Lecture 10 Math 241 10/13/16

Review
$X \sim \text{Bernoulli}(p) := p^x(1-p)^{1-x}$ param? Free variable?
$X \sim \text{Binomial}(n,p) := \binom{n}{x} p^x (1-p)^{n-x}$ what values could free variable take?
$X \sim \text{Hyper}(n,K,N) := \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$ $X \sim \text{Hyper}(n,p,N) := \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$

$\lim_{N\to\infty} \text{Hyper}(n,p,N) = \text{Binom}(n,p)$ Conceptually...

Property of PMF $\sum_{x \in Supp(X)} p(x) = 1$

$\sum_{x \in \{0,\dots,n\}} \binom{n}{x} p^x (1-p)^{n-x} = 1$ How?

Recall: $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$ binomial thm.

let... $a = p$, $b = 1-p$, $i = x$ $\quad (p + (1-p))^n = \sum_{i=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

1^n
$=$
1

DONE ... this is why the binomial is named so...

X_1 and X_2 are ind $\quad X_1, X_2 \overset{ind}{\sim}$

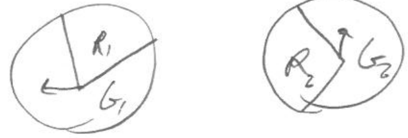
$P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$
$P(X_2 = x_2 \mid X_1 = x_1) = P(X_2 = x_2)$ $\quad \forall x_1 \in Supp(X_1)$,
$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1)\,P(X_2 = x_2)$ $\quad \forall x_2 \in Supp(X_2)$

Joint mass function (JMF)

the only way to know independence is to know Ω's and the process or be given the JMF

$X_1 \sim \text{Bern}\left(\frac{1}{3}\right)$ Ω_1 Ω_2

$X_2 \sim \text{Bern}\left(\frac{1}{3}\right)$



unconnected

$X_1 \stackrel{d}{=} X_2$? Yes. X_1, X_2 ind? Yes sure

$P(X_1 = 1 \mid X_2 = 0) = P(R_1 \mid G_2) = P(R_1) = P(X_1 = 1)$

$P(X_1 = 1 \mid X_2 = 1) = P(R_1 \mid R_2) = P(R_1) = P(X_1 = 1)$ ✓

$P(X_1 = 0 \mid X_2 = 0) = P(G_1 \mid G_2) = P(G_1) = P(X_1 = 0)$

$P(X_1 = 0 \mid X_2 = 1) = P(G_1 \mid R_2) = P(G_2) = P(X_1 = 0)$

so X_1, X_2 are $\stackrel{d}{=}$ & ind $\Rightarrow$ "iid" (Indep & Identically Distributed)

$X_1, X_2 \stackrel{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$ Most of statistics is based on the iid property

$X_1, X_2 \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$

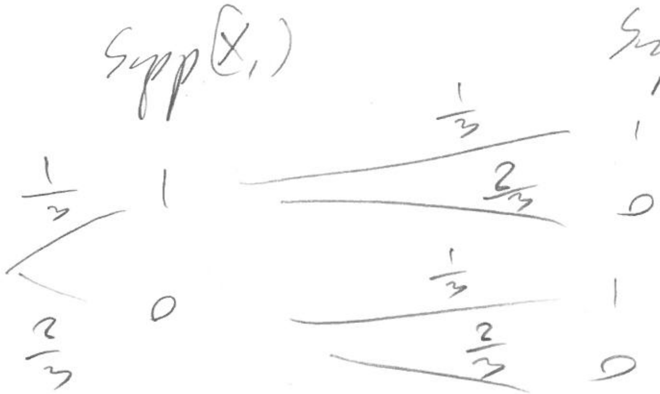
a function of two r.v.'s

let $T_2 := X_1 + X_2 = g(X_1, X_2)$ Conceptually... what is this?

a new r.v.

$\text{Supp}[T_2] = \{0, 1, 2\}$

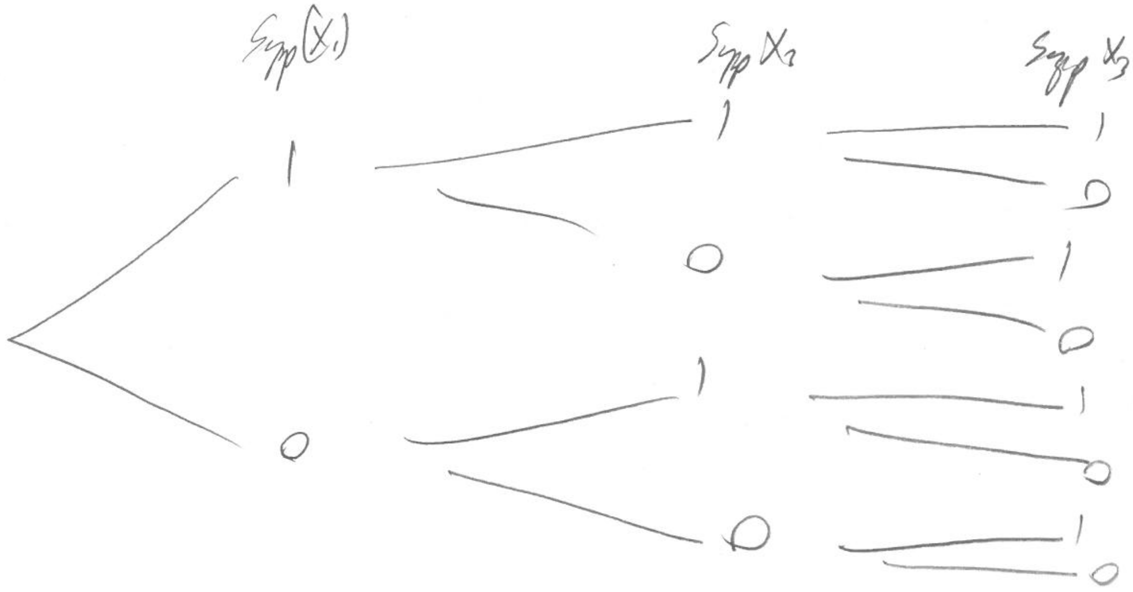
$T_2 \sim ?$ Let's use a tree to figure this out



	T_2
$P(X_1 = 1, X_2 = 1) = \frac{1}{9}$	2
$P(X_1 = 1, X_2 = 0) = \frac{2}{9}$	1
$P(X_1 = 0, X_2 = 1) = \frac{2}{9}$	1
$P(X_1 = 0, X_2 = 0) = \frac{4}{9}$	0
1	

$$\Rightarrow T_2 \sim \begin{cases} 0 & \text{wp } \frac{4}{9} \\ 1 & \text{wp } \frac{4}{9} \\ 2 & \text{wp } \frac{1}{9} \end{cases}$$

Let's do $T_3 := X_1 + X_2 + X_3$



	T
$\left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0$	3
$\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1$	2
$\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1$	2
$\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2$	1
$\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1$	2
$\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2$	1
$\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2$	1
$\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^3$	0

4

$$T_3 \sim \begin{cases} 0 & \text{w.p. } 1\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 \\ 1 & \text{w.p. } 3\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 \\ 2 & \text{w.p. } 3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 \\ 3 & \text{w.p. } 1\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 \end{cases}$$

1 - 3 - 3 - 1 pattern

$$\begin{array}{cccc} \underline{0}\ \underline{0}\ \underline{0} & \underline{1}\ \underline{0}\ \underline{0} & \underline{1}\ \underline{1}\ \underline{0} & \underline{1}\ \underline{1}\ \underline{1} \\ & \underline{0}\ \underline{1}\ \underline{0} & \underline{1}\ \underline{0}\ \underline{1} & \\ & \underline{0}\ \underline{0}\ \underline{1} & \underline{0}\ \underline{1}\ \underline{1} & \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \end{array}$$

$$\Rightarrow T_3 \sim \binom{3}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{3-x} = \text{Binom}\left(3, \frac{1}{3}\right)$$

$$X_1, X_2, \dots, X_n \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right) \qquad T = \sum_{i=1}^{n} X_i$$

$$T \sim \begin{cases} 0 & \text{w.p. } \binom{n}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^n \\ 1 & \text{w.p. } \binom{n}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{n-1} \\ 2 & \text{w.p. } \binom{n}{2} \\ \vdots & \quad \vdots \\ n-2 & \\ n-1 & \\ n & \end{cases} = \text{Binom}\left(n, \frac{1}{3}\right)$$

$$X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}(p) \qquad T = \sum_{i=1}^{n} X_i$$

$$T \sim \begin{cases} 0 & \text{w.p. } \binom{n}{0} \\ 1 & \text{w.p. } \binom{n}{1} \\ 2 & \text{w.p. } \binom{n}{2} p^2 (1-p)^{n-2} \\ \vdots & \quad \vdots \\ n-2 & \quad \vdots \\ n-1 & \quad \vdots \\ n & \end{cases}$$

do this first

Two ways to look at binomial

$$\lim_{N\to\infty} \text{Hyper}(n,p,N)$$

— or —

$X_1,\ldots,X_n \overset{iid}{\sim} \text{Bern}(p)$

$X_1+\ldots+X_n$

for binomial..

$$F(x) := P(X \le x) = \sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i}$$

no closed form

$$= I_{1-p}(n-k, 1+k)$$

regularized incomplete beta function

$$= (n-k)\binom{n}{k} \int_0^{1-p} t^{n-k-1} (1-t)^k \, dt$$

no closed form

not tested

$\underbrace{X_1, X_2, \ldots}$ $\overset{iid}{\sim}$ Bern(p)

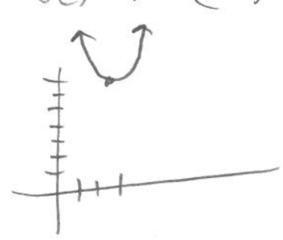
possibly infinite series of binary experiments w/ same prob.

independent / one another

min, max

argmin, argmax

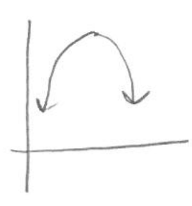
$f(x) = 7 + (x-3)^2$



max $\{f(x)\}$ undefined

argmax $\{f(x)\}$ undefined

$f(x) = 7 - (x-3)^2$



min $\{f(x)\} = 7$

argmin $\{f(x)\} = 3$

{3, 4, 7, 11, 12, 13, 18, ... }

let $T := \min\{t : X_t = 1\}$

T is the first time a success occurs

AKA the "stopping time"

$P(T=1) = p$ $\quad \frac{X=1}{t=1} \quad P(X=1)$

Why multiplication?

$P(T=2) = (1-p)p$ $\quad \begin{array}{ccc} X = & 0 & 1 \\ t = & 1 & 2 \end{array} \quad P(X_1=0, X_2=1) = P(X_1=0)P(X_2=1)$

Independence of the Bernoulli experiments

$P(T=3) = (1-p)^2 p$ $\quad \begin{array}{cccc} X & 0 & 0 & 1 \\ t & 1 & 2 & 3 \end{array}$

$P(T=t) = (1-p)^{t-1} p$

$\underbrace{\frac{0}{1}\ \frac{0}{2} \cdots \frac{0}{t-2}\ \frac{0}{t-1}}_{t-1 \text{ failures}}\ \underbrace{\frac{1}{t}}_{1 \text{ success}}$

$X \sim \text{Geometric}(p) := \underbrace{(1-p)^{x-1} p}_{p(x) \text{ PMF}}$

switch back to std notation

$\text{Supp}(X) = \{1, 2, \ldots\} = \mathbb{N}$

$0 \notin$ Why?

Param Space $p \in (0,1)$ why?

it's built from Bernoullis

If... $p=1$ $X \sim \text{Deg}(1)$

$$\sum_{x \in Supp(X)} p(x) = 1 \; ?$$

$$\sum_{i=1}^{\infty} (1-p)^{i-1} p \overset{?}{=} 1 \Rightarrow \sum_{i=1}^{\infty} (1-p)^{i-1} \overset{?}{=} \frac{1}{p} \Rightarrow \sum_{i=0}^{\infty} (1-p)^{i} \overset{?}{=} \frac{1}{p}$$

let $q := 1-p$ since $p \in (0,1) \Rightarrow q \in (0,1)$

$$S := \sum_{i=0}^{\infty} q^i \overset{?}{=} \frac{1}{1-q}$$

$= q^0 + q^1 + q^2 + q^3 + \ldots$ "geometric series"

$= 1 + q + q^2 + q^3 + \ldots$

$= 1 + q(1 + q + q^2 + \ldots$

$= 1 + q(S) \Rightarrow S - qS = 1 \Rightarrow S(1-q) = 1 \Rightarrow S = \frac{1}{1-q}$ ✓

This is how it gets its name

$$F(x) := P(X \le x) = \sum_{i=1}^{x} (1-p)^{i-1} p \quad \text{HARD}$$

— or —

$$F(x) = 1 - P(X > x)$$

the success is somewhere here

$$P(X>x) = P\Big(\underbrace{\underline{0}\ \underline{0}\ \_\ \_\ \_\ \cdots\ \underline{0}}_{\text{these are all 0's}}\ \overbrace{\underline{0}\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{0}\ \cdots}\Big)$$

$1\ 2\ 3\ 4\ 5 \quad x \quad x+1$

these are all 0's

$$P(X>x) = (1-p)^x$$

Seen another way

$$P(X>x) = P(X=x+1) + P(X=x+2) + P(X=x+3) + \ldots$$

$$= \sum_{i=x+1}^{\infty} (1-p)^{i-1} p$$

$$= \sum_{i=1}^{\infty} (1-p)^{i+x-1} p$$

$$= (1-p)^x \underbrace{\sum_{i=1}^{\infty} (1-p)^{i-1} p}_{1}$$