Lecture 12 10/29/15

of equivalent configurations, # success, # failure

$X \sim \text{Binomial}(n, p) := \binom{n}{x} p^{x} (1-p)^{n-x}$

Flip 97 coins $P(37\,H)$?

$X \sim \text{Binom}(97, \frac{1}{2})$

$P(X = 37) = \binom{97}{37} \frac{1}{2}^{37} \frac{1}{2}^{60}$

Bet 86 x on black. P(winning 52 times)

$X \sim \text{Binom}(86, \frac{18}{38})$

$P(X = 52) = \binom{86}{52} \left(\frac{18}{38}\right)^{52} \left(\frac{20}{38}\right)^{34}$

5000 houses in Belle Harbor

P(Hurricane flood damage) $= \frac{1}{120}$

P(1000 claims) ?

$X \overset{?}{\sim} \text{Binom}(5000, \frac{1}{120})$ Why not?

96,000 houses have theft insurance all over the nation,

P(burglary in NY in a given year) $= \frac{1}{250}$

$X \overset{?}{\sim} \text{Binom}(96000, \frac{1}{250})$ Why not?

iid!!

"If at first you don't succeed.."

→ eponymous proof again..

$$T \sim \text{Geometric}(p) := (1-p)^{x-1}\, p$$

$p \in (0,1)$ $\quad p = 0 \Rightarrow$ illegal

$\qquad\qquad\quad p = 1 \Rightarrow \text{Deg}(1)$ stops immediately

$\text{Supp}[T] = \mathbb{N}$ why $0 \notin \text{Supp}[T]$? Need at least one trial!

Consider $X_1, X_2, \ldots \stackrel{iid}{\sim} \text{Bern}(p)$ possibly infinite seq r.v.'s

Let $T = \min\limits_t \{X_t = 1\} = f(X_1, X_2, \ldots)$ a r.v. indicating stopping time

$P(T=1) = p \quad \underline{1}$

$P(T=2) = (1-p)p \quad \underline{0}\ \underline{1}$

$P(T=3) = (1-p)^2 p \quad \underline{0}\ \underline{0}\ \underline{1}$

$\vdots$

$$P(T=x) = (1-p)^{x-1} p \quad \overbrace{\underbrace{\underline{0}\ \underline{0}\ \cdots\ \underline{0}}_{x-1}\ \underline{1}}^{x}$$

$T \sim \text{Geom}(p) := (1-p)^{x-1}(p)$

Param space $p \in (0,1)$

Support

does $\sum_{x \in Supp(X)} p(x) = 1$?

Note $1-p \in (0,1)$ since $p \in (0,1)$

$$\sum_{x=1}^{\infty} p(1-p)^{x-1} \stackrel{?}{=} 1 \Rightarrow \sum_{x=1}^{\infty} (1-p)^{x-1} \stackrel{?}{=} \frac{1}{p} \Rightarrow \sum_{x=0}^{\infty} (1-p)^{x} \stackrel{?}{=} \frac{1}{p}$$

let $q := 1-p$

$\sum_{x=0}^{\infty} q^x$ What is this? $1+q+q^2+q^3+q^4+\dots$ "Geometric Series"

Converge? $\sum_{x=0}^{\infty} q^x < \int_0^{\infty} q^x = \frac{q^x}{\ln(q)}\Big]_0^{\infty} = \frac{1}{\ln(q)}\left(\lim_{x\to\infty} q^x - 1\right)$

$\Rightarrow$ only converge if $q < 1$

let $S = 1+q+q^2+q^3+\dots$

$= 1+q\left(1+q+q^2+q^3+\dots\right)$

$S = 1+qS$

$(1-q)S = 1 \Rightarrow S = \frac{1}{1-q}$

$$\sum_{x=0}^{\infty} (1-p)^x = \frac{1}{1-(1-p)} = \frac{1}{p} \checkmark$$ eponymous proof!

$F(x) := P(X \le x) = \sum_{i=1}^{x} (1-p)^{i-1} p$ HARD...

or $P(X \le x) = 1 - P(X > x)$ by complement rule... $P(X \in A) = 1 - P(X \in A^c)$

$A^c = Supp(X) \setminus A$!!

will not prove this... obvious...

if $X > x$ all Bernoulli trials up until x are fails $\underbrace{0\ 0\ 0\ \dots\ 0}_{x}$ $(1-p)^x$

$\Rightarrow F(x) = 1-(1-p)^x$

e.g. Flip coin until heads. P(~~H~~ Heads on 17^{th} flip)?

$X \sim \text{Geom}\left(\frac{1}{2}\right)$ $\quad P(X=17) = \left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)^{17-1} = \frac{1}{2^{17}}$

Defective chips are prob $\frac{1}{1000}$. P(you see 100^{th} chip fail)

$X \sim \text{Geom}\left(\frac{1}{1000}\right)$ $\quad P(X=100) = \frac{1}{1000}\left(\frac{999}{1000}\right)^{99}$

Are you sure?

What is prob you see a def chip before #500?

$F(500) = \boxed{1-\left(\frac{1}{1000}\right)^{500}}$

Someone works for you and they need to grade papers. What's the prob they take until 20^{th} paper to make a mistake?

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$?

Generalize this... $X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

$T := \min_t \left\{ \sum_{i=1}^{t} X_i = r \right\}$ i.e. wait until r successes

Again param space $p \in (0,1)$, $r \in \mathbb{N}$

$r=3$

0 0 0 1 0 0 1 0 1 $\quad T=9$ in this case

Can $T=1$? No

Can $T=2$? No

Can $T=3$? Yes $\quad$ 1 1 1 $\quad P(T=3) = p^3$

Can $T=4$? Yes $\quad 3\left\{\begin{matrix} 0\,1\,1 \\ 1\,0\,1 \\ 1\,1\,0 \end{matrix}\right.$ 1 $\quad P(T=4) = \underbrace{3}_{\binom{3}{2}}(1-p)p^3$

Last one **MUST** be a success!

$\text{Supp}(T) = \{3, 4, \ldots\}$