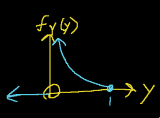


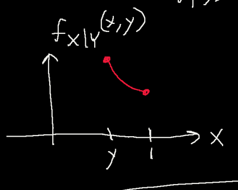
$f_X(x),\ f_{Y|X}(x,y)$, $f_{X,Y}(x,y)$, $f_Y(y)$, $f_{X|Y}(x,y)$

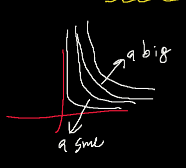
$0<y<x<1 \Rightarrow \mathbb{1}_{x\in[y,1]}$

$$f_{X|Y}(x,y) = \frac{f_{Y|X}(x,y)\, f_X(x)}{f_Y(y)} = \frac{\frac{1}{x}\mathbb{1}_{y\in[0,x]}\mathbb{1}_{x\in[0,1]}}{-\ln(y)\,\mathbb{1}_{y\in[0,1]}} = \frac{1}{-x\ln(y)}\mathbb{1}_{x\in[y,1]}\,\mathbb{1}_{y\in[0,1]}$$

$$= \left(-\frac{1}{\ln(y)}\right)\frac{1}{x}\ \ldots = a\frac{1}{x}$$

$f_{X|Y}(x,y)$





Mixture / Compound rv's / densities / masses

2/3 of the time
T ~ Exp(1/5)
and 1/3 of the time
T ~ Exp(1/20)

What is the distribution of waiting time unconditionally?

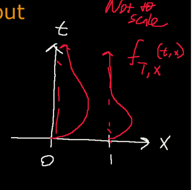
Define a new rv whose realization states are these two different states. X ~ Bern(2/3) and thus:

T | X = 1 ~ Exp(1/5) and T | X = 0 ~ Exp(1/20)

$$f_T(t) = \int_{Supp[X]} f(t,x)\,dx$$

but X is discrete... so we sum over the support to margin it out

Not to scale



$$f_T(t) = \sum_{x\in Supp[X]} f(t,x) = \sum_{x\in\{0,1\}} f_{T|X}(t,x)\,p(x)$$

$$= f_{T|X}(t,0)\frac{1}{3} + f_{T|X}(t,1)\frac{2}{3} = \frac{1}{3}Exp\left(\tfrac{1}{20}\right) + \frac{2}{3}Exp\left(\tfrac{1}{5}\right)$$

The unconditional T is called a "mixture model" but you are mixing multiple rv's where each consituent of the mixture is a discrete state of another rv called the "mixing model", i.e. X.

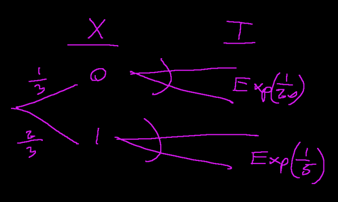
$$p_{X|T}(x,t) = \frac{f_{X,T}(x,t)}{f_T(t)} = \frac{f_{T|X}(t,x)\,p_X(x)}{f_T(t)}$$

$$= \frac{\left(\frac{1}{20}e^{-\frac{1}{20}t}\mathbb{1}_{x=0} + \frac{1}{5}e^{-\frac{1}{5}t}\mathbb{1}_{x=1}\right)\left(\frac{2}{3}\right)^x\left(\frac{1}{3}\right)^{x-1}\mathbb{1}_{x\in\{0,1\}}}{\frac{1}{3}\frac{1}{20}e^{-\frac{1}{20}t} + \frac{2}{3}\frac{1}{5}e^{-\frac{1}{5}t}}$$

$E[T|X=0] = 20$
$E[T|X=1] = 5$

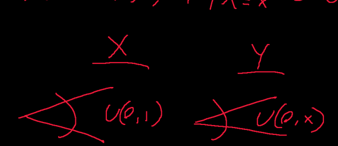
$$= \frac{\frac{1}{3}\frac{1}{20}e^{-\frac{1}{20}t}\mathbb{1}_{x=0} + \frac{2}{3}\frac{1}{5}e^{-\frac{1}{5}t}\mathbb{1}_{x=1}}{\frac{1}{3}\frac{1}{20}e^{-\frac{1}{20}t} + \frac{2}{3}\frac{1}{5}e^{-\frac{1}{5}t}}$$

$$X|T=t \sim Bern\left(\frac{\frac{2}{3}\frac{1}{5}e^{-\frac{1}{5}t}}{\frac{1}{3}\frac{1}{20}e^{-\frac{1}{20}t} + \frac{2}{3}\frac{1}{5}e^{-\frac{1}{5}t}}\right)$$



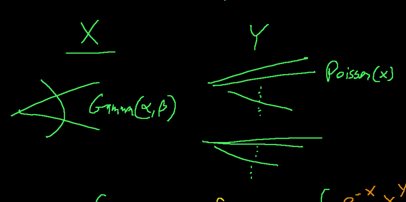
tree diagram for a mixture model

$X \sim U(0,1),\ Y|X=x \sim U(0,x)$ Tree diagram:



$f_Y(y)$ is called a "compound distribution" which really is just a mixture model where the mixing model is continuous. "Mixture" is when mixing model is discrete and "compound" is when mixing model is continuous.

p156 $X \sim Gamma(\alpha,\beta),\ Y|X=x \sim Poisson(\overset{\lambda}{x})$ Y is discrete



$$p_Y(y) = \int_{Supp[X]} p_{Y|X}(x,y)\,f_X(x)\,dx = \int_{x\in(0,\infty)} \frac{e^{-x}x^y}{y!}\mathbb{1}_{y\in\{0,1,\ldots\}}\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\,\cancel{\mathbb{1}_{x\in(0,\infty)}}\,dx$$

1

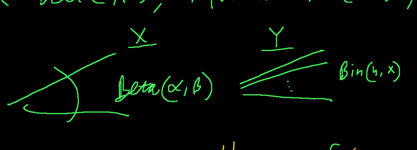
$$= \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{1}{y!}\mathbb{1}_{y\in\{0,1,\ldots\}}\int_0^\infty x^{y+\alpha-1}e^{-(\beta+1)x}\,dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)}\frac{1}{y!}\mathbb{1}_{y\in\{0,1,\ldots\}}\frac{\Gamma(y+\alpha)}{(\beta+1)^{y+\alpha}}$$

$$= \cdots = ExtNegBin\left(\alpha, \frac{\beta}{\beta+1}\right)$$

More flexible count model. Two parameters instead of one.

$X \sim Beta(\alpha,\beta),\ Y|X=x \sim Bin(n,x),\ Y \sim ?$



$\mathbb{1}_{y\in\{0,\ldots,n\}}$

$$p_Y(y) = \int_{Supp[X]} p_{Y|X}(x,y)\,\overset{pdf}{f_X(x)}\,dx = \int_{x\in(0,1)} \binom{n}{y}x^y(1-x)^{n-y}\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \binom{n}{y}\frac{1}{B(\alpha,\beta)}\mathbb{1}_{y\in\{0,\ldots,n\}}\underbrace{\int_0^1 x^{y+\alpha-1}(1-x)^{n-y+\beta-1}\,dx}_{\text{beta function}}$$

$$= \frac{B(y+\alpha,\,n-y+\beta)}{B(\alpha,\beta)}\binom{n}{y}\mathbb{1}_{y\in\{0,\ldots,n\}} = BetaBinom(n,\alpha,\beta)$$

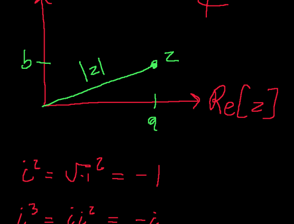
the betabinomial rv is a result of compounding a binomial with a beta. It is a more flexible count model for situations with a maximum count. It is an "overdispersed binomial" because it has higher variance so the rare counts are less rare.

$$X \sim Gamma(\alpha,\beta),\ Y|X=x \sim Exp(x) \overset{HW}{\Longrightarrow} Y \sim Lomax(\beta,\alpha)$$

Midterm II ↑

Final exam ↓

We will review complex numbers $z := a + bi \in \mathbb{C}$, $i := \sqrt{-1}$, $a,b \in \mathbb{R}$



Re[z] := a, the "real" piece
Im[z] := b, the "imaginary" piece

$|z| := \sqrt{a^2+b^2}$

Arg[z] := angle of z with the real axis = arctan(b/a)

usually

$i^2 = \sqrt{-1}^2 = -1$

$i^3 = i i^2 = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$i^5 = i i^4 = i$

$\vdots$

$i^n = i^{n \bmod 4}$

Taylor Series

$$e^{tx} = \sum_{x=0}^{\infty}\frac{(tx)^k}{k!} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{t^5x^5}{5!} + \ldots$$

$$\sin(tx) = \quad = tx - \frac{t^3x^3}{3!} + \frac{t^5x^5}{5!} - + - + \ldots$$

$$\cos(tx) = \quad = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - + - + \ldots$$

$$e^{itx} = \sum_{x=0}^{\infty}\frac{i^kt^kx^k}{k!} = 1 + itx - \frac{t^2x^2}{2!} - \frac{it^3x^3}{3!} + \frac{t^4x^4}{4!} + \frac{it^5x^5}{5!} - - + + \ldots$$

$i\sin(tx)$
$\|$

$$\sin(itx) = itx - \frac{it^3x^3}{3!} + \frac{it^5x^5}{5!} - + - \ldots$$

$$\cos(itx) = 1 - \frac{t^2x^2}{2!} + \frac{t^4x^4}{4!} - + - + \ldots$$

$\|$
$\cos(tx)$

Euler's Identity

$$\Rightarrow e^{itx} = \sin(itx) + \cos(itx) \Rightarrow e^{itx} = i\sin(tx) + \cos(tx)$$

if $tx = \pi$

$$\Rightarrow e^{i\pi} = -1 \Rightarrow e^{i\pi} + 1 = 0$$

$$\text{Define } L^1 := \left\{ f : \int_{\mathbb{R}} |f(x)|\,dx < \infty \right\}$$

This is called the set L1 or the set of "integrable" functions or the set of "absolutely integrable" functions. All PDFs ∈ L1

If $f \in L^1$, $\exists\, \hat{f}$ the "Fourier Transform" of f:

$$\hat{f}(\omega) := \int_{\mathbb{R}} e^{-i2\pi\omega x} f(x)\,dx$$

Taking f and deriving its $\hat{f}$ is called the "forward Fourier transformation operation" or "Fourier analysis".

If $\hat{f}$ ∈ L1 (which is not guaranteed), then we can reverse i.e. do an "inverse Fourier transformation operation" or "Fourier synthesis" to get the original function f back:

$$f(x) = \int_{\mathbb{R}} e^{i2\pi\omega x}\hat{f}(\omega)\,d\omega$$

Fourier inversion thm: if f, $\hat{f}$ ∈ L1, then f and $\hat{f}$ are 1:1.