9/17/15 Lecture #5 Math 241

$P(R) \neq \frac{|\{R\}|}{|\Omega|}$

$\mathbb{1}_{\omega_i \in A} = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{o/t} \end{cases}$

Since $\forall \omega \in \Omega, P(\{\omega\}) \neq \frac{1}{|\Omega|}$

astrologic?

Classical Definition (Laplace)

Why didn't ancient greeks develop prob???

Need new def of prob function..

$P(A) := \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\omega_i \in A}$ where $\omega_1, \ldots, \omega_n$ are draws from Ω.

Why is this the most popular? It's the easiest to understand?

Problems

① Assume Stationarity: ω is pulled from same Ω (AKA repeatable conditions).

② Never see $n = \infty$. So $P(A) \approx \frac{1}{n} \sum \mathbb{1}_{\omega_i \in A}$ if n is large but not exact. We will see this with conf. int's & hyp testing.

④ Not general! P(rain tomorrow)? P(OJ simpson guilty)

③ limit converges? Why?

Why is this most popular? Easier... Plus... prob began with analysis of gambling games which were repeated many, many times. Assume same process.

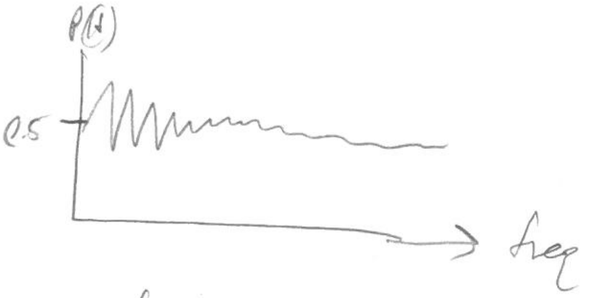
→ founding of prob.!

e.g. M. de Chevalier de Mere (1654) claimed in a letter to Pascal & Fermat that

$$P(\{\geq 1 \text{ 6-6 in 24 rolls of two dice}\}) < \frac{1}{2}$$

Turns out it's .4914! (We will prove this soon)

L.R.F. is <u>objective</u> i.e. property of the physical world. If humans weren't here, $P(H) = \frac{1}{2}$ still!



It seems that prob. is induced from a l.r.f.

l.r.f. $\Longrightarrow$ prob.

What about...

(II) Propensity Theory (1950's)

An object has inherent property to go one way or another which induces a l.r.f. Coin has innate tendency to flip Heads which is why it's about 50%.

So l.r.f. is an estimate of propensity

prob $\Rightarrow$ l.r.f.

Canonical Example

U238 has $\frac{1}{2}$ life of 4.5 Byrs. "Hardwired" is due to quantum mechanics. Does it matter what we think?

No... it would be the same whether us humans...

<u>Problems</u>

① Can't be calculated (save a few physical systems)

② <u>Still requires a physical object</u>

Both are objective theories. What about P(Trump wins in Nov)?

What about an "optimistic" day → P(US system was guilty)?

"degrees <u>of belief</u>" concerned with knowledge of humans

→ ~~Either w.r.t. or propensity: we still are concerned with randomness. Where does it come from?~~

Newton's Principia (1687) 3 laws of motion.

You know the starting conditions ⇒ you know the outcome

Major affect on western world thought. Laplace (1814)

philosophical essay on prob's (read)

(III) Logical Theory - given a set of evidence, all rational people would agree! Wrong... Next course...

(IV) Subjective - given a set of " ", people can disagree

P(OJ Simpson guilty) → objectively it's 0 or 1 since the event already occurred
→ subjectively it's a # $\in [0,1]$

If so, why not 0 or 1? Since we are ignorant and lack knowledge. It represents our "degree of belief." or degree of corroboration — P(F=ma)

⇒ Seems random but admittedly it's NOT!

Is the same for objective probability?

$P(H) = \frac{1}{2}$ → due to our ignorance of the system?

If we understood the system, would we know?

⇒ page 3 Is randomness real? LAPLACE: NO!!

And this is what we thought: All prob's are due to our own uncertainty:

Until...

Wasn't until the 1920's...

double-slit experiment demonstrated wave-particle duality

$\Rightarrow$ randomness seems to be a fundamental part of the universe $\Rightarrow$ ~~determinism~~ Einstein "dice with universe"

~~Summing up~~

~~Physical / dangers / large due to our~~

There is no definition of probability that we all are satisfied with.

Mathematics is not concerned with the "real" world. We will now define it mathematically. Assume Ω non-empty. ↑ technically wrong but okay for now.

P is a set function with domain all sets $A \subseteq \Omega$ s.t.

(a) $P(A) \geq 0 \;\; \forall A \subseteq \Omega$ (b) $P(\Omega) = 1$ (c) If $A_1, A_2, \ldots$ disjoint $\Rightarrow P\left(\bigcup_{i=1} A_i\right) = \sum P(A_i)$

<u>Thm I</u> $P(A) = 1 - P(A^c)$ "the complement rule"

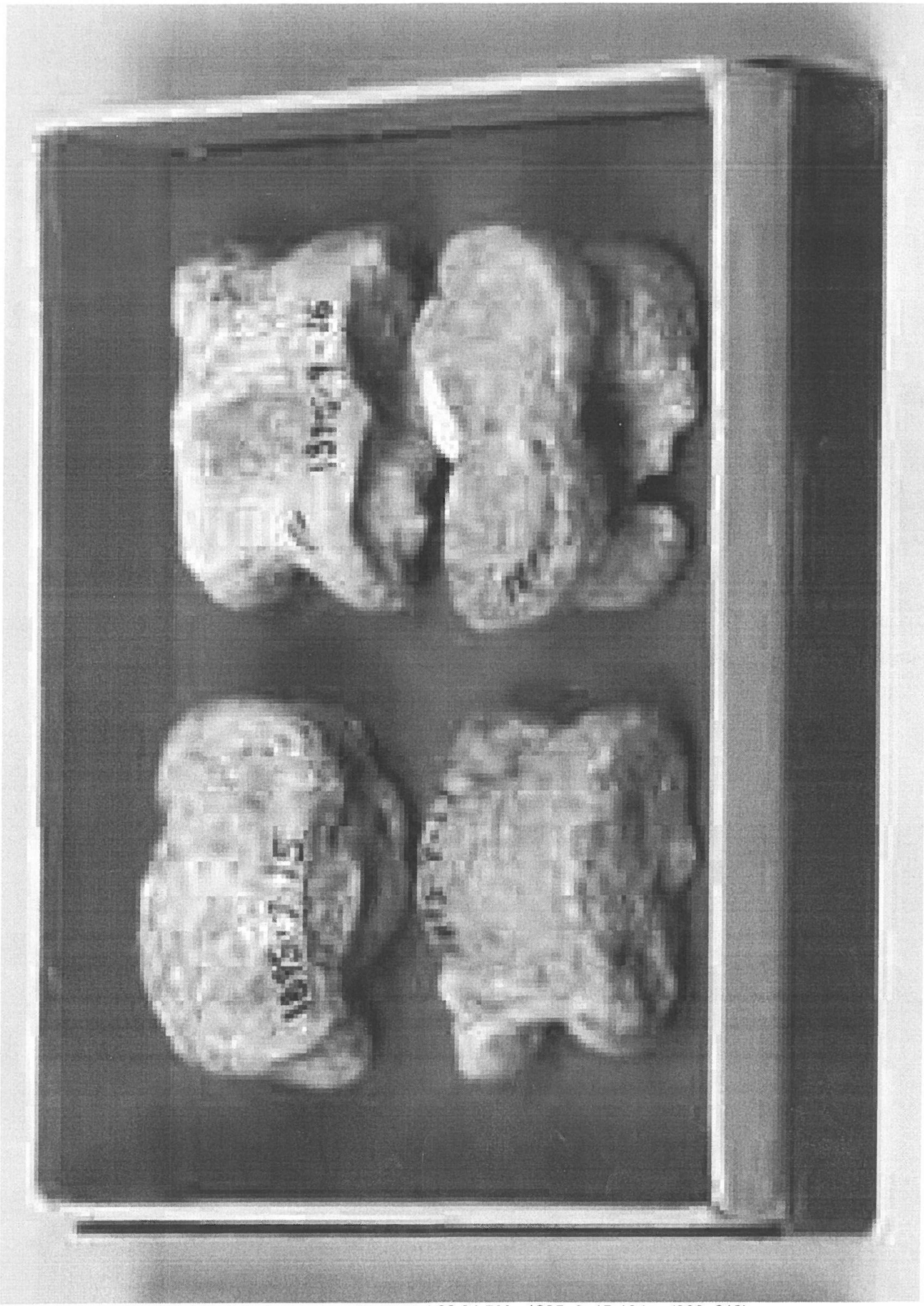
Recall $\Omega = A \cup A^c$ and $A \cap A^c = \phi$ (A, A^c disjoint)

$\Rightarrow P(\Omega) = P(A \cup A^c)$ by def. of function this is okay

$\Rightarrow 1 = P(A \cup A^c)$ (b)

$\Rightarrow 1 = P(A) + P(A^c)$ (c)

$\Rightarrow P(A) = 1 - P(A^c)$ algebra



sheep Knucklebones

astragali / tali

2 The classical theory

The classical theory of probability was a product of the thinking of the European Enlightenment, and it embodied many of the Enlightenment's characteristic ideas. In particular, we find the usual admiration for Newtonian mechanics, and the consequent belief in *universal determinism*. Indeed, Laplace's *Philosophical Essay on Probabilities* of 1814 gives one of the most famous formulations of the thesis of universal determinism. This is the formulation involving what is known as *Laplace's demon*. I will expound it in the next section.

Universal determinism and Laplace's demon

Laplace writes:

> We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it – an intelligence sufficiently vast to submit these data to analysis – it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past would be present to its eyes.
>
> (1814: 4)

The vast intelligence here described has come to be known as Laplace's demon. The idea is obviously founded on that of a human scientist (perhaps Laplace himself) using Newtonian mechanics to calculate the future paths of planets and comets. Extrapolating from this success, it was natural to suppose that a sufficiently vast intelligence could calculate the entire future course of the universe. Laplace himself relates his vast intelligence to human successes in astronomy. As he says:

> The human mind offers, in the perfection which it has been able to give to astronomy, a feeble idea of this intelligence. Its discoveries in mechanics and geometry, added to that of universal gravity, have enabled it to comprehend in the same analytical expressions the past and future states of the system of the world.
>
> (Laplace 1814: 4)

<u>Thm 2</u> $P(\phi) = 0$

~~Recall $\Omega^c = \phi$~~

By Thm 1, $P(A) = 1 - P(A^c)$

let $A = \Omega$

$\Rightarrow P(\Omega) = 1 - P(\Omega^c)$

Recall $\Omega^c = \phi$ by def. of complement

$\Rightarrow P(\Omega) = 1 - P(\phi)$

$\Rightarrow P(\phi) = 1 - P(\Omega) = 1 - 1 = 0$ ✓

(b)

<u>Thm 3</u> $p \to q \Leftrightarrow \sim q \to \sim p$

$p \wedge \sim q$

If $A \neq \phi \Rightarrow P(A) > 0$

$\Leftrightarrow P(A) \le 0 \Rightarrow A = \phi$

Since $P(A) \ge 0$

$\Rightarrow P(A) = 0 \Rightarrow A = \phi$ ✓

<u>Thm 4</u> $A \subset B \Rightarrow P(A) < P(B)$

If $A \subset B \Rightarrow B \setminus A = C \neq \phi$ by def of proper subset

$\Rightarrow A \cup C = B$ and A, C disjoint by construction

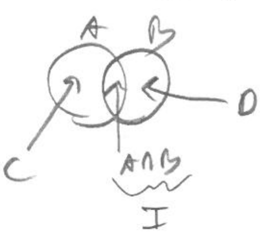
$P(A \cup C) = P(B)$

$P(A) + P(C) = P(B)$

$P(B) - P(A) = P(C) > 0$ by thm 3

$\Rightarrow P(B) > P(A) \Rightarrow P(A) < P(B)$

<u>Thm 5</u> $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$C := A \setminus B$

$D := B \setminus A$

$\Rightarrow A = C \cup I$ s.t. C, I disjoint $\Rightarrow P(A) = P(C) + P(I) \Rightarrow P(C) = P(A) - P(I)$

$B = D \cup I$ s.t. D, I disjoint $\Rightarrow P(B) = P(D) + P(I) \Rightarrow P(D) = P(B) - P(I)$

$A \cup B = C \cup I \cup D$

$P(A \cup B) = P(C \cup I \cup D) = P(C) + P(I) + P(D) = P(A) - P(I) + P(I) + P(B) - P(I) = P(A) + P(B) - P(I)$ ✓

Generally...

Law of Inclusion - Exclusion

$$P\left(\bigcup A_i\right) = \sum P(A_i) - \sum P(A_i \cap A_j) + P(A_i \cap A_j \cap A_k) - \dots + (-1)^{n+1} P\left(\bigcap A_i\right)$$

Thm 6

For $|\Omega| < \infty$,

If $P(\{\omega_i\}) = \frac{1}{|\Omega|}$ $\forall i$ then $P(A) = \frac{|A|}{|\Omega|}$

If $A = \emptyset \Rightarrow P(A) = 0$; If $A = \Omega \Rightarrow P(A) = 1$

So here, $A \subset \Omega$ and non-empty $\Rightarrow A = \{\omega_{(1)}, \dots \omega_{(n)}\} \Rightarrow |A| = n$

let $A_1 = \{\omega_{(1)}\}$, $A_2 = \{\omega_{(2)}\}, \dots, A_n = \{\omega_n\}$ note: all disjoint

$A = \bigcup_{i=1}^{n} A_i$ (c)

$$P(A) = P\left(\bigcup A_i\right) = \sum P(A_i) = \sum_{i=1}^{n} \frac{1}{|\Omega|} = \frac{n}{|\Omega|} = \frac{|A|}{|\Omega|}$$ ✓

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10 flowers 4R, 3G, 3B Ways to arrange?

$$\frac{10!}{4!3!3!} := \binom{10}{4,3,3} = \binom{10}{4}\binom{6}{3}\binom{3}{3} = \frac{10!}{4!\cancel{6}!} \frac{\cancel{6}!}{3!\cancel{3}!} \frac{\cancel{3}!}{3!\cancel{0}!}$$ ✓

$\binom{10}{4,3,3}$ ← multichoose notation

$\binom{10}{4}$: 4R, 6 Not Red

Balls & Urns n balls (indistinct), r urns (distinct)

and each urn must ~~be~~ not be empty

$r = 2, n = 4$

$X_1, \dots, X_r \in \mathbb{N}$
$X_1 + X_2 + \dots + X_r = n$
How many sol's?

$|o|o|o\,o|$ $\binom{n-1}{r-1}$