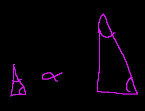
Kernels:

$$p(x) = c\,k(x) \Rightarrow p(x) \propto k(x)$$
$$f(x) = c\,k(x) \Rightarrow f(x) \propto k(x)$$



k(x) and p(x) / f(x) are 1:1

c > 0 but not a function of x, but scales the function so that its sum / integral is exactly 1.

$$1 = \sum_{\mathbb{R}} p(x) = \sum_{\mathbb{R}} c\,k(x) = c \sum_{\mathbb{R}} k(x) \Rightarrow c = \frac{1}{\sum k(x)}$$

$$1 = \int_{\mathbb{R}} f(x)\,dx = \int_{\mathbb{R}} c\,k(x)\,dx = c\int_{\mathbb{R}} k(x)\,dx \Rightarrow c = \frac{1}{\int_{\mathbb{R}} k(x)\,dx}$$

Problem solving technique: if you see a k(x) in the "wild" that you recognize as the k(x) for a rv you know, you know this k(x) is proportional to that rv's PMF/PDF.

$$X \sim Bin(n,p) := \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!} p^x (1-p)^n (1-p)^{-x} \mathbb{1}_{x\in\{0,\dots,n\}}$$
$$= \underbrace{n!(1-p)^n}_{c}\ \underbrace{\frac{1}{x!(n-x)!}\left(\frac{p}{1-p}\right)^x \mathbb{1}_{x\in\{0,\dots,n\}}}_{k(x)}$$

$$Y \sim Weibull(k,\lambda) := (k\lambda)(\lambda y)^{k-1} e^{-(\lambda y)^k}\mathbb{1}_{y>0}$$
$$= \underbrace{k\lambda^k}_{c}\ \underbrace{y^{k-1} e^{-(\lambda y)^k}\mathbb{1}_{y>0}}_{k(y)}$$

$$X \sim Gamma(\alpha,\beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\mathbb{1}_{x>0} = \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{c}\ \underbrace{x^{\alpha-1}e^{-\beta x}\mathbb{1}_{x>0}}_{k(x)}$$

$$k(y) = x^{(a+1)-1} e^{-bx} \propto Gamma(a+1, b)$$

X ~ Gamma(α_1, β) independent of Y ~ Gamma(α_2, β). T = X + Y ~ ?

$$f_T(t) = \int_{Supp[X]} f_X^{old}(x)\, f_Y^{old}(t-x)\,\mathbb{1}_{t-x\in Supp[Y]}\,dx$$

$$\underbrace{e^{-\beta t}\cancel{e^{\beta x}}} \qquad \underbrace{x\in(-\infty,t)} \qquad \underbrace{x-t\in(-\infty,0)}$$

$$= \int_{x\in(0,\infty)}^{x=t} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1-1}\cancel{e^{-\beta x}}\ \frac{\beta^{\alpha_2}}{\Gamma(\alpha_2)}(t-x)^{\alpha_2-1} e^{-\beta(t-x)}\mathbb{1}_{t-x\in(0,\infty)}\,dx$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta t}\mathbb{1}_{t>0}\int_{x=0}^{x=t} x^{\alpha_1-1}(t-x)^{\alpha_2-1}\,dx$$

let $u = \frac{x}{t}$ ($x = tu$), $\frac{du}{dx} = \frac{1}{t} \Rightarrow dx = t\,du$, $x = 0 \Rightarrow u = 0$, $x = t \Rightarrow u = 1$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta t}\mathbb{1}_{t>0}\int_{u=0}^{u=1} \underbrace{(tu)^{\alpha_1-1}(t-tu)^{\alpha_2-1}\,t}_{t^{\alpha_1-1}u^{\alpha_1-1}t^{\alpha_2-1}(1-u)^{\alpha_2-1}t^1}\,du$$

$$= \frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\beta t}\, t^{\alpha_1+\alpha_2-1}\mathbb{1}_{t>0}\underbrace{\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\,du}_{g(\alpha_1,\alpha_2)}$$

$$= \underbrace{\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\,du}_{c}\ \underbrace{t^{\alpha_1+\alpha_2-1}e^{-\beta t}\mathbb{1}_{t>0}}_{K(t)}$$

$$\propto Gamma(\alpha_1+\alpha_2, \beta) = \underbrace{\frac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)}}_{c} t^{\alpha_1+\alpha_2-1}e^{-\beta t}\mathbb{1}_{t>0}$$

$$\Rightarrow \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\,du = \frac{\cancel{\beta^{\alpha_1+\alpha_2}}}{\Gamma(\alpha_1+\alpha_2)}$$

$$\Rightarrow \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\,du = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)} = B(\alpha_1,\alpha_2)$$

define this famous integral as the "beta function"

$$B(\alpha_1,\alpha_2) := \int_0^1 u^{\alpha_1-1}(1-u)^{\alpha_2-1}\,du$$

$$\text{let } k(x) = \underbrace{x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x\in[0,1]}} \Rightarrow f(x) = ?$$

$$c = \frac{1}{\int_{\mathbb{R}} k(x)\,dx} = \frac{1}{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx} = \frac{1}{B(\alpha,\beta)}$$

$$\Rightarrow f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}_{x\in[0,1]} = Beta(\alpha,\beta)$$

$$1 = \int_{\mathbb{R}} f(x)\,dx = \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{1}{\cancel{B(\alpha,\beta)}}\overbrace{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}^{\cancel{B(\alpha,\beta)}}$$

Define the incomplete beta function: $B(a,\alpha,\beta) := \int_0^a u^{\alpha-1}(1-u)^{\beta-1}\,du$

Define the regularized incomplete beta function: $I_a(\alpha,\beta) := \frac{B(a,\alpha,\beta)}{B(\alpha,\beta)}$

$$X \sim Beta(\alpha,\beta),\quad F(x) = \int_0^x \frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}\,dy =$$
$$= \frac{1}{B(\alpha,\beta)}\int_0^x y^{\alpha-1}(1-y)^{\beta-1}\,dy = \frac{B(x,\alpha,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta)$$

Arbitrary Transformations in multiple dimensions.

let $g: \mathbb{R}^n \to \mathbb{R}^n$ and 1:1

Known: $\vec{X} \sim f_{\vec{X}}(\vec{x})$, $\vec{Y} = g(\vec{X}) \sim f_{\vec{Y}}(\vec{y}) = ?$

Recall what a multi-dimensional function is:

$$\vec{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} g_1(X_1,\dots,X_n) \\ g_2(X_1,\dots,X_n) \\ \vdots \\ g_n(X_1,\dots,X_n) \end{bmatrix}$$

Since it's 1:1, there must be an inverse vector function h:

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = \begin{bmatrix} h_1(Y_1,\dots,Y_n) \\ h_2(Y_1,\dots,Y_n) \\ \vdots \\ h_n(Y_1,\dots,Y_n) \end{bmatrix}$$

The change of variables formula is:

Proof is in a multi-variable calculus course

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h(\vec{y}))\,|J_h(\vec{y})|$$

the "Jacobian Determinant" $J_h(\vec{y}) := \det\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}$

Let's use the multivariable transformation formula to rederive the convolution formula.

Recipe to do these types of problems

(1) Find the set g_i's so that g_1 is the function you actually care about
(2) Find the set of h_i's
(3) Compute Jacob. Det.
(4) Substitute into change of variables formula
(5) Integrate out the nuisance dimensions

$$T = X_1 + X_2 = g_1(X_1, X_2)$$

1/2

$$Y_1 = X_1 + X_2 = g_1(X_1,X_2)$$
$$Y_2 = X_2 = g_2(X_1,X_2)$$
$$X_1 = Y_1 - X_2 = Y_1 - Y_2 = h_1(Y_1,Y_2)$$
$$X_2 = Y_2 = h_2(Y_1,Y_2)$$

3

$$J_h = \det\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1 \Rightarrow |J_h| = 1$$

4

$$f_{\vec{Y}}(\vec{y}) = f_{\vec{X}}(h_1(\vec{y}), h_2(\vec{y}))\,|J_h|$$
$$= f_{\vec{X}}(y_1 - y_2, y_2)$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} T \\ Nuisance \end{bmatrix}$$

5

$$f_T(t) = \int_{\mathbb{R}} f_{\vec{X}}(t-x, x)\,dx$$

this is exactly the convolution formula

$$\overset{X_1, X_2 \overset{ind}{\sim}}{=} \int_{\mathbb{R}} f_{X_1}(t-x)\,f_{X_2}(x)\,dx$$

$$\overset{X_1, X_2 \overset{iid}{\sim}}{=} \int_{\mathbb{R}} f(t-x)\,f(x)\,dx$$