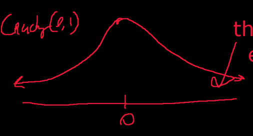
$R \sim \text{Cauchy}(0,1)$



the tails are "really thick" meaning extreme values occur regularly

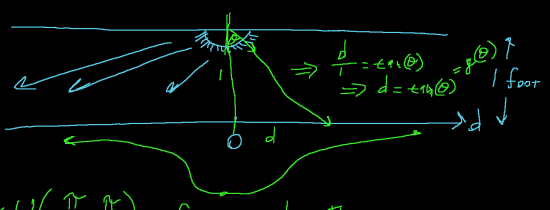
$$E[R] = \int_{\mathbb{R}} r \frac{1}{\pi} \frac{1}{1+r^2} dr = \infty$$ the expectation does not exist!

$$M_R(t) = E[e^{rt} f_R(r)] = \int_{\mathbb{R}} e^{rt} \frac{1}{\pi} \frac{1}{1+r^2} dr = \infty$$ the MGF doesn't exist!

$$\phi_R(t) = E[e^{irt}] = \int_{\mathbb{R}} e^{irt} \frac{1}{\pi} \frac{1}{1+r^2} dr = \underset{\text{complex analysis}}{\cdots} = e^{-|t|}$$

$$\phi_R'(t) = -\frac{t}{|t|} e^{-|t|}, \quad \phi_R'(0) =$$ doesn't exist i.e. there is no expectation

How did Cauchy derive his distribution? Imagine a light on a ceiling 1 foot above the ground. Consider one dimention. The light gives off rays uniformly over all 180 degrees of the ceiling. What is the distribution of the intensity of the light on the floor?



$$\Theta \sim U\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = f_\Theta(\theta) = \frac{1}{\pi} \mathbb{1}_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]}$$

$D \sim ?$

$$d = \tan(\theta) \iff \theta = \arctan(d) = g^{-1}(d) \quad 1:1$$

$$f_D(d) = f_\Theta(g^{-1}(d)) \left| \frac{d}{dd}[g^{-1}(d)] \right| \qquad \left| \frac{d}{dd}[\arctan(d)] \right| = \frac{1}{1+d^2}$$

$$= \frac{1}{\pi} \mathbb{1}_{\overbrace{\arctan(d) \in [-\frac{\pi}{2}, \frac{\pi}{2}]}^{d \in \mathbb{R}}} \frac{1}{1+d^2} = \frac{1}{\pi} \frac{1}{1+d^2} = \text{Cauchy}(0,1)$$

We now have all the tools to derive everything in an intro-to-stats course. For example:

$$X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2). \quad T_n = X_1 + \ldots + X_n, \quad \bar{X}_n = \frac{T}{n}$$

$$\Rightarrow T_n \sim N(n\mu, n\sigma^2)$$ proved using chf's

$$\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$ proved using shift and scale formula

this fact gives us the 1-sample z-test in Stats 101.

$$S_n^2 := \frac{1}{n-1} \sum (X_i - \bar{X}_n)^2 \sim ?$$

This is called the sample variance estimator.

$$= \frac{1}{n-1}\left((X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + \ldots + (X_n - \bar{X})^2\right) \sim ?$$

It looks like each $(X_i - \bar{X})^2$ term are iid. But... they're not since they both are functions of $\bar{X}$. So we're stuck...

Let's begin with something easier....

$$Z_1, \ldots, Z_n \overset{iid}{\sim} N(0,1). \quad \vec{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$$

$$\Rightarrow \vec{Z}^T \vec{Z} = \sum_{i=1}^n Z_i^2 \sim \chi^2_n \Rightarrow \sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$$

$$Z_i := \frac{X_i - \mu}{\sigma}$$ the standardized rv from above

$$X_i - \mu = X_i - \bar{X} + \bar{X} - \mu$$

$$\Rightarrow (X_i - \mu)^2 = ((X_i - \bar{X}) + (\bar{X} - \mu))^2 = (X_i - \bar{X})^2 + 2\underbrace{(X_i - \bar{X})(\bar{X} - \mu)}_{X_i\bar{X} - \bar{X}^2 - X_i\mu + \bar{X}\mu} + (\bar{X} - \mu)^2$$

$$\Rightarrow \sum_{i=1}^n (X_i - \mu)^2 = \sum (X_i - \bar{X})^2 + 2\left(\sum X_i\bar{X} - \sum \bar{X}^2 - \sum X_i\mu + \sum \bar{X}\mu\right) + \sum(\bar{X} - \mu)^2$$

$$= \underbrace{\sum (X_i - \bar{X})^2}_{(n-1)S^2} + 2\Big(\underbrace{\bar{X}\underbrace{\sum X_i}_{n\bar{X}}}_{\cancel{n\bar{X}^2}} - \cancel{n\bar{X}^2} - \underbrace{\mu\underbrace{\sum X_i}_{n\bar{X}}}_{\cancel{n\mu\bar{X}}} + \cancel{n\bar{X}\mu}\Big) + n(\bar{X} - \mu)^2$$

$$= (n-1)S^2 + n(\bar{X} - \mu)^2$$

$$\Rightarrow \vec{Z}^T\vec{Z} = \sum \left(\frac{X_i - \mu}{\sigma}\right)^2 = \underbrace{\frac{n-1}{\sigma^2} S^2}_{\sim \chi^2_{n-1}} + \underbrace{\frac{n(\bar{X} - \mu)^2}{\sigma^2}}_{\sim \chi^2_1} \sim \chi^2_n$$

We decomposed a χ^2_n rv into functions of the sample variance and sample average.

$$\frac{n(\bar{X} - \mu)^2}{\sigma^2} = \frac{(\bar{X} - \mu)^2}{\frac{\sigma^2}{n}} = \left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 = Z^2 \sim \chi^2_1$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

Conjecture:

$\frac{n-1}{\sigma^2} S^2 \sim \chi^2_{n-1}$ and indep. of $\bar{X}$.

Because...

$$U_1 \sim \chi^2_{k_1} \text{ indep of } U_2 \sim \chi^2_{k_2} \Rightarrow U_1 + U_2 \sim \chi^2_{k_1 + k_2}$$

$$\vec{Z}^T \vec{Z} \sim \chi^2_n$$

$$\Rightarrow \underbrace{\vec{Z}^T I_n \vec{Z}}_{\text{quadratic form}} \sim \chi^2_n \qquad B = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 0 & \\ & 0 & & \ddots \end{bmatrix}$$

quadratic form

$$\vec{Z}^T \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}}_{B_1} \vec{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix} \cdot \begin{bmatrix} Z_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Z_1^2 \sim \chi^2_1$$

indep

$$\vec{Z}^T \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & 0 & 0 & \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{B_2} \vec{Z} = Z_2^2 \sim \chi^2_1$$

$$\Rightarrow \vec{Z}^T B_1 \vec{Z} + \vec{Z}^T B_2 \vec{Z} + \ldots + \vec{Z}^T B_n \vec{Z} = \vec{Z}^T I \vec{Z} \sim \chi^2_n$$

$$\Rightarrow \vec{Z}^T (B_1 + \ldots + B_n) \vec{Z}$$ and $\vec{Z}^T B_i \vec{Z}$ are all indep. and χ^2_1

$$\text{rank}[B_1] = 1, \ \text{rank}[B_2] = 1, \ldots, \ \text{rank}[B_n] = 1$$

$$\sum_{i=1}^n \text{rank}[B_i] = n$$

Cochran's Thm

If $B_1 + B_2 + \ldots + B_n = I$ and $\text{rank}[B_1] + \text{rank}[B_2] + \ldots + \text{rank}[B_n] = n$ then

(a) $\vec{Z}^T B_i \vec{Z} \sim \chi^2_{\text{rank}[B_i]}$

(b) all $\vec{Z}^T B_i \vec{Z}$ are independent of each other

The example B_i's above is a special case of Cochran's thm. It is not a proof since it is only one example. We won't do the proof.

Cochran's thm gives us a geometric intuition about chi-squared constituent rv's. The degrees of freedom represent the number of dimensions out of the total n dimensional space of all n Z's.

Let's return to proving the conjecture.

$$\vec{Z}^T \vec{Z} = \sum Z_i^2 = \sum ((Z_i - \bar{Z}) + (\bar{Z}))^2$$

$$= \sum (Z_i - \bar{Z})^2 + 2(Z_i - \bar{Z})\bar{Z} + \bar{Z}^2$$

$$= \sum (Z_i - \bar{Z})^2 + 2\Big(\underbrace{\sum Z_i \bar{Z}}_{\underbrace{\bar{Z}\sum Z_i}_{\underbrace{n\bar{Z}}_{n\bar{Z}^2}}} - \underbrace{\sum \bar{Z}^2}_{n\bar{Z}^2}\Big) + \underbrace{\sum \bar{Z}^2}_{n\bar{Z}^2}$$

$$\text{let } \vec{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow \bar{Z} = \frac{Z_1 + \ldots + Z_n}{n} = \frac{1}{n} \vec{Z}^T \vec{1}$$

$$\Rightarrow n\bar{Z}^2 = n\left(\frac{1}{n} \vec{Z}^T \vec{1}\right)^2 = \frac{1}{n} \vec{Z}^T \vec{1}\, \vec{Z}^T \vec{1}$$

$$= \frac{1}{n} \vec{Z}^T \vec{1} (\vec{Z}^T \vec{1})^T = \frac{1}{n} \vec{Z}^T \vec{1}\, \vec{1}^T \vec{Z} = \vec{Z}^T \underbrace{\left(\frac{1}{n} J_n\right)}_{B_2} \vec{Z}$$

$$J_n := \vec{1}_n \vec{1}_n^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{bmatrix}$$

we have it now as a quadratic form