Lecture 17 Nov 20, 2014

$E(X+Y) = E(X) + E(Y) \quad \forall X, Y$

$Var(X+Y) = Var(X) + Var(Y) \quad \forall X, Y$ ind.

$E[aX+b] = aE(X) + b$

Same rules apply

Previously, $X \sim Geom(p)$

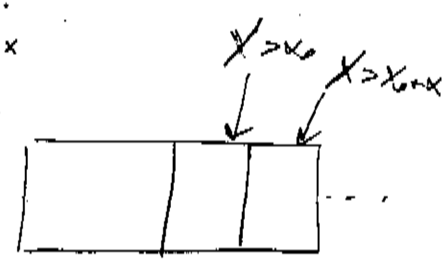
$$P(X = x) = P(X = x_0 + x \mid X > x_0) \quad \text{(memorylessness property)}$$

equivalent to:

$$P(X > x) = P(X > x_0 + x \mid X > x_0)$$

Now, $X \sim Exp(\lambda) := \lambda e^{-\lambda x}$, $F(x) = 1 - e^{-\lambda x}$

$\Rightarrow 1 - F(x) = e^{-\lambda x}$



$B \subset A \Rightarrow A \cap B = B$

W.T.S. $P(X > x) \overset{?}{=} P(X > x_0 + x \mid X > x_0)$

$$\Rightarrow 1 - F(x) = \frac{1 - F(x_0 + x)}{F(x_0)} \longrightarrow \frac{P(X > x_0 + x)}{P(X > x_0)}$$

$$\Rightarrow e^{-\lambda x} = \frac{e^{-\lambda(x_0 + x)}}{e^{-\lambda x_0}} = \frac{\cancel{e^{-\lambda x_0}} e^{-\lambda x}}{\cancel{e^{-\lambda x_0}}} = e^{-\lambda x} \checkmark$$

HW Hint

$$E[X] := \int_{x \in Supp(X)} x f(x)\, dx = \int_0^\infty x \lambda e^{-\lambda x}\, dx = \lambda \int_0^\infty x e^{-\lambda x}\, dx$$

let $u = x$, $dv = e^{-\lambda x}\, dx$

and use the fact $\int_a^b u\, dv = uv - \int_a^b v\, du$ integration by parts

$Var[X]$ ... same idea (E.C.)

Next "Brand Name" cont. r.v.

$X \sim U(a, b)$ idea: all #'s between $[a,b]$ are "equally likely" to be realized

Called the "uniform r.v." — it's the continuous analogue of the "discrete uniform r.v."

$Supp[X] = [a, b]$ Param space: $a \in \mathbb{R},\ b \in \mathbb{R}$

think about PDF, let $a = 0, b = 1$ so called std. uniform r.v.

very important in CS

Math.random(); Java

What is $P(X \in [0, 0.5])$? $= \frac{1}{2} = \int_0^{\frac{1}{2}} f(x)\,dx$

$P(X \in [0.4, 0.6]) = 0.2 = \int_{0.4}^{0.6} f(x)\,dx$

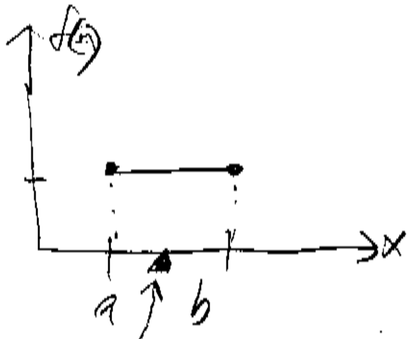
$\vdots$

$$\Rightarrow \int_{x_1}^{x_2} f(x)\,dx = \frac{x_2 - x_1}{1} \longleftarrow \text{length of support}$$

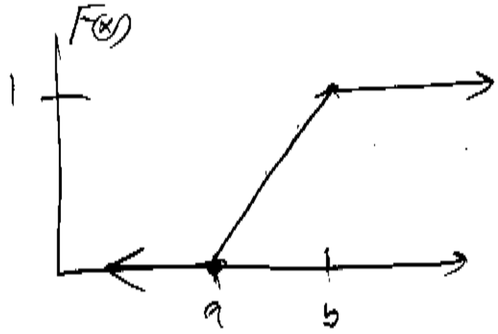
$X \sim U(a, b)$

$$\int_{x_1}^{x_2} f(x)\,dx = \frac{x_2 - x_1}{b - a} \Rightarrow f(x) = \frac{1}{b-a} \quad \forall x \in Supp[X] = [a, b]$$

$$= (x_2 - x_1)\left(\frac{1}{b-a}\right)$$



$\Rightarrow$



$F(x) = \int \frac{1}{b-a}\,dx = \left(\frac{1}{b-a}\right)x + C \Rightarrow$ what is C? We know $F(a)=0$

$$\Rightarrow \frac{a}{b-a} + C = 0 \Rightarrow C = -\frac{a}{b-a}$$

Also $F(b)=1 \Rightarrow \frac{b}{b-a} + C = 1 \Rightarrow C = 1 - \frac{b}{b-a} = -\frac{a}{b-a}$

$$\Rightarrow \boxed{F(x) = \frac{x-a}{b-a}}$$

Useful? Q64 bus runs every 15 min. in the morning. You show up at the busstop. What is P(you wait more than 10 min)?

$T \sim U(0,15)$ $\quad P(T>10) = 1 - P(T \le 10) = 1 - F(10) = 1 - \frac{10-0}{15-0} = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$

$X \sim U(a,b)$

$$E[X] = \int_{x \in Supp(x)} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{a+b}{2}$$

What's your intuition?

$$Var[X] = \int_{x \in Supp(x)} (x-\mu)^2 f(x)\,dx = E(X^2) - \mu^2 = \int_a^b x^2 f(x)\,dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a}\,\frac{b^3-a^3}{3} - \frac{(a+b)^2}{4} = \cancel{\frac{1}{b-a}}\,\frac{\cancel{(b-a)}(a+b)^2}{3} - \frac{(a+b)^2}{4}$$

$$= (a+b)^2\left(\frac{1}{3} - \frac{1}{4}\right) = \boxed{\frac{(a+b)^2}{12}}$$

This is actually (b-a)^2 / 12... and the proof is wrong...
See Google for a proof. This will not be tested on the final since
I made this mistake

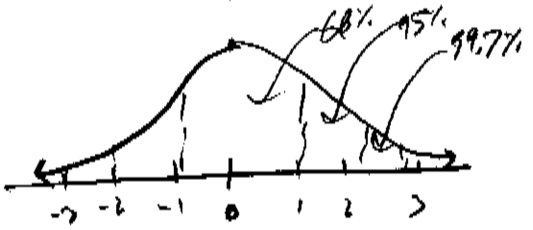
Last, $Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $Supp[Z] = \mathbb{R}$ (everywhere!!)

"the standard normal" distribution

Is this a PDF?

① $f(x) > 0 \ \forall x \in \mathbb{R}$? Yes $e^{-x^2} > 0 \ \forall x$ $\quad e^{-u^2} \to$ the "Gaussian function"

② $\int f(x)dx = 1$? $\quad \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \overset{?}{=} 1 \Rightarrow \int_{\mathbb{R}} e^{-\frac{x^2}{2}} = \sqrt{2\pi}$ ← Famous! The Gaussian integral or Euler-Poisson integral



This is true ... the question...

$$E[Z] = \int_{\mathbb{R}} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

let $u = \frac{-x^2}{2} \Rightarrow \frac{du}{dx} = -x \Rightarrow dx = -\frac{du}{x}$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x e^{u} \cdot -\frac{du}{x} = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{u} du = -\frac{1}{\sqrt{2\pi}} \left[e^{u}\right]_{-\infty}^{0} = \frac{1}{\sqrt{2\pi}} \left[-e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}(0-0) = 0$$

$$Var[Z] = \int_{\mathbb{R}} (x - E[Z])^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} dx$$

$x = u, \ dv = x e^{-\frac{x^2}{2}} \Rightarrow dx = du, \ v = \int x e^{-\frac{x^2}{2}} dx = -e^{-\frac{x^2}{2}}$

$$= \frac{1}{\sqrt{2\pi}} \left(\left[x e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} + \underbrace{\int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx}_{\sqrt{2\pi}} \right) = \frac{1}{\sqrt{2\pi}}(0 + \sqrt{2\pi}) = 1$$

$$\Rightarrow SE[Z] = 1$$

Which is why it's called the standard normal...

$\Phi(x) := F_Z(x) := P(Z \le x)$ ... what is $\Phi(0)$?

Since $f(x)$ symmetric, $f(x) = f(-x)$, expectation $= 0$, immediate $\Phi(0) = 0.5$

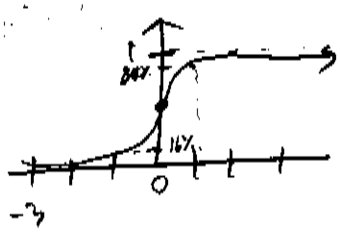
Three integrals to memorize

$$P(Z \in [-1,1]) = F(1) - F(-1) = \int_{-1}^{1} f(x)dx \approx 0.68$$

$P(Z \in [-2,2]) \approx 0.95$

$P(Z \in [-3,3]) \approx 0.997$ (must be memorized)

AKA "68-95-99.7 rule" or the "3σ rule" or the "empirical rule"



mark all here

$P(|Z|>1) \approx 32\% \rightarrow P(Z<-1)=16\%$, $P(Z>1)=16\%$

?

$P(|Z|>2) \approx 5\% \rightarrow P(Z>2)=2.5\%$, $P(Z<-2)=2.5\%$

$P(|Z|>3) \approx 0.3\%$

$X \sim \text{Binomial}(n,p)$, is $2X \sim \text{Binomial}$? No $\text{Supp}[2X]=\{0,2,4,\dots,2n\}$

Supp of binomial $\{0,\dots,n\}$

$2X \not\sim$ a binomial r.v. distribution...

How about

$X \sim \text{Exp}(\lambda)$, let $Y=2X$

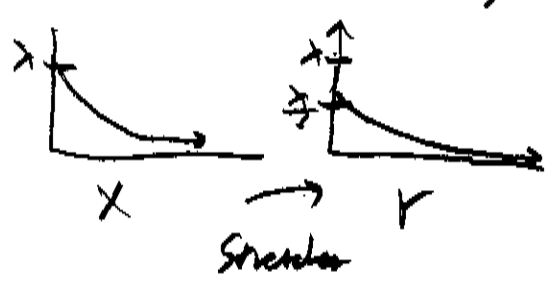
$$F_Y(x) = P(Y\le x) \underset{\text{by def}}{=} P(2X \le x) \underset{\text{algebra}}{=} P\left(X \le \tfrac{x}{2}\right) \underset{\text{by def}}{=} F_X\left(\tfrac{x}{2}\right) \Rightarrow f_Y(x) = \tfrac{1}{2} f_X\left(\tfrac{x}{2}\right)$$

it's closed

by def of PDF

$$= \tfrac{1}{2}\lambda e^{-\lambda \frac{x}{2}}$$

$$= \tfrac{\lambda}{2} e^{-\frac{\lambda}{2}x} = \text{Exp}\left(\tfrac{\lambda}{2}\right)$$



Kind of cool...
(not on test)

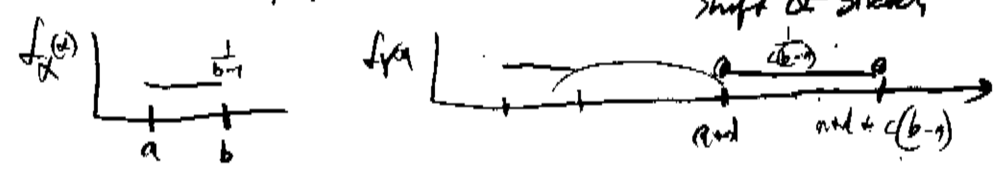
$X \sim U(a,b)$ $Y = cX+d$

$a = \frac{x-d}{c}$, $b = \frac{x-d}{c}$ $\Rightarrow a' = d+ac$, $b' = d+bc$

$$F_Y(x) = P(Y \le x) \underset{\text{by def}}{=} P(cX+d \le x) = P\left(X \le \tfrac{x-d}{c}\right) = F_X\left(\tfrac{x-d}{c}\right) \Rightarrow f_Y(x) = \tfrac{1}{c} f_X\left(\tfrac{x-d}{c}\right) =$$

$$\frac{1}{c(b-a)} \quad \forall x \in [d+ac, d+cb]$$

Shift & Stretch



$Y \sim U(d+ac, d+cb)$

$Z \sim N(0,1)$, $X = \mu + \sigma Z$ $\quad E[X] = E[\mu+\sigma Z] = \mu + \sigma \overset{\nearrow 0}{E[Z]} = \mu$

$Var[X] = Var[\mu+\sigma Z] = Var(\sigma Z) = \sigma^2 \overset{\nearrow 1}{Var[Z]} = \sigma^2$

but what is its distr?

$$F_X(x) = P(X \le x) = P(\mu+\sigma Z \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right) = F_Z\left(\frac{x-\mu}{\sigma}\right) \Rightarrow \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = f_X(x)$$

$f_X(x) > 0 \quad \forall x \in Supp[X] = \mathbb{R}$, $\int_{\mathbb{R}} f(x)\,dx = 1$ just make u subst.

$\Rightarrow f_X(x)$ is a PMF

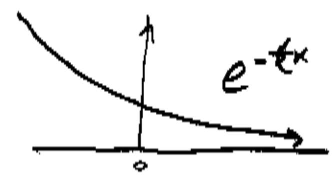
$X \sim N(\mu, \sigma^2)$ the "normal density" → can have arbitrary expectation & variance.

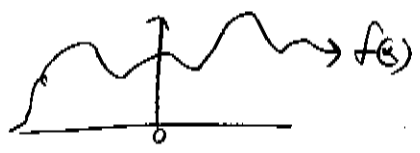
Who cares about the normal density? (r.v. w/ it) We need ... moment generating function first

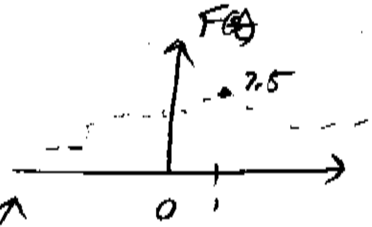
↓ Lec 18

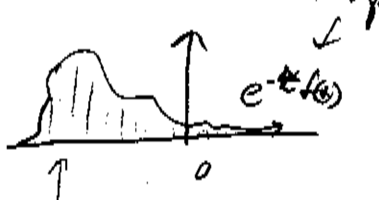
$$F(t) := \mathcal{L}[f(x)] := \int_{\mathbb{R}} e^{-tx} f(x)\,dx \longleftarrow$$ integrates out x to leave just a function of s

Bilateral Laplace Transform









let $t=1$

Repeat for all $s \in \mathbb{R}$ to get a full fn of s

$$\int_{\mathbb{R}} e^{-tx} f(x) = 7.5$$