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Lec 16

$$E(X_1+X_2) = \sum_{x_1} x_1 \sum_{x_2} p(x_1,x_2) + \sum_{x_2} x_2 \sum_{x_1} p(x_1, x_2)$$

$$= \sum_{x_1} x_1 p(x_1) + \sum_{x_2} x_2 p(x_2)$$

$$= E[X_1] + E[X_2]$$

$$\underbrace{E\left[\sum_{i=1}^{n} X_i\right]}_{E[T_n]} = \sum_{i=1}^{n} E[X_i] \overset{\text{if ident distri}}{=} \sum_{i=1}^{n} \mu = n\mu$$

$$E[\bar{X}_n] = E\left[\frac{T_n}{n}\right] = \frac{1}{n} E[T_n] \underset{\text{ident distri}}{=} \frac{1}{n} n\mu = \mu$$

$\bar{X} \to \mu$

$X_1, \ldots X_r \overset{iid}{\sim} Geom(p)$

$T = X_1 + X_2 \ldots X_r \sim NegBinom(r, p)$

$$E(T) = E[X_1 + \ldots X_r]$$

$$= \sum_{i=1}^{r} E(X_i)$$

$$= \frac{r}{p}$$

$X \sim \text{Hypergeom}(n, K, N)$

$X = X_1 + X_2 + \cdots X_n$

$X_1 \sim \text{Bernoulli}\left(\frac{K}{N}\right)$

$X_2 \sim \text{Bernoulli}\left(\frac{K}{N}\right)$ — all same.

$X_1 \ldots X_n \overset{\text{iden}}{\sim} \text{Ber}\left(\frac{K}{N}\right)$

$X_2 | X_1 = 1 \sim \text{Ber}\left(\frac{K-1}{N-1}\right)$

$E[X] = \sum_{i=1}^{n} E(X_i) = n\frac{K}{N}$

$\text{Var}[X] = E[(X-n)^2] = E[(X^2 - 2\mu X + \mu^2)]$

$= E(X^2) - 2\mu \underbrace{E(X)}_{\mu} + \mu^2$

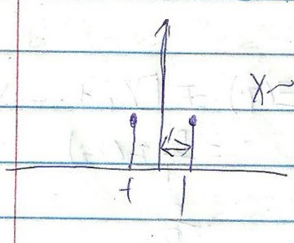
$\sigma^2 = E(X^2) - \mu^2$

$E(X^2) = \sigma^2 + \mu^2$

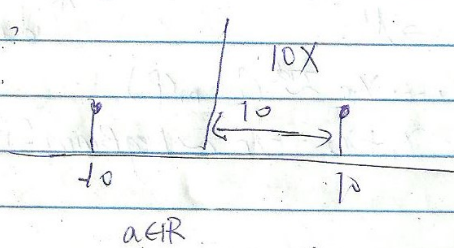
$\left.\begin{array}{l} E(X) \\ E(X^2) \\ E(X^3) \\ \vdots \end{array}\right\}$ "moments"

$\sigma^2 \cdot \left\{\begin{array}{l} E(X-\mu) \\ E[(X-\mu)^2] \\ E[(X-\mu)^3] \end{array}\right\}$ Central moments.

$\left.\begin{array}{l} \dfrac{E[X-\mu]}{\sigma} = 0 \\ \dfrac{E[(X-\mu)^2]}{\sigma^2} = 1 \\ \dfrac{E[(X-\mu)^3]}{\sigma^3} \\ \dfrac{E[(X-\mu)^4]}{\sigma^4} \end{array}\right\}$ Standard moments



$X \sim$ Radial Rademach?



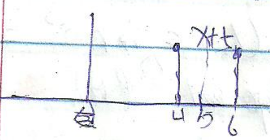
$a \in \mathbb{R}$

$\text{Var}(aX) = a^2\sigma^2$

$Var(ax) = E[(ax - E(ax))^2] = E[(ax - a\mu)^2]$

$= E[(a(x-\mu))^2] = E[a^2(x-\mu)^2]$

$= a^2 E[(x-\mu)^2] = a^2\sigma^2$



$c \in \mathbb{R}$

$Var[X+c]$

$= \sigma^2$

$Var[X+c] = E[((X+c) - E(X+c))^2]$

$= E[(X+c-\mu-c)^2] = \sigma^2$

$Var(aX+c) = a^2\sigma^2$

$SE(aX+c) = \sqrt{Var(aX+c)}$

$= \sqrt{a^2\sigma^2}$

$= |a|\sigma$

$Var[X_1+X_2] = E[(X_1+X_2 - \underbrace{E[X_1+X_2]}_{\mu_1+\mu_2})^2]$

$= E[(X_1+X_2-\mu_1-\mu_2)^2]$

$= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 - 2X_1\mu_1 - 2X_1\mu_2 - 2X_2\mu_1 - 2X_2\mu_2 + 2X_1X_2 + 2\mu_1\mu_2]$

$= E[X_1^2] + E(X_2^2) + \mu_1^2 + \mu_2^2 - 2\mu_1^2 - 2\mu_1\mu_2 - 2\mu_1\mu_2 - 2\mu_2^2 + 2E(X_1X_2] + 2\mu_1\mu_2$

$= \sigma_1^2 + \sigma_2^2 + 2(E(X_1X_2) - \mu_1\mu_2)$

$:= Cov[X_1, X_2]$, "Covarimace". ← definition

$E(X_1 X_2] = \sum_{x_1} x_1 \sum_{x_2} x_2 P(x_1, x_2)$.

if X_1, X_2 independent $\Rightarrow = \sum_{x_1} x_1 \sum_{x_2} x_2 P(x_1)P(x_2)$

$= \underbrace{\sum_{x_1} x_1 P(x_1)}_{E(X_1)} \underbrace{\sum_{x_2} x_2 P(x_2)}_{E(X_2)}$

$= \mu_1\mu_2$.

$\Rightarrow Cov(X_1 X_2) = 0$ if X_1, X_2 independent.

if X_1, X_2 independent.

if iid

$$Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var(X_i) = n\sigma^2$$

$$SE\left[\sum_{i=1}^{n} X_i\right] = \sqrt{\sum_{i=1}^{n} Var(X_i)} = \sqrt{n\sigma^2} = \sqrt{n}\,\sigma$$

if X_1, X_2 independent. if iid.

$$Var[\bar{X}] = Var\left[\frac{T}{n}\right] = \frac{1}{n^2} Var(T) = \frac{1}{n^2}\sum Var(X_i) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

if X_1, X_2 independent. if iid

$$SE[\bar{X}] = \sqrt{\frac{1}{n^2} Var(T)}$$

indeper $= \sqrt{\frac{1}{n^2}\sum Var(X_i)}$

iid $= \frac{\sigma}{\sqrt{n}}$

$X \sim Ber(p) = \sigma^2 = p(1-p)$

$X_1 \ldots X_n \overset{iid}{\sim} Ber(p)$

$T = X_1 + X_2 \cdots X_n \sim Binomial(n, p)$

$Var(T) = \sum Var(X) = \sum p(1-p) = np(1-p)$

$$Var(X) = \sum_{x \in supp(x)} (x-\mu)^2 p(x)$$

$$= \sum_{x=0}^{n} (x-np)^2 \binom{n}{x} p^x (1-p)^{n-x}$$

$$= E(X^2) - \mu^2$$

$$= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} - n^2p^2$$

$$\sum_{x=1}^{n} x^2 \frac{n!}{x!(x-n)!} p^x (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} x \underbrace{\frac{(n-1)!}{(x-1)!(n-x)!}}_{=\binom{n-1}{x-1}} p^{x-1} (1-p)^{n-x}$$

Let $y = x - 1$

$\Rightarrow x = y + 1$

$x = 1 \ldots n$

$y = 0, \ldots n-1$

$$= np \sum_{y=0}^{n-1} (y+1) \binom{n-1}{y} p^y (1-p)^{n-1-y}$$

Let $m = n - 1$

$$= np\Big(\underbrace{\sum_{y=0}^{m} y\binom{m}{y}p^y(1-p)^{m-y}}_{mp} + \overbrace{\sum_{y=0}^{m}\binom{m}{y}p^y(1-p)^{m-y}}^{1}\Big)$$

$\Downarrow$ Binomial (m, p)

$$= np((n-1)p+1) = np(np-p+1) = n^2p^2 - np^2 + np$$

$$= -np^2 + np \quad = n(-p^2+p) = np(1-p)$$

$$Var(X) = \sum_{x=r}^{\infty}\left(x-\frac{r}{p}\right)^2\binom{x-1}{r-1}(1-p)^{x-r}p^r = \frac{r(1-p)}{p^2}$$

$X \sim NegBin(r, p)$

$X \sim Geom(p) = (1-p)^{x-1}p$

$$Var(X) = E(X^2) - \mu^2) = \sum_{x=1}^{\infty} x^2(1-p)^{x-1}p - \frac{1}{p^2}$$

$$= \frac{1-p}{p^2}$$

$F(X) = 1-(1-p)^x$

$P(X \le x) = 1 - P(X > x)$

$P(X > x) = (1-p)^x$

$P(X = 17 \mid X > 10)$

$$= \frac{P(X=17 \text{ \& } X>10)}{P(X>10)}$$

$$= \frac{P(X=17)}{(1-p)^{10}}$$

$$= \frac{(1-p)^{16}p}{(1-p)^{10}} = (1-p)^6 p = P(X=7)$$

$P(X = b+x \mid X > b) = P(X = x)$ "memorylessness"