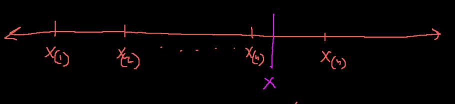
Let's begin with



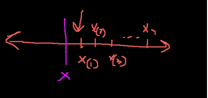
$$F_{X_{(n)}}(x) := P(X_{(n)} \le x) = P(X_1 \le x, X_2 \le x_2, \dots, X_n \le x)$$

$$\overset{X_1,\dots,X_n \overset{ind}{\sim}}{=} P(X_1 \le x) \cdot P(X_2 \le x) \cdot \ldots \cdot P(X_n \le x)$$

$$= F_{X_1}(x) F_{X_2}(x) \cdots F_{X_n}(x) = \prod_{i=1}^{n} F_{X_i}(x)$$

$$\overset{X_1,\dots,X_n \overset{iid}{\sim}}{=} F_X(x)^n$$

Let's derive the CDF then PDF of the minimum.



$$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x)$$

$$= 1 - P(X_1 > x, X_2 > x, \dots, X_n > x)$$

$$\overset{X_1,\dots,X_n \overset{ind}{\sim}}{=} 1 - P(X_1 > x) P(X_2 > x) \cdot \ldots \cdot P(X_n > x)$$

$$= 1 - (1 - F_{X_1}(x))(1 - F_{X_2}(x)) \cdots (1 - F_{X_n}(x))$$

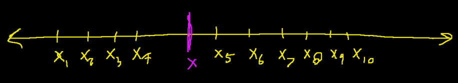
$$\overset{X_1,\dots,X_n \overset{iid}{\sim}}{=} 1 - (1 - F_X(x))^n$$

For iid only...

$$f_{X_{(n)}}(x) = \frac{d}{dx}\left[F_X(x)^n\right] = n f_X(x) F_X(x)^{n-1}$$

$$f_{X_{(1)}}(x) = \frac{d}{dx}\left[1 - (1 - F_X(x))^n\right] = n f_X(x)(1 - F_X(x))^{n-1}$$

Let's derive the CDF and then PDF of the kth order statistic.
Let n =10, k = 4 as an example.



What is the probability of such an event occuring?

$$P(X_1 \le x, \dots, X_4 \le x, X_5 > x, \dots, X_{10} > x)$$

$$\overset{X_1,\dots,X_n \overset{ind}{\sim}}{=} \prod_{i=1}^{4} P(X_i \le x) \prod_{i=5}^{10} P(X_i > x) = \prod_{i=1}^{4} F_{X_i}(x) \prod_{i=5}^{10} 1 - F_{X_i}(x)$$

$$\overset{X_1,\dots,X_n \overset{iid}{\sim}}{=} F_X(x)^4 (1 - F_X(x))^6$$

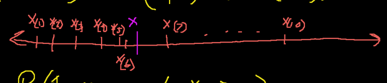
What is P(any 4 x's ≤ x and the other 6 x's being > x)

$$= \sum P(X_{S_1} \le x, \dots, X_{S_4} \le x, X_{S_5} > x, \dots, X_{S_{10}} > x)$$

all ways S of choosing 4 from
{1, 2, ..., 10}

$$\overset{X_1,\dots,X_n \overset{ind}{\sim}}{=} \sum_{\text{all } S} \prod_{i=1}^{4} F_{X_{S_i}}(x) \prod_{i=5}^{10} 1 - F_{X_{S_i}}(x)$$

$$\overset{X_1,\dots,X_n \overset{iid}{\sim}}{=} \sum_{\text{all } S} F_X(x)^4 (1 - F_X(x))^6 = \binom{10}{4} F_X(x)^4 (1 - F_X(x))^6$$



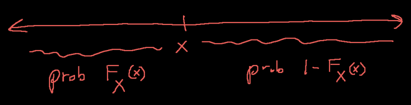
$$F_{X_{(4)}}(x) = P(X_{(4)} \le x) = P(4 \text{ x's} \le x,\ 6 \text{ x's} > x)$$
$$+ P(5 \text{ x's} \le x,\ 5 \text{ x's} > x)$$
$$+ P(6 \text{ x's} \le x,\ 4 \text{ x's} > x)$$
$$+ \dots$$
$$+ P(10 \text{ x's} \le x)$$

$$\overset{X_1,\dots,X_n \overset{iid}{\sim}}{=} \sum_{j=4}^{10} \binom{10}{j} F_X(x)^j (1 - F_X(x))^{n-j}$$

This is easily generalized for arbitrary n and k for the iid case:

$$F_{X_{(k)}}(x) = \sum_{j=k}^{n} \binom{n}{j} F_X(x)^j (1 - F_X(x))^{n-j} \overset{k=n}{=} F_X(x)^n \checkmark$$

$$\overset{k=1}{=} 1 - \binom{n}{0} F_X(x)^0 (1 - F_X(x))^n = 1 - (1 - F_X(x))^n \checkmark$$



$$f_{X_{(k)}}(x) = \frac{d}{dx}\left[\sum_{j=k}^{n} \binom{n}{j} F(x)^j (1 - F(x))^{n-j}\right] = \sum_{j=k}^{n} \binom{n}{j} \frac{d}{dx}\left[F(x)^j (1 - F(x))^{n-j}\right]$$

$$= \sum_{j=k}^{n} \binom{n}{j} \left(F(x)^j (n-j)(-f(x))(1 - F(x))^{n-j-1} + (1 - F(x))^{n-j} j f(x) F(x)^{j-1} \right)$$

If j=n, this is zero

$$= \sum_{j=k}^{n} \underbrace{\binom{n}{j} j}_{\frac{n!}{(j-1)!(n-j)!}} f(x) F(x)^{j-1} (1 - F(x))^{n-j} - \sum_{j=k}^{n} \underbrace{\binom{n}{j} (n-j)}_{\frac{n!}{j!(n-j-1)!}} f(x) F(x)^j (1 - F(x))^{n-j-1}$$

$$= \sum_{j=k}^{n} \frac{n!}{(j-1)!(n-j)!} j f(x) F(x)^{j-1} (1 - F(x))^{n-j} - \sum_{j=k}^{n-1} \frac{n!}{j!(n-(j+1))!} f(x) F(x)^j (1 - F(x))^{n-(j+1)}$$

let $\ell = j+1 \Leftrightarrow j = \ell - 1$

$$= \text{'' '' '' '' '' ''} - \sum_{\ell=k+1}^{n} \frac{n!}{(\ell-1)!(n-\ell)!} f(x) F(x)^{\ell-1} (1 - F(x))^{n-\ell}$$

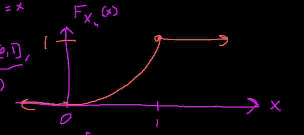
Note that l and j are free variables which means that these two terms are exactly the same except the sum in the first term goes from k, ..., n and in the second term it goes from k+1, ..., n so the difference is only where j = k

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F(x)^{k-1} (1 - F(x))^{n-k} \overset{k=n}{=} n f(x) F(x)^{n-1} \checkmark$$

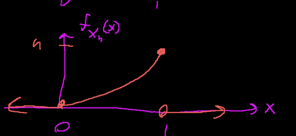
$$\overset{k=1}{=} n f(x)(1 - F(x))^{n-1} \checkmark$$

$$X_1, \dots, X_n \overset{iid}{\sim} U(0,1) = \underbrace{\mathbb{1}_{x \in [0,1]}}_{f(x)}, \quad F(x) = x$$

$$F_{X_{(n)}}(x) = F(x)^n = x^n$$



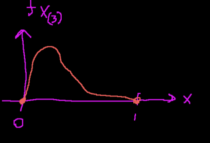
$$f_{X_n}(x) = n f(x) F(x)^{n-1} = n x^{n-1} \mathbb{1}_{x \in [0,1]}$$



$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} \mathbb{1}_{x \in [0,1]}$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\Gamma(n-k+1)} x^{k-1} (1-x)^{n-k+1-1} \mathbb{1}_{x \in [0,1]}$$

$$= \text{Beta}(k, n-k+1)$$



We will return to the beta rv and explore it more fully after we do another topic.

X ~ Erlang(k_1, λ) indep of Y ~ Erlang(k_2, λ) ⇒ X + Y ~ Erlang(k_1+k_2, λ)

You would then conjecture that
X ~ Gamma(α_1, β) indep of Y ~ Gamma(α_2, β)
⇒ X + Y ~ Gamma(α_1 + α_2 , β)

This isn't really easy to prove without another concept called "kernels".
