Lecture 8 Math 241 10/1/15

~~All probs cond.~~

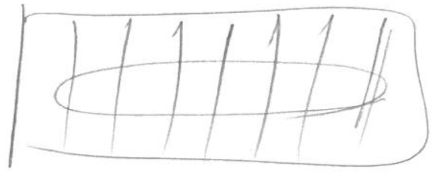
~~$P(H) = \frac{1}{2}$?~~ ~~$P(H|\Omega)$~~

~~fair coin~~

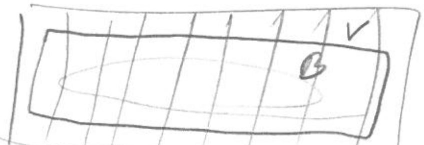
~~equally li~~

Bayes thm: $A_1, \ldots, A_n$ mut. excl. coll. exh.

$$P(A_\ell | B) = \frac{P(B|A_\ell)\, P(A_\ell)}{\sum_{i=1}^{n} P(B|A_i)\, P(A_i)}$$



We can further condition



no need for two bars

$$P(A_\ell | B, V) = \frac{P(B|A_\ell, V)\, P(A_\ell|V)}{\sum_{i=1}^{n} P(B|A_i, V)\, P(A_i|V)}$$

V: Pick D2 initially , B: Gameshow host opens D3

A_1: Car is in D1

A_2: Car is in D2

A_3: Car is in D3

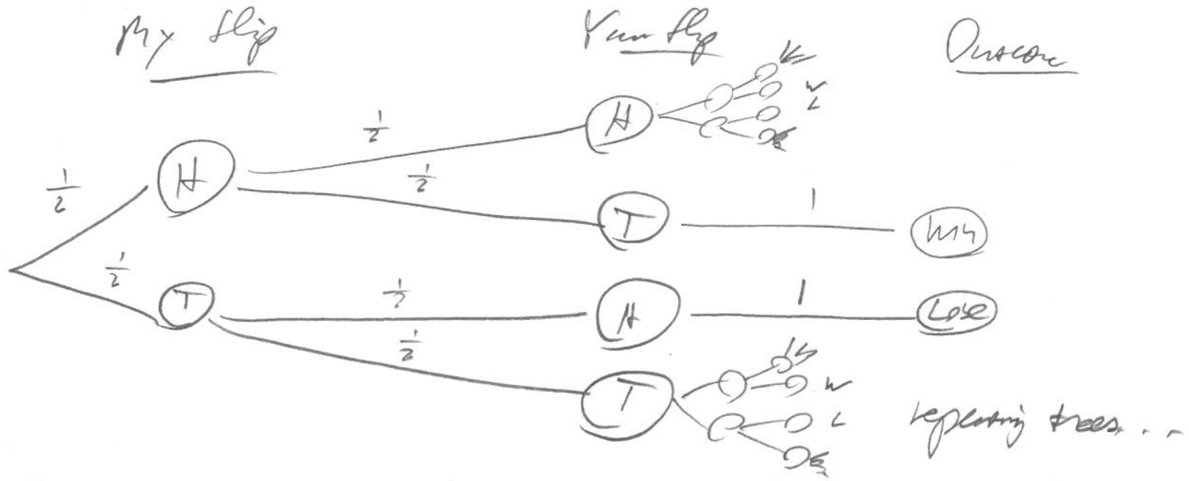
P(Car in D1 | Gameshow host opens D3 & Pick D2 initially) =

$$\frac{P(\text{Open D3} | \text{Car in D1, Pick } D_2)}{\underbrace{P(\text{Open D3} | \text{Car in D1, Pick } D_2)}_{1} + \underbrace{P(\text{Open D3} | \text{Car in D2, Pick } D_2)}_{\frac{1}{2}} + \underbrace{P(\text{Open D3} | \text{Car in } D_3\text{, pick } D_2)}_{0}}$$

$= \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}}$ do you like trees better?

DEMO w/ 4 cups or doors

Imagine a game: I flip, you flip. If I get H, you get T, I win, if you get H, I get T, you win. Otherwise, play again



$$P(\text{win}) = \frac{1}{4} + \frac{1}{4}P(\text{win}\,|\,\text{Tie}) + \frac{1}{4}P(\text{win}\,|\,\text{Tie})$$

$$= \frac{1}{4} + \frac{1}{2}P(\text{win}\,|\,\text{Tie})$$

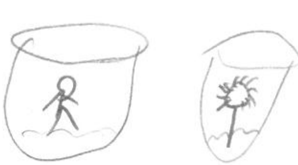
But $P(\text{win}\,|\,\text{Tie}) = P(\text{win})$ since Tie, win Independent!

$$\Rightarrow P(\text{win}) = \frac{1}{4} + \frac{1}{2}P(\text{win})$$

$$= \frac{1}{2}P(\text{win}) = \frac{1}{4} \Rightarrow \boxed{P(\text{win}) = \frac{1}{2}}$$

Balls & Urns

Imagine you have 10 balls (~~distinct~~) and 2 urns distinct



How many ways to put those 10 balls in the 2 urns? Restriction: there must be at least one ball in each urn.

1, 9
2, 8
3, 7
⋮
9, 1

$\left.\begin{matrix}1,9\\2,8\\3,7\\\vdots\\9,1\end{matrix}\right\}$ 9 ways

$\binom{9}{1}$ $\left[\, o \mid o \mid o \mid o\ o\ o\ o\ o\ o \mid o \,\right]$

How about 3 urns?

1, < 1,8 / 2,7 / ⋮ / 8,1 $\binom{8}{1}$

2 — 1,7 / ⋮ / 7,1 $\binom{7}{1}$

3

⋮

8 — 1 — 1

doesn't scale.. better way?

$$\left[\, \overbrace{o\ o\ o \mid o\ o\ o \mid o\ o\ o\ o}^{2} \,\right]$$
$$9$$

9 positions, dividers are indistinct, dividers can't be in same place

$\Rightarrow \binom{9}{2}$

In general, n balls, k urns $\Rightarrow \binom{n-1}{k-1}$

$x_1 + x_2 + \dots + x_k = n$ where $x_1, x_2, \dots, x_k \in \mathbb{N}$

How many sol's? $\binom{n-1}{k-1}$

Now let there not be the restriction: urns can be empty!

$x_1 + x_2 + \dots + \cancel{x_k} = n$ s.t. $x_1, \dots x_k \in \mathbb{N}_0$

let $x_1' = x_1 + 1, \dots, x_k' = x_k + 1 \Rightarrow x_1, \dots, x_k \in \mathbb{N}$

$\Rightarrow (x_1' - 1) + (x_2' - 1) + \dots + (x_k' - 1) = n$

$\Rightarrow x_1' + x_2' + \dots + x_k' = n + k$ s.t. $x_1, \dots, x_k \in \mathbb{N}$

How my sols'?

$$\binom{n+k-1}{k-1}$$

How is this possible?

10 balls, 2 urns, urns can be empty

$[|0|0|0|0|0|0|0|0|0|0|]$ $\binom{11}{1} = \binom{10+2-1}{2-1}$

10 balls 3 urns

$[0\ 0\ 0\ 0||0\ 0\ 0\ 0\ 0\ 0]$

Same as 10 Red flowers, 2 blue flowers

$$\frac{12!}{10!\,2!} = \binom{12}{2} = \binom{10+3-1}{3-1}$$

$i_1, \dots, i_k \in \mathbb{N}_0$

Recall: $(a_1 + a_2 + \dots a_k)^n = \sum\limits_{i_1 + \dots + i_k = n} \binom{n}{i_1, i_2 \dots i_k} a_1^{i_1} \cdot \dots \cdot a_k^{i_k}$

k^n terms but how my unique terms? $\binom{n+k-1}{k-1}$

e.g.

$$(a+b)^4 = (1)a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + (1)b^4$$

$2^4 = 16$ $\quad 1+4+6+4+1 = 16$ since $2^4 = \sum_{i=0}^{4} \binom{4}{i}$

but 5 unique terms!

$$\binom{4+2-1}{2-1} = \binom{5}{1} = 5$$

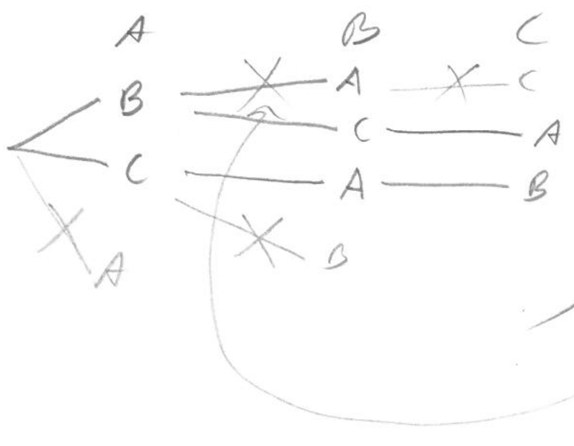
4 balls placed in a urn or b urn and urns allowed to be empty

$\longrightarrow a^2b^2$ term

Hat problem / Matching problem

n people walk into the room and toss their hat on the table. Everyone chooses a random hat. What is P(no one gets their hat back)?

$n = 3$



$$\frac{2}{3!} = \boxed{\frac{1}{3}}$$

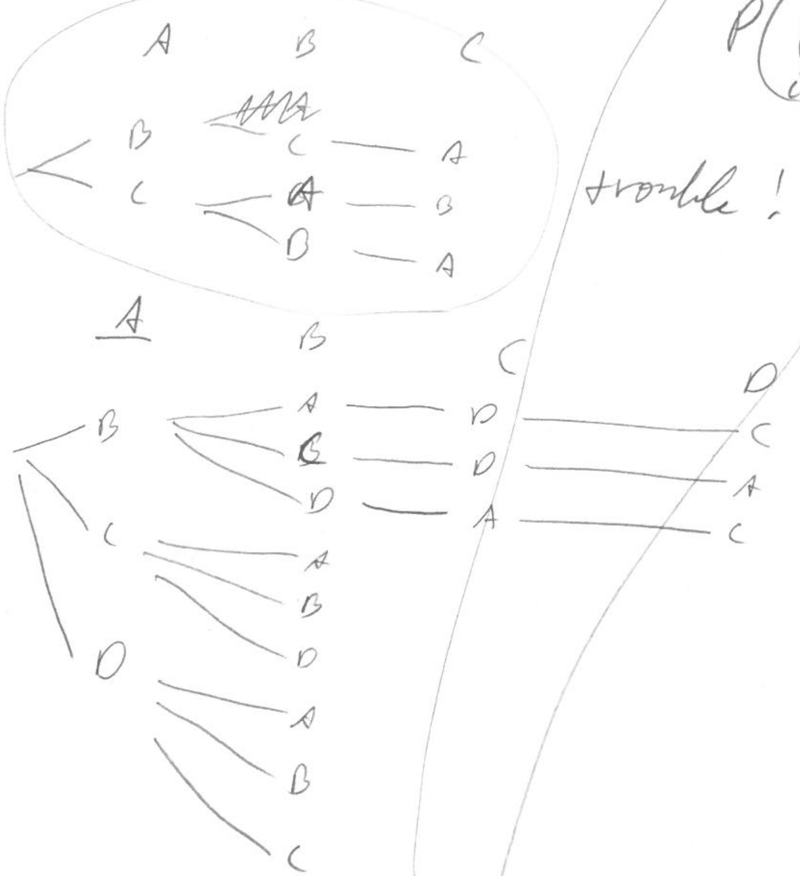
doesn't scale well to n large...

3 people

P(no one selects their hat) $= 1 - P(\geq 1$ of them selects their own hat$)$

$\overline{}$

$3!$



trouble!

$= P(A_1 \text{ or } A_2 \text{ or } \ldots \text{ or } A_n)$

$$P\left(\bigcup_{i=1}^{n} A_i\right)$$

let A_i denote the probability man i gets his hat

if another guy gets his hat... I don't care

$$= \sum P(A_i) - \sum_{i \neq j} P(A_i \cap A_j) + \sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k) - \ldots (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

$$P(A_1) = \frac{1 \cdot \underline{n-1} \cdot \underline{n-2} \ldots \underline{1}}{n!} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$\sum P(A_i) = 1$$

$$P(A_1 \cap A_2) = \frac{\underline{1} \cdot \underline{1} \cdot \underline{n-2} \ldots \underline{1}}{n!} = \frac{(n-2)!}{n!}$$

$\cancel{\binom{n}{2} = \frac{n!}{(n-2)!\,2!}} \cancel{\frac{n!}{2!\,n!}}$

$$\sum_{i \neq j} P(A_i \cap A_j) = \binom{n}{2} \frac{(n-2)!}{n!} = \frac{n!}{(n-2)!\,2!} \frac{(n-2)!}{n!} = \boxed{\frac{1}{2!}}$$

$$P(A_1 \cap A_2 \cap A_3) = \frac{1 \cdot 1 \cdot 1 \cdot (n-3) \cdots}{n!} = \frac{(n-3)!}{n!}$$

$$\sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k) = \binom{n}{3} \frac{(n-3)!}{n!} = \frac{\cancel{n!}}{\cancel{(n-3)!}\,3!} \frac{\cancel{(n-3)!}}{\cancel{n!}} = \frac{1}{3!}$$

$$P(A_1 \cap A_2 \cap \ldots A_n) = \frac{1 \cdot 1 \cdots 1}{n!}$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots (-1)^n \frac{1}{n!}$$

$$1 - \downarrow = 1 - (\quad) = \underbrace{1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots}_{\text{What is this?}}$$

Taylor Series

We will see later this..

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!}(x-c)^i \quad \forall c \quad \text{with restrictions}$$

If $c=0$

$$e^x := 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad (e^0,\ e^0,\ e^0)$$

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + - + \ldots$$

$$\Rightarrow \boxed{P(\text{no hats match in } n) \approx e^{-1} = 0.368} \quad \text{for large } n$$

will be ~~on~~ test

$$\text{Odds}(A) := \frac{1-P(A)}{P(A)} \rightarrow \text{Odds Against}$$

$$\text{Odds}(H) = 1:1$$

$$\text{Odds}(\text{get a } 6) = \frac{5/6}{1/6} = 5:1$$

→ win \$5 → bet \$1 to be an even game

Not on Exam!

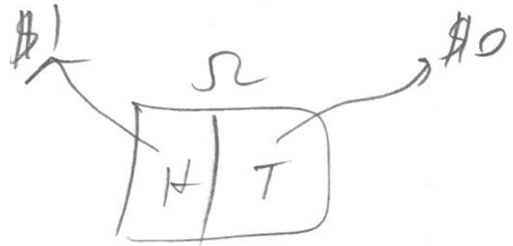
MIDTERM 1 ↑

MIDTERM 2 ↓

R.V. Theory $\Omega = \{H, T\}$ but H, T are not #'s!

What if you want to model the outcome numerically?

Just assign a # and a unit (if needed)



equal law (the r.v.)

dist as

$$X \sim \begin{cases} \$1 & \text{w.p. } \frac{1}{2} \\ \$0 & \text{w.p. } \frac{1}{2} \end{cases}$$

$P(\text{win } \$1) = \frac{1}{2}$

H, T not here! We don't care if it was a H or T outcome, we just care about the numerical outcome.

X is a function

$X : \Omega \to \mathbb{R}$

so $X(\omega) \to x_1$, $\to x_2$ not allowed!

One outcome → One # (not necessarily unique)

$X(H) = \$1$

$X(T) = \$0$

"ω is chosen" and X operates as a dumb switchboard!

$P : 2^{\Omega} \to [0,1]$ but I want to ask $P(X = \$1) = \frac{1}{2}$!