Lecture 11 Math 241 10/20

$X \sim \text{Geom}(p) := (1-p)^{x-1} p$ Valid PMF $\text{Supp}(X) = \mathbb{N}$, param space $p \in (0,1)$

you can think of this as the stopping time of $X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

e.g. prob of getting royal flush $= \dfrac{4}{\binom{52}{5}} = 1.53 / \text{million}$
$= .00000153$ ("p")

Scenario...
Play poker until you get a Royal Flush. On which # hand do you get it for the first time?

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(.00000153)$ stopping time

$X \sim \text{Geom}(.00000153)$

What is the prob. I get it on the millionth hand?

$P(X = 1000000) = (.9999985)^{999999} \cdot .00000153$

What is the prob I get it on the millionth time or sooner?

$F(x) = P(X \le x) = 1 - (1-p)^x$ powerful...

$P(X \le 1000000) = 1 - .9999985^{1000000} = .777 \approx 80\%$

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

Instead of querying the time of the first success, I want to query the time of getting r successes.

$$T = \min \left\{ t : \sum_{i=1}^{t} X_i = r \right\}$$

let's say $r=2$

0	1	0	0	~~1~~	0	0	0	1	$\Rightarrow x=3$
1	2	3	4	5	6	7	8	9	

$P(T=1)$ ✗

$P(T=2)$ ✗

$P(T=3) = p^3$

$P(T=4) = 3p^3(1-p)$

$P(T=5) = 6p^3(1-p)^2$

$r=3$

1 1 1

$r=3$

1 1 0 1
1 0 1 1
0 1 1 1

1 1 0 0 1
1 0 1 0 1
1 0 0 1 1
0 1 1 0 1
0 1 0 1 1
0 0 1 1 1

$P(T=x)$

$= p^3(1-p)^{x-3}\binom{x-1}{2}$

$$\underbrace{0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0}_{x-1}\ \ \underset{x}{1}$$

ways is

Generalize ... look for r

\# failures

$(x-1)-(r-1) = x-r$

\# successes

$$\underbrace{0\ 0\ \overbrace{1\ 0\ 1\ 0\ 1\ 0\ 1}^{r-1}\ 0\ 0\ 0}_{x-1}\ \ \underset{x}{1}$$

Stop at r successes...

$$P(X=x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$$

$X \sim \text{NegBin}(r, p)$ the negative binomial

$\text{Supp}[X] = \{r, r+1, \ldots\}$

parameter space

$p \in (0,1)$ from the Bernoulli

$r \in \mathbb{N}$ negative? fractional? 0?

$r=1 \Rightarrow X \sim Geom(p)$

$$X \sim NegBin(1,p) = \underset{\substack{\parallel \\ \binom{x-1}{0}}}{\binom{x-1}{1-1}} \cdot (1-p)^{x-1} \cdot p^1 = (1-p)^{x-1}p = Geom(p)$$

$$\sum p(x) = 1 ?$$

$$\sum_{x=r}^{\infty} \binom{x-1}{r-1}(1-p)^{x-r}p^r = 1$$

Recall:

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q} \quad \text{if } q \in (0,1)$$

$$\sum_{x=0}^{\infty} (1-p)^x = \frac{1}{p}$$

$$\sum_{x=1}^{\infty} (1-p)^{x-1} = p^{-1} \qquad \text{now take derivative...}$$

$$\sum_{x=1}^{\infty} (x-1)(1-p)^{x-2} = (-1)\,p^{-2} \qquad 1^{st}$$

$$\sum_{x=2}^{\infty} (x-1)(1-p)^{x-2} = (-1)\,p^{-2}$$

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$$\sum_{x=3}^{\infty} (x-1)(x-\overset{=2}{2})(1-p)^{x-3 = 2+1} = (-1)(-2)p^{-3} \quad 2^{nd}$$

$$\sum_{x=4 = 3+1}^{\infty} (x-1)(x-2)(x-\overset{=3}{3})(1-p)^{x-4 = 3+1} = (-1)(-2)(-3)p^{-4 = 3+1} \quad 3^{rd}$$

$\vdots \qquad r-1$ times

$$\sum_{x=r}^{\infty} \underbrace{(x-1)(x-2)\cdots(x-r+1)}_{10\cdot 9\cdot 8\cdot 7 = \frac{10!}{6!}}(1-p)^{x-r = r-1+1} = (r-1)!\,p^{-r = r-1+1}$$

$x = r$ ← $(r-1)+1$

$$\frac{(x-1)!}{(x-r)!}$$

Multiply

$$\sum_{x=r}^{\infty} \frac{(x-1)!}{(x-r)!\,(r-1)!}(1-p)^{x-r} = p^{-r} \quad (p^r)$$

Note: $(r-1)! = (x-1)-(x-r)$

$$\Rightarrow \sum_{x=r}^{\infty} \binom{x-1}{r-1}(1-p)^{x-r}p^r = 1 \quad \checkmark$$

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Equivalent Parameterization $X \sim NegBin(r,p)$ But...

Let $X = \#$ of failures...

$$\underbrace{\overbrace{\overbrace{\_\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0}\ \underline{0} \cdots}^{x}\ \_}^{x+r-1}\ \underline{1}}_{x+r}$$

$$P(X=x) = \binom{x+r-1}{r-1}(1-p)^x p^r$$

$$\binom{x+r-1}{r-1} = \binom{x+r-1}{x}$$

Param Space:

$p \in (0,1)$

$r \in \mathbb{N}$

FYI...

$$\frac{(x+r-1)!}{x!\,(r-1)!} = \frac{(x+r-1)\cdot(x+r-2)\cdots r}{x!} = (-1)^x\,\frac{(-r)\cdot(-r-1)(-r-2)\cdots(-r-x+1)}{x!}$$

$$= (-1)^x\binom{-r}{x}$$

negative binomial

this is where it gets its name

$$p(x) = \binom{-r}{x}(1-p)^x p^r$$

$X_1, X_2, \dots, X_r \overset{iid}{\sim} \text{Geom}(p)$

$X_1 + X_2 + \dots + X_r \sim \text{NegBin}(r, p)$

wait for first, wait for 2nd, ...

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Roll dice until you get 17 6's

What's the prob. it takes until the 107th roll?

$X \sim \text{NegBin}(17, \frac{1}{6})$

$P(X=107) = \binom{106}{16} \left(\frac{5}{6}\right)^{90} \left(\frac{1}{6}\right)^{17}$

$$
\begin{tikzcd}
 & & \text{Hyp} \arrow[dl, bend right, "{n=1 \text{ Special case}}"'] \arrow[d, "{\text{Lim } N\to\infty}"] \\
 & \text{Bern} \arrow[dl, "\text{Stop @ 1}"'] \arrow[d, "{\text{Stop @ } r}"] \arrow[r, bend right, "\text{Sum iid}"'] & \text{Binom} \arrow[l, bend right, "{n=1 \text{ Special case}}"'] \\
\text{Geom} \arrow[r, bend right, "\text{Sum iid}"'] & \text{NegBin} \arrow[l, bend right, "{\text{Special case } r=1}"'] &
\end{tikzcd}
$$

Philosophical Jumps

r.v. $\rightarrow$ $X \sim \text{Bern}(p)$

realization of r.v. (so unreal) $\rightarrow$ $X=0,\ X=1$

datum: realization of a r.v.

data: realizations of r.v.'s

iid data: " " " iid "

$P(X=x)$ ← prob of what we just saw

datum $\in \text{Supp}(X)$??

In class demos

$X_1, \ldots X_6 \overset{iid}{\sim} \text{Bern}(\tfrac{1}{2})$

$X_1, \ldots, X_6 \overset{iid}{\sim} \text{Hyper}(3, 4, 8)$

$X_1, \ldots, X_6 \overset{iid}{\sim} \text{Bin}(8, \tfrac{1}{2})$

$X_1, \ldots X_6 \overset{iid}{\sim} \text{Geom}(\tfrac{1}{2})$

$X_1, \ldots, X_6 \overset{iid}{\sim} \text{NegBin}(2, \tfrac{1}{2})$

$X_1, \ldots X_{60} \overset{iid}{\sim} \text{Rademacher}$

Record Dem

~~sol~~

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