MATH 241 Fall 2015 Homework #3

Professor Adam Kapelner

Sunday 20$^{\text{th}}$ September, 2015

Problem 1

A little bit of a philosophy.

(a) [easy] What are some problems with the long run frequency definition of probability?

(b) [easy] How did Chevalier de Mere in 1654 know that the $\mathbb{P}\left(\text{one or more double sixes in 24 rolls of two dice}\right) < \frac{1}{2}$?

(c) [easy] What are some problems with the propensity definition of probability?

(d) [easy] What idea(s) inspired Karl Popper to invent the propensity definition?

(e) [easy] What is the main problem with the subjective definition of probability?

(f) [easy] What is the difference between probabilities that are objective and probabilities that are epistemic?

(g) [easy] Why would you call an objective probability "random" but not an epistemic probability? Explain.

(h) [easy] If all information was known, would there still be epistemic probabilities? Yes/no is fine.

(i) [easy] According to Laplace (and his interpretation of Newton), if all information was known about physical systems including all laws and all initial conditions, would there be randomness? Yes/no and discuss.

(j) [difficult] According to Laplace, what is randomness? I've uploaded LaPlace's quote in lecture 5 on the course homepage. You can answer this in a few words.

(k) [difficult] Knowing what we known in the 21st century, if all information was known about physical systems including all laws and all initial conditions, would there be randomness? If so, what theory has demonstrated evidence for this?

(l) [difficult] What upset Einstein in 1926 to say "God does not play dice with the universe?"

(m) [difficult] What is the prevailing theory about why probability wasn't formalized using mathematics prior to the 1600's?

(n) [E.C.] Why is the logical definition of probability "silly" (in my words)?

Problem 2

We will get our feet wet with basic "axioms" and theorems. Assume all capital letters are sets. If the problem asks you to prove a fact, you may only use your knowledge of set theory and the definition of $\mathbb{P}(\cdot)$ given in the book / lecture. Most of the answers are in the book or in my lecture notes. Try to do them yourself and only use the book if you are having trouble.

(a) [easy] List (1) all assumptions prior to and (2) the three conditions that make $\mathbb{P}(\cdot)$, the set function that returns a probability. These latter three conditions are also known as the "axioms of probability."

(b) [E.C.] Explain why the three conditions are not technically axioms.

(c) [easy] Prove that if A_1 and A_2 are disjoint (mutually exclusive), $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2)$.

(d) [easy] Prove that $\mathbb{P}(\varnothing) = 0$.

(e) [harder] Prove that $\mathbb{P}(A) \leq 1$.

(f) [difficult] Assuming the previous theorem that $\mathbb{P}(A) \leq 1$, prove that $\mathbb{P}(A) \in [0, 1]$.

(g) [difficult] Prove that if $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

(h) [difficult] Prove the law of inclusion-exclusion for two arbitrary sets: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A, B)$ (in the notes).

(i) [harder] State the general law of inclusion-exclusion for arbitrary n i.e. $\mathbb{P}(\cup_{i=1}^{n} A_i) = ?$

(j) [E.C.] On a separate sheet of paper, prove the general law of inclusion-exclusion.

(k) [difficult] Some authors for condition (a), which is $\mathbb{P}(A) \geq 0 \; \forall A$, use instead the condition $\mathbb{P}(A) \in [0, 1]$. Why is this addition detail unncessary?

(l) [difficult] Prove that if $A = \{\omega_1, \omega_2, \ldots\}$ possibly infinite in cardinality, then

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(\{\omega_i\})$$

Hint: look at the proof of why $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$ in the case of equally likely outcomes.

(m) [E.C.] Prove what I bungled in class: if A is non-empty then $\mathbb{P}(A) > 0$.

(n) [E.C.] Describe a sequence of sets $A_1, A_2, \ldots$ which are all non-empty where $\sum_{i=1}^{\infty} \mathbb{P}(A_i) = 1$. "Describe" means to explicitly state the elements in each of the sets.