Lecture 11 Oct 21, 2018

Previously

$X \sim \text{Binom}(n,p) := \binom{n}{x} p^x (1-p)^{n-x}$, $\text{Supp}(X) = \{0, \ldots, n\}$, $n \in \mathbb{N}, p \in (0,1)$

n: fixed # of trials, how many successes x if prob of success is p

$X \sim \text{NegBin}(r,p) := \binom{x-1}{r-1} p^r (1-p)^{x-r}$, $\text{Supp}(X) = \{r, r+1, \ldots\}$, $r \in \mathbb{N}, p \in (0,1)$

r: fixed # of successes, how many trials x if prob of success is p

"n" is now "x", "x" is now "r"

~~How about the following question~~

Is the neg bin. PMF a PMF?

$$\underset{\geq 1}{\binom{x-1}{r-1}} \underset{>0}{p^r} \underset{>0}{(1-p)^{x-r}} \checkmark$$

$$\sum_{\text{Supp}(X)} p(x) \overset{?}{=} 1$$

$$\sum_{x=r}^{\infty} \binom{x-1}{r-1} p^r (1-p)^{x-r} = 1$$

HARD... E.C. (on google) on HW...

~~$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$~~

~~let $y = x - r$~~ $\Rightarrow x = y + r$

~~$\sum_{y=0}^{\infty} \binom{y+r-1}{r-1} p^r (1-p)^y$~~

~~$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n\,(n-1)!}{k\,(k-1)!\,(n-k)!} = \frac{n}{k}\binom{n-1}{k-1}$~~

~~$\frac{(n-1)!}{(k-1)!\,(n-1-(k-1))!}$~~

~~$\binom{n}{k}\frac{k}{n} = \binom{n-1}{k-1}$~~

~~$p^r \sum_{y=0}^{\infty} \frac{(y+r-1)!}{(r-1)!\,y!} (1-p)^y$~~

~~$\frac{10!}{5!} = 10 \cdot 9 \cdots 7$~~

~~$\frac{p^r}{(r-1)!} \sum_{y=0}^{\infty} (y+r-1)(y+r-2) \cdots r$~~

Imagine I was only looking for one success.

$$X \sim \text{NegBin}(1,p) = \binom{x-1}{r-1} p^r (1-p)^{x-r} = \binom{x-1}{0} p^1 (1-p)^{x-1} = (1-p)^{x-1} p$$

$$\overbrace{\underbrace{0\quad 0\quad 0\quad \ldots\ldots\quad 0}_{x-1}\quad 1}^{x}$$

$P(\text{fail})^{x-1} P(\text{success})$

$X \sim \text{Geometric}(p) = \text{NegBin}(1,p)$ geometric is a special case of the Neg Bin.

$\quad\quad \overset{\shortparallel}{(1-p)^{x-1} p}$

$\text{Supp}(X) = \mathbb{N}$

$p \in (0,1)$, $p=1$ is the degenerate case $X \sim \text{Deg}(1)$ $\Rightarrow$ I will not consider it in the support

Is $f(x) = \underbrace{(1-p)}_{>0}{}^{x-1} p$ a PMF?

$\underbrace{(1-p)^{x-1}}_{>0} \underbrace{p}_{>0}$ ✓

$$\sum_{x=1}^{\infty} (1-p)^{x-1} p \overset{?}{=} 1 \quad \text{let } q = 1-p \Leftrightarrow 1-q = p$$

$$\sum_{x=1}^{\infty} (1-p)^{x-1} \overset{?}{=} \frac{1}{p} \qquad \sum_{x=1}^{\infty} q^{x-1} = \sum_{x=0}^{\infty} q^x = 1 + q + q^2 + q^3 + q^4 + \ldots$$

lots of algebra only works for #s ∈ ℝ

only works if $B < \infty$...

$$\sum_{x=0}^{\infty} q^x = B \Rightarrow 1 + q + q^2 + q^3 + q^4 + \ldots \Rightarrow B - 1 = q + q^2 + q^3 + q^4 + \ldots$$

$$\Rightarrow B - 1 = q\underbrace{(1 + q^2 + q^3 + \ldots)}_{} \Rightarrow B - 1 = qB \Rightarrow B - qB = 1 \Rightarrow B(1-q) = 1$$

$$\Rightarrow B = \frac{1}{1-q}$$

$$\Rightarrow \sum_{x=0}^{\infty} q^x = \frac{1}{1-q} \Rightarrow \sum_{x=1}^{\infty} (1-p)^{x-1} = \overset{B}{\frac{1}{1-q}} = \frac{1}{p}$$ ✓

How do we know $1+q+q^2+\dots < \infty$?

Note $\beta = 1+q+q^2+\dots = \lim_{N\to\infty} 1+q+q^2+\dots+q^N$ anytime you see a ∞, it's a limit!

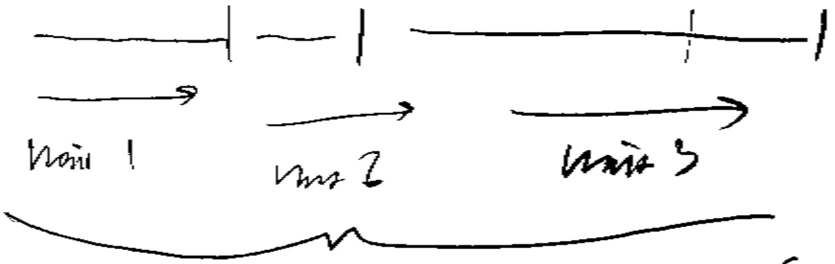
Note $(1+q+q^2+\dots+q^N)(1-q) = 1-q^{N+1} \Rightarrow 1+q+q^2+\dots+q^N = \frac{1-q^{N+1}}{1-q}$

$$\Rightarrow \beta = \lim_{N\to\infty} \frac{1-q^{N+1}}{1-q} = \frac{1}{1-q}\left(1-\lim_{N\to\infty} q^{N+1}\right)$$

which converges if $|q|<1$ Since $q=1-p$ and $p\in(0,1) \Rightarrow q\in(0,1) \Rightarrow |q|<1$ ✓

What is Geometric(p)? Wait until 1 success

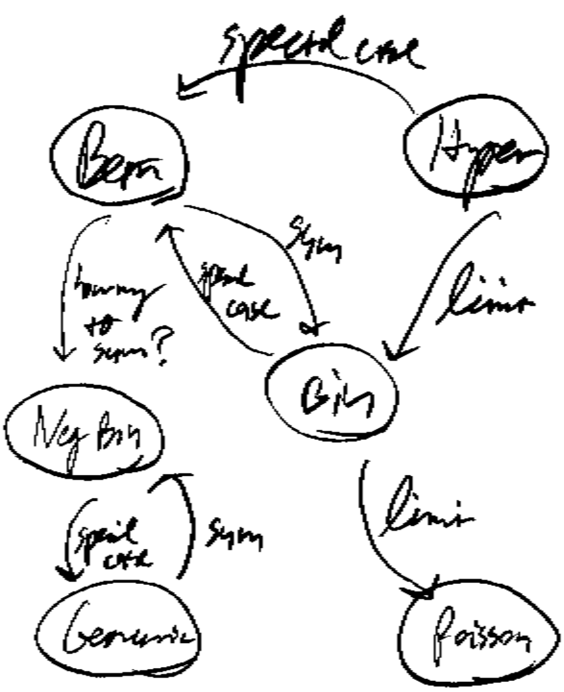
What is NegBin(r,p)? Wait until r successes
$r=3$



this length is distributed $\sim$ NegBin$(3,p)$

$$\underset{T\sim}{\Rightarrow} \text{NegBin}(r,p) \Rightarrow T = \sum_{i=1}^{r} X_i \text{ where } X_1,\dots,X_r \overset{iid}{\sim} \text{Geometric}(p)$$

We will prove this later



last discrete r.v. we will study

Recall $T_n = X_1 + X_2 + \dots + X_n$ where $X_1, \dots, X_n \overset{iid}{\sim}$ Bernoulli(p)

$\Rightarrow T_n \sim$ Binomial(n, p)

let $n = 100$, $p = 0.01 = \frac{1}{100}$

$P(X=1) = \binom{100}{1}(0.01)^1(0.99)^{99} \approx 0.3697$

let $n = 1000$, $p = 0.001 = \frac{1}{1000}$

$P(X=1) = \binom{1000}{1}(0.001)^1(0.999)^{999} \approx 0.3681$

let $n = 1{,}000{,}000$, $p = 0.000001$

$P(X=1) = \binom{1{,}000{,}000}{1}(0.000001)^1(0.99999)^{999{,}999} \approx 0.3679$

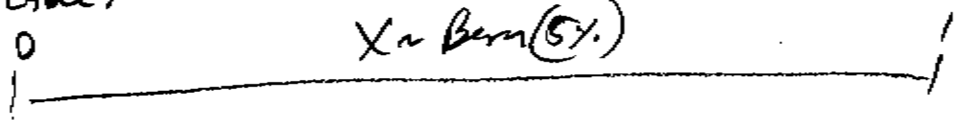
reasonably, this is approaching a limit

as $n \to \infty$ and $p \to 0$ but np remains constant at $\overset{\lambda = np =}{1}$, we are converging to something.

ch 2.6 Poisson distribution

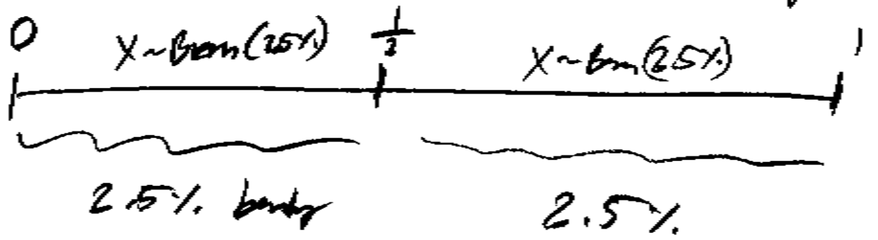
Does this happen in reality? Sure $n \to \infty$ # callers in Bell tech suppose $p \to 0$ prob that they call is between 9AM-10AM Tues morning. "Poisson distribution"

Or it can be thought of as a time limit. Imagine a wire 1 mi long. A power cable.

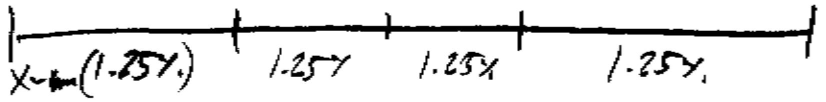


the wire has a 5% chance of being defective in 10 yrs anywhere along its mile length. But how many places will it break?

assume each piece of wire is independent



T = # breaks $\quad T \sim \text{binomial}(2, 2.5\%)$



$T \sim \text{Binomial}(4, 1.25\%)$

⋮

$T \sim \text{Binomial}\left(n, \frac{5\%}{n}\right)$

⋮

eventually the Bernoulli's have length 0, $p = 0$

"Poisson process"

L6

Need PMF of Poisson.

$$\lim_{\substack{n\to\infty \\ \text{or } p\to 0}} \binom{n}{x} p^x (1-p)^{n-x}$$

while $\lambda = np$

let $p = \frac{\lambda}{n}$

$$\lim_{n\to\infty} \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{n!}{(n-x)!\,x!}(\lambda^x)\left(\frac{1}{n^x}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\underbrace{\frac{n!}{(n-x)!}}\,\underbrace{\frac{1}{n^x}}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\frac{10!}{(10-4)!} = \underbrace{10\cdot 9\cdot 8\cdot \overset{10-4+1}{7}}_{4\text{ terms}}$$

$$\frac{\underbrace{(n)(n-1)\cdots(n-x+1)}_{x\text{ terms}}}{\underbrace{(n)(n)\cdots(n)}_{x\text{ terms}}}$$

$$= \frac{\lambda^x}{x!}\left(\lim_{n\to\infty}\frac{n}{n}\right)^{\nearrow 1}\left(\lim_{n\to\infty}\frac{n-1}{n}\right)^{\nearrow 1}\cdots\left(\lim_{n\to\infty}\frac{n-x+1}{n}\right)^{\nearrow 1}\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)\underbrace{\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}\right)}_{x\text{ is a const}}$$

$$\lim_{n\to\infty}(f(n))^a = \left(\lim_{n\to\infty} f(n)\right)^a$$

$$\left(1-\lim_{n\to\infty}\frac{\lambda}{n}\right)^{-x} = (1-0)^{-x} = 1^{-x} = 1$$

$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$ interesting limit

what is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$n=10 \rightarrow 2.5937$

$n=100 \rightarrow 2.2048$

$n=1000 \rightarrow 2.7169$

$n=1,000,000 \rightarrow 2.7183.$

$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$\uparrow$

special constant

also ... e is defined as $1=\int_1^e \frac{1}{x}\,dx$

or $e := \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots = \sum_{i=0}^{\infty}\frac{1}{i!}$ Taylor Series

$\Rightarrow$ here get $\lim_{n\to\infty}\left(1+\frac{c}{n}\right)^n$ where c is a constant, $c=-\lambda$

let $\frac{1}{m}=\frac{c}{n} \Rightarrow n=cm$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{m}\right)^{cm} = \left(\lim_{n\to\infty}\left(1+\frac{1}{m}\right)^m\right)^c$$ if $n\to\infty \Rightarrow m\to\infty$

$$=\left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^c = e^c$$

$$\Rightarrow \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

binomial with

iteration $N\to\infty$

$\text{Supp}(X)=\{0,\dots,n\}$

$$\Rightarrow \boxed{X\sim \text{Poisson}(\lambda) := \frac{\lambda^x e^{-\lambda}}{x!}}$$

$\text{Supp}(X) = \mathbb{N}_0 = \{0,1,2,\dots\}$

$= \mathbb{N}\cup\{0\}$

Parameter space $\lambda \in (0,\infty)$ If $\lambda = 0$ $X \sim \text{Poisson}(0)$ $\rightarrow$ behaves like $p=0$ for Bernoulli/Binomial $\Longrightarrow$ not meaningful!

Is it a PMF?

$f(x) \geq 0$?

$$f(x) = \frac{\overset{>0}{\lambda^x}\ \overset{>0}{e^{-\lambda}}}{\underset{\geq 1}{x!}} \quad \forall x \in \text{Supp}(X) \quad ✓$$

~~STOP~~

$$\sum_{x \in \text{Supp}(X)} f(x) \overset{?}{=} 1$$

$$\sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} \overset{?}{=} 1 \Rightarrow \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \overset{?}{=} \frac{1}{e^{-\lambda}} = e^{\lambda}$$

Remember, Taylor series can approximate the value of a function close to a value "a".

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

approximating $f(x)$ close to $a=0$ yields. Different a's allow for better accuracy without having to compute the infinite sum

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) = e^x = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$f'(y) = e^y$, $f''(y) = e^y$, $f'''(y) = e^y$ .... $f^{(n)}(y) = e^y$ $\forall n \in \mathbb{N}$

$e^0 = 1$

$$\Rightarrow f(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda} \quad ✓$$