Lecture 13 Math 241

1

Can $T=5$? Yes

$$6\left\{\begin{array}{l} \underline{1}\ \underline{1}\ \underline{0}\ \underline{0}\ \underline{1} \\ \underline{1}\ \underline{0}\ \underline{1}\ \underline{0} \\ \underline{1}\ \underline{0}\ \underline{0}\ \underline{1} \\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{1} \\ \underline{0}\ \underline{1}\ \underline{1}\ \underline{0} \\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{1} \end{array}\right.$$

$\underbrace{\qquad\qquad}_{X-1}$ $\to r-1$

$$P(T=5) = 6\,(1-p)^2\,p^3$$

$6 = \binom{4}{2}$

Always p^3 why? Always 3 successes!

Always $X-3$ failures

$$X \sim \text{NegBin}(r,p) := \binom{x-1}{r-1}(1-p)^{x-r}p^r = \underbrace{\binom{x-1}{r-1}(1-p)^{x-r}p^{r-1}}_{X\sim \text{Binomial}(x-1,\,p),\ P(X=r-1)}\,p$$

$$\text{Supp}(X) = \{r, r+1, r+2, \dots\} = \mathbb{N}\setminus\{1,\dots,r-1\}$$

$$\sum_{x\in \text{Supp}(X)} p(x) = 1\ ?$$

$i=6$

$i-1=5$

$i-1 \to x^5$

$5x^4$

$5\cdot 4\, x^3$

$5\cdot 4\cdot 3\, x^2$

$\uparrow$ $i-1$

$r-1=$ diff 3 times $\Rightarrow r=4$

Note the geometric series ← totally legal relation

$$\frac{1}{1-x} = \sum_{i=1}^{\infty} x^{i-1}$$ Let's take $r-1$ derivatives of each side

if $|x|<1$

if $x=1-p$ ✓

$$(r-1)!\,(1-x)^{-r} = \sum_{i=r}^{\infty} \underbrace{(i-1)(i-2)\cdots(i-r+1)}_{\frac{(i-1)!}{(i-r)!}}\,x^{i-r}$$

$\uparrow$ $i<r$ has terms $=0$

$$\Rightarrow (1-x)^{-r} = \sum_{i=r}^{\infty}\binom{i-1}{r-1}x^{i-r}$$ mult. both sides by $(1-x)^r$

$$1 = \sum_{i=r}^{\infty}\binom{i-1}{r-1}(1-x)^r x^{i-r}$$

let $x=1-p$ Done...

Equivalent parametrization

$r = \#$ successes $r \in \mathbb{N}$

$X = \#$ failures

$r=3$

$$\underbrace{\underline{0}\ \underline{1}\ \underline{0}\ \underline{0}\ \underline{1}}_{x+r-1}\ \underline{1}$$

$$P(T=x) = \binom{x+r-1}{x}(1-p)^x p^r \qquad \text{Supp}(X) = \mathbb{N}_0$$

Some textbooks do this... Why? Easier to think about # failures since you know stopping time is this $+ r$.

FYI... Not responsible for this...

$$= \frac{(x+r-1)!}{x!\,(r-1)!} = \frac{(x+r-1)(x+r-2)\cdots r}{x!} = (-1)^x \frac{(-r)(-r-1)\cdots(-r-x+1)}{x!}$$

$$= (-1)^x \binom{-r}{x}$$

Hence Negative binomial

looks like binomial with a negative sign

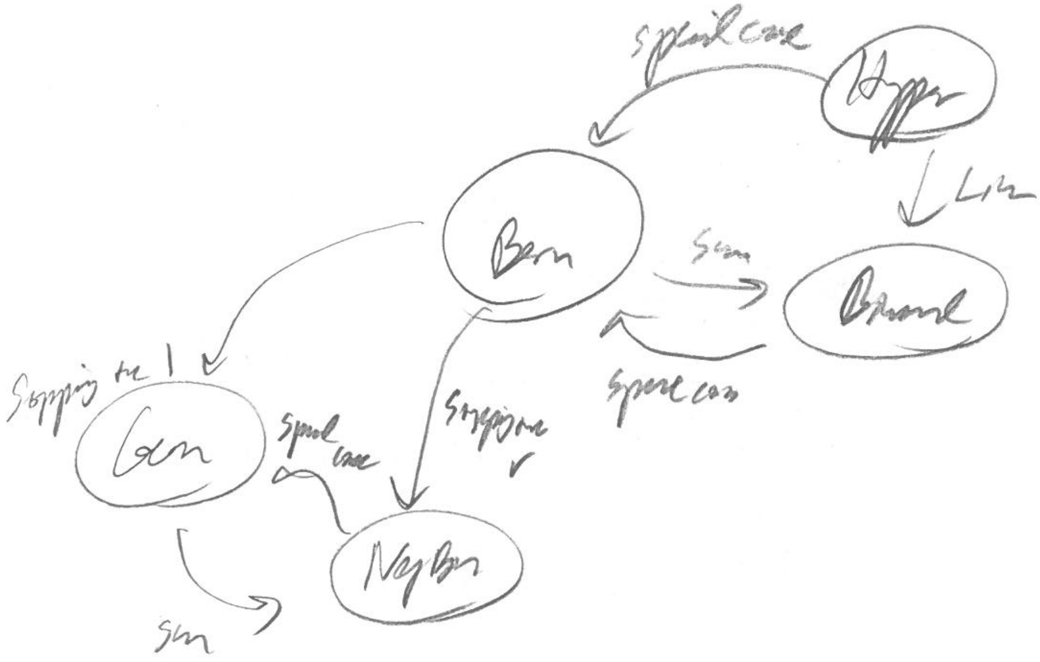
Roll Dice until you get 17 6's. Prob you get this on 107th roll.

$$X \sim NegBin\left(17, \tfrac{1}{6}\right) \qquad P(X=107) = \binom{106}{16}\left(\tfrac{5}{6}\right)^{90}\left(\tfrac{1}{6}\right)^{17}$$

$$NegBin(1,p) = \underbrace{\binom{x-1}{1-1}}_{1}(1-p)^{x-1}p = Geom(p) \quad \text{Why?}$$

$X_1, \dots, X_r \overset{iid}{\sim} Geom(p)$ Same reason $Binom(1,p) = Bern(p)$

$X_1 + \dots + X_r \sim NegBin(r,p)$ Why??



"Special function"

$$F(x) = P(X \le x) = \sum_{i=r}^{x} \binom{i-1}{r-1}(1-p)^{i-1} p^r$$

$$= 1 - P(X > x) = 1 - \sum_{i=x+1}^{\infty} \binom{i-1}{r-1}(1-p)^{i-1} p^r$$

$= 1 - P(< r$ successes between $1 \dots x)$

$$= 1 - \Big(\underbrace{P(0 \text{ success " } 1..x)}_{X \sim \text{Binomial}(x,p)} + P(1 \text{ success}) + \dots + P(r-1 \text{ success})\Big)$$

$P(X=0)$ $\qquad P(X=1)$ $\qquad P(X=r-1)$

$$= 1 - \sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i}$$

$$= 1 - F(x)$$

Binomial CDF

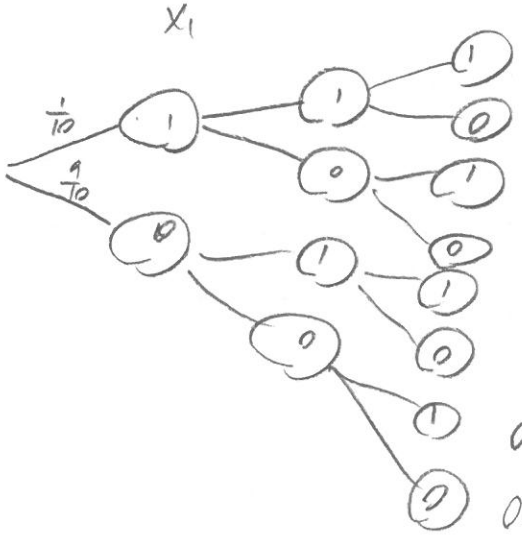
incomplete regu gamma function from before or manually compute

$X_1, \dots, X_n \overset{iid}{\sim} p(x)$

"the r.v. of the avg."

$T_n := X_1 + \dots + X_n, \quad \bar{X}_n := \dfrac{X_1 + \dots + X_n}{n} = \dfrac{T_n}{n}$

e.g. $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}\left(\frac{1}{10}\right)$



prob	T	$\bar{X}$
0.001	3	1
0.0081	2	2/3
0.081	2	2/3
0.009	1	1/3
0.081	2	2/3
0.009	1	1/3
0.009	1	1/3
0.729	0	0

$T \sim \begin{cases} 0 & \text{wp } 0.729 \\ 1 & \text{wp } 0.027 \\ 2 & \text{wp } 0.243 \\ 3 & \text{wp } 0.001 \end{cases} \qquad \bar{X} \sim \begin{cases} 0 \\ 1/3 \\ 2/3 \\ 1 \end{cases}$

Philosophical Jump: $X_1, X_2 \overset{iid}{\sim} \text{Bern}\left(\frac{1}{2}\right)$ Coin flip

r.v. is a process or a "dgp"

$X_1 = 0, \; x_2 = 1$

realization: to make real

datum

↳ def: realization of r.v

draw: " " r.v's

iid draw: " " " iid "

a datum is $\in \text{Supp}[X]$ $\quad P(X = x)$ prob of what we just saw...

who is X? $X \sim$?

Height 5'8'' $P(X = 5'8'') = $? need model

If we don't have model, doesn't mean it doesn't exist

Def: sample average

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is a realization from $\bar{X}$

In class demos

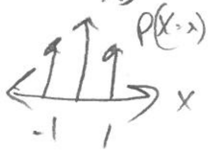
$X_1, \dots, X_8 \stackrel{iid}{\sim} \text{Hyper}(\overset{n}{3}, \overset{K}{8}, \overset{N}{8})$ $x_1 =$ ⋮ $x_8 =$ $\bar{x}$

$X_1, \dots X_8 \stackrel{iid}{\sim} \text{Binomial}(8, \frac{1}{2})$ $x_1 =$ ⋮ $x_8 =$ $\bar{x}$

$X_1, \dots, X_8 \stackrel{iid}{\sim} \text{Geom}(\frac{1}{2})$ $\bar{x}$

$X_1, \dots, X_8 \stackrel{iid}{\sim} \text{NegBin}(3, \frac{1}{2})$ $\bar{x}$

$X_1, \dots X_{80} \stackrel{iid}{\sim} \text{Rademacher}$ $\bar{x} = \frac{\#H\text{'s} - \#T\text{'s}}{80}$



← Draw histogram (use bars)

Pivot "where"