Math 241 Fall 2014-2015
Midterm Examination One Solutions

Professor Adam Kapelner

October 2, 2014

Full Name ______________________________ Section (A or B) _____

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using a cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

______________________________ ______________

signature date

Instructions

This exam is seventy five minutes and closed-book. You are allowed one page (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If I say "compute," this means the solution will be a number. I advise you to skip problems marked
qu[Extra Credit] until you have finished the other questions on the exam, then loop back and plug in all the holes. I also advise you to use pencil.

The exam is 100 points total. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. Good luck!

Problem 1 You are playing a game with six balls. On each ball, a large letter is printed:

(O) (O) (P) (W) (W) (W)

Also on each ball there is also a unique 36 digit alphanumeric serial number (not illustrated above). Imagine the balls were thrown into a urn and the urn was shook and the balls were taken out one-by-one without looking at which ball is which until all six balls are picked.

(a) [3 pt / 3 pts] How many ways are there to order the six balls if the serial numbers are distinct? You do not have to compute the answer numerically.

$$6!$$

(b) [4 pt / 7 pts] How many ways are there to order the balls if you disregard the serial number but only care about the letter displayed on the ball? You must compute the answer explicitly.

$$\frac{6!}{3!\,2!} = \frac{\overset{3}{\cancel{6}}\cdot 5\cdot 4\cdot \cancel{3}\cdot\cancel{2}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{2}} = 60$$

(c) [6 pt / 13 pts] What is the probability of spelling "POWWOW" (in that order) from your draw of six balls by using counting methods only. The next problem requires the conditional probability calculation. Compute explicitly.

arrangements of W

$$\frac{|A|}{|\Omega|} = \frac{3!\,2!}{6!} = \boxed{\frac{1}{60}}$$

arrangements of O

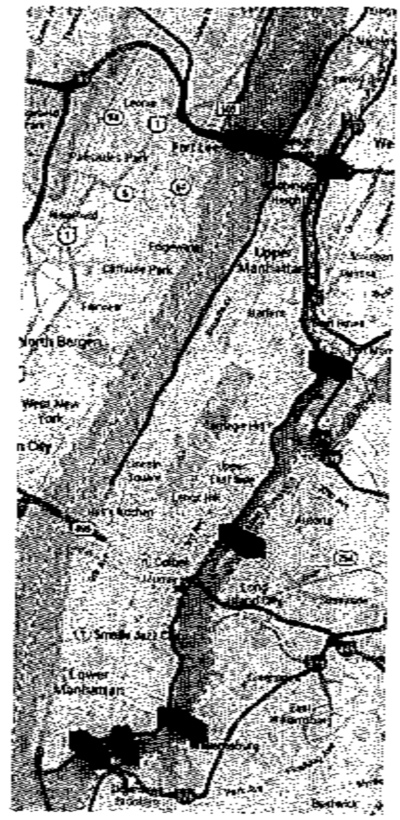
Powwow is one of the 60 $\Rightarrow \boxed{\frac{1}{60}}$

(d) [6 pt / 19 pts] Instead of drawing all six balls, you draw four without replacement. What is the probability of getting one "P," one "O," and two "W"'s in no specific order? You do not need to compute explicitly; you can leave your answer in choose notation.

$$\frac{\binom{1}{1}\binom{2}{1}\binom{3}{2}}{\binom{6}{4}} = \frac{1\cdot 2\cdot 3}{\frac{\overset{3}{\cancel{6}}\cdot 5\cdot\cancel{4}\cdot\cancel{3}\cdot 2\cdot 1}{(2\cdot 1)(\cancel{4\cdot 3\cdot 2\cdot 1})}} = \boxed{\frac{6}{15}}$$

(choose notation is fine, no need to compute)

Problem 2 In this problem, you are one of the military commanders defending Manhattan island from an attacking entourage. You are in charge of defending the island from a fleet of tanks which intelligence has informed you can enter the island *only* through its large bridges (tunnels or minor bridges cannot accomodate their tanks). The map to the right marks their strategic points of entry. You do not know of the attacker's plans, but you know your best chance of defending the city is to line the points of entry with anti-tank battalions. You have 20 such battalions at your disposal.



There are seven point of entry bridges marked in black. Clockwise from the top middle:

- the George Washington bridge
- the Alexander Hamilton and Washington bridge
- the Triborough (Robert F Kennedy) bridge
- the Queensboro bridge
- the Williamsburg bridge
- the Manhattan bridge
- the Brooklyn bridge

Assuming that the attacker will attack through all points of entry, you need to place *at least* one anti-tank battalion at each bridge.

(a) [5 pt / 24 pts] How many different ways to arrange your battalions exist? Assume each of your battalions are the same but each bridge is different. Remember, you need at least one battalion at each entry point. You do not need to compute the answer explicitly.

20 balls, 7 urns $\Rightarrow \binom{20-1}{7-1} = \boxed{\binom{19}{6}}$

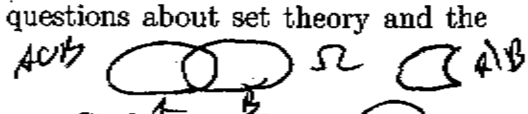
(b) [3 pt / 27 pts] [Extra Credit] Now assume each of your battalions are *not* the same and each bridge is still different. How many different ways to arrange your battalions exist now? You do not need to compute the answer explicitly.

$\boxed{\binom{19}{6} 20!}$

# of arrangements battalions indistinct

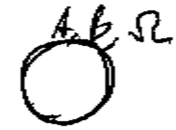
for each arrangement, # of ways to order the distinct battalions

Problem 3 This short-answer section will ask basic questions about set theory and the mathematical definition of probability.



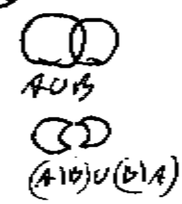
(a) [3 pt / 30 pts] If $A \cup B = \Omega$ then $A \backslash B = \varnothing$ always. Circle one: True **False**

(b) [3 pt / 33 pts] If $A \cap B = \Omega$ then $A \backslash B = \varnothing$ always. Circle one: **True** False



(c) [3 pt / 36 pts] If $A \cup B = (A \backslash B) \cup (B \backslash A)$ always. Circle one: True **False**

(d) [2 pt / 38 pts] Below is Boole's inequality (AKA Bonferroni's inequality):

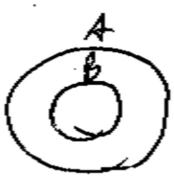


$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i).$$

Under what condition(s) is Boole's inequality an equality (*i.e.* = and not <)? If A_1, A_2, ... are ________. Fill in the blank below using the terminology we learned in class.

disjoint OR mutually exclusive

(e) [6 pt / 44 pts] Prove that if $A \subset B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$. If you use some of the three conditions (axioms) or probabilitiy, you must state that axiom clearly and define it. If you use a fact from set theory, you must say so as well.



$A \subset B \Rightarrow \exists\, C = A \backslash B$ s.t. $A \cup C = B$ and $A \cap C = \phi$ (from set theory)

$P(A \cup C) = P(B)$ (from logic since $A \cup C = B$)

$P(A) + P(C) = P(B)$ (Axiom 3: prob's of unions of disjoint sets can be added $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$ if $A_i \cap A_j \neq 0$ $\forall i \neq j$)

$P(B) - P(A) = P(C)$ (algebra)

$P(B) - P(A) \geq 0$ since $P(C) \geq 0$ (Axiom 2: prob of all sets is ≥ 0)

$P(A) \leq P(B)$ (algebra)

Problem 4 We will be considering the event $A = \{\text{Prof. Kapelner's favorite color is purple}\}$.

(a) [2 pt / 46 pts] Posit a numeric solution for the value of $\mathbb{P}(A)$. There are many correct answers but many incorrect answers as well. You do not need to justify your answer.

0.5 any $\# \in (0,1)$ is acceptable

(b) [4 pt / 50 pts] Explain why the long run frequency definition of probability (the one provided by the textbook) would be useless in providing a solution for $\mathbb{P}(A)$. Answer *in English.*

this does not come from a sample space where outcomes can be drawn. -OR-
this "experiment" of witnessing the professor's favorite color cannot be done even once
(lots of answers are acceptable here)

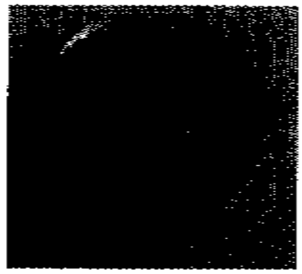
(c) [3 pt / 53 pts] Which of the four definitions of probability we discussed in class did you use to answer part (a)? The next question asks why. You can just write the one word answer here.

Subjective

(d) [4 pt / 57 pts] Why is this the definition you used? Answer *in English.*

We have used our own notions of truth and knowledge to decide a level of certainty. Additionally, the subjective definition allows others to have different opinions given the same evidence.

Problem 5. You play a game with your friend. You take a bunch of coins and tape them together.



Now, when tossing this bundle of coins, there are three possibilities: heads, tails and side, $\Omega = \{H, T, S\}$, but they are not equally likely. You estimate that $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = 5/11$ and $\mathbb{P}(\{S\}) = 1/11$. Use these numbers to answer the questions below.

(a) [3 pt / 60 pts] What is the probability you toss the coin bundle three times and get HTS in that order? No need to compute the solution explicitly.

$$P(H,T,S) = P(H)\,P(T)\,P(S) \quad \text{due to independence}$$
$$= \boxed{\frac{5}{11}\cdot\frac{5}{11}\cdot\frac{1}{11}}$$

(b) [3 pt / 63 pts] What is the probability you toss the coin bundle and get a "side" given that you previously got a head and before that got a tail? That is, find $\mathbb{P}(S_3 \mid H_1, T_2)$ where the subscripts reference the chronological order of the three tosses. Compute explicitly.

$$P(S_3 \mid H_2, T_1) = P(S_3) \quad \text{by independence}$$
$$= \boxed{\frac{1}{11}}$$

(c) [4 pt / 67 pts] What is the probability you toss the coin bundle three times and get at least one head or at least one tail? The word "or" here means non-exclusive or which in English is commonly phrased "and/or." No need to compute the solution explicitly.

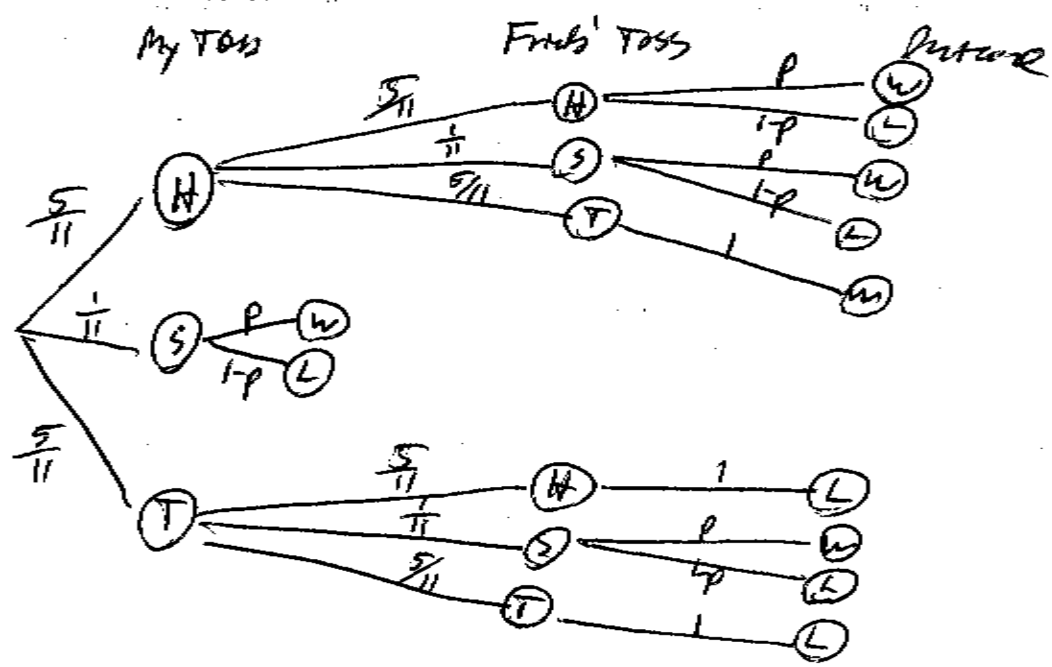
de Morgan's Rule

$$P(\text{at least one } H \text{ -OR- at least one } T) \underbrace{=}_{\text{complement rule}} 1 - P(\text{no } H \text{ and no } T)$$
$$= 1 - P(\text{all 3 tosses are side})$$
$$= 1 - P(S)^3 \quad \text{independence}$$
$$= \boxed{1 - \left(\tfrac{1}{11}\right)^3}$$

(d) [6 pt / 73 pts] You play a game with your friend using this coin bundle. Here are the rules:

- If you get a heads and he gets a tail, you win.
- If you get a tail and he gets a head, he wins.
- If you both get a head, you play again.
- If you both get a tail, he wins.
- If there is a side flipped, immediately flip again.

Draw a tree for this game. The tree should have three levels (1) your toss, (2) your friend's toss and (3) the outcome. For the outcome, you should mark "win" with "W" and "lose" with "L". Make sure you indicate the probabilities at all branches between all levels. For tree recursions, denote the probability of winning as p. Do not calculate marginal probabilities for final disjoint outcomes. Remember, $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = 5/11$ and $\mathbb{P}(\{S\}) = 1/11$.



(e) [3 pt / 76 pts] [Extra Credit] What are fair "odds against" for you winning this game? This is the dollar amount you would get if you wager $1 and you win the game. Remember, you must answer in $x:1$ format where x is a number. Compute explicitly.

$$p = \frac{5}{11}\cdot\frac{5}{11} + \frac{5}{11}\cdot\frac{5}{11}\cdot p + \frac{5}{11}\cdot\frac{1}{11}\cdot p + \frac{1}{11}\cdot p + \frac{5}{11}\cdot\frac{1}{11}\cdot p$$

$$p = \frac{25}{121} + \frac{25}{121}p + \frac{5}{121}p + \frac{11}{121}p + \frac{5}{121}p$$

$\frac{46}{75}$

$$p = \frac{25}{121} + \frac{46}{121}p$$

$$p\left(1 - \frac{46}{121}\right) = \frac{25}{121}$$

$$p\frac{75}{121} = \frac{25}{121}$$

$$\Rightarrow p = \frac{25}{75} = \frac{1}{3}$$

$$\text{Odds} = \frac{P(\text{Lose})}{P(\text{Win})} = \frac{1-\frac{1}{3}}{\frac{1}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = \boxed{2:1}$$

Problem 6 This short-answer section will ask basic questions about random variables.

(a) [3 pt / 79 pts] You are given $Y \sim \text{Uniform}(\{\text{zebra, giraffe, lion, gazelle}\})$. Is this a random variable? Answer yes or no; no need for justification.

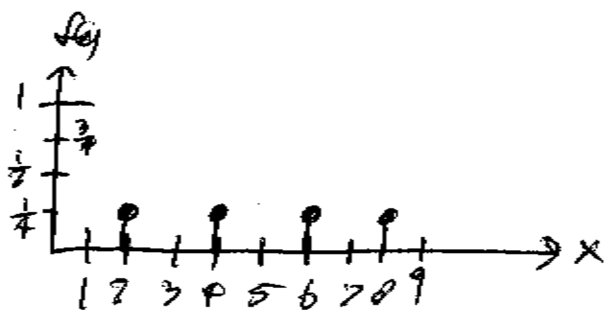
$\boxed{\text{No}}$: the support of all r.v.'s must be $\subseteq \mathbb{R}$

the support of Y seems to be a subset of Safari animals $\not\subseteq \mathbb{R}$

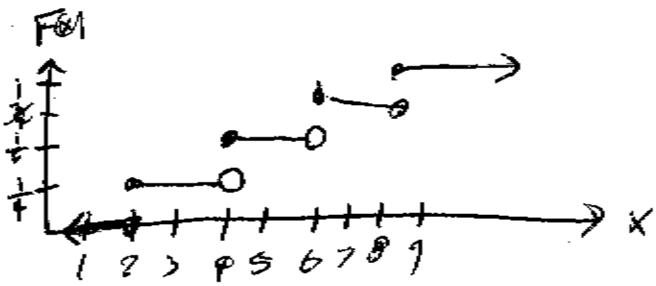
(b) [3 pt / 82 pts] You are given $X \sim \text{Uniform}(\{2, 4, 6, 8\})$. What is the support of this r.v.?

$\text{Support}(X) = \boxed{\{2, 4, 6, 8\}}$

(c) [4 pt / 86 pts] You are given $X \sim \text{Uniform}(\{2, 4, 6, 8\})$. Draw its PMF.



(d) [4 pt / 90 pts] You are given $X \sim \text{Uniform}(\{2, 4, 6, 8\})$. Draw its CDF.



(e) [3 pt / 93 pts] You are given $X \sim \text{Uniform}(\{2, 4, 6, 8\})$. Does this r.v. have any parameters? Write "yes" or "no." There is no need for explanation.

$\boxed{\text{No.}}$ You do not need to explain, but basically, the uniform discrete is found to give equal prob to all elements in its support. Nothing can be varied.

Yes is also an acceptable answer, so this question was not graded

Problem 7 A drivers education website requires students to read an essay on drunk driving as part of its curriculum. Below is an excerpt:

> As more alcohol is consumed the risk of getting into a vehicular accident if the person drives grows. For example, a man that weighs about 160 pounds would have a BAC of 0.04 an hour after drinking two beers. It's still way below the limit of driving under the influence but the likelihood of getting into an accident is 1.4 times more probable than [the national average]. Add two more beers then the probability goes up tenfold. Make it a six pack with two more beers, the drinker reaches the limit of 0.10 BAC and the risk is now 48 times more than [the national average] Add two more for the road and you reach 0.15 BAC well above the legal limit and the risk is now 380 times than the [the national average]. Drunk driving is never an option...

During another part of the curriculum, they read excerpts of the National Safety Council's (NSC) report on traffic fatalities countrywide:

> The motor-vehicle death rate per 100,000,000 vehicle-miles was 1.54 in 2005...

The typical distance between a friend's apartment and the home apartment is 10 miles. Denote the event of getting into an accident during these 10 miles as "A." Denote the event of driving while mostly drunk on a weekend night (*i.e.* 0.10 BAC) as "D." Please use this notation going forward for full credit.

Using the NSC report's figures, we are going to crudely approximate the probability of someone getting into an accident within those typical 10 miles as:

$$p := \mathbb{P}(A) \approx \frac{1.54}{100,000,000} \times 10 = 1.54 \times 10^{-7}.$$

We will also need to know how many people are drunk and behind the wheel. According to the National Highway Traffic Safety Administration (NHTSA) report,

> ...In 2007 over two percent of weekend night-time drivers had blood alcohol concentrations (BAC) above the legal limit (greater than 0.08g/dL)...

Since we are interested in a BAC of 0.10 (mostly drunk), we will be conservative and halve this number and round. Thus, we approximate

$$\mathbb{P}(D) = 1\%.$$

A staggering number, but one that is most likely about right. Be careful out there on the road...

See previous page (highlighted text)

(a) [5 pt / 98 pts] Use our crude approximations of $\mathbb{P}(A)$ and $\mathbb{P}(D)$ with the risk multiple found in the driver's ed manual essay to compute the probability of getting into a fatal accident while driving home from a friend's party on Saturday night after the driver drank heavily. Part of this question is defining what we are looking for using our notation of $A,\ D,\ A^C,\ D^C$ and the conditional probability function. No need to compute explicitly: you can leave your answer in terms of $p := \mathbb{P}(A)$.

$$P(A \mid D) = \boxed{48p}$$

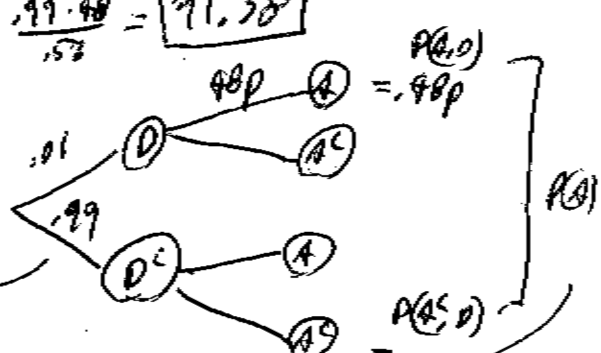
(some tests have 37x on page 9)

(b) [8 pt / 106 pts] If you got into an accident, what was the probability you were drinking heavily? Use our notation of $A,\ D,\ A^C,\ D^C$ and the conditional probability function. No need to compute explicitly: you can leave your answer in terms of $p := \mathbb{P}(A)$.

$$P(D \mid A) = \frac{P(A,D)}{P(A)} = \frac{P(A \mid D)\,P(D)}{P(A)} = \frac{(48\cancel{p})\,0.01}{\cancel{p}} = \boxed{.48}$$

(c) [3 pt / 109 pts] [Extra Credit] Compute explicitly the accident risk of driving drunk.

$$\frac{P(A \mid D)}{P(A \mid D^c)} = \frac{48p}{\frac{P(A,D^c)}{P(D^c)}} = \frac{48\cancel{p}}{\frac{.52\cancel{p}}{.99}} = \frac{.99 \cdot 48}{.52} = \boxed{91.38}$$



Since $P(A) = P(A,D) + P(A^c, D)$

$\Rightarrow p = .48p + x \Rightarrow x = .52p$