Lecture 16 Nov 10

Recall $X \sim \text{Geometric}(p) \Rightarrow \cancel{f(x) = (1-p)^{x-1} p}$

$t=1$ $t=2$ ...

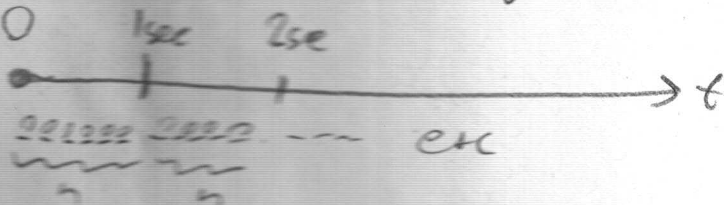
$\underline{0}$ $\underline{0}$ $\underline{0}$ $\underline{0}$ $\underline{1}$ individual, discrete Bernoulli realizations

Bern(p) Bern(p) ...

What if we imagine as a function of time, t?

$$f(t) = (1-p)^{t-1} p$$ switched free variable

But now each unit of time, let's say each second, is composed by many many Bernoulli experiments, n of them



where each has very low prob. So $n \approx \infty$, $p \approx 0$ but $\lambda = np$

(Same setup as Poisson) $\Rightarrow p = \frac{\lambda}{n}$

Now at $t=2$, we have $2n$ experiments

$$f(t) = (1-p)^{nt-1} p = \left(1 - \frac{\lambda}{n}\right)^{nt-1} \frac{\lambda}{n}$$ → and get PMF for any continuous time!

Same game... let $n \to \infty$

$$\lim_{n\to\infty} f(t) = \overset{e^{-\lambda t}}{\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{nt}} \; \overset{1}{\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-1}} \; \overset{0}{\lim_{n\to\infty} \frac{\lambda}{n}} = 0 \quad \forall t$$

$\Rightarrow$ PMF d.n.e. since $f(x) > 0 \; \forall x \in Supp[X]$

Problem! PMF is 0. There is zero prob that you stop at any amount of time. File that away...

How about CDF? $X \sim Geometric(p)$

the "success" is after x, meaning 1, 2, ..., x all fail!

$F(t) = P(X \leq t) = 1 - P(X > t) = 1 - (1-p)^t$

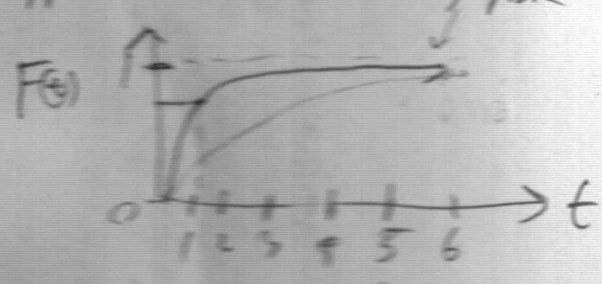
back to continuous time

$F(t) = P(X \leq t) = 1 - P(X > t) = 1 - \left(1 - \frac{\lambda}{n}\right)^{nt}$

to get $F(t)$ for any t where the experiment happens "continuously"...

$$\lim_{n \to \infty} F(t) = \lim_{n \to \infty} 1 - \left(1 - \frac{\lambda}{n}\right)^{nt} = 1 - \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^{nt} = 1 - \left(\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^{n}\right)^t = 1 - e^{-\lambda t}$$

$Supp[X] = [0, \infty)$ $\lambda \in (0, \infty)$



Valid CDF?

$\lim_{t \to -\infty} F(t) \overset{?}{=} 0 \Rightarrow F(0) = 0 \Rightarrow 1 - e^{-\lambda(0)} = 1 - 1 = 0$ ✓

$\lim_{t \to \infty} F(t) \overset{?}{=} 1 \Rightarrow 1 - \lim_{t \to \infty} e^{-\lambda t} = 1 - \lim \frac{1}{e^{\lambda t}} = 1 - 0 = 1$ ✓

Monotonically increasing?

$\frac{d}{dt} F(t) > 0 \; \forall t \in Supp[X]$ $\quad \frac{dF}{dt} = \lambda e^{\lambda t} > 0 \; \forall t > 0$ ✓

$\lambda = 1$

what's the prob you get a success/step before 1 sec?

$F(1) = 1 - e^{-1} = 0.63$

" " " 6 sec?

$F(6) = 1 - e^{-6} = 0.998$

What's the prob you "stop" between 3, 4 sec?

$$P(X \in (3,4)) = F(4) - F(3) = (1-e^{-4}) - (1-e^{-3}) = .031$$

What about stopping exactly at 3 sec?

$f(3) = 0$ because the PMF $= 0$ always! Why?

Because continuous time doesn't exist! It's a mathematical illusion! What does it even mean?

$\longrightarrow \mathbb{R}$ $\qquad |\mathbb{R}| = c$ ← the kind of cont!

$\vdash\!\!+\!\!+\!\!+\!\!+\!\!\longrightarrow \mathbb{N}$ $\qquad |\mathbb{N}| = \aleph_0$ ← we can grasp as time

Does time really move continuously?

Is space continuous? Greeks thought things were infinitely divisible. Atomic theory of matter. But atoms are divisible? How far?

Planck length $\vdash\!\!\!-\!\!\!-\!\!\!-\!\!\!\dashv$ length of distance where anything < it, you cannot tell the difference. Then since c speed of light is the fastest known velocity, the amount of time it takes a photon to traverse it is the smallest unit of time "Planck time" 5.3×10^{-44} s

$\Rightarrow$ time is discrete and so is space (as we believe right now)

So this r.v is "fake" ... purely a model.

Another reason... $t \Rightarrow$ see step... $f(3)=0$

$t = 3.000000\ldots$ infinite information! Can't write it down!

but if you say $t \in [2.9999999, 3.0000001]$

Now we have a real probability... $F(b) - F(a)$

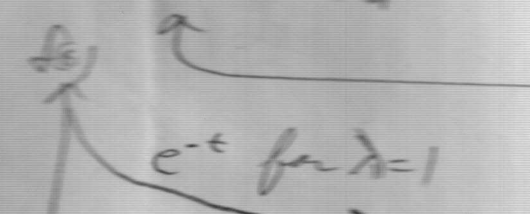
Now... question — how does $F(t)$ change?

$F'(t) = \frac{dF}{dt} = \lambda e^{-\lambda t}$ represents how dense the prob is

since $F(b) - F(a) = \int_a^b F'(t)\, dt$ fundamental thm of calc.

Def: $f(t) := \frac{dF}{dt}$ the derivative of the CDF, is called the prob density function PDF

SAME notation as PMF... oh well!!
It just really sucks!!!



$f(3) = .05 \neq P(X=3) = 0$!!!!!!!

The PDF does <u>NOT</u> spit out probs!

It's an abstract metric which informs you of how likely the prob is relative to other places... it is not REAL!!!

Def

X is a continuous r.v (not a discrete r.v)

if the support is continuous i.e. $|Supp(X)| = c \neq \aleph_0$

You cannot "enumerate" elements in the support

$Supp(X) \neq \{x_1, x_2, \dots\}$

Rules / Conditions (0) $Supp(X) \subseteq \mathbb{R}$

(1) $f(x) > 0 \quad \forall x \in Supp(X)$

PDF NOT PMF

(2) $\int_{x \in Supp(X)} f(x)\, dx = 1$

(3) $P(X \in [a,b]) = F(b) - F(a) = \int_a^b f(x)\, dx$

$\forall a,b \in \mathbb{R}$ s.t. $b > a$, $a,b \in Supp(X)$

Properties

(1) $f(x) = F'(x)$

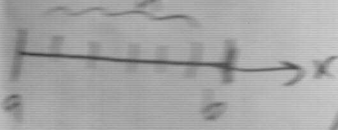
(2) $F(x_0) = \int_{\inf\{Supp(X)\}}^{x_0} f(x)\, dx \quad \forall x_0 \in Supp(X)$

be useful with cases!

inf $\rightarrow$ g.l.b.

sup $\rightarrow$ l.u.b.

Why can't we use sums? Because sums go over discrete supports!



Remember let $\Delta x = \frac{b-a}{n}$, $x_i = a + i\frac{b-a}{n}$

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} = \int_a^b f(x)\, dx$$

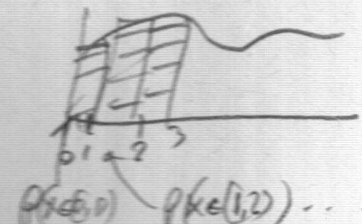
$\int$ is a sum over continuums

letting $\lim_{n\to\infty}$ converts X space from discrete to cont.

PMF $f(x) = 0$ always... why $P(X=a) = \int_a^a f(x)\, dx = 0$ No area!!!

What is E[X]? for X discrete $\sum_{x \in Supp(X)} x \underbrace{f(x)}_{PMF}$

Here...



$\Rightarrow \quad E(X) = \int_{x \in Supp(X)} x \underbrace{f(x)}_{PDF} \, dx$

$E(g(X)) = \int g(x) f(x) \, dx$

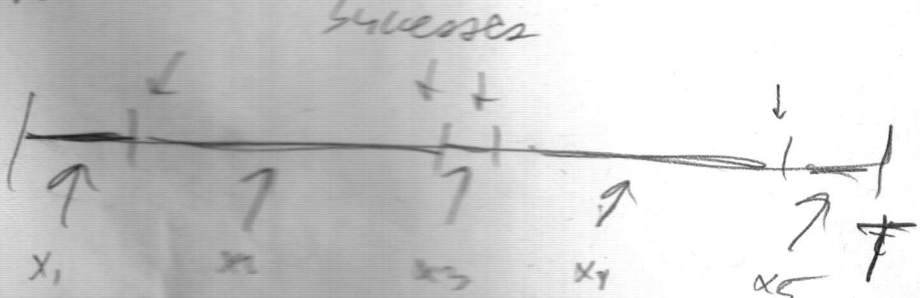
$Var(X) = \int (x-\mu)^2 f(x) \, dx$

Exponential r.v (what we've been studying all along)

$X \sim Exp(\lambda) := \lambda e^{-\lambda x}$ ← use x as free variable now, not t

$:= f(x)$ ← the PDF is the footprint now (CDF always is footprint)

Waiting time in a continuous support where each "unit" of support is a bernoulli experiment with very low p.



Poisson Process

How many events to t? $N \sim Poisson(\lambda t)$

Waiting time between events $X \sim Exp(\lambda)$

Distribution theory is so cool!!!

Geometric: discrete waiting time r.v for 1 success

Exponential: continuous waiting time r.v "

NegBin: discrete waiting time r.v for number success

Gamma/Erlang: continuous " " " " (242)

Special property: memorylessness

$F(x_0) = \int_{x \in S} f(x) \, dx$

$= \int_0^{x_0} \lambda e^{-\lambda x} \, dx$

$F(0) = e^{\lambda 0} + C$

$F(0) = 1 + C = 0$

$C = 1$

$= \lambda \int_0^{x_0} e^{-\lambda x} \, dx$

$= \lambda \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_0^{x_0} + C$

$= -e^{-\lambda x} \Big|_0^{x_0} + C$

$= -(e^{-\lambda x_0} - 1) + C$

$= 1 - e^{-\lambda x_0} + C$

at $x_0 = 0$, $F(x_0) = 0 \Rightarrow 1 - e^{-\lambda(0)} + C = 0 = C$

EC on problem... $X \sim \text{Geometric}(p)$

$$P(X = x) = P(X = x_0 + x \mid X > x)$$

Waiting time Geometric(p)

given that you waited this far, the waiting time is still Geometric(p)!

x_0

Supp(X)

Proof

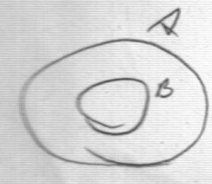
$P(X=x) = (1-p)^{x-1} p$, $\quad P(X = x_0 + x \mid X > x) = \dfrac{P(X = x_0 + x \;\&\; X > x)}{P(X > x)}$

$$= \frac{P(X = x_0 + x)}{P(X > x)}$$

$$= \frac{(1-p)^{x_0 + x - 1} p}{(1-p)^{x_0}}$$

$$= \frac{\cancel{(1-p)^{x_0}} (1-p)^{x-1} p}{\cancel{(1-p)^{x_0}}}$$

Why?



$A \cap B = B$

Since $B \subset A$ (proper subset)

↓ lecture 17

$X \sim \text{Exp}(\lambda)$ $\quad 1 - F(t) = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$

$$P(X > t) = P(X > t_0 + t \mid X > t_0)$$

$$e^{-\lambda t} = \frac{P(X > t_0 + t \;\&\; X > t_0)}{P(X > t_0)}$$

$$= \frac{P(X > t_0 + t)}{P(X > t_0)} = \frac{e^{-\lambda(t_0 + t)}}{e^{-\lambda t_0}} = \frac{\cancel{e^{-\lambda t_0}} e^{-\lambda t}}{\cancel{e^{-\lambda t_0}}}$$