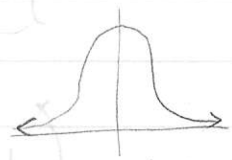
11/17/2016.

	PMF	PDF	CDF	$E[X]$	$Var[X]$	$\|Supp[X]\|$
Discrete	$P(x) \subset [0,1]$ $\sum_{x \in Supp[X]} P(x) = 1$	not exist	exists	$\sum_{x \in Supp[X]} x\,p(x)$	$\sum (x-\mu)^2 p(x)$	$\leq \|\mathbb{N}\|$
Continuus	not exist	$f(x) \geq 0$ $\int_{Supp[X]} f(x)dx = 1$	exists	$\int_{Supp[X]} x f(x) dx$	$\int (x-\mu)^2 f(x) dx$	$= \|\mathbb{R}\|$

	Quantile$[X, p]$
Discrete	$\min_x \{x : F(x) \geq p\}$
Continuus.	x s.t. $F(x) = p$. $x = F^{-1}(p)$



EX: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is this a PDF

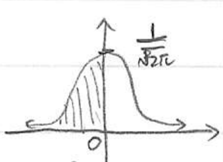
a) $f(x) \geq 0$ $\quad \frac{1}{\sqrt{2\pi}}$ is $\oplus$, $\frac{1}{e^{x^2/2}}$ is $\oplus$. ✓

b) $\int_{\mathbb{R}} f(x)\,dx = 1 \Rightarrow \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1 \Rightarrow \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$

let $n = \frac{1}{\sqrt{2}} x \Rightarrow n^2 = \frac{x^2}{2}$, $dn = \frac{1}{\sqrt{2}} dx \Rightarrow dx = \sqrt{2}\, dn$

$\int_{\mathbb{R}} e^{-n^2} \sqrt{2}\, dn = \sqrt{2\pi} \Rightarrow \underbrace{\int_{\mathbb{R}} e^{-n^2} dn = \sqrt{\pi}}_{\text{重要}} \Rightarrow \left(\int_{\mathbb{R}} e^{-n^2} dn\right)^2 = \pi \Rightarrow$

$\int e^{-n^2} dn \int e^{-n^2} dn = \pi \Rightarrow \int e^{-x^2} dx \int e^{-y^2} dy = \pi \Rightarrow \iint_{x \in \mathbb{R}, y \in \mathbb{R}} e^{-(x^2+y^2)} \underbrace{dx\,dy}_{dA} = \pi$



$v = r^2$, $dv = 2r\,dr \Rightarrow \frac{dv}{2} = r\,dr \Rightarrow \int_0^\infty e^{-v} \frac{dv}{2} = \frac{1}{2} = 1$

$[e^{-v}]_0^\infty = e^{-0} - \lim_{v \to \infty} e^{-v} = 1$ ✓.

This is PDF. $Z \sim N(0,1) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

"standard normal r.v.", "bell curve" "Gaussian r.v."

$F(0) = \frac{1}{2}$

This is one case of the general normal dist.

Let $u = \frac{x^2}{2} \Rightarrow du = x\,dx$

$$E[Z] = \int_{supp[X]} x f(x)\,dx : \int_{x=-\infty}^{x=\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} e^{-u} du = \frac{1}{\sqrt{2\pi}} \left[-e^{-u}\right]_{x=-\infty}^{x=\infty} = \frac{1}{\sqrt{2\pi}} \left[-e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\lim_{x \to -\infty} e^{-\frac{x^2}{2}} - \lim_{x \to \infty} e^{-\frac{x^2}{2}} \right) = 0 \quad \Rightarrow \boxed{\mu = 0.}$$

$$Var[Z] = E[Z^2] - \mu^2 = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = (\text{work}) = 1$$

↑ by integration by part.

$SE[Z] = 1$, $\boxed{\sigma^2 = \sigma = 1}$

This has no closed form expression, we need a computer to do this.

△ $$F(x) = \int f(x)\,dx + C = \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + C}_{\text{not possible}}$$

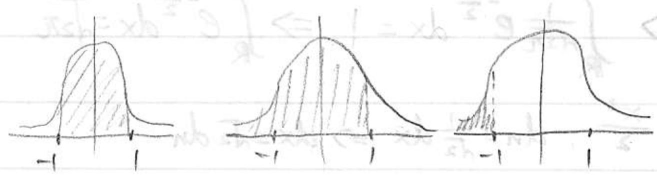
"Risch Algorithm"

△ $$F(x) = P(Z \le x) = \int_{-\infty}^{x} \frac{1}{2\pi} e^{-\frac{x^2}{2}} dx$$

What's $F(0)$? the amount of probability up until 0.

$\frac{1}{2}$ because we proved that the bell $=1$, since the bell is symmetrical, at $x=0$, the CDF $= \frac{1}{2}$.

△ $P(Z \in [-1,1]) = F(1) - F(-1) \approx \underline{0.68}$

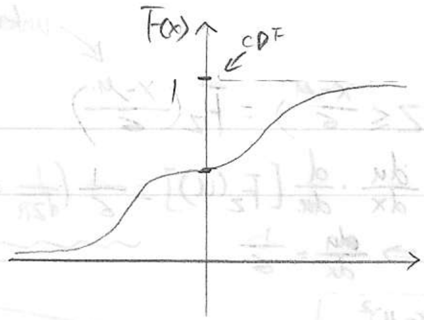


$P(Z \in [-2,2]) = F(2) - F(-2) \approx \underline{0.95}$

$P(Z \in [-3,3]) = F(3) - F(-3) \approx \underline{\underline{0.997}}$

"3σ rule"
or "Erpincal Rule"
or "68-95-997 rule"

"The amount of prob. between -1 SE and 1 SE is 0.68"



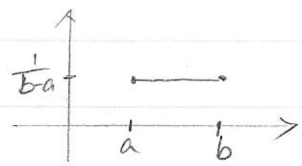
Review midterm 2.

$T \sim Geom(p)$, $E(T) = \frac{1}{p}$, $\lim_{n \to \infty} P(t) = 0$ has no PMF

$F(x) = 1 - e^{-\lambda y}$

$X \sim Exp(\lambda) = f(x) = \lambda e^{-\lambda x}$ expounital r.v. $E[X] = \frac{1}{\lambda}$

$X \sim U(a,b)$, $f(x) = \frac{1}{b-a}$



例: $X \sim Exp(\lambda)$ $Y = 2X \sim ?$ Let $\lambda' = \frac{\lambda}{2}$

$F_Y(x) = P(Y \le x) = P(2X \le x) = P(X \le \frac{x}{2}) = F_X(\frac{x}{2}) = 1 - e^{-\lambda \frac{x}{2}} = 1 - e^{-\lambda' x} = Exp(\lambda') \Rightarrow$

$Exp(\frac{\lambda}{2}) \Rightarrow Y \sim Exp(\frac{\lambda}{2})$

例: $Y = d + cX$

$F_Y(x) = P(Y \le x) = P(cX + d \le x) = P(X \le \frac{x-d}{2}) = F_X(\frac{x-d}{2}) = \frac{\frac{x-d}{c} - a}{b-a} \cdot \frac{c}{c} \Rightarrow$

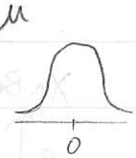
$\frac{x-(d+ca)}{cb-ca} = \frac{x-(d+ca)}{(d+cb)-(d+ca)} = \frac{x-a'}{b'-a'} = U(a',b') = U(d+ca, d+cb)$

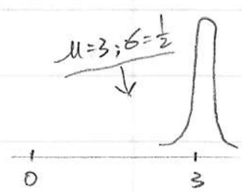
(Let $a' = d+ca$, $b' = d+cb$.)

例: $Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, let $X := \sigma Z + \mu$

$E[X] = \sigma \overset{0}{\cancel{E(Z)}} + \mu = \mu$

$SE[X] = SE[\sigma Z + \mu] = |\sigma| \overset{1}{\cancel{SE[Z]}} = |\sigma|$





unknowable

$$F_X(x)=P(X\le x)=P(\sigma Z+\mu\le x)=P\left(Z\le \frac{x-\mu}{\sigma}\right)=F_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$f(x)=\frac{d}{dx}F_X(x)=\frac{d}{dx}F_Z\left(\frac{x-\mu}{\sigma}\right)=\frac{du}{dx}\cdot\frac{d}{du}\left[F_Z(u)\right]=\frac{1}{\sigma}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{\sigma}}\right)\Rightarrow$$

let $u=\frac{x-\mu}{\sigma}\Rightarrow\frac{du}{dx}=\frac{1}{\sigma}$

$$\boxed{X\sim N(\mu,\sigma)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{1}{2\sigma}(x-\mu)^2}}$$