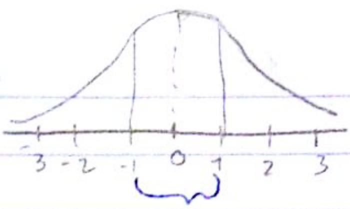
Nov 19

Lec 19

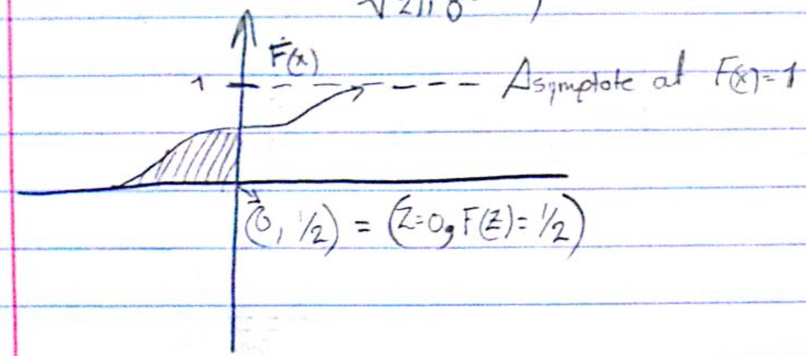
(...Continued)

Normal Dist.

$$Z \sim N(0,1) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$



$$X \sim N(\mu,\sigma^2) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}, \quad \mu \in \mathbb{R},\ \sigma^2 \in (0,\infty)$$



$Z = +1$

$Z = -2$

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ex. $X \sim N(70'', (3'')^2)$ ⟹ P(American Male has height more than 73''?)

Average Male Height

$\sigma = 3''$

Standardize 70'' & $(3'')^2$

$$P(X > 73'') = P\left(\frac{X - 70''}{3''} > \frac{73'' - 70''}{3''}\right) = P(Z > 1) = 16\%$$

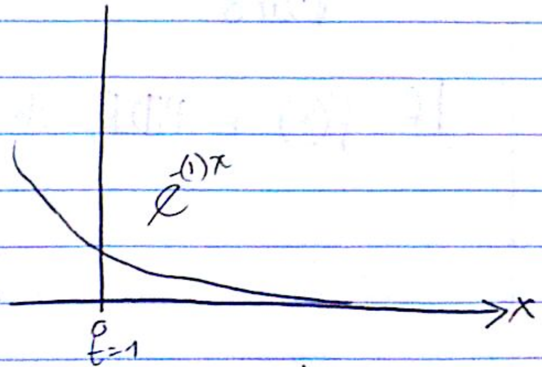
↳ Hw Answer $\underline{16}$ %

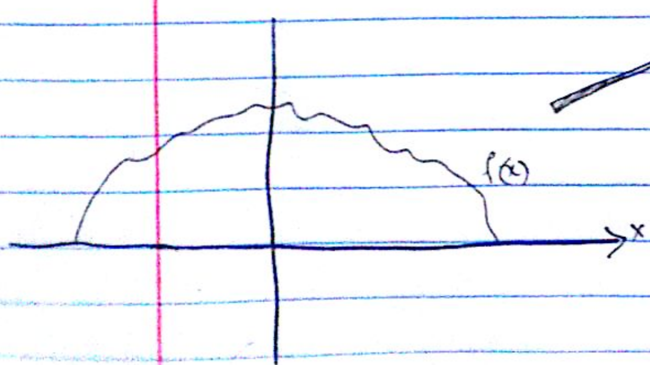
Units Wipe Out

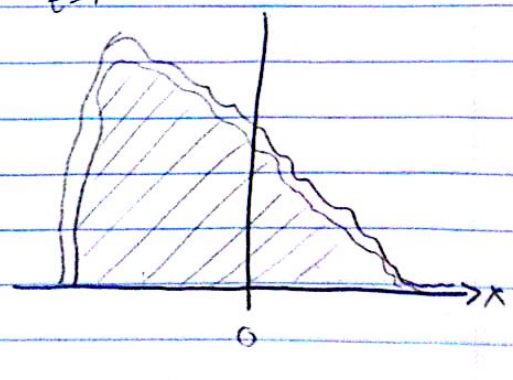
$$L(t) = \int_{\mathbb{R}} e^{-tx} f(x)\,dx$$

Laplace Transformation

Multiplication Of 2 functions







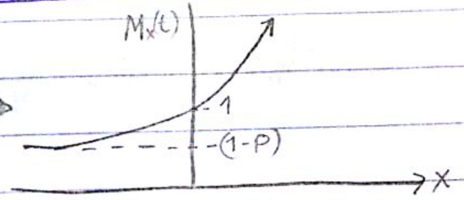
$L(t)$ → If it exists is One-to-One with $f(x)$

Why we care about this? $\int_{\mathbb{R}} e^{tx} f(x)\,dx$ "Continuous"

$$L(-t) = \boxed{M_x(t)} = E[e^{tX}]$$

$$\boxed{\text{If } X \overset{d}{=} Y \Rightarrow M_X(t) = M_Y(t)}$$

"discreete" $\sum_{\text{all } x} e^{tx} p(x)$



→ Moment Generating Function (MGF)

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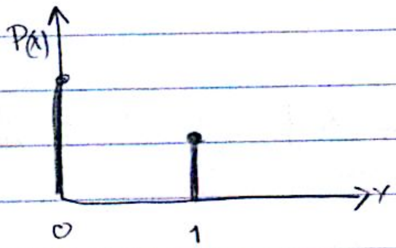
ex $X \sim \text{Bernoulli}(p)$



$$\underbrace{M_x(t) = E[e^{tX}] = \sum_{x=0}^{1} e^{tx} p^x (1-p)^{1-x} = 1-p+e^{t}p}_{e^{t(1)} p^{(1)} (1-p)^{1-(1)}}$$

$X \sim \text{Binomial}(n, p)$

$E[X]$
$E[X^2]$
$\vdots$
$E[X^{17}]$ → Painful to do! (without new trick)

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots = \sum_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!}(x-c)^i$$

Centered at "0" → $f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \dots + = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!}(x)^i$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$e^{tx} = 1 + tX + \frac{t^2X^2}{2!} + \dots$$

$$L(-t) = M_x(t) = E\left[e^{tX}\right] = E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \dots\right]$$

$$= 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \frac{t^4}{4!}E[X^4] + \dots$$

$$M_x'(t) = E[X] + \frac{t}{1!}E[X^2] + \frac{t^2}{2!}E(X^3) + \frac{t^3}{3!}E[X^4] + \dots$$

1st Derivative

At zero

$$M_x'(0) = E[X]$$

$$M_x''(t) = E[X^2] + \frac{t}{1!}E[X^3] + \frac{t^2}{2!}E[X^4] + \dots$$

2nd Derivative

At Zero

$$M_x''(0) = E[X^2] \dots \quad \text{I} \quad M_x^{(k)}(0) = E[X^k]$$

$$M_x(t) = 1 - p + e^t p$$

$$M_x'(t) = e^t p, \Rightarrow M_x'(0) = p = E[X]$$

$T = X_1 + X_2$ such that X_1, X_2 are independent.

$$M_T(t) = M_{X_1+X_2}(t) = E\left[e^{t(X_1+X_2)}\right] = E\left[e^{tX_1}e^{tX_2}\right] = E\left[e^{tX_1}\right]E\left[e^{tX_2}\right]$$

Only When Independent $= M_{X_1}(t)\,M_{X_2}(t)$

$$T = \sum_{i=1}^{n} X_i \qquad M_T(t) = \prod_{i=1}^{n} M_{X_i}(t) = \left(M_X(t)\right)^n$$

all independent

Only When Identically Distributed

$Y = aX + c$

$$M_Y(t) = M_{aX+c}(t) = E\left[e^{t(aX+c)}\right] = E\left[e^{taX+tc}\right] = E\left[e^{taX}e^{tc}\right] = e^{tc}E\left[e^{taX}\right]$$

$$= e^{tc}M_X(ta)$$

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ex

$X \sim \text{Binomial}(n, p)$

$$M_X(t) = E\left[e^{tx}\right] = \sum_{x=0}^{n} \underbrace{e^{tx}}_{(e^t)^x}\binom{n}{x}p^x(1-p)^{n-x} = \underbrace{\sum_{x=0}^{n}\binom{n}{x}(e^tp)^x(1-p)^{n-x}}_{\text{Binomial Theorem}}$$

$$= (1-p+e^tp)^n$$

$$M_Y(t) = (M_X(t))^n = \underbrace{(1-p+e^tp)^n}_{\text{This must be a binomial.}}$$

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ex.

$X \sim Exp(\lambda) := \lambda e^{-\lambda x}$

$$M_X(t) = E\left[e^{tx}\right] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx = \lambda\int_0^\infty e^{(t-\lambda)x}\,dx = \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)x}\right]\Big|_0^\infty$$

$$= \frac{\lambda}{t-\lambda}\left[\lim_{x\to\infty} e^{(t-\lambda)x} - 1\right]$$

$$= \frac{\lambda}{\lambda-t} \quad \text{Iff } t < \lambda$$

Watch "The Grandfather"

Rule # 3

$$M_Y(t) = M_X(at) = \frac{\lambda\,(1/a)}{\lambda - at(1/a)} = \frac{\lambda/a}{\underbrace{\lambda/a}_{\lambda'} - t} = \frac{\lambda'}{\lambda' - t} \Longrightarrow Y \sim Exp(\lambda')$$

$$= Exp\left(\frac{\lambda}{a}\right)$$

MGF of

$$M_Z(t) = E(e^{tZ}) = \int_{\mathbb{R}} e^{tx} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \int_{\mathbb{R}} \frac{e^{-1/2(x-t)^2}}{\sqrt{2\pi}} e^{t^2/2} dx =$$

$$tx = \frac{t^2}{2}, \quad -1/2(x^2 - 2tx) = -1/2((x-t)^2 - t^2)$$

$$e^{t^2/2} \int \underbrace{\frac{e^{-1/2(x-t)^2}}{\sqrt{2\pi}}}_{} dx = \boxed{e^{\frac{t^2}{2}}}$$

PDF of the General Normal R.V. with $N(t, 1^2)$

$X \sim N(t, 1)$

$$:= \frac{e^{-1/2(x-t)^2}}{\sqrt{2\pi}}$$

$$E[Z] = M_Z'(0) = \left[t e^{t^2/2}\right] = 0$$

$$E[Z^2] = M_Z''(0) = t^2 e^{t^2/2} + e^{t^2/2}\Big]_0 = 1$$

$X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ Some Distribution

$m+n, \mu, \sigma^2 < \infty$

$$\bar{X} = \frac{X_1 + \ldots + X_n}{n}, \quad E[\bar{X}] = \mu, \quad SE[\bar{X}] = \frac{\sigma}{\sqrt{n}}$$

$$C_n := \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}, \quad E[C_n]=0, \quad St[C_n]=1, \quad Var[C_n]=1, \quad E[C_n^2]=1$$

How is C_n distributed if n is large?

$$\frac{\frac{X_1+\dots+X_n}{n} - \frac{\overbrace{n\mu}^{\mu_1+\mu_2+\dots+\mu_n}}{\mu}}{\frac{\sigma}{\sqrt{n}}} = \frac{\frac{(X_1-\mu)+\dots+(X_n-\mu)}{n}}{\frac{\sigma}{\sqrt{n}}} \frac{\sqrt{n}}{\sqrt{n}} = \frac{(X_1-\mu)+\dots+(X_n-\mu)}{\sigma\sqrt{n}}$$

$$= \frac{1}{\sqrt{n}}\left(\frac{X_1-\mu}{\sigma}+\dots+\frac{X_n-\mu}{\sigma}\right) = \frac{1}{\sqrt{n}}\left(Z_1+\dots+Z_n\right) = \frac{Z_1}{\sqrt{n}}+\dots+\frac{Z_n}{\sqrt{n}}$$