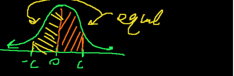
$Z \sim N(0,1)$ let $Y = Z^2 \sim ?$




We can't use our tranformation of variable formula to find the density of Y since $g(z) = z^2$ is an invertible function since $z \in \mathbb{R}$. So we need to solve this another way

$$F_Y(y) := P(Y \le y) = P(Z^2 \le y) = P\left(Z \in [-\sqrt{y}, \sqrt{y}]\right)$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_Z(z)\,dz = 2\int_0^{\sqrt{y}} f_Z(z)\,dz = 2\left(F_Z(\sqrt{y}) - F_Z(0)\right)$$

Do we know the CDF of the standard normal?

$$F_Z(z) = \int_{-\infty}^{v} \frac{1}{\sqrt{2\pi}} e^{-v^2/2}\,dv$$

unfortunately this not solvable in closed form (proven)

Obviously the quantile function isn't either.

$F_Z(0) = \frac{1}{2}$

$$= 2F_Z(\sqrt{y}) - 1, \quad f_Y(y) = \frac{d}{dy}\left[\;\right] = \cancel{2}\frac{1}{\cancel{2}\sqrt{y}} \cdot f_Z(\sqrt{y})$$

$$= \frac{1}{\sqrt{y}}\frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2}\,\mathbb{1}_{\underbrace{\sqrt{y}\in\mathbb{R}}_{y>0}}$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}\,\mathbb{1}_{y>0}$$

$$\propto y^{\frac{1}{2}-1} e^{-\frac{1}{2}y}\,\mathbb{1}_{y>0} \propto \boxed{Z^2 \sim \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)}$$

$Z_1, \ldots, Z_k \overset{iid}{\sim} N(0,1), \quad k \in \mathbb{N}$

$$Z_1^2 + \ldots + Z_k^2 \sim \text{Gamma}\left(\frac{k}{2}, \frac{1}{2}\right) = \chi^2_k$$

This Gamma is special and called the "chi-squared with k degree of freedom". The "degrees of freedom" is its only parameter.

$$\chi^2_k := \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}\,\mathbb{1}_{x>0}$$

$$\chi^2_1 = \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} e^{-\frac{x}{2}}\,\mathbb{1}_{x>0}$$

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$X \sim \chi^2_k, \quad Y = \sqrt{X} \sim ?$

$$\Downarrow$$

$$X = Y^2 = g^{-1}(Y) \qquad \frac{d}{dy}\left[g^{-1}(y)\right] = 2y$$

$$f_Y(y) = f_X(y^2)\,|2y| = \frac{1}{2^{k/2}\Gamma\left(\frac{k}{2}\right)} (y^2)^{\frac{k}{2}-1} e^{-\frac{y^2}{2}}\,\mathbb{1}_{y^2>0}\,(2y)$$

$$= \frac{1}{2^{k/2-1}\Gamma\left(\frac{k}{2}\right)} y^{k-1} e^{-\frac{y^2}{2}}\,\mathbb{1}_{y>0} = \chi_k$$

This is the "chi distribution" with k degrees of freedom

$Z \sim N(0,1), \; Y = |Z| \sim ? \quad Y = |Z| = \sqrt{Z^2} \sim \chi_1$

Two times the density of the std normal

$$2\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right) = \sqrt{\frac{2}{\pi}}\, e^{-y^2/2}\,\mathbb{1}_{y>0}$$

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$$X \sim \text{Gamma}(\alpha, \beta), \quad Y = cX \sim \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c}\frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{y}{c}\right)^{\alpha-1} e^{-\frac{\beta}{c}y}\,\mathbb{1}_{\frac{y}{c}>0} \qquad c>0$$

$$= \frac{(\beta/c)^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{\beta}{c}y}\,\mathbb{1}_{y>0}$$

$$= \text{Gamma}\left(\alpha, \frac{\beta}{c}\right)$$

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$X_1 \sim \chi^2_{k_1}$ indep of $X_2 \sim \chi^2_{k_2}$

$$R = \frac{X_1/k_1}{X_2/k_2} \sim ?$$

$$U = \frac{1}{k_1}X_1 \sim \text{Gamma}\left(\underbrace{\frac{k_1}{2}}_{a}, \frac{k_1}{2}\right) \qquad \frac{\frac{1}{2}}{\frac{1}{k_1}}$$

$$V = \frac{1}{k_2}X_2 \sim \text{Gamma}\left(\underbrace{\frac{k_2}{2}}_{b}, \frac{k_2}{2}\right)$$

$$R = \frac{U}{V} \sim \int_{\mathbb{R}} f_U(ru) f_V(u)\,|u|\,du$$

$$= \mathbb{1}_{r>0}\int_0^\infty f_U^{old}(ru) f_V^{old}(u)\,u\,du$$

$$= \mathbb{1}_{r>0}\int_0^\infty u \frac{a^a}{\Gamma(a)}(ru)^{a-1} e^{-aru} \frac{b^b}{\Gamma(b)} u^{b-1} e^{-bu}\,du$$

$$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1}\,\mathbb{1}_{r>0}\int_0^\infty u^{a+b-1} e^{-(ar+b)u}\,du$$

$$= \frac{a^a b^b}{\Gamma(a)\Gamma(b)} r^{a-1}\,\mathbb{1}_{r>0}\,\frac{\Gamma(a+b)}{(ar+b)^{a+b}}$$

$$= \frac{a^a b^b}{B(a,b)} r^{a-1} (ar+b)^{-(a+b)}\,\mathbb{1}_{r>0}$$

$$= \frac{a^a \cancel{b^b}}{B(a,b)} r^{a-1} \frac{1}{b^a \cancel{b^b}}\left(\frac{a}{b}r+1\right)^{-(a+b)}\,\mathbb{1}_{r>0}$$

$$= \frac{\left(\frac{a}{b}\right)^a}{B(a,b)} r^{a-1}\left(\frac{a}{b}r+1\right)^{-(a+b)}\,\mathbb{1}_{r>0}$$

$$= \frac{\left(\frac{k_1}{k_2}\right)^{k_1/2}}{B\left(\frac{k_1}{2}, \frac{k_2}{2}\right)} r^{\frac{k_1}{2}-1}\left(\frac{k_1}{k_2}r+1\right)^{-\frac{k_1+k_2}{2}}\,\mathbb{1}_{r>0} = F_{k_1,k_2}$$

$k_1, k_2 \in \mathbb{N}$

The Fisher-Snecedor distribution with k1 numerator degrees of freedom and k2 denominator degrees of freedom. It's a special case of the BetaPrime distribution. It's found in "linear regression" a common statistical modeling technique.

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$Z \sim N(0,1)$ indep. of $X \sim \chi^2_k$

$$W = \frac{Z}{\sqrt{X/k}} \sim ?$$

$$\Rightarrow W^2 = \frac{U\left\{\frac{Z^2}{1}\right.}{V\left\{\frac{X}{k}\right.} \sim F_{1,k} \qquad U \sim \text{Gamma}\left(\tfrac{1}{2}, \tfrac{1}{2}\right), \quad V \sim \text{Gamma}\left(\tfrac{k}{2}, \tfrac{k}{2}\right)$$

$$F_{W^2}(w^2) := P(W^2 \le w^2) = P\left(W \in [-w, w]\right)$$

$$= F_W(w) - F_W(-w)$$

Take d/dw of both sides: $\frac{d}{dw}\left[F_{W^2}(w^2)\right] = \frac{d}{dw}\left[F_W(w) - F_W(-w)\right]$

$$\Rightarrow \cancel{2}w\, f_{W^2}(w^2) = f_W(w) - -f_W(w) = \cancel{2} f_W(w)$$

$$\Rightarrow f_W(w) = \cancel{w}\frac{\left(\frac{1}{k}\right)^{\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{k}{2}\right)} \cancel{(w^2)^{\frac{1}{2}-1}}\left(\frac{1}{k}w^2+1\right)^{-\frac{1+k}{2}}\,\mathbb{1}_{w>0}$$

$$B\left(\tfrac{1}{2}, \tfrac{k}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)}, \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma\left(\frac{k}{2}\right)}\left(1+\frac{w^2}{k}\right)^{-\frac{k+1}{2}}\,\mathbb{1}_{w>0} = T_k$$

This is called Student's T distribution with k degrees of freedom and $k \in \mathbb{N}$.

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$Z_1, Z_2 \overset{iid}{\sim} N(0,1)$

$$R = \frac{Z_1}{Z_2} \sim \int_{\mathbb{R}} f(ru) f(u)\,|u|\,du = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2u^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,|u|\,du \qquad e^{-\frac{1+r^2}{2}u^2}$$

$|-5| = 5 = --5$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^0 e^{-\left(\frac{1+r^2}{2}\right)u^2}(-u)\,du + \int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} u\,du\right)$$

let $v = -u \quad du = -dv \Rightarrow v^2 = u^2 \Rightarrow u = -\infty \Rightarrow v = \infty$, $\Rightarrow u = 0 \Rightarrow v = 0$

$$= \frac{1}{2\pi}\left(-\int_\infty^0 e^{-\left(\frac{1+r^2}{2}\right)v^2} v\,dv + \int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} u\,du\right)$$

$$= \frac{1}{\cancel{2}\pi}\left(\cancel{2}\int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)u^2} u\,du\right)$$

let $t = u^2 \Rightarrow \frac{dt}{du} = 2u \Rightarrow du = \frac{1}{2u}dt$

$u = 0 \Rightarrow t = 0, \quad u = \infty \Rightarrow t = \infty$

$$= \frac{1}{\pi}\int_0^\infty e^{-\left(\frac{1+r^2}{2}\right)t}\,\cancel{u}\,\frac{1}{2\cancel{u}}\,dt = \frac{1}{2\pi}\left[\frac{1}{-\frac{1+r^2}{2}} e^{-\frac{1+r^2}{2}t}\right]_0^\infty$$

$$= \frac{1}{\cancel{2}\pi}\frac{1}{\frac{1+r^2}{\cancel{2}}} = \frac{1}{\pi}\frac{1}{1+r^2} = \text{Cauchy}(0,1)$$

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$R \sim \text{Cauchy}(0,1)$

$$X = c + \sigma R \sim \frac{1}{\sigma\pi}\frac{1}{1+\left(\frac{x-c}{\sigma}\right)^2} = \text{Cauchy}(c, \sigma)$$

$\sigma > 0, \; c \in \mathbb{R}$