Math 368 / 650 Fall 2021
Midterm Examination One

Professor Adam Kapelner

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Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

By taking this exam, you acknowledge and agree to uphold this Code of Academic Integrity.

Instructions

This exam is 70 minutes (variable time per question) and closed-book. You are allowed **one** page (front and back) of a "cheat sheet", blank scrap paper and a graphing calculator. Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 [10min] (and 10min will have elapsed) These are questions about indicator functions. Let $a, b \in \mathbb{R}$ and $b > a$. And let X be a discrete rv.

- [18 pt / 18 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

(a) $\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{a\}} = a$

(b) $\sum_{x \in \mathbb{R}} \mathbb{1}_{x \in \{a\}} = 1$

(c) $\sum_{x \in \mathbb{R}} a\mathbb{1}_{x \in \{a\}} = a$

(d) $\sum_{x \in \mathbb{R}} a\mathbb{1}_{x \in \{a\}} = 1$

(e) $\prod_{x \in \mathbb{R}} a\mathbb{1}_{x \in \{a\}} = a$

(f) $\prod_{x \in \mathbb{R}} a\mathbb{1}_{x \in \{a\}} = 1$

(g) $\int_{\mathbb{R}} a\mathbb{1}_{x \in [a,b]} = a$

(h) $\int_{\mathbb{R}} a\mathbb{1}_{x \in [a,b]} = b$

(i) $\int_{\mathbb{R}} a\mathbb{1}_{x \in [a,b]} = b - a$

(j) $\int_0^1 \mathbb{1}_{x \in [a,b]} = b - a$

(k) $p(x) = p^{old}(x)\mathbb{1}_{x \in \text{Supp}[X]}$

(l) $\sum_{x \in \mathbb{R}} p^{old}(x) = 1$

(m) $\sum_{x \in \text{Supp}[X]} p^{old}(x) = 1$

(n) $\sum_{x \in \mathbb{N}} p(x) = 1$

(o) $\sum_{x \in \mathbb{Z}} p(x) = 1$

(p) $\int_{\mathbb{R}} p^{old}(x) = 1$

(q) $\int_{\text{Supp}[X]} p^{old}(x) = 1$

(r) $\sum_{x \in \text{Supp}[X]} p^{old}(x)^2 = 1$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

$\boxed{\text{bckm}}$

Problem 2 [11min] (and 21min will have elapsed) Let $X \sim U(\{1,2,3\})$ and $Y \sim U(\{-1,-2,-3\})$ and $T = X + Y$.

- [19 pt / 37 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

  (a) $\sum_{x \in \mathbb{R}} p_{X,Y}(x,y) = 1$
  (b) $p_{X,Y}$ has at most 9 x, y input pairs that produce nonzero values
  (c) $p_X^{old}(x) = 1/3$
  (d) $p_Y^{old}(y) = 1/3$
  (e) $p_T^{old}(t) = 1/9$ if X, Y are independent.
  (f) $p_T^{old}(2) = 1/9$ if X, Y are independent.
  (g) $p_T(t) = p_{X,Y}(x,y) \star p_{X,Y}(x,y)$
  (h) $p_T(t) = \sum_{x \in \mathbb{R}} \sum_{y \in \mathbb{R}} p_{X,Y}(x,y)$
  (i) The expectation of T is 0 if X, Y are independent.
  (j) The expectation of T is 0 regardless of the dependence relationship of X, Y.
  (k) $\mathbb{S}\mathrm{upp}[T] = \{-2,-1,0,1,2\}$ if X, Y are independent.
  (l) $\mathbb{S}\mathrm{upp}[T] = \{-2,-1,0,1,2\}$ regardless of the dependence relationship of X, Y.
  (m) T could be a degenerate rv.
  (n) You can compute $p_T(t)$ for all $t \in \mathbb{R}$ if X, Y are independent given the information provided.
  (o) You can compute $p_T(t)$ for all $t \in \mathbb{R}$ if X, Y are dependent given the information provided.
  (p) $\mathbb{C}\mathrm{ov}[X, Y] = 0$ if X, Y are independent.
  (q) $\mathbb{C}\mathrm{ov}[X, T] = 0$ if X, Y are independent.
  (r) $\mathbb{C}\mathrm{ov}[Y, T] = \mathbb{E}[YT] - \mathbb{E}[Y]\,\mathbb{E}[T]$ if X, Y are independent.
  (s) $\mathbb{C}\mathrm{ov}[X, Y] = \mathbb{C}\mathrm{ov}[Y, X]$ if X, Y are dependent.

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

$\boxed{\text{abcdfgijkmnprs}}$

Problem 3 [10min] (and 31min will have elapsed) Let $X \sim \text{Geometric}(p_x)$ independent of $Y \sim \text{Geometric}(p_y)$ and $T = X + Y$.

- [15 pt / 52 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

  (a) The PMF of T can be derived using one of the discrete convolution formulas

  (b) $\mathbb{S}\text{upp}[X] = \mathbb{S}\text{upp}[T]$

  (c) If $p_x > p_y$ it is likely that $X > Y$

  (d) $T \sim \text{Geometric}(p_x + p_y)$

  (e) $T \sim \text{NegBin}(2, p_x + p_y)$

  (f) If $p_x = p_y = 1$, T is a degenerate rv

  (g) If $p_x = p_y = 0$, T is a degenerate rv

  (h) If $p_x = p_y = \frac{1}{2}$ then $T \sim \text{NegBin}\left(2, \frac{1}{2}\right)$

  (i) If $p_x = p_y = \frac{1}{2}$ then $\mathbb{P}(X = Y) > 0$

  (j) If $p_x = p_y = \frac{1}{2}$ then $\mathbb{P}(X = Y) = \frac{3}{4}$

  (k) If $p_x = p_y = \frac{1}{2}$ then $\mathbb{P}(X = Y) = \frac{1}{3}$

  (l) If $p_x = p_y = \frac{1}{2}$ then $\mathbb{P}(X = Y) = \frac{1}{2}$

  (m) If $p_x = p_y = \frac{1}{2}$ then $\mathbb{P}(X = Y) = \frac{1}{4}$

  (n) If $p_x = p_y = \frac{1}{2}$ then $\mathbb{P}(X = Y) = \frac{1}{8}$

  (o) If $p_x = p_y = \frac{1}{2}$ then $\mathbb{P}(X = Y) = \frac{1}{16}$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

abfhik

Problem 4 [10min] (and 41min will have elapsed) Let $X \sim \text{Geometric}(p)$ independent of $Y \sim \text{Geometric}(p)$ and $T = X + Y$.

- [13 pt / 65 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

(a) $p_{X\,|\,T}(x,t) = p_{X,T}(x,t)$

(b) $p_{X\,|\,T}(x,t) = p_{X,Y}(x,t)/p_T(t)$

(c) $p_{X\,|\,T}(x,t) = p_{X,Y}(x,t-x)/p_T(t)$

(d) $p_{X\,|\,T}(x,t) = \dfrac{(1-p)^x(1-p)^{t-x}}{(t+1)(1-p)^t}$

(e) $p_{X\,|\,T}(x,t) = \dfrac{(1-p)^x(1-p)^{t-x}}{(t+1)(1-p)^t p^2}$

(f) $p_{X\,|\,T}(x,t) = \dfrac{(1-p)^x(1-p)^{t-x}}{t(1-p)^{t+1}}$

(g) $p_{X\,|\,T}(x,t) = \dfrac{(1-p)^x(1-p)^{t-x}}{t(1-p)^{t+1}p^2}$

(h) $p_{X\,|\,T}(x,t)$ is a geometric rv

(i) $p_{X\,|\,T}(x,t)$ is a negative binomial rv

(j) $p_{X\,|\,T}(x,t)$ is a binomial rv

(k) $p_{X\,|\,T}(x,t)$ is a poisson rv

(l) $p_{X\,|\,T}(x,t)$ is a uniform discrete rv

(m) $p_{X\,|\,T}(x,t)$ is a degenerate rv

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

cdl

Problem 5 [7min] (and 48min will have elapsed) Let $\boldsymbol{X} = [X_1, \ldots, X_n]^\top \sim p(\boldsymbol{x})$ and $\mathbb{E}[\boldsymbol{X}] = \boldsymbol{0}_n$.

- [10 pt / 75 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

(a) If $\boldsymbol{A}$ be an $m \times n$ matrix of constants, then $\mathbb{E}[\boldsymbol{A}\boldsymbol{X}] = \boldsymbol{0}_m$

(b) $\mathbb{V}\text{ar}[\boldsymbol{X}]$ is an $n \times n$ matrix with entries $\mathbb{E}[X_i X_j]$ at row i and column j

(c) $\mathbb{V}\text{ar}[\boldsymbol{X}]$ can be written as a quadratic form

(d) $\mathbb{V}\text{ar}[\boldsymbol{X}]$ can be written as the expectation of an inner product

(e) $\mathbb{V}\text{ar}[\boldsymbol{X}]$ can be written as the expectation of an outer product

(f) If $X_1, \ldots, X_n \overset{ind}{\sim}$ then $\mathbb{V}\text{ar}[\boldsymbol{X}] = \boldsymbol{I}_n$

(g) There exists a $p(\boldsymbol{x})$ where $\mathbb{V}\text{ar}[\boldsymbol{X}]$ is *not* symmetric

(h) There exists a $p(\boldsymbol{x})$ where $\mathbb{V}\text{ar}\left[\boldsymbol{1}^\top \boldsymbol{X}\right] > \sum_{i=1}^n \mathbb{V}\text{ar}[X_i]$

(i) There exists a $p(\boldsymbol{x})$ where $\mathbb{V}\text{ar}\left[\boldsymbol{1}^\top \boldsymbol{X}\right] = \sum_{i=1}^n \mathbb{V}\text{ar}[X_i]$

(j) There exists a $p(\boldsymbol{x})$ where $\mathbb{V}\text{ar}\left[\boldsymbol{1}^\top \boldsymbol{X}\right] < \sum_{i=1}^n \mathbb{V}\text{ar}[X_i]$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

abehij

Problem 6 [10min] (and 58min will have elapsed) Let $X_1 \sim \text{Binomial}(n_1, p_1)$ independent of $X_2 \sim \text{Binomial}(n_2, p_2)$ and consider the difference $D = X_1 - X_2 \sim p_D(d)$. Let $Y = -X_2$.

- [10 pt / 85 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

(a) If n_1 was large and $p_1 \approx 0$, then the PMF of X_1 can be approximated with low error by $\text{Poisson}(n_1 p_1)$

(b) $X_1 \sim \binom{n_1}{x} p_1^x (1-p_1)^{n_1-x} \mathbb{1}_{x \in \{0,1,\ldots,n_1\}}$

(c) $X_2 \sim \binom{n_2}{x} p_2^x (1-p_2)^{n_2-x} \mathbb{1}_{x \in \{0,1,\ldots,n_2\}}$

(d) $Y \sim \binom{n_2}{-y} p_2^{-y} (1-p_2)^{n_2+y} \mathbb{1}_{y \in \{0,1,\ldots,n_2\}}$

(e) $p_D(d) = \sum_{x \in \mathbb{R}} p_{X_1}(x) p_{X_2}(d-x)$

(f) $p_D(d) = \sum_{x \in \mathbb{S}\text{upp}[X_1]} p_{X_1}^{old}(x) p_{X_2}^{old}(d-x) \mathbb{1}_{d-x \in \{0,1,\ldots,n_2\}}$

(g) $p_D(d) = \sum_{x=0}^{n_1} \binom{n_1}{x} p_1^x (1-p_1)^{n_1-x} \binom{n_2}{x-d} p_2^{x-d} (1-p_2)^{n_2+d-x} \mathbb{1}_{x-d \in \{0,1,\ldots,n_2\}}$

(h) $p_D(d) = \sum_{x=d}^{n_1} \binom{n_1}{x} p_1^x (1-p_1)^{n_1-x} \binom{n_2}{x-d} p_2^{x-d} (1-p_2)^{n_2+d-x}$

(i) D is a binomial rv

(j) D is a poisson rv

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

abcgh

Problem 7 [12min] (and 70min will have elapsed) A large factory produces marbles with the following color distribution:

Color	Blue	Red	Green	Yellow	Orange
Percentage	20	20	30	5	25

You sample 100 marbles randomly from the assembly line. Let X_b, X_r, X_g, X_y, X_o denote the rv's modeling the counts of Blue, Red, Green, Yellow and Orange marbles respectively in your sample and let $\boldsymbol{X}$ denote the column vector of those rv's stacked.

- [15 pt / 100 pts] Record the letter(s) of all the following that are **true** in general. At least one will be true.

(a) $X_b \sim \text{Binomial}\,(100,\, 0.2)$ and $X_r \sim \text{Binomial}\,(100,\, 0.2)$

(b) $X_b + X_r \sim \text{Binomial}\,(200,\, 0.2)$

(c) $\mathbb{E}\,[X_b + X_r] = 40$.

(d) $\boldsymbol{x} \in \mathbb{R}$

(e) $\boldsymbol{x}^\top \mathbf{1} = 100$ for all $\boldsymbol{x} \in \mathbb{S}\text{upp}\,[\boldsymbol{X}]$

(f) $|\mathbb{C}\text{orr}\,[X_b, X_r]\,| = |\mathbb{C}\text{ov}\,[X_b, X_r]\,|$

(g) $|\mathbb{C}\text{orr}\,[X_b, X_r]\,| > |\mathbb{C}\text{orr}\,[X_y, X_g]\,|$

(h) $p_{X_b \,|\, X_r}(x, y)$ is undefined for $y > 100$

(i) Generally speaking, the more blue marbles in the sample, the less yellow marbles in the sample.

(j) If it is known that there are 8 blue marbles in the sample, then the number of yellow marbles is expected to be lower as compared to if you have no information about the number of blue marbles in the sample.

For the remaining questions, assume that we are told there are 8 blue marbles in the sample.

(k) The number of yellow marbles will be a drawn from a $\text{Binomial}\,(92,\, 0.05)$ rv.

(l) The number of yellow marbles will be a drawn from a $\text{Binomial}\,(92,\, 0.0625)$ rv.

(m) The number of yellow marbles will be a drawn from a $\text{Binomial}\,(75,\, 0.05)$ rv.

(n) The number of yellow marbles will be a drawn from a $\text{Binomial}\,(75,\, 0.0625)$ rv.

(o) $[X_r\, X_g\, X_y\, X_o]^\top \sim \text{Multinom}_4\,(92,\, \boldsymbol{p})$ where the vector $\boldsymbol{p}$ can be computed using information provided in this problem.

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

aceghilo