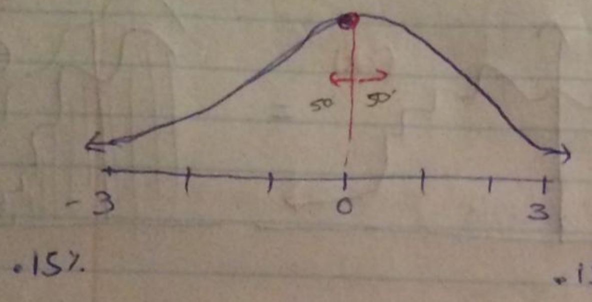
$$Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

no CDF in close Form
no possible todo integration




$$X \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

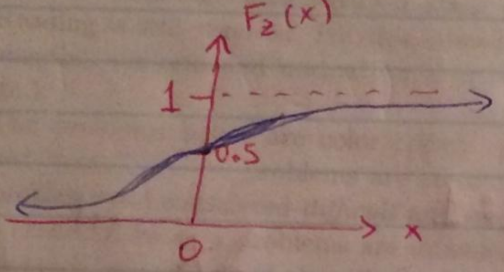
shift scale

Parameter Space

$\mu \in \mathbb{R}$

$\sigma^2 \in (0,\infty)$

CDF:




only know .5 point $F_Z(0)$

.0 varance is degenerative

+ # above mean

− # below mean

$* \; X \sim N(70'', 3''^2)$ Male height

height $\sigma$

SE 3"

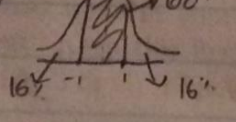
$3''^2$ is the varance

$$P\left(\begin{array}{l}\text{someous male}\\ \text{is taller than } 73''\end{array}\right) = P(X > 73'') = P\left(\underbrace{\frac{X-70''}{3''}}_{\text{standarized}} > \frac{73''-70''}{3''}\right)$$

mean from

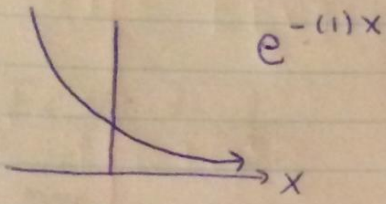
−1 and 1 is 68% 32% is left over 16% & 16%

$$= P(Z > 1) = \underline{16\%}$$

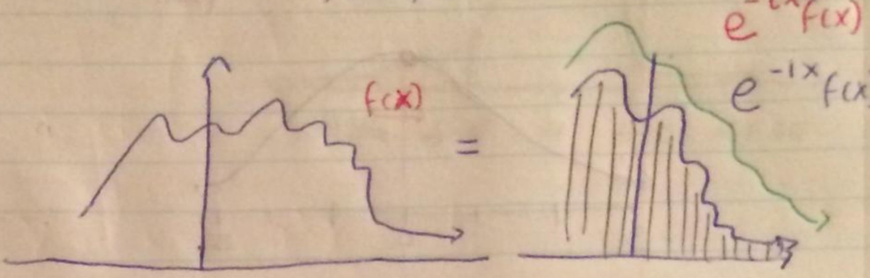


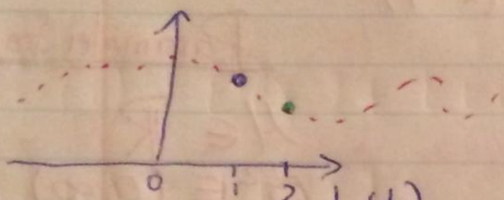
$$L(t) := \int_{\mathbb{R}} \underset{\sim}{e^{-tx}}\, \underset{\sim}{f(x)}\, dx$$

Bilade Laplace Transform




$t = 1$




$L(t)$ and $f(x)$ are 1:1

## Define

$$\underbrace{L(-t) = M_x(t)}_{\text{DNA}} := E[e^{tx}] = \int e^{tx} f(x)\, dx \quad \text{pdf} \quad 1:1$$

$$= \sum_{x \in Supp[x]} e^{tx} p(x) \quad \text{PMF}$$

Discrete

Body

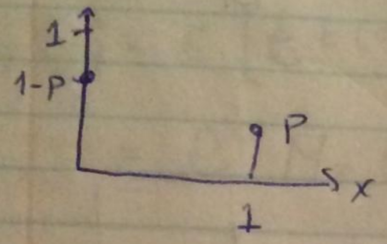
$M_x(t)$ & $\Sigma$ or $\int$ are 1:1

Moment generating function

$X \overset{d}{=} Y$

| | Contin | Discrete |
|---|---|---|
| Body | $f_x(x) = f_y(x)$ | $P_x(x) = P_y(x)$ |
| DNA | $M_x(t) = M_y(x)$ | " |

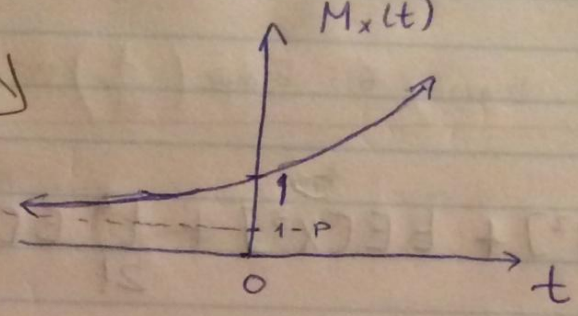
$X \sim \text{Bern}(p)$




$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{1} e^{tx} p^x (1-p)^{1-x}$$

$$= \underbrace{\overset{e^{t(0)} p^0 (1-p)^{1-0}}{1-p}}_{0} + \underbrace{e^t p}_{1} \leftarrow e^{t(1)} p^{(1)} (1-p)^{1-1}$$

same




$X \sim \text{Bino}(n,p)$ $\qquad E[X]$, $E[X^2]$, $E[X^{97}]$

no possible

$$f(x) \approx f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots = \sum_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!}(x-c)^i$$

Taylor Series

$$\approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

$$e^x \approx 1 + \underset{\frac{1}{1!}}{x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$M_x(t) = E[e^{tx}] = E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \dots\right]$$

$$= 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \dots$$

derivative

$$M_x'(t) = E[X] + tE[X^2] + \frac{t^2}{2!}E[X^3] + \frac{t^3}{3!}E[X^4] + \cdots$$

$$M_x'(0) = E[X]$$

Bernulli

$M_x'(t) = 1 - p + e^t p$ denvative with respect to 0 $e^t p$

$(0) = 0$

$E[X] = 0$

$$M_x''(t) = E[X^2] + tE[X^3] + \frac{t^2}{2!}E[X^4] + \cdots$$

$$M_x''(0) = E[X^2]$$

Sheet cheat

I $M_x^{(k)}(0) = E[X^k]$ Get any moment I want

$T = X_1 + X_2$ if $X_1, X_2$ independent

$$M_T(t) = M_{X_1+X_2}(t) = E\left[e^{t(X_1+X_2)}\right] = E\left[e^{tX_1}e^{tX_2}\right]$$

$$= E[e^{tX_1}]\,E[e^{tX_2}] = M_{X_1}(t)\,M_{X_2}(t)$$

II if $X_1, \ldots, X_n$ independent $M_T(t) \overset{\text{by independence}}{=} \prod_{i=1}^{n} M_{X_i}(t)$

$= (M_x(t))^n$

if identical distibuted

$Y = aX + c$

$$M_Y(t) = E\left[e^{t(aX+c)}\right] = E[e^{tax} e^c] = e^c E[e^{\overset{t'}{ta}x}]$$

$$= e^c M_X(t') = e^c M_X(at)$$

III

- $X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(p)$ Sum of iid bernu is binomial

$$M_T(t) = (M_X(t))^n = (1-p+e^t p)^n$$

$T \sim \text{Bino}(n, p)$

$$E[e^{tT}] = \sum_{x=0}^{n} \underbrace{e^{tx}}_{(e^t)^x} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (\overset{b}{e^t p})^x (\overset{a}{1-p})^{n-x} = \underbrace{(1-p+e^t p)^n}$$

* Sum of iid bernulies is a binomial ← conclusion

- $X \sim \text{Exp}(\lambda)$ Know support & PDF

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \overset{PDF}{\lambda e^{-\lambda x}}\, dx = \lambda \int_0^\infty e^{(t-\lambda)x}\, dx$$

$$= \frac{\lambda}{t-\lambda}\left[\underbrace{e^{(t-\lambda)x}}_{(0) = 1}\right]_0^\infty = \frac{\lambda}{t-\lambda}\left(\lim_{x\to\infty} e^{(t-\lambda)x} - 1\right) = \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

if $(t-\lambda)$ is (+) then $\infty$

if $(t-\lambda)$ is (−) then $0$

• $Y = aX$ $\quad \cancel{e^c} M_x(at)$

$$M_Y(t) = M_X(at) = \frac{\lambda}{\lambda - at}\left(\frac{\frac{1}{a}}{\frac{1}{a}}\right) = \frac{\overbrace{\frac{\lambda}{a}}^{\lambda'}}{\underbrace{\frac{\lambda}{a}}_{\lambda'} - t} = \underbrace{\frac{\lambda'}{\lambda' - t}}_{\text{looks like}} \rightarrow$$

looks like $X \sim Exp(\lambda)$

$M_X(\lambda) = \frac{\lambda}{\lambda - t}$

$$\Rightarrow Y \sim Exp(\lambda')$$

$$= Exp\left(\frac{\lambda}{a}\right)$$

• $Z \sim N(0,1)$

$e^{-\frac{1}{2}(x^2 - 2tx)}$ $\quad (x-t)^2 = x^2 - 2xt + t^2$

$e^{-1/2((x-t)^2 - t^2)} = \underbrace{e^{-\frac{1}{2}(x-t)^2} e^{t^2/2}}_{*}$

$$M_Z(t) = \int_{\mathbb{R}} e^{tx} \overbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}^{\text{PDF}} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{\overbrace{tx - \frac{x^2}{2}}} dx$$

$$* \quad = e^{t^2/2} \int_R \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx$$

$f(x)$ for $N(t, 1^2)$

$$= e^{t^2/2} \quad M_Z(t)$$

Moment generate function for N

General N pdf

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$$

• integrat density over support $= 1$

$E[Z] = 0$

$$M_Z'(0) = t e^{t^2/2} \Big]_0 = 0$$

$Var[Z] = 1$ $\quad E[Z^2] = 1$

$$M_Z''(0) = t^2 e^{t^2/2} + e^{t^2/2} \Big]_0 = \underline{1}$$

- $X_1, \ldots, X_n \overset{iid}{\sim}$ Some distr with $\mu, \sigma^2 < \infty$ Finite

$$\bar{X} := \frac{X_1 + \ldots + X_n}{n}, \quad \underbrace{E[\bar{X}] = \mu, \quad SE[\bar{X}] = \frac{\sigma}{\sqrt{n}}}_{\text{Final \& how to derive}}$$

$$C_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$$ $\mu = 0 \quad SE = 1$

substract the meand and divide by SE get standarization

$E[C_n] = 0$

$SE[C_n] = 1$

Goal: Find distribution of $C_n$ if n large.

$$C_n := \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}\quad \frac{X_1+\dots+X_n}{n} - \frac{n\mu}{n}}{\cancel{\sqrt{n}}\quad \frac{\sigma}{\sqrt{n}}} = \frac{(X_1-\mu)+\dots+(X_n-\mu)}{\sigma\sqrt{n}}$$

$$= \frac{1}{\sqrt{n}}\left(\left(\frac{X_1-\mu}{\sigma}\right)+\dots+\left(\frac{X_n-\mu}{\sigma}\right)\right)$$

$$\frac{1}{\sqrt{n}}\left(Z_1+\dots+Z_n\right) = \frac{Z_1}{\sqrt{n}}+\dots+\frac{Z_n}{\sqrt{n}}$$