$X_1 \sim Poisson(\lambda_1)$ indep. of $X_2 \sim Poisson(\lambda_2)$, $T = X_1 + X_2 \sim ?$

$$\phi_{X_1+X_2}(t) \overset{P3}{=} \phi_{X_1}(t)\,\phi_{X_2}(t) = e^{\lambda_1(e^{it}-1)}\, e^{\lambda_2(e^{it}-1)} = e^{(\lambda_1+\lambda_2)(e^{it}-1)}$$

$$\overset{P1}{\Longrightarrow} T \sim Poisson(\lambda_1+\lambda_2)$$

Consider $X_1, \ldots, X_n \overset{iid}{\sim}$ unknown PMF/PDF with expectation μ and variance σ²

let $T_n = X_1 + \ldots + X_n \Rightarrow \phi_{T_n}(t) \overset{P3}{=} (\phi_X(t))^n$

let $\bar{X}_n = \frac{X_1+\ldots+X_n}{n} = \frac{T_n}{n} \Rightarrow \phi_{\bar{X}_n}(t) \overset{P2}{=} \phi_{T_n}\left(\frac{t}{n}\right) = \left(\phi_X\left(\frac{t}{n}\right)\right)^n$

From Math 241, $E[\bar{X}_n] = \mu$ and $Var[\bar{X}_n] = \frac{\sigma^2}{n}$

let $Z_n = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}}{\sigma}\bar{X}_n - \frac{\sqrt{n}}{\sigma}\mu \Rightarrow \phi_{Z_n}(t) \overset{P2}{=} e^{-\frac{it\mu\sqrt{n}}{\sigma}}\phi_{\bar{X}_n}\left(\frac{\sqrt{n}}{\sigma}t\right)$

$$= e^{-\frac{it\mu\sqrt{n}}{\sigma}}\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n$$

From Math 241, $E[Z_n] = 0$ and $Var[Z_n] = SD[Z_n] = 1$

$\varphi^n = e^{n\ln(\varphi)}$

$$= e^{-\frac{it\mu\sqrt{n}}{\sigma} + n\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}$$

$$= e^{n\left(-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)\right)}$$

We have a sequence of rv's and by (P8), if we find a limiting characteristic function, we know there is a limiting rv model

$$= e^{\frac{-\frac{it\mu}{\sigma\sqrt{n}} + \ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right)}{\frac{1}{n}} \cdot \frac{t^2}{\sigma^2} \cdot \frac{t^2}{\sigma^2}}$$

So let's take the limit and see what happens

$$\phi_{Z_n}(t) = e^{\frac{t^2}{\sigma^2}\left(\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{n\sigma^2}}\right)}$$

$$\lim_{n\to\infty}\phi_{Z_n}(t) = e^{\frac{t^2}{\sigma^2}\lim_{n\to\infty}\frac{\ln\left(\phi_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right) - \frac{it\mu}{\sigma\sqrt{n}}}{\frac{t^2}{n\sigma^2}}}$$

$\phi_X(0) \overset{P2}{=} 1$

let $u = \frac{t}{\sigma\sqrt{n}} \Rightarrow n\to\infty \Rightarrow u \to 0$

$$= e^{\frac{t^2}{\sigma^2}\lim_{u\to 0}\frac{\ln(\phi_X(u)) - i\mu u}{u^2}}$$

$\phi_X'(0) \overset{P4}{=} iE[X] = i\mu$

$\phi_X(0) = 1$

L'Hopital

$$= e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{\frac{\phi_X'(u)}{\phi_X(u)} - i\mu}{u}}$$

$$= e^{\frac{t^2}{2\sigma^2}\lim_{u\to 0}\frac{\phi_X(u)\phi_X''(u) - \phi_X'(u)^2}{\phi_X(u)}}$$

$(i\mu)^2 = -\mu^2$

$\phi_X''(0) \overset{P4}{=} i^2E[X^2] = -E[X^2]$

$$= e^{\frac{t^2}{2\sigma^2}\frac{\phi_X(0)\phi_X''(0) - \phi_X'(0)^2}{\phi_X(0)}}$$

$$= e^{\frac{t^2}{2\sigma^2}\left(-\left(\underbrace{E[X^2] - \mu^2}_{\sigma^2}\right)\right)} = e^{-t^2/2} = \phi_Z(t)$$

Let's now attempt to use (P6) inversion to convert that limiting ch.f. into the density of the underlying rv

$\phi_Z(t) \overset{?}{\in} L^1$ $\quad \int_{\mathbb{R}} e^{-t^2/2}dt \overset{?}{<} \infty \quad \int_{\mathbb{R}} e^{-t^2/2}dt = \sqrt{2\pi} < \infty$

Yes!

Gaussian integral and it is proved in mult. var. calculus class

We can now use (P6) to "invert" the ch.f. AKA Fourier synthesis

$$f_Z(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz}\phi_Z(t)\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itz}e^{-t^2/2}dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(itz + \frac{t^2}{2}\right)}dt$$

$(a+b)^2 = a^2 + 2ab + b^2$

$$\frac{t^2}{2} + itz = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2 - \left(\frac{iz}{\sqrt{2}}\right)^2 = \left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2 + \frac{z^2}{2}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{\sqrt{2}iz}{2}\right)^2}e^{-\frac{z^2}{2}}dt = \frac{1}{2\pi}e^{-\frac{z^2}{2}}\int_{\mathbb{R}} e^{-\left(\frac{t}{\sqrt{2}} + \frac{iz}{\sqrt{2}}\right)^2}dt$$

let $u = \frac{t}{\sqrt{2}} + \frac{iz}{\sqrt{2}} \Rightarrow \frac{du}{dt} = \frac{1}{\sqrt{2}} \Rightarrow dt = \sqrt{2}\,du$ $\quad t=\infty \Rightarrow u=\infty$, $t=-\infty \Rightarrow u=-\infty$

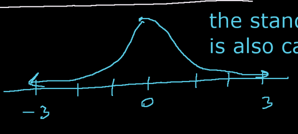
$$= e^{-z^2/2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-u^2}\sqrt{2}\,du = e^{-\frac{z^2}{2}}\frac{1}{\pi\sqrt{2}}\underbrace{\int_{\mathbb{R}} e^{-u^2}du}_{\sqrt{\pi}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}} = N(0,1)$$

This concludes the proof of the central limit theorem:

$X_1, \ldots, X_n \overset{iid}{\sim}$ mean μ and variance σ² then...

$$\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

$Z \sim N(0,1)$



the standard normal is also called the standard Gaussian or the standard "bell curve"

$E[Z] \overset{P4}{=} i\phi_Z'(0)$

$= i\,0\,e^{-0^2/2} = 0$

$\phi_Z'(t) = \frac{d}{dt}\left[e^{-t^2/2}\right] = -te^{-t^2/2}$

$SD[Z] = 1$

$Var[Z] = E[Z^2] - \cancel{E[Z]}^2 \overset{P4}{=} -\phi_Z''(0) = -(0^2e^{0^2/2} - e^{0^2/2}) = 1$

$\phi_Z''(t) = \frac{d}{dt}\left[-te^{-t^2/2}\right] = -\left(-t^2e^{-t^2/2} + e^{-t^2/2}\right) = t^2e^{-t^2/2} - e^{-t^2/2}$

$X = \mu + \sigma Z$ where $\mu \in \mathbb{R}$, $\sigma > 0$

$$f_X(x) = \frac{1}{\sigma}f_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2} = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$= N(\mu, \sigma^2)$

$E[X] = \mu$, $Var[X] = \sigma^2$

general normal rv

$$\phi_X(t) \overset{P2}{=} e^{it\mu}\phi_Z(\sigma t) = e^{it\mu}e^{-\sigma^2t^2/2} = e^{it\mu - \frac{\sigma^2t^2}{2}}$$

$X_1 \sim N(\mu_1, \sigma_1^2)$ indep of $X_2 \sim N(\mu_2, \sigma_2^2)$, $T = X_1 + X_2 \sim ?$

$$\phi_T(t) \overset{P3}{=} \phi_{X_1}(t)\phi_{X_2}(t) = e^{it\mu_1 - \frac{\sigma_1^2t^2}{2}}e^{it\mu_2 - \frac{\sigma_2^2t^2}{2}}$$

$$= e^{it\mu_1 - \frac{\sigma_1^2t^2}{2} + it\mu_2 - \frac{\sigma_2^2t^2}{2}} = e^{it(\mu_1+\mu_2) - \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}$$

$$\overset{P1}{\Longrightarrow} T \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

To do this via the convolution formula if really hard....

$X \sim N(\mu, \sigma^2)$, $Y = e^X \sim ?$ $\quad X = \ln(Y) = g^{-1}(Y)$

$\frac{d}{dy}\left[g^{-1}(y)\right] = \frac{1}{y}$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{1}{y}\right| = f_X(\ln(y))\frac{1}{|y|}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2}\mathbb{1}_{\ln(y)\in\mathbb{R}}\frac{1}{|y|}$$

$$= \frac{1}{y}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2}\mathbb{1}_{y>0}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2y^2}}e^{-\frac{1}{2\sigma^2}(\ln(y)-\mu)^2}\mathbb{1}_{y>0} = LogN(\mu, \sigma^2)$$

the "log normal" rv