MATH 241 Fall 2014 Homework #4

Professor Adam Kapelner

Due 5PM in my office, Tues Sept 30, 2014

(this document last updated Wednesday 15^{th} October, 2014 at 9:06am)

Instructions and Philosophy

Once again, the path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out". Feel free to "work out" with others; I want you to work on this in groups.

Reading is still *required*. For this homework set, please read the conditional probability section of Chapter 1 of Hogg, Tanis and Zimmerman (H, T & Z) as well as the random variables section of Chapter 2.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]"; and purple problems are for *extra credit* which are also marked "[E.C.]." The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems.

This homework is worth 100 points but the point distribution will not be determined until after the due date. Late homework will be penalized 10 points per day up to a maximum of three days.

15 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You may also use `writelatex.com` which is a web service (you don't have to install or configure anything on your local computer). If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. You are free to handwrite on separate paper if you wish but I STRONGLY recommend to write on a printout of this document since you will always have the questions handy to study from (and it is easier for me to grade accurately). Keep this page printed for your records. Write your name and section below where section A is if you're registered for the 9:15AM–10:30AM lecture and section B is if you're in the 12:15PM-1:30PM lecture.

NAME: ____________________________________ SECTION (A or B): ______

Conditional Probability We will solve more problems using conditional probability.

Problem 1

We will follow up here with questions on the Monte Hall game.

(a) [easy] In class, we used the Bayes Theorem (the law of total probability with many applications of Bayes Rule) to show that if you pick door 1 and door 2 opens, then the probability that the car is in door 3 is 2/3. Repeat that calculation here.

(b) [harder] Now imagine a variant of the game is played in the following way: there are four doors, you pick one and the game show host opens up two doors to reveal two goats. You now have the option to keep the door you selected initially or switch to the other door that remains closed. What is the probability of winning if you switch? You can use Bayes Theorem as in (a) or draw a tree like we did in class.

(c) [difficult] [OPTIONAL] Imagine the variant where there are now n doors. You choose 1 and the game show host opens up $n-2$ doors to reveal $n-2$ goats. You have the option to keep the door you selected initially or switch to the other closed door. What is the probability of winning if you switch?

Trees We will solve problems using trees and introduce the Bernoulli random variable.

Problem 2

You play a game with your friend. You both roll a die. Whoever rolls higher wins. If you roll the same number, you tie.

(a) [easy] What is the probability you tie?

(b) [easy] What is the probability you win? Draw a tree to figure this out. The first branch is the numerical value of your roll, the second branch is whether you Win (W), Tie (T) or Lose (L).

(c) [harder] Imagine upon ties, the game continues: you both roll again. You play until someone has a higher roll than the other. What is the probability you win this game? Use the algebraic trick we talked about in class.

(d) [E.C.] What are fair odds on the following game? Consider the same game as described above with one rule change: your friend automatically wins if you both tie on rolling a 1 and a 1.

Random Varibles We now begin question about the second unit of this class: r.v.'s.

Problem 3

In this problem, you use coins to create randomness. Your goal is to design a random variable X which is distributed in the following way:

$$X \sim \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

which we discussed in class is called a "Bernoulli random variable" with *parameter* p and is denoted $X \sim \text{Bernoulli}\,(p)$. Any number of coin flips are allowed and any arbitrary rules to create outcomes is allowed. For instance, flip 3 coins and if ω =HTH is observed, let $X(\omega) = 1$, etc. Feel free to describe your system using a tree structure or if it is very simple, you can describe it *in English*.

(a) [easy] Design a coin flipping system which yields $p = \frac{1}{2}$.

(b) [easy] Design a coin flipping system which yields $p = \frac{1}{4}$.

(c) [easy] Design a coin flipping system which yields $p = \frac{1}{8}$.

(d) [easy] Design a coin flipping system which yields $p = 0$.

(e) [easy] Design a coin flipping system which yields $p = 1$.

(f) [difficult] Design a coin flipping system which yields $p = \frac{1}{3}$.

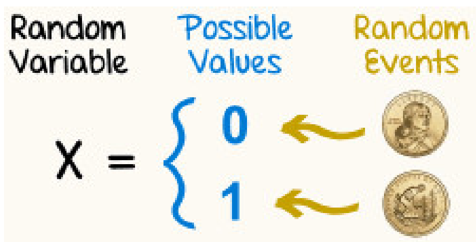
(g) [E.C.] Design a coin flipping system which yields $p = \frac{1}{7}$.

(h) [E.C.] Design a coin flipping system which yields $p = \frac{1}{5}$.

(i) [E.C.] Prove or disprove that you can design a coin flipping system which yields an arbitrary $p \in \mathbb{Q} \cap [0,1]$. Disclaimer: I tried this problem for a half hour and could not solve it.

Problem 4

In class we spoke about how random variables map outcomes from the sample space to a number *i.e.* $X : \Omega \to \mathbb{R}$. That is they are set functions, just like the probability function which is $\mathbb{P} : \Omega \to [0,1]$. We will be investigating this concept here.



(a) [easy] In the previous problem, you used coins to design the random variable $X \sim$ Bernoulli $\left(\frac{1}{2}\right)$ using the Ω from a coin flip. There are other ways to create X. For instance: roll a die and map outcomes 1,2,3 to 0 and outcomes 4,5,6 to 1. This works because

$$\mathbb{P}(X=0) = \mathbb{P}(\{\omega : X(\omega)=0\}) = \mathbb{P}(\{1\} \cup \{2\} \cup \{3\}) = 1/2 \text{ and}$$
$$\mathbb{P}(X=1) = \mathbb{P}(\{\omega : X(\omega)=1\}) = \mathbb{P}(\{4\} \cup \{5\} \cup \{6\}) = 1/2.$$

Describe three other scenarios or devices that produce their own Ω's that also result in $X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$

(b) [harder] We talked about in class how the sample space no longer needs to be considered once the random variable is described. Why? Use your answer to (a) to inspire this answer. Write it *in English* below.

(c) [difficult] Back to philosophy... Let's say X models the price difference that IBM stock moves in one day of trading. For instance, if the stock closed yesterday at \$56.24 and today it closed at \$57.24, the random variable would be \$1 for today. According to our definition of a random variable, there is a sample space with outcomes being drawn ($\omega \in \Omega$) that is "controlling" the value of X. Describe it the best you can *in English*. There are no right or wrong answers here, but your answer must be coherent and demonstrate you understand the question.

Problem 5

We will now study probability mass functions (PMF's) denoted as $f(x)$ and cumulative distribution functions (CDF's) denoted as $F(X)$ and review the r.v.'s we did in class.

(a) [easy] Draw the PMF for $X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$.

(b) [easy] Draw the PMF for $X \sim \text{Bernoulli}\left(p\right)$.

(c) [easy] Draw the CDF for $X \sim \text{Uniform}\left(\{1, 3, 4, 9\}\right)$.

(d) [easy] Using the r.v. from the previous question, what is $\mathbb{P}\left(X \in (3, 9)\right)$?

(e) [easy] Take a r.v. X with $\text{Supp}\,(X) = [0, 1]$. Is this a "discrete r.v.?" Explain.

(f) [difficult] In class we defined the Bernoulli r.v. as:

$$X \sim \begin{cases} 1 & \text{w.p. } \; p \\ 0 & \text{w.p. } \; 1-p \end{cases}$$

but we did not put its PMF on the board. Write $f(x)$ for $X \sim \text{Bernoulli}\,(p)$ that is only valid on $\text{Supp}\,(X)$.

(g) [E.C.] Write a PMF for $X \sim \text{Bernoulli}\,(p)$ that is valid always $\forall x \in \mathbb{R}$.

(h) [difficult] Sometimes knowing the Ω matters a little bit. Let's say $X_1 \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ is generated from one coin and $X_2 \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ is generated from another coin independently tossed. Create a new r.v. $T = X_1 + X_2$. Describe its PMF using the $\sim$ notation like in the previous problem.

(i) [difficult] It may seem to you that we use F for CDF and f for PMF because there's a connection to derivatives and anti-derivatives. We are going to see if this is true here? Consider the PMF we discussed for $X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$. Does $\int f(x)\,\mathrm{d}x = F(x) + C$ where the constant $C \in \mathbb{R}$? Explain.

(j) [difficult] How about the opposite? Consider the CDF we discussed for $X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$. Does $\mathrm{d}/\mathrm{d}x[F(x)] = f(x)$? Explain.

Problem 6

We now combine the concept of recursing trees with random variables. This r.v. is called the geometric r.v. and will not be tested on the upcoming midterm. I do expect you to be able to do this type of reasoning on the midterm though.

Imagine you play a game. You flip a coin with the probability of heads is not necessarily fair. Define $p := \mathbb{P}(\{H\})$. If you get heads, you stop. If you get tails, you keep flipping.

(a) [easy] How many flips of the coins can you potentially flip? Describe the set of number of flips that could occur. Use the notation from lecture 1.

(b) [easy] What is the probability you flip three times? That is two tails and a head.

(c) [harder] What is the probability you flip x times?

(d) [easy] Let X be the r.v. that models the number of flips. What is the $\text{Supp}(X)$? Is it finite?

(e) [easy] What is the parameter space for p?

(f) [difficult] Find the PMF for X. This PMF only has to be valid $\forall x \in \text{Supp}(X)$.

(g) [difficult] Find the CDF for X.

(h) [harder] In class we spoke about the two properties of PMFs. The first property is that $f(x) \in [0, 1]$. Show that this is true $\forall x \in \text{Supp}(X)$ for this r.v.

(i) [difficult] In class we spoke about the two properties of PMFs. The second property is that

$$\sum_{x \in \text{Supp}(X)} f(x) = 1$$

which is similar to the concept in probability of $\mathbb{P}(\Omega) = 1$. Show that this holds for this r.v.

(j) [easy] In class we spoke a lot about "successes" and "failures." In English, explain the resulting outcomes of this random variable using the words "successes" and "failures."