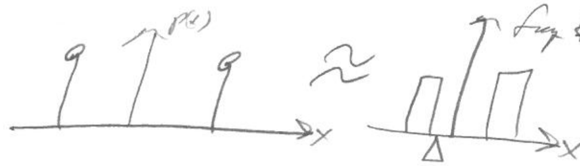
Lecture 14 Math 241 10/27/15

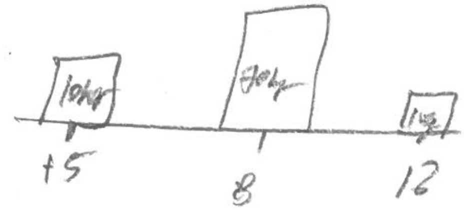


The pivot is a special point: called the mean/expectation/exp. val.

Denoted $\mu := E(X)$ operator r.v. Def ?? How...

$\bar{X}_n \longrightarrow E(X)$ Similar to $P(A) = \lim_{n\to\infty} \frac{\#}{\cdots}$

But we don't need to! We know $p(x)$! So we can just find the pivot of the PMF.



How to get pivot? Remember physics...

Weighted avg = pivot

prob-weighted avg of support

Useless usually

$$\Rightarrow \mu := E(X) := \sum_{x \in Supp(X)} x\, p(x) \Leftarrow \int_{\omega \in \Omega} X(\omega)\, dP(\omega)$$

$\bar{X}_n \longrightarrow \mu$ is a thm Law of Large #'s (LLN).

Fact $\int_0^{10} f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \dots + \int_9^{10} f(x)\,dx$

Consider... $Supp(X) = \{x_1, x_2, \dots\}$ discrete

$$\int_{\omega \in \Omega} X(\omega)\,dP(\omega) = \int_{\{\omega : X(\omega) = x_1\}} X(\omega)\,dP(\omega) + \int_{\{\omega : X(\omega) = x_2\}} X(\omega)\,dP(\omega) + \dots$$

$$= x_1 \int dP(\omega) + x_2 \int dP(\omega) + \dots$$

$$= x_1 P(X_1) + x_2 P(X_2) + \dots = \sum_{x \in Supp(X)} x\,p(x)$$

not responsible for this

Return to this once more

$X \sim$ Rademacher $\quad E(X) = \sum_{x \in Supp(X)} x\,p(x) = (-1)p(-1) + (1)p(1)$

$$= -1 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{2} = \boxed{0}$$

$X \sim$ Bernoulli$\left(\frac{1}{3}\right)$

" " " " $= 0\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right) = \boxed{\frac{1}{3}}$

Make sense...

$X \sim$ Bern(p) $\quad E(X) = 0(1-p) + 1(p) = \boxed{p}$

$X \sim$ Unif$(1,5,6)$ $\quad E(X) = 1\,\frac{1}{3} + 5\,\frac{1}{3} + 6\,\frac{1}{3} = \boxed{4}$

$X \sim$ Unif(A) $\quad E(X) = \sum_{x \in Supp(X)} x\,p(x) = \sum_{x \in A} x \frac{1}{|A|} = \boxed{\frac{1}{|A|} \sum_{x \in A} x}$

$X \sim \text{Binomial}(8, \frac{1}{2})$

$$E(X) = \ldots = \sum_{x=0}^{8} x \binom{8}{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{8-x} = \frac{1}{2^8}\left(0\binom{8}{0} + 1\binom{8}{1} + \ldots 8\binom{8}{8}\right)$$

$\frac{1}{2^8}$

$$= \frac{1}{2^8}\left(8 + 2\cdot 28 + 3\cdot 56 + 4\cdot 70 + 5\cdot 56 + 6\cdot 28 + 7\cdot 8 + 8\right)$$

$$= \frac{1}{256}(1024) = \boxed{4}$$

Make sense?

$X \sim \text{Binomial}(n, p)$

Why?

$$E(X) = \sum x\, p(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=\boxed{1}}^{n} x \frac{n!}{x!\,(n-x)!}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} (1-p)^{n-x} = np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-y-1} = np \underbrace{\sum_{y=0}^{m} \binom{m}{y} p^y (1-p)^{m-y}}_{1}$$

let $y = x-1$

$\Rightarrow x = y+1$

$x = 1 \ldots n$

$y = 0 \ldots n-1$

let $m = n-1$

$= \boxed{np}$ Make sense?

Compare

$X \sim \text{Hyper}(n, K, N)$

$$E(X) = \sum_{x \in \text{Supp}(X)} x \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

HARD! E.C. ...

$$= \boxed{n\frac{K}{N}}$$

We will see an easy proof later... next class

Compare

$X \sim \text{Geom}(p)$ Recall $|q|<1$ $\sum_{x=0}^{\infty} q^x = \frac{1}{1-q}$

$$E(X) = \sum_{x=1}^{\infty} x(1-p)^{x-1} p$$

let $y = x-1 \Rightarrow x = y+1$
if $x \in \{1, \dots, \infty\} \Rightarrow y = \{0, \dots\}$

$$= \sum_{y=0}^{\infty} (y+1)(1-p)^y p = \sum_{y=0}^{\infty} y(1-p)^y p + \sum_{y=0}^{\infty} (1-p)^y p$$

$$= \sum_{y=1}^{\infty} y(1-p)^y p + p \underbrace{\sum_{y=0}^{\infty} (1-p)^y}_{\frac{1}{p}}$$

$$= (1-p) \sum_{y=1}^{\infty} y(1-p)^{y-1} p + 1$$

$$\mu = (1-p)\mu + 1$$

$$\Rightarrow \mu = \mu - p\mu + 1$$

$$\Rightarrow p\mu = 1 \Rightarrow \boxed{\mu = \frac{1}{p}}$$

$X \sim \text{NegBin}(r, p)$

$$E(X) = \sum_{x=r}^{\infty} x \binom{x-1}{r-1} (1-p)^{x-r} p^r$$ HARD

we will see easier way... $\boxed{\mu = \frac{r}{p}}$

$\mu \in \text{Supp}(X)$? $X \sim \text{Bern}\left(\frac{1}{2}\right)$ $\frac{1}{2} \overset{?}{\in} \{0, 1\}$ No...

$$\text{Quantile}[X, p] := \min_{x \in \text{Supp}[X]} \{F(x) \geq p\}$$

it's the first ~~value~~ in the support that captures more than p prop of the whole ~~probability~~ support

$X \sim \text{Bin}(10, .4)$

x	$p(x)$	$F(x)$
0	.0060	.0060
1	.0403	.0464
2	.1209	.1673
3	.2150	.3823
4	.2508	.6331
5	.2007	.8338
6	.1115	.9452
7	.0425	.9877
8	.0106	.9983
9	.0016	.9999
10	.0001	1
	1	?

0, 25%, 50%, 75%, 100%ile } quartiles

0, 33.33, 66.67, 100 } tertiles

0, 10%ile?, 20%ile, 30%ile, ..., 90%ile, 100 } deciles

0, 20%ile, 40%ile, 60%ile, 80%ile, 100 } quintiles

$\text{Median}(X) := \text{Quantile}[X, 0.5]$ halfway mark

$\stackrel{?}{=} E(X)$

$X \sim \begin{cases} 0 & \text{wp } 99\% \\ 1 & \text{wp } 1\% \end{cases}$ Skewed

$\mu = .01 \neq \text{median} = 0$

$\text{IQR}[X] := \text{Quantile}[X, 0.75] - \text{Quantile}[X, 0.25]$

$= 5 - 3 = 2$

6

$$\text{Mode}(X) = \underset{x}{\text{argmax}} \{p(x)\}$$ most likely value...

$= \$1$ here..

Roulette in America. Bet \$1 on Black... Payout 1:1

$$X \sim \begin{cases} \$1 & \text{w.p. } \frac{18}{38} \\ -\$1 & \text{w.p. } \frac{20}{38} \end{cases} \quad E(X) = \quad \ldots -\frac{2}{38} = -\$0.053$$

its interpretation
In the long run...
your average winnings will be..

Bet on #7 Payout 35:1

$$X \sim \begin{cases} \$35 & \text{w.p. } \frac{1}{38} \\ -\$1 & \text{w.p. } \frac{37}{38} \end{cases} \quad E(X) = \ldots$$

Bet on First Dozen Payout 2:1

$$X \sim \begin{cases} \$2 & \text{w.p. } \frac{12}{38} \\ -\$1 & \text{w.p. } \frac{26}{38} \end{cases} \quad E(X) = \ldots$$

All bets in roulette the same ... story

Europe Bet on Black

$$X \sim \begin{cases} \$1 & \text{w.p. } \frac{18}{37} \\ -\$1 & \text{w.p. } \frac{19}{37} \end{cases} \quad E(X) = \quad -\$0.027 \text{ "much fairer"}$$