

# MATH 241 Fall 2014 Homework #6

Professor Adam Kapelner

Due 5PM in my office, Tues Oct 21, 2014

(this document last updated Wednesday 15<sup>th</sup> October, 2014 at 11:08am)

## Instructions and Philosophy

Once again, the path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual “working out”. Feel free to “work out” with others; **I want you to work on this in groups**. This one is a short workout.

Reading is still *required*. For this homework set, please read the binomial and negative binomial variable section of Chapter 2.

The problems below are color coded: **green** problems are considered *easy* and marked “[easy]”; **yellow** problems are considered *intermediate* and marked “[harder]”, **red** problems are considered *difficult* and marked “[difficult]”; and **purple** problems are for *extra credit* which are also marked “[E.C.]”. The *easy* problems are intended to be “giveaways” if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems.

This homework is worth 100 points but the point distribution will not be determined until after the due date. Late homework will be penalized 10 points per day up to a maximum of three days.

15 points are given as a bonus if the homework is typed using L<sup>A</sup>T<sub>E</sub>X. Links to installing L<sup>A</sup>T<sub>E</sub>X and program for compiling L<sup>A</sup>T<sub>E</sub>X is found on the syllabus. You may also use [writelatex.com](http://www.writelatex.com) which is a web service (you don’t have to install or configure anything on your local computer). If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the “\vspace” command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. You are free to handwrite on separate paper if you wish but I STRONGLY recommend to write on a printout of this document since you will always have the questions handy to study from (and it is easier for me to grade accurately). Keep this page printed for your records. Write your name and section below where section A is if you’re registered for the 9:15AM–10:30AM lecture and section B is if you’re in the 12:15PM–1:30PM lecture.

NAME: \_\_\_\_\_ SECTION (A or B): \_\_\_\_\_

**Independence and equality of distribution of r.v.'s** Since we haven't covered much else, this majority of this assignment will be about this distribution.

### Problem 1

Imagine two Bernoulli r.v.'s  $X_1$  and  $X_2$  which model two fair coin flips where Heads is mapped to 1 and tails is mapped to 0. The probability of heads is  $1/2$ .

- (a) [easy] Given no other information, explain using the definition of r.v. independence why these two r.v.'s are independent.
- (b) [easy] Given no other information, explain using the definition of equality in distribution why  $X_1 \stackrel{d}{=} X_2$ .
- (c) [easy] Are  $X_1, X_2 \stackrel{iid}{\sim}$  Bernoulli( $p$ )?
- (d) [harder] Now imagine these two coins were linked using some sort of sorcery. They are flipped at the same time but are guaranteed to flip the same way. That is, if the first coin goes heads, the second coin must go heads (and if the first coin goes tails, the second coin must go tails).



Explain using the definition of r.v. independence why these two r.v.'s are *dependent*.

(e) [harder] Using the same two sorcery-controlled coins, explain using the definition of equality in distribution why or why not  $X_1 \stackrel{d}{=} X_2$ .

(f) [easy] Are  $X_1, X_2 \stackrel{iid}{\sim}$  Bernoulli( $p$ ) if they are modeled by these two sorcery-controlled coins?

**The Binomial Distribution** This is one of the most useful distributions on the planet, so it pays to put in time to practice using it.

## Problem 2

Imagine you are flipping the same bundle of coins from the midterm. The probability of the coin bundle landing on its side is  $\mathbb{P}(S) = 1/11$ . Let's call landing on its side a "success."



(a) [easy] I flip the coin bundle once. Model a success as a "1." Show that the r.v. modeling this event outcome is Bernoulli and define its parameter.

- (b) [easy] Show that this event outcome can also be modeled by a Binomial and define its parameters.
- (c) [easy] Let's say we flip 10 times. What is the probability that we get one (and only one) success?
- (d) [easy] Let's say we flip 10 times. What is the probability that we get 5 (and only 5) successes?
- (e) [easy] Let's say we flip 10 times. What is the probability that we get 8 (and only 8) successes?
- (f) [harder] Let's say we flip 10 times. What is the probability we get one or two successes?

- (g) [difficult] Let's say we flip 10 times. What is the probability we get 3 or less successes? That is, solve for  $\mathbb{P}(X \leq 3) = F(3)$ .

### Problem 3

Imagine you are playing roulette again this time in America. The probability of winning a bet on black is  $18/38$ . Call this a “success.”



- (a) [easy] Let's say we spin 15 times. What is the probability that we get 10 successes?
- (b) [harder] Let's say we spin 30 times. Write a summation expression for getting 15 or more successes. Do not compute the answer explicitly.
- (c) [difficult] Preview of statistics. You are now the casino floor manager for roulette. You witness 40 spins and it comes out black 18 times. Is this a “weird” or “unexpected” outcome? Explain using a calculation and a few sentences *in English*.

(d) [difficult] You witness 40 spins and it comes out black 38 times. Is this a “weird” or “unexpected” outcome? Explain using a calculation and a few sentences *in English*.

(e) [E.C.] You witness 40 spins. How many times should black occur “normally?” At which large values of number of blacks do get concerned by? At which small values of number of blacks do you get concerned by?

#### **Problem 4**

We will now look at the binomial in general.

(a) [easy] Show that success and failure is arbitrary by letting the number of successes  $x$  equal the number of failures  $n - x$  and the probability of success  $p$  equals the probability of failure  $1 - p$  using the PMF of the binomial distribution. This is similar to last homework where we illustrated the same fact for the hypergeometric distribution.

(b) [harder] Show using the definition of equals in distribution that  $X_1 \stackrel{d}{=} X_2$  if  $X_1 \sim \text{Bernoulli}(p)$  and  $X_2 \sim \text{Binomial}(1, p)$ .

(c) [difficult] Let  $X_1, X_2, X_3, X_4 \stackrel{iid}{\sim} \text{Bernoulli}(p)$  and  $T_4 = \sum_{i=1}^4 X_i$ . Use a tree structure like we did in class to show that  $\mathbb{P}(T_4 = 2) = \binom{4}{2} p^2 (1-p)^2$ .

(d) [easy] In (c) explain why you need the  $\binom{4}{2}$  term *in English*.

(e) [easy] In (c) explain what the function of the  $p^2(1-p)^2$  term is *in English*.

(f) [easy] Let  $S_n = X_1 + \dots + X_n$  where  $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Bernoulli}(p)$ . How is  $S_n$  distributed?

### Problem 5

Now that we understand both the binomial and the concept of  $\stackrel{iid}{\sim}$ , we will ask some conceptual questions.

(a) [harder] Recall  $X_1, X_2$  from problem 1(d) which were the two sorcery-controlled coins. Let  $T_2 = X_1 + X_2$ . Is  $T_2 \sim \text{Binomial}(2, \frac{1}{2})$ ? Why or why not?

(b) [harder] The human mouth has 32 teeth. If the probability of a cavity at some point in a lifetime is 5%, is it possible to calculate the probability of 7 cavities during a lifetime using a binomial r.v. model  $X \sim \text{Binomial}(32, 5\%)$  and computing  $\mathbb{P}(X = 7)$ ? Why or why not?



**Negative Binomial** A related distribution to the binomial but still different.

### **Problem 6**

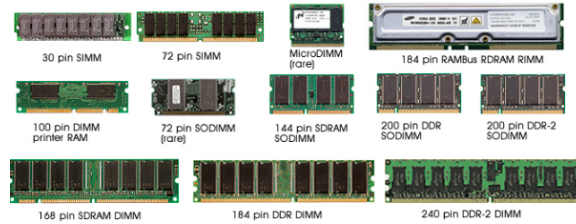
We will rederive the negative binomial PMF as we did in class. The probability of success is  $p$  and the number of successes we wish to find is  $r$ .

- (a) [easy] If we are waiting  $x$  trials to finally see exactly  $r$  successes, what does the outcome result of the last trial *need* to be?
  
  
  
  
  
  
  
  
  
  
- (b) [easy] How many trials do we witness in order to witness  $r - 1$  successes not counting the last trial?
  
  
  
  
  
  
  
  
  
  
- (c) [easy] Can these  $r - 1$  successes happen anywhere within these  $x - 1$  trials?
  
  
  
  
  
  
  
  
  
  
- (d) [easy] If you get  $r - 1$  successes in  $x - 1$  trials, how many failures do you get?
  
  
  
  
  
  
  
  
  
  
- (e) [easy] How many ways is there to get  $r - 1$  successes among  $x - 1$  trials?
  
  
  
  
  
  
  
  
  
  
- (f) [easy] What is the probability of getting  $r - 1$  successes and  $x - r$  failures *in that order* if successes and failures are independent?

- (g) [easy] Use the answers in (e) and (f) to find the probability of getting  $r - 1$  successes in  $x - 1$  trials.
- (h) [easy] Use the answers in (g) and the probability of a final success to finally derive the full PMF of the Negative Binomial distribution.
- (i) [easy] Let  $X \sim \text{NegBin}(r, p)$ . What is the support of  $X$ ?
- (j) [harder] What is the parameter space of  $r$  and  $p$ ? Be careful not to allow degenerate cases.

### **Problem 7**

You are testing RAM. The manufacturing process is near perfect. The probability of finding faulty RAM is about 1 in 300. We assume all RAM chips are independent with respect to whether they are faulty.



- (a) [easy] What is the probability you get three faulty RAM chips in a row?
- (b) [harder] What is the probability you have to investigate 100 RAM chips in order to find exactly 3 faulty chips? Compute explicitly.
- (c) [harder] What is the probability you have to investigate 500 RAM chips in order to find exactly 3 faulty chips? You can leave in choose notation and use exponents as well.
- (d) [difficult] What is the probability you have to investigate more than 500 RAM chips to see exactly 3 faulty chips? You can leave in choose notation and use exponents as well.