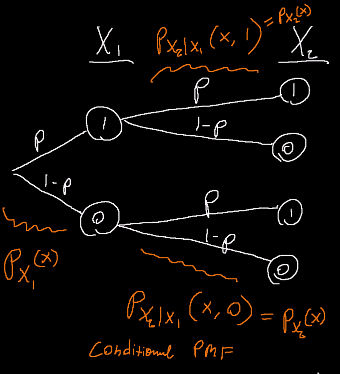
$$\sum_{x \in A} g(x) \mathbb{1}_{x \in B} = \sum_{x \in A \cap B} g(x) \qquad \int_A g(x) \mathbb{1}_{x \in B}\, dx = \int_{A \cap B} g(x)\, dx$$

$X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$. $T_2 = X_1 + X_2 \sim p_{T_2}(t)$

Goal: derive the PMF of T_2. First, what is the Supp[T_2]?
Supp[T_2] = {0, 1, 2} = Supp[X_1] + Supp[X_2] = {0, 1} + {0, 1}

A + B = {a + b, a ∈ A, b ∈ B}

Let's visualize this situation with a "tree"



JMF $p_{\vec{X}}(\vec{x})$	T
p^2	2
$p(1-p)$	1
$(1-p)p$	1
$(1-p)^2$	0
1	

$$T \sim \begin{cases} 2 & \text{w.p. } p^2 \\ 1 & \text{w.p. } 2p(1-p) \\ 0 & \text{w.p. } (1-p)^2 \end{cases}$$

$(1-p)^2 + (1-p)p$
$1 - 2p + \cancel{p^2} + p - \cancel{p^2}$

$p(1-p) + p^2$
$p - \cancel{p^2} + \cancel{p^2}$

$p_{\vec{X}}(\vec{x})$

Supp[X_1] \ Supp[X_2]	0	1	
0	$(1-p)^2$	$(1-p)p$	$1-p$
1	$p(1-p)$	p^2	p
	$1-p$	p	1

$$p_{X_2|X_1}(x, 0) = \begin{cases} \frac{(1-p)^2}{1-p} = \frac{p_{X_2,X_1}(0,0)}{p_{X_1}(0)} = 1-p & \text{if } x = 0 \\ \frac{(1-p)p}{1-p} = \frac{p_{X_2,X_1}(1,0)}{p_{X_1}(0)} = p & \text{if } x = 1 \end{cases}$$

$$X_1 \sim \begin{cases} 1 & \text{w.p. } \frac{1}{6} \\ 2 & \\ 3 & \vdots \\ 4 & \\ 5 & \\ 6 & \end{cases} = U(\{1,2,3,4,5,6\})$$ uniform discrete rv

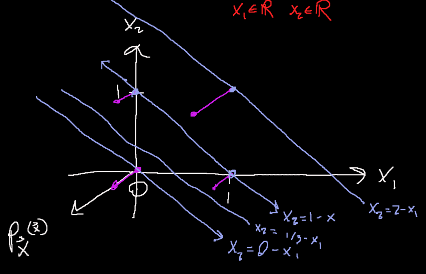
$$X_2 \sim \begin{cases} 1 & \text{w.p. } 1/5000 \\ 2 & \\ \vdots & \\ 5000 & \end{cases} = U(\{1, 2, \ldots, 5000\})$$

$T_2 = X_1 + X_2 \sim \,?$ The tree method really wouldn't be practical. We need a more general powerful formula.

In general this is termed a "discrete convolution" sometimes denoted: $p_{T_2}(t) = p_{X_1}(x) * p_{X_2}(x)$

$$p_{T_2}(t) = P(T_2 = t) = \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1,X_2}(x_1, x_2) \mathbb{1}_{x_1 + x_2 = t}$$

$x_1 + x_2 = 1 \Rightarrow x_2 = 1 - x_1$



$p_{T_2}\left(\frac{1}{3}\right) = 0$
$p_{T_2}(2) =$
$p_{T_2}(1) =$
$p_{T_2}(0)$

$$= \sum_{x_1 \in \mathbb{R}} \sum_{x_2 \in \mathbb{R}} p_{X_1,X_2}(x_1, x_2) \mathbb{1}_{\underline{x_2 = t - x_1}} \quad x_2 \in \{t - x_1\}$$

$$= \sum_{x_1 \in \mathbb{R}} p_{X_1,X_2}(x_1, t - x_1) = \sum_{x \in \mathbb{R}} p_{X_1,X_2}(x, t-x)$$ general convolution formula

$$\overset{X_1, X_2 \text{ ind}}{=} \sum_{x \in \mathbb{R}} p_{X_1}(x) p_{X_2}(t-x) = \sum_{x \in \mathbb{R}} p^{old}_{X_1}(x) \mathbb{1}_{x \in Supp[X_1]} p^{old}_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}$$

$$= \sum_{x \in Supp[X_1]} p^{old}_{X_1}(x) p^{old}_{X_2}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}$$ independent convolution formula we will use

$$\overset{X_1, X_2 \text{ iid}}{=} \sum_{x \in \mathbb{R}} p(x) p(t-x) = \sum_{x \in \mathbb{R}} p_{old}(x) \mathbb{1}_{x \in Supp[X]} p_{old}(t-x) \mathbb{1}_{t-x \in Supp[X]}$$

$$= \sum_{x \in Supp[X]} p_{old}(x) p_{old}(t-x) \mathbb{1}_{t-x \in Supp[X]}$$ iid convolution formula we will use

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$X_1, X_2 \overset{iid}{\sim} \text{Bern}(p)$,

$$p_{T_2}(t) = \sum_{x \in \{0,1\}} p^x (1-p)^{1-x} p^{t-x} (1-p)^{1-t+x} \mathbb{1}_{t-x \in \{0,1\}}$$

$$= \sum_{x \in \{0,1\}} p^t (1-p)^{2-t} \mathbb{1}_{t \in \{x, x+1\}} = p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \mathbb{1}_{t \in \{x, x+1\}}$$

$$= p^t (1-p)^{2-t} \left(\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}} \right) = \binom{2}{t} p^t (1-p)^{2-t} = \text{Binom}(2, p)$$

binomial rv

$$\mathbb{1}_{t \in \{0,1\}} + \mathbb{1}_{t \in \{1,2\}} = \begin{cases} 1 & \text{if } t = 0 \\ 2 & \text{if } t = 1 \\ 1 & \text{if } t = 2 \end{cases} = \binom{2}{t}, \quad \binom{n}{k} := \frac{n!}{k!(n-k)!} \mathbb{1}_{k \in \{0, 1, \ldots, n\}}$$

Again...

$$X_1, X_2 \overset{iid}{\sim} \text{Bern}(p) = p^x (1-p)^{1-x} \mathbb{1}_{x \in \{0,1\}} = \binom{1}{x} p^x (1-p)^{1-x}$$

$$\binom{1}{x} = \frac{1!}{x!(1-x)!} \mathbb{1}_{x \in \{0,1\}}$$

$$p_{T_2}(t) = \sum_{x \in \mathbb{R}} \binom{1}{x} p^x (1-p)^{1-x} \binom{1}{t-x} p^{t-x} (1-p)^{1-t+x}$$

$$= p^t (1-p)^{2-t} \sum_{x \in \mathbb{R}} \binom{1}{x} \cdot \binom{1}{t-x} = p^t (1-p)^{2-t} \sum_{x \in \{0,1\}} \binom{1}{t-x}$$

$$\binom{1}{x} = \mathbb{1}_{x \in \{0,1\}}$$

$$= p^t (1-p)^{2-t} \left(\binom{1}{t} + \binom{1}{t-1} \right) = \binom{2}{t} p^t (1-p)^{2-t}$$

Pascal's Identity:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \text{if } n = 2, k = t \quad \binom{2}{t} = \binom{1}{t} + \binom{1}{t-1}$$

$$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p) \quad T_3 = \overline{X_1 + X_2} + X_3 \sim p_{T_3}(t) = \,?$$
$$= X_3 + T_2$$