MATH 368/621 Fall 2021 Homework #2

Professor Adam Kapelner

Due by email 11:59PM Thursday, September 30, 2021

(this document last updated Friday 10^{th} September, 2021 at 11:27am)

Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out." Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, review from math 241 about conditional probability, expectation and variance then read on your own about the multinomial distribution, conditional vector expectation, covariances, variance-covariance matrices.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]" and purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems.

This homework is worth 100 points but the point distribution will not be determined until after the due date. See syllabus for the policy on late homework.

Up to 7 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. The easiest way to use overleaf is to copy the raw text from hwxx.tex and preamble.tex into two new overleaf tex files with the same name. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. I do not accept homeworks which are *not* on this printout. Keep this first page printed for your records.

NAME: ____

Problem 1

These exercises introduce probabilities of conditional subsets of the supports of multiple r.v.'s.

(a) [difficult] Let $X \sim \text{Geometric}(p_x)$ independent of $Y \sim \text{Geometric}(p_y)$. Find $\mathbb{P}(X > Y)$ using the method we did in class. Note that p_x and p_y are now different.

(b) [easy] [MA] Prove this a different way by finding $\mathbb{P}(X = Y)$ and then using the law of total probability.

(c) [easy] [MA] As both p_x and p_y are reduced to zero, but $r = \frac{p_x}{p_y}$, what is the asymptotic probability you found in (a)?

(d) [difficult] Let $X \sim \text{Poisson}\,(\lambda)$ independent of $Y \sim \text{Poisson}\,(\lambda)$. Find an expression for $\mathbb{P}\,(X > Y)$ *as best as you are able to answer*. Part of this exercise is identifying where you cannot go any further.

Problem 2

These exercises will introduce the Multinomial distribution.

(a) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = k$, what is the parameter space for both n and $\boldsymbol{p}$?

(b) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = k$, what is the $\text{Supp}\left[\boldsymbol{X}\right]$?

(c) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = k$, what is $\dim\left[\boldsymbol{p}\right]$?

(d) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = 2$, express p_2 as a function of p_1.

(e) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = 2$, how are both X_1 and X_2 distributed?

(f) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(n, \boldsymbol{p}\right)$ and $n = 10$ and $\dim\left[\boldsymbol{X}\right] = 7$ as a column vector, give an example value of $\boldsymbol{x}$, a realization of the r.v. $\boldsymbol{X}$.

(g) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\left(9,\ [0.1\ 0.2\ 0.7]^\top\right)$, find $\mathbb{P}\left(\boldsymbol{X} = [3\ 2\ 4]^\top\right)$ to the nearest two decimal places.

(h) [difficult] [MA] If $\boldsymbol{X}_1 \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ and independently $\boldsymbol{X}_2 \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$ where $\dim\,[\boldsymbol{X}_1] = \dim\,[\boldsymbol{X}_2] = k$. Find the JMF of $\boldsymbol{T}_2 = \boldsymbol{X}_1 + \boldsymbol{X}_2$ from the definition of convolution. This looks harder than it is! First, use the definition of convolution and factor out the terms that are not a function of $x_1, \ldots, x_K$. Finally, use Theorem 1 in this paper: [click here] for the summation.

Problem 3

These exercises will introduce review expectation and variance and introduce covariance as well as expectation and variance of multidimensional (vector) r.v.'s.

(a) [harder] Consider a sequence of independent r.v.'s $X_1, \ldots, X_n$ and prove that

$$\mathbb{E}\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} \mathbb{E}\,[X_i]\,.$$

(b) [easy] Prove that $\mathbb{C}\text{ov}\left[X_1, X_2\right] = \mathbb{E}\left[(X_1 - \mu_1)(X_2 - \mu_2)\right]$.

(c) [easy] Prove that $\mathbb{C}\text{ov}\left[X, X\right] = \mathbb{V}\text{ar}\left[X\right]$.

(d) [easy] Prove that $\mathbb{C}\text{ov}\left[X_1, X_2\right] = \mathbb{C}\text{ov}\left[X_2, X_1\right]$.

(e) [easy] Prove that $\mathbb{C}\text{ov}\left[a_1 X_1, a_2 X_2\right] = a_1 a_2 \mathbb{C}\text{ov}\left[X_1, X_2\right]$.

(f) [easy] Prove that $\mathbb{C}\text{ov}\left[X_1 + X_3, X_2\right] = \mathbb{C}\text{ov}\left[X_1, X_2\right] + \mathbb{C}\text{ov}\left[X_2, X_3\right]$.

(g) [harder] [MA] Prove that

$$\mathbb{C}\text{ov}\left[\sum_{i \in A} X_i, \sum_{j \in B} Y_j\right] = \sum_{i \in A} \sum_{j \in B} \mathbb{C}\text{ov}\left[X_i, Y_j\right]$$

.

(h) [difficult] Prove that

$$\mathbb{V}\text{ar}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{C}\text{ov}\left[X_i, X_j\right]$$

without using the vector formulas.

(i) [easy] Prove $\mathbb{E}\left[a\boldsymbol{X} + \boldsymbol{c}\right] = a\boldsymbol{\mu} + \boldsymbol{c}$ where the following are constants: $a \in \mathbb{R}, \boldsymbol{c} \in \mathbb{R}^K$.

(j) [easy] Prove $\mathbb{V}\text{ar}\left[\boldsymbol{c}^\top \boldsymbol{X}\right] = \boldsymbol{c}^\top \Sigma \boldsymbol{c}$ where $\boldsymbol{c} \in \mathbb{R}^K$, a constant and $\Sigma := \mathbb{V}\text{ar}\left[\boldsymbol{X}\right]$, the variance-covariance matrix of the vector r.v. $\boldsymbol{X}$. This is marked easy since it's in the notes.

(k) [easy] Why is $\boldsymbol{c}^\top \Sigma \boldsymbol{c}$ called a "quadratic form?" Read about it on wikipedia.

Problem 4

These exercises are about the Multinomial distribution.

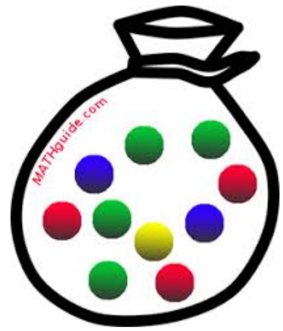
(a) [easy] Explain in English why $\boldsymbol{B} \sim \text{Multinomial}\,(1, \boldsymbol{p})$ is the multidimensional generalization of the Bernoulli r.v.

(b) [easy] Explain in English why the following should be true. Remember how the sampling from the bag works.

$$\binom{n}{x_1, x_2, \ldots, x_K} = \binom{n}{x_1}\binom{n - x_1}{x_2}\binom{n - (x_1 + x_2)}{x_3} \cdot \ldots \cdot \binom{n - (x_1 + x_2 + \ldots x_{K-1})}{x_K}$$

(c) [harder] [OPTIONAL] Prove the combinatorial identity in (b).

(d) [easy] Consider the following bag of 4 green, 3 red, 2 blue and 1 yellow marbles:



Draw one marble with replacement 37 times. What is the probability of getting 10 red, 17 green, 6 blue and 4 yellow? Compute explicitly to the nearest two significant digits.

(e) [difficult] [MA] If $\boldsymbol{X} \sim \text{Multinomial}\,(n, \boldsymbol{p})$, prove that its JMF sums to one, i.e. $\sum_{\boldsymbol{x} \in \text{Supp}[\boldsymbol{X}]} p_{\boldsymbol{X}}(\boldsymbol{x}) = 1$.

(f) [difficult] [MA] If $\boldsymbol{X} \sim \text{Multinomial}(n,\, \boldsymbol{p})$, prove that any marginal distribution is binomial with n and p_j as parameters i.e.

$$p_{X_j}(x_j) = \text{Binomial}(n,\, p_j)$$

We only assumed this in class because it makes sense conceptual given balls being sampled from an urn, but it was never explicitly proven.

(g) [E.C.] [MA] If $\boldsymbol{X} \sim \text{Multinomial}(n,\, \boldsymbol{p})$, find the JMF of any subset of $X_1, \ldots, X_k$. Is it technically multinomial? This is not much harder than the previous problem if formulated carefully.

(h) [harder] Explain in English why the following should be true. Remember how the sampling from the bag works.

$$\boldsymbol{B}_1, \ldots, \boldsymbol{B}_n \overset{iid}{\sim} \text{Multinomial}\,(1,\, \boldsymbol{p}) \quad \text{then} \quad \boldsymbol{X} := \sum_{i=1}^{n} \boldsymbol{B}_i \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$$

(i) [harder] Find the answer by reasoning in English. No need to prove mathematically.

$$\boldsymbol{X}_1, \ldots, \boldsymbol{X}_r \overset{iid}{\sim} \text{Multinomial}\,(n,\, \boldsymbol{p}) \quad \text{then} \quad \boldsymbol{T} := \sum_{i=1}^{r} \boldsymbol{X}_i \sim \ ?$$

(j) [easy] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$, find $p_{\boldsymbol{X}_{-j}|X_j}(\boldsymbol{x}_{-j}, x_j)$. This is marked easy since it's in the notes.

(k) [E.C.] [MA] If $\boldsymbol{X} \sim \text{Multinomial}\left(n,\, \boldsymbol{p}\right)$, find a proof for $\mathbb{C}\text{ov}\left[X_i, X_j\right] = -np_ip_j$ that is qualitatively different than the one we did in class.

(l) [harder] If $\boldsymbol{X} \sim \text{Multinomial}\left(n,\, \boldsymbol{p}\right)$ where $\dim\left[\boldsymbol{X}\right] = K$ and $\boldsymbol{p} = \frac{1}{K}\mathbf{1}_K$. What is the limit of $\mathbb{C}\text{ov}\left[X_i, X_j\right]$ as K gets large but n is fixed. Why does this make sense?

(m) [easy] Correlation ρ is a unitless measure bounded between $[-1, 1]$ and is a type of normalized covariance metric. It is defined for two r.v.'s as

$$\rho_{1,2} := \mathbb{C}\text{orr}\left[X_1,\, X_2\right] := \frac{\sigma_{1,2}}{\sigma_1\sigma_2} = \frac{\mathbb{C}\text{ov}\left[X_1, X_2\right]}{\mathbb{S}\text{D}\left[X_1\right]\mathbb{S}\text{D}\left[X_2\right]} = \frac{\mathbb{C}\text{ov}\left[X_1, X_2\right]}{\sqrt{\mathbb{V}\text{ar}\left[X_1\right]\mathbb{V}\text{ar}\left[X_2\right]}}$$

where $\mathbb{S}\text{D}\left[\cdot\right]$ denotes the standard deviation of a r.v., the square root of its variance. Find $\mathbb{C}\text{orr}\left[X_i,\, X_j\right]$ for two arbitrary elements in the r.v. vector $\boldsymbol{X} \sim \text{Multinomial}\left(n,\, \boldsymbol{p}\right)$.

(n) [easy] If $\boldsymbol{c} = [1\ 2\ 3\ 4]^\top$, compute the inner product $\boldsymbol{c}^\top\boldsymbol{c}$ and the outer product $\boldsymbol{c}\boldsymbol{c}^\top$.

Problem 5

These exercises will give you more practice with indicator functions.

(a) [easy] Resolve as best as possible: $\sum_{x\in\mathbb{Z}} \mathbb{1}_{x\in[0,c]}$ where $c \in \mathbb{N}_0$.

(b) [easy] Resolve as best as possible: $\sum_{x\in\{0,1,\ldots,d\}} \mathbb{1}_{x\in[0,c]}$ where $c, d \in \mathbb{N}_0$.

(c) [easy] Resolve as best as possible: $\int_{\mathbb{R}} \mathbb{1}_{x\in[0,c]}\mathrm{d}x$ where $c \in \mathbb{R}$.

(d) [easy] Resolve as best as possible: $\int_{-\infty}^{d} \mathbb{1}_{x\in[0,c]}\mathrm{d}x$ where $c, d \in \mathbb{R}$.

(e) [easy] Resolve as best as possible: $\int_{d}^{d+1} \mathbb{1}_{x\in[0,c]}\mathrm{d}x$ where $c, d \in \mathbb{R}$.

(f) [harder] Resolve as best as possible: $\int_{d}^{d+1} \mathbb{1}_{x\in[c,c+1]}\mathrm{d}x$ where $c, d \in \mathbb{R}$.

Problem 6

We will get some practice with the simple transformation $Y = g(X) = -X$ for discrete r.v.'s.

(a) [easy] If $X \sim \text{Bern}\,(p)$, find the PMF of $Y = -X$. Make sure the PMF is valid $\forall y \in \mathbb{R}$.

(b) [easy] If $X \sim \text{NegBin}\,(r,\, p)$, find the PMF of $Y = -X$. Make sure the PMF is valid $\forall y \in \mathbb{R}$.

(c) [harder] If $\boldsymbol{X} \sim \text{Multinomial}\,(n,\, \boldsymbol{p})$, find the JMF of $\boldsymbol{Y} = -\boldsymbol{X}$. Make sure the JMF is valid $\forall \boldsymbol{y} \in \mathbb{R}^K$.