Math 368 / 650 Fall 2021
Midterm Examination Two

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Thursday, November 11, 2021

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

By taking this exam, you acknowledge and agree to uphold this Code of Academic Integrity.

Instructions

This exam is 70 minutes (variable time per question) and closed-book. You are allowed **one** page (front and back) of a "cheat sheet", blank scrap paper and a graphing calculator. Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 [12min] (and 12min will have elapsed) Let $X_1 \sim \text{Exp}(\lambda_1)$ independent of $X_2 \sim \text{Exp}(\lambda_2)$ where $\lambda_1 \neq \lambda_2$ but are both valid values in the parameter space of the exponential rv. Let $T = X_1 + X_2$.

- [17 pt / 17 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

(a) X_1 does not have a PMF

(b) X_1 does not have a CDF

(c) $\mathbb{P}(T \leq x) = \mathbb{P}(X_1 \leq x, X_2 \leq x)$

(d) $\mathbb{P}(X_1 > x, X_2 > x) = e^{-(\lambda_1+\lambda_2)x}$

(e) $\mathbb{S}\text{upp}[T] = (0, \infty)$

(f) T is Erlang-distributed

(g) T is Gamma-distributed

(h) $T \sim \int_{\mathbb{S}\text{upp}[X_1]} f_{X_1}(x) f_{X_2}(x) dx$

(i) $T \sim \int_{\mathbb{S}\text{upp}[X_1]} f_{X_1}(x) f_{X_2}(t-x) dx$

(j) $T \sim \int_{\mathbb{S}\text{upp}[X_1]} f_{X_1}^{old}(x) f_{X_2}^{old}(t-x) \mathbb{1}_{t-x \in \mathbb{S}\text{upp}[X_1]} dx$

(k) $T \sim \int_0^\infty \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2(t-x)} \mathbb{1}_{x-t \in (-\infty, 0)} dx$

(l) $T \sim \lambda_1 \lambda_2 \int_0^t e^{-\lambda_1 x} e^{-\lambda_2(t-x)} dx$

(m) $T \sim \lambda_1 \lambda_2 e^{-\lambda_2 t} \mathbb{1}_{t>0} \int_0^t e^{(\lambda_2-\lambda_1)x} dx$

(n) $T \sim \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_2 t} \mathbb{1}_{t>0} \left[e^{(\lambda_2-\lambda_1)x}\right]_0^t$

(o) $T \sim \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_2 t} e^{(\lambda_2-\lambda_1)t} \mathbb{1}_{t>0}$

(p) $T \sim \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) \mathbb{1}_{t>0}$

(q) If (p) were to be true, then the density of T would have a kernel given by $k(t) = e^{-\lambda_1 t} - e^{-\lambda_2 t}$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 2 [9min] (and 21min will have elapsed) Let $X_1, X_2 \overset{iid}{\sim} \text{Lomax}(\alpha, \lambda) := \overbrace{\alpha\lambda^{\alpha}(x+\alpha)^{-(\alpha+1)}}^{f^{old}(x)} \mathbb{1}_{x>0}$ with parameter space $\alpha, \lambda > 0$. Let $T = X_1 + X_2$, $R = X_1/X_2$ and $N = X_1/(X_1+X_2)$

- [10 pt / 27 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

  (a) If the density of X_i were to be decomposed into $c \times k(x)$ then $c = \alpha\lambda^{\alpha}$

  (b) $f_T(t) = \mathbb{1}_{t>0} \int_0^t f^{old}(x) f^{old}(t-x)dx$

  (c) $f_T(t) \propto \mathbb{1}_{t>0} \int_0^t f^{old}(x) f^{old}(t-x)dx$

  (d) $f_T(t) \propto \mathbb{1}_{t>0} \int_0^t ((x+\alpha)(t-x+\alpha))^{-(\alpha+1)} dx$

  (e) $f_R(r) \propto \mathbb{1}_{r>0} \displaystyle\int_0^t \frac{f^{old}(x)}{f^{old}(r)} dx$

  (f) $f_R(r) = \mathbb{1}_{r>0} \int_0^{\infty} x f^{old}(rx) f^{old}(x)dx$

  (g) $f_R(r) \propto \mathbb{1}_{r>0} \int_0^{\infty} x((rx+\alpha)(x+\alpha))^{-(\alpha+1)} dx$

  (h) $f_N(n) \propto \mathbb{1}_{n>0} \displaystyle\int_0^t \frac{f^{old}(x)}{f^{old}(x) + f^{old}(n)} dx$

  (i) $f_N(n) = \mathbb{1}_{n>0} \int_0^{\infty} x f^{old}(nx) f^{old}(x-nx)dx$

  (j) $f_N^{old}(n) \propto \int_0^{\infty} x((nx+\alpha)(x-nx+\alpha))^{-(\alpha+1)} dx$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 3 [8min] (and 29min will have elapsed) Let $X_1, X_2 \overset{iid}{\sim} \text{Gamma}\,(\alpha,\, \beta)$. Let $T = X_1 + X_2$, $R = X_1/X_2$ and $N = X_1/(X_1 + X_2)$.

- [11 pt / 38 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

(a) $F_X(x) = P(\alpha, \beta x)$

(b) $F_X(x) \propto \gamma(\alpha, \beta x)$

(c) $T \sim \text{Gamma}\,(2\alpha,\, \beta)$

(d) $T \sim \text{Erlang}\,(2\alpha,\, \beta)$ if $2\alpha \in \mathbb{N}$

(e) $R \sim \text{BetaPrime}(\alpha, \alpha)$

(f) $N \sim \text{Beta}(\alpha, \alpha)$

(g) $N \sim \text{Beta}(\beta, \alpha)$

Assume (f) is true for the remainder of this problem

(h) $N \sim \text{U}\,(0,\, 1)$ if $\alpha = 1$

(i) $F_N(n) = \int_0^n (u(1-u))^{\alpha-1} du$

(j) $F_N(n) = B(n, \alpha, \alpha)/B(\alpha, \alpha)$

(k) $F_N(n) = I_n(\alpha, \alpha)$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 4 [13min] (and 42min will have elapsed) Let $X_1, X_2 \overset{iid}{\sim} \text{Gamma}(\alpha, \beta)$. Let $M = X_1 X_2$, $\boldsymbol{g} : \mathbb{R}^2 \to \mathbb{R}^2$, $\boldsymbol{h} : \mathbb{R}^2 \to \mathbb{R}^2$ which denotes the inverse of $\boldsymbol{g}$, $\boldsymbol{X} := \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $\boldsymbol{x} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\boldsymbol{Y} := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ and $\boldsymbol{y} := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

- [14 pt / 52 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

  (a) $\mathbb{S}\text{upp}[M] = (0, \infty)$

  (b) The function $\boldsymbol{g}(x_1, x_2) = \begin{bmatrix} x_1 x_2 \\ x_1 x_2 \end{bmatrix}$ is invertible.

  (c) The function $\boldsymbol{g}(x_1, x_2) = \begin{bmatrix} x_1 x_2 \\ x_2 \end{bmatrix}$ is invertible.

  (d) If $\boldsymbol{h}(y_1, y_2) = \begin{bmatrix} y_1/y_2 \\ y_2 \end{bmatrix}$, then the Jacobian determinant is $\begin{bmatrix} 1/y_2 & -y_1/y_2^2 \\ 0 & 1 \end{bmatrix}$.

  (e) If $\boldsymbol{h}(y_1, y_2) = \begin{bmatrix} y_1/y_2 \\ y_2 \end{bmatrix}$, then the Jacobian determinant is $1/y_2$.

  (f) If $\boldsymbol{x} = \boldsymbol{h}(y_1, y_2) = \begin{bmatrix} y_1/y_2 \\ y_2 \end{bmatrix}$ then $f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{X}}(y_1/y_2, y_2)/|y_2|$

  (g) $f_M(m) = f_{\boldsymbol{X}}(m, m)$

  (h) $f_M(m) = f_{X_1}(m) f_{X_2}(m)$

  (i) $f_M(m) = \int_{\mathbb{R}} f_{X_1}(um) f_{X_2}(m) du$

  (j) $f_M(m) = \int_{\mathbb{R}} f_{X_1}(m/u) f_{X_2}(u)/u \, du$

  (k) $f_M(m) = \int_{\mathbb{R}} f_{X_1}(m/u) f_{X_2}(u)/|u| \, du$

  (l) $f_M(m) = \int_{\mathbb{R}} f_{\boldsymbol{X}}(m/u, u)/|u| \, du$

  (m) $f_M(m) = \dfrac{\beta^{2\alpha}}{\Gamma(\alpha)^2} \mathbb{1}_{m>0} \displaystyle\int_0^\infty \frac{1}{u} \left(\frac{m}{u}\right)^{\alpha-1} e^{-\beta m/u} u^{\alpha-1} e^{-\beta u} \, du$

  (n) $f_M(m) = \dfrac{\beta^{2\alpha}}{\Gamma(\alpha)^2} m^{\alpha-1} \mathbb{1}_{m>0} \displaystyle\int_0^\infty \frac{e^{-\beta(m/u+u)}}{u} \, du$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 5 [9min] (and 51min will have elapsed) Consider a sequence of rv's $X_1, \ldots, X_n \overset{iid}{\sim} \text{Logistic}\,(0,\,1) := \dfrac{e^{-x}}{(1+e^{-x})^2}$ whose expectation is zero and let $X_{(1)}, \ldots, X_{(n)}$ denote this sequence's order statistics and $R := X_{(n)} - X_{(1)}$.

- [18 pt / 70 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

(a) For all i, X_i is an "error distribution"

(b) There exist nonzero constants a, b such that for all i, $aX_i + b$ is an "error distribution"

(c) $F_{X_{(n)}} = F(x)^n$

(d) If $X_1, \ldots, X_n$ were not independent, the formula in (c) could be different

(e) $X_{(1)}, \ldots, X_{(n)} \overset{iid}{\sim} \text{Logistic}\,(0,\,1)$

(f) $X_{(1)}, \ldots, X_{(n)}$ are all independent

(g) $X_{(1)}$ has most of its mass near zero

(h) $X_{(n)}$ has most of its mass near zero

(i) $\mathbb{S}\text{upp}\left[X_{(k)}\right] = \mathbb{R}$ for all k and all n

(j) $\mathbb{E}\left[R\right] = 0$

(k) $\mathbb{P}\left(R \in [-a, +a]\right)$ for $a \in \mathbb{R}$ increases as n gets larger

(l) $F_{X_{(1)}}(x) = 1 - \left(\dfrac{e^{-x}}{1+e^{-x}}\right)^n$

(m) $F_{X_{(1)}}(x) = 1 - n\left(\dfrac{e^{-x}}{1+e^{-x}}\right)^n$

(n) $F_{X_{(1)}}(x) = 1 - \left(\dfrac{e^{-x}}{(1+e^{-x})^2}\right)^n$

(o) $f_{X_{(1)}}(x) = n\left(\dfrac{e^{-x}}{1+e^{-x}}\right)^n$

(p) $f_{X_{(k)}}(x) = n\dfrac{e^{-x(n-k+1)}}{(1+e^{-x})^{n+1}}$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 6 [8min] (and 59min will have elapsed) Let $X \sim \text{Exp}(\lambda)$ and $Y \mid X = x \sim \text{U}(0,\, x)$

- [10 pt / 80 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

  (a) The support of the joint density of X and Y is in the first quadrant of the cartesian plane, below the line $y = x$ and above the x-axis.

  (b) $\mathbb{S}\text{upp}[Y] = [0, 1]$

  (c) $\mathbb{P}(Y > 1/2) = 1/2$ for all λ in the parameter space of the exponential rv

  (d) For positive y, $f_Y(y)$ is monotonically decreasing

  (e) $f_X(x) = \lambda e^{-\lambda x}$

  (f) $f_Y(y) = \lambda e^{-\lambda y}$

  (g) $f_Y(y) = \mathbb{1}_{y>0} \displaystyle\int_y^\infty \frac{e^{-\lambda x}}{x} dx$

  (h) $\mathbb{S}\text{upp}[X \mid Y = y] = [y, \infty)$

  (i) $X \mid Y = y$ is a uniform rv

  (j) $\displaystyle\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{x} \mathbb{1}_{y \in [0,x]}\ \lambda e^{-\lambda x} \mathbb{1}_{x>0}\ dx dy = 1$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 7 [11min] (and 70min will have elapsed) Let $X \sim \text{Exp}(\lambda)$ and $Y \mid X = x \sim \text{Exp}(x)$

- [12 pt / 92 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

  (a) $f_X(x) = \lambda e^{-\lambda x}$
  (b) $f_{Y \mid X}(y, x) = x e^{-xy} \mathbb{1}_{y>0}$
  (c) $\lambda \int_0^\infty \int_0^\infty x e^{-xy} \, e^{-\lambda x} \, dx dy = 1$
  (d) The rv Y is considered a "compound distribution"
  (e) $\mathbb{S}\text{upp}[Y] = \mathbb{R}$
  (f) Y is not a valid rv
  (g) $f_Y(y) = \int_{\mathbb{R}} f_{Y \mid X}(y, x) f_X(x) dx$
  (h) $f_Y(y) = e^{-\lambda y} / y \mathbb{1}_{y>0}$
  (i) $f_Y(y) = \lambda e^{-\lambda y} / (y + \lambda) \mathbb{1}_{y>0}$
  (j) $f_Y(y) = \lambda y / (y + \lambda) \mathbb{1}_{y>0}$
  (k) $f_Y(y) = \lambda / (y + \lambda)^2 \mathbb{1}_{y>0}$
  (l) $f_Y(y) = \lambda e^{-\lambda y} / y \mathbb{1}_{y>0}$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Some of these antiderivatives (from Wolfram Alpha) may help you with the above problem:

$$
\begin{aligned}
\int x^3 e^{-ax} dx &= -\frac{e^{-ax}(a^3x^3 + 3a^2x^2 + 6ax + 6)}{a^4} + C \\
\int x^2 e^{-ax} dx &= -\frac{e^{-ax}(a^2x^2 + 2ax + 2)}{a^3} + C \\
\int x e^{-ax} dx &= -\frac{e^{-ax}(ax + 1)}{a^2} + C
\end{aligned}
$$