Lecture 10

Oct 14

$X \sim \text{Hypergeometric}(n, K, N)$

	sample size less than or eq. # failures $n \le N-K$	sample size greater than # of $n > N-K$
sample size less than or equal to # successes $n \le K$	min: 0 max: n (I)	min: $n-(N-K)$ max: n (II)
sample size greater than # of successes $n > K$	min: 0 max: K (III)	min: $n-(N-K)$ max: K (IV)

(I) $n=3,\ N=20,\ K=11,\ N-K=9$

I can at most get 3 successes

I can at most get 3 failures

$\Rightarrow$ 0 successes

(III) $n=3,\ N=20,\ K=2,\ N-K=18$

I can at min get 0 successes

but I can most get 2 successes

(II) $n=3,\ N=20,\ K=18,\ N-K=2$

I can at min get 1 success (2 get failure)

I can at most get 3 successes

(IV) $n=18,\ N=20,\ K=11,\ N-K=9$

I can at min get 9 successes because

I can at most get 9 failures.

I can at most get 11 successes because

that's all there are

Notes:

① if $n \le N-K \Rightarrow n-(N-K) \le 0$

$$\Rightarrow \text{Supp}(X) = \{\max\{0, n-(N-K)\}, \ldots, \min\{n, K\}\}$$

$$\boxed{\lim_{N\to\infty} \text{Hypergeometric}(n, pN, N) = \text{Binomial}(n,p)}$$

if your sample is taken from an infinite population where the prob of success in one sample is p, then $X \sim \text{Binomial}(n,p)$

with PMF as we just described. Incredibly useful!

Imagine 10 coin flips, what's prob 5 H?

$X \sim \text{Binomial}(10, \frac{1}{2})$ success H, prob(S) $= \frac{1}{2}$

$$P(X=5) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{10-5} = \frac{10!}{5!\,5!}\,\frac{1}{2^{10}} = \cancel{\frac{10\cdot9\cdot8\cdot7\cdot6}{5\cdot4\cdot3\cdot2\cdot1}\cdot\frac{1}{2\cdot2\cdots2}} = \frac{63}{256} = 0.24$$

24 rolls of two dice prob of at least one 6,6

$$= 1 - P(\text{no } 6,6 \text{ in } 24) = 1 - \left(\frac{35}{36}\right)^{24} = 0.491$$

$X \sim \text{Binom}(24, \frac{1}{36})$

$$P(X=0) = \binom{24}{0}\left(\frac{1}{36}\right)^0\left(\frac{35}{36}\right)^{24}$$

success is a 6 $P(S) = \frac{1}{6} = p$

15 dice rolls. Prob(two 6's) $\Rightarrow X \sim \text{Binom}(15, \frac{1}{6})$

$$P(X=2) = \binom{15}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{13} = 0.27$$

prob (three 6's)

$$P(X=3) = \binom{15}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{12} = 0.23$$

prob (10 6's)

$$= P(X=10) = \binom{15}{10}\left(\frac{1}{6}\right)^{10}\left(\frac{5}{6}\right)^5 =$$

$\text{Supp}(X) = \{0, 1, \ldots, n\}$ Verify $f(x) > 0 \;\; \forall x \in \text{Supp}$

Parameter space $n \in \mathbb{N}$, $p \in [0,1]$

$$\underbrace{\binom{n}{x}}_{\geq 1} \underbrace{p^x}_{\geq 0} \underbrace{(1-p)^{n-x}}_{\geq 0} \checkmark$$

$$\sum_{x \in \text{Supp}(X)} f(x) \stackrel{?}{=} 1 \qquad \underbrace{\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}}_{\text{binomial expansion}} = (p + (1-p))^n = 1^n = 1 \checkmark$$

~~Geometric r.v. How many Bernoulli outcomes does it take to~~

p137 ch 4.1 Independence of r.v.'s

two r.v.'s (X_1, X_2) are said to be "independent" if

~~$P(X=x, Y=y) = P(X=x)\,P(Y=y)$~~

~~(or $P(X=x$~~

$$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1)\,P(X_2 = x_2) \quad \forall x_1 \in \text{Supp}(X_1) \text{ and } \forall x_2 \in \text{Supp}(X_2)$$

or $P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$

or $P(X_2 = x_2 \mid X_1 = x_1) = P(X_2 = x_2)$

The same as def of $P(AB) = P(A)\,P(B)$

but here it's for all A, B that could happen.

Ex 4: $X_1 \sim \text{Bernoulli}(\tfrac{1}{2})$ from Ω_1, coin flip

$X_2 \sim \text{Bernoulli}(\tfrac{1}{2})$ from Ω_2, coin flip

$P(X_2 = H \mid X_1 = T) \overset{?}{=} P(X_2 = H)$ $\quad X_1, X_2$ "independent"

We say $X_1 \overset{d}{=} X_2$ since $f_1(x_1) = f_2(x_2)$ or $F_1(x) = F_2(x)$

f or F are the "footprints" of a r.v.

If X_1, X_2 ind. & $X_1 \overset{d}{=} X_2$ we say $X_1, X_2 \overset{iid}{\sim}$ ind. & iden. dist.

We said $T_2 = X_1 + X_2$ (new r.v.) what is PMF of T_2?

Supp(X)
usually $f: \mathbb{R} \to [0,1]$ ~~xxx~~

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Back to trees

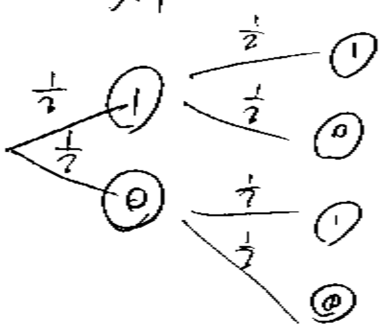
$P(X_2 = 1 \mid X_1 = 1)$

why?

$f(x_1, x_2)$

joint PMF Supp(X₁) × Supp(X₂)
$f: \mathbb{R} \times \mathbb{R} \to [0,1]$ joint CDF... ~~the~~ not covered in this class

X_1 X_2 $P(X_1 = x_1, X_2 = x_2)$ $T_2 = X_1 + X_2$



$P(X_1 = 1, X_2 = 1) = \frac{1}{4}$ — 2

" $\frac{1}{4}$ — 1

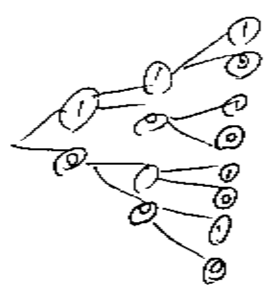
" $\frac{1}{4}$ — 1

" $\frac{1}{4}$ — 0

		X_2: 0	1
X_1	0	0	1
	1	1	2

$P(T = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ $T \not\sim \text{Uniform}(\{0,1,2\})$

How about $T_3 = X_1 + X_2 + X_3$



	T
$\frac{1}{8}$	3
"	2
"	2
"	1
"	2
"	1
"	1
"	0

$T \sim \begin{cases} 3 & P(T=3) = \frac{1}{8} \\ 2 & P(T=2) = \frac{3}{8} \\ 1 & P(T=1) = \frac{3}{8} \\ 0 & \frac{1}{8} \end{cases} \not\sim \text{Uniform}(\{0,1,2,3\})$

$X_3 = 0$: X_1 / X_2 grid [0, 1; 1, 2]
$X_3 = 1$: X_1 / X_2 grid [1, 2; 2, 3]

which illustration is better?

Pattern?

$P(T_3 = 2)$ — — — How many ways to have 2 1's and 1 0?

$\binom{3}{2} = 3$

$X_1, \ldots, X_4 \overset{iid}{\sim} \text{Bern}(\frac{1}{2})$

$T_4 = \sum_{i=1}^{4} X_i$ $\text{Supp}(T_4) = \{0, \ldots, 4\}$

draw tree

$T_4 \sim \begin{cases} 4 & 1/16 \\ 3 & 4/16 \\ 2 & 6/16 \\ 1 & 4/16 \\ 0 & 1/16 \end{cases}$

$X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(\frac{1}{2})$

$T_n = \sum_{i=1}^{n} X_i$ $T_n \sim \begin{cases} n & \binom{n}{n} \frac{1}{2^n} \\ n-1 & \binom{n}{n-1} \frac{1}{2^n} \\ \vdots \\ 1 & \binom{n}{1} \frac{1}{2^n} \\ 0 & \binom{n}{0} \frac{1}{2^n} \end{cases}$

In general if $p \neq \frac{1}{2}$ ...

$$T_n \sim \begin{cases} n & \binom{n}{n} p^n (1-p)^0 \\ n-1 & \binom{n}{n-1} p^{n-1}(1-p)^1 \\ \vdots & \\ 1 & \binom{n}{1} p^1 (1-p)^{n-1} \\ 0 & \binom{n}{0} p^0 (1-p)^n \end{cases}$$

$$\Rightarrow T \sim \text{Binomial}(n,p) := \binom{n}{k} p^k (1-p)^{n-k}$$

So the binomial is the (a) limit of Hypergeometric and (b) Sum of iid Bernoulli's

how many ways to get 7 1's and $n-7$ 0's? $\binom{n}{7}$

_ _ _ _ ... _ $\qquad P\left(\begin{matrix}7 \text{ 1's}, n-7 \text{ 0's} \\ \text{in any specific order}\end{matrix}\right) = p^7(1-p)^{n-7}$

$N \sim \text{Binom}(n,p) \qquad P(N=7) = \underset{\text{to order}}{\# \text{ways}} \cdot \text{Prob}$

$$= \binom{n}{7} p^7 (1-p)^{n-7}$$

Now ask a slightly different question ...

Imagine you keep realizing a success/failure until you get r of them. How many times do you need to keep going.

Flip coins until you get 3 H. Let X = # times flipped

$Prob(H) = p$

$Supp(X) = \{3, 4, \dots\}$

What is $P(X=10)$?

$$\overbrace{\underbrace{1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0}_{9}\ \ 1}^{10}$$

last toss must be p

gotta get 2 success $\Rightarrow$ 7 failures in any order

$= P\left(\begin{array}{c}\text{2 success \& 7 failures in any order}\\ \text{out of 9 flips}\end{array}\right) \cdot p$

$= \binom{9}{2} p^2 (1-p)^7 \cdot p \Rightarrow \binom{9}{2} p^3 (1-p)^7$

$P(X=x)$ generally $P\left(\begin{array}{c}\text{2 success and } x-3 \text{ failures out}\\ \text{of } x-1 \text{ flips}\end{array}\right) \cdot p$

$= \binom{x-1}{2} p^2 (1-p)^{x-3} \cdot p$

$= \binom{x-1}{2} p^3 (1-p)^{x-3}$ r success $x-r$ fails

$X \sim NegBin(r, p)$ $\quad P(X=x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$

any # ways to get $r-1$ success in $x-1$ flips.