Math 241 Lec 18 11/17/15

Recall cont r.v.'s $\left(|Supp(X)| = |\mathbb{R}| > |\mathbb{N}|\right)$

$P(X=x) = 0 \quad \forall x \in Supp(X) \Rightarrow$ No PMF!

But $F'(x) = f(x)$ (the PDF)

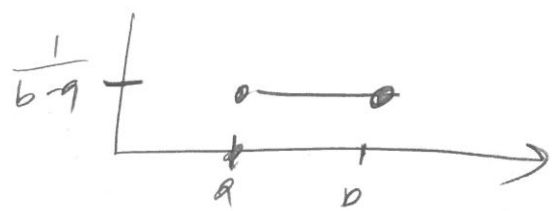
Recall $X \sim Unif(\{1, 7, 10\})$ discrete uniform

Now... $X \sim Unif(a,b) := \frac{1}{b-a}$ cont. uniform

$f(x) = \frac{1}{b-a}$ $\quad Supp(X) = [a,b]$ $\quad$ Param space $a, b \in \mathbb{R}$ s.t. $a<b$

Is this a PDF?

$\int_{x \in Supp(X)} f(x)\,dx = 1$? - - - - -



$\frac{1}{b-a} \overset{?}{>} 1$ Yes

$b = 1.1,\ a = 1.0$

$\Rightarrow f(x) = 10$!

What is CDF?

$F(x) = \int f(x)\,dx + C = \int \frac{1}{b-a}\,dx + C = \frac{x}{b-a} + C$

c? $F(a) = 0$

$\frac{a}{b-a} + C = 0 \Rightarrow C = -\frac{a}{b-a} \Rightarrow F(x) = \frac{x}{b-a} + -\frac{a}{b-a} = \boxed{\frac{x-a}{b-a}}$

$X \sim U(0,1)$ =: the "standard uniform"

parameter vals people call std ... depends on the r.v.

$f(x) = 1, \quad F(x) = x$ (super simple!)

gives me a random # between 0,1 ... most important to CS.

$$\mu = E(X) = \int_a^b x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{x^2}{2(b-a)}\Big]_a^b = \frac{b^2-a^2}{2(b-a)} = \boxed{\frac{b+a}{2}}$$ makes sense on graph

median $= \{x : F(x) \geq 0.5\}$ ~~In cont r.v.'s~~

$$F(x) = 0.5 \Rightarrow \frac{x-a}{b-a} = \frac{1}{2} = x-a = \frac{b-a}{2} \Rightarrow x = \frac{b-a}{2} + a = \frac{b-a}{2} + \frac{2a}{2} = \frac{b+a}{2}$$

$$\sigma^2 = \mathrm{Var}(X) = \int_a^b (x-\mu)^2 f(x)\,dx = E(X^2) - \mu^2 = \int x^2 f(x)\,dx - \mu^2 = \int_a^b \frac{x^2}{b-a}\,dx - \left(\frac{b+a}{2}\right)^2$$

$\underset{g(X)}{}$

$$= \frac{x^3}{3(b-a)}\Big]_a^b - \left(\frac{b+a}{2}\right)^2 = \frac{b^3-a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2$$

$$\begin{array}{l} \quad\; b^2+ab+a^2 \\ b-a \,\overline{)\, b^3 - a^3} \\ \quad -(b^3-ab^2) \\ \hline \quad\; ab^2 - a^3 \\ \quad -(ab^2 - a^2 b) \\ \hline \quad\; a^2 b - a^3 \\ \quad -(a^2 b - a^3) \\ \hline \quad\; 0 \end{array}$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4(b^2+ab+a^2) - 3(b^2+2ab+a^2)}{12} = \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12} \Rightarrow \boxed{\sigma = \frac{b-a}{\sqrt{12}}}$$

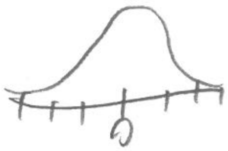
"Student" "normal" "Gaussian" "Bell"

$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ ← Bell Curve

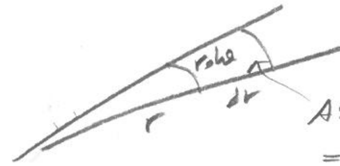
$> 0 \ \forall x$

why Z?

$Supp(Z) = \mathbb{R}$



$dA = dx\,dy$



$A \approx$ rectangle $= r\,dr\,d\theta$

Verify $\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$ $\cancel{= 2\int_0^\infty e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}}$ $\Rightarrow \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$

$\cancel{\Rightarrow \int_0^\infty e^{-\frac{x^2}{2}} dx = \frac{\sqrt{2\pi}}{2}}$ $\cancel{\Rightarrow \int_0^\infty e^{-u^2} \sqrt{2}\, du} = \frac{\sqrt{2\pi}}{2} = \int_{\mathbb{R}} e^{-u^2} du = \sqrt{\pi}$

~~let u =~~

let $u = \frac{x}{\sqrt{2}} \Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{2}} \Rightarrow dx = \sqrt{2}\,du \Rightarrow \frac{x^2}{2} = u^2$

$$\int_{\mathbb{R}} e^{-u^2} \sqrt{2}\, du = \sqrt{2\pi} \Rightarrow \underbrace{\int_{\mathbb{R}} e^{-u^2} du = \sqrt{\pi}}_{\text{Gaussian Integral}}$$

Consider

let $x^2 + y^2 = r^2$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x^2+y^2)} dx\,dy = \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy \overset{\text{if...}}{=} \pi$$

$x = r\cos\theta$
$y = r\sin\theta$

$$= \int_\theta \int_r e^{-r^2} r\,dr\,d\theta$$

$$dA = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta = r\,dr\,d\theta$$

$= r\cos^2\theta + r\sin^2\theta = r(1) = r$

let $u = r^2 \Rightarrow \frac{du}{dr} = 2r \Rightarrow dr = \frac{du}{2r}$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = \int_0^{\infty} e^{-r^2} r\,dr \int_0^{2\pi} d\theta = 2\pi \int_0^{\infty} e^{-r^2} r\,dr = 2\pi \int_0^{\infty} e^{-u} \cancel{r} \frac{du}{2\cancel{r}} = \pi\left[-e^{-u}\right]_0^{\infty}$$

$$= -\pi\left(\lim_{u\to\infty} e^{-u} - e^{0}\right) = \pi \checkmark$$

let $u = \frac{x^2}{2} \Rightarrow \frac{du}{dx} = x \Rightarrow dx = \frac{du}{x}$

$$\mu = E[Z] = \int_{x \in \text{Supp}(X)} x f(x)\,dx = \int_{\mathbb{R}} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{u_1}^{u_2} \cancel{x} e^{-u} \frac{du}{\cancel{x}} = -\frac{1}{\sqrt{2\pi}}\left[e^{-u}\right]_{u_1}^{u_2} = -\frac{1}{\sqrt{2\pi}}\left[e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty}$$

$$= -\frac{1}{\sqrt{2\pi}}(0-0) = 0$$

$$\sigma^2 = Var[Z] = E[Z^2] - \cancel{\mu^2} = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \underbrace{\ldots}_{\text{integration by parts}} = 1 \Rightarrow \sigma = 1$$

$\mu = 0, \sigma = 1 \Rightarrow Z$ is already standardized

$F(x)$? Need $\int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ indefinite integral... not available in closed form.
(See Risch algorithm)

for our purposes $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$ we use numerical integration... your calculator does this

$F(0) = 0.5$

$P(Z \in [-1,1]) = 0.68$ "68-95-99.7 Rule"
$P(Z \in [-2,2]) = 0.95$ "empirical rule"
$P(Z \in [-3,3]) = 0.997$ "3σ rule"

Recall $X \sim Exp(\lambda) := \lambda e^{-\lambda x}$

let $Y = 2X$ How is Y distributed? Never did this before!

$$F_Y(x) = P(Y \le x) = P(2X \le x) = P\left(X \le \frac{x}{2}\right) = F_X\left(\frac{x}{2}\right)$$

$$f_Y(x) = F_Y'(x) = F_Y'\left(\frac{x}{2}\right) = \frac{d}{dx}\left[1 - e^{-\lambda \frac{x}{2}}\right] = \frac{\lambda}{2} e^{-\frac{\lambda}{2}x} = Exp\left(\frac{\lambda}{2}\right)$$

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let $X \sim U(a,b)$, let $Y = cX + d$ distr of Y?

$E(Y) = cE(X) + d = c\left(\frac{a+b}{2}\right) + d$

$Var(Y) = c^2 Var(X) = c^2 \frac{(b-a)^2}{12}$

$$F_Y(x) = P(Y \le x) = P(cX + d \le x) = P\left(X \le \frac{x-d}{c}\right) = F_X\left(\frac{x-d}{c}\right)$$

$$f_Y(x) = F_Y'(x) = F_X'\left(\frac{x-d}{c}\right) = \frac{d}{dx}\left[\frac{\frac{x-d}{c} - a}{b-a}\right] = \frac{d}{dx}\left[\frac{\frac{a(x-d)}{ca} - \frac{c}{ca}}{b-a}\right] = \frac{d}{dx}\left[\frac{a(x-d) - c}{ca(b-a)}\right] = \frac{a}{ca(b-a)} = \frac{1}{c(b-a)}$$

$Supp(Y) = [ca + d, cb + d] \Rightarrow Y \sim Unif(ca + d, cb + d)$

not a function of $x \Rightarrow$ Unif

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let $Z \sim N(0,1)$, $X = \sigma Z + \mu$

$E(X) = \sigma E(Z) + \mu = \mu$

$Var(X) = \sigma^2 Var(Z) = \sigma^2 \Rightarrow SD(X) = \sigma$

How is X distr?

General normal density $N(\mu, \sigma^2)$!!

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$F_X(x) = P(X \le x) = P(\sigma Z + \mu \le x) = P\left(X \le \frac{x-\mu}{\sigma}\right) = F_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$f_X(x) = F_X'(x) = F_Z'\left(\frac{x-\mu}{\sigma}\right) = F_Z'(u)$$

let $u = \frac{x-\mu}{\sigma}$, $\frac{du}{dx} = \frac{1}{\sigma}$

$$\frac{d}{dx}\left[F_Z(u)\right] = \frac{d}{du}\frac{du}{dx}\left(F_Z(u)\right) = \frac{d}{du}\left(F_Z(u)\right)\frac{du}{dx} = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \frac{1}{\sigma}$$