}_1 = 29.15$. Find a $\hat{CI}_{y_1, 95\%}$ where y_1 is the response value for a new census tract with the same measurements as \mathbf{x}_1 . to the nearest two digits. Regardless of the truly appropriate t value, use 1.96 as the t value.

$$\begin{aligned}
 \hat{CI}_{y_1, 95\%} &= \left[\hat{y}_1 \pm t_{1-\alpha/2, n-(p+1)} \cdot s_e \sqrt{1 + \mathbf{x}_1^\top (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{x}_1} \right] \\
 &= \left[\hat{y}_1 \pm t_{1-\alpha/2, n-(p+1)} \cdot s_e \sqrt{1 + h_{1,1}} \right] \\
 &= \left[29.15 \pm 1.96 \cdot 5.289 \sqrt{1 + 0.0053} \right] = [18.76, 39.54]
 \end{aligned}$$

We now model `medv` using `rm + lstat` via an OLS. The RMSE for this regression is 5.540 and here are the slope estimates:

(Intercept)	rm	lstat
-1.36	5.09	-0.64

- (k) [7 pt / 62 pts] Calculate the F-statistic for $H_0 : \beta_{\text{zn}} = \beta_{\text{nox}} = \beta_{\text{dis}} = 0$ to the nearest two digits.

$$\hat{f} := \frac{\frac{SSE_A - SSE}{k}}{\frac{SSE}{df_{\text{error}}}} = \frac{\frac{15437.87 - 13986.76}{3}}{\frac{13986.76}{500}} = 17.29$$

The value of k is the number of features we are setting to zero in H_0 which is 3. The value of SSE we take from part (h). To obtain the value of SSE_A , see the calculation in part (h). $SSE_A = df_A s_{e_A}^2$ where the quantities on the rhs are now applicable to the reduced model with features in set A . Following part (a), $df_A = n - ((p - k) + 1) = 506 - ((5 - 3) + 1) = 503$. And $s_{e_A}^2$ can be found in the text above this question. Thus, $SSE_A = 503 \cdot 5.54^2 = 15437.87$.

Below is $(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \hat{\mathbf{D}} \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1}$, a matrix where \mathbf{X} is the design matrix and $\hat{\mathbf{D}}$ is the diagonal matrix with the residuals squared along its diagonal.

	(Intercept)	rm	lstat
(Intercept)	29.20	-4.14	-0.26
rm	-4.14	0.59	0.03
lstat	-0.26	0.03	0.00

- (l) [5 pt / 67 pts] Assume the errors are independent, mean centered but neither homoskedastic nor normally distributed. Create a $\hat{CI}_{\beta_{\text{rm}}, 95\%}$, the confidence interval for the true slope parameter of the variable **rm** to the nearest two digits.

$$\begin{aligned}\hat{CI}_{\beta_{\text{rm}}, 1-\alpha} &= \left[b_{\text{rm}} \pm z_{1-\alpha/2} \sqrt{\left((\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \hat{\mathbf{D}} \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} \right)_{\text{rm}, \text{rm}}} \right] \\ \hat{CI}_{\beta_1, 95\%} &= \left[b_1 \pm 1.96 \sqrt{\left((\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \hat{\mathbf{D}} \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} \right)_{2,2}} \right] \\ &= \left[5.09 \pm 1.96 \sqrt{0.59} \right] = [3.58, 6.60]\end{aligned}$$

- (m) [1 pt / 68 pts] The confidence interval in the previous question is... circle one:
exact / approximate

Problem 3 Consider a subset of the vocab data in the `carData` package. The response is a person's score on a vocabulary test. This score ranges in $\{0, 1, 2, \dots, 10\}$ and features: **gender** (categorical: male/female), **nativeBorn** (categorical: yes/no), **age** (continuous: measured in years) and **educ** (continuous: measured in years). We will use a negative binomial glm with the standard exponential link-to-linear function for its mean. Below is the output:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	0.7761238	0.0165403	46.923	< 2e-16 ***
gendermale	-0.0267524	0.0049960	-5.355	8.57e-08 ***
nativeBornyes	0.1603976	0.0094713	16.935	< 2e-16 ***
age	0.0021438	0.0001438	14.907	< 2e-16 ***
educ	0.0582323	0.0008373	69.548	< 2e-16 ***
Theta: 172454				
Std. Err.: 143423				
2 x log-likelihood: -115304.3				

- (a) [5 pt / 73 pts] Is there any reason why we should not model this response metric using the negative binomial model with mean log-linear in the covariates?

The response metric doesn't have the support of a true count model as it only ranges from 0 to 10. The model may give nonsensical predictions (i.e., vocabulary scores > 10). Thus, the inference will also be suspect.

Despite what you wrote in (a), we will ignore any concerns about the appropriateness of this model going forward.

- (b) [5 pt / 78 pts] Considering all other covariate values the same, what would be the predicted *percent* difference in mean score of a male versus a female to the nearest two digits?

Since $\hat{y} = e^{b_0} e^{b_1 x_1} \dots e^{b_p x_p}$, the prediction is affected by not an addition, but a multiple, $e^{b_j x_j}$, thus the percent effect is $(e^{b_j x_j} - 1) \times 100$. For the male-vs-female coefficient, we then get

$$(e^{-0.0267524} - 1) \times 100 = -2.64\%$$

- (c) [6 pt / 84 pts] Compute $\hat{CI}_{\beta_{\text{educ}}, 95\%}$, the confidence interval for the slope parameter within the link function for the variable `educ` to the nearest four digits.

$$\begin{aligned} \hat{CI}_{\beta_{\text{educ}}, 1-\alpha} &= [b_{\text{educ}} \pm z_{1-\alpha/2} \cdot s_{b_{\text{educ}}}] \\ \hat{CI}_{\beta_4, 95\%} &= [b_4 \pm 1.96 \cdot s_{b_4}] \\ &= [0.0582323 \pm 1.96 \cdot 0.0008373] = [0.0566, 0.0599] \end{aligned}$$

- (d) [1 pt / 85 pts] The confidence interval in the previous question is... circle one:
 exact / approximate

- (e) [6 pt / 91 pts] Predict the vocabulary score of a female, foreign-born, age 25 with 17yr of education. Round the score to the nearest whole number.

$$\begin{aligned}
 \hat{y} &= \text{round} \left(e^{b_0} e^{b_1 x_1} \cdot \dots \cdot e^{b_p x_p} \right) \\
 &= \text{round} \left(e^{b_0} e^{b_1(0)} e^{b_2(0)} e^{b_3(25)} e^{b_4(17)} \right) \\
 &= \text{round} \left(e^{0.7761238} e^{0.0021438(25)} e^{0.0582323(17)} \right) \\
 &= \text{round}(6.169809) = 6
 \end{aligned}$$

We now run the same model but this time omitting features `gender` and `nativeBorn`. Below is the output

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	0.9130409	0.0139108	65.64	<2e-16 ***
age	0.0022436	0.0001438	15.61	<2e-16 ***
educ	0.0578375	0.0008323	69.49	<2e-16 ***
Theta: 173175				
Std. Err.: 146404				
2 x log-likelihood: -115635.4				

Here are some values of the inverse CDF of the χ^2_{df} distribution:

df	Probability less than the critical value				
	0.90	0.95	0.975	0.99	0.999
1	2.706	3.841	5.024	6.635	10.828
2	4.605	5.991	7.378	9.210	13.816
3	6.251	7.815	9.348	11.345	16.266
4	7.779	9.488	11.143	13.277	18.467
5	9.236	11.070	12.833	15.086	20.515
6	10.645	12.592	14.449	16.812	22.458
7	12.017	14.067	16.013	18.475	24.322

- (f) [2 pt / 93 pts] For the test of $H_0 : \beta_{\text{gender}} = \beta_{\text{nativeBorn}} = 0$ at $\alpha = 1\%$, would would be the critical value of the likelihood ratio test that the test statistic is compared to?

The degrees of freedom of this test is 2 because we are knocking out 2 features. Since $\alpha = 1\%$, we are looking for the 99%ile of the χ^2_2 which according we find in the table above on the second row and the fourth column: 9.210.

- (g) [7 pt / 100 pts] Run the test of $H_0 : \beta_{\text{gender}} = \beta_{\text{nativeBorn}} = 0$ at $\alpha = 1\%$ and record your decision and write one sentence that interprets the result of the decision.

Since $\hat{\Lambda} = 2 \ln(\mathcal{L}_{\text{full}}) - 2 \ln(\mathcal{L}_{\text{reduced}}) = -115304.3 - -115635.4 = 331.1 > \chi^2_{2,99\%} = 9.210$, we reject H_0 . We can conclude that **gender** and **nativeBorn** are important in predicting vocabulary score in the context of the other variables and assuming the negative binomial model log-linear in the covariates.