"Algebra of sets"

associative
$A \cup (B \cup C) = (A \cup B) \cup C$ like addition
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ similar to addition operation
distributive

Lecture 2 ~~Mark 24~~ Sept 2, 2014

- Sept 23 pending 9:15-12:15
- HW spacing / no spacing

"sample space"
"outcome space"

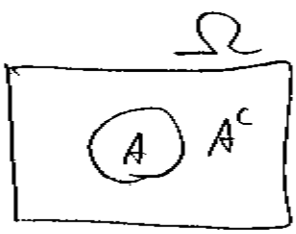
We ended off with the special set Ω the universe. This allows us to define the last set operation we need.

"Complement" for set A, A^c is all elements <u>not</u> in set A → denoted A' in the book

$$A^c := \Omega \setminus A$$

$F^c = \{Bob, Joe, Max\}$

ϕ^c, Ω^c



$\Omega \setminus A^c$, $A \cap A^c$, $A^c \cap \Omega \Rightarrow \forall A_i\ A_i \cap \Omega = A_i$

$A \cup A^c$ $A \cup \Omega \Rightarrow \forall A_i\ A_i \cup \Omega = \Omega$, $A^c \subseteq \Omega$?

$(A \cup B)^c = \Omega \setminus (A \cup B) = A^c \cap B^c$ De-Morgan's
$(A \cap B)^c = \Omega \setminus (A \cap B) = A^c \cup B^c$ Laws

Some more set theory

finite / infinite

Gödel $|\mathbb{R}| > |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$ "countable"

$\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$, $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\}$ all ratios

$\mathbb{R} = \mathbb{Q} \cup \{\text{all irrationals}\}$ eg $\sqrt{2}, \pi, e \dots$

"set builder notation" Everything $\in \mathbb{R}$ is assumed if Ω is clear

$[1,2] := \{x \in \mathbb{R} : x \geq 1\ \&\ x \leq 2\}$, $[a,b] := \{x \in \mathbb{R} : x \geq a\ \&\ x \leq b\}$ $(-\infty, \infty) = \{x \in \mathbb{R} ; x < \infty, x > -\infty\} = \mathbb{R}$

13.50 = \$13.50 if in USA

$(a,b] := \{x \in \mathbb{R} : x > a, x \leq b\}$ $(a,b) := \{x \in \mathbb{R} : x > a, x < b\}$ $[-\infty, \infty]$?

$,$, $\&$, $\cap$ all the same $\Rightarrow$ mean AND

From now on, elements of a set represent "outcomes". → and it represents something that happens.

An outcome is now $\omega \in \Omega$. For example, heads a coin toss.

$\omega_1 = H$, $\omega_2 = T$, $\Omega = \{H, T\}$, $2^\Omega = \{\phi, \{H\}, \{T\}, \{H, T\}\}$ $\boxed{H \mid T}\ \Omega$ mut. excl.? coll. exh.?

Elements of 2^{Ω} are all subsets of Ω. Subsets of Ω are not "outcomes", they're "compound outcomes" called "events" but they are "measured" in relation to the size of Ω to determine likelihood or prob. of occurring.

$\boxed{\Omega: \text{set of outcomes},\ 2^{\Omega}: \text{set of events}}$

$|\Omega| = 2$ does $|H|$ make sense? No, $|\{H\}|$ makes sense, $\{H\}$ is the event, H is the outcome

$|\phi|$ makes sense $|\{T\}|$ makes sense and $|\{H,T\}|$ makes sense, all elements of 2^{Ω}. We can then probabilities of sets of events which are subsets of the "sample space" or the universe Ω.

Here, what are "outcomes"? They allow this event to occur?

"all events"

not defining this yet

H	T

$$P(\{H\}) = \frac{|\{H\}|}{|\Omega|} = \frac{1}{2} = P(\{T\}) \quad , P(\phi) = \frac{|\phi|}{|\Omega|} = 0$$

$$\underbrace{P(\{H,T\}) = P(\{H\} \cup \{T\})}_{\text{are equal}} = \frac{|\{H,T\}|}{|\Omega|} = \frac{2}{2} = 1$$

$P: A \subseteq \Omega \to [0,1]$

what is $P(\{5\}) = ?$ undefined!
$f(x) = x, x > 0$ $f(-1)$ undefined!

$P(H)$ is technically meaningless, P only operates on sets. It's a "set function".

More interesting sample space: two coin tosses

Ω_2, $|\Omega| = 4 \leftarrow \omega_1 = HH,\ \omega_2 = HT,\ \omega_3 = TH,\ \omega_4 = TT$

HH	HT
TH	TT

$2^{\Omega} = \{\phi, \dots\}$ How big? How many sets of events can we make "what's the prob of ...?"

Now it's easy to see "$P(HH)$" $= P(\{HH\}) = \frac{|\{HH\}|}{|\Omega|} = 1/4$

"abuse of notation"

let $A = \{\omega : \omega \text{ has at least one } H\} = \{HH, HT, TH\}$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{3}{4}$$

How are these different?
different events!
$\omega_2 \neq \omega_3$ otherwise they couldn't both be in Ω. Elements of Ω are unique (it's a set).

Let $B = \{\omega : \omega$ has at least one T$\}$ $P(B) = 3/4$

$P(A \cup B) = ?$ $P(A \cap B) =$

$P(A \setminus B)$, A, B mutually exclusive?

$A_i \cap A_j = \emptyset \ \forall i,j$

A, B collectively exhaustive?

$\bigcup_i A_i = \Omega$

Four tosses (redux)

"independence" 2nd coin toss doesn't "remember" 1st coin toss

Ω

HHHH			

$|\Omega| = 16$

Why is $P(HHHH) = P(HTTH)$?

Some think l.h.s < r.h.s. since 4 H's is a "rare" event.

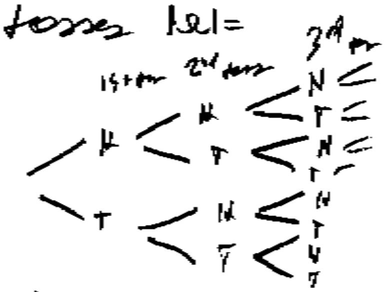
No! each event is equally rare but random mind impressed by 4 H's. It's like an optical illusion.

Five tosses $|\Omega| = ?$

Ten tosses $|\Omega| = ?$

20? 30? 40? $\rightarrow$ Reasonable # of tosses but an unreasonable sample space size.

n tosses $|\Omega| =$



... "probability tree"

How to count

$|\Omega| = \underset{\substack{\text{1st toss}\\(H/T)}}{2} \cdot \underset{\substack{\text{2nd toss}\\(H/T)}}{2} \cdots \underset{\substack{\text{nth toss}\\(H/T)}}{2} = 2^n$

One coin toss, One die roll

Ω

H1	H2	H3	H4	H5	H6
T1	T2	T3	T4	T5	T6

$|\Omega| = \underset{\substack{\text{toss}\\(H/T)}}{2} \cdot \underset{\text{roll}}{6} = 12$

Go event-by-event and count # possibilities and then multiply. Why??

Imagine Jan, Amy, Susan in front of you. Let sample space be all unique arrangements.
How many ways to arrange them?

$\Omega = \{\langle J, M, S\rangle, \langle J, S, M\rangle, \langle M, J, S\rangle, \langle M, S, J\rangle, \langle S, M, J\rangle, \langle S, J, M\rangle\}$

$|\Omega| = 6 \neq 3 \cdot 3 \cdot 3 = 3^3 = 27$ Why? Different type of event. No duplicates/clones allowed. Ω's can be very different! Gotta be careful.

But our "counting" method from before works:

$$\underset{\text{factorial}}{3!} := \underset{1^{st}\ \cancel{\text{seat}}\ \text{chair}}{\underline{3}} \cdot \underset{2^{nd}\ \cancel{\text{seat}}\ \text{chair}}{\underline{2}} \cdot \underset{3^{rd}\ \cancel{\text{seat}}\ \text{chair}}{\underline{1}} = 6$$

How about 5 people?

$$\underset{1^{st}}{\underline{5}}\ \ \underline{4}\ \ \underline{3}\ \ \underline{2}\ \ \underline{1} = 120$$

10 people: $10! \approx 3.6M$ truly big!!!!

30 people: $30! \approx 2.65 \cdot 10^{32}$ $= |\Omega|$ $= \dim(\text{universe})$ in ft.

4 people: $4!$ once again a reasonable # of people, unreasonable $\cancel{\#}$ $|\Omega|$.

These are called "permutations": # of ways to order a fixed collection.

$\rightarrow 10 P_{10}$ — # of "chairs"; # of "people"

What if # chairs less than # of people? 10 people, 1 chair

$\underset{1^{st}\ \text{chair}}{\underline{\quad}}$

10 people, 2 chairs

$$\underset{1^{st}}{\underline{10}}\ \underset{2^{nd}}{\underline{9}}$$

10 people, 5 chairs $\underset{1^{st}}{\underline{10}} \cdot \underset{2^{nd}}{\underline{9}} \cdot \underset{3^{rd}}{\underline{8}} \cdot \underset{4^{th}}{\underline{7}} \cdot \underset{5^{th}}{\underline{6}} = 30,240$

$= \frac{10!}{5!}$

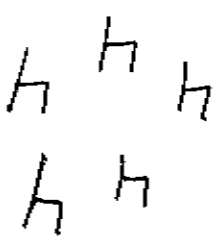
What's the rule?

$${}_nP_k := \frac{n!}{(n-k)!} = \frac{n}{1}\ \frac{n-1}{2}\ \frac{n-2}{3} \cdots \frac{n-k+2}{k-1}\ \frac{n-k+1}{k} =$$

What about 10 people, 20 chairs? I'm not thinking...

$$\underset{1^{st}\text{ person}}{\underline{20}} \cdot \underset{2^{nd}\text{ person}}{\underline{19}} \cdot \ldots \cdot \underset{10^{th}\text{ person}}{\underline{11}} = \frac{20!}{10!} = {}_{20}P_{10}$$

5 people, 5 chairs, but chairs arranged in a circle



Still five chairs: 5!

But, we don't care about rotation. How many rotations? 5

$$\underset{1^{st}\text{ chair}}{\underline{1}}\ \underset{2^{nd}\text{ chair}}{\underline{4}}\ \frac{3}{3}\ \frac{2}{2}\ \frac{1}{1} = 4! \qquad \Rightarrow \frac{5!}{5} = 24$$

$\{M, J, S\}$

M J / S	M S / J	S M / J	S J / M	J M / S	J S / M

$\Rightarrow 2$ arrangements

5 people, "M, J" have to sit next to each other

$\underline{M}\ \underline{J}\ \underline{3}\ \underline{2}\ \underline{1}$ or $\underline{3}\ \underline{M}\ \underline{J}\ \underline{2}\ \underline{1}$

How many locations for M, J? 4. Arrange M, J? $2 \cdot 4 \cdot 3! = 48$

5B, 5G, 10 Chairs.

why?

$\underline{5}\ \underline{5}\ \underline{4}\ \underline{4} - - - - \underline{1}\ \underline{1} = 5!\,5! \cdot 2$

Don't care about order of boys? Don't care about order of B&G?

$\frac{5!\,5! \cdot 2}{5!}$ $\qquad\qquad$ $\frac{5!\,5!\,2}{5!\,5!} = 2$

Decks of Cards

Suits: C, S, D, H

Ranks, A, 2, ..., 10, J, Q, K

Suits × Ranks $= \Omega$ $\quad |\Omega| = 52$

How many ways of getting & arranging 5 card hand?

What if we don't care about the arrangement order?