$X_1, X_2 \overset{iid}{\sim} \text{Poisson}(\lambda)$, let $Y = -X_2 \sim \frac{\lambda^{-y} e^{-\lambda}}{(-y)!} \mathbb{1}_{-y \in \{0,1,\ldots\}} = \frac{\lambda^{-y} e^{-\lambda}}{(-y)!} \mathbb{1}_{y \in \{\ldots,-1,0\}}$

$D = X_1 + Y \sim p_D(d) = ?$ ($= X_1 - X_2$)

$$p_D(d) = \sum_{x \in Supp[X_1]} p_{X_1}^{old}(x)\, p_Y^{old}(d-x)\, \mathbb{1}_{d-x \in Supp[Y]}$$

$$= \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\lambda^{x-d} e^{-\lambda}}{(x-d)!} \mathbb{1}_{d-x \in \{\ldots,-1,0\}}$$

$x - d \in \{0,1,\ldots\}$

$x \in \{d, d+1, \ldots\}$

$$= e^{-2\lambda} \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} \mathbb{1}_{x \in \{d,d+1,\ldots\}}$$

$$= e^{-2\lambda} \begin{cases} \sum_{x \in \{0,1,\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d \le 0 \\ \sum_{x \in \{d,d+1,\ldots\}} \frac{\lambda^{2x-d}}{x!(x-d)!} & \text{if } d > 0 \end{cases}$$

let $d' = -d$, $x' = x - d = x + d'$

next time

$$= e^{-2\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{2(x'+d)-d}}{x!(x+d')!} = e^{-2\lambda} \sum_{x=0}^{\infty} \frac{\left(\frac{2\lambda}{2}\right)^{2x+|d|}}{x!(x+|d|)!}$$

$$= e^{-2\lambda} I_{|d|}(2\lambda) = \text{Skellam}(\lambda, \lambda)$$

Modified Bessel Function of the first kind denoted $I_{|d|}(2\lambda)$

this was discovered in 1946

Mid I ↑ we're done with discrete rv's

Mid II ↓

Recall $B_1, B_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

let $X := \{\#$ of zeroes before the first one$\} = \text{Geom}(p) = (1-p)^x p \mathbb{1}_{x \in \{0,1,\ldots\}}$

disjoint

$$P(X>0) = P(X=1) + P(X=2) + P(X=3) + \ldots$$

$$= P(B_1=0, B_2=1) + P(B_1=0, B_2=0, B_3=1) + P(B_1=0, B_2=0, B_3=0, B_4=1)$$

law of total probability

$$= P(B_1 = 0) = (1-p)^1$$

$$P(X>1) = P(X=2) + P(X=3) + \ldots$$

$$= P(B_1=0, B_2=0, B_3=1) + P(B_1=0, B_2=0, B_3=0, B_4=1)$$

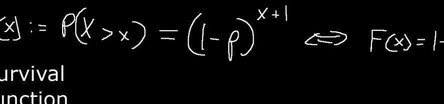
law of total probability

$$= P(B_1=0, B_2=0) = (1-p)^2$$

⋮

$$S(x) := P(X > x) = (1-p)^{x+1} \iff F(x) = 1 - S(x) = 1 - (1-p)^{x+1}$$

survival function




Let's instead run n experiments per time unit.

$X_n = \{$ # of experimental zeroes before the first one $\}$

$$p_{X_n}(t) = (1-p)^{nt} p \, \mathbb{1}_{t \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots\}}$$

$$F_{X_n}(t) = 1 - (1-p)^{nt}$$

If $n \to \infty$ as before with the Poisson we also want p → 0 so let λ = np which remains constant

$p = \frac{\lambda}{n}$

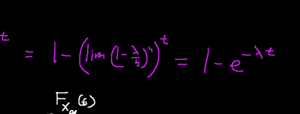
$$p_{X_\infty}(t) = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{nt} \frac{\lambda}{n} \mathbb{1}_{t \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots\}}$$

$$= \left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^t \left(\lim_{n\to\infty} \frac{\lambda}{n}\right) \lim \mathbb{1}_{t \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots\}}$$

$$= e^{-\lambda t} (0) \mathbb{1}_{t \in [0, \infty)}$$ set of all non-negative reals

$= 0$ thus there is no PMF!

$$F_{X_\infty}(t) = \lim_{n\to\infty} F_{X_n}(t) = \lim_{n\to\infty} 1 - \left(1 - \frac{\lambda}{n}\right)^{nt} = 1 - \left(\lim\left(1-\frac{\lambda}{n}\right)^n\right)^t = 1 - e^{-\lambda t}$$




legal CDF! It's zero as t → -∞ and one as t → ∞ and monotonically increasing

Note: | Supp[$X_\infty$] | = |ℝ| > |ℕ| thus this new rv is *not* discrete. Since it's support has the same cardinality of the continuum, it is called a "continuous" rv. There is no PMF! The PMF is zero everywhere. Thus P(X = a) = 0 for all values a. We now define for a continuous rv X,

$$f_X(x) = \frac{d}{dx}\left[F_X(x)\right]$$ which is called the probability density function (PDF). Then by the fundamental theorem of calculus,

$$P(X \in [a,b]) = \underbrace{F(b) - F(a)}_{P(X \le b) - P(X \le a)} = \int_a^b f_X(x)\,dx.$$

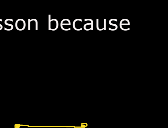
Properties of the PDF: $\int_{\mathbb{R}} f(x)\,dx = 1$, $f(x) \ge 0$ i.e. the CDF is monotonically increasing

$$1 = P(X \in [-\infty, \infty]) = \int_{-\infty}^{\infty} f(x)\,dx$$

The support of a continuous rv X is Supp[X] = {x : f(x) > 0}

exponential rv:

$$X \sim Exp(\lambda) := \underbrace{f(x)}_{f^{dd}(x)} = \lambda e^{-\lambda x} \mathbb{1}_{x \ge 0}$$




Parameter space: λ ∈ [0, ∞) same as Poisson because it is the same conceptually

uniform rv:

$$X \sim U(a,b) = \frac{1}{b-a} \mathbb{1}_{x \in [a,b]}$$


Supp[X] = [a, b]
Parameter space: b > a and a, b ∈ ℝ

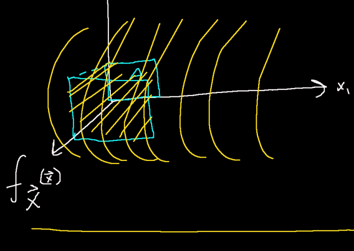
if X_1, ..., X_K are independent

$$\vec{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix} \sim f_{\vec{X}}(\vec{x}) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_K}(x_k) = f(x_1) f(x_2) \cdots f(x_k)$$ if iid

joint density function (JDF)

$$\int \cdots \int_{\mathbb{R}^k} f_{\vec{X}}(\vec{x})\, dx_1 \cdots dx_k = 1$$

To get probabilities from a JDF, we integrate over regions:




$$P(A) = \iint_A f_{\vec{X}}(\vec{x})\, dx_1 dx_2$$

let $T_2 = X_1 + X_2 \sim f_T(t) = ?$

This is now a continuous convolution.