Lecture 8 & 9 ~~Sept~~ 23 2014

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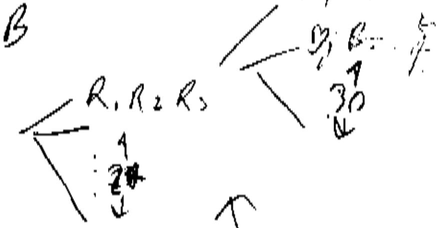
10 cards, 4 R, 6 B

P(3R, then 2B)



not a prob. tree, a "counting" tree

$4P_3$ $\quad \dfrac{4P_3 \, 6P_2}{10P_5}$

not who it happened ... counting the # of ways explicitly!

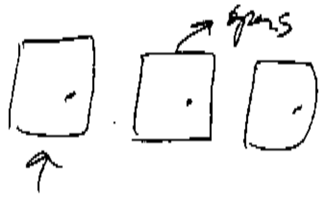
P(2B then 3R)

$\dfrac{6P_2 \, 4P_3}{10P_5}$ order matters here but it's okay because order matters on bottom too

arrangements of 3R's, 2B's & pick position of the 3 R's

$$\stackrel{?}{=} \frac{\binom{4}{3}\binom{6}{2}}{\binom{10}{5}} = \frac{\frac{4P_3}{3!} \cdot \frac{6P_2}{2!}}{\frac{10P_5}{5!}} = \frac{4P_3 \, 6P_2 \cdot \binom{5}{3}}{10P_5}$$

$$P(B_e|A) = \frac{P(A|B_e)\,P(B_e)}{\sum_{i=1}^{k} P(A|B_i)\,P(B_i)}$$



Pick D_1

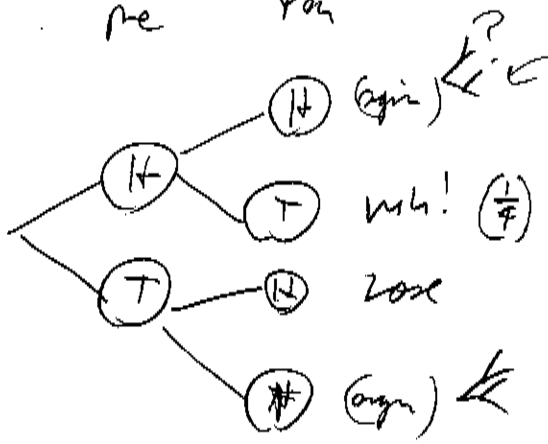
$$\frac{1}{3} = P(\text{Is } D_3 \,|\, D_2 \text{ open}) = \frac{P(D_2 \text{ open} \,|\, \text{Is } D_3)\,P(\text{Is } D_3)}{P(D_2 \text{ open} \,|\, \text{Is } D_1)\,P(\text{Is } D_1) + P(D_2 \text{ open} \,|\, \text{Is } D_2)\,P(\text{Is } D_2) + P(D_2 \text{ open} \,|\, \text{Is } D_3)\,P(\text{Is } D_3)} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{3}{6}} = \frac{2}{3}$$

$$P(B_e|A, C) = \frac{P(A|B_e, C)\,P(B_e|C)}{\sum_{i=1}^{k} P(A|B_i, C)\,P(B_i|C)}$$

$$= \frac{P(D_2 \text{ open} \,|\, \text{Is } D_1, \text{Pick } D_1)\;P(\text{Is } D_1 \,|\, \text{Pick } D_1)}{P(D_2 \text{ open} \,|\, \text{Is } D_1, \text{Pick } D_1)\,P(\text{Is } D_1|\text{Pick } D_1) + P(D_2 \text{ open} \,|\, \text{Is } D_2, \text{Pick } D_1)\,P(\text{Is } D_2|\text{Pick } D_1) + P(D_2 \text{ open} \,|\, \text{Is } D_3, \text{Pick } D_1)\,P(\text{Is } D_3|\text{Pick } D_1)}$$

More fun prob calc. Let's say we play a game – both flip coins. If I get H you get T, I win; if I get T you get H, you win; if we both get T,T, ~~toss~~ again; if we both get H,H, toss again:

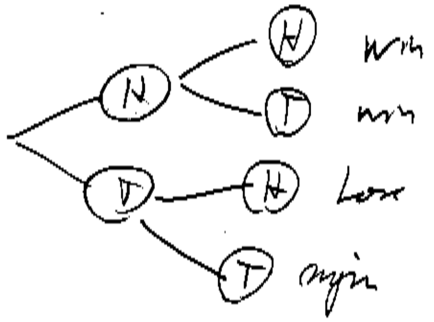
Obviously... $\frac{1}{2}$. But why?



$p := P(\text{win})$

$$p = \frac{1}{4} + \frac{1}{4}(p) + \frac{1}{4}(p) \Rightarrow p = \frac{1}{4} + \frac{1}{2}p \Rightarrow \frac{1}{2}p = \frac{1}{4} \Rightarrow \boxed{p = \frac{1}{2}}$$

Same game but I win if HH



$$p = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}p \Rightarrow \frac{3}{4}p = \frac{1}{2} \Rightarrow \boxed{p = \frac{2}{3}}$$

See this on HW...

Order and pick ind $\Rightarrow$ $P(\text{Is } D_x \mid \text{Pick } D_1) = P(\text{Is } D_x)$

$$P(D_2 \text{ open} \mid \text{Is } D_1, \text{Pick } D_1) = \frac{1}{2}$$
$$P(D_2 \text{ open} \mid \text{Is } D_2, \text{Pick } D_1) = 0$$
$$P(D_2 \text{ open} \mid \text{Is } D_3, \text{Pick } D_1) = 1$$

$$P(\text{Is } D_2 \mid D_2 \text{ open}, \text{Pick } D_1) = \ldots = \frac{2}{3}$$

$$P(\text{Is } D_1 \mid D_2 \text{ open}, \text{Pick } D_1) = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3}} = \frac{\frac{1}{6}}{\frac{3}{6}} = \boxed{\frac{1}{3}}$$

→ Other page!!!!!

Prob used to make calculations about chance or odds.

What if you want to model an event's results, so there an outcome associated with an "outcome result"

$$\boxed{H \mid T} \qquad H \to +\$1, \quad T \to \$0$$

$P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$ there's $\frac{1}{2}$ chance I make \$1, $\frac{1}{2}$ " I get nothing

I can say this, define a random variable, r.v.

$$X \sim \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ 0 & \text{w.p. } \frac{1}{2} \end{cases}$$

Capital letter notation — distributed as — resulting outcome — chance of getting it

Prob is great to make calculations... but what if you want to model an event which has a real outcome which matters?

IBM stock ↑ ⇒ make money, IBM stock ↓ ⇒ lose money.

Outcomes (ω's), the outcomes now have a numeric value. The function that connects the two is called a random variable, most generally denoted

$X: \quad x = X(\omega)$ — outcome goes in — real # comes out

$$X : \underset{(\text{domain})}{\omega} \longrightarrow \underset{(\text{range})}{\mathbb{R}}$$

$y = f(x)$ — # in — # out

$$f : \underset{(\text{domain})}{\mathbb{R}} \longrightarrow \underset{(\text{range})}{\mathbb{R}}$$

just like a function,

$$X : \omega \begin{array}{l} \nearrow \mathbb{R} \\ \searrow \mathbb{R} \end{array}$$

Only one result allowed

$\Omega = \{H, T\} \qquad X(H) = 1, \quad X(T) = -1$

Can be thought of as if H flipped, make \$1, if T flipped, lose \$1.

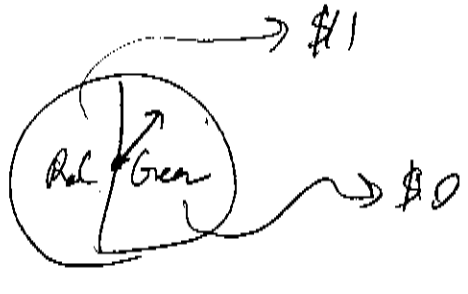
What is $P(X=1)$ here? Remember "P" is a set function defined on $A \subseteq \Omega$. Thus, this is shorthand for $P(\{\omega : X(\omega) = 1\})$

$P : \Omega \to [0,1]$

How many ω's can $X(\omega)=1$? $P(\{H\}) = \frac{1}{2} \Rightarrow P(X=1) = \frac{1}{2}$,

$P(X=-1) = \frac{1}{2}$, $P(X=0) \overset{?}{=} 0 \qquad \{x : X(\omega) = x, \ \omega \in \Omega\}$ is the "range" of X.

Ω doesn't matter anymore!



$X \sim \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ 0 & \text{w.p. } \frac{1}{2} \end{cases}$

map $\omega \in \{1,2,3\} \to 1$, $\omega \in \{4,5,6\} \to 0$ $\Rightarrow X \sim \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ 0 & \text{w.p. } \frac{1}{2} \end{cases}$

As long as I know the r.v. and its "distribution" and only care about the "resulting outcome", I have no need to know Ω or the ω's.

This r.v. is called a "Bernoulli r.v."

$X \sim \text{Bernoulli}\left(\frac{1}{2}\right) := \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ 0 & \text{w.p. } \frac{1}{2} \end{cases}$

more generally,

$X \sim \text{Bernoulli}(p) := \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$ Valid values of p? $p \in [0,1]$

where p is called a "parameter". Parametric Theory?

Couple definitions:

① the "support" of X is $\{0,1\} \subset \mathbb{R}$

generally $\{x : P(X=x) > 0\}$ $\quad$ $\text{Support}(X) \overset{?}{=} \text{Domain}(X)$

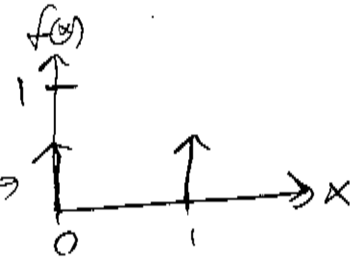
$\text{Support}(X) :=$

"all values that can be produced by X"

If $\left| \text{Support}(X) \right| \le |\mathbb{N}|$

that means there are either a finite # of things that can happen or an infinite but "ctbl" # of things, then X is called a "discrete r.v."

let $f(x) := P(X = x)$, $x \in \mathbb{R}$ which is called the "probability mass function"



drawn like so

Rules: $f(x) \le 1 \quad \forall x$

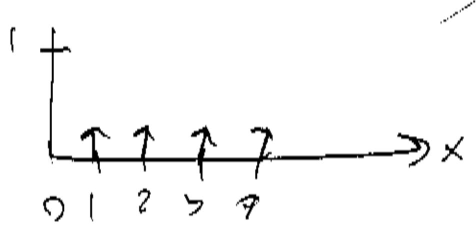
$$\sum_{x \in \mathbb{R}} f(x) = 1 \quad \text{or} \quad \sum_{x \in \text{Support}(X)} f(x) = 1$$

Bernoulli: $\sum_{i=0}^{1} f(x) = 1$

And $P(X \in A) = \sum_{x \in A} f(x)$

Let $X \sim \begin{cases} 1 & \text{w.p. } 1/4 \\ 2 & \text{"} \\ 3 & \text{"} \\ 4 & \text{"} \end{cases}$

"line graph" we will NOT be using prob. histograms. Why?



$$P(X \in \{2, 3\}) = f(2) + f(3) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Verify $f(x)$ is a valid PMF $\quad f(x) \le 1 \;\; \forall x$ ✓

$$\sum_{x \in \text{Support}(X)} f(x) = 1 \;\; ✓$$

Uniform discrete r.v.'s:

$$X \sim U(\{1,2,3,4\})$$

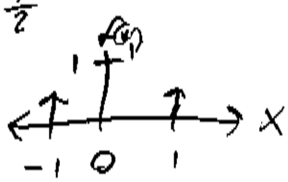
$$\Rightarrow X \sim x \text{ w.p. } \frac{1}{|\text{Support}(X)|} \quad \forall x \in \text{Support}(X)$$

this just means 20 elements $P(\text{each element}) = 1/20$

Rademacher r.v. s.

$$X \sim \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$$

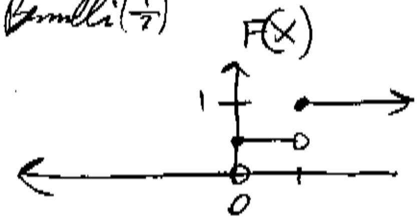
Why is this called the 1-d random walk?



distribution function AKA cumulative distribution function

$$F(x) := P(X \leq x), \quad x \in \mathbb{R}$$

$X \sim \text{Bernoulli}(\frac{1}{2})$



$F(-32) = 0, \quad F(17) = 1$

$F(0.5793) = \frac{1}{2}, \quad F(0) = \frac{1}{2}, \quad F(1) = 1$

Make sure you draw # lines for a reason!

10 cards, 4R, 6B

What is the prob of selecting x red cards on a draw of 3 cards?

$$f(x) := P(X=x) = \frac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$$

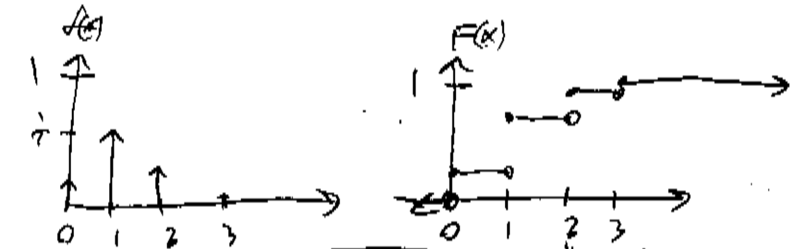
$\text{Support}(X) \stackrel{?}{=} \{0,1,2,3\}$

$$= \frac{\frac{4!}{(4-x)!\,x!}\;\frac{6!}{(3+x)!\,(3-x)!}}{\frac{10!}{3!}}$$

no pattern...

If Reds are "successes", this describes the number of successes (x) in "n" trials "without replacement!" "success" vs "failure"!

$f(0) = \frac{1}{6}$, $f(1) = \frac{1}{2}$, $f(2) = \frac{3}{10}$, $f(3) = \frac{1}{30}$



$X \sim \text{Hypergeometric}(3, 4, 10)$

max # of draws

successes

of total samples...

generally

$X \sim \text{Hypergeom}(n, K, N)$

samples

max successes

total samples

$\text{Support}(X) = \{0, 1, \ldots, n\}$

$$f(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$ common

Valid values? $N \in \mathbb{N}$

$K = \{0, 1, \ldots, N\}$

$n = \{0, 1, \ldots, N\}$

$\text{Support}(K)$ - depends...