Nov 24

Lecture 20

$$Z \sim N(0,1) \Rightarrow M_Z(t) = e^{t^2/2}$$

$X_1, \ldots, X_2 \overset{iid}{\sim}$ Some Distribution μ, σ^2

Goal!
How is C_n distributed if n is large?

C's of m

$$C_n = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

→ The Standardized $\bar{X}$

SE = 1

mean = 0 = μ

$$C_n = \frac{Z_1}{\sqrt{n}} + \cdots + \frac{Z_n}{\sqrt{n}}$$ such That:

$$\boxed{Z_i := \frac{X_i - \mu}{\sigma}}$$

$$M_{C_n}(t) \overset{iid}{=} \left(M_{\frac{Z}{\sqrt{n}}}(t)\right)^n = \left(M_Z\left(\tfrac{t}{\sqrt{n}}\right)\right)^n = \left(1 + \frac{t}{\sqrt{n}}\underbrace{E[Z]}_{\to 0} + \underbrace{\left(\frac{t^2}{\sqrt{n^2}}\right)\frac{E[Z^2]}{2!}}_{\to =1} + \underbrace{\frac{t^3}{n^{3/2}}\frac{E[Z^3]}{3!} + \ldots}_{\to = e}\right)^n$$

$$M_X(t) = E[e^{tX}] = E\left[1 + tX + \frac{t^2X^2}{2!} + \ldots\right]$$

Taylor Series

$E[Z] = 0$

$$E[Z^2] = 1 \Rightarrow \begin{cases} 1 - 0 - E[X^2] \\ \sigma^2 - (E[X])^2 = E[X^2] \end{cases}$$

Recall $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Goal:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$\frac{1}{n^2}$ → Becomes 0 very quickly

$$\lim_{n\to\infty}\left(1+\frac{t^2/2}{n}+o\left(\frac{1}{n}\right)\right)^n$$

n	lim
10^2	5.04
10^3	4.48
⋮	3.49
10^{12}	2.718... $\approx e$

$\boxed{\frac{1}{n^2} \text{ is } O\left(\frac{1}{n}\right)}$

$\boxed{\frac{1}{n^2} \le \frac{1}{n}}$ Because $\frac{1/n^2}{1/n} = \boxed{\frac{1}{n} \to 0 \text{ as } n\to\infty}$

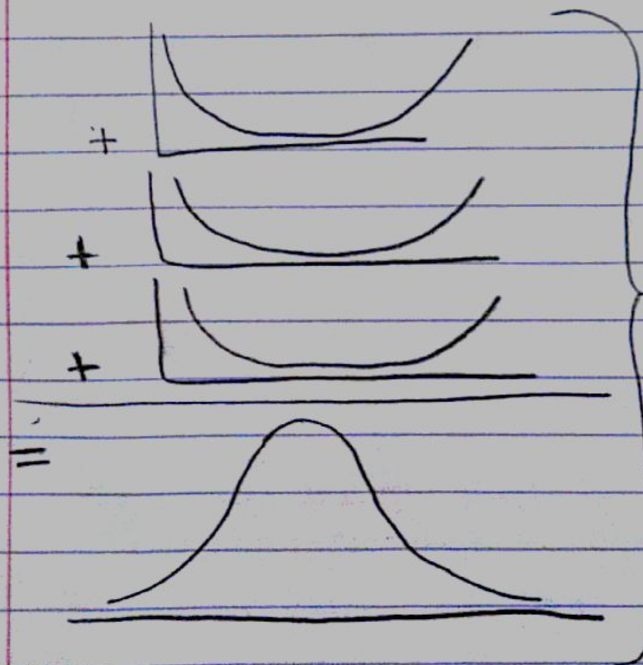
Any function $\le \frac{1}{n}$ Does Not affect $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$$\lim_{n\to\infty}\left(1+\frac{t^2/2}{n}+\underbrace{O\left(\frac{1}{n}\right)}_{f(n)\le\frac{1}{n}}\right)^n = e^{t^2/2}$$

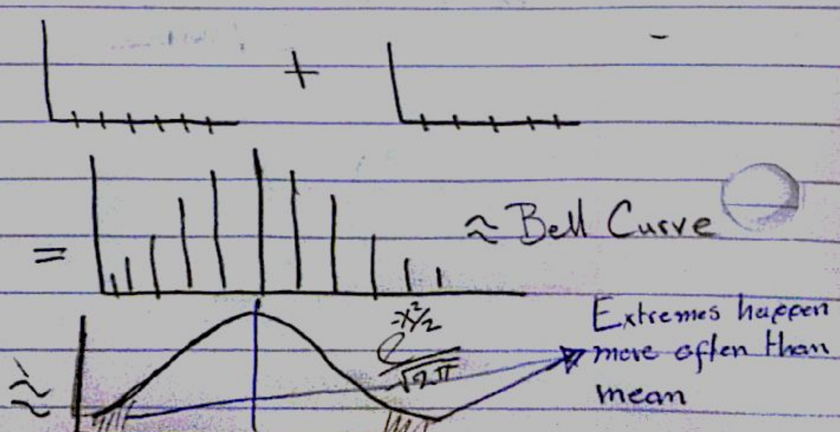
Holy Grial Of This Class

Central Limit Theorem $\boxed{C_n = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \text{ if } n \text{ large}}$

$$\bar{X} = \frac{\sigma}{\sqrt{n}}C_n+\mu \sim N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right), \quad T = n\bar{X} \sim N\left(n\mu, (\sqrt{n}\sigma)^2\right)$$



similar to:



say "flipping Coins"

$X_1, \ldots, X_{30} \overset{iid}{\sim} Geom(1/2)$

$P\left(\begin{matrix}\text{you wait more than 2.75} \\ \text{flips on average?}\end{matrix}\right) \rightarrow P(\bar{X} > 2.75)$

average? → $\bar{X}$

$\bar{X} \sim N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) =$

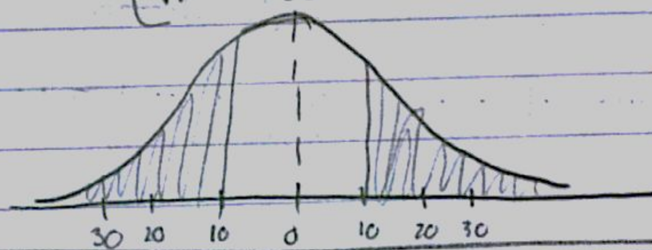
By central Limit Theorem

$\mu = \frac{1}{p} = 1/\tfrac{1}{2} = 2$

$\sigma = \sqrt{\frac{1-p}{p^2}} = \sqrt{\frac{1-1/2}{\left(\frac{1}{2}\right)^2}} = \sqrt{2}$

$\frac{\sigma}{\sqrt{n}} = \sqrt{\frac{2}{30}} = 0.258$

$= N(2, (0.258)^2)$



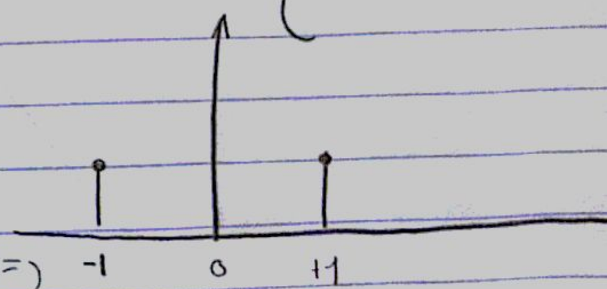
$P(\bar{X} > 2.75) = P\left(\frac{\bar{X}-2}{0.258} > \frac{2.75-2}{0.258}\right) \approx P(Z > 3) = 0.0015$

Random Walk forward/backward:
for 100 steps.

$X_1, \ldots, X_{100} \overset{iid}{\sim} \begin{cases} +1 & \text{wp } 1/2 \\ -1 & \text{wp } 1/2 \end{cases}$

$P\left(\begin{matrix}\text{You see more than 10 steps away} \\ \text{from your starting point?}\end{matrix}\right)$

$T \sim N\left(\mu n, (\sqrt{n}\sigma)^2\right) = N(0, 10^2)$



$P(|T| > 10) = 2P(T > 10) = 2P\left(\frac{T-0}{10} > \frac{10-0}{10}\right) =$

$\mu = 0$

$\sigma = 1$

$2P(Z > 1) = 2 * (0.16) = 0.32$

$$P(\text{A shipment being late}) = 2\% = 0.02$$

$$P\left(\text{That on average, more than } 3\% \text{ are late among } 10{,}000 \text{ orders}\right)$$

$$X_1, \ldots, X_{10{,}000} \overset{iid}{\sim} \text{Bernoulli}(0.02)$$

$$\underline{P(\bar{X} > 0.03)}, \quad \bar{X} \overset{\text{by CLT}}{\sim} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) = N(0.02, 0.0014^2)$$

$$\mu = p = 0.02$$

$$\frac{\sigma}{\sqrt{n}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.02(0.98)}{10{,}000}} \approx 0.0014$$

$$P(\bar{X} > 0.03) = P\left(\frac{\bar{X} - 0.02}{0.0014} > \frac{0.03 - 0.02}{0.0014}\right) = P(\bar{X} > 7.14) = 13 \times 10^{-12} \approx 0$$

$$X_1, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$$

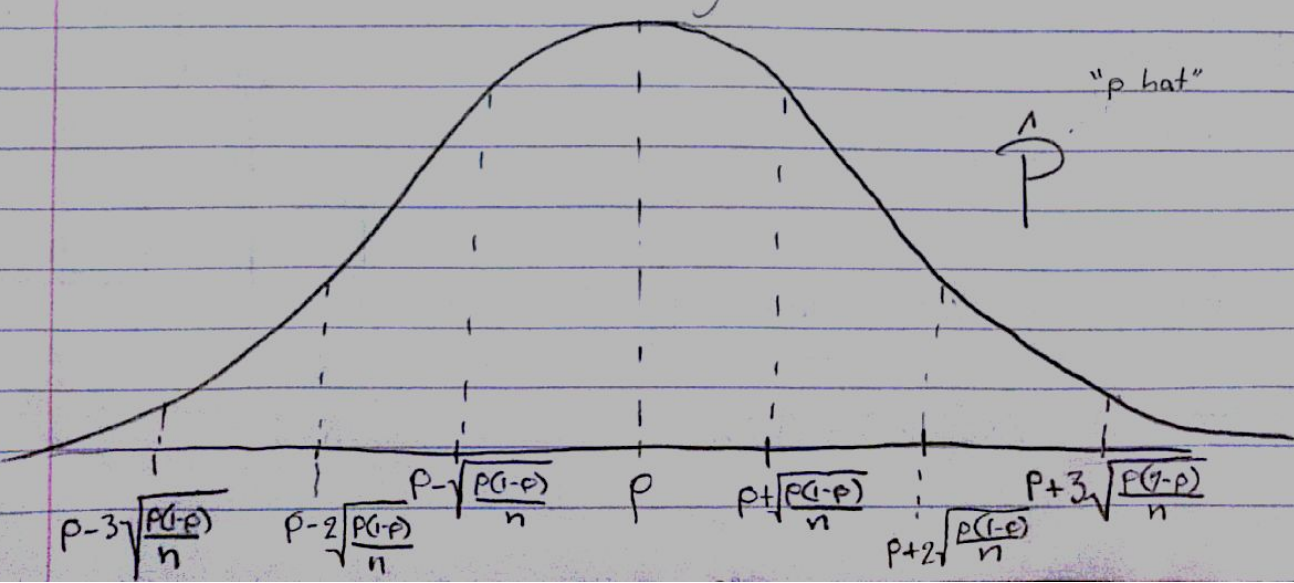
$\frac{\text{How many 1's}}{n}$

$$\bar{X} = \frac{\# 1\text{'s}}{n} = \hat{p}$$

$\hat{p}$ → "Sample Proportion" → "p hat"

$\bar{X}$ → is also a realization of (big) $\bar{X}$

$$\Rightarrow \hat{P} = \bar{X} \sim N\left(p, \left(\sqrt{\frac{p(1-p)}{n}}\right)^2\right)$$



end of Prob.

Begin of Stats.

$\hat{P} = \frac{7}{18} = 0.39$

#Of ppl. who lives mushrooms in this class

$\hookrightarrow n =$ # Of people in this class.

What is probability?

1. Prob: Model Parameters.
Probability That Someone Lives Mushrooms

2. Population Proportion
Imagine a "real percentage" of mushroom Lovers worldwide, humanwide.

a) Goal: $\boxed{\text{Find}}\ \boxed{p}$ $\Rightarrow$ CAN'T = Impossible P is unknowable

b) Goal: Estimate p using a sample $\Rightarrow$ It is possible. How? $\rightarrow$ Use $\hat{p}$ to guess/infer p.
$\hat{P}$ is knowable
$\hookrightarrow$ "p hat"

Population Parameters are unknowable!

Best Guess of p? $\rightarrow \hat{p}$ is the best guess, since $\hat{p} \approx p$. Because $\hat{P} \xrightarrow[\text{by CLT}]{} p$
(By Central Limit Thm)

Interval Guess of p?