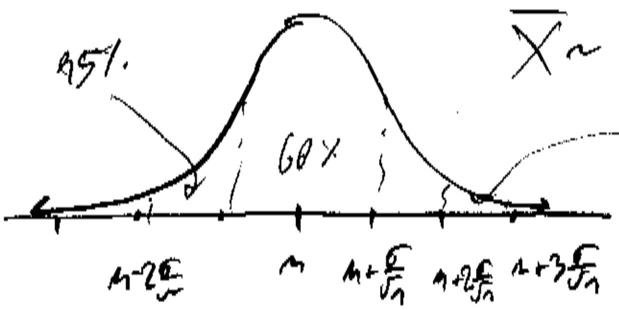
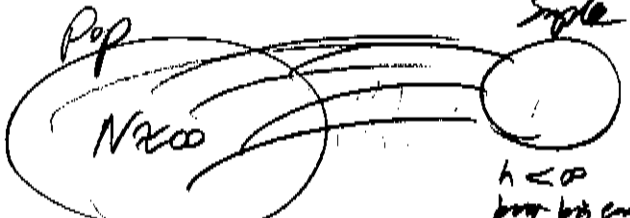
Lecture 20 Dec 4, 2014

Recall

by CLT

$$\bar{X} \sim N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$$

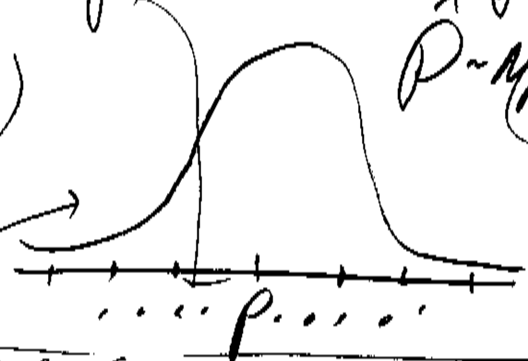




possible realization

by CLT

$$\hat{P} \sim N\left(p, \left(\sqrt{\frac{p(1-p)}{n}}\right)^2\right)$$



$T_n \sim \text{Binom}(n = 594, p)$

$n = 594$ # blues $= 116 = \sum \mathbb{1}_{x_i = 1}$ — we don't know p, the param. use $\hat{p}$ as its proxy!

$$\hat{p} = \frac{\sum \mathbb{1}_{x_i=1}}{n} = \frac{116}{594} = 0.1953$$

get normal → compute $\hat{p}$ drawn from $\hat{P}$

Imagine if we construct intervals.. "margin of error"

$$\left[\hat{p} - \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right] =: \left[\hat{p} \pm \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$P(|Z| > 3) \approx \frac{1}{1000}$

$P(|Z| > 4) \approx \frac{6}{100{,}000}$

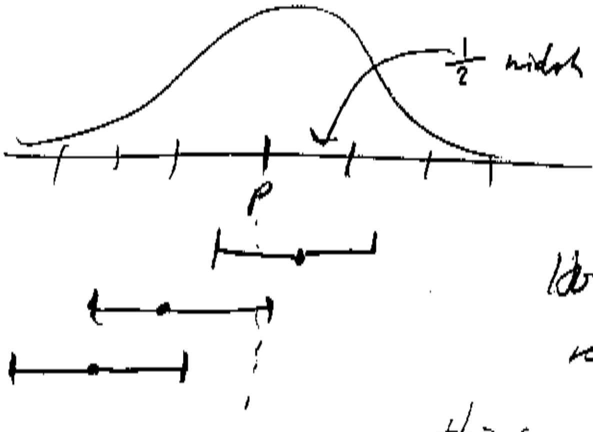
$P(|Z| > 5) \approx \frac{6}{10 \text{ million}}$

$P(|Z| > 6) \approx \frac{2}{1 \text{ billion}}$

$P(|Z| > 7) \approx 3/\text{trillion}$

$P(|Z| > 8) \approx 1/\text{quadrillion}$

$P(|Z| > 9) \approx$ R fails.. too small.



How often do I "capture" the real p in this interval?

this covers as long as $\hat{p}$ is within one SD.

$$\Rightarrow P(\text{coverage}) = P\left(\hat{p} \in \left[p - \sqrt{\frac{p(1-p)}{n}}, p + \sqrt{\frac{p(1-p)}{n}}\right]\right) = P\Big(\underbrace{\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}}}_{Z} \in [-1, 1]\Big)$$

$$= P(|Z| \leq 1) = 68\%$$

68% is small..

Imagine you consider a more general interval.

$$\left[\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right] \rightarrow P(p \in \text{an interval})$$

Coverage? $\mathbb{P}\left(\hat{P} \in \left[p - z_{\frac{\alpha}{2}}\sqrt{\frac{p(1-p)}{n}},\ p + z_{\frac{\alpha}{2}}\sqrt{\frac{p(1-p)}{n}}\right]\right)$

$$= \mathbb{P}\left(\frac{\hat{P}-p}{\sqrt{\frac{p(1-p)}{n}}} \in \left[z_{\frac{\alpha}{2}}, z_{\frac{\alpha}{2}}\right]\right) = \mathbb{P}\left(|Z| \le z_{\frac{\alpha}{2}}\right)$$

if $z_{\frac{\alpha}{2}} = 2 \Rightarrow$ Coverage is 95%

if $z_{\frac{\alpha}{2}} = 3 \Rightarrow$ Coverage is 99.7%

Select a coverage you are comfortable with $1-\alpha$ where "α" is the prob of <u>no</u> coverage

Solve $1 - \Phi(z_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$ ✓ need $\frac{\alpha}{2}$ on left and $\frac{\alpha}{2}$ on right

e.g. $\alpha = 5\% \Rightarrow 1-\alpha = 95\%$ coverage

Need 2.5% in the right tail. So calculate

$$2.5\% = \int_{z_{2.5\%}}^{\infty} f(x)\,dx \quad \text{------} \quad \approx 2$$

PDF std. norm

e.g. $\alpha = 1\% \Rightarrow 1-\alpha = 99\%$ coverage Need 0.5% on right tail.

$$0.5\% = \int_{z_{0.5\%}}^{\infty} f(x)\,dx \quad \text{----} = 2.56$$

Define

usually bad using $\sqrt{\frac{p(1-p)}{n}}$. Assume

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \approx \sqrt{\frac{p(1-p)}{n}}$$

Ph.D. theses have been written about this, but will assume it's golden...

$$CI_{p,1-\alpha} := \left[\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

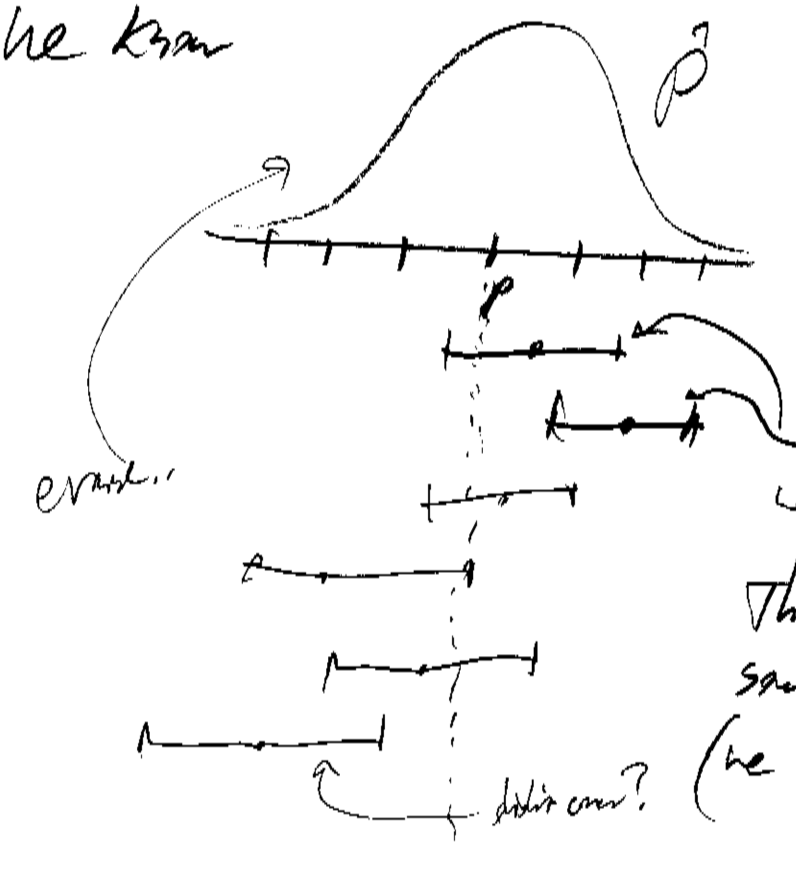
to be a $(1-\alpha)$ "two-sided" confidence interval (CI) for the parameter p, the "true" binomial proportion.

Why do we care? There is a true p parameter. We do not know it! We use $\hat{p}$ to estimate p.

$E[\hat{p}] = p$. So at least we're on the right track.

But $SE[\hat{p}] = \sqrt{\frac{p(1-p)}{n}}$ which means $\hat{p} \neq p$ exactly due to variance.

We know



but $\hat{p}$ can come from anywhere on this distribution!

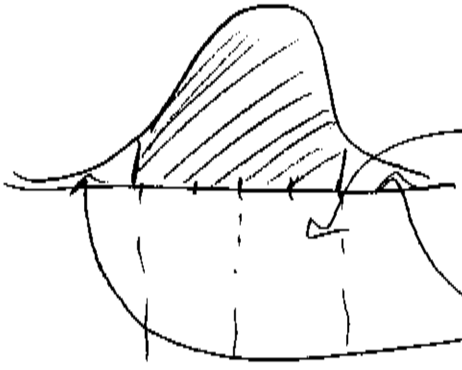
And we don't know p. All we see is

we don't even see (- - -)!

This is the blessing (we know something about p) and the curse (we may not cover p)!

(4)

Remember Coverage = $P(p \in$ your interval$)$. Can you see it from this picture?



$z_{\frac{\alpha}{2}} = 2 \Rightarrow 1-\alpha \overset{=}{} 95\%$ coverage

if $\hat{p}$ lands here, it covers

if $\hat{p}$ lands here it does not cover.

Interpretations

(1) If many samples are taken, a $1-\alpha$ two-sided CI will "capture" the true pop. proportion p, $1-\alpha$ of the time. "Repeated experiment" interpretation

(2) Before the experiment, the $1-\alpha$ two-sided CI has a $1-\alpha$ prob of capturing the true pop. proportion.

Problem

(1) It does not mean for a given interval that there is a $1-\alpha$ prob it covers p. Imagine $P(p \in [0.41, 0.59])$. The prob statement must be made about a r.v.! p is a constant. Does not make sense. It's degenerate, either 0 or 1.

$\Rightarrow$ CI's do not give you what you really want!!!

$1-\alpha$ Confidence $\neq$ $1-\alpha$ probability!! "Confidence" is a technical term and it only means one of these 2.

Give a 95% CI for p, the true prop. of blue M&Ms.

$Z_{2.5\%} = 2$

$$CI_{p, 95\%} := \left[\hat{p} \pm Z_{2.5\%}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$$= \left[0.1953 \pm 2 \cdot \sqrt{\frac{0.1953(1-0.1953)}{594}}\right]$$

$$= [.1628, .2278]$$

Interpret: If many sampled experiments were made, this interval captures p 95% of the time

Give a 99% CI for p, the ~~true~~ prop. of blue M&M's.

$Z_{0.5\%} = 2.56$

$$CI_{p, 99\%} = \left[\hat{p} \pm Z_{0.5\%}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$$= \left[0.1953 \pm 2.56\sqrt{\frac{0.1953(1-0.1953)}{594}}\right]$$

$$= [.1537, .2369]$$

Interpret!

Idea of CI's: don't know p, make inference ~~for~~ on it using $\hat{p}$. Now... what if you hypothesize p is a certain value, and you want to see if your sampling is consonant with the expectation?