Lecture 3

Math 241 9/3/15

Previously... J, M, S sitting where order matters and no replacement

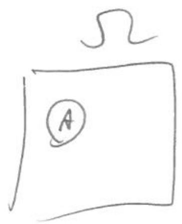
$_3P_3$ $\quad 3\cdot 2\cdot 1$

$= \frac{3!}{(3-3)!}$ $\qquad$ generally $\quad {}_nP_k = \frac{n!}{(n-k)!}$

This gets its own notation because it happens a lot. Let's talk about one off situations!

Bob & Jane, Max & May, Joe & Susan } married

the 6 people
If they sit randomly,
What is probability that all married couples are sitting together?



$P(A) = \frac{|A|}{|\Omega|}$ each arrangement is equally likely

$|\Omega|? = {}_6P_6 = 6! = 720$ $\quad \Rightarrow 6.7\%$

$|A|$ $\quad \underline{6}\ \underline{1}\ \underline{4}\ \underline{1}\ \underline{2}\ \underline{1} = 48$

$\underset{HH}{\underline{3}}\ \underset{HH}{\underline{2}}\ \underset{HH}{\underline{1}} \cdot (2\cdot 2\cdot 2) = 3!\, 2^3$

What is prob of alt. boy-girl-boy-girl?

$(\underline{3}\ \underline{3}\ \underline{2}\ \underline{2}\ \underline{1}\ \underline{1})\ 2$

BBB GGG $\quad 3!3!$

How about only ~~Bob/Jim~~ Max/Mary sit togeth? How?

Max-Mary as a unit

$\boxed{MM}$ B Ja Jo S

$\underline{\underline{5! \cdot 2}}$

100 ~~people~~ balls, 3 ~~chairs~~ position order matters no replacement

$$_{100}P_3 = \frac{100!}{(100-3)!} = \underset{\#1}{\underline{100}} \cdot \underset{\#2}{\underline{99}} \cdot \underset{\#3}{\underline{98}} = 970{,}200$$

100 balls, 3 positions with replacement ??

$\underline{100}\ \underline{100}\ \underline{100} = 1{,}000{,}000$

10,000 balls, 3 position no replacement

$\underline{10000}\ \underline{9999}\ \underline{9998}$

" yes " : $\frac{.9997}{}$

$\underline{10000}\ \underline{10000}\ \underline{10000}$

$$\frac{\#\ n \text{ balls } k \text{ positions no repl}}{\#\ n \text{ balls } k \text{ '' yes repl}} = ?\quad k \text{ is fixed } n \text{ is large}$$

If n is large, replacement / no replacement ⇒ no diff

$$\lim \frac{{}_nP_k}{n^k} = \frac{\overbrace{n(n-1)\cdot\ldots\cdot(n-k+1)}^{k}}{\underbrace{(n)(n)\cdot\ldots\cdot(n)}_{k}}$$

recall $\lim\limits_{x\to a} f(x)\,g(x) = \lim f(x) \lim g(x)$

if f, g cont

$$\lim \frac{n}{n} \lim \underbrace{\frac{n-1}{n}}_{1-\frac{1}{n}} \cdots \lim \underbrace{\frac{n-k+1}{n}}_{1-\frac{k-1}{n}} = 1 \cdot 1 \cdots 1 = \boxed{1}$$

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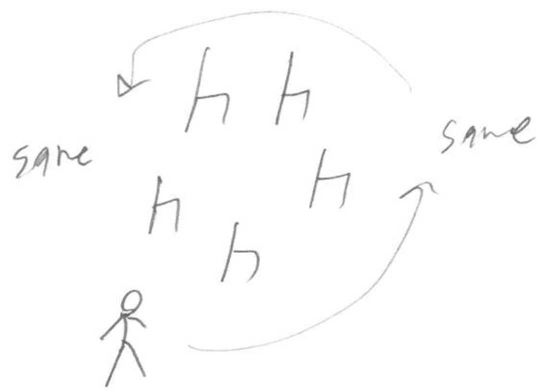
5 people, 5 chairs in a circle but circle is rotationally invariant



$$\frac{{}_5P_5}{5} = 24$$

5 rotations

dividing out invariance principle

9 flowers 5 blue, 3 red

each blue is distinct; each red is distinct

$B_1 B_2 B_3 B_4 B_5 R_1 R_2 R_3$ diff

How many ways to order? $8!$

But what if the Red are indistinct? As in R_1, R_2, R_3 appear the same or you don't care about the diff?

Imagine	B_1	B_2	R_1	B_3	R_2	B_4	R_3	B_5
=		"	R_1	"	R_3	"	R_2	"
=		"	R_3	"	R_2	"	R_1	"
=		"	R_3		R_1		R_2	
=			R_2		R_1		R_3	
			R_2		R_3		R_1	

For each config of flowers we collapse $3! = 6$ to 1 hence we divide by $3!$

$= \frac{8!}{3!}$

4

How about blues also mixed?

B B R B R B R B $\longrightarrow$ 5! 3! possibilities have the need to be collapsed

$\Rightarrow \dfrac{8!}{5!3!} = 56$

Let's see it...

How about 3B, 2R $\longrightarrow \dfrac{5!}{3!2!} = 10$

BBBRR

$B_1 B_2 B_3 RR$ — R_1R_2, R_2R_1

$B_1 B_3 B_2 RR$

$B_2 B_1 B_3 RR$

$B_2 B_3 B_1 RR$

$B_3 B_1 B_2 RR$

$B_3 B_2 B_1 RR$

12

BBRBR
BBRRB
BRBBR
BRBRB
BRRBB
RBBBR
RBBRB
RBRBB
RRBBB

10

$10 \cdot 12 = 120 = 5!$

What's the $P(2H\ 2T)$? $= \dfrac{|A|}{|\Omega|} = \dfrac{|A|}{16}$

$H_1 H_2 T_1 T_2 \longrightarrow \dfrac{4!}{2!2!} = 6$

100 flips

What's $P(50H, 50T) = \dfrac{|A| \; \frac{50!}{25!25!}}{|\Omega^{50}| = 2^{50}} = 11\%$

1000 flips

$P = P(500H, 500T) = \dfrac{|A|}{\Omega^{100}} = \dfrac{\frac{1000!}{(500!)^2}}{2^{1000}}$ Blow up!

Logs help? $\ln(1000!) - 2\ln(500!) - 1000\ln(2)$

$$n! = \prod_{i=1}^{n} i$$

$$\ln(n!) = \sum_{i=1}^{n} \ln i$$ no blow up... but 2n operations!

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\Rightarrow \ln(n!) \approx \frac{1}{2}\ln(2\pi) + \left(n+\frac{1}{2}\right)\ln(n) - n$$

$$\ln(p) = \frac{1}{2}\ln(2\pi) + 1000.5\ln(1000) - 1000 - \ln(2\pi) - 500.5\ln(500) + 500 - 1000\ln(2) = -3.6798$$

$$\Rightarrow p = e^{-3.6798} = .0252$$

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4 people 2 chairs $n = \{J, B, M, D\}$

How many ways to order them? ${}_4P_2 = \frac{4!}{2!} = 12$

What if we don't care about order?

$$\frac{{}_4P_2}{2!} = \frac{\frac{4!}{2!}}{2!} = \frac{4!}{2!2!} = \frac{24}{2\cdot 2} = 6$$

$\{\{J,B\}, \{J,M\}, \{J,D\}, \{B,M\}, \{B,D\}, \{M,D\}\}$

dividing out where we don't care about

combinations of 4 people taken 2 at a time

combinations of n ~~[illegible]~~ ~~[illegible]~~ objects taken k at a time

$${}_nC_k := \frac{{}_nP_k}{k!} = \frac{n!}{(n-k)!\,k!}, \quad \binom{n}{k} := {}_nC_k$$

notation I'll use "choose notation" "n choose k"

for set A, s.t. $|A| = n$, let $C_{A,k} := \{B : B \subseteq A \ \&\ |B| = k\}$

$$\binom{n}{k} = |C_{A,k}|$$

52 cards in a deck.

$R = \{A, 2, \ldots, K\}$

$S = \{S, H, C, D\}$ $\qquad \Omega = R \times S$

5-card draw poker ..... $|\Omega| = \binom{52}{5}$

order doesn't matter

Shuffled randomly... each comb. equally likely

$P(\text{Royal Flush}) = \frac{|\text{Royal Flush}|}{\quad}$

Identities

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n!\,(n-k+1)}{(n-k+1)(n-k)!\,k(k-1)!}$

$= \frac{n!}{(n-k+1)!\,(k-1)!}\left(\frac{n-k+1}{k}\right)$

$= \binom{n}{k-1}\left(\frac{n-k+1}{k}\right)$

Trivialities

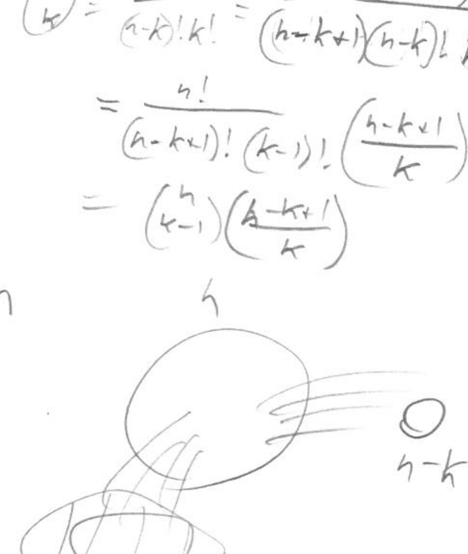
① $\binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n$

② $\binom{n}{n-1} = \frac{n!}{(n-(n-1))!\,(n-1)!} = n$

③ $\binom{n}{n} = 1$

④ $\binom{n}{0} = 1$

⑤ $\binom{n}{n-k} = \frac{n!}{(n-(n-k))!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$



$\binom{n}{k} = \frac{n!}{(n-k)\,k!} = \frac{n(n-1)!}{(n-k)\,k!}$

$= \frac{(n-1)!\,n}{(n-k)(n-k-1)!\,k!}$

$= \binom{n-1}{k}\frac{n}{n-k}$

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

$= \frac{n(n-1)!}{(n-k)!\,k(k-1)!}$

$= \binom{n-1}{k-1}\frac{n}{k}$

$(n-1)-(k-1)$