From end of lec 6 and beginning of lec 7...

Skellam

$X_1, X_2 \overset{iid}{\sim} Poisson(\lambda)$, $\quad D := X_1 - X_2 = X_1 + {}^-X_2 \sim p_D(d)$

$$Supp[D] = Supp[X_1] + Supp[-X_2] = Supp[X_1] + {}^-Supp[X_2]$$

$$= \{0,1,2,\dots\} + \{\dots,-2,-1,0\} = \mathbb{Z} \text{ (integers)}$$

$X_1, X_2 \overset{iid}{\sim} Poisson(\lambda)$, let $Y = {}^-X_2 \sim \frac{\lambda^{-y} e^{-\lambda}}{(-y)!} \mathbb{1}_{-y \in \{0,1,\dots\}} = \frac{\lambda^{-y} e^{-\lambda}}{(-y)!} \mathbb{1}_{y \in \{\dots,-1,0\}}$

$$D \overset{= X_1 - X_2}{=} X_1 + Y \sim p_D(d) = ?$$

$$p_D(d) = \sum_{x \in Supp[X_1]} p_{X_1}^{old}(x)\, p_Y^{old}(d-x)\, \mathbb{1}_{d-x \in Supp[Y]}$$

$$= \sum_{x \in \{0,1,\dots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{x-d} e^{-\lambda}}{(x-d)!} \mathbb{1}_{\underbrace{d-x \in \{\dots,-1,0\}}_{\substack{x \in \{d, d+1,\dots\} \\ x-d \in \{0,1,\dots\}}}}$$

$$= e^{-2\lambda} \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} \mathbb{1}_{x \in \{d,d+1,\dots\}}$$

$$= e^{-2\lambda} \begin{cases} \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d \le 0 \\ \sum_{x \in \{d,d+1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d > 0 \end{cases}$$

$x \overset{=}{} x'+d$

let $x' = x - d \iff x = x' + d$

$$= e^{-2\lambda} \begin{cases} \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d \le 0 \\ \sum_{x' \in \{0,1,\dots\}} \frac{\lambda^{\overbrace{2(x'+d)-d}^{2x'+d}}}{(x'+d)!\,x'!} & \text{if } d > 0 \end{cases}$$

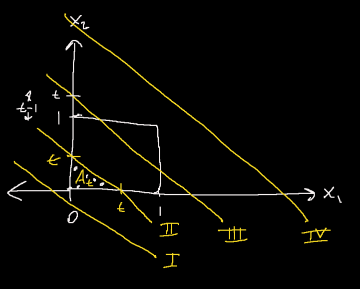
Note: if $d > 0 \Rightarrow |d| = d$, $d \le 0 \Rightarrow -|d| = d$

$$= e^{-2\lambda} \begin{cases} \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!} & \text{if } d \le 0 \\ \sum_{x' \in \{0,1,\dots\}} \frac{\lambda^{2x'+|d|}}{x'!(x'+|d|)!} & \text{if } d < 0 \end{cases} = e^{-2\lambda} \sum_{x \in \{0,1,\dots\}} \frac{\lambda^{2x+|d|}}{x!(x+|d|)!}$$

$$= e^{-2\lambda} \sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!} = e^{-2\lambda} I_{|d|}(2\lambda) = Skellam(\lambda, \lambda)$$

$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim f_{\vec{X}}(\vec{x})$, $X_1, X_2 \overset{iid}{\sim} U(0,1)$, $\quad T = X_1 + X_2 \sim f_T(t) = f_{X_1}(x) \underset{\text{convolution}}{*} f_{X_2}(x) = ?$

CDF method: we will derive the CDF and then take its derivative.

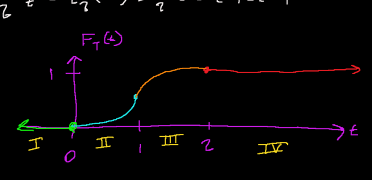


$$F_T(t) := P(T \le t) = P(X_1 + X_2 \le t) = P(\vec{X} \in A_t)$$

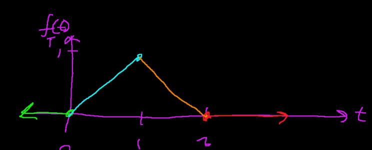
$$A_t := \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \underset{x_2 \le t - x_1}{x_1 + x_2 \le t}\right\}$$

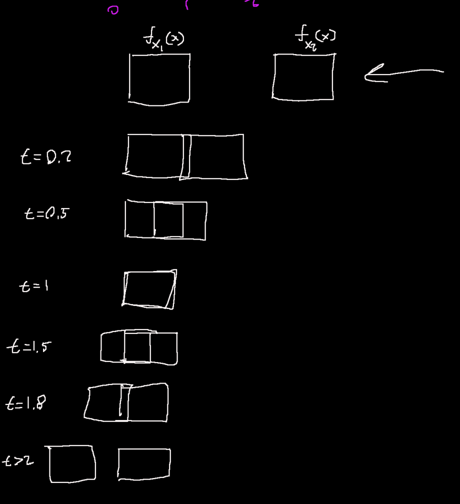
$$= \begin{cases} 0 & , t < 0 & \text{I} \\ \frac{1}{2}t^2 & , t \in [0,1] & \text{II} \\ -\frac{t^2}{2} + 2t - 1 & , t \in (1,2] & \text{III} \\ 1 & , t > 2 & \text{IV} \end{cases}$$

$$\frac{1}{2}t^2 - 2\frac{1}{2}(t-1)^2 = \frac{1}{2}t^2 - t^2 + 2t - 1$$

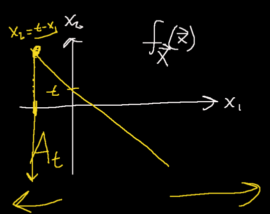


$$f(t) = F'(t) = \begin{cases} 0 & , t < 0 \\ t & , t \in [0,1] \\ 2-t & , t \in (1,2] \\ 0 & , t > 2 \end{cases}$$





Let's derive a general formula



$$F_T(t) = P(\vec{X} \in A_t) = \iint_{A_t} f_{\vec{X}}(\vec{x})\, dx_1 dx_2$$

$$= \int_{\mathbb{R}} \int_{-\infty}^{t-x_1} f_{X_1,X_2}(x_1, x_2)\, dx_2\, dx_1$$

$x_2 = t - x \Rightarrow v - x = t - x \Rightarrow v = t$

let $x_1 = x$, $x_2 = v - x \iff v = x_2 + x$

$\Rightarrow dx_1 = dx \qquad \Rightarrow dx_2 = dv$

$x_2 = -\infty \Rightarrow v = -\infty$

$$= \int_{\mathbb{R}} \int_{-\infty}^{t} f_{X_1,X_2}(x, v-x)\, dv\, dx$$

$$\boxed{F_T(t) = \int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1,X_2}(x, v-x)\, dx\, dv}$$

$$\overset{X_1, X_2 \overset{ind}{\sim}}{=} \int_{-\infty}^{t} \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(v-x)\, dx\, dv \overset{X_1, X_2 \overset{iid}{\sim}}{=} \int_{-\infty}^{t} \int_{\mathbb{R}} f(x) f(v-x)\, dx\, dv$$

$$f_T(t) = \frac{d}{dt}\left[\ \right]$$

Leibnitz's Rule:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} g(x,y)\, dy\right] = g(x, b(x))\, b'(x) + g(x, a(x))\, a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}\left[g(x,y)\right] dy$$

Corr: if the outer derivative and bounds are a 3rd variable:

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} g(x,y)\, dy\right] = g(x, b(t))\, b'(t) + g(x, a(t))\, a'(t) + \cancel{\int_{a(t)}^{b(t)} \frac{\partial}{\partial t} g(x,y)\, dy}^{\,0}$$

Corr: b(t) = t, a(t) = -∞

$$\frac{d}{dt}\left[\int_{-\infty}^{t} g(x,y)\, dy\right] = g(x,t)$$

$$= \int_{\mathbb{R}} f_{X_1,X_2}(x, t-x)\, dx \qquad \text{general convolution formula}$$

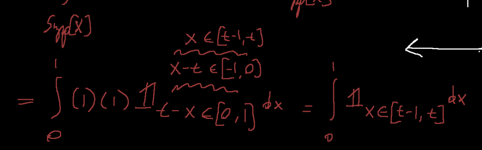
$$\overset{X_1, X_2 \overset{ind}{\sim}}{=} \int_{\mathbb{R}} f_{X_1}(x) f_{X_2}(t-x)\, dx \overset{\text{if } X_1, X_2 \overset{iid}{\sim}}{=} \int_{\mathbb{R}} f(x) f(t-x)\, dx$$

$$= \int_{\mathbb{R}} f_{X_1}^{dd}(x) \mathbb{1}_{x \in Supp[X_1]} f_{X_2}^{dd}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}\, dx$$

$$= \int_{Supp[X_1]} f_{X_1}^{dd}(x) f_{X_2}^{dd}(t-x) \mathbb{1}_{t-x \in Supp[X_2]}\, dx$$

$$= \int_{\mathbb{R}} f^{dd}(x) \mathbb{1}_{x \in Supp[X]} f^{dd}(t-x) \mathbb{1}_{t-x \in Supp[X]}\, dx$$

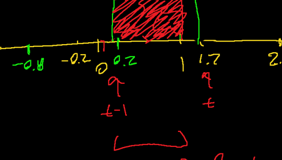
$$= \int_{Supp[X]} f^{dd}(x) f^{dd}(t-x) \mathbb{1}_{t-x \in Supp[X]}$$



$$\int_{\mathbb{R}} f(x) \mathbb{1}_{x \in A}\, dx = \int_A f(x)\, dx$$

$X_1, X_2 \overset{iid}{\sim} U(0,1)$, $T = X_1 + X_2 \sim f_T(t) = ?$

$$f_T(t) = \int_{Supp[X]} f^{dd}(x) f^{dd}(t-x) \mathbb{1}_{t-x \in Supp[X]}\, dx$$



$$= \int_0^1 (1)(1) \mathbb{1}_{\underbrace{t-x \in [0,1]}_{\substack{x-t \in [-1,0] \\ x \in [t-1,t]}}}\, dx = \int_0^1 \mathbb{1}_{x \in [t-1,t]}\, dx$$

$$= \begin{cases} 0, & t < 0 \\ t, & t \in [0,1] \\ 2-t, & t \in [1,2] \\ 0, & t > 2 \end{cases}$$