Fourier Analysis / Synthesis goes back to 1807. Informally, his idea was that functions can be decomposed into a sum of sines and cosines where f(x) is called the "time domain" of the signal and $\hat{f}(\omega)$ is the "frequency domain". Further, $|\hat{f}(\omega)|$ provides the amplitudes of the sine/cosines and Arg[$\hat{f}(\omega)$] provides their phase shifts.

Back to probability theory... Let X be a rv an define its "characteristic function" (chf) as

$$\phi_X(t) := E\left[e^{itX}\right] = \int_{\mathbb{R}} e^{itx} f(x)\,dx \quad \text{this is the Fourier analysis in the unit } t = -2\pi\omega$$

$$= \sum_{x \in \mathbb{R}} e^{itx} p(x) \quad \text{and this is "discrete Fourier analysis"}$$

Properties of the chf:

(P0) $\phi_X(0) = E\left[e^{it(0)}\right] = E[1] = 1$

(P1) $\phi_X(t) = \phi_Y(t) \iff X \stackrel{d}{=} Y$ (uniqueness)

(P2) $Y = aX + b \Rightarrow \phi_Y(t) = E\left[e^{it(aX+b)}\right]$

$$= E\left[e^{iatX} e^{itb}\right]$$

$$= e^{itb} E\left[e^{it'X}\right] \qquad \text{let } t' = at$$

$$= e^{itb} \phi_X(t') = e^{itb}\phi_X(at)$$

(P3) $X_1, X_2 \stackrel{ind}{\sim}$, $T = X_1 + X_2$

$$\phi_T(t) = E\left[e^{it(X_1+X_2)}\right] = E\left[e^{itX_1} e^{itX_2}\right] = E\left[e^{itX_1}\right]E\left[e^{itX_2}\right]$$

$$= \phi_{X_1}(t)\,\phi_{X_2}(t)$$

(P4) Moment generation

technical conditions fulfilled

$$\phi_X'(t) = \frac{d}{dt}\left[E\left[e^{itX}\right]\right] = E\left[\frac{d}{dt}\left[e^{itX}\right]\right] = E\left[iXe^{itX}\right]$$

$$\phi_X'(0) = E\left[iXe^{i(0)X}\right] = E[iX] = iE[X] \Rightarrow E[X] = \frac{\phi_X'(0)}{i}$$

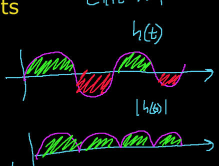
$$\phi_X''(t) = E\left[iX\frac{d}{dt}\left[e^{itX}\right]\right] = E\left[i^2X^2e^{itX}\right]$$

$$\phi_X''(0) = E\left[i^2X^2e^{i(0)X}\right] = E\left[i^2X^2\right] = i^2E\left[X^2\right] \Rightarrow E\left[X^2\right] = \frac{\phi_X''(0)}{i^2}$$

$\vdots$

$$E\left[X^n\right] = \frac{\phi_X^{(n)}(0)}{i^{n \bmod 4}} \quad \text{if the moment exists}$$

Calc 101



(P5) Existence and boundedness

$$|\phi_X(t)| = \left|E\left[e^{itX}\right]\right| = \left|\int_{\mathbb{R}} e^{itx} f(x)\,dx\right|$$

$$\leq \int_{\mathbb{R}} \left|e^{itx} f(x)\right| dx$$

$$= \int_{\mathbb{R}} \left|e^{itx}\right| f(x)\,dx = \int_{\mathbb{R}} f(x)\,dx = 1$$

$$\left|e^{itx}\right| = \left|\underbrace{i\sin(tx)}_{bi} + \underbrace{\cos(tx)}_{a}\right| = \sqrt{a^2+b^2} = \sqrt{\sin^2(tx) + \cos^2(tx)} = \sqrt{1} = 1$$

$$\Rightarrow |\phi_X(t)| \leq 1 \Rightarrow \phi_X(t) \in [-1, 1]$$

(P6) Inversion (consequence of Fourier inversion thm)

$$\text{If } \phi_X(t) \in L^1 \Rightarrow f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\phi_X(t)\,dt$$

(P7) Levy's CDF thm (we won't use this in this class)

$$P\left(X \in [a,b]\right) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it}\phi_X(t)\,dt$$

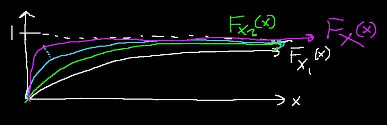
No need for $\phi_X(t) \in L^1$!

(P8) Levy's Continuity Thm. Consider a sequence of rv's $X_1, X_2, \ldots, X_n$

$$\text{If } \lim_{n\to\infty} \phi_{X_n}(t) = \phi_X(t) \Rightarrow X_n \xrightarrow{d} X$$

Convergence in distribution (denoted $\xrightarrow{d}$) is defined as pointwise convergence of the sequence of rv's CDF i.e.

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \forall x$$



Define the moment generating function (MGF)

$$M_X(t) := E\left[e^{tX}\right]$$

(P0) $M_X(0) = 1$

(P1) $M_X(t) = M_Y(t) \Longrightarrow X \stackrel{d}{=} Y$

(P2) $Y = aX + b \Rightarrow M_Y(t) = e^{tb}M_X(at)$

(P3) $X_1, X_2 \stackrel{ind}{\sim}$, $T = X_1 + X_2 \Rightarrow M_T(t) = M_{X_1}(t)M_{X_2}(t)$

(P4) $E\left[X^n\right] = M_X^{(n)}(0)$

There is no (P5) since MGF's may not exist and may not be bounded for all t. There is a limited form of (P6). There is no (P7). There is a limited form of (P8).

We will use chf's over mgf's because they are more powerful. They can do everything mgf's can do and much more.

$X \sim \text{Gamma}(\alpha, \beta)$

$$\phi_X(t) := E\left[e^{itX}\right] = \int_0^\infty e^{itx} \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}}_{f(x)} dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta - it)x}\,dx$$

$$= \frac{\beta^\alpha}{\cancel{\Gamma(\alpha)}} \frac{\cancel{\Gamma(\alpha)}}{(\beta - it)^\alpha} = \left(\frac{\beta}{\beta - it}\right)^\alpha$$

1:1

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ indep. of $X_2 \sim \text{Gamma}(\alpha_2, \beta)$, $T = X_1 + X_2 \sim ?$

$$\phi_{X_1+X_2}(t) \underset{(P3)}{=} \phi_{X_1}(t)\,\phi_{X_2}(t) = \left(\frac{\beta}{\beta - it}\right)^{\alpha_1}\left(\frac{\beta}{\beta - it}\right)^{\alpha_2}$$

$$= \left(\frac{\beta}{\beta - it}\right)^{\alpha_1 + \alpha_2} \overset{(P1)}{\Longrightarrow} X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \beta)$$

$X \sim \text{Poisson}(\lambda)$

$$\frac{e^{-\lambda e^{it}}}{e^{-\lambda e^{it}}}$$

$$\phi_X(t) = \sum_{x \in \{0,1,\ldots\}} e^{itx} \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^\infty \left(e^{it}\right)^x \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^\infty \frac{\left(e^{it}\lambda\right)^x e^{-\lambda}}{x!}$$

$$= \frac{e^{-\lambda}}{e^{-\lambda e^{it}}} \sum_{x=0}^\infty \underbrace{\frac{\left(e^{it}\lambda\right)^x e^{-\lambda e^{it}}}{x!}}_{\text{PMF for Poisson}(e^{it}\lambda)} = e^{-\lambda}e^{\lambda e^{it}}$$

$$= \boxed{e^{\lambda(e^{it} - 1)}}$$

$$1$$