Math 2?1 Lecture 5 Sept 11, 2014

Balls & Urns

n balls, r urns, both distinguishable

r^n

$\underline{\ \ }_1\ \underline{\ \ }_2\ \underline{\ \ }_3 \cdots \underline{\ \ }_r$

multinomial ~~coefficient~~
recall ZipCar by me
2 vans, 7 coupes, 3 SUVs, 1 minivan
2 luxury cars. How many ways to
distribute them if car type distinguishable?

Deal out all cards out from players Hearts, Bridge.
How many hands? $\binom{52}{13,13,13,13} \approx 5.4E28$

balls are no longer distinguishable and each urn must have at least one ball

Imagine 6 balls, 2 urns

→ 4 balls some urns

0|0 0 0
0 0|0 0
0 0 0|0

0|0|0|0|0|0 $\Rightarrow 5 \Rightarrow n-1$

6 balls 3 urns

0|0 0|0 0 0 0| $\Rightarrow \binom{5}{2} = \binom{6-1}{3-1}$

n balls r urns

0 0|0 0|...|- 0 $\binom{n-1}{r-1}$

How many ways to make

$X_1 + X_2 + \dots + X_r = n$

s.t. $X_i \in N \ \forall i$ reminder $N = \{1, 2, \dots\}$

~~Allow urns to be empty...~~

~~6 balls on urns~~

~~|0|0|0|0|0|0|0| $\Rightarrow$ n+1~~

~~allow $x_i = 0$ let $y_i = x_i + 1$~~

~~$\Rightarrow y_1 + y_2 + \dots + y_r = n + r$~~

~~$y_i \in N$~~

~~$\Rightarrow \binom{n+r-1}{r-1}$~~

~~Distinguish?~~

we should also be aware the definition of probability still follow some fundamental rules

$P(A) = 1 - P(A^c)$ Complement Rule ... very useful

$P(A \cup B) = P(A) + P(B) - P(AB)$ inclusion exclusion

If asked what is $P(A)$. You can also solve for $P(A^c)$.

Chevalier de Mere game: win if 6-6 at least once in 24.

What is $P(\text{win})$? $\{1,2,\ldots,6\} \times \{1,2,\ldots,6\}$

Ω

1,1	1,2	.	.
2,1	2,2	.	.
6,1	6,2	. .	$\boxed{6,6}$

$= \frac{|\text{win}|}{|\Omega|} = \frac{1}{36}$

Do this 24 times

$P(A \text{ or spade})$

$= P(A) + P(\text{spade}) - P(A, \text{spade})$

$= \frac{1}{13} + \frac{1}{4} - \frac{1}{52}$

$= \frac{16}{52}$

$P(A \cup B) \leq P(A) + P(B)$ Boole's Inequality

$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$

equal only if disjoint

Birthday Problem — What do you think?

$P(\text{at least one of your share the same bday})$

$= P(\text{one pair}) + P(2 \text{ pairs}) + P(3 \text{ pairs}) + \ldots + P\left(\binom{28}{2} \text{ pairs}\right)$ disjoint

Lots of stuff to compute
also very hard! Impossible? I couldn't do it...

$= 1 - P(\text{no pairs})$

no one shares same bday! Imagine 3 people. Do Ω first.

$\frac{365}{1^{st} \text{ person}}$ $\frac{365}{2^{nd} \text{ person}}$ $\frac{365}{3^{rd} \text{ person}}$ $|\Omega| = \frac{365^3}{3!}$

Do A

$\frac{365}{1^{st} \text{ person}}$ $\frac{364}{2^{nd} \text{ person}}$ $\frac{363}{3^{rd} \text{ person}}$ ← Sample without replacement

$|A| = \frac{{}_{365}P_3}{3!}$

$\Rightarrow P(\text{no pair})_{3 \text{ people}} = \frac{{}_{365}P_3}{365^3} = 0.992 \Rightarrow P(\geq 1 \text{ pair}) = 0.008$

$P(\text{no pair in 26 people}) = \dfrac{_{365}P_{26}}{365^{26}} = .346 \Rightarrow P(\geq 1 \text{ pair in } 26) = .654$

where is 50% mark?

$$\frac{_{365}P_x}{365^x} = \frac{1}{2} = \frac{\frac{365!}{(365-x)!}}{365^x} = \frac{1}{2} \quad \text{use computer} \ldots \quad x = 23$$

"Odds" is another way to express prob of A.

What are the odds of rolling a six?

$$Odds(A) := \frac{P(A)}{1-P(A)} = \frac{P(A)}{P(A^c)} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5} \text{ or } 1:5$$

Usually, we use odds to mean "odds against"

Redefine $Odds(A) := \dfrac{P(A^c)}{P(A)} = \dfrac{\frac{5}{6}}{\frac{1}{6}} = 5:1$ always the fraction is divided and you miss :1 why?

when you bet \$1, win \$5 if A

This means, playing a "fair game" you'll break even after many games

$\frac{5}{6}$ → -1, $\frac{1}{6}$ → 5 expectation .. $(-1\$)\frac{5}{6} + (\$5)\frac{1}{6} = 0$ (fair for both parties)

(We will do expectation next week most likely)

"I'll give you 7:1 odds" What does the bookie think the $P(A)$ is?

Fair would be $Odds(A) = \dfrac{1-P(A)}{P(A)} = 7 \Rightarrow 1-P(A) = 7P(A) \Rightarrow 1 = 8P(A) \Rightarrow P(A) = \frac{1}{8} = .125$

Bookie believes $P(A) < .125$ so he makes money. How much? Next week...

Event Independence   p29-33

$P(\{H, H\})$ personally

HH	HT
TH	TT

$\frac{|\{H,H\}|}{|\Omega|} = \frac{1}{4}$

But $P(H$ knowing first flip is $H) = \frac{1}{2}$

→ knowing the first flip is "informationally irrelevant"

this idea of "knowing" or being "given" is a general concept and it has its own notation "|"

$P(A|B)$ is prob of A given B happened

if A, B are "informationally irrelevant", then knowing B doesn't matter at all

$P(A|B) = P(A)$ imagine $P($IBM stock ↑ tomorrow | rains in Buenos Aires tomorrow$)$
$= P($IBM stock ↑ tomorrow$)$ effectively discard the other info.

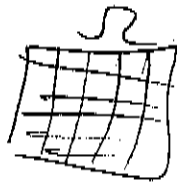
think about 4 flips...

$$P(H_1) \cdot P(H_2 | H_1) \cdot P(H_3 | H_2, H_1) \; P(H_4 | H_3, H_2, H_1)$$

irrel.   irrel.   irrel.

$= P(H_1) \cdot P(H_2) \cdot P(H_3) \cdot P(H_4)$

prob. heads doesn't change!

$= P(H)^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$ ... much easier than  drawing...

How about rolling a 6 twice?

$P(\{6\})^2 = \frac{1}{6^2} = \frac{1}{36}$

How about rolling a 6-6 twice

$P(\{6,6\})^2 = \frac{1}{36^2}$

How about at least one double 6 in 24 rolls

$$P(\geq 1 \;\; 6,6 \text{ in } 24) = P(1 \;\; 6,6 \text{ in } 24) + P(2 \;\; 6,6\text{'s in } 24)$$
$$+ \ldots + \underbrace{P(23 \;\; 66\text{'s in } 24) + P(24 \;\; 6\text{'}6\text{'s in } 24)}$$

HARD - what can we do?

$$= 1 - \underbrace{P(0 \;\; 6,6\text{'s in } 24)}$$
$$= 1 - P(24 \text{ non } 6\text{'}6\text{'s in } 24) \qquad 1 - P(6,6) = 1 - \tfrac{1}{36} = \tfrac{35}{36}$$
$$= 1 - P(\text{not } 6,6)^{24}$$
$$= 1 - \left(\tfrac{35}{36}\right)^{24} = .4914 < \tfrac{1}{2} \quad (\text{Chevalier de Mere})$$

if A, B are informally unrelated, we say A and B are "independent":

$$P(A, B) = P(A \text{ and } B) = P(A \cap B) = P(A)\,P(B)$$

the general rule... if $A_1, A_2, \ldots$ are independent...

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \prod_{i=1}^{\infty} P(A_i)$$

If $P(A|B)$ is $P(A^c|B)$?
$\stackrel{?}{=} P(A^c)$
$= 1 - P(A|B) = 1 - P(A)$ ✓

are disjoint events independent?

H, T are disjoint
$P(H \text{ if } T) = 0 \neq \frac{1}{2} = P(H)$

"A dependent coin"

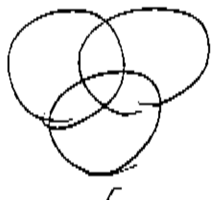
H, T correlated to be the same

$P(H_1) = \frac{1}{2}$, $P(H_2) = \frac{1}{2}$ $\quad P(H_1 H_2) = \frac{1}{2} \neq \frac{1}{4}$ $\quad (\geq)$ next class...

dependent... $P(\text{lung cancer} \mid \text{smoking}) \neq P(\text{lung cancer})$

$A \subset \Omega$ (F) $A = \Omega$

A B $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



C

$\Omega_E = \{9\diamondsuit, 10\diamondsuit, \dots, A\diamondsuit, 9\clubsuit, 10\clubsuit, \dots, A\clubsuit,$
$9\spadesuit, 10\spadesuit, \dots, A\spadesuit, 9\heartsuit, 10\heartsuit, \dots, A\heartsuit\}$

5 Euchre cards ordered $_{24}P_5$

$$\bigcup_{i=1}^{\infty} [0, \tfrac{1}{2^i}] = [0, \tfrac{1}{2}] \cup [0, \tfrac{1}{4}] \cup \dots = [0, \tfrac{1}{2}]$$

$$\bigcap_{i=1}^{\infty} [0, \tfrac{1}{2^i}] = (0, \tfrac{1}{2}] \cap [0, \tfrac{1}{4}] \cap \dots$$

$$= \lim_{i \to \infty} (0, \tfrac{1}{2^i}] = (0, 0] = \{0\}$$

$\mathbb{R} \setminus \mathbb{Q}$ irrationals... $\pi \in \mathbb{R} \setminus \mathbb{Q}$ $0, 1, -1, \emptyset$?

R, S if $R \geq 4$

$\Omega = \{1, 2, 3, \langle 4,1\rangle, \langle 4,2\rangle, \dots, \langle 4,8\rangle, \langle 5,1\rangle, \langle 5,2\rangle, \dots, \langle 5,8\rangle,$
$\langle 6,1\rangle, \langle 6,2\rangle, \dots, \langle 6,8\rangle\}$, $|\Omega| = 27$

But $P(\{1\}) = \frac{1}{6} \neq \frac{1}{27}$ Not eq. likely