From a previous class, let the DGP be iid Gumbel(θ, 1). The data is: <2.15, 1.91, 3.66, 4.85, 3.03, 1.03, 3.58> and n = 7. We derived the MLE and I(θ) as follows:

$$\hat{\theta}^{MLE} = \ln\left(\frac{n}{\sum e^{-X_i}}\right), \quad \hat{\hat{\theta}}^{MLE} = 2.26$$

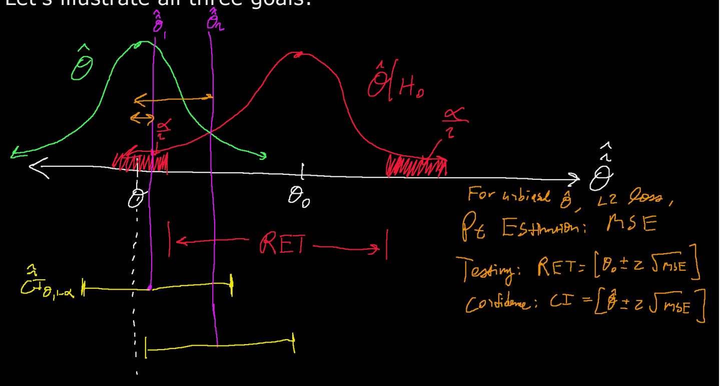
$$I(\theta)^{-1} = 1 \qquad I(\hat{\hat{\theta}}^{MLE})^{-1} = 1$$

$$\hat{CI}_{\theta,95\%} \approx \left[2.26 \pm 1.96\sqrt{\frac{1}{7}}\right] = [1.52, 3.00]$$

Let the DGP be unknown iid mean θ. The data is: <2.15, 1.91, 3.66, 4.85, 3.03, 1.03, 3.58> and n = 7. Find a 95% CI for θ.

$$CI_{\theta,95\%} = \left[\bar{x} \pm 1.96\frac{s}{\sqrt{n}}\right] = \left[2.89 \pm 1.96\frac{1.28}{\sqrt{7}}\right] = [1.94, 3.84]$$

Let's illustrate all three goals:



Note: The first experiment's CI "covered" θ but the second experiment's CI did "not cover" θ. Question: when you create a CI, do you know if it covered? NO! You don't know θ.

$$P(\theta \in \hat{CI}_{\theta,1-\alpha}) \neq 1-\alpha$$

since this probability is zero or one (either it's not inside or it is inside). Thus, any individual CI is technically meaningless. This philosophical problem has been discussed for a very long time. The Bayesian approach offers another strategy.

The problem with point estimation is you get a thetahahat and you don't know if that specific thetahathat is close to θ or not!

The problem with testing is that you have no idea if your decision is an error (if you reject, it could be a type I error; if you retain, it could be a type II error).

Yes these problems exist. But this is the best we got! It's better than making up stuff.

In the AF study, $\hat{\hat{\theta}}_1$ = 27/81 = 0.333, $\hat{\hat{\theta}}_2$ = 12/79 = 0.152

$$\phi := \underbrace{\text{Odds Against}(\theta)}_{g(\theta)} = \frac{1-\theta}{\theta}$$

Odds against is a 1:1 function with θ and thus is an equivalent way of thinking about probability.

Can we provide an estimate for odds against?

$$\hat{\hat{\phi}}_1 = \frac{1-\hat{\hat{\theta}}_1}{\hat{\hat{\theta}}_1} = \frac{1-0.333}{0.333} = 2$$

"odds against getting AF in the control group are 2:1".

If you are thinking about the iid Bern DGP with φ as the parameter and not θ as the parameter, maybe you want a means to run valid hypothesis tests and build CI's?

We will now learn the "delta method" p240-243 of C&B.

Let $\hat{\varphi} = g(\hat{\theta})$ where
* $\hat{\theta}$ is asymptotically normal
* g is a differentiable function with no critical points

Then,

$$\frac{g(\hat{\theta}) - g(\theta)}{|g'(\theta)|\, SE[\hat{\theta}]} \xrightarrow{d} N(0,1) \Rightarrow \frac{g(\hat{\theta}) - g(\theta)}{|g'(\theta)|\, SE[\hat{\theta}]} \overset{\cdot}{\sim} N(0,1)$$

Slutsky's $\Downarrow$

$$\frac{g(\hat{\theta}) - g(\theta)}{|g'(\hat{\theta})|\, \hat{SE}[\hat{\theta}]} \xrightarrow{d} N(0,1) \quad \text{'' '' '' '' '' ''}$$

$$\Rightarrow \underset{\overset{\|}{g(\theta)}}{CI_{\phi,1-\alpha}} \approx \left[g(\hat{\theta}) \pm z_{1-\frac{\alpha}{2}} |g'(\hat{\theta})|\, \hat{SE}[\hat{\theta}]\right]$$

Proof of the delta method thm:

$$g(\hat{\theta}) \approx g(\theta) + (\hat{\theta} - \theta)\, g'(\theta)$$

this is the first order Taylor series approx.

$$\Rightarrow g(\hat{\theta}) - g(\theta) \approx (\hat{\theta} - \theta)\, g'(\theta)$$

$$\frac{g(\hat{\theta}) - g(\theta)}{g'(\theta)\, SE[\hat{\theta}]} \approx \frac{(\hat{\theta} - \theta)\, \cancel{g'(\theta)}}{\cancel{|g'(\theta)|}\, SE[\hat{\theta}]} = \frac{\hat{\theta} - \theta}{SE[\hat{\theta}]} \xrightarrow{d} N(0,1)$$

In our example, $|g'(\theta)| = \left|\frac{d}{d\theta}\left[\frac{1-\theta}{\theta}\right]\right| = |-\theta^{-2}| = \frac{1}{\theta^2}$

$$CI_{\phi,95\%} = \left[\underbrace{\frac{1-\hat{\hat{\theta}}}{\hat{\hat{\theta}}}}_{\hat{\hat{\phi}}} \pm 1.96 \cdot \underbrace{\frac{1}{\hat{\theta}^2}}_{|g'(\hat{\theta})|} \underbrace{\sqrt{\frac{\hat{\hat{\theta}}(1-\hat{\hat{\theta}})}{n}}}_{SE[\hat{\theta}]}\right]$$

$$= \left[2 \pm 1.96 \frac{1}{0.333^2} \sqrt{\frac{0.333 \cdot 0.667}{81}}\right] = [1.07, 2.93]$$