Lecture 11 Oct 15, 2015 Math 241

L4

let $N=100, p=0.5, n=6$ Hyper$(6, 0.5, 100 \ldots)$

$$P(X=3) = \frac{\binom{50}{3}\binom{50}{3}}{\binom{100}{6}} = 0.3223$$

$N=1000$

$$\ldots = 0.3174$$

$N=10{,}000$

$$= 0.3126$$

$\vdots$

converge?

What is the limiting r.v.? In the limit sampling without replacement is sampling with replacement...

$$\lim_{N\to\infty} \text{Hypergeometric}(n,p,N) = \lim_{N\to\infty} \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}} = \lim_{N\to\infty} \frac{\frac{(pN)!}{(pN-x)!\,x!}\,\frac{((1-p)N)!}{((1-p)N-(n-x))!\,(n-x)!}}{\frac{N!}{(N-n)!\,n!}}$$

$$\lim_{x} a f(x) = a \lim f(x)$$

$$= \frac{n!}{x!(n-x)!} \lim_{N\to\infty} \frac{(N-n)!}{N!} \frac{(pN)!}{(pN-x)!} \frac{((1-p)N)!}{((1-p)N-(n-x))!}$$

$$= \binom{n}{x} \lim_{N\to\infty} \frac{\overbrace{(pN)\cdot(pN-1)\cdots(pN-x+1)}^{x}\,\overbrace{((1-p)N)((1-p)N-1)\cdots((1-p)N-(n-x)+1)}^{n-x}}{\underbrace{N\cdot(N-1)(N-2)\cdots(N-n+1)}_{n \text{ terms}}}$$

$\lim f(x)g(x) = \lim f(x) \lim g(x)$ if f,g conv.

$$\binom{n}{x} \lim \frac{pN}{N} \lim \frac{pN-1}{N-1} \cdots \lim \frac{pN-x+1}{N-x+1} \cdot \lim \frac{(1-p)N}{N-x} \lim \frac{(1-p)N-1}{N-x-1} \cdots \lim \frac{(1-p)N-(n-x)+1}{N-n+1}$$

$$\binom{n}{x} \underbrace{p \cdots p}_{x} \underbrace{(1-p) \cdots (1-p)}_{n-x} = \binom{n}{x} p^x (1-p)^{n-x}$$

prob success is p

Sample n balls w/ ~~replacement~~ count # successes...

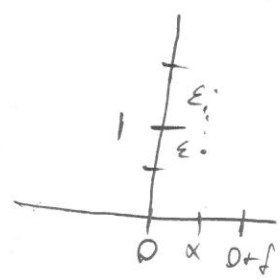
$X \sim \text{Binomial}(n,p) := \binom{n}{x} p^x (1-p)^{n-x} = p(x)$ Is $p(x) > 0$

Parameter Space

p? $p \in [0,1]$?

$p = 0$ $p(x) = \binom{n}{x} 0^x 1^{n-x} = 0$ $\forall x$ except 0

Since $0^0 := 1$ by def



$\cancel{f(x) = x^x}$

$\cancel{\ln f(x) = x \ln(x) \Rightarrow f(x) = e^{x \ln(x)}}$

$\cancel{= e^{x\left((x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots\right)}}$

$\cancel{= e^{x(x-1)} e^{-x\frac{(x-1)^2}{2}} e^{x\frac{(x-1)^3}{3}} \cdots}$ $\nexists$ slightly okay..

Since $\lim_{x \to 0} x^x = 1$

↑

HARD

proof (EC)

$p = 0 \Rightarrow$ ~~p(x)~~ $X \sim \text{Deg}(0)$

$p = 1 \Rightarrow p(x) = \binom{n}{x} 1^x 0^{n-x} \Rightarrow X \sim \text{Deg}(n)$

$\Rightarrow p \in (0,1)$ n? $n = 0$? No... $n \in \mathbb{N}$ Same $N \to \infty$ Assm before

$$\sum_{x \in Supp(X)} p(x) \stackrel{?}{=} 1 \qquad \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

$$Supp(X) = \{0, \dots, n\}$$

$$(a+b)^n = \sum_{\cancel{x}=0}^{n} \binom{n}{x} a^{n-x} b^x$$

$$1 = 1^n = ((1-p)+p)^n = \sum_{i=0}^{n} \binom{n}{x} (1-p)^{n-x} p^x \checkmark$$

this is why it's called the "Binomial r.v."

Special Case

$$X \sim \text{Binomial}(1,p) = \binom{1}{x} p^x (1-p)^{1-x} = p^x (1-p)^{1-x} = \text{Bernoulli}(p)$$

$Supp(X) = \{0,1\}$ $\qquad \binom{1}{x} = 1$ if $x=0$ & $x=1$ $\qquad$ of course!

Independence of r.v.'s

X_1 is said to be indep of X_2 if:

$$P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$$

or

$$P(X_2 = x_2 \mid X_1 = x_1) = P(X_2 = x_2) \qquad \forall\, x_1 \in Supp(X_1),\ x_2 \in Supp(X_2)$$

or

$$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1)\,P(X_2 = x_2)$$

a joint ~~mass~~ mass function (jmf)

To prove independence, you need to know about the coins or know the JMF (soon)

$X_1 \sim \text{Bern}\left(\frac{1}{3}\right)$ from Ω of coin A

$X_2 \sim \text{Bern}\left(\frac{1}{3}\right)$ " " B

$X_1 \stackrel{d}{=} X_2$ for sure

$P(X_1 = 1 \mid X_2 = 1) = P(H_1 \mid H_2) = P(H_1) = P(X_1 = 1)$ ✓ $= \frac{1}{3}$

$P(X_1 = 1 \mid X_2 = 0) = P(H_1 \mid T_2) = P(H_1) = P(X_1 = 1)$ ✓ $= \frac{1}{3}$

$P(X_1 = 0 \mid X_2 = 1) = P(T_1 \mid H_2) = P(T_1) = P(X_2 = 0)$ ✓ $= \frac{2}{3}$

$P(X_1 = 0 \mid X_2 = 0) = P(T_1 \mid T_2) = P(T_1) = P(X_2 = 0)$ ✓ $= \frac{2}{3}$

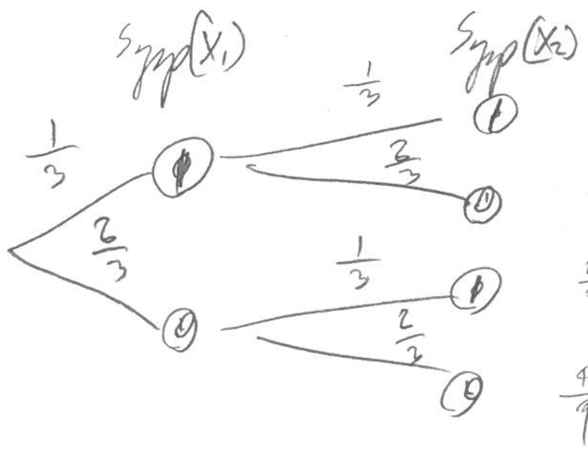
So X_1 and X_2 are independent

In fact X_1 and X_2 are indep & $\overbrace{\text{identically}}^{\text{equal in distr}}$ distributed iid

Denoted $X_1, X_2 \stackrel{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$

Now consider $T_2 := X_1 + X_2$ ~~what is~~ How is T distr?

new r.v.



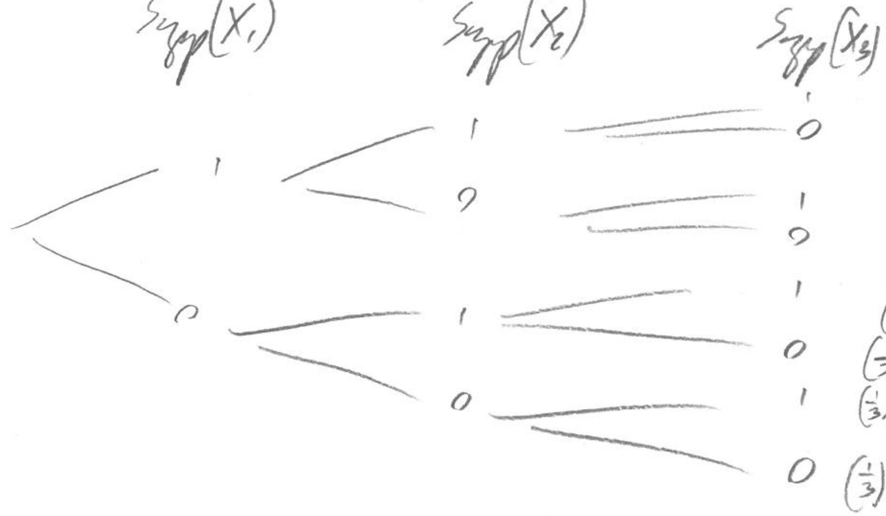
	T
$\frac{1}{9}$	2
$\frac{2}{9}$	1
$\frac{2}{9}$	1
$\frac{4}{9}$	0

$\text{Supp}(T) = \{0, 1, 2\}$

$T \sim \begin{cases} 2 & \text{wp } \frac{1}{9} \\ 1 & \text{wp } \frac{4}{9} \\ 0 & \text{wp } \frac{4}{9} \end{cases}$

$\neq \text{Unif}(\{0,1,2\})$

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$, $\quad T_3 := X_1 + X_2 + X_3$ $\qquad \text{Supp}[X] = \{0,1,2,3\}$



	T
$\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0$	3
$\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1$	2
$\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1$	2
$\left(\frac{1}{2}\right)^1\left(\frac{2}{3}\right)^2$	1
$\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1$	2
$\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2$	1
$\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2$	1
$\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3$	2

Now we gonna add up

$$T \sim \begin{cases} 0 & \text{wp } (1)\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 \\ 1 & \text{wp } 3\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 \\ 2 & \text{wp } 3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 \\ 3 & \text{wp } 1\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 \end{cases}$$

1-3-3-1 seen before!

$\underline{0}\,\underline{0}\,\underline{0}$ $\binom{3}{0}$ $\qquad$ $\begin{matrix}\underline{1}\,\underline{0}\,\underline{0}\\ \underline{0}\,\underline{1}\,\underline{0}\\ \underline{0}\,\underline{0}\,\underline{1}\end{matrix}$ $\binom{3}{1}$ $\qquad$ $\begin{matrix}\underline{1}\,\underline{1}\,\underline{0}\\ \underline{1}\,\underline{0}\,\underline{1}\\ \underline{0}\,\underline{1}\,\underline{1}\end{matrix}$ $\binom{3}{2}$ $\qquad$ $\underline{1}\,\underline{1}\,\underline{1}$ $\binom{3}{3}$

$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$ $\qquad T = \sum_{i=1}^{n} X_i \qquad \text{Supp}(X) = \{0, \ldots, n\}$

$$T \sim \begin{cases} 0 & \binom{n}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^n \\ 1 & \binom{n}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{n-1} \\ 2 & \binom{n}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{n-2} \\ 3 & \binom{n}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{n-3} \\ \vdots & \\ n-2 & \binom{n}{n-2}\left(\frac{1}{3}\right)^{n-2}\left(\frac{2}{3}\right)^2 \\ n-1 & \binom{n}{n-1}\left(\frac{1}{3}\right)^{n-1}\left(\frac{2}{3}\right)^1 \\ n & \binom{n}{n}\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^0 \end{cases}$$

$X_1, \dots, X_n \overset{iid}{\sim} Bern(p) \quad T_n \sim \sum_{i=1}^{n} X_i \quad Supp[X] = \{0, \dots, n\}$

$$T_n \sim \begin{cases} 0 & \binom{n}{0} p^0 (1-p)^n \\ 1 & \binom{n}{1} p^1 (1-p)^{n-1} \\ \vdots & \\ n-1 & \binom{n}{n-1} p^{n-1} (1-p)^1 \\ n & \binom{n}{n} p^n (1-p)^0 \end{cases}$$

$T_n \sim \text{Binomial}(n, p) := \binom{n}{x} p^x (1-p)^{n-x}$

Two ways to think about binomial: limit of hypergeometric w/ n, p fixed - or - sum of n iid Bern(p)'s.

$P(X \le x) = \sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i}$

$F(x)$? Regularized Incomplete Beta Function

$= I_{1-p}(n-k, 1+k) = (n-k)\binom{n}{k} \underbrace{\int_0^{1-p} t^{n-k-1} (1-t)^k \, dt}_{\text{no closed form solution}}$

What if $X_1, X_2, \dots \overset{iid}{\sim} Bern(p)$ (possibly infinite sequence of r.v.'s)

let $T = \min_t \{X_t = 1\}$ a r.v. indicating "stopping time" (first success)

$P(T=1)$ $\underline{1}$ p $\quad P(T=3)$ $\underline{0}\,\underline{0}\,\underline{1}$ $(1-p)^2 p$

$P(T=2)$ $\underline{0}\,\underline{1}$ $(1-p)p$ $\quad P(T=4)$ $\underline{0}\,\underline{0}\,\underline{0}\,\underline{1}$ $(1-p)^3 p$

$p(x) := P(T=x) = \underbrace{\underline{0}\,\underline{0} \dots \underline{0}}_{x-1}\,\underline{1}$

$= (1-p)^{x-1} p$