Lecture 6 9/13/16 Math 271

"P" is a ~~set~~ function s.t.

(a) $\exists\ \Omega \neq \phi$ s.t. $P(\Omega) = 1$

(b) $P(A) \geq 0 \quad \forall A \subseteq \Omega$

(c) If $A_1, A_2, \ldots$ disjoint $\Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

you can add disjoint probs of

Thm I $\quad P(A) = 1 - P(A^c)$

$\Omega = A \cup A^c$ set theory

$P(\Omega) = P(A \cup A^c)$ by def of function

$P(\Omega) = P(A) + P(A^c)$ via (c)

$1 = P(A) + P(A^c)$ via (a)

$\Rightarrow P(A) = 1 - P(A^c)$ (algebra)

Thm 2 $\quad P(\phi) = 0$

$P(\Omega) = 1 - P(\Omega^c) = 1 - P(\phi) \Rightarrow 1 = 1 - P(\phi) \Rightarrow P(\phi) = 0$

Thm 3

$A \subseteq B \Rightarrow P(A) \leq P(B)$

$\Rightarrow C := B \setminus A$

$\Rightarrow A \cup C = B \ \& \ A \cap C = \phi$

$P(A \cup C) = P(B)$

$P(A) + P(C) = P(B)$ (c)

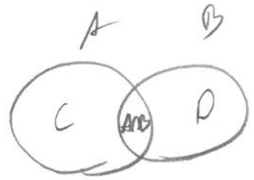
$P(C) = P(B) - P(A) \geq 0$ (b)

$P(B) \geq P(A)$ ✓

Thm 5

Inclusion - exclusion

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$C = A \setminus B$

$D = B \setminus A$

$I = A \cap B$

$P(A) = P(C) + P(I) \Rightarrow P(C) = P(A) - P(I)$

$P(B) = P(D) + P(I) \Rightarrow P(D) = P(B) - P(I)$

$$P(A \cup B) = P(C) + P(D) + P(I)$$

$$= \left(P(A) - P(I)\right) + \left(P(B) - \cancel{P(I)}\right) + \cancel{P(I)}$$

$$= P(A) + P(B) - P(A \cap B)$$

~~Generally...~~ $\cancel{P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i \neq j} P(A_i \cap}$

HW: Booles inequality

Thm 6

$$|\Omega| < \infty, \text{ if } P(\{\omega_i\}) = \frac{1}{|\Omega|} \ \forall \omega \Rightarrow P(A) = \frac{|A|}{|\Omega|}$$

let $n = |A| < \infty$ — since $|\Omega| < \infty$ $A \subseteq \Omega \Rightarrow |A| < \infty$

$$A = \{\omega_1, \omega_2, \dots, \omega_n\}$$

$$A = \{\omega_1\} \cup \{\omega_2\} \cup \dots \cup \{\omega_n\}$$

$$\Rightarrow A = \bigcup_{i=1}^{n} \{\omega_i\}$$

(c)

$$P(A) = P(\ ) = \sum_{i=1}^{n} P(\{\omega_i\}) = \sum_{i=1}^{n} \frac{1}{|\Omega|} = \frac{n}{|\Omega|} = \frac{|A|}{|\Omega|}$$ ✓

Imagine $n = 1000$ people (Ω)

200 smokers (A)

60 lung cancer (B)

36 s & l.c (AB)

$A \cap B$, A, B or $A \& B$

Via LRF Def...

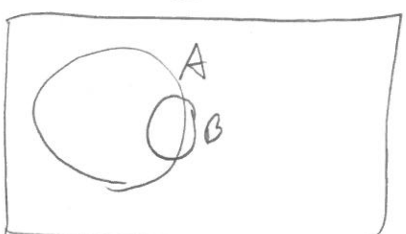
$P(A) = \frac{200}{1000} = 0.2$

$P(B) = \frac{60}{1000} = 0.06$

$P(AB) = \frac{36}{1000} = 0.036$

What if I want to know the "probability of l.c. only among smokers"

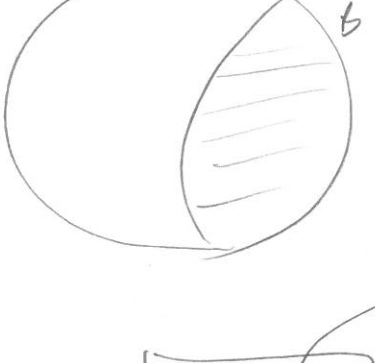
given smoking, what is $P(\text{l.c.})$? Denoted...



$P(B|A)$

Then you just need to look at A and ignore the rest of the Ω.

$\Omega' := A \subset \Omega$



$P(B|A) = \frac{36}{200} = 0.18$

What if we only had prob's?

Same shape but zoomed in...

just like

Note

$$P(B|A) = P(B,A) \cdot \text{Zoom}$$

Let's take a look at zooming

Zoom: ?

$$\text{Zoom} = \frac{\text{prior scope size}}{\text{new scope size}} = \frac{1}{0.5} = 2 \qquad \text{Zoom} = \frac{P(\Omega)}{P(A)} = \frac{1}{P(A)}$$

$$\boxed{P(B|A) := \frac{P(B,A)}{P(A)}}$$ Def. of cond. prob.

$$\Rightarrow P(B,A) = P(B|A)P(A)$$ corollary

$$\text{if } P(A|B) = \frac{P(A,B)}{P(B)} = \frac{P(B,A)}{P(B)}$$

$$\Rightarrow \boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$ "Bayes Rule" (1763)

$$P(B|A) = \frac{.036}{.2} = .18 \approx 20\%$$ smoke is corr. with l.c.

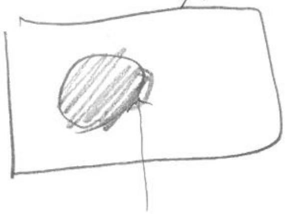
low? high?

$$P(\text{smoke}|\text{l.c.}) = P(A|B) = \frac{P(A,B)}{P(B)} = \frac{.036}{.06} = .6$$ good chance he was smoker...

$$P(\text{l.c.} \mid \text{didn't smoke}) = \frac{P(B, A^c)}{P(A^c)} = \frac{?}{1-0.2} = \frac{.024}{.8} = .03$$

high? low?

A^c



B

$P(B) = P(B, A) + P(B, A^c)$

$\Rightarrow P(B, A^c) = P(B) - P(B, A)$ proof of this coming soon..

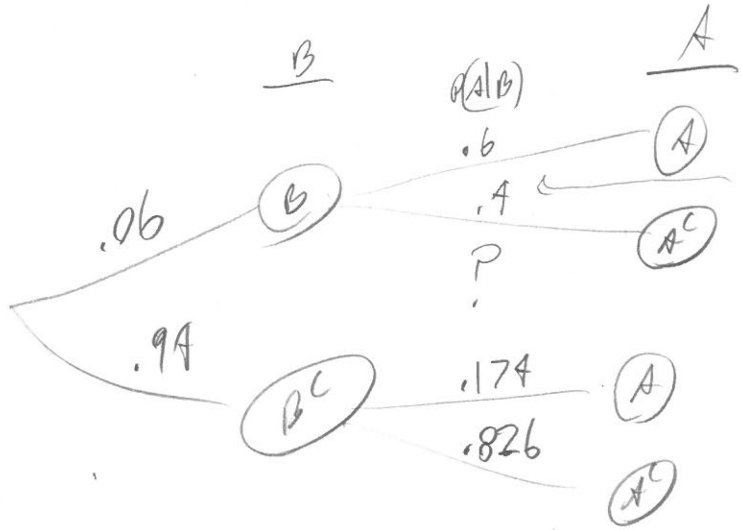
$= .024$

$$\frac{P(B|A)}{P(B|A^c)} = \frac{.18}{.03} = 6 \quad \text{Risk ratio}$$

$RR = 6?$ $P(B|A) = 0.18?$

How to make decision???

$P(B|A)$? $P(B|A^c)$? How many questions are there? So let's build a tree to answer all questions



joint sum

$P(AB) = .036$

$P(A^cB) = .024$

$P(AB^c) = .164$

$P(A^cB^c) = .776$

mut excl, coll. exh

$= 1$

$P(A|B) \overset{?}{=} 1 - P(A^c|B)$

$\frac{P(AB)}{P(B)} \overset{?}{=} 1 - \frac{P(A^c, B)}{P(B)}$

$\Rightarrow P(AB) \overset{?}{=} P(B) - P(A^c, B)$

$\Rightarrow P(B) \overset{?}{=} P(AB) + P(A^cB)$ ✓

$P(A) = P(B^c, A) + P(B, A)$

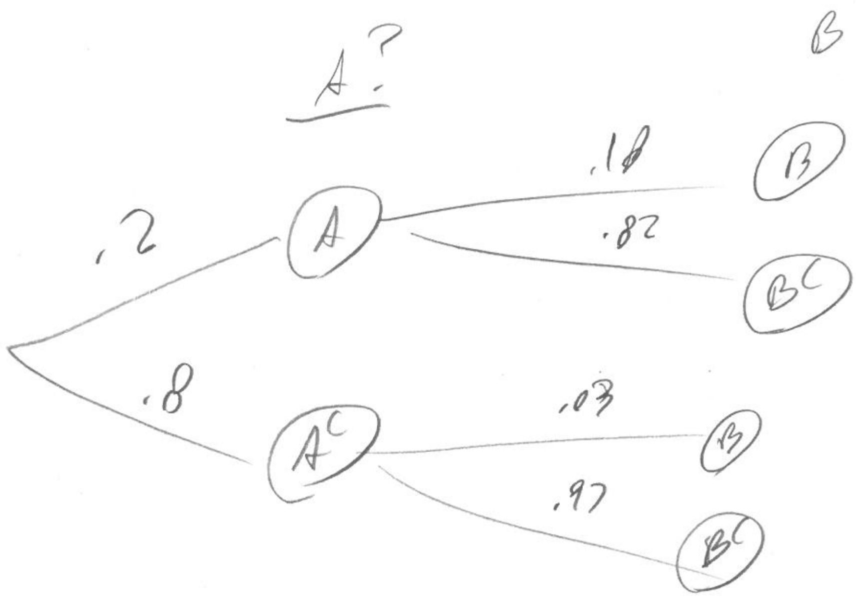
$\qquad\quad \shortparallel$

$.2 - .036 = .164$

makes sense but not useful

$P(A^c|B^c) = .826$

What are we missing? $P(B|A)$... Need to "invert the tree"



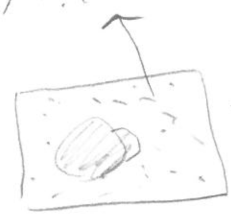
$P(AB) = .036$

$P(A, B^C) = .164$

$P(A^C B) = .024$

$P(A^C B^C) = .776$

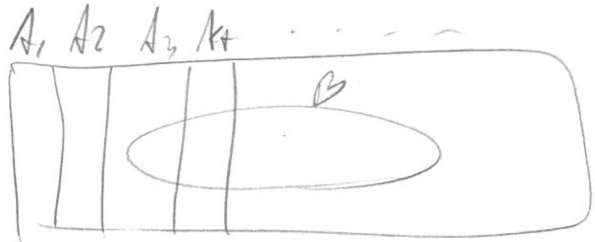
$P(B^C | A^C) = .97$



$\Omega = A^C$

Prove that idea

Consider event B and mut. excl, coll exh. events $A_1, A_2, \ldots$



$P(B) = P(B \cap \Omega)$ HW 1

$= P(B \cap (A_1 \cup A_2 \cup A_3 \cup \ldots))$ mut. excl. coll. exh.

$= P((B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3) \cup \ldots)$ How to prove?

Are $B \cap A_i$ and $B \cap A_j$ mut. excl?

mut. excl. and

$(B \cap A_i) \cap (B \cap A_j)$

$= B \cap A_i \cap A_j$

$= B \cap \phi = \phi$ Her 1 $\Rightarrow$ Yes

$$\Rightarrow \boxed{P(B) = \sum_{i=1}^{\infty} P(B, A_i)}$$ Law of Total Prob

with $A_1 = A$, $A_2 = A^c$ $\Rightarrow P(B) = P(B, A) + P(B, A^c)$

$$P(B) = \sum_{i=1}^{\infty} P(B|A_i)P(A_i)$$

$$P(A_\ell|B) = \frac{P(B|A_\ell)\,P(A_\ell)}{P(B)}$$ Bayes Rule

$$P(A_\ell|B) = \frac{P(B|A_\ell)\,P(A_\ell)}{\sum_{i=1}^{\infty} P(B|A_i)P(A_i)}$$ Bayes Thm.

Cond. Prob. Super weird!!

I know this

You have two kids. One is a girl. $P(\text{other is girl}) \overset{?}{=} \frac{1}{2}$

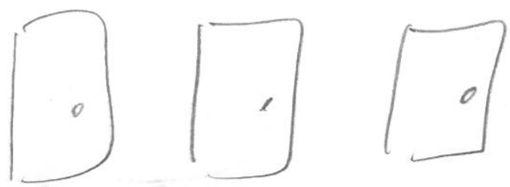
Ω

GG	GB
BG	BB

$P(\{GG\} \mid \text{one is a girl})$

$= P(\{GG\} \mid \{GG, GB, BG\}) = \frac{|\{GG\}|}{|\{GG, GB, BG\}|} = \boxed{\frac{1}{3}}$ Crazy.....

Monte Hall



two goats and a car

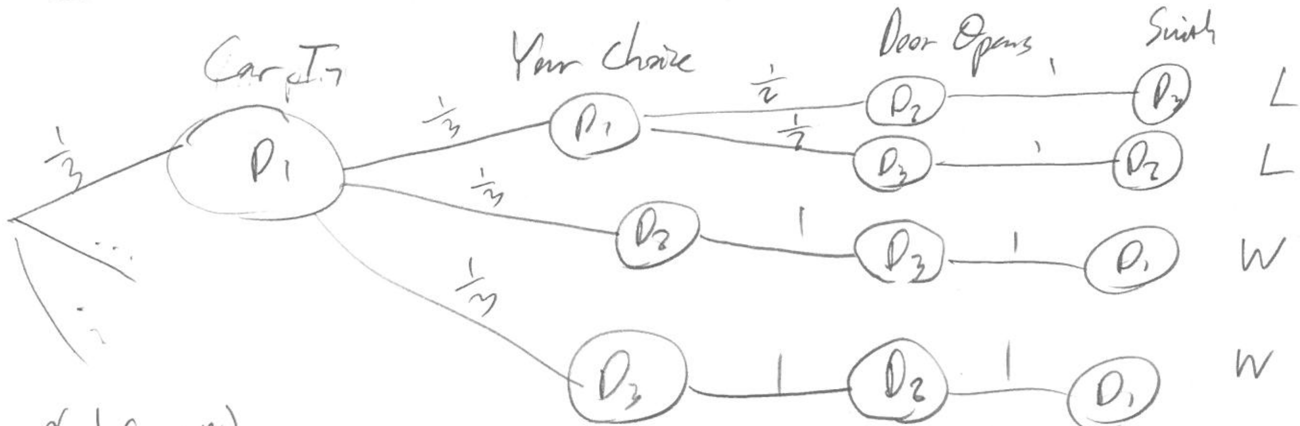
Game

① You pick a door

② Gameshow host opens one of the other two doors to show goat.

③ You have the option of keeping your door or switching

P(winning if switch)? Draw Tree



$P(W) = P(W \mid \text{Car in } D_1) + P(W \mid \text{Car in } D_2) + P(W \mid \text{Car in } D_3)$

$P(W \mid \text{Car in } D_1) = \frac{\frac{1}{3}\left(\frac{1}{3} + \frac{1}{3}\right)}{\frac{1}{3}} = \boxed{\frac{2}{3}}$