Let $X_1, \ldots, X_n \overset{iid}{\sim}$ DGP with mean θ > 0 and unknown variance σ²
where we seek inference for φ := g(θ) = ln(θ).
Due to CLT, we know that for $\hat{\theta} = \bar{X}$,

$$\frac{\hat{\theta} - \theta}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{d} N(0,1) \xRightarrow{\text{Slutsky's}} \frac{\hat{\theta} - \theta}{\frac{S}{\sqrt{n}}} \xrightarrow{d} N(0,1)$$

$$g'(\theta) = \frac{1}{\theta}, \quad |g'(\theta)| = \frac{1}{|\theta|} = \frac{1}{\theta}$$

$$\underbrace{\ln(\hat{\theta}) - \ln(\theta)}_{\phi} \Big/ \frac{1}{\theta}\frac{\sigma}{\sqrt{n}} \xrightarrow{d} N(0,1)$$ by Delta Method thm

$$\Downarrow \text{Slutsky's}$$

$$\frac{\ln(\hat{\theta}) - \ln(\theta)}{\frac{1}{\hat{\theta}} \frac{S}{\sqrt{n}}} \xrightarrow{d} N(0,1) \Rightarrow \hat{CI}_{\phi, 1-\alpha} \approx \left[\ln(\hat{\theta}) \pm z_{1-\frac{\alpha}{2}} \frac{1}{\hat{\theta}} \frac{S}{\sqrt{n}}\right]$$

Example from Midterm last year. x̄ = 2.57, s = 1.00, n = 30.
Make a 95% CI for log mean rat survival.

$$\hat{CI}_{\ln(\theta), 95\%} \approx \left[\ln(2.57) \pm 1.96 \frac{1}{2.57} \frac{1}{\sqrt{30}}\right] = [0.805, 1.083]$$

Back to the AF study where $\hat{\theta}_1 = 0.333$, $n_1 = 81$, $\hat{\theta}_2 = 0.152$, $n_2 = 79$

How much more likely is getting AF in the control group vs. the treatment group?

The canonical way of measuring this is $\hat{\theta}_1 - \hat{\theta}_2 = .181$ and asking is this significantly different from zero? Use the "risk differentce" and a 2-prop z-test! This is in the paper.
Another way of measuring this is the ratio of these probabilities called the "risk ratio" (RR) defined as:

$$RR = \frac{\theta_1}{\theta_2}, \quad \hat{RR} = \frac{\hat{\theta}_1}{\hat{\theta}_2}, \quad \hat{RR} = \frac{\hat{\theta}_1}{\hat{\theta}_2} = \frac{.333}{.152} = 2.192$$

$$\theta_1 = 19\%, \ \theta_2 = 1\% \Rightarrow \theta_1 - \theta_2 = 18\%, \ RR = \frac{19\%}{1\%} = 19$$

$$\theta_1 = 98\%, \ \theta_2 = 80\% \Rightarrow \theta_1 - \theta_2 = 18\%, \ RR = \frac{98\%}{80\%} = 1.2$$

It would be nice to run tests on φ = RR and get CI's for φ = RR.

Multivariate Delta Method. Proof of this is beyond scope of course.

let $g : \mathbb{R}^K \to \mathbb{R}$ with some technical restrictions

let $\Sigma := Var\left[\begin{bmatrix}\hat{\theta}_1 \\ \vdots \\ \hat{\theta}_K\end{bmatrix}\right]$ the variance-covariance matrix (in Math 368)

and $$\frac{\Sigma^{-1}}{\sqrt{n}}\left(\begin{bmatrix}\hat{\theta}_1 \\ \vdots \\ \hat{\theta}_K\end{bmatrix} - \begin{bmatrix}\theta_1 \\ \vdots \\ \theta_K\end{bmatrix}\right) \xrightarrow{d} N_K(\vec{0}_K, I_K)$$

This is called an asymptotically normal vector estimator.

$$\Rightarrow \frac{g(\hat{\theta}_1, \ldots, \hat{\theta}_K) - g(\theta_1, \ldots, \theta_K)}{\sqrt{\vec{\nabla}_g^T \Sigma \vec{\nabla}_g}} \xrightarrow{d} N(0,1)$$

$$\xRightarrow{K=2,\ \hat{\theta}_1, \hat{\theta}_2 \text{ ind}} \frac{g(\hat{\theta}_1, \hat{\theta}_2) - g(\theta_1, \theta_2)}{\sqrt{\left(\frac{\partial g}{\partial \theta_1}\right)^2 Var[\hat{\theta}_1] + \left(\frac{\partial g}{\partial \theta_2}\right)^2 Var[\hat{\theta}_2]}} \xrightarrow{d} N(0,1)$$

In this class, we will only see k = 2 and independent estimators since they'll be functions of data from different populations. So...

$$\Sigma = \begin{bmatrix} Var[\hat{\theta}_1] & 0 \\ 0 & Var[\hat{\theta}_2] \end{bmatrix}, \quad \vec{\nabla}_g = \begin{bmatrix} \frac{\partial g}{\partial \theta_1} \\ \frac{\partial g}{\partial \theta_2} \end{bmatrix}$$

$$\Rightarrow \vec{\nabla}_g^T \Sigma \vec{\nabla}_g = \begin{bmatrix} \frac{\partial g}{\partial \theta_1} & \frac{\partial g}{\partial \theta_2} \end{bmatrix} \Sigma \begin{bmatrix} \frac{\partial g}{\partial \theta_1} \\ \frac{\partial g}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial \theta_1} & \frac{\partial g}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} Var[\hat{\theta}_1] \frac{\partial g}{\partial \theta_1} \\ Var[\hat{\theta}_2] \frac{\partial g}{\partial \theta_2} \end{bmatrix}$$

$$= \left(\frac{\partial g}{\partial \theta_1}\right)^2 Var[\hat{\theta}_1] + \left(\frac{\partial g}{\partial \theta_2}\right)^2 Var[\hat{\theta}_2]$$

For the risk ratio, $\varphi = g(\theta_1, \theta_2) = \frac{\theta_1}{\theta_2}, \quad \frac{\partial g}{\partial \theta_1} = \frac{1}{\theta_2}, \quad \frac{\partial g}{\partial \theta_2} = -\frac{\theta_1}{\theta_2^2}$

$\begin{bmatrix}\hat{\theta}_1 \\ \hat{\theta}_2\end{bmatrix}$ will be asymptotically normal since each element is asymptotically normal and the elements are independent.

The CI formula via the multivariate delta method is:

$$\hat{CI}_{\phi, 1-\alpha} = \left[g(\hat{\theta}_1, \hat{\theta}_2) \pm z_{1-\frac{\alpha}{2}} \sqrt{\left(\frac{\partial g}{\partial \theta_1}\right)^2 Var[\hat{\theta}_1] + \left(\frac{\partial g}{\partial \theta_2}\right)^2 Var[\hat{\theta}_2]}\right]$$

In our case... using this plus Slutsky's thm...

$$CI_{\phi, 95\%} = \left[\frac{\hat{\theta}_1}{\hat{\theta}_2} \pm 1.96 \sqrt{\frac{1}{\hat{\theta}_2^2} \frac{\hat{\theta}_1(1-\hat{\theta}_1)}{n_1} + \frac{\hat{\theta}_1^2}{\hat{\theta}_2^4} \frac{\hat{\theta}_2(1-\hat{\theta}_2)}{n_2}}\right]$$

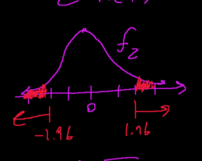
$$= \left[\frac{0.333}{0.152} \pm 1.96 \sqrt{\frac{1}{0.152^2} \frac{0.333(1-0.333)}{81} + \frac{0.333^2}{0.152^4} \frac{0.152(1-0.152)}{79}}\right]$$

$$= [1.020, 3.362]$$

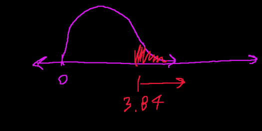
Usually you want to test against the null RR = 1 which means the drug is the same as the control / placebo.

Some facts from distribution theory (covered in Math 368):

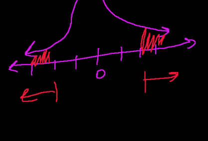
$Z \sim N(0,1)$



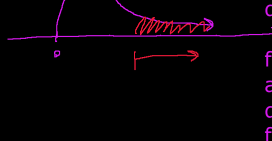
let $Y = Z^2 \sim \chi^2_1$ Chi-squared rv with 1 degree of freedom



$X \sim T_{df}$



let $Y = X^2 \sim F_{1,df}$ Fisher-Snecedor rv with numerator degrees of freedom 1 and denom. degree of freedom df.



* Every Z test (approx or exact) is equivalent to a chi-squared-1 test (approx or exact)
* Every T-df test (approx or exact) is equivalent to a F-1-df test (approx or exact)

For example...

If $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 known, $\hat{\theta} = \bar{X}$, $H_0: \theta = \theta_0$

$$\frac{\hat{\theta} - \theta_0}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \Rightarrow \frac{(\hat{\theta} - \theta_0)^2}{\frac{\sigma^2}{n}} \sim \chi^2_1$$

If $X_1, \ldots, X_n \overset{iid}{\sim} Bern(\theta)$, $H_0: \theta = \theta_0$, $\hat{\theta} = \bar{X}$

$$\frac{\hat{\theta} - \theta_0}{\sqrt{\frac{\theta_0(1-\theta_0)}{n}}} \overset{\circ}{\sim} N(0,1) \Rightarrow \frac{(\hat{\theta} - \theta_0)^2}{\frac{\theta_0(1-\theta_0)}{n}} \overset{\circ}{\sim} \chi^2_1$$

If $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 unknown, $\hat{\theta} = \bar{X}$, $H_0: \theta = \theta_0$

$$\frac{\hat{\theta} - \theta_0}{\frac{S}{\sqrt{n}}} \sim T_{n-1} \Rightarrow \frac{(\hat{\theta} - \theta_0)^2}{\frac{S^2}{n}} \sim F_{1, n-1}$$

If $X_1, \ldots, X_n \sim f(\theta)$. $\hat{\theta}^{MLE}$ is derived, $H_0: \theta = \theta_0$

$$\frac{\hat{\theta}^{MLE} - \theta_0}{\sqrt{\frac{I(\theta_0)^{-1}}{n}}} \overset{\circ}{\sim} N(0,1) \Rightarrow \frac{\hat{\theta}^{MLE} - \theta_0}{\frac{I(\theta_0)^{-1}}{n}} \overset{\circ}{\sim} \chi^2_1$$

In many textbooks, you'll see the Wald test as a chi-squared-1 test. This is the reason why: the two are equivalent.

Consider the case where you roll a 6-sided die n times and you want to test if the die is unfair. What is the null? The parameters are:

$\theta_1 := P(\text{rolling } 1), \ \theta_2 := P(\text{rolling } 2), \ldots, \ \theta_6 = P(\text{rolling } 6)$

$H_0: \theta_1 = \theta_2 = \ldots = \theta_6 = \frac{1}{6}$

H_a: At least one of the θ's ≠ 1/6.

Now we need a test statistic / estimator that measures deviance / departure from the null hypothesis. Then we need its sampling distribution (at least its approximate sampling distribution). Then we can run the test.

Remember our data will look like <4,3,6,..., 3>, the numbers that come up on the die in n rolls.

$x_1 = 4, \ x_2 = 3, \ x_3 = 6, \ x_n = 3$