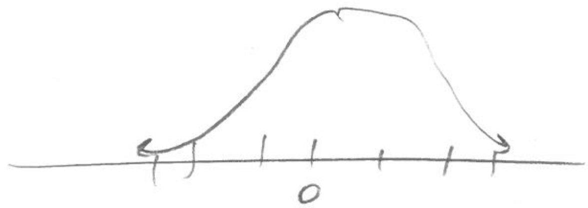
Math 201 Lec 19 11/17/15

$Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

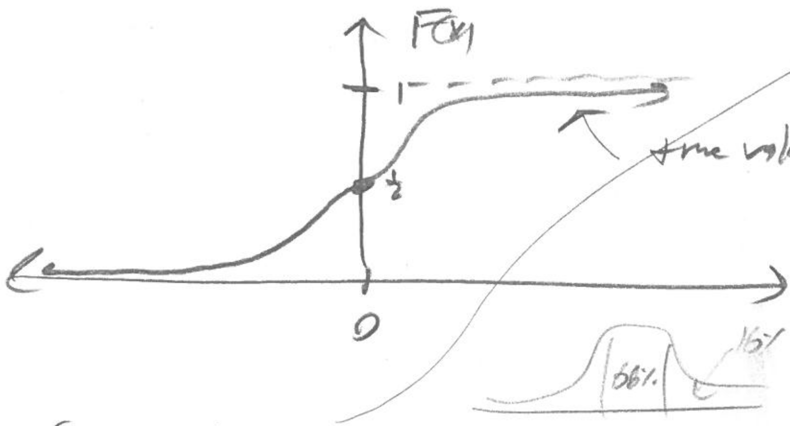


$X \sim N(\mu,\sigma^2) := \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$Supp(Z) = \mathbb{R}$

$Supp(X) = \mathbb{R}$

$\mu \in \mathbb{R}, \quad \sigma^2 \in (0,\infty)$



true values impossible to know

Z scores

$-1 \Rightarrow$ 1 SE below mean

$+2.5 \Rightarrow$ 2SE above mean

$P(Z \leq -1) = .16$

$P(Z \leq -2) = .025$

$P(Z \leq -3) = .0015$

Male Height

$X \sim N(70'', 3''^2)$

$P(\text{height more than } 73'')$?

$Supp(X) = \mathbb{R}$, Param space $\mu \in \mathbb{R}$, $\sigma^2 \in (0, \infty)$

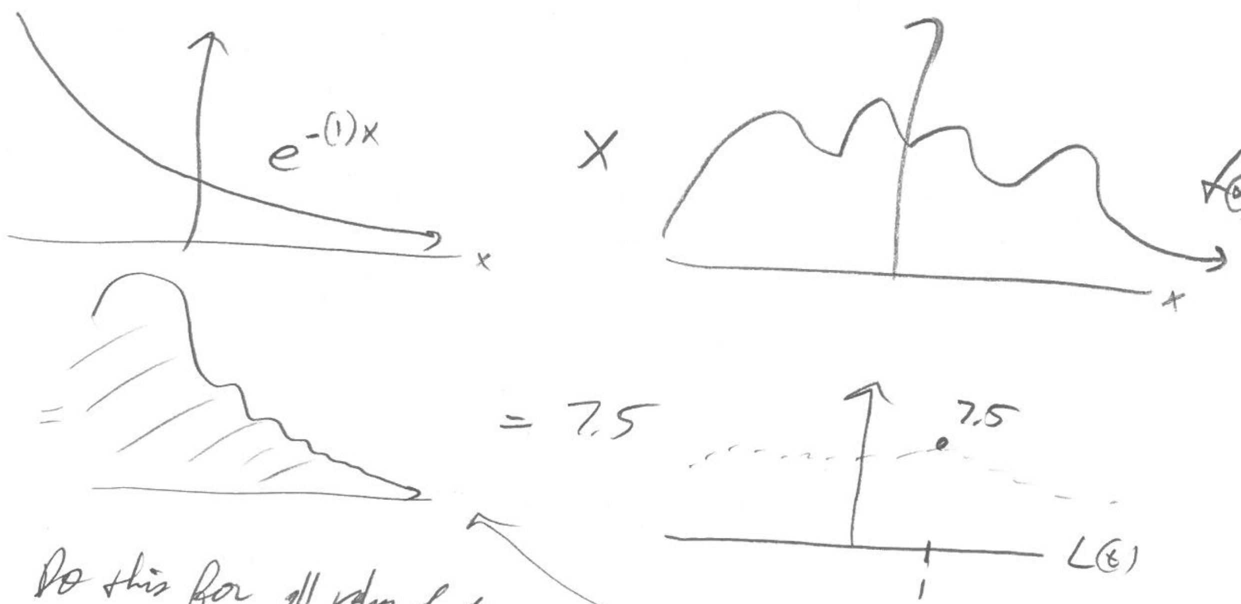
Why is normal density so important??? Wait 3 lectures...

Let $L(t) := \int_{\mathbb{R}} e^{-tx} f(x)\, dx$ "Bilateral Laplace Transform"

a function of a function (operator)

"integrates x out!"

$L(1) = \int e^{-x} f(x)\, dx$



Do this for all values of t..

$L(t)$ and $f(x)$ are 1:1 if $L(t)$ exists (could be ∞)

Deep proof... can't even do it..

Thus $L(t)$ and $f(x)$ are two ways to look at the same thing.

Define

$$M_X(t) := E[e^{tX}] = \int_{x \in Supp(X)} e^{tx} f(x)\,dx \quad \text{(cont)}$$

$$= \sum_{x \in Supp X} e^{tx} p(x) \quad \text{(discrete)}$$

"moment generating function (mgf)" for r.v. X

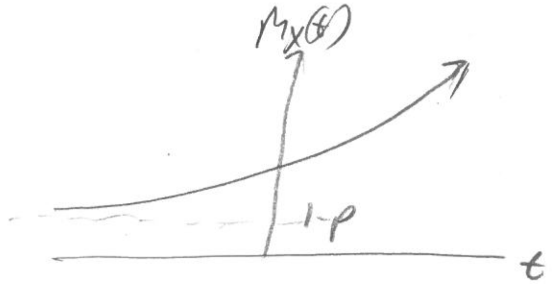
$M_X(t)$ is 1:1 with $f(x)$ (or $p(x)$ if discrete) so if $X \overset{d}{=} Y$

$\Rightarrow M_X(t) = M_Y(t)$ (as long as they exist)

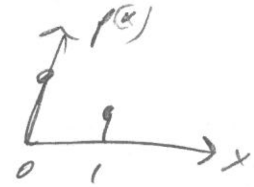
$X \sim Bern(p)$

$$M_X(t) = E(e^{tX}) = e^{t(0)}p(0) + e^{t(1)}p(1) = (1)\,p(0) + e^t p(1)$$

$$= \boxed{1-p+e^t p}$$



$\Longleftrightarrow$



These two things are the same in some way...

What is the more reasonable case?

$$X \sim \text{Binom}(n,p) \qquad E(X^{17}) = \sum_{x=0}^{n} x^{17}\binom{n}{x} p^x (1-p)^{n-x} \quad \text{Impossible!}$$

What is e^x about $\approx$?

x close to c

$$f(x) \approx f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots$$

x close to 0

$$e^x = \underbrace{e^0}_{1} + \frac{e^0}{1!}x + \frac{e^0}{2!}x^2 + \ldots \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$e^{tX} = 1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \ldots$$

$$M_X(t) = E(e^{tX}) = E[\quad] = E(1) + \underbrace{E\left(\frac{tX}{1!}\right)}_{\frac{t}{1!}E(X)} + \underbrace{E\left[\frac{t^2X^2}{2!}\right]}_{\frac{t^2}{2!}E(X^2)} + E\left[\frac{t^3X^3}{3!}\right] + \ldots \sum_{i=0}^{\infty} \frac{t^i}{i!} E[X^i]$$

$$M_X'(t) = \frac{E(X)}{1!} + \frac{t}{1!}E(X^2) + \frac{t^2}{2!}E(X^3) + \frac{t^3}{3!}E(X^4) + \ldots$$

$$M_X'(0) = E(X)$$

$$M_X''(t) = E(X^2) + \frac{t}{1!}E(X^3) + \frac{t^2}{2!}E(X^4) + \ldots$$

$$M_X''(0) = E(X^2)$$

⋮

(1) $\Rightarrow M_X^{(k)}(0) = E(X^k)$

"moment generating function"

Other properties. Let's say $T = X_1 + X_2$ where X_1, X_2 indep.

(2) $M_T(t) = M_{X_1+X_2}(t) = E\left[e^{t(X_1+X_2)}\right] = E\left(e^{tX_1}e^{tX_2}\right) = E\left(e^{tX_1}\right)E\left(e^{tX_2}\right)$

$= M_{X_1}(t)\, M_{X_2}(t)$

$E\left(g(X_1)g(X_2)\right)$

$= Eg(X_1)\, Eg(X_2)$ when indep

(3) $Y = aX + c$

$M_Y(t) = E\left[e^{t(aX+c)}\right] = E\left[e^{taX}e^{c}\right] = e^c E\left[e^{taX}\right] = e^c M_X(at)$

$X \sim Bin(n,p)$

Binom Thm again

$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x(1-p)^{n-x} = \sum_{x=0}^{n}\binom{n}{x}\underbrace{(e^t p)^x}_{b}\underbrace{(1-p)^{n-x}}_{a} = (e^t p + 1-p)^n$

$e^{tx} = (e^t)^x$

$X_1, \ldots, X_n$ iid Bern

$T = X_1 + \ldots + X_n \sim Bin(n,p)$ since identical

(2) since ind.

$M_T = M_{X_1}(t) \cdot \ldots \cdot M_{X_n}(t) = \left(M_{X_1}(t)\right)^n = \left(1-p+pe^t\right)^n = M_T(t)$ for binomial

easy proof!!!

$X \sim Exp(\lambda)$

$$M_X(t) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx = \lambda\int_0^\infty e^{\overbrace{tx-\lambda x}^{x(t-\lambda)}}\,dx = \lambda\left[\frac{1}{t-\lambda}e^{x(t-\lambda)}\right]_0^\infty = \frac{\lambda}{t-\lambda}$$

if $t<\lambda$ o/w not defined

$Y = aX \sim Exp\left(\frac{\lambda}{a}\right)$

Let's see $\quad M_Y(t) = M_X(at) = \frac{\lambda}{at-\lambda} = \dfrac{\overbrace{\frac{\lambda}{a}}^{\lambda'}}{\underbrace{t-\frac{\lambda}{a}}_{\lambda'}} \iff Exp(\lambda') = Exp\left(\frac{\lambda}{a}\right)$

easier!

$$Z \sim N(0,1) \quad M_Z(t) = E[e^{tZ}] = \int_{\mathbb{R}} e^{tx}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx = \int \frac{1}{\sqrt{2\pi}}e^{\overbrace{-\frac{x^2}{2}+tx}^{-\frac{1}{2}(x^2-2tx)}}\,dx$$

Note $\quad (x-t)^2 = x^2-2tx+t^2 \quad$ so $\quad (x-t)^2-t^2 = x^2-2tx$

$$= \int\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left((x-t)^2-t^2\right)}\,dx = \int\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-t)^2}e^{-\frac{t^2}{2}}\,dx = e^{-t^2/2}\overbrace{\int\underbrace{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-t)^2}}_{\text{PDF for } N(t,1)}\,dx}^{1} = \boxed{e^{-t^2/2}}$$

$E(Z) = M_Z'(0) = te^{-t^2/2} = 0$

$E(Z^2) = M_Z''(0) = \frac{d}{dz}\left[te^{-t^2/2}\right]_0 = t(te^{-t^2/2}) + e^{-t^2/2}(1) = 1$

Let $X_1, \dots, X_n \overset{iid}{\sim}$ something with mean μ, $\sigma^2 < \infty$

$\bar{X} := \frac{X_1 + \dots + X_n}{n}$ $\quad E(\bar{X}) = \mu, \; Var(\bar{X}) = \frac{\sigma^2}{n} \Rightarrow SE(\bar{X}) = \frac{\sigma}{\sqrt{n}}$

let $C_n := \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$ $\quad$ we know $E(C_n) = 0$, $SE(C_n) = 1$

since C_n is $\bar{X}$ standard (see thm 8)

Goal: Find distribution of C_n as n gets big...

$$C_n := \frac{\frac{X_1 + \dots + X_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\frac{X_1 + \dots + X_n}{n} - \frac{\overbrace{n\mu}^{(\mu + \dots + \mu)}}{n}}{\frac{\sigma}{\sqrt{n}}} \frac{\sqrt{n}}{\sqrt{n}} = \frac{(X_1 - \mu) + \dots + (X_n - \mu)}{\sigma \sqrt{n}}$$

$$= \frac{1}{\sqrt{n}} \left(\left(\frac{X_1 - \mu}{\sigma}\right) + \dots + \left(\frac{X_n - \mu}{\sigma}\right) \right) = \frac{1}{\sqrt{n}} (Z_1 + \dots + Z_n) = \frac{Z_1}{\sqrt{n}} + \dots + \frac{Z_n}{\sqrt{n}}$$

iid

let $Z_i := \frac{X_i - \mu}{\sigma}$ ... Z_i is X_i standard... iid r.v.'s still

STOP

$$M_{C_n}(t) = M_{\frac{Z}{\sqrt{n}}}(t) \cdots M_{\frac{Z}{\sqrt{n}}}(t) = M_Z\left(\frac{t}{\sqrt{n}}\right) \cdots M_Z\left(\frac{t}{\sqrt{n}}\right) = \left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

Rule? $\qquad$ Rule? $\qquad$ Why?