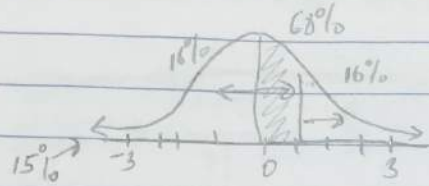
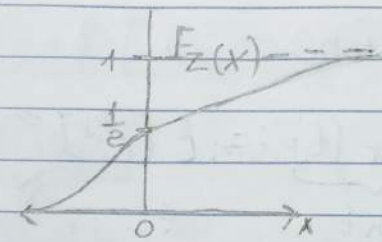
$$Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$Z = +1$

$Z = -1.5$



$$X \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$



$X \sim N(70'', 3''^2)$

$P(\text{some US male is taller than } 73'') = P(X > 73'') = P\left(\frac{X-70}{3''} > \frac{73''-70}{3''}\right)$

$= P(Z > 1)$ ← standard normal

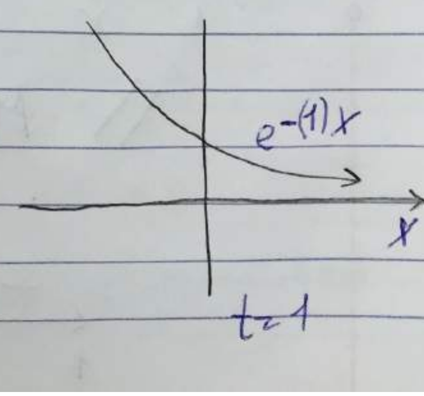
Parameter space

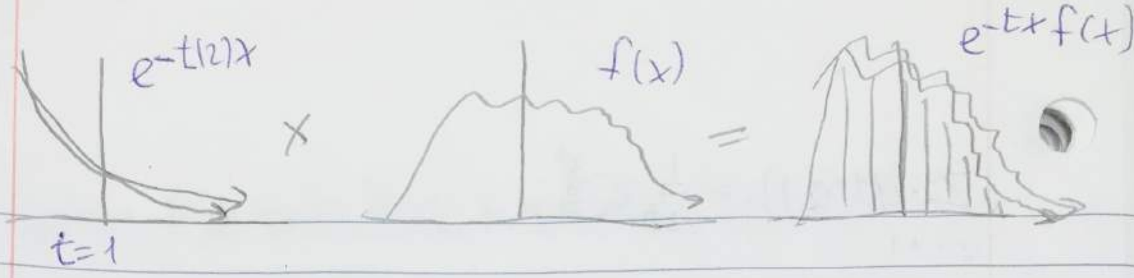
$\mu \in \mathbb{R}$

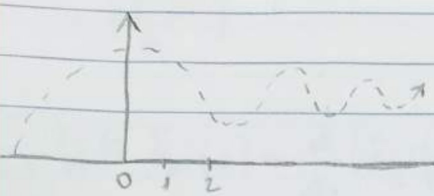
$\sigma^2 \in (0, \infty)$

$$L(t) := \int_{\mathbb{R}} e^{-tx} f(x)\, dx$$

Biland
laplace
Transform







$L(t)$ and $f(x)$ are 1:1

Define

$$L(-t) = \underbrace{M_X(t)}_{\substack{\text{moment}\\ \text{generating}\\ \text{function (mgt)}}} := E[e^{tX}] \overset{\text{continuous}}{=} \int_{\mathbb{R}} e^{tx} f(x)\,dx$$

$$\overset{\text{discrete}}{=} \sum_{x \in supp(X)} e^{tx} p(x)$$

$$X \overset{d}{=} Y$$

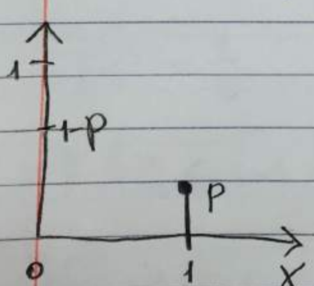
continuous case

$f_X(x) = f_Y(y)$

$M_X(t) = M_Y(t)$

discrete

$p_X(x) = p_Y(y)$

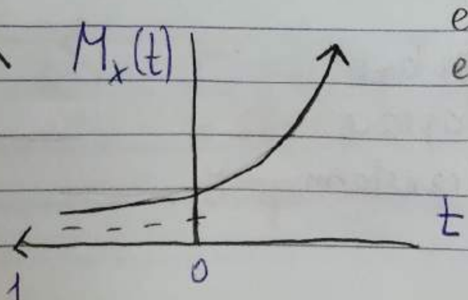


$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{1} e^{tx} p^x (1-p)^{1-x} =$$

$$= 1 - p + e^t p$$

$e^{t(0)} p^0 (1-p)^{1-0}$

$e^{t(1)} p^1 (1-p)^{1-1}$



$\downarrow$ not possible

geometric $X \sim \text{Binom}(n,p)$ $E[X]$, $E[X^2]$, $E[X^{97}]$

$$\boxed{f(x) \approx f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots = \sum_{i=0}^{\infty} \frac{f^{i}(c)}{i!}(x-c)^{i}}$$

$$\approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

- $L(-t) = M_X(t) := E[e^{tX}] = E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \dots\right]$

$$= 1 + tE(X) + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \dots$$

$$M_X'(t) = E[X] + tE[X^2] + \frac{t^2}{2!}E[X^3] + \frac{t^3}{3!}E[X^4] + \dots$$

$$m_X'(0) = E[X]$$

$$m_X''(t) = E[X^2] + tE[X^3] + \frac{t^2}{2!}E[X^4] + \dots$$

I $m_X^{k}(0) = E[X^k]$

$T = X_1 + X_2$ if X_1, X_2 independent

$$M_T(t) = M_{X_1+X_2}(t) = E\left[e^{t(X_1+X_2)}\right] = E\left[e^{tX_1}e^{tX_2}\right] =$$

$$= E\left[e^{tX_1}\right]E\left[e^{tX_2}\right] = M_{X_1}(t) \cdot M_{X_2}(t)$$

II $M_T(t) = \prod_{i=1}^{n} m_{X_i}(t) = (M_X(t))^n$

by indipendence

if identically distributed

III $Y = aX + c$

$$M_Y(t) = E\left[e^{t(aX+c)}\right] = E\left[e^{taX} e^{c}\right] =$$

$$= e^{c} E\left[e^{\overset{t'}{(ta)}X}\right] = e^{c} M_X(t') = e^{c} M_X(at)$$

- $X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(p)$

$$M_T(t) = \left(M_X(t)\right)^n = \left(1 - p + e^{t} p\right)^n$$

- $T \sim \text{Binom}(n, p)$

$$E(e^{tT}) = \sum_{x=0}^{n} \overset{(e^t)^x}{e^{tx}} \binom{n}{x} p^{x} (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} \underset{b}{(e^{t}p)^{x}} \cdot \underset{a}{(1-p)^{n-x}} = (1 - p + e^{t} p)^{n}$$

(Binomial is iid Bernoullis)

- $X \sim \text{Exp}(\lambda)$

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tX} \lambda e^{-\lambda x}\, dx = \lambda \int_0^\infty e^{(t-\lambda)x}\, dx =$$

$$= \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)x}\right]_0^\infty = \frac{\lambda}{t-\lambda}\left(\overset{0}{\lim_{x\to\infty} e^{(t-\lambda)x}} - 1\right) =$$

$$= \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

* $Y = aX$

$$M_Y(t) = \overset{\cancel{e}\, M_X(at)}{M_X(at)} = \frac{\lambda}{\lambda - at}\left(\frac{\frac{1}{a}}{\frac{1}{a}}\right) = \frac{\frac{\lambda}{a}}{\underbrace{\frac{\lambda}{a}}_{\lambda'} - t} =$$

$$= \frac{\lambda'}{\lambda' - t} \Rightarrow Y \sim \text{Exp}(\lambda') = \text{Exp}\left(\frac{\lambda}{a}\right)$$

$$(x-t)^2 = x^2 - 2xt + t^2$$

$$e^{-\frac{1}{2}\left((x-t)^2 - t^2\right)} = e^{-\frac{1}{2}(x-t)^2} \cdot e^{\frac{t^2}{2}}$$

$$\underbrace{e^{-\frac{1}{2}(x^2 - 2tx)}}$$

$Z \sim N(0,1)$ PDF for Z

$$(t) = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \overbrace{e^{tx - \frac{x^2}{2}}} dx = e^{\frac{t^2}{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx$$

$f(x)$ for $N(t, 1^2)$

$\frac{\text{density}}{\text{support}} = 1$

$$= \boxed{e^{\frac{t^2}{2}}}$$

$E(Z) = 0$

$$M_Z'(0) = te^{\frac{t^2}{2}} \Big]_0 = 0$$

$Var[Z] = 1$

$E[Z^2] = 1$

$$M_Z''(0) = t^2 e^{\frac{t^2}{2}} + e^{\frac{t^2}{2}} \Big]_0 = 1$$

- $X_1, \dots, X_n \overset{iid}{\sim}$ some distribution with $\mu, \sigma^2 < \infty$

$$\overline{X} = \frac{X_1 + \dots + X_n}{n}, \; E(\overline{X}) = \mu, \; SE[\overline{X}] = \frac{\sigma}{\sqrt{n}}$$

Goal: find distr. of C_n if n is large

$$C_n := \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \frac{\sqrt{n}}{\sqrt{n}} = \frac{\frac{X_1 + \dots + X_n}{n} - \frac{n\mu}{n}}{\frac{\sigma}{\sqrt{n}}} = \frac{(X_1 - \mu) + \dots + (X_1 - \mu)}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}}\left(\left(\frac{X_1 - \mu}{\sigma}\right) + \dots + \left(\frac{X_n - \mu}{\sigma}\right)\right)$$

$E(C_n) = 0$

$E(C_n) = 1$

$$\frac{1}{\sqrt{n}}(Z_1 + \dots + Z_n) = \frac{Z_1}{\sqrt{n}} + \dots + \frac{Z_n}{\sqrt{n}}$$