Math 368 / 650 Fall 2021
Final Examination

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Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

By taking this exam, you acknowledge and agree to uphold this Code of Academic Integrity.

Instructions

This exam is 110 minutes (variable time per question) and closed-book. You are allowed **three** pages (front and back) of a "cheat sheet", blank scrap paper and a graphing calculator. Please read the questions carefully. Within each problem, I recommend considering the questions that are easy first and then circling back to evaluate the harder ones. No food is allowed, only drinks.

Problem 1 [20min] (and 20min will have elapsed) Let $X \sim \text{Logarithmic}(p) := \overbrace{-\frac{1}{\ln(1-p)}\frac{p^x}{x}}^{p^{old}(x)} \mathbb{1}_{x \in \mathbb{N}}$, a rv with parameter space $p \in (0,1)$ that Sir RA Fisher introduced in 1943 to model relative species abundance.

- [23 pt / 23 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

(a) $\sum_{x \in \mathbb{R}} p^{old}(x) = 1$

(b) X has the same support as the rv $Y \sim \text{Poisson}(\lambda)$

(c) $F(x) = -\frac{1}{\ln(1-p)} \sum_{x=1}^{\infty} \frac{p^x}{x}$

(d) $-\sum_{x=1}^{\infty} \frac{1}{\ln(1-p)} \frac{p^x}{x} = 1$

(e) $\mathbb{E}[X] = -\frac{1}{\ln(1-p)} \sum_{x=1}^{\infty} \frac{p^x}{x}$

(f) $\mathbb{E}[X] = -\frac{1}{\ln(1-p)} \frac{p}{1-p}$

(g) $\mathbb{E}[X] = -\frac{1}{\ln(1-p)} \frac{1}{1-p}$

(h) $\mathbb{E}[X] = -\frac{1}{\ln(1-p)} e^p$

(i) $\mathbb{E}[X] = 1$ for all $p \in (0,1)$

(j) If (i) were to be true then $e \leq \mathbb{E}\left[e^X\right]$

(k) If (i) were to be true then $e \geq \mathbb{E}\left[e^X\right]$

(l) $\phi_X(t) = -\frac{1}{\ln(1-p)} \sum_{x=1}^{\infty} \frac{p^x}{x}$

(m) $\phi_X(t) = -\frac{1}{\ln(1-p)} \sum_{x=1}^{\infty} e^{tx} \frac{p^x}{x}$

(n) $\phi_X(t) = -\frac{1}{\ln(1-p)} \sum_{x=1}^{\infty} \frac{(e^{it}p)^x}{x}$

(o) $\phi_X(t) = \frac{\ln(1-e^{it}p)}{\ln(1-p)}$

(p) $\phi_X(t)$ is finite for all $t \in \mathbb{R}$

(q) $\phi_X(t) \in L^1$

(r) $M_X(t)$ is finite for all $t \in \mathbb{R}$ where $M_X(t)$ is the moment generating function of X

(s) $f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \frac{\ln(1-e^{it}p)}{\ln(1-p)} dt$

Let $Y = aX$ where $a > 0$.

(t) $Y \sim -\frac{a}{\ln(1-p)} \frac{p^y}{y} \mathbb{1}_{y \in \mathbb{N}}$

(u) $Y \sim -\frac{1}{\ln(1-p)} \frac{p^{y/a}}{y/a} \mathbb{1}_{y/a \in \mathbb{N}}$

(v) $Y \sim -\frac{1}{\ln(1-p)} \frac{p^{ay}}{ay} \mathbb{1}_{ay \in \mathbb{N}}$

(w) $Y \sim -\frac{a}{\ln(1-p)} \frac{p^{y/a}}{y} \mathbb{1}_{y \in \{a, 2a, 3a, \ldots\}}$

Problem 2 [20min] (and 40min will have elapsed) Let $X \sim \text{Logarithmic}(p) := \overbrace{-\frac{1}{\ln(1-p)}\frac{p^x}{x}}^{p^{old}(x)} \mathbb{1}_{x \in \mathbb{N}}$, a rv with parameter space $p \in (0,1)$ that Sir RA Fisher introduced in 1943 to model relative species abundance. It can be shown that $\phi_X(t) = \frac{\ln(1-e^{it}p)}{\ln(1-p)}$, $\mu := \mathbb{E}[X] = -\frac{1}{\ln(1-p)}\frac{p}{1-p}$ and $\sigma^2 := \mathbb{V}\text{ar}[X] = -\frac{p^2 + p\ln(1-p)}{(1-p)^2 \ln(1-p)^2}$. Let $X_n \sim \text{Logarithmic}(1/n)$ where X_i is independent of X_j if $i \neq j$.

- [14 pt / 37 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

(a) X_n is a legal rv for all $n \in \mathbb{N}$

(b) $\phi_{X_n}(t) = \frac{\ln(1-e^{it}/n)}{\ln((n-1)/n)}$

(c) $\mathbb{S}\text{upp}[X_n] = \{1/n, 2/n, 3/n, \ldots\}$

(d) If $T = X_1 + \ldots + X_n$ then $\phi_T(t) = (\phi_{X_n}(t))^n$

(e) If $\bar{X} = \frac{1}{n}(X_1 + \ldots + X_n)$ then $\phi_{\bar{X}}(t) = \phi_T(t/n)$

(f) If $\bar{X} = \frac{1}{n}(X_1 + \ldots + X_n)$ then $\bar{X} \xrightarrow{p} -\frac{1}{\ln(1-p)}\frac{p}{1-p}$

(g) If you can show that the PMF of X_n converges to the PMF of W, some other rv, then $X_n \xrightarrow{d} W$

(h) $X_n \xrightarrow{d} \text{Poisson}(\lambda)$ where $\lambda = np = n\frac{1}{n} = 1$

(i) $X_n \xrightarrow{d} 1$

(j) $X_n \xrightarrow{p} 1$

(k) X_n does not converge in distribution to any legal rv

(l) X_n does not converge in probability to any constant

(m) $\mathbb{C}\text{ov}[X_i, X_j] = \frac{1}{i} - \frac{1}{j}$ if $i < j$

(n) If the PMF of any general sequence X_n (not necessarily the Logarithmic rv sequence in this problem) does not converge to any legal PMF, then X_n cannot converge in distribution to any rv

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 3 [14min] (and 54min will have elapsed) Let $X \sim \text{Logarithmic}(p) := \overbrace{-\frac{1}{\ln(1-p)}\frac{p^x}{x}}^{p^{old}(x)} \mathbb{1}_{x\in\mathbb{N}}$, a rv with parameter space $p \in (0,1)$ that Sir RA Fisher introduced in 1943 to model relative species abundance. It can be shown that $\phi_X(t) = \frac{\ln(1-e^{it}p)}{\ln(1-p)}$, $\mu := \mathbb{E}[X] = -\frac{1}{\ln(1-p)}\frac{p}{1-p}$ and $\sigma^2 := \mathbb{V}\text{ar}[X] = -\frac{p^2 + p\ln(1-p)}{(1-p)^2\ln(1-p)^2}$. Let $X_1, \ldots, X_n \sim \text{Logarithmic}(p)$ and $T_n = X_1 + \ldots + X_n$ and $\bar{X}_n = T_n/n$.

- [14 pt / 51 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

(a) $\mathbb{S}\text{upp}[T_2] = \mathbb{N}$

(b) $T_2 \sim \sum_{x=1}^{t-1} p^{old}(x)p^{old}(t-x)$

(c) $T_2 \sim \sum_{x=1}^{t} p^{old}(x)p^{old}(t-x)$

(d) $T_2 \sim \frac{p^t}{\ln(1-p)^2}\sum_{x=1}^{t-1}\frac{1}{x(t-x)}$

(e) $T_2 \sim \text{Logarithmic}(2p)$

(f) $X_1 - X_2 \sim \text{Skellam}(2p)$

(g) $X_1 - X_2 \sim \text{Deg}(0)$

(h) $\mathbb{P}(T_2 = 2) = \frac{p^2}{\ln(1-p)^2}$

(i) $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,\, 1)$

(j) $\frac{T_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,\, 1)$

(k) $\frac{T_n - n\mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,\, 1)$

(l) The central limit theorem does not apply in the case of a sum of iid logarithmic rv's

(m) $\bar{X} \xrightarrow{p} -\frac{1}{\ln(1-p)}\frac{p}{1-p}$

(n) The weak law of large numbers does not apply ~~for~~ in the case of a sum of iid logarithmic rv's since $\mathbb{V}\text{ar}[X]$ can be undefined for some values of p in the parameter space

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 4 [19min] (and 73min will have elapsed) Let $X \sim \text{Logarithmic}(p) := \overbrace{-\frac{1}{\ln(1-p)}\frac{p^x}{x}}^{p^{old}(x)} \mathbb{1}_{x \in \mathbb{N}}$, a rv with parameter space $p \in (0,1)$ that Sir RA Fisher introduced in 1943 to model relative species abundance. It can be shown that $\phi_X(t) = \frac{\ln(1-e^{it}p)}{\ln(1-p)}$, $\mu := \mathbb{E}[X] = -\frac{1}{\ln(1-p)}\frac{p}{1-p}$ and $\sigma^2 := \mathbb{V}\text{ar}[X] = -\frac{p^2 + p\ln(1-p)}{(1-p)^2\ln(1-p)^2}$. Let $Y \mid X = x \sim \text{Logarithmic}(x)$ and $X \sim \text{Beta}(\alpha, \beta)$.

- [14 pt / 65 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

(a) The rv Y is called a "compound distribution"

(b) The rv X is continuous

(c) The rv Y is continuous

(d) $f_Y(y) = \int_0^1 -\frac{1}{\ln(1-p)}\frac{p^x}{x}\mathbb{1}_{x\in\mathbb{N}} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx$

(e) $f_Y(y) = \int_0^1 -\frac{1}{\ln(1-y)}\frac{y^x}{x}\mathbb{1}_{x\in\mathbb{N}} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx$

(f) $f_Y(y) = \int_0^1 -\frac{1}{\ln(1-x)}\frac{x^y}{y}\mathbb{1}_{y\in\mathbb{N}} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx$

(g) $f_Y(y) = \int_0^1 -\frac{1}{\ln(1-x)}\frac{x^y}{y}\mathbb{1}_{y\in\mathbb{N}} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dy$

(h) $\mathbb{E}[Y] = \mathbb{E}_X\left[-\frac{1}{\ln(1-X)}\frac{X}{1-X}\right]$

(i) $\mathbb{E}[Y] = \mathbb{E}_X\left[-\frac{1}{\ln(1-p)}\frac{X}{1-X}\right]$

(j) $\mathbb{E}[Y] = -\frac{1}{\ln(1-p)}\frac{p}{1-p}$

(k) $\mathbb{V}\text{ar}[Y] = \mathbb{V}\text{ar}_X\left[\frac{1}{\ln(1-X)}\frac{X}{1-X}\right] - \mathbb{E}_X\left[\frac{X^2 + X\ln(1-X)}{(1-X)^2\ln(1-X)^2}\right]$

(l) $\mathbb{V}\text{ar}[Y] = \mathbb{V}\text{ar}_X\left[\frac{1}{\ln(1-X)}\frac{X}{1-X}\right] - \frac{p^2 + p\ln(1-p)}{(1-p)^2\ln(1-p)^2}$

(m) $\mathbb{V}\text{ar}[Y] = \frac{1}{\ln(1-p)}\frac{p}{1-p} - \frac{p^2 + p\ln(1-p)}{(1-p)^2\ln(1-p)^2}$

(n) $f_{X|Y}(x, y)$ can be written in either closed or not closed form given the information provided here

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 5 [9min] (and 82min will have elapsed) Let $Z_1, Z_2 \overset{iid}{\sim} \mathcal{N}(0,\, 1)$, $M = Z_1 Z_2$ and let $U = Z_2$.

- [8 pt / 73 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

  (a) $f_{M,U}(m,u) = f_{Z_1,Z_2}(m,u)$

  (b) $f_{M,U}(m,u) = f_{Z_1,Z_2}(m/u,u)$

  (c) $f_{M,U}(m,u) = f_{Z_1,Z_2}(m/u,u)/u$

  (d) $f_M(m) = \displaystyle\int_{\mathbb{R}} f_{M,U}(m,u)du$

  (e) $f_M(m) = \displaystyle\int_{\mathbb{R}} f_{Z_1}(m/u) f_{Z_2}(u) \frac{1}{|u|} du$

  ~~(f)~~ $\cancel{f_M(m) = \displaystyle\int_{\mathbb{R}} f_{Z_1}(m/u) f_{Z_2}(u) \frac{1}{|u|} du}$

  (g) $f_M(m) = \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}} e^{-(m^2/u^2+u^2)/2} \frac{1}{|u|} du$

  (h) $f_M(m) = \dfrac{1}{\pi} \displaystyle\int_0^{\infty} e^{-(m^2/u^2+u^2)/2} \frac{1}{u} du$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 6 [15min] (and 97min will have elapsed) Let $Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{2n} \overset{iid}{\sim} \mathcal{N}(0, 1)$ and $\boldsymbol{Z} := [Z_1 \; Z_2 \; ... \; Z_{2n}]^\top$. Let $\bar{Z}_1 = \frac{1}{n}\sum_{i=1}^{n} Z_i$ and $\bar{Z}_2 = \frac{1}{n}\sum_{i=n+1}^{2n} Z_i$. Note that $\mathbb{E}[Z_i^4] = 3$.

- [19 pt / 92 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

(a) Markov's inequality proves that $\mathbb{P}(|Z_1| > 2) \leq 1/2$

(b) Markov's inequality proves that $\mathbb{P}(|Z_1| > 2) \leq 1/4$

(c) Chebyshev's inequality proves that $\mathbb{P}(|Z_1| > 2) \leq 2/9$

(d) Chebyshev's inequality proves that $\mathbb{P}(|Z_1| > 2) \leq 1/8$

(e) $\text{Med}[Z_1^2] \leq 2$

(f) Z_1^2/Z_2^2 is Fisher's F-distributed

(g) Z_1^2/Z_2^2 is Student's T-distributed

(h) Z_1^2/Z_2^2 is Cauchy-distributed

(i) $Z_1^2/|Z_2|$ is Fisher's F-distributed

(j) $Z_1^2/|Z_2|$ is Student's T-distributed

(k) $Z_1^2/|Z_2|$ is Cauchy-distributed

(l) $\boldsymbol{Z}^\top \mathbf{1}$ is a quadratic form

(m) $\boldsymbol{Z}^\top \boldsymbol{Z}$ is a quadratic form

(n) $\boldsymbol{Z}^\top \mathbf{1} \sim \mathcal{N}(0, 2n)$

(o) $\boldsymbol{Z}^\top \mathbf{1} \sim \mathcal{N}(0, 1/(2n))$

(p) $\boldsymbol{Z}^\top \boldsymbol{Z} \sim \mathcal{N}(n, 2n)$

(q) $\boldsymbol{Z}^\top \boldsymbol{Z} \sim \mathcal{N}(0, 1/(2n))$

(r) $\bar{Z}_1$ and $\bar{Z}_2$ are independent

(s) $\bar{Z}_1$ and $\bar{Z}_2$ are $\overset{iid}{\sim}$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.

Problem 7 [13min] (and 110min will have elapsed) Let $Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{2n} \overset{iid}{\sim} \mathcal{N}(0, 1)$ and $\boldsymbol{Z} := [Z_1 \; Z_2 \; \ldots \; Z_{2n}]^\top$. Let $\bar{Z}_1 = \frac{1}{n}\sum_{i=1}^{n} Z_i$ and $\bar{Z}_2 = \frac{1}{n}\sum_{i=n+1}^{2n} Z_i$. Also let $W_1 := \sum_{i=1}^{n}(Z_i - \bar{Z}_1)^2$ and $W_2 := \sum_{i=n+1}^{2n}(Z_i - \bar{Z}_2)^2$

- [14 pt / 106 pts] Record the letter(s) of all the following that are **true**. At least one will be true.

(a) $W_1 \sim \chi^2_k$ where you have enough information to compute k

(b) $W_1 \sim \text{Gamma}(k_1, k_2)$ where you have enough information to compute k_1 and k_2

(c) W_1 and $\bar{Z}_1$ are independent

(d) W_1 and $\bar{Z}_1$ are $\overset{iid}{\sim}$

(e) If (c) was true, it could be proved with Cochran's theorem

(f) If $W_1 = \boldsymbol{Z}^\top \boldsymbol{B}_1 \boldsymbol{Z}$ then $\boldsymbol{B}_1 = \boldsymbol{I}_{2n}$

(g) If $W_1 = \boldsymbol{Z}^\top \boldsymbol{B}_1 \boldsymbol{Z}$ then $\boldsymbol{B}_1 = \boldsymbol{J}_{2n}$

(h) If $W_1 = \boldsymbol{Z}^\top \boldsymbol{B}_1 \boldsymbol{Z}$ then $\boldsymbol{B}_1$ has rank $2n - 1$

(i) If $W_1 = \boldsymbol{Z}^\top \boldsymbol{B}_1 \boldsymbol{Z}$ then $\boldsymbol{B}_1$ has rank $n - 1$

(j) If $W_1 = \boldsymbol{Z}^\top \boldsymbol{B}_1 \boldsymbol{Z}$ then $\boldsymbol{B}_1$ has rank 1

Let $Z_{(1)}, Z_{(2)}, \ldots, Z_{(2n)}$ be the order statistics for the sequence $Z_1, Z_2, \ldots, Z_{2n}$.

(k) $Z_{(i)}$ is beta-distributed for all i

(l) $\mathbb{S}\text{upp}\left[Z_{(i)}\right] = \mathbb{R}$ for all i

(m) $\mathbb{E}\left[Z_{(i)}\right] < \mathbb{E}\left[Z_{(j)}\right]$ for all $i < j$

(n) The support of the range of the sequence $Z_1, Z_2, \ldots, Z_{2n}$ is $(0, \infty)$

Your answer will consist of a lowercase string (e.g. `aebgd`) where the order of the letters does not matter.