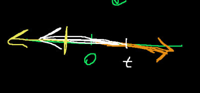
p146 $X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda)$. $\quad T_2 = X_1 + X_2 \sim f_{T_2}(t) = ?$

$$f_T(t) = \int_{Supp[X]} f^{old}(x)\, f^{old}(t-x)\, \mathbb{1}_{t-x \in Supp[X]}\, dx$$

$$(-\infty, t) \cap (0,\infty) = \begin{cases} (0,t) & \text{if } t \geq 0 \\ \phi & \text{if } t < 0 \end{cases}$$



$$= \int_{x\in(0,\infty)} \underbrace{\lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)}}_{\lambda^2 e^{-\lambda t}} \mathbb{1}_{t-x\in(0,\infty)}\, dx \qquad \overset{-\lambda t + \lambda x}{} \qquad t-x\in(0,\infty),\ x-t\in(-\infty,0),\ x\in(-\infty,t)$$

$$= \lambda^2 e^{-\lambda t} \int_0^t dx\, \mathbb{1}_{t\geq 0} = t\lambda^2 e^{-\lambda t}\mathbb{1}_{t\geq 0}$$

$$T_3 = X_1 + X_2 + X_3 \sim f_T(t) = ?$$
$$= T_2 + X_3$$

$$f_T(t) = \int_{Supp[T_2]} f^{old}_{T_2}(x)\, f_{X_3}(t-x)\, \mathbb{1}_{t-x\in Supp[X_3]}\, dx$$

$$= \int_{x\in(0,\infty)} x\lambda^2 \cancel{e^{-\lambda x}}\, \lambda e^{-\lambda(t-x)} \mathbb{1}_{t-x\in(0,\infty)}\, dx \qquad e^{-\lambda t}\cancel{e^{\lambda x}} \qquad x\in(-\infty,t)$$

$$= \lambda^3 e^{-\lambda t}\int_0^t x\, dx\, \mathbb{1}_{t\geq 0} = \frac{1}{2}t^2\lambda^3 e^{-\lambda t}\mathbb{1}_{t\geq 0}$$

$$f_{T_4}(t) = \int_0^\infty \frac{1}{2}x^2\lambda^3 e^{-\lambda x}\lambda e^{-\lambda(t-x)}\mathbb{1}_{x\in(-\infty,t)}\, dx$$

$$= \frac{1}{2}\lambda^{3+1} e^{-\lambda t}\int_0^t x^2\, dx\, \mathbb{1}_{t\geq 0}$$

$$= \left(\frac{1}{3}\right)\frac{1}{2}\lambda^{3+1} e^{-\lambda t} t^{2+1}\mathbb{1}_{t\geq 0}$$

$\vdots$

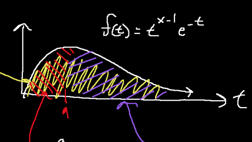
$$f_{T_k}(t) = \frac{1}{(k-1)!}\lambda^k e^{-\lambda t} t^{k-1}\mathbb{1}_{t\geq 0} = Erlang(k,\lambda)$$

Support of an Erlang = $(0, \infty)$, parameter space $\lambda \in (0, \infty)$, $k \in \mathbb{N}$

We're going to do some boring math now with definitions.

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\, dt$$

This is the "gamma function" and no closed form antiderivative exists. Its values are approximated by a computer.



$$= \underbrace{\int_0^a t^{x-1}e^{-t}\, dt}_{\gamma(x,a)} + \underbrace{\int_a^\infty t^{x-1}e^{-t}\, dt}_{\Gamma(x,a)}, \quad a\in(0,\infty)$$

lower incomplete gamma function | upper incomplete gamma function

$$\Rightarrow P(x,a) := \frac{\gamma(x,a)}{\Gamma(x)}, \quad Q(x,a) := \frac{\Gamma(x,a)}{\Gamma(x)} \Rightarrow P(x,a) + Q(x,a) = 1$$

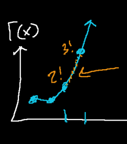
lower regularized gamma function | upper regularized gamma function

Why is the gamma function so important? It comes up a lot but there's also another cool reason. Watch:

$$\Gamma(1) = \int_0^\infty t^{1-1}e^{-t}\, dt = 1, \quad \Gamma(x+1) \overset{HW}{=} \cdots = x\Gamma(x)$$

$$\Gamma(2) = 1\cdot 1 = 1,\ \Gamma(3) = 2\cdot 1 = 2!,\ \Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2! = 3!$$

$$\Rightarrow \Gamma(n) = (n-1)!,\ \Gamma(n+1) = n!$$



real extension to the discrete factorial function

$$X \sim Erlang(k,\lambda) = \frac{\lambda^k e^{-\lambda x}x^{k-1}}{(k-1)!}\mathbb{1}_{x>0} = f(x)$$

$$F(x) = P(X\leq x) = \int_0^x f(y)\, dy = \int_0^x \frac{\lambda^k e^{-\lambda y}y^{k-1}}{(k-1)!}\mathbb{1}_{y>0}\, dy$$

$$= \frac{\lambda^k}{(k-1)!}\int_0^x y^{k-1}e^{-\lambda y}\mathbb{1}_{y>0}\, dy = \frac{\lambda^k}{(k-1)!}\int_0^x y^{k-1}e^{-\lambda y}\, dy\, \mathbb{1}_{x>0}$$

Useful integrals related to the gamma function family:

$$\int_0^\infty t^{x-1}e^{-ct}\, dt \overset{c>0}{=} \int_0^\infty \left(\frac{u}{c}\right)^{x-1}e^{-u}\frac{1}{c}\, du = \frac{1}{c^x}\int_0^\infty u^{x-1}e^{-u}\, du = \frac{\Gamma(x)}{c^x}$$

let $u = ct \Rightarrow t = \frac{u}{c} \Rightarrow dt = \frac{1}{c}du$, $t = 0 \Rightarrow u = 0$, $t = \infty \Rightarrow u = \infty$, $t = a \Rightarrow u = ac$

$$\int_0^a t^{x-1}e^{-ct}\, dt = \int_0^{ac}\left(\frac{u}{c}\right)^{x-1}e^{-u}\frac{1}{c}\, du = \frac{1}{c^x}\int_0^{ac} u^{x-1}e^{-u}\, du = \frac{\gamma(x,ac)}{c^x}$$

$$= \frac{\cancel{\lambda^k}}{(k-1)!}\frac{\gamma(k,\lambda x)}{\cancel{\lambda^k}} = \frac{\gamma(k,\lambda x)}{\Gamma(k)} = P(k,\lambda x)$$

If $n\in\mathbb{N}$ $\quad \Gamma(n,a) = \int_a^\infty \underbrace{t^{n-1}}_{u}\underbrace{e^{-t}\, dt}_{dv} = [uv]_a^\infty - \int_a^\infty v\, du$

$$\frac{du}{dt} = (n-1)t^{n-2} \Rightarrow du = (n-1)t^{n-2}dt, \quad v = \int e^{-t}dt = -e^{-t}$$

$$\lim_{t\to\infty}\frac{t^b}{e^t} = 0$$

$$= \left[-t^{n-1}e^{-t}\right]_a^\infty - \int_a^\infty(-e^{-t})(n-1)t^{n-2}\, dt$$

$$= a^{n-1}e^{-a} + (n-1)\int_a^\infty t^{n-2}e^{-t}\, dt = a^{n-1}e^{-a} + (n-1)\Gamma(n-1,a)$$

$$= a^{n-1}e^{-a} + (n-1)\left(a^{n-2}e^{-a} + (n-2)\Gamma(n-3,a)\right)$$

$\vdots$

how does this end? It ends when the first argument to the upper incomplete gamma function is 1 i.e. $\Gamma(1,a)$:

$$\Gamma(1,a) = \int_a^\infty t^{1-1}e^{-t}\, dt = \int_a^\infty e^{-t}\, dt = \left[-e^{-t}\right]_a^\infty = 0 - -e^{-a} = e^{-a}$$

$$= a^{n-1}e^{-a} + (n-1)a^{n-2}e^{-a} + (n-2)(n-1)a^{n-3}e^{-a} + (n-3)(n-2)(n-1)a^{n-4}e^{-a} +$$
$$\ldots(n-1)(n-2)\cdots(1)a^0e^{-a}$$

$$= e^{-a}(n-1)!\left(\frac{a^{n-1}}{(n-1)!} + \frac{a^{n-2}}{(n-2)!} + \frac{a^{n-3}}{(n-3)!} + \ldots + \frac{a^0}{1!}\right)$$

$$= e^{-a}(n-1)!\sum_{i=0}^{n-1}\frac{a^i}{i!} \iff \Gamma(n+1,a) = e^{-a}n!\sum_{i=0}^{n}\frac{a^i}{i!}$$

getting indicator functions correct for CDF's is very difficult so I will ignore it in Math 368

$$X \sim Poisson(\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}\mathbb{1}_{x\in\{0,1,\ldots\}}$$

$$F(x) = P(X\leq x) = \sum_{y=0}^x \frac{e^{-\lambda}\lambda^y}{y!}\mathbb{1}_{y\in\{0,1,\ldots\}} = \mathbb{1}_{x\geq 0}\sum_{y=0}^x\frac{e^{-\lambda}\lambda^y}{y!}$$

$$= e^{-\lambda}\sum_{y=0}^x\frac{\lambda^y}{y!} = \frac{1}{x!}e^{-\lambda}x!\sum_{y=0}^x\frac{\lambda^y}{y!} = \frac{\Gamma(x+1,\lambda)}{x!}$$

$$= \frac{\Gamma(x+1,\lambda)}{\Gamma(x+1)} = Q(x+1,\lambda)$$

This means there is an intimate relationship between the Erlang and the Poisson called the "Poisson process" which we will explore next class.