9/8/15 Math 241 Lecture #4

Review 5 flowers, 3 B, 2R

$\overbrace{3!\,2!}$ → each arrangement has this many (12)

$$P(RBRBB) = \cancel{RBR} \; \frac{\underline{2}\;\underline{3}\;\underline{1}\;\underline{2}\;\underline{1}}{5!} = \frac{1}{10}$$

$5!$ ← lets universe be all permutations

$$= \frac{1}{\frac{5!}{3!\,2!}} = \frac{1}{10}$$

1 ← of which it's 1

$\frac{5!}{3!\,2!}$ ← lets universe be all ~~colorings~~ permutations where you don't R,B

$$P(R \text{ is first}) = \frac{2(4!)}{5!} = \frac{2}{5}$$

$$P(R \text{ is first}) = \frac{?}{\frac{5!}{3!\,2!}} = \frac{4}{10} = \frac{2}{5}$$

$\underline{R}\;(\underbrace{BRBB})$

$\frac{4!}{3!\,1!} = 4$

$|A| = n$

$|2^A| = 2^{|A|} = 2^n$

$= \{B : B \subseteq A\} = \{B : B \subseteq A, |B| = 0\} \cup$
$\{B : B \subseteq A, |B| = 1\} \cup$
$\vdots$

$$2^A = \bigcup_{i=0}^{n} \{B : B \subseteq A, |B| = i\} \quad \text{all disjoint}$$

$$|2^A| = |\cup| = \sum_{i=0}^{n} |\{B : B \subseteq A, |B| = i\}|$$

$$= \sum_{i=0}^{n} \binom{n}{i} = 2^n$$

e.g. $2^4 = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$
$= 1 + 4 + 6 + 4 + 1$
$= 16$ ✓

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = (a+b)(a+b)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

$2 \times 2 \times 2 = 8$ terms $\quad (a+b)^n$ has 2^n terms

$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b) = \binom{4}{4}a^4 + \binom{4}{3}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{1}ab^3 + \binom{4}{0}b^4$$

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \quad \text{(Binomial Thm.)}$$

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} x^i = \sum_{i=0}^{n} \binom{n}{i} x^i$$

$$= \underbrace{\binom{n}{0} x^0}_{1} + \sum_{i=1}^{n-1} \binom{n}{i} x^i + \underbrace{\binom{n}{n} x^n}_{x^n}$$

$$= (1+x)(1+x)^{n-1}$$

$$= (1+x) \sum_{i=0}^{n-1} \binom{n-1}{i} x^i$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{i} x^i + \sum_{i=0}^{n-1} \binom{n-1}{i} x^{i+1}$$

$$\cancel{\sum_{i=0}^{n} \left(\binom{n-1}{i-1} + \binom{n-1}{i} \right) x^i}$$

$$\cancel{= \sum_{i=0}^{n} \binom{n-1}{i-1} x^i + \sum_{i=0}^{n} \binom{n-1}{i} x^i}$$

$$\cancel{= \sum_{i=0}^{n-1} \binom{n-1}{i-1} x^i + x^n + \sum}$$

~~let $j = i+1$~~

~~$\Rightarrow i = j-1$~~

~~$i=0 \Leftrightarrow j=1$~~

~~$i=n-1 \Leftrightarrow j=n$~~

$$\cancel{\sum_{j=1}^{n} \binom{n-1}{j-1} x^j}$$

$$\cancel{= \sum_{j=0}^{n} \binom{n-1}{j-1} x^j}$$

$$\underbrace{\binom{n-1}{0}}_{1} x^0 + \binom{n-1}{1} x^1 + \binom{n-1}{2} x^2 + \dots + \binom{n-1}{n-2} x^{n-2} + \binom{n-1}{n-1} x^{n-1}$$

$$+ \binom{n-1}{0} x^1 + \binom{n-1}{1} x^2 + \cancel{\binom{n-1}{2} x^3} + \dots + \binom{n-1}{n-2} x^{n-1} + \underbrace{\binom{n-1}{n-1} x^n}_{x^n}$$

$$\cancel{1} + \sum_{i=1}^{n-1} \binom{n}{i} x^i + \cancel{x^n} = \cancel{1} + \sum_{i=1}^{n-1} \left(\binom{n-1}{i} + \binom{n-1}{i-1} \right) x^i + \cancel{x^n}$$

Polynomial equal
Implies $\Rightarrow$
Coefficients
equal

$$\Rightarrow \binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$$

~~$\vec{a} \cdot \vec{x} = 0 \quad \forall x \Rightarrow \vec{a} = \emptyset$~~

~~$\|a\| \|x\| \cos(\theta) = 0$ Assume not $\|a\| > 0$~~

~~let $\vec{a} \neq \vec{x} = \vec{1} \Rightarrow \underset{>0}{\|a\|} \, \underset{>0}{\|x\|} \, \underset{\neq 0}{\cos\theta} = 0 \quad \perp$~~

~~$x \neq 0$~~

o/t big algebra mess

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & + & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & + & \binom{3}{1} & = & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

Note $\binom{3}{1} = \binom{2}{0} + \binom{2}{1}$ by above formula

thus the Δ of #'s can be built simply by adding the two #'s above it

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$\binom{5}{3}$

Pascal's Δ

Poker! 52 cards 5 card hands, order doesn't matter

# of royal flushes

$$P(\text{Royal Flush}) = \frac{|\text{Royal Flush}|}{\binom{52}{5}}$$

$\binom{52}{5} \to 2,598,960$

10, J, Q, K, A (SAME SUIT)

Only 4 suits $\Rightarrow$ only 4

$\binom{4}{1}\binom{5}{5}$

$$\frac{4}{\binom{52}{5}} \approx \frac{1}{600,000}$$

$P(\text{Straight flush}) = \dfrac{|\quad|}{\binom{52}{5}}$

A2345
23456
⋮
678910
78910J
910JQK
10JQKA

} Straight but same suit

choose the suit
↓
$\binom{4}{1}\binom{10}{1} - 4$ ↙ 10 starting pos's for str

↑ # royal flushes
(don't double count)

D ≡ } # straights
S ≡ } ✓
C ≡ } ↙ same!
H ≡ } S

St fl
roy fl

ranks, all suits, all remaining ranks, one remaining suit

$P(4\text{-of-a-kind}) = \dfrac{|\quad|}{\binom{52}{5}} \to \underbrace{\binom{13}{1}\binom{4}{4}}_{4\text{-of-a-kind}}\ \underbrace{\binom{12}{1}\binom{4}{1}}_{\text{remaining card}}$

AAAA
2222
⋮
KKKK

$P(\text{full house}) = \dfrac{|\quad|}{\binom{52}{5}} = \underbrace{\binom{13}{1}\binom{4}{3}}_{3\text{ of a kind}}\ \underbrace{\binom{12}{1}\binom{4}{2}}_{2\text{ of a kind}}$ ≠ come back to this ...

$\binom{13}{2}\binom{4}{3}\binom{4}{2}$?

$13\binom{4}{3}12\binom{4}{2} \neq \underbrace{\frac{13\cdot 12}{2}\binom{4}{3}\binom{4}{2}}$

↳ undercounting by 50%!

$\overbrace{\text{AAA}}^{3\text{-of-a-kind}}\underbrace{22}_{2\text{-of-a-kind}}$

AAA22 ≠ AA222!

choose suit

choose 5 ranks of 13

$$P(\text{flush}) = \frac{|\cdot|}{\binom{52}{5}} \longrightarrow \binom{4}{1}\binom{13}{5}$$

All cards same suit

starting position   Str. fl's

$$P(\text{straight}) = \frac{|\cdot|}{\binom{52}{5}} \longrightarrow \binom{10}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1} - \underbrace{\binom{10}{1}\binom{4}{1}}_{\text{Str. fl's}}$$

3 of a kind   2 remaining cards

$$P(3\text{ of a kind}) = \frac{|\cdot|}{\binom{52}{5}} \longrightarrow \underbrace{\binom{13}{1}\binom{4}{3}}_{\text{3 of a kind}}\underbrace{\binom{12}{2}\binom{4}{1}\binom{4}{1}}_{\text{2 remaining cards}}$$

Why not $\binom{48}{2}$?

$$\frac{12\cdot 11}{2}\cdot 4\cdot 4 < \frac{48\cdot 47}{2}$$

full house!

why not $\binom{12}{1}\binom{4}{1}\binom{11}{1}\binom{4}{1}$

AAA 2D 3C $\neq$ AAA 3C 2D   order doesn't matter

$$P(2\text{ pair}) = \frac{|\cdot|}{\binom{52}{5}}$$

$$\underbrace{\binom{13}{2}\binom{4}{2}\binom{4}{2}}_{\text{two pair}}\underbrace{\binom{11}{1}\binom{4}{1}}_{\text{remaining card}}$$

$$P(\text{no pair}) = \frac{|\cdot|}{\binom{52}{5}} \longrightarrow \binom{13}{5}\binom{4}{1}^5 - \underbrace{\text{straights}}_{\binom{10}{1}\binom{4}{1}^5} - \underbrace{\text{flushes}}_{\binom{4}{1}\binom{13}{5}} + \underbrace{\text{str fl's}}_{\binom{4}{1}\binom{10}{1}}$$

Previously, $\Omega = \{H, T\}$ $P(\{H\}) = \frac{|\{H\}|}{|\Omega|}$ ω was chosen out of Ω and each ω was equally likely

Now, $\Omega = \{R, G, B\}$

Def outcomes are equally likely if

$$P(\{\omega\}) = \frac{1}{|\Omega|} \quad \forall \omega \in \Omega$$

coin flips, dice rolls, cards, ...

$$P(\{R\}) \neq \frac{|\{R\}|}{|\Omega|} = \frac{1}{3}$$

So $P(A) = \frac{|A|}{|\Omega|}$ is NOT a good definition!

day is ...

$\Omega = \{\text{sunny}, \text{cloudy}, \text{rainy}, \text{snowy}\}$

$P\{\text{snowy}\} \neq \frac{1}{4}$

Further ... $\Omega = \{H, T\}$ How do I even know it's equally likely??

$P(\{H\}) = P(\{T\}) = \frac{1}{2}$?

Maybe its a bad assumption?

Need a better def. of prob!

(I) Limiting Frequency Def.

let $\mathbb{1}_{\omega \in A} = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$

$$P(A) := \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \mathbb{1}_{\omega_i \in A}}{n}$$ } prop of time the event A occurs given many, many outcomes of the process

Problems

1) ~~Who chooses $\omega \in \Omega$ and~~ is it guaranteed to be the same draw each time?
Nature is predictable? Nature works the same way all times?

2) We never see $n = \infty$! So we only get approx probs

$P(A) \approx \frac{\sum \mathbb{1}_{\omega_i \in A}}{n}$ for n as large as we can afford

3) Why does this limit converge?  $\lim_{x \to \infty} \sin(x)$ d.n.e. (Can be proven if #1 is assumed)

4) Non-general! P(OJ Simpson guilty)? P(raining tomorrow at 3PM in 11367)