Math 281 Lecture 2 9/1/15

Special set called Ω (Omega) universe, sample space, space of discourse. You define it!

let $\Omega = F \cup M$, obviously $F \subseteq \Omega$, $M \subseteq \Omega$ / Joe $\in \Omega$, Mary $\in \Omega$

In fact $\forall A$, $A \subseteq \Omega$! ($\forall \omega$, $\omega \in \Omega$

sets — all sets subset of universe — elems — " $\uparrow$ "lowercase omega"

$F \cup \Omega = \Omega$, $F \cap \Omega = F$ rules!

Select a name "at random", what is prob. of female?

$$P(F) = \frac{|F|}{|\Omega|}$$

$|F|$ ← size of set we're looking for

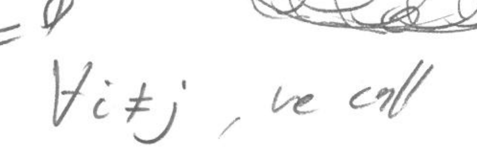
$|\Omega|$ ← size of universe

prob is $\uparrow$ a set function

If $A_1, \ldots, A_n$ are sets, and $\bigcup_{i=1}^{n} A_i = \Omega$, we call set of sets $\{A_1, \ldots, A_n\}$ collectively exhaustive.

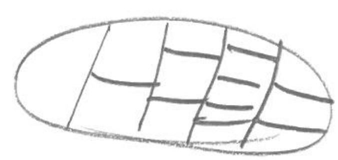
Ω

" " " $A_i \cap A_j = \phi$ $\forall i \neq j$, we call $\{A_1, \ldots, A_n\}$ mutually exclusive (or "disjoint")



Ω

Mutually exclusive & collectively exhaustive



No double counting

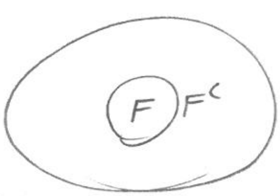
$$|\Omega| = \sum_{i=1}^{n} |A_i| \quad \text{(for finite sets)}$$

First set operation: Complement!

$A^c := \Omega \setminus A$ the universe save the elements of A

$F^c = \{Bob, Joe, Max\}$

Ω



$F \cup F^c = \Omega$ $\{F, F^c\}$ are coll. exhaus.

$F \cap F^c = \emptyset$ $\{F, F^c\}$ are mut. excl.

$\Omega^c = \emptyset$ $F \cap \Omega = F$

$\emptyset^c = \Omega$

$A, B \subset \Omega$

$(A \cup B)^c = A^c \cap B^c$ $(A \cap B)^c = A^c \cup B^c$

Ω

De Morgan's Laws

Recall $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{Z} := \{\dots, -1, 0, 1, \dots\}$

$\mathbb{Q} := \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\}$ rationals

$|\mathbb{N}| = \aleph_0$ ctbl. ∞ $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$

But $\mathbb{Q}$ has holes!

For instance 

? $\sqrt{2}$ $\quad \nexists\, p \in \mathbb{Z}, q \in \mathbb{N}$ st $\frac{p}{q} = \sqrt{2}$! & p, q share no factors

Imagine they did!

$$\left(\frac{p}{q}\right)^2 = 2$$

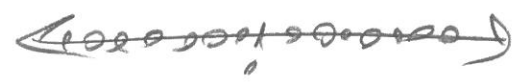
$$\Rightarrow p^2 = 2q^2 \Rightarrow \frac{p^2}{2} = q^2 = A \in \mathbb{N}$$

$p = p_1 p_2 \cdots p_k$ (unique factors)

$$p_1^2 p_2^2 \cdots p_k^2 = 2q^2 \Rightarrow 2 \in \{p_1, \dots, p_k\}$$

$$\frac{p_1^2 p_2^2 \cdots p_{k-1}^2\, \cancel{2} \cdot 2}{\cancel{2}} = q^2$$

$$B = p_1^2 p_2^2 \cdots p_{k-1}^2 = \frac{q^2}{2} \Rightarrow 2 \in \{q_1, \dots, q_k\} \quad \perp$$



$\in \mathbb{N}$

$$\mathbb{R} := \mathbb{Q} \cup \{\overbrace{\text{all holes in}}^{\text{irrationals}}\ \mathbb{Q}\} \qquad \text{1870's} \qquad \text{real \#'s!!}$$

$$|\mathbb{R}| \overset{= c \text{ uncntbl. infinity}}{>} |\mathbb{Q}|$$

$[1,2] := \{x : x \geq 1 \ \& \ x \leq 2\}$

$(1,2) := \{x : x > 1 \ \& \ x < 2\}$

$\max([1,2]) = 2$

$\max((1,2))$ d.n.e.

$\min([1,2]) = 1$

$\min((1,2)) =$ d.n.e.

$[a,b] = \{x : x \geq a \ \& \ x \leq b\} \subset \mathbb{R}$

$(a,b) = \{x : x > a \ \& \ x < b\} \subset \mathbb{R}$

$(-\infty, \infty) = \mathbb{R}$ they are the same

~~New set operation: Cartesian Product~~

~~First define~~ ordered pair := $\langle a,b\rangle$ is a set where order matters

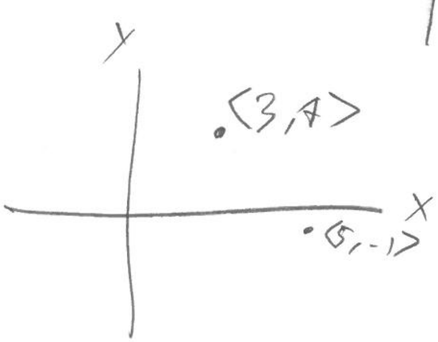
previously $\{a,b\} = \{b,a\}$

but now $\langle a,b\rangle \neq \langle b,a\rangle$ → Descartes

define $A \times B$ as the "Cartesian product": it is the set of all ordered pairs $\{\langle a,b\rangle : a \in A, b \in B\}$

e.g. $\{1,2\} \times \{3,4\} = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle\}$

$|A \times B| = |A||B|$



Cartesian Plane := $\mathbb{R} \times \mathbb{R}$

$A^2 := A \times A \qquad |A^2| = |A|^2$

Set theory is over!

$A^n := A \times \cdots \times A$

$|A^n| = |A|^n$

~~Set theory is over!~~

From now on: all sets we consider have elements called "outcomes" i.e. things that actually happen!

Ω is the set of all things that happen

$\Omega = \{H, T\}$ $\quad$ H, T mutually exclusive? Be careful..

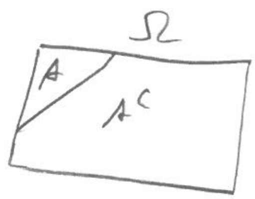
$\{\{H\}, \{T\}\}$ mut. excl.

$2^\Omega = \{\emptyset, \{H\}, \{T\}, \{H,T\}\}$ the powerset of Ω are the events.

Prob of events is defined, prob of outcomes are not. This is a mathematical convenience. Outcomes important because they belong to events.

"for now" $P(A) := \dfrac{|A|}{|\Omega|}$ where $A \in 2^\Omega$ $\quad P: \underbrace{2^\Omega}_{\text{domain}} \to \underbrace{[0,1]}_{\text{range}}$

just like $f(x) = x^3$ $\quad f: \mathbb{R} \to \mathbb{R}$



$P(A)$ is the proportion of the sample space

$$P(\{H\}) = \frac{|\{H\}|}{|\{H,T\}|} = \frac{1}{2}$$

$P(H)$ is meaningless but I will abuse the notation

$$P(\emptyset) = \frac{|\emptyset|}{|\Omega|} = \frac{0}{|\Omega|} = 0$$

$$P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$$

What is $|2^\Omega|$ represent?

All things we can ask what is the prob of?

$P(\emptyset)$ $\quad P(\{H\})$ $\quad P(\{T\})$ $\quad P(\{H,T\})$

Two coin flips!

$\Omega^2 := \Omega \times \Omega$

Ω

HH	HT
TH	TT

$P(\{HH\}) = \frac{\{\langle H, H\rangle\}}{|\Omega^2|} = \frac{1}{4}$

Prob of at least one H ?

$A := \{\text{at least one } H\} = \{HH, HT, TH\}$

$P(A) = \frac{|\quad|}{|\Omega^2|} = \frac{3}{4}$

P at least one T ?

$B = \{\quad\} = \{HT, TH, TT\}$ $\quad P(B) = \frac{3}{4}$

$P(A \cup B) \overset{=1}{\neq} \frac{3}{4} + \frac{3}{4}$

P(at least one tail and one Head) $= P(A \cap B) = P(\{HT, TH\}) = \frac{1}{2}$

$|2^{\Omega^2}| = 2^{|\Omega^2|} = 2^4 = 16$ only 16 things to ask what is prob of ... ?

4 tosses!

HHHH	HHHT	HHTT	HTTT
HTHH	THHH	HHTH	TTTT

Is $P(HHHH) = P(HTHT)$?

Yes $= \frac{1}{16}$

Seems like $P(HTHT) > P(HHHH)$

so equality is like an optical illusion!

What are you really thinking?

$$P(2H, 2T) = \frac{6}{|\Omega^4|} = \frac{6}{16} > \frac{1}{16} \ !$$

gotta be careful... how many ways?

HHTT, HTHT, THHT, THTH, HTTH, TTHH $\Rightarrow 6$

$2^{10} \approx 1000$, $2^{20} \approx 1000000$, ...

$|2^{\Omega^4}| \approx 64000$

$2^x \approx 1000^{x/10}$ CS majors...

$x \ln 2 = \frac{x}{10} \ln 1000$

$2 \approx 1000^{.1} = 1.9953$ cool!

$\Omega :=$ { flip coin and roll dice }

$\Omega = \{H, T\} \times \{1, 2, \ldots, 6\}$

H1	H2	H3	H4	H5	H6
T1	T2	T3	T4	T5	T6

$$P(\text{even and } H) = \frac{|\{H2, H4, H6\}|}{|\Omega|} = \frac{3}{12}$$

What if Ω isn't as simple as $\Omega_1 \times \Omega_2$?

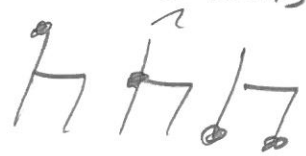
Need to learn how to count size of weird sample spaces!

A common type of set!

Jane Mark Susan

let $\{J, M, S\}$ be studying in front of three distinct seats.

How many ways to seat them?



$$\underset{\text{seat 1}}{\underline{3}} \cdot \underset{\text{seat 2}}{\underline{2}} \cdot \underset{\text{seat 3}}{\underline{1}} = 6$$

seat 1	seat 2	seat 3
J	M	S
	S	M
M	J	S
	S	J
S	J	M
	M	J

Tree diagram easy to see

Distinct Orderings are called "permutations". Sample space whose outcomes are permutations of an underlying set.

$\Omega = \{(JMS), JSM, MJS, MSJ, SJM, SMJ\}$

$|\Omega| = 6 \neq 3^3 = 27$ Why?

~~JJJ~~

duplicates not allowed in this setup

~~[illegible]~~ Sampling without replacement where order matters = permutation

Sampling with replacement is the familiar $\Omega^3 = 27$

J J J

When does this happen? Decks of cards for instance

Let's say I sample n objects without replacement. How many ways to order them?

$$\frac{n}{\#1}\ \frac{n-1}{\#2}\ \frac{n-2}{\#3}\ \cdots\cdots\ \frac{1}{\#n}$$

$$n! = n\cdot(n-1)(n-2)\cdots\cdots 1 = \prod_{i=1}^{n} i$$

"n-factorial"

How many ways to seat 5 people? $5! = 120$

10 people? $10! = 3{,}600{,}000$

20 people? $2.7\times10^{32} \approx$ diam(universe) in ft.

What about 10 people and 3 chairs?

$$\underline{10}\cdot\underline{9}\cdot\underline{8} = \frac{10!}{7!} = \frac{10!}{(10-3)!}$$

General notation $_{10}P_3$ stands for "permutation"

total objects — # of place

$$_nP_k := \frac{n!}{(n-k)!}$$

10 people, 10 chairs $_{10}P_{10} = \frac{10!}{(10-10)!} = \frac{10!}{0!}$

$\boxed{0! := 1}$ in order to make this sensical