$$P_{T_3}(t) = \sum_{x \in Supp[X_3]} P^{old}_{X_3}(x)\, P^{old}_{T_2}(t-x)\, \mathbb{1}_{t-x \in Supp[T_2]}$$

$$= \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} \binom{2}{t-x} p^{t-x} (1-p)^{2-t+x}\, \mathbb{1}_{t-x \in \{0,1,2\}}$$

Note: the indicator function is part of the combinatorial term hence it could be dropped.

$$= \sum_{x \in \{0,1\}} p^t (1-p)^{3-t} \binom{2}{t-x} = p^t (1-p)^{3-t} \sum_{x \in \{0,1\}} \binom{2}{t-x}$$

$$= p^t (1-p)^{3-t} \left(\binom{2}{t} + \binom{2}{t-1} \right) \underset{\text{by Pascal's Identity}}{=} \binom{3}{t} p^t (1-p)^{3-t} = Binom(3, p)$$

On the HW you'll prove that the sum of n iid Bernoulli(p) rvs is a Binom(n, p).

$X_1 \sim Binom(n_1, p)$ indep. of $X_2 \sim Binom(n_2, p)$, $T = X_1 + X_2 \sim p_T(t) = ?$

This is intuitively a Binom(n_1 + n_2, p). Let's show this:

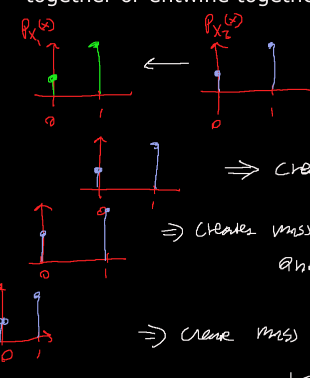
$$p_T(t) = \sum_{x \in \mathbb{R}} p_{X_1}(x)\, p_{X_2}(t-x) = \sum_{x \in \mathbb{R}} \binom{n_1}{x} p^x (1-p)^{n_1 - x} \binom{n_2}{t-x} p^{t-x} (1-p)^{n_2 - t + x}$$

$$= p^t (1-p)^{n_1 + n_2 - t} \sum_{x \in \mathbb{R}} \binom{n_1}{x}\binom{n_2}{t-x} = \binom{n_1+n_2}{t} p^t (1-p)^{n_1+n_2-t} = Binom(n_1 + n_2, p)$$

by Vandermonde's Identity

$X_1, X_2 \overset{iid}{\sim} Bern(p)$ $\quad T_2 = X_1 + X_2 \sim Binom(2, p)$ $\quad p_{T_2}(t) = p_{X_1}(x) * p_{X_2}(x)$

What does "convolution" mean? "Convolve" means roll or coin together or entwine together.



$\Rightarrow$ create mass $t = 1 + 0$ w.p. $p_{X_1}(1)\, p_{X_2}(0) = p(1-p)$

$\Rightarrow$ creates mass $t = 0 + 0$ w.p. $p_{X_1}(0)\, p_{X_2}(0) = (1-p)^2$
and $t = 1 + 1$ w.p. $p_{X_1}(1)\, p_{X_2}(1) = p^2$

$\Rightarrow$ create mass $t = 0 + 1$ w.p. $p_{X_1}(0)\, p_{X_2}(1) = (1-p)p$

$$T_2 \sim \begin{cases} 0 & \text{w.p. } (1-p)^2 \\ 1 & \text{w.p. } 2(1-p)p \\ 2 & \text{w.p. } p^2 \end{cases}$$

let $B_1, B_2, \ldots \overset{iid}{\sim} Bern(p)$

Let X := # of zero realizations before the first one realization (the number of failures before you succeed)

X := $\min_t$ {t : B_t = 1} - 1

Supp[X] = {0, 1, 2, ...}. Let's calculate the PMF:

$p(0) = p$
$p(1) = (1-p)p$
$p(2) = (1-p)^2 p$
$\vdots$
$p(x) = \underbrace{(1-p)^x p}_{p^{old}(x)}\, \mathbb{1}_{x \in \{0,1,\ldots\}}$

This is called a "geometric" rv.
X ~ Geom(p) =

$X_1, X_2 \overset{iid}{\sim} Geom(p)$, $T_2 = X_1 + X_2 \sim p_{T_2}(t) = ?$

$$p_{T_2}(t) = \sum_{x \in \{0,1,\ldots\}} (1-p)^x p\, (1-p)^{t-x} p\, \mathbb{1}_{t-x \in \{0,1,\ldots\}}$$

$\underbrace{t - x \in \{0, 1, \ldots\}}$
$\underbrace{x - t \in \{\ldots, -1, 0\}}$
$x \in \{\ldots, t-1, t\}$

$$= \sum_{x \in \{0,1,\ldots,t\}} (1-p)^t p^2 = (1-p)^t p^2 \sum_{x \in \{0,\ldots,t\}} (1) = (t+1)(1-p)^t p^2 = p^{old}_{T_2}(t)$$

$Supp[T_2] = Supp[X_1] + Supp[X_2] = \{0, 1, 2, \ldots\}$

$X_1, X_2, X_3 \overset{iid}{\sim} Geom(p)$, $T_3 = X_1 + X_2 + X_3 = X_3 + T_2 \sim p_{T_3}(t) = ?$

$$p_{T_3}(t) = \sum_{x \in \{0,1,\ldots\}} (1-p)^x p\, (t - x + 1)(1-p)^{t-x} p^2\, \mathbb{1}_{t-x \in \{0,1,\ldots\}}$$

$$= \sum_{x \in \{0,1,\ldots,t\}} (1-p)^t p^3 (t - x + 1) = (1-p)^t p^3 \sum_{x \in \{0,1,\ldots,t\}} \underbrace{t - x + 1}_{(t+1) - (x)}$$

$$= (1-p)^t p^3 \left(\sum_{x \in \{0,\ldots,t\}} (t+1) - \sum_{x \in \{0,\ldots,t\}} x \right)$$

$$= (1-p)^t p^3 \left((t+1) \sum_{x \in \{0,\ldots,t\}} 1 - \sum_{x \in \{0,\ldots,t\}} x \right)$$

$$= (1-p)^t p^3 \left((t+1)^2 - \frac{t(t+1)}{2} \right)$$

$$(t+1)\left((t+1) - \frac{t}{2}\right) = (t+1)\left(\frac{t}{2} + 1\right) = \frac{(t+1)(t+2)}{2} = \frac{(t+2)!}{t!\,2!} = \binom{t+2}{2}$$

$$= \binom{t+2}{2} (1-p)^t p^3 = NegBin(3, p)$$

This is the "negative binomial" rv.

$X_1, \ldots, X_r \overset{iid}{\sim} Geom(p) \Rightarrow T_r \sim NegBin(r, p) = \binom{t+r-1}{r-1} (1-p)^t p^r$

you'll prove this by induction on the homework. Why does this PMF make sense?

$(1-p)^t \longrightarrow$ you are failing t times (there are t zeroes). This is the probability of t zeroes.

$p^r \longrightarrow$ you are succeeding r times (there are r ones). This is the probability of r ones.

$\binom{n}{k} = \binom{n}{n-k}$

$\binom{t+r-1}{r-1} \longrightarrow$ this is the number of ways of getting r-1 successes among t+r-1 total trials OR the number of ways of getting t failures out of t+r-1 total trials

$$\underbrace{\underset{1}{\underline{0}}\ \underset{2}{\underline{1}}\ \underline{1}\ \underline{0}\ \underline{0} \ldots \underset{t}{\underline{1}}\ \underset{t+1}{\underline{1}} \ldots \underset{t+r-1}{\underline{0}}}_{r-1 \text{ successes}}\ \underset{t+r}{\underline{1}}$$

$t = 17$
$r = 5$

$X \sim Bin(n, p) = \binom{n}{x} p^x (1-p)^{n-x}$ $\quad p = \frac{\lambda}{n}$

Let n → ∞ and p → 0 but λ = np, a fixed constant. What is the PMF of the rv under this situation?

$$\lim_{n \to \infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} = \lim_{n \to \infty} \frac{n!}{x!\,(n-x)!} \frac{\lambda^x}{n^x} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x} \cdot \mathbb{1}_{x \in \{0,\ldots,n\}}$$

$$= \frac{\lambda^x}{x!} \lim \frac{n!}{(n-x)!\, n^x} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x} \mathbb{1}_{x \in \{0,\ldots,n\}}$$

$$= \frac{\lambda^x}{x!} \lim \frac{\overbrace{n(n-1)\cdots(n-x+1)}^{x \text{ terms}}\cancel{(n-x)}\cdots\cancel{(1)}}{\cancel{(n-x)}\cancel{(n-x-1)}\cdots\cancel{(1)}\underbrace{(n)(n)\cdots(n)}_{x \text{ terms}}} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x} \mathbb{1}_{x \in \{0,\ldots,n\}}$$

$$= \frac{\lambda^x}{x!} \lim \frac{n}{n} \lim \frac{n-1}{n} \cdots \lim \frac{n-x+1}{n}\, \underbrace{\lim \left(1 - \frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \lim \left(1 - \frac{\lambda}{n}\right)^{-x} \lim \mathbb{1}_{x \in \{0,\ldots,n\}}$$

$$= \frac{\lambda^x}{x!} e^{-\lambda}\, \mathbb{1}_{x \in \{0,1,\ldots\}} = Poisson(\lambda)$$

$X_1, X_2 \overset{iid}{\sim} Poisson(\lambda) \Rightarrow X_1 + X_2 \sim Poisson(2\lambda)$

using the iid convolution formula and the identity from 241:

$$\sum_{x \in \{0,\ldots,t\}} \binom{t}{x} = 2^t$$