Lecture 15 Math 241 10/29/15

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~~Talked about $E[X]$ what about $Y = g(X)$ some~~

I take Uber. Van Wyck: 7 min, Jewel Ave: 12 min
(no traffic) (if traffic)

$P(\text{traffic}) = 0.7$

$$W \sim \begin{cases} 7\text{min} & \text{wp } 0.7 \\ 12\text{min} & \text{wp } 0.3 \end{cases} \qquad E(W) = 7 \cdot 0.7 + 12 \cdot 0.3 = 7.8\text{min}$$

Interpret...

Uber charges \$0.40/min

$B = \$0.40/\text{min} \cdot W = g(W)$

$E[B]$? We need to figure out ~~[illegible]~~
if $Y = g(X)$ what is $E[Y]$? Assume Y is discrete r.v.

$$E(Y) := \int_\Omega Y(\omega)\, dP(\omega) = \int_\Omega g(X(\omega))\, dP(\omega) = \int_{\{\omega : X(\omega) = x_1\}} g(X(\omega))\, dP(\omega) + \int_{\{\omega : X(\omega) = x_2\}} g(X(\omega))\, dP(\omega) + \ldots$$

$$= g(x_1) \int dP(\omega) + g(x_2) \int dP(\omega) + \ldots$$

$$\boxed{E(g(X)) = \sum_{x \in \text{Supp}(X)} g(x)\, p(x)}$$

$$E(B) = E[0.40\,W] = \sum_{w \in Sp(W)} 0.40\,w\,p(w) = 2.8\,p(7) + 4.8\,p(12)$$
$$= 2.8 \cdot 0.7 + 4.8 \cdot 0.3$$
$$= \$3.40$$

interpret

Note

let $g(X) = aX$ s.t. $a \in \mathbb{R}$

$$E(aX) = \sum_{x \in Sp(X)} a x\,p(x) = a \sum x\,p(x) = a\,E(X)$$

What is $P(B=b)$ i.e. the PMF of B? Can't cover this... sorry!

There is also a \$3 base fee so..

$T = \$3 + \underbrace{B}_{g(w)}$ What is $E[T]$?

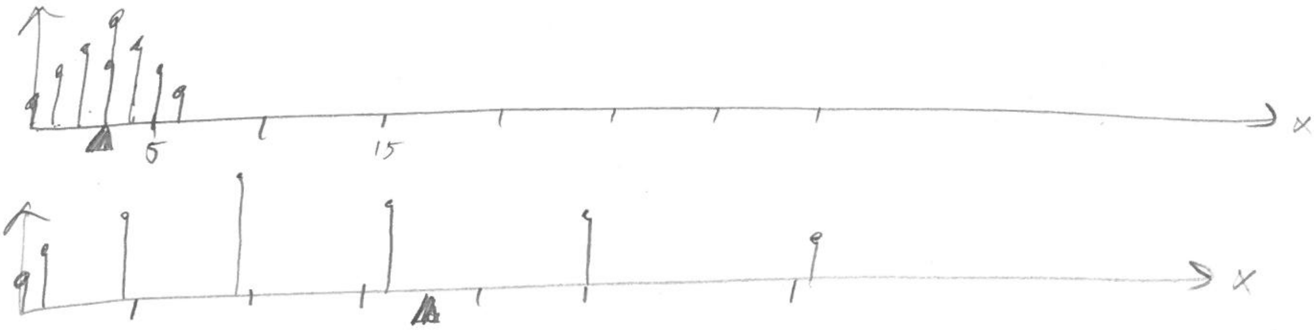
$$E(T) = E[\$3 + B] = \sum_{w \in Sp\,W} (\$3 + 0.40 \cdot w)\,p(w) = 5.80\,p(7) + 7.8\,p(12)$$
$$= 5.8 \cdot (0.7) + 7.8\,(0.3)$$
$$= \$6.12$$

let $g(X) = X + c$

$$E(g(X)) = E(X + c) = \sum (x + c)\,p(x) = \sum x\,p(x) + \overbrace{\sum c\,p(x)}^{c\overbrace{\sum p(x)}^{1}}$$
$$= E(X) + c$$

$$\boxed{g(X) = aX + c \Rightarrow E(g(X)) = a\,E(X) + c}$$

let $X \sim Bin(6, \frac{1}{2})$ $\quad Y = X^2 = g(X) \quad E(Y)$?



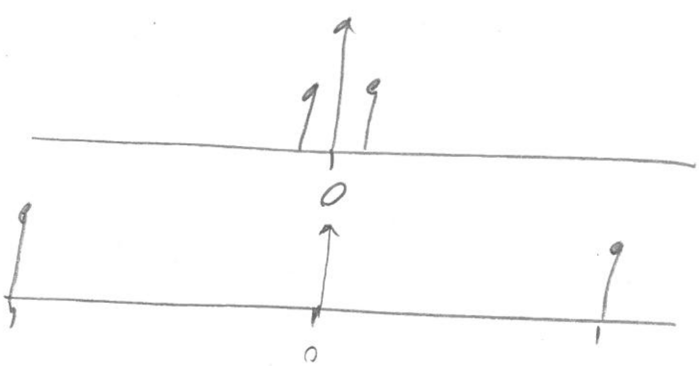
$0 \to 0$
$1 \to 1$
$2 \to 4$ $\qquad Eg(X) = \sum_{x=0}^{6} x^2 \binom{6}{x} \frac{1}{2^6} = 17.5$ no easy way!

$6 \to 36$

$E(X) = 3$, $E(X^2) = 17.5$ any pattern? No...

Only if $g(X)$ is linear...

$X \sim$ Rademacher, $\quad Y = 10X \sim \begin{cases} -10 & \text{wp } \frac{1}{2} \\ 10 & \text{wp } \frac{1}{2} \end{cases}$



$E[X] = 0$

$E(Y) = E(10X) = 10E(X) = 10 \cdot 0 = 0$

obviously $E(X) = E(Y) \not\Rightarrow X \overset{d}{=} Y$

Y is more "dispersed" than X around its pivot, 0.

Can we capture this concept of dispersion?

Need to define dispersal about the pivot

penalty function
↓

$e(X,\mu) = X - \mu$ distance

But we don't care if its ⊕ or ⊖ so..

$e(X,\mu) = |X - \mu|$ but abs value hard to work with $e'(X,\mu)$? piecewise..

$e(X,\mu) = (X-\mu)^2$ "squared error loss" very natural.. further away, the higher the fee.

Any penalty function is arbitrary!

Let $L := (X-\mu)^2 = g(X)$ a r.v. representing loss

$E[L]$ is how far away from the pivot in sqd distance on avg..

for Rademacher

$L = (X-\mu)^2 = X^2 \sim \{1 \text{ wp } 1 \Rightarrow E[L] = 1$

for $10X$ (Rademacher $\times 10$) more dispersed!

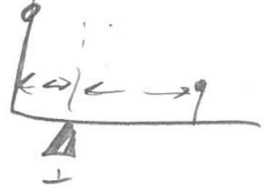
$L = (10X-\mu)^2 = (10X)^2 = 100X^2 \sim \{100 \text{ wp } 1 \Rightarrow E(L) = 100$

ET

$\sigma^2 := Var(X) :=$

$E(L) := E[(X-\mu)^2]$ and call it "variance" of r.v. X

$X \sim Bern\left(\frac{1}{3}\right)$



$\frac{1}{3}$

$\frac{1}{3}$ of the time you're $\frac{2}{3}$ away or $\frac{4}{9}$ sqd. dist away

$\frac{2}{3}$ of the time you're $\frac{1}{3}$ away or $\frac{1}{9}$ sqd dist away

$L \sim \begin{cases} \frac{1}{9} \text{ wp } \frac{2}{3} \\ \frac{4}{9} \text{ wp } \frac{1}{3} \end{cases}$ $Var(X) = \frac{1}{9} \cdot \frac{2}{3} + \frac{4}{9} \cdot \frac{1}{3} = \frac{7}{27} = .259$

$$Var[X] = \sum_{x=0}^{1}(x-\mu)^2 p(x) = \left(0-\tfrac{1}{3}\right)^2 p(0) + \left(1-\tfrac{1}{3}\right)^2 p(1)$$
$$= \tfrac{1}{9}\cdot\tfrac{2}{3} + \tfrac{4}{9}\cdot\tfrac{1}{3} = .259$$

$X \sim Bern(p)$

$$Var(X) = \sum_{x=0}^{1} \ldots\ldots\ldots\ p(1-p)$$

Roulette: Bet on lucky #7

$$X \sim \begin{cases} \$35 \text{ w.p. } \frac{1}{38} \\ -\$1 \text{ w.p. } \frac{37}{38} \end{cases} \quad \mu = -0.053 \text{ (from last class)}$$

$$Var(X) = E\left[(X-\mu)^2\right] = (\$35 - -\$0.053)^2 p(\$35) + (-\$1 - -\$0.053)^2 p(-\$1)$$
$$= 33.207\ \$^2$$

On avg... you're winnings will be 33.207 $\$^2$ away from the center as measured by sqd distance.

$\$^2$? What is the interpretation?

Let $\sigma := \sqrt{\sigma^2} = \sqrt{Var(X)} =: SE(X)$ the "standard error" or "standard deviation"

same units as Supp(X)

Here $SE(X) = \$5.76$

synonym for variance

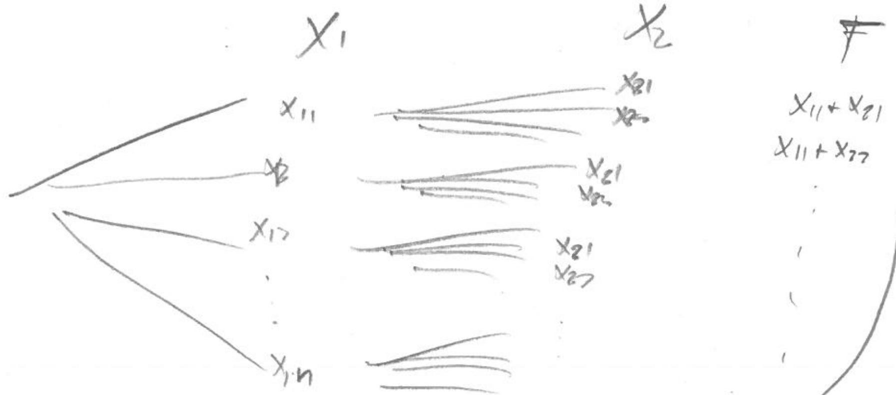
Interpretation? None...
Just a measure of spread... Soon will see that it's useful..

Recall $T = X_1 + X_2$

not so clear

What is $E[T]$? $= \sum_{t \in Supp(T)} t \, p(t)$

not so clear



jmf

$$\frac{p(x_1, x_2)}{P(X_1 = x_{11}) \cdot P(X_2 = x_{21} \mid X_1 = x_{11})}$$

$T = g(X_1, X_2)$

$$E[T] = E[g(X_1, X_2)] = \sum_{\langle x_1, x_2 \rangle \in Supp(X_1) \times Supp(X_2)} g(x_1, x_2)\, p(x_1, x_2)$$

the proof is a generalization of the previous proof using a double integral

$$= \sum_{x_1 \in Supp(X_1)} \sum_{x_2 \in Supp(X_2)} g(x_1, x_2)\, p(x_1, x_2)$$

Imagine if X_1, X_2 independent...

PMF for X_1

PMF for X_2

$$p(x_1, x_2) = p_1(x_1)\, p_2(x_2)$$

$$E[T] = \sum_{x_1} \sum_{x_2} (x_1 + x_2)\, p(x_1) p(x_2) = \sum_{x_1}\sum_{x_2} x_1 p(x_1) p(x_2) + \sum_{x_1}\sum_{x_2} x_2 p(x_1) p(x_2)$$

$$= \underbrace{\sum_{x_1} x_1 p(x_1)}_{E(X_1)} \underbrace{\sum_{x_2} p(x_2)}_{1} + \underbrace{\sum_{x_1} p(x_1)}_{1} \underbrace{\sum_{x_2} x_2 p(x_2)}_{E(X_2)}$$

$E[X_1 + X_2] = E(X_1) + E(X_2)$ if X_1, X_2 independent