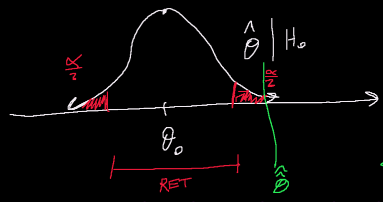
If the null hypothesis is true, what do the pvals look like?



$$\hat{\hat{z}} = \frac{\hat{\hat{\theta}} - \theta_0}{SE[\hat{\theta}]} \overset{?}{\in} RET$$

You can prove that the pvals are distributed U(0,1)
This is a fact from MATH 368.

We return now to general statistical testing theory. We previously proved the monster MLE thm:

$$RET_\alpha \approx \left[\theta_0 \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{I(\theta_0)^{-1}}{n}}\right]$$

$$\frac{\hat{\theta}^{MLE} - \theta}{\sqrt{\frac{I(\theta)^{-1}}{n}}} \xrightarrow{d} N(0,1)$$

Wald test $H_0: \theta = \theta_0$

Wald CI

$$CI_{\theta, 1-\alpha} = \left[\hat{\theta}^{MLE} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{I(\hat{\theta}^{MLE})^{-1}}{n}}\right]$$

We're now going to derive another means of testing $H_a: \theta \neq \theta_0$ which uses the MLE theory from a different perspective.
Recall for any iid DGP,

$$s(\theta; X_1, \ldots, X_n) \overset{def}{=} \sum_{i=1}^{n} \underbrace{\ell'(\theta; X_i)}_{W_i}$$

$$\Rightarrow \frac{1}{n} s(\theta; X_1, \ldots, X_n) = \bar{W} \qquad E[W_i] = 0 \quad \text{Fact 1b}$$

$$Var[W_i] = I(\theta)$$

CLT

$$\Rightarrow \frac{\bar{W} - E[\bar{W}]}{\sqrt{Var[\bar{W}]}} \xrightarrow{d} N(0,1)$$

$$\Rightarrow \frac{\frac{1}{n} s(\theta; X_1, \ldots, X_n) - 0}{\sqrt{\frac{I(\theta)}{n}}} \xrightarrow{d} N(0,1)$$

$$\Rightarrow \frac{s(\theta; X_1, \ldots, X_n)}{\sqrt{n I(\theta)}} \xrightarrow{d} N(0,1) \Rightarrow \frac{s(\theta; X_1, \ldots, X_n)}{\sqrt{n I(\theta)}} \overset{\cdot}{\sim} N(0,1)$$

$$\overset{H_0: \theta = \theta_0}{\Longrightarrow} \frac{s(\theta_0; X_1, \ldots, X_n)}{\sqrt{n I(\theta_0)}} \overset{\cdot}{\sim} N(0,1)$$

We can use this to test. We calculate the lhs and check if it's inside of RET = [-1.96, 1.96] if alpha = 5% for example.
This is called the "Score Test" or "Lagrange Multiplier Test" (Rao, 1948).

This is like no test we've ever seen! It's like we magically have a standardized test statistic without ever specifying an estimator and calculating an estimate, calculating the std error of the estimator to standardize the estimate (to get z).

Chances are you will find the same thing as the Wald test but not always. Consider iid Bern(θ). Derive the score test statistic:

$$\mathcal{L}(\theta; x_1, \ldots, x_n) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i} = \theta^{\sum x_i} (1-\theta)^{n - \sum x_i}$$

$$\ell(\theta; x_1, \ldots, x_n) = \sum x_i \ln(\theta) + (n - \sum x_i) \ln(1-\theta)$$

$$\underset{\|}{s(\theta; x_1, \ldots, x_n)}$$

$$\ell'(\theta; x_1, \ldots, x_n) = \frac{\sum x_i}{\theta} - \frac{n - \sum x_i}{1-\theta}, \quad I(\theta) \overset{notes}{= \cdots\cdots =} \frac{1}{\theta(1-\theta)}$$

$$= \frac{(1-\theta)\sum x_i - \theta(n - \sum x_i)}{\theta(1-\theta)} = \frac{(1-\theta) n\bar{x} - \theta(n - n\bar{x})}{\theta(1-\theta)}$$

$$= \frac{n\bar{x} - \cancel{\theta n \bar{x}} - \theta n + \cancel{\theta n \bar{x}}}{\theta(1-\theta)} = \frac{n(\bar{x} - \theta)}{\theta(1-\theta)}$$

$$\Rightarrow z = \frac{s(\theta; x_1, \ldots, x_n)}{\sqrt{n I(\theta)}} = \frac{\frac{n(\bar{x}-\theta)}{\theta(1-\theta)}}{\sqrt{\frac{n}{\theta(1-\theta)}}} = \frac{n(\bar{x}-\theta)}{\sqrt{n\theta(1-\theta)}} = \frac{\bar{x} - \theta_0}{\sqrt{\frac{\theta_0(1-\theta_0)}{n}}}$$

same as the Wald statistic

Here's an example where the score test will be a different test. Let X1,...,Xn are iid Logistic(θ, 1) $:= \frac{e^{-(x_i - \theta)}}{\left(1 + e^{-(x_i - \theta)}\right)^2} = f(x_i; \theta) = \mathcal{L}(\theta; x_i)$

Recall from the HW that there is no closed form MLE for θ!
Of course you can use Xbar which will be consistent by Law of Large Numbers and will be approx normal by CLT. It could be that score test will be more powerful than the CLT test.

$$\mathcal{L}(\theta; x_1, \ldots, x_n) = \frac{e^{-\overbrace{\sum x_i}^{n\bar{x}} - n\theta}}{\prod_{i=1}^{n} \left(1 + e^{-(x_i - \theta)}\right)^2}, \quad \ell(\theta; x_1, \ldots, x_n) = -n(\bar{x} - \theta) - 2\sum_{i=1}^{n} \ln\left(1 + e^{-(x_i - \theta)}\right)$$

$$s(\theta; x_1, \ldots, x_n) = \frac{d}{d\theta}[\;\;] = n - 2\sum_{i=1}^{n} \frac{e^{-(x_i - \theta)}}{1 + e^{-(x_i - \theta)}}$$

$$\frac{d}{d\theta}\left[e^{-(x_i - \theta)}\right] = \frac{d}{d\theta}\left[e^{-x_i} e^{\theta}\right] = e^{-x_i} \frac{d}{d\theta}\left[e^{\theta}\right] = e^{-x_i} e^{\theta} = e^{-(x_i - \theta)}$$

$$\Downarrow$$

$$\ell'(\theta; x_i) = 1 - 2\frac{e^{-(x_i - \theta)}}{1 + e^{-(x_i - \theta)}}$$

$$\ell''(\theta; x_i) = -2\frac{\left(1 + e^{-(x_i - \theta)}\right) e^{-(x_i - \theta)} - \left(e^{-(x_i - \theta)}\right)^2}{\left(1 + e^{-(x_i - \theta)}\right)^2}$$

$$= -2\frac{e^{-(x_i - \theta)} + \cancel{\left(e^{-(x_i - \theta)}\right)^2} - \cancel{\left(e^{-(x_i - \theta)}\right)^2}}{\left(1 + e^{-(x_i - \theta)}\right)^2}$$

$$I(\theta) = E_X\left[-\ell''(\theta; X)\right] = 2\int_{\mathbb{R}} \frac{e^{-(x - \theta)}}{\left(1 + e^{-(x - \theta)}\right)^2} \overbrace{\frac{e^{-(x - \theta)}}{\left(1 + e^{-(x - \theta)}\right)^2}}^{f(x_i; \theta)} dx$$