$$\Sigma := \mathrm{Var}[\vec{X}] := \begin{bmatrix} \overset{\mathrm{Cov}[X_1,X_1]}{\mathrm{Var}[X_1]} & \mathrm{Cov}[X_1,X_2] & \cdots & \mathrm{Cov}[X_1,X_n] \\ \mathrm{Cov}[X_2,X_1] & \mathrm{Var}[X_2] & & \\ & & \ddots & \\ & & & \mathrm{Var}[X_n] \\ \mathrm{Cov}[X_n,X_1] & & & \end{bmatrix} := \{\mathrm{Cov}[X_i,X_j]\} \overset{HW}{=} \underbrace{E[\vec{X}\vec{X}^T]}_{\text{Outer product}} - \vec{\mu}\vec{\mu}^T \in \mathbb{R}^{n\times n}$$

$n\times 1 \; 1\times n$

Symmetric matrix

variance-covariance matrix

$$\vec{\mu} := E[\vec{X}] := \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{bmatrix} \in \mathbb{R}^n$$

$$\text{If } X_1, X_2, \ldots, X_n \overset{ind}{\sim} \Rightarrow \mathrm{Var}[\vec{X}] = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \ddots \; \sigma_n^2 \end{bmatrix}$$

diagonal matrix

$$\text{If } X_1, X_2, \ldots, X_n \overset{iid}{\sim} \Rightarrow \mathrm{Var}[\vec{X}] = \sigma^2 I_n$$

the n x n identity matrix

Rules for expectation and variance of vector rv's . let $\vec{a} \in \mathbb{R}^K$ constant

I

$$E[\vec{X} + \vec{a}] = \begin{bmatrix} E[X_1 + a_1] \\ E[X_2 + a_2] \\ \vdots \\ E[X_K + a_K] \end{bmatrix} = \begin{bmatrix} \mu_1 + a_1 \\ \mu_2 + a_2 \\ \vdots \\ \mu_K + a_K \end{bmatrix} = \vec{\mu} + \vec{a}$$

II

$$E[\vec{a}^T\vec{X}] = E[a_1X_1 + a_2X_2 + \ldots + a_KX_K] = \sum_{i=1}^K E[a_iX_i] = \sum_{i=1}^K a_i\mu_i = \vec{a}^T\vec{\mu}$$

$A \in \mathbb{R}^{L\times K}$ constants

III

$$E[A\vec{X}] = E\left[\begin{bmatrix} a_{11}X_1 + a_{12}X_2 + \ldots + a_{1K}X_K \\ a_{21}X_1 + a_{22}X_2 + \ldots + a_{2K}X_K \\ \vdots \\ a_{L1}X_1 + a_{L2}X_2 + \ldots + a_{LK}X_K \end{bmatrix}\right] = E\left[\begin{bmatrix} \vec{a}_{1\cdot}\vec{X} \\ \vec{a}_{2\cdot}\vec{X} \\ \vdots \\ \vec{a}_{L\cdot}\vec{X} \end{bmatrix}\right] = \begin{bmatrix} E[\vec{a}_{1\cdot}\vec{X}] \\ E[\vec{a}_{2\cdot}\vec{X}] \\ \vdots \\ E[\vec{a}_{L\cdot}\vec{X}] \end{bmatrix}$$

$$\overset{II}{=} \begin{bmatrix} \vec{a}_{1\cdot}\vec{\mu} \\ \vec{a}_{2\cdot}\vec{\mu} \\ \vdots \\ \vec{a}_{L\cdot}\vec{\mu} \end{bmatrix} = \begin{bmatrix} \vec{a}_{1\cdot} \\ \vec{a}_{2\cdot} \\ \vdots \\ \vec{a}_{L\cdot} \end{bmatrix}\vec{\mu} = A\vec{\mu}$$

IV

$$\mathrm{Var}[\vec{a}^T\vec{X}] = \mathrm{Var}[a_1X_1 + a_2X_2 + \ldots + a_KX_K]$$

by rule 4 of previous lecture

by rule 3 of previous lecture

$$= \sum_{i=1}^K\sum_{j=1}^K \mathrm{Cov}[a_iX_i, a_jX_j] = \sum_{i=1}^K\sum_{j=1}^K a_ia_j\,\mathrm{Cov}[X_i,X_j]$$

$$= \vec{a}^T\,\mathrm{Var}[\vec{X}]\,\vec{a} = \boxed{\vec{a}^T\Sigma\vec{a}}$$

let $V \in \mathbb{R}^{K\times K}$

$$\underset{(1\times K)(K\times K)(K\times 1)}{\vec{a}^TV\vec{a}} = [a_1 \cdots a_K]\begin{bmatrix} V_{11} & \cdots & V_{1K} \\ \vdots & & \vdots \\ V_{K1} & \cdots & V_{KK} \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_K \end{bmatrix} = [a_1 \cdots a_K]\begin{bmatrix} a_1V_{11} + \ldots + a_KV_{1K} \\ \vdots \\ a_1V_{K1} + \ldots + a_KV_{KK} \end{bmatrix}$$

scalar

$$= (a_1a_1V_{11} + a_1a_2V_{12} + \ldots + a_1a_KV_{1K}) +$$
$$(a_2a_1V_{21} + a_2a_2V_{22} + \ldots + a_2a_KV_{2K}) +$$
$$\vdots$$
$$(a_Ka_1V_{K1} + a_Ka_2V_{K2} + \ldots + a_Ka_KV_{KK}) = \sum_{i=1}^K\sum_{j=1}^K a_ia_jV_{ij} = \vec{a}^TV\vec{a}$$

This expression is called a "quadratic form".

A tangent from finance. Let X_1, ..., X_K be the returns of assets. The assets have means μ_1, ..., μ_K and Σ is their variance-covariance matrix. Let w_1, ..., w_K be a set of weights on each asset s.t. they sum to one. Your portfolio is $F = \vec{w}^T\vec{X}$

$$\mu_F = \vec{w}^T\vec{\mu}$$

Target a mean return of $\mu_0 = \mu_F$ and find the portfolio allocation w_1, ..., w_K that minimizes Var[F].

$$\mathrm{Var}[F] = \vec{w}^T\Sigma\vec{w} \qquad \min_{\vec{w}} \vec{w}^T\Sigma\vec{w} \text{ s.t. } \vec{w}^T\vec{\mu} = \mu_0 \text{ and } \vec{w}^T\vec{1} = 1,$$

This optimization problem is easily solved. We won't do it here.

$$\vec{X} \sim \mathrm{Multinom}(n, \vec{p}) \qquad E[\vec{X}] = n\vec{p}$$

$$\text{Each } X_j \sim \mathrm{Bin}(n, p_j)$$

$$\Rightarrow E[X_j] = np_j, \quad \mathrm{Var}[X_j] = np_j(1-p_j)$$

$$\mathrm{Var}[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & & \mathrm{Cov}[X_i,X_j] \\ & np_2(1-p_2) & \\ & & \ddots \\ & & np_K(1-p_K) \end{bmatrix} \qquad \mathrm{Cov}[X_i,X_j] < 0$$

$$p_{X_1,X_2,\ldots,X_n}(x_1,\ldots,x_n) = \binom{n}{x_1 \ldots x_n} p_1^{x_1}\cdots p_n^{x_n} \quad \text{multinomial JMF}$$

$$E[g(X_i,X_j)] = \sum_{x_i \in \mathrm{Supp}[X_i]}\sum_{x_j \in \mathrm{Supp}[X_j]} g(x_i,x_j)\,p_{X_i,X_j}(x_i,x_j)$$

$$\mathrm{Cov}[X_i,X_j] := E[X_iX_j] - \mu_i\mu_j$$

$$= \sum_{x_i \in \{0,\ldots,n\}}\sum_{x_j \in \{0,\ldots,n\}} x_ix_j\,p_{X_i,X_j}(x_i,x_j) - \mu_i\mu_j$$

too difficult!

arbitrary 2-dimensional subset of the K-dimensional rv

$$X_i \sim \mathrm{Bin}(n,p_i) \Leftrightarrow X_i = X_{1i} + X_{2i} + \ldots + X_{ni} \text{ s.t. } X_{1i},\ldots,X_{ni} \overset{iid}{\sim} \mathrm{Bern}(p_i)$$

$$X_j \sim \mathrm{Bin}(n,p_j) \Leftrightarrow X_j = X_{1j} + X_{2j} + \ldots + X_{nj} \text{ s.t. } X_{1j},\ldots,X_{nj} \overset{iid}{\sim} \mathrm{Bern}(p_j)$$

$\vec{X} \sim \mathrm{Multi}(n, \vec{p})$

$$\Leftrightarrow \vec{X} = \vec{X}_1 + \vec{X}_2 + \ldots + \vec{X}_n \text{ s.t. } \vec{X}_1,\ldots,\vec{X}_n \overset{iid}{\sim} \mathrm{Multi}(1,\vec{p})$$

$$\mathrm{Cov}[X_i,X_j] = \mathrm{Cov}[X_{1i} + \ldots + X_{ni}, X_{1j} + \ldots + X_{nj}]$$

$$= \sum_{l=1}^n\sum_{m=1}^n \mathrm{Cov}[X_{li}, X_{mj}]$$

if $l \neq m$ Cov = 0 due to indep.

$$= \sum_{l=1}^n \mathrm{Cov}[X_{li}, X_{lj}]$$

$$= \sum_{l=1}^n E[X_{li}X_{lj}] - \mu_{li}\mu_{lj}$$

expectation of Bernoulli rv

$$= \sum_{l=1}^n \sum_{x_{li} \in \{0,1\}}\sum_{x_{lj} \in \{0,1\}} x_{li}x_{lj}\,p_{X_{li}X_{lj}}(x_{li},x_{lj}) - p_ip_j$$

$x_{li} = x_{lj} = 1$

$$= \sum_{l=1}^n (1)(1)\,p_{X_{li}X_{lj}}(1,1) - p_ip_j$$

the probability of getting an apple and a banana at the same time which is 0

$$= \sum_{l=1}^n -p_ip_j = \boxed{-np_ip_j}$$

$$\mathrm{Var}[\vec{X}] = \begin{bmatrix} np_1(1-p_1) & -np_1p_2 & \cdots & -np_1p_K \\ & np_2(1-p_2) & \cdots & -np_2p_K \\ & & \ddots & \\ & & & np_K(1-p_K) \end{bmatrix}$$