Lecture 3 MATH 241 Sept 4, 2018

• Sept 23?

→ Some confusion about $HT = TH$

More slow down. Recall $\Omega = \{H, T\}$. Two coin flips

Review

$\Omega_2 := \Omega \times \Omega$ Cartesian product $\mathbb{R} \times \mathbb{R}$

all sets of 2-tuples → ordered

100 marbles in a bag

$\langle H,H\rangle, \langle H,T\rangle, \langle T,H\rangle, \langle T,T\rangle$

labeled $\{1, \ldots, 100\}$ thus $HT \neq TH$

Picking them "with replacement", how many arrangements of 5 # draws?

$$100^5 \Rightarrow n^k.$$

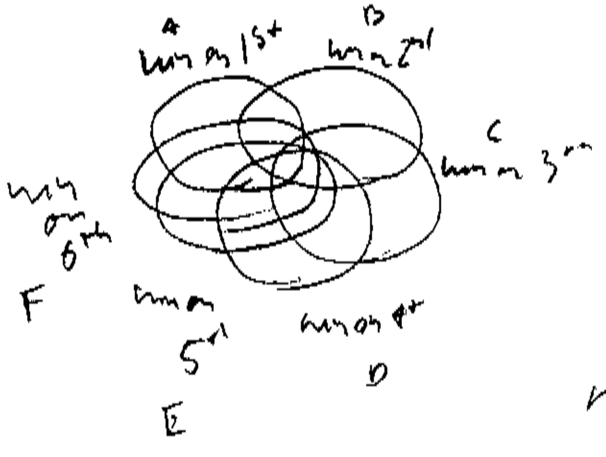
" without replacement, " " ?

$$100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 = \frac{100!}{(100-5)!} = \frac{100!}{95!} \Rightarrow \frac{n!}{(n-k)!}$$

dice rolls, coins, spinners, lotteries, etc. → with replacement

cards in a hand, people & chairs etc → without replacement

More counting... Roll a die 6 X. How many ways can there be where you get at least one "win" where "win" is defined as a 1 on the 1st roll, or a 2 on the 2nd roll, ... or a 6 on the 6th roll? Let's count.



Looking for all outcomes in

$W := A \cup B \cup C \cup D \cup E \cup F$

not mutually exclusive! why?

Why is this wrong

$$\underline{1}\ \underline{6}\ \underline{6}\ \underline{6}\ \underline{6}\ \underline{6} = 6^5 \quad (A)$$

1st 2nd 3rd 4th 5th 6th

$$\underline{6}\ \underline{1}\ \underline{6}\ \underline{6}\ \underline{6}\ \underline{6} = 6^5 \quad (B)$$

$\vdots$

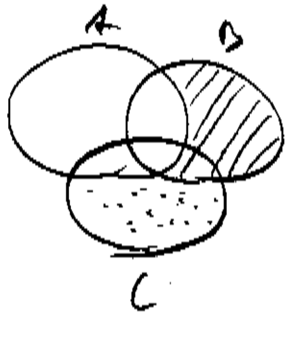
$$\underline{6}\ \underline{6}\ \underline{6}\ \underline{6}\ \underline{6}\ \underline{1} = 6^5 \quad \Rightarrow 6 \cdot 6^5 \ ?$$

Can't be right!

$|A \cup B| \neq |A| + |B|$ double counting if not disjoint / mut. excl.

How many ways can you get 6 rolls? $6^6 = 6 \cdot 6^5$

Can we split up W into a set of disjoint subsets so we can add up their sizes to get the answer?



$$A \cup B \cup C = A \cup (B \setminus A) \cup (C \setminus (A \cup B))$$

disjoint by construction

A: $\underline{1}\ \underline{6}\ \underline{6}\ \underline{6}\ \underline{6}\ \underline{6} = 6^5$

$B \setminus A$: $\underline{5}\ \underline{1}\ \underline{6}\ \underline{6}\ \underline{6}\ \underline{6} = 5 \cdot 6^4$

$C \setminus (A \cup B)$: $\underline{5}\ \underline{5}\ \underline{1}\ \underline{6}\ \underline{6}\ \underline{6} = 5^2 \cdot 6^3$ $\Rightarrow 6^5 + 5 \cdot 6^4 + 5^2 6^3 + 5^3 6^2 + 5^4 6 + 5^5 = 31{,}031$

$\vdots$

What's this? $\rightarrow \dfrac{\alpha}{6^6} \approx .665$

Where we left off Thurs...

52 cards, choose 5 → how many arrangements?

$$_{52}P_5 = \frac{52!}{(52-5)!} = \frac{52!}{47!}$$

How about if we don't care about the order of the cards?

Divide out what we don't care about. Def of "combination"

$$\frac{_{52}P_5}{5!} = \frac{52!}{47!\,5!}$$

\# arrangements ϕ only 1 way and $\binom{n}{n}$

Generally, n elements, $k \le n$ $\quad {}_nP_k \Rightarrow k$ is an "ordered" sample

$$\binom{n}{k} := {}_nC_k \Rightarrow k \text{ is an "unordered" sample}$$

$$= \frac{_nP_k}{k!} = \frac{n!}{(n-k)!\,k!}$$

Traditions → $0! = 1$ why?

① $\binom{n}{0} = 1$ ③ $\binom{n}{n} = 1$

② $\binom{n}{1} = n$ ④ $\binom{n}{n-1} = n$

⑤ $\binom{n}{k} = \binom{n}{n-k}$

$\binom{52}{5} =$ 2,598,960 # of poker hands

Let's say 10 people, How many ways to pick 3 people?

$$\binom{10}{3} = \frac{10!}{7!\,3!} = \frac{10\cdot\cancel{9}^{3}\cdot\cancel{8}^{4}}{\cancel{3}\cdot\cancel{2}} = 40$$

Binomial coefficients

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

draw this out....

Previously $P(\langle H,H,H,H\rangle) = P(\langle H,T,H,T\rangle)$

What about prob of getting 2T2H? How many ways?

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{\cancel{4}^{2}\cdot 3\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}} = 6 \Rightarrow 6 \text{ times more likely than } \langle H,H,H,H\rangle$$

Prob of getting 500H on 1000 flips? Estimate it

$$p = P(A) = \frac{|A|}{|\Omega|} = \frac{\frac{1000!}{500!\,500!}}{2^{1000}}$$

Computation error! Big #'s ⇒ work with logs!

Stirling's Approx

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \Rightarrow \ln(n!) = \frac{1}{2}\ln(2\pi n) + n\ln\left(\frac{n}{e}\right)$$
$$= \frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln n + n\ln n - n$$
$$= \frac{1}{2}\ln(2\pi) + \left(n+\frac{1}{2}\right)\ln(n) - n$$

$$\ln(p) = \ln(1000!) - 2\ln(500!) - 1000\ln 2$$
$$\approx \frac{1}{2}\ln(2\pi) + 1000.5\ln(1000) - \cancel{1000} - \ln(2\pi) - 1001\ln(500) + \cancel{1000} - 1000\ln(2)$$
$$= -3.6797 \Rightarrow p \approx \exp(-3.6797) = 0.0252$$ correct to the 10,000's place.

Kinda weird... don't you "expect" half the coins to be heads.
⇒ Basis of r.v. theory and inference which is my lectures down the road.

Back to decks of cards. What is the probability of {2C, 8H, KD, AS, 3C}?
How many ways to get this hand?

$$\frac{|A|}{|\Omega|} = \frac{1}{\binom{52}{5}} \approx 1 \text{ in } 2.6\text{ M}$$

How about a royal flush? $4/\binom{52}{5}$ $\xleftarrow{\binom{1}{1}}$ 10 J Q K A $\binom{4}{1}\binom{5}{5}$

" straight flush? $\binom{10}{1}\binom{4}{1} - 4$ A2345, 23456, 34567, ..., 5, 6, 7, 8, 9, 10

How about 4-of-a-kind? $\binom{4}{4}\binom{13}{1}\binom{48}{1}$

Straight? $\binom{10}{1}\binom{4}{1}^5$

3-of-a-kind $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}$

Full House? $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$ AA KKK

why not $\binom{12}{1}\binom{11}{1}$?

Flush? $\binom{4}{1}\binom{13}{5} - \underbrace{\left(\binom{10}{1}\binom{4}{1} - 4\right) - 4}_{40}$

2-pair $\binom{13}{2}\binom{4}{2}^2\binom{44}{1}$ or $\rightarrow \binom{11}{1}\binom{4}{1}$

1-pair $\binom{13}{1}\binom{4}{2}\cancel{\binom{48}{4}}$ ← why wrong? $\binom{12}{3}\binom{4}{1}^3$

straight? flush?

No pair $\binom{13}{5}\binom{4}{1}^5 \underbrace{-\binom{10}{1}\binom{4}{1}^5}_{St} \underbrace{-\binom{4}{1}\binom{13}{5}}_{fl} + \binom{10}{1}\binom{4}{1}$

A-high $\binom{4}{1}\binom{12}{4}\binom{4}{1}^4$ ... but possible straight, possible flush, straight flush

2, 3, 4, 5

St flush

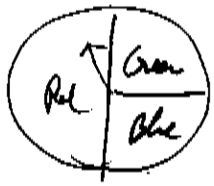
$\Omega = \{H, T\}$

H	T

previously, $P(A) = \frac{|A|}{|\Omega|}$ this only works for universes with "equally likely outcomes" (p 9)

i.e. $P(\omega_i) = \frac{1}{|\Omega|} \; \forall i$

Imagine...



$\Omega = \{Red, Green, Blue\}$

$P(\{Red\}) \neq \frac{|\{Red\}|}{|\Omega|} = \frac{1}{3}$

thus $P(A) = \frac{|A|}{|\Omega|}$ is NOT a good definition of probability of any A.

How is it defined? In the book it's defined as the "limiting frequency" II

let $\mathbb{1}_{\omega \in A} = \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A \end{cases}$ the "indicator function"

$$P(A) := \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \mathbb{1}_{\omega_i \in A}}{n}$$ what is the frequency of A over many pulls from $\omega \in \Omega$