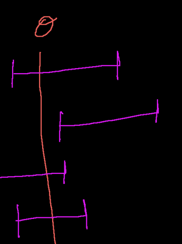


dataset 1

dataset 2

dataset 3

dataset 4

the coverage probability estimate here is 75%

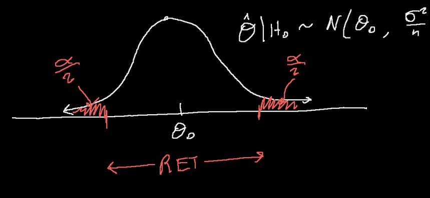
A "confidence interval" (CI) with coverage probability 1 - α is denoted

$$\hat{CI}_{\theta,1-\alpha}, \quad \hat{\hat{CI}}_{\theta,1-\alpha}$$

where the first denotes the interval estimat*or* and the second denotes the interval estimat*e*. We need to define this. Given a α, compute the lower and upper bound of the interval.

Let's begin with an example. Let the DGP be iid N(θ, σ²) where σ² is known and θ is the inferential target. Consider the two sided test where the alternative hypothesis is θ ≠ θ_0 where the estimator is $\bar{X}$ and the size / level is α. This was our picture:

$$\hat{\theta} \mid H_0 \sim N\left(\theta_0, \frac{\sigma^2}{n}\right) \qquad H_0: \theta = \theta_0$$



$$P(\hat{\theta} \in RET \mid H_0) = 1-\alpha$$

inversion

$$\Rightarrow P\left(\hat{\theta} \in \left[\theta_0 \pm z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right] \,\middle|\, \theta = \theta_0\right) = 1-\alpha$$

$$\Rightarrow P\left(\hat{\theta} - \theta_0 \in \left[\pm z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right] \,\middle|\, \theta = \theta_0\right) = 1-\alpha$$

$$\Rightarrow P\left(\theta_0 - \hat{\theta} \in \left[\pm z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right] \,\middle|\, \theta = \theta_0\right) = 1-\alpha$$

$$\Rightarrow P\left(\theta_0 \in \left[\hat{\theta} \pm z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right] \,\middle|\, \theta = \theta_0\right) = 1-\alpha$$

$$\Rightarrow P\left(\theta_0 \in \left[\underbrace{\hat{\theta} - z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}}_{w_L(X_1,\ldots,X_n)}, \underbrace{\hat{\theta} + z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}}_{w_U(X_1,\ldots,X_n)}\right] \,\middle|\, \theta = \theta_0\right) = 1-\alpha$$

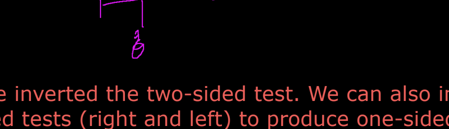
Let $CI_{\theta,1-\alpha} = \left[\hat{\theta} \pm z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right]$

which has coverage probability 1 - α.

We constructed this CI by "inverting the test"

$$\hat{\hat{\theta}} \in RET_{\theta_0,\alpha} \iff \theta_0 \in \hat{CI}_{\theta,1-\alpha}$$

p421 C&B: Both hypothesis testing and interval construction look for "consistency" between the sample statistic $\hat{\hat{\theta}}$ and the population parameter θ. Hypothesis test fix the parameter θ and ask "which $\hat{\hat{\theta}}$'s are consistent?" Confidence sets fix the sample statistic and ask "which parameter θ's are consistent?"



Note: we inverted the two-sided test. We can also invert the one-sided tests (right and left) to produce one-sided CI's e.g.

$$\hat{CI}_{L,\theta,1-\alpha} = [w_L(X_1,\ldots,X_n), \infty)$$

$$\hat{CI}_{R,\theta,1-\alpha} = (-\infty, w_R(X_1,\ldots,X_n)]$$

This class will almost exclusively focus on the two-sided CI's.

Sometimes the sampling distribution is known exactly. Inverting the test here yields *exact* CI's. Sometimes the sampling distribution is known only approximately. Inverting the test here yields *approximate* or *asymptotic* CI's meaning the coverage probabilities are approximately what you specified and converge to what you specified as n → ∞.

Let the DGP be iid N(θ, σ²) where σ² is unknown and θ is the inferential target.

$$\hat{CI}_{\theta,1-\alpha} = \left[\hat{\theta} \pm t_{1-\frac{\alpha}{2}, n-1} \frac{s}{\sqrt{n}}\right] \quad \text{exact!}$$

Let the DGP for sample 1 be iid N(θ_1, σ_1^2) and sample 2 be independent and iid N(θ_2, σ_2^2)

if $\sigma_1^2 = \sigma_2^2 = \sigma^2$ and known

$$\hat{CI}_{\theta,1-\alpha} = \left[(\hat{\theta}_1 - \hat{\theta}_2) \pm z_{1-\frac{\alpha}{2}} \sigma \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right] \quad \text{exact}$$

if $\sigma_1^2 \neq \sigma_2^2$ but known

$$= \left[(\hat{\theta}_1 - \hat{\theta}_2) \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right] \quad \text{exact}$$

if $\sigma_1^2 = \sigma_2^2$ but unknown

$$= \left[(\hat{\theta}_1 - \hat{\theta}_2) \pm t_{1-\frac{\alpha}{2}, n_1+n_2-2} s_{pooled} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right] \quad \text{exact}$$

if $\sigma_1^2 \neq \sigma_2^2$ and unknown

$$\approx \left[(\hat{\theta}_1 - \hat{\theta}_2) \pm t_{1-\frac{\alpha}{2}, df} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}\right] \quad \text{approx}$$

where df is the Satterthwaite approximation

t

$H_1: \theta \neq \theta_0$

$$\frac{\hat{\theta} - \theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \xrightarrow{d} N(0,1) \Rightarrow \frac{\hat{\theta} - \theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \overset{\sim}{\sim} N(0,1)$$

by the CLT or by the monster MLE thm. Let's do the inversion now:

$$P\left(\frac{\hat{\theta} - \theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \in \left[-z_{1-\frac{\alpha}{2}}, z_{1-\frac{\alpha}{2}}\right] \,\middle|\, \theta = \theta_0\right) \approx 1-\alpha$$

$$\Rightarrow P\left(\frac{\theta - \hat{\theta}}{\sqrt{\frac{\theta(1-\theta)}{n}}} \in \left[-z_{1-\frac{\alpha}{2}}, z_{1-\frac{\alpha}{2}}\right] \,\middle|\, \theta = \theta_0\right) \approx 1-\alpha$$

$$\Rightarrow P\left(\theta - \hat{\theta} \in \left[\pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\theta(1-\theta)}{n}}\right] \,\middle|\, \theta = \theta_0\right) \approx 1-\alpha$$

$$\Rightarrow P\left(\theta \in \left[\underbrace{\hat{\theta} - z_{1-\frac{\alpha}{2}} \sqrt{\frac{\theta(1-\theta)}{n}}}_{w_L ?}, \underbrace{\hat{\theta} + z_{1-\frac{\alpha}{2}} \sqrt{\frac{\theta(1-\theta)}{n}}}_{w_U ?}\right] \,\middle|\, \theta = \theta_0\right) \approx 1-\alpha$$

Are we done? We're not since the w_L and w_U need θ to be computed. This is illegal. We're stuck. But we know by Slutsky's thm that:

$$\frac{\hat{\theta} - \theta}{\sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}} \xrightarrow{d} N(0,1)$$

but it just goes slower so you need more n and it's more approximate

$$\Rightarrow P\left(\theta \in \left[\hat{\theta} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}\right]\right) \approx 1-\alpha$$

$$\Rightarrow \hat{CI}_{\theta,1-\alpha} \approx \left[\hat{\theta} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}\right]$$

This is known as the "confidence interval for the binomial proportion" and is still being researched today. When θ is close to zero or one, it's not very good.

Pop 1 DGP is iid Bern(θ_1) independent of Pop 2 DGP is iid Bern(θ_2), two sided test then

$$\frac{(\hat{\theta}_1 - \hat{\theta}_2) - (\cancel{\theta_1 - \theta_2})^{0}}{\sqrt{\frac{\theta_1(1-\theta_1)}{n_1} + \frac{\theta_2(1-\theta_2)}{n_2}}} \xrightarrow{d} N(0,1)$$

We can't invert using the lhs's standardized statistic since we'll get the same problem where we don't know the two values of θ when we try to compute the interval estimator. So once again, we use Slutsk'y thm and:

$$\frac{\hat{\theta}_1 - \hat{\theta}_2}{\sqrt{\frac{\hat{\theta}_1(1-\hat{\theta}_1)}{n_1} + \frac{\hat{\theta}_2(1-\hat{\theta}_2)}{n_2}}} \xrightarrow{d} N(0,1)$$

$$\Rightarrow \hat{CI}_{\theta_1-\theta_2,1-\alpha} \approx \left[\hat{\theta}_1 - \hat{\theta}_2 \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{\theta}_1(1-\hat{\theta}_1)}{n_1} + \frac{\hat{\theta}_2(1-\hat{\theta}_2)}{n_2}}\right]$$

Let the DGP be iid with mean θ where σ² is unknown and θ is the inferential target. By the CLT and Slutsky's thm,

$$\frac{\hat{\theta} - \theta}{\frac{s}{\sqrt{n}}} \xrightarrow{d} N(0,1) \Rightarrow \hat{CI}_{\theta,1-\alpha} \approx \left[\hat{\theta} \pm z_{1-\frac{\alpha}{2}} \frac{s}{\sqrt{n}}\right]$$

Let the DGP be iid f(θ) and θ is the inferential target and $\hat{\theta}^{MLE}$ is known. By the monster MLE thm and Slutsky's thm,

$$\frac{\hat{\theta}^{MLE} - \theta}{\sqrt{\frac{I(\hat{\theta}^{MLE})^{-1}}{n}}} \xrightarrow{d} N(0,1) \Rightarrow CI_{\theta,1-\alpha} \approx \left[\hat{\theta}^{MLE} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{I(\hat{\theta}^{MLE})^{-1}}{n}}\right]$$