HALF OF MATH 241 Fall 2014 Homework #10

Professor Adam Kapelner

Due 5PM in my office, Tues Nov 25, 2014

(this document last updated Thursday 20$^{\text{th}}$ November, 2014 at 11:36am)

Instructions and Philosophy

Once again, the path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out". Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, please start reading Chapter 3. Avoid the parts that deal with "moment generating functions" for now (we will get to this soon enough).

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]"; and purple problems are for *extra credit* which are also marked "[E.C.]." The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems.

This homework is worth 100 points but the point distribution will not be determined until after the due date. Late homework will be penalized 10 points per day.

15 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You may also use `writelatex.com` which is a web service (you don't have to install or configure anything on your local computer). If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. **Handing it in without the printout incurs a penalty of 10 points.** Keep this page printed for your records. Write your name and section below where section A is if you're registered for the 9:15AM–10:30AM lecture and section B is if you're in the 12:15PM-1:30PM lecture.

NAME: ______________________________ SECTION (A or B): ______

Fundamentals of Continuous r.v.'s We will learn about this other type of r.v.

Problem 1

This problem will focus on the continuous exponential r.v. and you will see how it's built from the discrete geometric r.v.

(a) [easy] Let $X \sim \text{Geometric}\,(p)$ and use t to indicate the free variable. In each unit of time, we have n experiments now. Write the PMF for this r.v.

(b) [easy] Let $n \to \infty$ and $p \to 0$ but keep thier product pinned at the constant $\lambda = np$. Show that the PMF of this new r.v. T is zero everywhere.

(c) [harder] Find the CDF of T by taking the same limit as the last problem.

(d) [easy] Let $\lambda = 2.92$. What is $\mathbb{P}\,(T = 2)$?

(e) [easy] Let $\lambda = 3.12$. What is $\mathbb{P}\,(T \leq 2)$?

(f) [easy] Let $\lambda = 4.56$. What is $\mathbb{P}\,(T \in [2, 2.7])$?

(g) [easy] What is $\text{Supp}[T]$?

(h) [harder] What is $|\text{Supp}[T]|$? That is, what is the size of this set?

(i) [easy] What is the parameter space of T?

(j) [easy] Run the following in `R`. It will generate 5 realizations from $T_1, \ldots, T_t \overset{iid}{\sim} \text{Exp}(\lambda = 4.56)$: `rexp(5, 4.56)` and write them below.

(k) [easy] Look at the first draw. Is this number really a draw? Or is it rounded?

(l) [difficult] Assume it's rounded from the decimal after the last decimal you see. Find the probability the computer spits out that number when realizing the r.v.

(m) [easy] Derive the PDF $f(t)$ from the CDF.

(n) [easy] Let $\lambda = 4.56$. Compute $f(0.1)$ using the PMF and $f(0.1)$ using the PDF.

(o) [harder] Interpret the PDF at 0.1, $f(0.1)$. What does that number mean?

(p) [harder] In the last problem you got an answer greater than 1. Why should it be possible that the PDF can yield numbers greater than 1?

(q) [difficult] Derive the CDF from the PDF. This will involve anti-differentiation. And you have to worry about the constant of integration and solve for it (somehow). Justify how you solve for the constant to arrive at the same CDF you found in (c).

(r) [easy] Run the following lines in `R` *one at a time* which will plot the PDF for $T \sim \text{Exp}(0.1)$, $T \sim \text{Exp}(1)$, $T \sim \text{Exp}(10)$ and $T \sim \text{Exp}(100)$. Print out this image and attach it to your homework.

```
par(mfrow = c(2, 2))
ts = seq(0, 4, 0.01)
plot(ts, dexp(ts, 0.1), type = "l", ylim = c(0, 1))
plot(ts, dexp(ts, 1), type = "l", ylim = c(0, 1))
plot(ts, dexp(ts, 10), type = "l", ylim = c(0, 1))
plot(ts, dexp(ts, 100), type = "l", ylim = c(0, 1))
#last line placeholder
```

How do you design an exponential r.v. to give large numbers — should λ be big or small?

(s) [easy] Let $\lambda = 4.56$, compute $\mathbb{P}(T \in [0, 1])$ using integration on the PDF.

(t) [easy] Let $\lambda = 4.56$, compute $\mathbb{P}(T \in [0, 1])$ using a difference of CDF values.

(u) [easy] What theorem describes why the last two problems should be equal? Write the statement of this theorem below.

(v) [easy] Let $\lambda = 4.56$, what is $\mathbb{P}(T = 1)$ using the integral definition of probability?

(w) [difficult] Let $T \sim \text{Exp}(\lambda)$. Show that $\mathbb{E}[T] = 1/\lambda$ using the definition of expectation for continuous r.v.'s. Infinitely googlable.

(x) [easy] Let's say $\lambda = 2$, what is $\mathbb{E}[T]$? Pretend the unit of time is seconds.

(y) [difficult] Pretend you are approximating the exponential with a geometric r.v.. Let $n = 1000$ and $p = 0.002$ to have $\lambda = 2$. Show that the expectation of that geometric r.v. is the same as $\mathbb{E}[T]$ in the previous problem.

(z) [E.C.] Let $T \sim \text{Exp}(\lambda)$. Show that $\mathbb{V}\text{ar}[T] = 1/\lambda^2$ using the definition of expectation for continuous r.v.'s. Infinitely googlable.

(aa) [easy] Let $T \sim \text{Exp}(\lambda)$. Show that $\mathbb{S}\text{E}[T] = 1/\lambda$. If you can't do the last question, do this one by assuming the answer of the last question.

(bb) [difficult] For a discrete r.v., we defined the mode as:

$$\text{Mode}[X] := \underset{x \in \text{Supp}[X]}{\arg\max} \{f(x)\}$$

where $f(x)$ was the PMF. For continuous r.v.'s, keep the definition the same but replace the PMF with the PDF. Using this definition, find the mode of $T \sim \text{Exp}(\lambda)$. Does this make sense and why?

(cc) [difficult] Let $T \sim \text{Exp}(\lambda)$. Show that the median $[T] = \ln(2)/\lambda$.

(dd) [easy] Let's say in one unit of time, the number of "events" is distributed as $X \sim \text{Poisson}(\lambda)$. How is the *inter-arrival* time between the events distributed? That is, how long do you wait in between events? You just need to say $T \sim$ something below.

(ee) [harder] Prove the memorylessness property of the geometric r.v.

(ff) [harder] Prove the memorylessness property of the exponential r.v.

(gg) [E.C.] We previously discussed the convolution for discrete r.v.'s. The convolution of two continuous r.v.'s occurs when you are trying to find the density of $T = X_1 + X_2$. The formula is below:

$$f_T(x) = f_{X_1}(x) * f_{X_2}(x) := \int_{\mathbb{R}} f_{X_1}(s) f_{X_2}(x-s)ds$$

For X_1 and X_2 only supported on $[0, \infty)$ such as the exponential r.v., the formula becomes:

$$f_T(x) = f_{X_1}(x) * f_{X_2}(x) = \int_0^x f_{X_1}(s) f_{X_2}(x-s)ds$$

We know that when adding waiting time in the discrete case, the sums of $r \overset{iid}{\sim}$ Geometric (p) r.v.'s are distributed as a NegBin $(r,\, p)$.

Prove that the sum of $\overset{iid}{\sim}$ Exp (λ) r.v.'s are distributed as an Erlang(r, λ) which is the continuous analogue of the discrete negative binomial distribution.

That is, prove the convolution of r $\overset{iid}{\sim}$ Exp (λ) has the Erlang "footprint," defined by its density:

$$f_T(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$$

You should use induction.