$$\vec{X} \sim \text{Multi}_{2}\left(n, \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}\right) = \binom{n}{x_1, x_2} p_1^{x_1} p_2^{x_2} = p_{X_1, X_2}(x_1, x_2)$$

$$\underset{K=2}{\uparrow} \qquad \underset{\vec{p}}{\uparrow}$$

$$p_{X_2}(x_2) = \sum_{x_1 \in \text{Supp}[X_1]} p_{X_1, X_2}(x_1, x_2)$$

$$= \sum_{x_1 \in \{0,1,\ldots,n\}} \frac{n!}{x_1!\, x_2!} p_1^{x_1} p_2^{x_2} \mathbb{1}_{\underbrace{x_1 + x_2 = n}_{x_1 \in \{n - x_2\}}} \cancelto{1}{\mathbb{1}_{x_1 \in \{0,\ldots,n\}}} \mathbb{1}_{x_2 \in \{0,\ldots,n\}}$$

$$= \frac{n!}{x_2!} p_2^{x_2} \mathbb{1}_{x_2 \in \{0,\ldots,n\}} \sum_{x_1 \in \{n - x_2\}} \frac{1}{x_1!} p_1^{x_1} = \frac{n!}{x_2!} p_2^{x_2} \mathbb{1}_{x_2 \in \{0,\ldots,n\}} \frac{1}{(n-x_2)!} p_1^{n-x_2}$$

$p_1 = 1 - p_2$

$$\stackrel{\downarrow}{=} \binom{n}{x_2} p_2^{x_2} (1-p_2)^{n-x_2}$$

$$= \text{Bin}(n, x_2)$$

$$p_{X_1|X_2}(x_1, x_2) = \frac{p_{X_1, X_2}(x_1, x_2)}{p_{X_2}(x_2)}$$

$$= \frac{\binom{n}{x_1, x_2} p_1^{x_1} \cancel{p_2^{x_2}}}{\binom{n}{x_2} \cancel{p_2^{x_2}} (\underbrace{1-p_2}_{p_1})^{n-x_2}}$$

$$= \frac{\frac{\cancel{n!}}{x_1!\, \cancel{x_2!}} \mathbb{1}_{\underbrace{x_1 + x_2 = n}_{x_1 = n - x_2}} \cancelto{1}{\mathbb{1}_{x_1 \in \{0,\ldots,n\}}} \mathbb{1}_{x_2 \in \{0,\ldots,n\}} p_1^{x_1}}{\frac{\cancel{n!}}{\cancel{x_2!}(n-x_2)!} \mathbb{1}_{x_2 \in \{0,\ldots,n\}} p_1^{n-x_2}}$$

$$\mathbb{1}^{u}_{A} := \begin{cases} 1 & \text{if } A \\ \text{dne.} & \text{if } A^c \end{cases} \Downarrow \quad \mathbb{1}^{u}_{A} = \frac{\mathbb{1}_A}{\mathbb{1}_A}$$

$$= \frac{(n-x_2)!}{x_1!} \mathbb{1}_{x_1 = n - x_2} p_1^{x_1 + x_2 - n} \mathbb{1}^{u}_{x_2 \in \{0,\ldots,n\}}$$

$$= \mathbb{1}^{u}_{x_2 \in \{0,\ldots,n\}} \begin{cases} \frac{(n-x_2)!}{x_1!} p_1^{x_1 + x_2 - n} & \text{if } x_1 = n - x_2 \\ 0 & \text{if not} \end{cases}$$

$$= \mathbb{1}^{u}_{x_2 \in \{0,\ldots,n\}} \boxed{\begin{cases} 1 & \text{if } x_1 = n - x_2 \\ 0 & \text{if not} \end{cases}}$$

$$= \text{Deg}(n - x_2) \mathbb{1}^{u}_{x_2 \in \{0,\ldots,n\}}$$

$\vec{X} \sim \text{Multi}_K(n, \vec{p})$ let $\vec{X}_{-j}$ denote all elements of X-vec *except* the jth component

$$\Rightarrow \vec{X}_{-j} \mid X_j \sim p_{\vec{X}_{-j}|X_j}(\vec{x}, x_j) \underset{\underset{\text{intuition}}{\uparrow}}{=} \text{Multi}_{K-1}(n - x_j, \vec{p}\,')$$

$$= \frac{p_{\vec{X}}(\vec{x}, x_j)}{p_{X_j}(x_j)}$$

$$= \frac{\binom{n}{x_1 \ldots x_j \ldots x_K} p_1^{x_1} \cdot \ldots \cdot p_j^{x_j} \cdot \ldots \cdot p_K^{x_K}}{\binom{n}{x_j} p_j^{x_j} (1-p_j)^{n-x_j}}$$

$$\mathbb{1}^{u}_{x_j \in}$$

$$= \frac{\frac{\cancel{n!}}{x_1! \cdots \cancel{x_j!} \cdots x_K!} \mathbb{1}_{x_1 \in} \cdots \cancel{\mathbb{1}_{x_j \in}} \cdots \mathbb{1}_{x_K \in} \mathbb{1}_{x_1 + \ldots + x_j + \ldots + x_K = n} p_1^{x_1} \cdots \cancel{p_j^{x_j}} \cdots p_K^{x_K}}{\frac{\cancel{n!}}{\cancel{x_j!}(n-x_j)!} \cancel{\mathbb{1}_{x_j \in}} \cancel{p_j^{x_j}} (1-p_j)^{n-x_j}}$$

$$= \underbrace{\frac{(n-x_j)!}{x_1! \cdots x_{j-1}!\, x_{j+1}! \cdots x_K!} \mathbb{1}_{x_1 \in} \cdots \mathbb{1}_{x_{j-1} \in} \mathbb{1}_{x_{j+1} \in} \cdots \mathbb{1}_{x_K \in} \mathbb{1}_{x_1 + \ldots + x_{j-1} + x_{j+1} + \ldots + x_K = n - x_j}} \frac{p_1^{x_1} \, p_{j-1}^{x_{j-1}} \, p_{j+1}^{x_{j+1}} \cdots p_K^{x_K} \, \mathbb{1}^{u}_{x_j \in}}{(1-p_j)^{n-x_j}}$$

let $n' := n - x_j = x_1 + \ldots + x_{j-1} + x_{j+1} + \ldots + x_K$

$$= \binom{n'}{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_K} \frac{p_1^{x_1}}{(1-p_j)^{x_1}} \cdot \ldots \cdot \frac{p_{j-1}^{x_{j-1}}}{(1-p_j)^{x_{j-1}}} \cdot \frac{p_{j+1}^{x_{j+1}}}{(1-p_j)^{x_{j+1}}} \cdot \ldots \cdot \frac{p_K^{x_K}}{(1-p_j)^{x_K}}$$

$$\text{let } \vec{p}\,' = \begin{bmatrix} p_1/(1-p_j) \\ \vdots \\ p_{j-1}/(1-p_j) \\ p_{j+1}/(1-p_j) \\ \vdots \\ p_K/(1-p_j) \end{bmatrix}$$

$$= \text{Multi}_{K-1}(n', \vec{p}\,') \mathbb{1}^{u}_{x_j \in \{0,\ldots,n\}}$$

$$X \sim U(\{0,1,2,3\}) = \begin{cases} 0 & \text{wp } \frac{1}{4} \\ 1 & \text{wp } \frac{1}{4} \\ 2 & \text{wp } \frac{1}{4} \\ 3 & \text{wp } \frac{1}{4} \end{cases}$$

this is called the "uniform discrete" rv and in general...

$$X \sim U(A), \quad \text{Supp}[X] = A, \quad A \subset \mathbb{R} \text{ s.t. } A \text{ is finite}$$

$$\overset{\shortparallel}{\frac{1}{|A|} \mathbb{1}_{x \in A}}$$

let $Y = g(X) = -X$

$$\begin{aligned} X = 0 &\Leftrightarrow Y = 0 \\ X = 1 &\Leftrightarrow Y = -1 \\ X = 2 &\Leftrightarrow Y = -2 \\ X = 3 &\Leftrightarrow Y = -3 \end{aligned}$$

$$\Rightarrow P(X=2) = P(Y=-2)$$

$$p_Y(y) := P(Y = y) = P(-X = y) = P(X = -y) = p_X(-y)$$

$$\text{Supp}[Y] = \{z : p_Y(z) > 0\} = \{z : p_X(-z) > 0\} \overset{z' = -z}{=} \{-z' : p_X(z') > 0\}$$

$$= -\{z' : p_X(z') > 0\} = -\text{Supp}[X].$$

$$X \sim \underset{\binom{n}{x} p^x (1-p)^{n-x}}{\text{Bin}(n,p)}, \quad Y = -X \sim \binom{n}{-y} p^{-y} (1-p)^{n+y}$$

$$X \sim \text{Poisson}(\lambda), \quad Y = -X \sim \frac{\lambda^{-y} e^{-\lambda}}{(-y)!}$$

$$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda) \Longrightarrow T = X_1 + X_2 \sim \text{Poisson}(2\lambda)$$

$$p_{X_1|T}(x, t) = \frac{p_{X_1, T}(x, t)}{p_T(t)} \qquad \begin{bmatrix} X_1 \\ T \end{bmatrix} \text{ is 1:1 with } \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$= \frac{p_{X_1, X_2}(x, t - x)}{p_T(t)}$$

since $X_1, X_2 \overset{ind}{\sim}$

$$= \frac{p_{X_1}(x)\, p_{X_2}(t - x)}{p_T(t)}$$

$$= \frac{\frac{\lambda^x \cancel{e^{-\lambda}}}{x!} \frac{\lambda^{t-x} \cancel{e^{-\lambda}}}{(t-x)!}}{\frac{(2\lambda)^t \cancel{e^{-2\lambda}}}{t!}}$$

$$= \frac{\lambda^t\, t!}{(2\lambda)^t\, x!\, (t-x)!}$$

$$= \binom{t}{x} \left(\frac{1}{2}\right)^t = \binom{t}{x} \left(\frac{1}{2}\right)^x \left(1 - \frac{1}{2}\right)^{t-x} = \text{Bin}\left(t, \frac{1}{2}\right)$$

Skellam

$$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda), \qquad D := X_1 - X_2 = X_1 + {}^{-}X_2 \sim p_D(d)$$

$$\text{Supp}[D] = \text{Supp}[X_1] + \text{Supp}[-X_2] = \text{Supp}[X_1] + {}^{-}\text{Supp}[X_2]$$

$$= \{0, 1, 2, \ldots\} + \{\ldots, -2, -1, 0\} = \mathbb{Z} \text{ (integers)}$$