Previously $Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $X \sim N(\mu,\sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

why so mysterious? In order to show you why, we need to do some weird math!

L1

Lecture 10 Nov 25, 2017

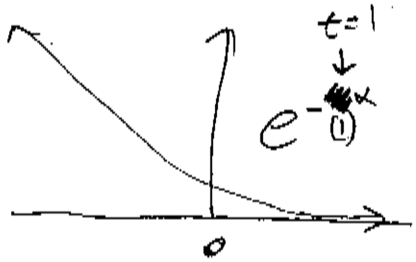
Imagine $f(x)$

$$F(t) := \mathcal{L}\left[f(x)\right] := \int_{\mathbb{R}} e^{-tx} f(x)\,dx$$

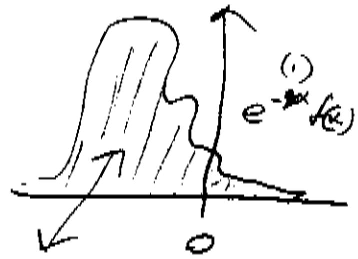
Bilateral Laplace transform

"integrate x away" to leave a function of t

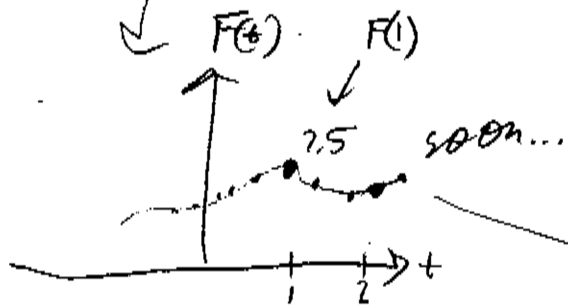
Want $F(1)$



=



Now $\int_{\mathbb{R}} e^{-(1)x} f(x) = 7.5$



Now do for all t

integrating for general t gives us the function for all t

Theorems are $F(t) \longleftrightarrow f(x)$ are 1:1. You give me an $f(x)$, I get $F(t)$. If I get the same $F(t)$ for $f_1(x)$ and $f_2(x) \Rightarrow f_1(x) = f_2(x)$.

The moment generating function (mgf) $M_X(t)$ is defined for a cont. r.v. X as follows

the mirror image of the bilateral Laplace transform

$$M_X(t) = F(-t) := \int_{\mathbb{R}} \underbrace{e^{tx}}_{g(x)} f(x)\,dx = E\left[e^{tX}\right] = E(g(X)) \qquad g(X) = e^{tX}$$

clear support since $f(x) = 0$ if $x \notin Supp(X)$

and for a discrete r.v.

$$M_X(t) = E\left[e^{tX}\right] = \sum_{x \in Supp(X)} e^{tx} f(x)$$

must know for test!

All it is is an expectation of a carefully-chosen $g(x)$. Why is e^{tX} so special ???? Two reasons...

(I) $X_1 + X_2$ ind.. Hard to analyze! Convolutions!

$$M_{X_1+X_2}(t) = E\left[e^{t(X_1+X_2)}\right] = E\left[e^{tX_1} e^{tX_2}\right] = E(e^{tX_1}) E(e^{tX_2}) = M_{X_1}(t) M_{X_2}(t)$$

$E[g(X) g(Y)] = E(g(X)) E(g(Y))$ if X, Y ind.

Addition magically became multiplication!! Exam question

(II) $X \sim Binom(n,p)$

$E(X) = np$ — hard to get $\sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$ gets lucky..

$Var(X) = E(X^2) - \mu^2 = np(1-p)$

(III) Uniqueness $M_X(t) \leftrightarrow f(x)$

What about $E[X^{17}]$? $\sum x^{17}\binom{n}{x}p^x(1-p)^{n-x}$ ... good luck!!

Need easy way to get moments...

Remember $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots = \sum_{i=0}^{\infty} \frac{y^i}{i!}$

all moments

$M_X(t) := E[e^{tX}] = E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \ldots\right] = 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \frac{t^4}{4!}E(X^4) \ldots$

this generally will assume is justified... (upper level math ....)

$M_X'(t) = \frac{d}{dt}\left[M_X(t)\right] = 0 + E(X) + \frac{t}{1!}E(X^2) + \frac{t^2}{2!}E(X^3) + \frac{t^3}{3!}E(X^4)$

$M_X'(0) = E(X)$

$M_X''(t) = E(X^2) + tE[X^3] + \frac{t^2}{2!}E(X^4) + \ldots$

$M_X''(0) = E(X^2)$

$\vdots$ k^{th} derivative

$M_X^{(k)}(0) = E[X^k]$ Cool!

Fact: if $X \overset{d}{=} Y \Rightarrow$

$M_X(t) = M_Y(t)$

$\Rightarrow 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \ldots$

$= 1 + tE(Y) + \frac{t^2}{2!}E(Y^2) + \ldots$

$\Rightarrow$ all moments are the same $\Rightarrow X \overset{d}{=} Y$

$X \sim \text{Bernoulli}(p)$ $M_X(t) = E(e^{tX}) = \sum_{x=0}^{1} e^{tx} \underbrace{f(x)}_{\text{PMF}} = \underbrace{e^{t(0)}}_{1} f(0) + \underbrace{e^{t(1)}}_{e^t} f(1)$

$= (1-p) + pe^t$

$X \sim \text{Binom}(n,p)$ $M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} \underbrace{e^{tx}}_{(e^t)^x}\binom{n}{x}p^x(1-p)^{n-x} = \sum \binom{n}{x}(pe^t)^x(1-p)^{n-x}$

$= (pe^t + 1 - p)^n$ (binom. thm again)

$T_n = X_1 + \dots + X_n$ where $X_1, \dots, X_n \overset{iid}{\sim}$ Bernoulli(p), $T_n \sim$ Binom(n,p)

$$M_{T_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \cdot \dots \cdot M_{X_n}(t)$$
$$= (1-p+pe^t)(1-p+pe^t) \cdot \dots \cdot (1-p+pe^t)$$
$$= (1-p+pe^t)^n =$$ the mgf for a binom r.v with n,p!!!

thus $X_1 + \dots + X_n$ must be binom!

$$Q = T_n + T_m = (X_1 + \dots + X_n) + (X_1' + \dots + X_m')$$ all iid Bernoulli(p) $\sim$ Binom$(n+m, p)$

$$M_Q(t) = M_{T_n}(t)\, M_{T_m}(t) = (1-p+pe^t)^n \cdot (1-p+pe^t)^m = (1-p+pe^t)^{n+m}$$

mgf for Binom$(n+m, p)$

$\sum$ Geometrics ~~Hy~~ Binom $\Rightarrow$ On Hw!

$\sum$ Poissons $\Rightarrow$ on Hw!

$$Z \sim N(0,1), \quad M_Z(t) = E[e^{tZ}] = \int_{\mathbb{R}} e^{tx} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{f_Z(x)} dx$$ PDF for std. normal

$$(x-t)^2 = x^2 - 2tx + t^2$$
$$\Rightarrow (x-t)^2 - t^2 = x^2 - 2tx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{tx - \frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2 - 2tx}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(x-t)^2 - t^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{+\frac{t^2}{2}} \int_{\mathbb{R}} e^{-\frac{(x-t)^2}{2}} dx$$ Gaussian integral with $u = x - t \Rightarrow \sqrt{2\pi}$

$$= e^{t^2/2}$$

$X \sim N(\mu,\sigma^2) \Rightarrow X = \mu + \sigma Z$

let $t' = \sigma t$

$M_X(t) = E(e^{tX}) = E[e^{t(\mu+\sigma Z)}] = E[e^{t\mu} e^{\sigma t Z}] = e^{t\mu} E[e^{\sigma t Z}] = e^{t\mu} M_Z(t')$

$= e^{t\mu} M_Z(\sigma t) = e^{t\mu} e^{(\sigma t)^2/2} = e^{t\mu} e^{\sigma^2 t^2/2} = e^{t\mu + \frac{\sigma^2 t^2}{2}}$

$X_1 \sim N(\mu_1, \sigma_1^2)$, $(\perp)$ $X_2 \sim N(\mu_2, \sigma_2^2)$ ... HW: $X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$ easy with m.g.f's!

Now you can read ch 2 & 3

Exist?

	PMF	PDF	CDF	mgf
discrete	Yes	No	Yes	mostly
cont.	No	Yes	Yes	mostly

→ Sometimes $\sum e^{tx} f(x) = \infty$ — OR — $\int e^{tx} f(x)\,dx = \infty \Rightarrow$ mgf d.n.e.

always exists but hardest to work with...

· Elementary transformations AND mgfs...

· Shift: $Y = X + c$ what is $M_Y(t)$?

$M_Y(t) = E[e^{t(X+c)}] = E[e^{tX+tc}] = E[e^{ct} e^{tX}] = e^{ct} E(e^{tX}) = \boxed{e^{ct} M_X(t)}$

Scale: $Y = aX$

$M_Y(t) = E[e^{taX}] = \boxed{M_X(ta)}$

Both: $Y = aX + c$

$M_Y(t) = e^{ct} M_X(at)$

$X_1, X_2, \ldots, X_n \overset{iid}{\sim} f(x)$ (either discrete or cont. doesn't matter) $\mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}$

not std normals just std. form of X

let $Z_1 = \frac{X_1 - \mu}{\sigma}, Z_2 = \frac{X_2 - \mu}{\sigma}, \ldots, Z_n = \frac{X_n - \mu}{\sigma}$

Remember... $E[Z_i] = 0, \quad SE[Z_i] = 1$

$$\bar{Z} := \frac{Z_1 + Z_2 + \ldots + Z_n}{n} = \frac{Z_1}{n} + \frac{Z_2}{n} + \ldots + \frac{Z_n}{n}, \text{ let } C_n := \sqrt{n}\,\bar{Z} = \frac{Z_1}{\sqrt{n}} + \frac{Z_2}{\sqrt{n}} + \ldots + \frac{Z_n}{\sqrt{n}}$$

Note that $\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\frac{X_1 + \ldots + X_n}{n} - \frac{n\mu}{n}}{\frac{\sigma}{\sqrt{n}}} = \sqrt{n}\left(\frac{(X_1 - \mu)}{n\sigma} + \frac{(X_2 - \mu)}{n\sigma} + \ldots + \frac{X_n - \mu}{n\sigma}\right) = \sqrt{n}\left(\frac{Z_1}{n} + \frac{Z_2}{n} + \ldots + \frac{Z_n}{n}\right) = \sqrt{n}\,\bar{Z}$

$E[\;] = 0$

$SE[\;] = 1$

What is this? As $n \to \infty$ gets closer and closer to μ with less and less variance!

Note that $\frac{T_n - n\mu}{\sigma\sqrt{n}} = \frac{(X_1 + \ldots + X_n) - n\mu}{\sigma\sqrt{n}} = \frac{(X_1 - \mu) + (X_2 - \mu) + \ldots + (X_n - \mu)}{\sigma\sqrt{n}} = \frac{Z_1}{\sqrt{n}} + \frac{Z_2}{\sqrt{n}} + \ldots + \frac{Z_n}{\sqrt{n}}$

$E[\;] = 0$

$SE[\;] = 1$

So... $\frac{T_n - n\mu}{\sigma\sqrt{n}} = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \sqrt{n}\,\bar{Z}$ purely by retarded algebra!

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