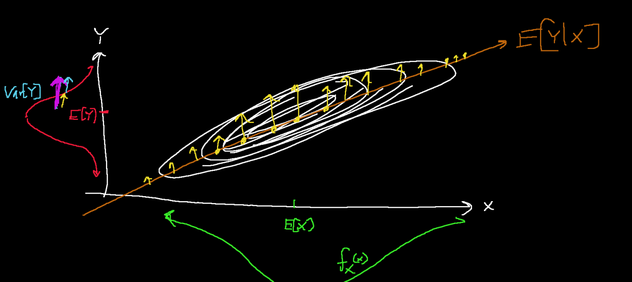
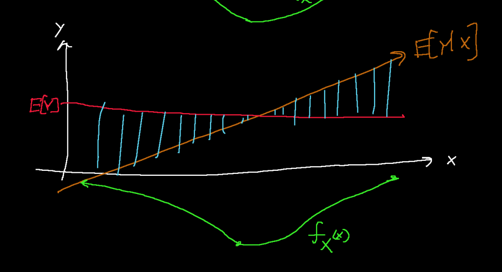
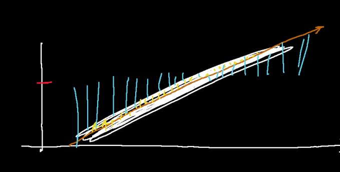
$$\mathrm{Var}_Y[Y] = E_Y[Y^2] - E_Y[Y]^2$$
$$= E_X\left[E_Y[Y^2|X]\right] - E_X\left[E_Y[Y|X]\right]^2$$
$$= E_X\left[\mathrm{Var}_Y[Y|X] + E_Y[Y|X]^2\right] - E_X\left[E_Y[Y|X]\right]^2$$
$$= E_X\left[\mathrm{Var}_Y[Y|X]\right] + E_X\left[\underbrace{E_Y[Y|X]}_{Q}{}^2\right] - E_X\left[\underbrace{E_Y[Y|X]}_{Q}\right]^2 \qquad E[R^2]-E[R]^2$$
$$\mathrm{Var}_Y[Y] = \underbrace{E_X\left[\mathrm{Var}_Y[Y|X]\right]}_{\mathrm{I}} + \underbrace{\mathrm{Var}_X\left[E_Y[Y|X]\right]}_{\mathrm{II}}$$

Law of Total Variance

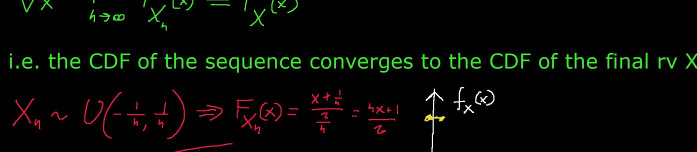




Here's an example with nearly zero term I and a lot of term II



Here's an example with zero term II and a lot of term I

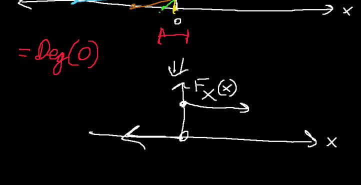


Convergence in distribution. $X_n \xrightarrow{d} X$ This means that:

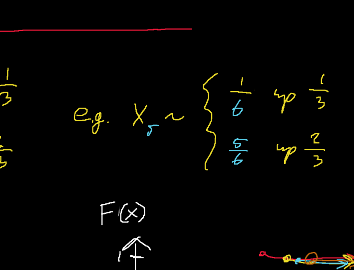
$$\forall x \quad \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

i.e. the CDF of the sequence converges to the CDF of the final rv X

$$X_n \sim U\left(-\frac{1}{n}, \frac{1}{n}\right) \Rightarrow F_{X_n}(x) = \frac{x+\frac{1}{n}}{\frac{2}{n}} = \frac{nx+1}{2}$$



$$= \begin{cases} 0 & \text{if } x < -\frac{1}{n} \\ \frac{nx+1}{2} & \text{if } x \in \left[\frac{1}{n}, \frac{1}{n}\right] \\ 1 & \text{if } x > \frac{1}{n} \end{cases}$$

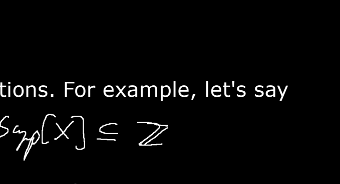


$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} = \mathrm{Deg}(0)$$

$$X_n \xrightarrow{d} \underbrace{\mathrm{Deg}(0)}_{X}$$

$$X_n \sim \begin{cases} \frac{1}{n+1} & \text{wp } \frac{1}{3} \\ 1-\frac{1}{n+1} & \text{wp } \frac{2}{3} \end{cases} \qquad \text{e.g. } X_5 \sim \begin{cases} \frac{1}{6} & \text{wp } \frac{1}{3} \\ \frac{5}{6} & \text{wp } \frac{2}{3} \end{cases}$$

$$\lim_{n\to\infty} X_n \sim \begin{cases} 0 & \text{wp } \frac{1}{3} \\ 1 & \text{wp } \frac{2}{3} \end{cases}$$



The PMF converged. Are PMF and CDF convergence equivalent?

$$\underbrace{X_n \xrightarrow{d} X}_{\lim F_{X_n}(x) = F_X(x)} \overset{?}{\Longleftrightarrow} \lim p_{X_n}(x) = p_X(x)$$

This is true under certain conditions. For example, let's say

$$\mathrm{Supp}[X_n] \subseteq \mathbb{Z} \text{ and } \mathrm{Supp}[X] \subseteq \mathbb{Z}$$

Let's prove that CDF convergence implies PMF convergence

$$p_{X_n}(x) = F_{X_n}\left(x+\frac{1}{2}\right) - F_{X_n}\left(x-\frac{1}{2}\right)$$

$$\lim p_{X_n}(x) = \lim F_{X_n}\left(x+\frac{1}{2}\right) - \lim F_{X_n}\left(x-\frac{1}{2}\right) = F_X\left(x+\frac{1}{2}\right) - F_X\left(x-\frac{1}{2}\right) = p_X(x) \checkmark$$

Let's prove that PMF converges implies CDF convergence

$$F_{X_n}(x) := P(X_n \le x) = \sum_{y=-\infty}^{x} p_{X_n}(y)$$

$$\lim F_{X_n}(x) = \lim \sum_{y=-\infty}^{x} p_{X_n}(y) = \sum_{y=-\infty}^{x} \lim p_{X_n}(y) = \sum_{y=-\infty}^{x} p_X(y) = P(X \le x) = F_X(x) \checkmark$$

We used this thm to prove that the limiting Binomial is Poisson

$$\lim X_n \sim \mathrm{Bin}\left(n, \frac{\lambda}{n}\right) = X \sim \mathrm{Poisson}(\lambda)$$

Is PDF and CDF convergence equivalent? No. Only PDF convergence implies CDF convergence. Here's a counter-example of CDF convergence without PDF convergence:

$$X_n \sim U\left(-\frac{1}{n}, \frac{1}{n}\right) = \underbrace{\frac{n}{2}\mathbb{1}_{x\in\left[-\frac{1}{n},\frac{1}{n}\right]}}_{f_{X_n}(x)} \qquad \lim f_{X_n}(x) = \infty\mathbb{1}_{x=0}$$

not a valid PDF $\neq$ PDF of

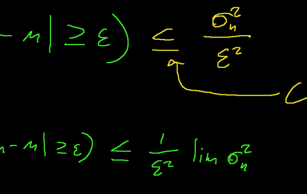
We know from before that $\lim F_{X_n}(x) = F_X(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{o/t} \end{cases} \Rightarrow X \sim \mathrm{Deg}(0)$

$c \in \mathbb{R}$,

$$X_n \xrightarrow{d} \underbrace{c}_{\mathrm{Deg}(c)} \Rightarrow F_X(x) = \begin{cases} 1 & \text{if } x \ge c \\ 0 & \text{o/t} \end{cases}$$

"Convergence in probability to a constant" $X_n \xrightarrow{p} c$ means:

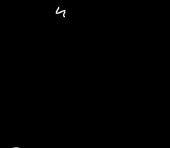
$$\forall \varepsilon > 0 \quad \lim P(|X_n - c| \ge \varepsilon) = 0$$



If X_n has a finite variance σ_n^2 and $E[X_n] = \mu$ then:

$$P(|X_n - \mu| \ge \varepsilon) \le \frac{\sigma_n^2}{\varepsilon^2}$$

Chebyshev's Ineq.



$$\Rightarrow \lim_{n\to\infty} P(|X_n - \mu| \ge \varepsilon) \le \frac{1}{\varepsilon^2}\lim \sigma_n^2$$

Let $X_1, \ldots, X_n \overset{iid}{\sim}$ mean μ, variance σ^2, $\bar{X}_n := \frac{X_1 + \ldots + X_n}{n}$

$$\overset{241}{\Longrightarrow} E[\bar{X}_n] = \mu, \quad \mathrm{Var}[\bar{X}_n] = \frac{\sigma^2}{n}$$

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| \ge \varepsilon) \le \frac{1}{\varepsilon^2}\lim_{n\to\infty}\frac{\sigma^2}{n} = 0$$

$$\Longrightarrow \boxed{\bar{X}_n \xrightarrow{p} \mu}$$

"Weak" Weak Law of Large Numbers (WLLN)

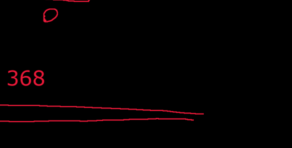
The first "weak" is because we assumed that the X_i's have a finite variance (we don't need this, see homework).

The implication of any of the LLN theorems (which there are a lot) is that the average cannot escape the expectation as the sample size grows larger.

$$X_n \sim U\left(-\frac{1}{n}, \frac{1}{n}\right) = \frac{n}{2}\mathbb{1}_{x\in\left[-\frac{1}{n},\frac{1}{n}\right]} \qquad \text{Prove } X_n \xrightarrow{p} 0$$

$$\lim P(|X_n| \ge \varepsilon) = 0$$

sufficient n



$$\lim P(|X_n| \ge \varepsilon) = 2\lim P(X_n \ge \varepsilon) = 2\lim 0\,\mathbb{1}_{\varepsilon \ge \frac{1}{n}} + \left(\frac{1}{n}-\varepsilon\right)\frac{n}{2}\mathbb{1}_{\varepsilon < \frac{1}{n}}$$

$$= \lim \left(\frac{1}{n}-\varepsilon\right) n \underbrace{\mathbb{1}_{\varepsilon < \frac{1}{n}}}_{0} = 0 \checkmark$$

End of Math 368

Topics we didn't get a chance to do:
* convergence in Law
* multivariate normal
* Holder's Inequalities