Lecture 4 MATH 201 Sept 9, 2014 → Sep 23 KH 270 9:15-noon

Previously... Full House

$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$ ✓ $\binom{13}{2}\binom{4}{3}\binom{4}{2}$ ×

$13\cdot 12$ $\frac{13\cdot 12}{2}$ Half! 444QQ ≠ QQQ44

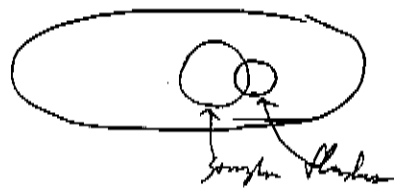
3-of-a-kind

$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2$ ✓ $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{1}\binom{11}{1}\binom{4}{1}$

$\frac{12\cdot 11}{2}$ 777 4Q ≠ 777 Q4 $12\cdot 11$

1-pair $\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3$ not $\binom{48}{4}$, not $\binom{12}{1}\binom{11}{1}\binom{10}{1}\binom{4}{1}^3$

No-pair $\binom{13}{5}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}^5 - \binom{4}{1}\binom{13}{5} + \binom{10}{1}\binom{4}{1}$

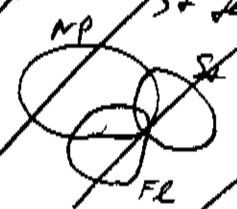


~~2-pair $\binom{13}{2}\binom{4}{2}^2\binom{40}{1}$ or $\rightarrow \binom{11}{1}\binom{4}{1}$~~

~~1-pair $\binom{13}{1}\binom{4}{2}\binom{48}{4}$ way too many? $\binom{12}{3}\binom{4}{1}^3$~~

~~No pair $\binom{13}{5}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}^5 - \binom{4}{1}\binom{13}{5} + \binom{10}{1}\binom{4}{1}$ straight? flush? St fl~~

~~A-high $\binom{4}{1}\binom{12}{4}\binom{4}{1}^4$ ... but possible straights, possible flush, st flush~~
~~2, 3, 4, 5~~



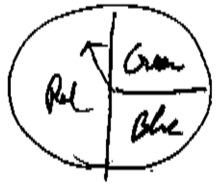
$\Omega = \{H, T\}$

H	T

previously, $P(A) = \frac{|A|}{|\Omega|}$ this only works for universes with "equally likely outcomes" (p 9)

i.e. $P(\{\omega_i\}) = \frac{1}{|\Omega|} \ \forall i$

Imagine...



$\Omega = \{Red, Green, Blue\}$

$P(\{Red\}) \neq \frac{|\{Red\}|}{|\Omega|} = \frac{1}{3}$

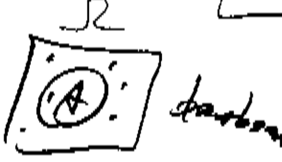
thus $P(A) = \frac{|A|}{|\Omega|}$ is NOT a good definition of probability of any A.

How is it defined? In the book it's defined as the "limiting frequency" II

let $\mathbb{1}_{\omega \in A} = \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A \end{cases}$ the "indicator function"

$P(A) := \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \mathbb{1}_{\omega_i \in A}}{n}$ what is the frequency of A over many pulls from $\omega \in \Omega$

Is this satisfying? Who chooses $\omega \in \Omega$?



dartboard?

We never get to witness $\lim_{n\to\infty}$! What is the prob. defendant is guilty?

Historical: prob. seems to originate in the study of gambling games. They spent hours playing and n gets really big, gamblers learned $P(A)$ and if they assume it better than the house, it is profitable.

For instance M. le Chevalier de Mere 1654 claimed that

$P(\{\text{at least one double } 6\text{'s in } 24 \text{ rolls of } 2 \text{ dice}\}) < \frac{1}{2}$

$1-\left(\frac{35}{36}\right)^{24} \approx 0.4914$ a difference of 0.0086!

Frequency definition of prob is "objective" meaning it is a property of the physical material world. If there were no humans, $P(H)$ would still be $\frac{1}{2}$.

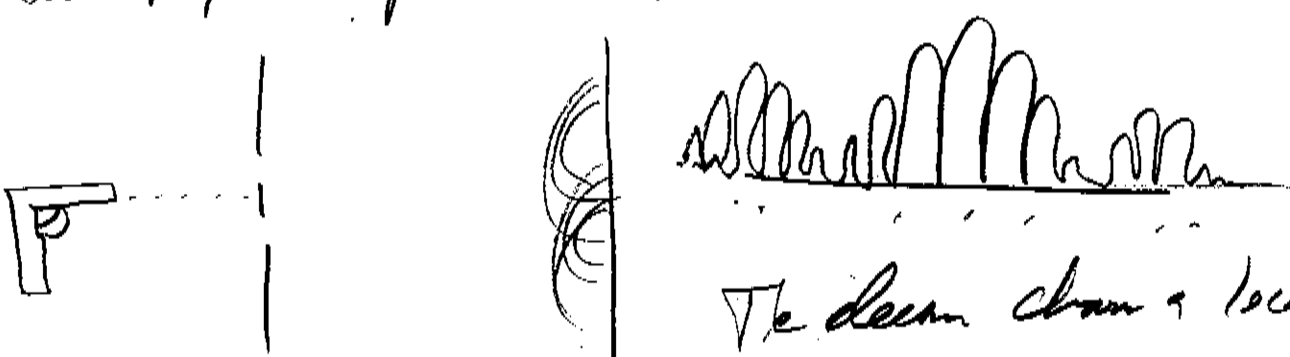
Another objective definition is the propensity theory (II). A coin has an inherent propensity hardwired into it to have a limiting frequency $\frac{1}{2}$. Calculating $P(H) = \frac{\sum_{i=1}^{n} \mathbb{1}_{\omega \in \{H\}}}{n}$ for n large is a strategy to elicit this prob, not define it. For example: uranium decomposition

Other definitions are "epistemic" which take prob. to be concerned with knowledge or belief of human beings. The logical theory (III) says that if all human beings were given the same information, they would arrive at the same prob. value which is "degree of belief." Prob. of rain tomorrow.

The subjective theory (IV) is "degree of belief" associated with each person and differences of opinion are allowed. P(OJ is guilty) Probability can be assigned to things you know nothing about. Pascal's Wager (1670).

But who cares? What is chance? What is randomness? How is it? Why does it exist? Determinism: ~~European~~ Enlightenment Newton's Principia (1687) had a big effect $F = MA$, law of gravity, ... $\Rightarrow$ the universe operates on a set of fixed rules. You know the rules and take the measurements, you predict the future. E.g. You know the air pressure, the force, the metallic properties, the wood properties, etc.... Would you not know if it is going to land (exactly)?

Laplace's 1814 Philosophical Essay on Probabilities... arguing that prob doesn't exist... Terrible crash above the 1920's with the discovery of wave-particle duality



The electron chooses a location!

$\omega \in \Omega$

Uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$ Can't know where something is and its speed simultaneously

Laplace writes:

> We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it – an intelligence sufficiently vast to submit these data to analysis – it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past would be present to its eyes.
>
> (1814: 4)

The vast intelligence here described has come to be known as Laplace's demon. The idea is obviously founded on that of a human scientist (perhaps Laplace himself) using Newtonian mechanics to calculate the future paths of planets and comets. Extrapolating from this success, it was natural to suppose that a sufficiently vast intelligence could calculate the entire future course of the universe. Laplace himself relates his vast intelligence to human successes in astronomy. As he says:

> The human mind offers, in the perfection which it has been able to give to astronomy, a feeble idea of this intelligence. Its discoveries in mechanics and geometry, added to that of universal gravity, have enabled it to comprehend in the same analytical expressions the past and future states of the system of the world.
>
> (Laplace 1814: 4)

> The regularity which astronomy shows us in the movements of the comets doubtless exists also in all phenomena.
>
> The curve described by a simple molecule of air or vapor is regulated in a manner just as certain as the planetary orbits; …
>
> (Laplace 1814: 6)

Einstein didn't love this... quote 1926

It seems that randomness is built-in to the universe...

Regardless...

Mathematical definition: ~~[illegible]~~ P is a set function on all sets $A_i \subseteq \Omega$.

(a) $P(A_i) \geq 0 \ \forall i$

(b) $P(\Omega) = 1$

(c) If $A_1, A_2, A_3, \ldots$ are disjoint

$\Rightarrow P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \quad \text{"countable"}$$

Axioms of prob.. But they're not really axioms. They're descriptions in a definition

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<u>Thm 1</u> $P(A_i) = 1 - P(A_i^c) \quad \forall i$

Pf: We know $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ if A_1, A_2 disjoint (c)

A, A^c are disjoint by def of complement.. and $A \cup A^c = \Omega$ by def of comp

$$\underbrace{1 = P(\Omega)}_{(b)} = P(A \cup A^c) = \underbrace{P(A) + P(A^c)}_{c} \Rightarrow P(A^c) = 1 - P(A)$$

<u>Thm(2)</u> $P(\phi) = 0$

$$\underbrace{P(\phi) = 1 - P(\phi^c)}_{\text{Thm 1}} = \underbrace{1 - P(\Omega)}_{\text{Def of } \Omega} = 1 - \underbrace{1}_{b} = 0$$

Similar to 0!

$A = \phi$

$\nexists\, \omega \in \phi = 0$ always

Ensure $A \subseteq B \Rightarrow P(A) \le P(B)$

If $\underbrace{A \subseteq B \Rightarrow A \cup (B \setminus A) = B}_{\text{from set theory}}$, A, $B\setminus A$ are disjoint i.e. $P(A) + P(B\setminus A) = P(B)$
$\Rightarrow P(A) \le P(B)$

(p17)

Back to counting... How many ways to arrange 4 purple flowers and 3 white flowers where all purple are considered the same and all white are considered the same

PPPP WWW

Remember $P_1 P_2 P_3 P_4 W_1 W_2 W_3 \Rightarrow 7!$ arrangements

Here $\frac{7!}{4!\,3!} = \binom{7}{4}$ How? How is 4 being chosen from 7?

$\overline{}_1\ \overline{}_2\ \overline{}_3\ \overline{}_4\ \overline{}_5\ \overline{}_6\ \overline{}_7$ Choose 4 pots to put the purples in (others get W's)

-or- choose 3 pots to put W's in, others get P's

Now add two yellows. How many arrangements?

$\binom{9}{4,3,2}$ $\frac{9!}{4!\,3!\,2!}$ or Choose 4 pots of 9, then choose 3 pots of 5, others have to be fixed

$= \binom{9}{4}\binom{5}{3} = \frac{9!}{4!\,5!}\,\frac{5!}{3!\,2!} =$

Add 6 reds...

$\binom{15}{4,3,2,6}$ $\frac{15!}{4!\,3!\,2!\,6!} = \binom{15}{4}\binom{11}{3}\binom{8}{2}$

"K categories"

n flowers x_1 first color, x_2 second color, ..., x_K $\Rightarrow \binom{n}{x_1, x_2, \ldots, x_K} = \frac{n!}{x_1!\,x_2!\cdots x_K!}$

Multinomial coefficients

n choose $x_1, x_2, \ldots, x_K$