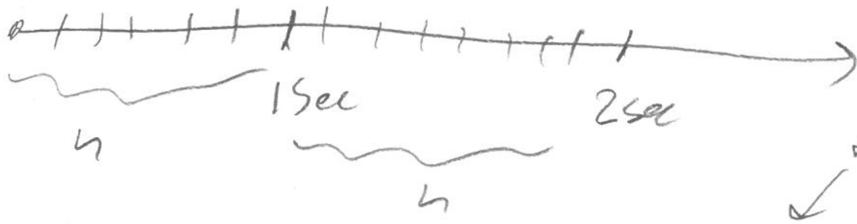
Math 241 Lec 17 11/5/15

t is a function of the num

$T \sim \text{Geometric}(p)$ $\quad p(t) = (1-p)^{t-1} p \quad F(t) = 1-(1-p)^t$

What if every second we run n separate Bernoullis?



total # of experiments run until stopping

As a function of time $p(t) = (1-p)^{nt-1} p$, $F(t) = 1-(1-p)^{nt}$

$$\underset{\text{num}}{E[T]} = \frac{1}{p} \leftarrow \text{\# of experiments}$$

$$E(T) = \frac{\frac{1}{p} \text{ exp}}{n \text{ exp/sec}} = \frac{1}{pn} \text{ sec}$$

in seconds

So if n is large... immediately stop!

But what if p is really really small?

$p \approx 0$ and $n \approx \infty$

let $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$$\Rightarrow p(t) = (1-p)^{nt-1} p = \left(1-\frac{\lambda}{n}\right)^{nt-1} \frac{\lambda}{n}$$

Now let $n \to \infty$ i.e. pretend experiments happen "continuously."

$$\lim f(x)^a = \left(\lim f(x)\right)^a$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nt-1}\frac{\lambda}{n} = \left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^t \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-1}\underbrace{\lim_{n\to\infty}\frac{\lambda}{n}}_{0} = 0$$

$p(t) = 0 \quad \forall t$ So is T a geometric still? No!

It's not even a discrete r.v.! It doesn't have a PMF!

$$\sum p(t) = 0 \neq 1$$

But does it have a CDF?

$$\lim_{n\to\infty} F(t) = \lim_{n\to\infty} 1-\Big(1-\overset{\frac{\lambda}{n}}{\cancel{p}}\Big)^{nt} = 1-\lim\left(1-\frac{\lambda}{n}\right)^{nt} = 1-\left(\lim\left(1-\frac{\lambda}{n}\right)^n\right)^t$$

What is

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

10	2.594
100	2.705
1000	2.717
10000	2.718
	⋮
	?

$e :=$

or $e := \sum_{i=0}^{\infty}\frac{1}{i!}$ or $\int_1^e \frac{1}{x}\,dx = 1$

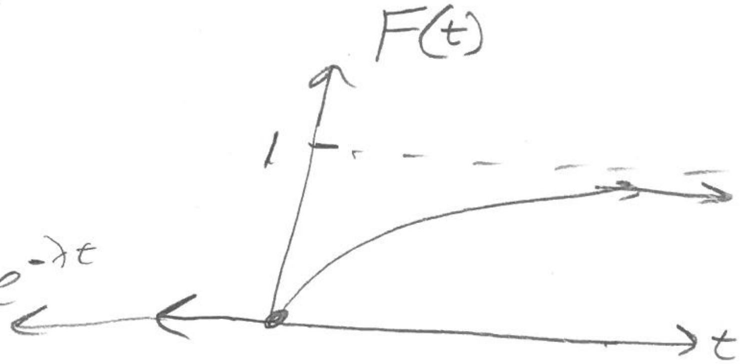
How about

$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$ let $\frac{a}{n}=\frac{1}{m}$ $\Rightarrow m=\frac{n}{a}$ if $n\to\infty \Rightarrow m\to\infty$

$\Rightarrow \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{ma}=\left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^a=e^a$

So...

$\lim F(t)=1-\left(\lim\left(1-\frac{\lambda}{n}\right)^n\right)^t=1-e^{-\lambda t}$



Is this a CDF? $F(t)\in[0,1]$? $1-\frac{1}{e^{\lambda t}}$ $t>0$

at $t=0$

$\lim_{t\to\infty} 1-e^{-\lambda t}\stackrel{?}{=}1$

$\Rightarrow 1-\underbrace{\lim_{t\to\infty}\frac{1}{e^{\lambda t}}}_{0}=1$ ✓

~~$0\le 1-e^{-\lambda t}\le 1$~~

~~$-1\le -e^{-\lambda t}\le 0$~~

~~$0\le e^{-\lambda t}\le 1$~~

~~$-\infty\le -\lambda t\le 0$~~

~~$0\le\lambda t\le\infty$~~ $\lambda\in\mathbb{R}$

~~$0\le t\le\infty$~~ ✓

$\lim_{t\to-\infty} 1-e^{-\lambda t}\stackrel{?}{=}0$

$1-e^{-\lambda(0)}=0$ ← since $t>0$

$1-1=0$ ✓

It's a CDF!

$F'(t)\ge 0\ \forall t$? $F'(t)=\lambda e^{-\lambda t}=\frac{\overset{+}{\lambda}}{\underset{+}{e^{\lambda t}}}>0$ ✓ $t>0$

CDF but no PMF! It is a r.v. but not a discrete r.v.

$Supp[T] = (0, \infty)$

$|Supp| = |\mathbb{R}| > |\mathbb{N}| \Rightarrow$ Not discrete!

Def: a cont r.v. has a support which is the $\underbrace{\text{size of the "continuum"}}_{|\mathbb{R}|}$

"uncntb. ∞"

How did this happen?

$\lim n \to \infty$ the experiments happen continuously... not discretely... no way to even count them

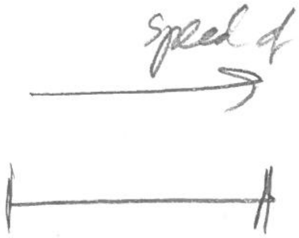
Infinite divisions mean no missing #'s ..

Is time actually continuous?

Ancient Greeks thought so... but quantum mechanics strikes again!

Planck Length 1.62×10^{-35} m

not possible to tell diff between two things two Planck length apart



$\frac{l}{c} = 5.3 \times 10^{-44}$ sec   time cannot be distinguishable

time seems to be discrete? Open question!

$p(3) = 0$ why?

infinite resolution / ideal

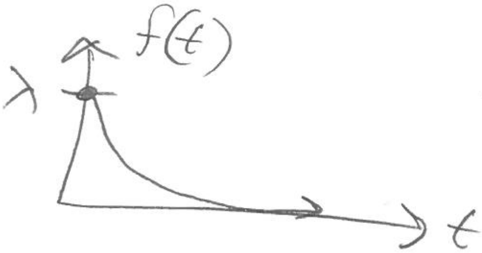
$P(X = 3.00000000\ldots\ldots) = 0$

But $P(X = 3.00000)$

$= P(X \in [2.99999, 3.00001]) = F(\ ) - F(\ ) > 0$

New Question: how fast does the CDF change?

$f(t) := F'(t) = \frac{d}{dt}\left[1 - e^{-\lambda t}\right] = \lambda e^{-\lambda t}$



prob density function (PDF) $\neq$ PMF which is 0 always

$P(T \in [a,b]) = F(b) - F(a) = \int_a^b f(t)\,dt$ F.T.C.

how dense is the prob in a certain region of the support?

$P(X=3) = P(X \in [3,3]) = \int_3^3 f(x)\,dx = 0$

Let $\lambda = 1$ $f(3) = 0.05 \neq p(3) = 0$ what does this mean??

PDF is an abstract metric. It is good for

① Comparing two places on X and indexing likelihood relation

② Integrating to find $P(X \in A)$

Def: X is a cont r.v. if $|Supp(X)| = |\mathbb{R}|$

Properties

① $Supp(X) \subseteq \mathbb{R}$

② $F(x)$ is a CDF

③ PMF d.n.e

④ $f(x) := F'(x)$ exists and $\int_{Supp(X)} f(x)\,dx = 1$ Can $f(x) > 1$? Yes

~~two~~ cont. r.v.'s ~~are iden~~ $X_1 \overset{d}{=} X_2$ iff $f_1(x) = f_2(x)$ or $\underbrace{F_1(x) = F_2(x)}_{\text{always works}}$

What is $E(X)$?

Previously

$E(X) = \sum_{x=1}^{n} x\, p(x)$

Now... Use Riemann sum to express..

$$E(X) = \int_{supp(X)} x f(x)\, dx \qquad E[g(X)] = \int_{supp(X)} g(x) f(x)\, dx$$

$$Var(X) = \int_{supp(X)} (x-\mu)^2 f(x)\, dx$$

All rules apply:

$$E(aX+c) = aE(X)+c$$

$$Var(aX+c) = a^2\sigma^2$$

$$SD(aX+c) = |a|\sigma$$

$$E(\Sigma X_i) = \Sigma E(X_i) \underset{\text{if ident. distr.}}{=} n\mu$$

$$Var(\Sigma X_i) \underset{\text{if indep}}{=} \Sigma Var(X_i) \underset{\text{if iid}}{=} n\sigma^2$$

Which r.v. have we been studying?

defined by its PDF now..

$X \sim Exp(\lambda) := \lambda e^{-\lambda x}$

$Supp(X) = (0, \infty)$ or $[0, \infty)$ not $[0, \infty]$ not ∞

this makes sense given $\lambda = np$, $p \in (0,1)$, $n \to \infty$

Param Space

$\lambda \in (0, \infty)$ $\quad \lambda = 0 \Rightarrow$ not a PDF illegal (not degenerate)

$$E[X] = \int_{Supp(X)} x f(x)\, dx = \int_0^\infty x \lambda e^{-\lambda x}\, dx =$$

$$\int u\, dv = uv - \int v\, du$$

let $u = x$, let $dv = e^{-\lambda x} dx$ $\quad v = \int dv = -\frac{1}{\lambda} e^{-\lambda x}$

$du = dx$

$$\int v\, du = \int -\frac{1}{\lambda} e^{-\lambda x} dx = \frac{1}{\lambda^2} e^{-\lambda x}$$

$$\Rightarrow \lambda \left[x\left(-\frac{1}{\lambda} e^{-\lambda x}\right) - \frac{1}{\lambda^2} e^{-\lambda x} \right]_0^\infty = -\left[\frac{x}{e^{-\lambda x}} + \frac{1}{\lambda e^{-\lambda x}} \right]_0^\infty$$

$$\left(\frac{0}{e^{-\lambda(0)}} + \frac{1}{\lambda e^{-\lambda 0}} \right) - \left(\lim_{x \to \infty} \frac{x}{e^{+\lambda x}} + \frac{1}{e^{+\lambda x}} \right) = \boxed{\frac{1}{\lambda}}$$

Makes sense

$X \sim Geom(p) \quad E(X) = \frac{1}{p}$

Now $p = \frac{\lambda}{n} \quad E(X) = \frac{n}{\lambda}$ exp's $\cdot \frac{1 \text{ sec}}{n \text{ exp}} = \boxed{\frac{1}{\lambda} \text{ sec}}$

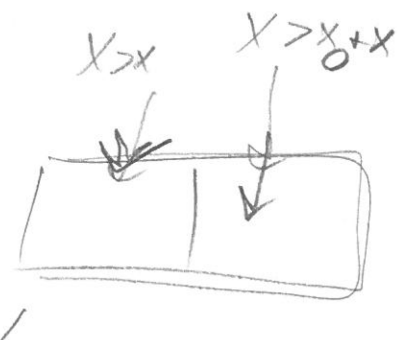
Var(X) ... HW...

Stopping Time

	Single	Multiple
Discrete	Geom	NegBin
Cont	Exp	Erlang

(Next 2x2)

	PMF	PDF	CDF	
Discrete	Y	N	Y	
Cont	N	Y	Y	



Is exponential memoryless?

$$P(X > x_0 + x \mid X > x) = \frac{P(X > x_0 + x)}{P(X > x)} = \frac{1 - e^{-\lambda(x_0 + x)}}{e^{-\lambda x}} = e^{-\lambda x} = P(X > x)$$

$$F(x) := P(X \le x) = 1 - P(X > x) = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$$