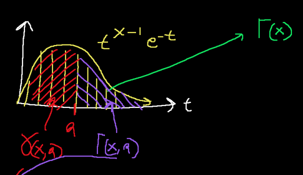


normalize / regularize (by dividing by the entire area)

$P(x,a) + Q(x,a) = 1$

$W_1, \ldots, W_k \overset{iid}{\sim} \text{Exp}(\lambda)$, $T_k = W_1 + \ldots + W_k \sim \text{Erlang}(k, \lambda)$ $\underline{\text{Unit}}$ time t

$$= \frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!} \mathbb{1}_{t \geq 0}$$

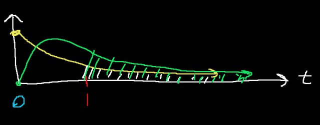
$$P(T_k \leq t) = F_{T_k}(t) = P(k, \lambda t)$$
$$P(T_k > t) = S_{T_k}(t) = Q(k, \lambda t)$$

$N \sim \text{Poisson}(\lambda) := \frac{e^{-\lambda}\lambda^n}{n!}$ # of events n

$$P(N \leq n) = F_N(n) = Q(n+1, \lambda)$$
$$P(N > n) = S_N(n) = P(n+1, \lambda)$$

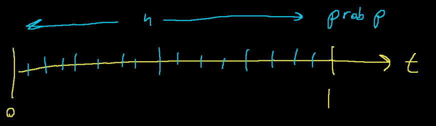
The "Poisson Process" is the equivalence of these two events: waiting and counting. Let's start with k=1.

$T_2 \sim \text{Erlang}(2, \lambda)$



$$P(T_1 > 1) = S_{T_1}(1) = Q(1, \lambda) = Q(0+1, \lambda) = F_N(0) = P(N \leq 0) = P(N=0)$$
$$P(T_2 > 1) = S_{T_2}(1) = Q(2, \lambda) = Q(1+1, \lambda) = F_N(1) = P(N \leq 1) = P(N \in \{0, 1\})$$
$$P(T_3 > 1) = S_{T_3}(1) = Q(3, \lambda) = Q(2+1, \lambda) = F_N(2) = P(N \leq 2) = P(N \in \{0, 1, 2\})$$
$$\vdots$$
$$P(T_k > 1) = S_{T_k}(1) = Q(k, \lambda) = Q((k-1)+1, \lambda) = F_N(k-1) = P(N \leq k-1) = P(N \in \{0, 1, \ldots, k-1\})$$

N models the number of events that occur in the first second.



The construction of the Poisson and the construction of the exponential were the same. We imagine the infinite n binomial's experiments being all in one second.

This table is a nice summary of many of the rv's we've seen:

Experiment Type	# of events in fixed time (Unit)	waiting time
Discrete	Binomial (Bernoulli)	NegBin (Geometric)
Continuous	Poisson	Erlang (Exponential)

Is there analogous relationship between the Binomial and the Negative Binomial? Yes (HW)

Let's "invent" two new rv's.
Recall $\Gamma(x) = (x-1)!$ or $\Gamma(x+1) = x!$ If $x \in \mathbb{N}$
The gamma function is a commonly used extension of the factorial function to $\mathbb{R}$.

$$T \sim \text{Erlang}(k, \lambda) := \frac{\lambda^k e^{-\lambda t} t^{k-1}}{(k-1)!} \mathbb{1}_{t \geq 0} \underset{k \in \mathbb{N}}{=} \frac{\lambda^k e^{-\lambda t} t^{k-1}}{\Gamma(k)} \mathbb{1}_{t \geq 0} = f(t)$$

$$T \sim \text{NegBin}(k, p) := \binom{k+t-1}{k-1}(1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0} = \frac{\Gamma(k+t)}{\Gamma(k)\, t!}(1-p)^t p^k \mathbb{1}_{t \in \mathbb{N}_0} = p(t)$$

Although Erlang(3.516, λ) and NegBin(3.516, p) are "illegal", do they make any conceptual sense? Yes. You can imagine averaging multiple waiting times for different legal k values.
I can also prove that f(t) and p(t) are legal for all k ∈ (0, ∞).
Thus we now have two new rv's just by extending the parameter space of old rv's and employing the gamma function extension.

$$X \sim \text{Gamma}(k, \lambda) \overset{\text{usually parameterized}}{=} \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x \geq 0}, \quad \alpha > 0, \ \beta > 0$$

$X \sim \text{ExtNegBin}(k, p) :=$ same

extended negative binomial

Transformations of Discrete rv's

$$X \sim \text{Bern}(p) = p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} = \begin{cases} 1 & \text{wp } p \\ 0 & \text{wp } 1-p \end{cases}$$

$$Y = g(X) = X + 3 \sim \begin{cases} 4 & \text{wp } p \\ 3 & \text{wp } 1-p \end{cases} = p^{y-3}(1-p)^{\overset{4-y}{1-(y-3)}} \mathbb{1}_{y \in \{3,4\}}$$

$y = g(x) = x+3$
$y - 3 = g^{-1}(y) = x$

conjecture: $p_Y(y) = p_X(g^{-1}(y))$ $\qquad \text{Supp}[Y] = g(\text{Supp}[X])$

$$p_Y(y) = P(Y=y) = P(g(X)=y) = P(X = g^{-1}(y)) = p_X(g^{-1}(y))$$

Assume the inverse function exists on Supp[X]

What if the inverse doesn't exist? Consider:

X ~ U({1,2,3,4,5,6,7,8,9,10}) = 1/10, the old-style PMF
Y = g(X) = min{X, 3} which is not a 1:1 function

y	$p_Y(y)$
1	1/10
2	1/10
3	8/10

$$p_Y(y) = \sum_{\{x : y = g(x)\}} p_X(x) \overset{g^{-1} \text{ exists on the Supp}[X]}{=} \sum_{\{x : g^{-1}(y) = x\}} p_X(x) = p_X(g^{-1}(y))$$

$\{x : g^{-1}(y) = x\} = x \in g^{-1}(y)$

$$X \sim \text{Bin}(n, p) := \binom{n}{x} p^x (1-p)^{n-x} \mathbb{1}_{x \in \{0,1,2,\ldots,n\}}, \quad Y = X^3 \sim \binom{n}{y^{\frac{1}{3}}} p^{y^{\frac{1}{3}}} (1-p)^{n - y^{\frac{1}{3}}} \mathbb{1}_{y^{\frac{1}{3}} \in \{0,1,2,\ldots,n\}}$$

$x = g^{-1}(y) = y^{\frac{1}{3}}$

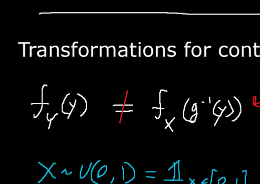
$y \in \{0, 1, 8, \ldots, n^3\}$

If Y = g(X) = X², would this strategy work? Yes... because on Supp[X] it is invertible (and that's all that matters).

Transformations for continuous rv's when g-inverse exists on Supp[X]

$f_Y(y) \neq f_X(g^{-1}(y))$ NO!

$X \sim U(0,1) = \mathbb{1}_{x \in [0,1]}$, $Y = g(X) = 2X \sim U(0,2) \Rightarrow X = \frac{Y}{2} = g^{-1}(Y)$



$$f_Y(y) = f_X\left(\frac{y}{2}\right) = \mathbb{1}_{\frac{y}{2} \in [0,1]} = \mathbb{1}_{y \in [0,2]} \neq \frac{1}{2}\mathbb{1}_{y \in [0,2]}$$