## Lecture 12 Oct 23

Recall

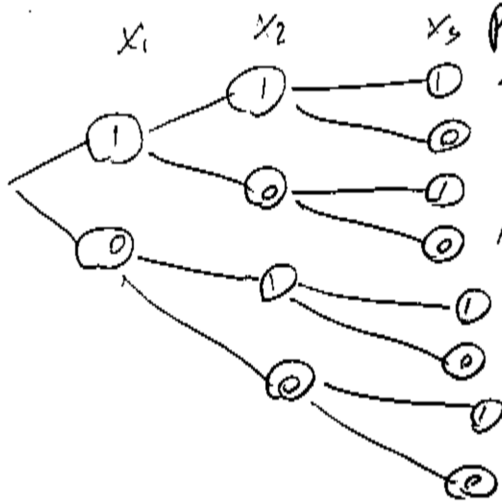
$T_n = X_1 + X_2 + \dots + X_n$ is the sum r.v.

Define

$$\bar{X}_n = \frac{T_n}{n} = \frac{X_1 + \dots + X_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$ is the "average" r.v.

What is its PMF?

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bernoulli}(\tfrac{1}{2})$

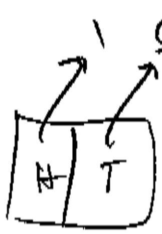



| pdf | $T_n$ | $\bar{X}_n$ |
|---|---|---|
| 1/8 | 3 | 1 |
| " | 2 | 2/3 |
| " | 2 | 2/3 |
| " | 1 | 1/3 |
| " | 2 | 2/3 |
| " | 1 | 1/3 |
| " | 1 | 1/3 |
| " | 0 | 0 |

$$\bar{X}_3 \sim \begin{cases} 1 & \text{w.p. } \frac{1}{8} \\ \frac{2}{3} & \text{w.p. } \frac{3}{8} \\ \frac{1}{3} & \text{w.p. } \frac{3}{8} \\ 0 & \text{w.p. } \frac{1}{8} \end{cases}$$

Philosophical Jump .. you're ready for it...

$X \sim \text{Bernoulli}(\tfrac{1}{2})$ is a r.v. but it is also a "data generating process" (or "dgp")



not really or concrete

"$X$" is the model but "$x$" is the "realization."

is the world and

$X \sim \text{Bernoulli}(\frac{1}{2})$ (coin flipping)

$x = 1$ or $0$ (coin flipped)

$X_1, \ldots, X_4 \overset{iid}{\sim} \text{Bernoulli}(\frac{1}{2})$

flip four coins

$x_1 = 1$
$x_2 = 0$
$x_3 = 0$
$x_4 = 1$ "Datum" " " " a r.v

__Definition:__ "Data" is the realization of r.v.'s

"iid data" " " iid r.v.'s.

You've seen little $x$ before

$P(X = x)$ What is the probability the model $X$ "generates" the value "$x$"? of course $x \in \text{Supp}(X)$ which represents all valid values the model can generate.

~~Definition: "Sample size", "realization"~~

__Definition__: the "average" is a realization from $\bar{X}_n$ where "$n$" is the # of r.v.'s realized.

$\bar{x} := \frac{1}{n} \sum_{i=1}^{n} x_i$ you've seen this in high school...

In class demos

$X_1,\dots,X_6 \overset{iid}{\sim}$ Hypergeometric$(3,4,8)$ — Bag of 8 coins. 4 marked. Draw 3 at random

$\{\cdot\cdot\cdot\cdot\cdot\cdot\}$ $\bar{X} =$ $=$

$X_1,\dots,X_6 \overset{iid}{\sim}$ Binomial$(8,\tfrac{1}{2})$ — Flip 8 coins ...

$\{\cdots\}$ $\bar{X} =$

$X_1,\dots,X_6 \overset{iid}{\sim}$ Geometric$(\tfrac{1}{2})$ — Flip coins until one head

$\{\cdots\}$ $\bar{X} =$

$X_1,\dots,X_6 \overset{iid}{\sim}$ NegBin$(\cancel{4}\,2,\tfrac{1}{2})$ — Flip coin until three heads

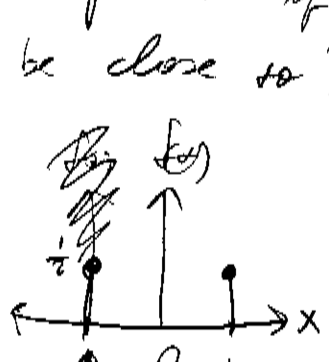
$\{\cdots\}$ $\bar{X} =$

$X_1,\dots,X_{Big} \overset{iid}{\sim}$ Rademacher$(\tfrac{1}{2})$ $= \begin{cases} 1 & \text{w.p. } \tfrac12 \\ -1 & \text{w.p. } \tfrac12 \end{cases}$ the random walk r.v.

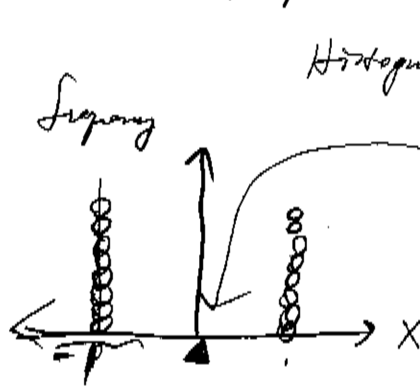
$\bar{X} =$

Large experiment: if I did many, many flips, what do I expect $\bar{X}$ to be close to?

pmf drawn.




Histogram




the average looks like a pivot, balancing all realizations

In the limit, where should this pivot be?

This pivot is a very important property of the r.v.
It is called the "expected value" or "expectation".
It is denoted $\mathbb{E}(X)$ special symbol you'll see in the book I'll just use $E$

Two definitions

(I) $\bar{X}_n \xrightarrow{n \to \infty} E[X]$

As $n$ goes large, the r.v. $\bar{X}_n$ becomes degenerate with one value

This is sort of like the long run freq. def. of prob:

$$P(A) := \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\omega_i \in A}$$ this is an "average" as well

(II) Since we know the model, we know what the probabilities are of each outcome result $x$ $\forall x \in Supp(X)$.

we know so prove the pivot rule!
It's a weighted average

Thus, $$\mathbb{E}(X) := \sum_{x_i \in Supp(X)} x_i f(x_i)$$

by common way

This is sort of like the propensity theory of probability. "p" is known and causes the thing to be realized following from pdf def.

Thus $\overline{X}_n \xrightarrow{n\to\infty} E[X]$ becomes a property and it is called the "Law of Large Numbers." Proof is beyond scope of course. But we will use it.

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$X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$

What is $E[X]$?

$$E(X) = \sum_{x \in \text{Supp}(X)} x f(x) = \sum_{x \in \{0,1\}} x f(x) = 0 \cdot f(0) + 1 \cdot f(1)$$
$$= 0 \cdot (1-p) + 1 \cdot p$$
$$= \boxed{p}$$

Does this make sense?

Imagine $X$ models winnings in \$ and $p = 0.3$

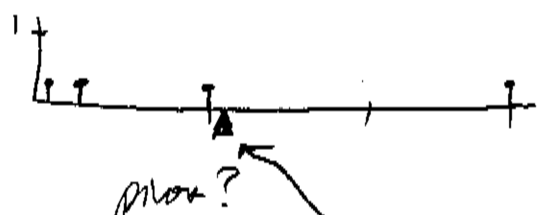
If I play many times, I expect to get 30¢ on average.

I can't get 30¢ on one play! $E(X) = p \notin \text{Supp}(X)$.

The expectation does not have to be a valid outcome result.

$X \sim \text{Unif}(\{1, 3, 10, 30\})$

$$E(X) = \sum_{x \in \text{Supp}(X)} x f(x)$$



pivot?

$$= 1 \cdot f(1) + 3 \cdot f(3) + 10 \cdot f(10) + 30 \cdot f(30)$$
$$= 1 \cdot \tfrac{1}{4} + 3 \cdot \tfrac{1}{4} + 10 \cdot \tfrac{1}{4} + 30 \cdot \tfrac{1}{4}$$
$$= \tfrac{1}{4}(1 + 3 + 10 + 30)$$
$$= \tfrac{1}{4} 44 = \boxed{11}$$

In general....

$$X \sim \text{Uniform}(\{c_1, c_2, \dots, c_K\}) \qquad E(X) = \frac{1}{K}\sum_{i=1}^{K} c_i$$

$$X \sim \text{Binomial}\left(8, \tfrac{1}{2}\right)$$

$$E[X] = \sum_{x \in \text{Supp}(X)} x f(x) = \cancel{0 \cdot f(0)} + 1 \cdot f(1) + 2f(2) + 3f(3) + 4f(4) + 5f(5) + 6f(6) + 7f(7) + 8f(8)$$

$$= \binom{8}{1}\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^7 + 2\binom{8}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^6 + 3 \dots\dots$$

$$= \frac{1}{2^8}\left(\binom{8}{1} + 2\binom{8}{2} + 3\binom{8}{3} + 4\binom{8}{4} + 5\binom{8}{5} + 6\binom{8}{6} + 7\binom{8}{7} + 8\binom{8}{8}\right)$$

$8 \cdot 7 \cdot 6 \qquad \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2}$

$$= \frac{1}{256}\left(8 + 2 \cdot 28 + 3 \cdot 56 + 4 \cdot 70 + 5 \cdot 56 + 6 \cdot 28 + 7 \cdot 8 + 8\right)$$

$$= \frac{1}{256}\left(8 + 56 + 168 + 280 + 280 + 168 + 56 + 8\right)$$

$$= \frac{1}{\cancel{256}_{128}}\left(\cancel{512} \cdot \cancel{2}\right) = \boxed{4}$$

Does this match your intuition?
You flip 8, $\frac{1}{2}$ chance each, so half of them should succeed, $8 \cdot \frac{1}{2} = 4$.

$$X \sim \text{Binomial}(n, p)$$

$$E(X) = \sum_{x \in \text{Supp}(X)} x f(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$\sum_{Y=0}^{n} \underbrace{x\binom{n}{x}} p^x(1-p)^{n-x} = \sum_{x=1}^{n}\binom{n}{x}p^x(1-p)^{n-x}$$

$$\underbrace{x\,\frac{n!}{x!\,(n-x)!}}$$

$$\underbrace{\frac{n!}{(x-1)!\,(n-x)!}} \rightarrow n\,\frac{(n-1)!}{(x-1)!\,(n-x)!} = n\binom{n-1}{x-1}$$

$$n\sum_{x=1}^{n}\binom{n-1}{x-1}p^x(1-p)^{n-x}$$

Binomial Thm.

$$np\sum_{x=1}^{n}\binom{n-1}{x-1}p^{x-1}(1-p)^{n-x} \qquad \sum_{x=0}^{n}\binom{n}{x}p^x(1-p)^{n-x} = 1$$

$y = x-1 \Rightarrow x = y+1 \qquad x = 1\ldots n \Rightarrow y = 0\ldots n-1$

$$np\sum_{y=0}^{n-1}\binom{n-1}{y}p^y(1-p)^{n-y-1}$$

let $m = n-1$

$$np\sum_{y=0}^{m}\binom{m}{y}p^y(1-p)^{m-y}$$

$np(1) = \boxed{np}$ Is this aligned with your intuition?