11/11/16

✗ | $T \sim$ Geometric $(p) := (1-p)^{t-1} p$ $\quad\quad \mathcal{F}(t) = 1-(1-p)^t$

$$\underset{\underbrace{}_{seconds}}{\longleftarrow|||||||\quad|||||||||\quad||||||||\longrightarrow} t$$
seconds   1   2   3 $\quad p(t)$

$1 - f(t) = (1-p)^t$
$E(t) = \frac{1}{p}$
$\hookrightarrow$ # of experiments

$p(t) = (1-p)^{nt-1} p$, $f(t) = 1-(1-p)^{nt} \to \mathcal{F}(t) = 1$

$E[t] = \frac{1}{p}$ exp. 1 sec $= \frac{1}{np}$ sec $\quad\quad$ 1

$\quad\quad \frac{1}{nexpp}$

Imagine n large but p small.

let $\lambda = np \Rightarrow p := \frac{\lambda}{n}$ reparametrization

$p(t) = \left(1-\frac{\lambda}{n}\right)^{nt-1} \frac{\lambda}{n} \cdots$

$f(t) = 1-\left(1-\frac{\lambda}{n}\right)^{nt}$

let $n \to \infty$, but λ remains λ

$\lim_{n\to\infty} p(t) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt} \frac{\lambda}{n} = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nt} \lim_{n\to\infty} \frac{\lambda}{n} = 0 \,\forall t$

$\underbrace{}_{\Theta}$

$\sum_{t\in supp[t]} p(t) = 0 \quad \to p(t)$ is not valid

T is not a discrete r.v.

$\lim_{x\to} f(x)^a = (\lim f(x))^a$
$a \in \mathbb{R}$

$\lim_{n\to\infty} \mathcal{F}(t) = \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nt} = 1-\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nt}$

$\hookrightarrow e := 1-\left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n\right)^t$

$\underbrace{}_{e^{-\lambda}}$

$\boxed{= 1-e^{-\lambda t}}$

$e = \sum_{i=0}^{\infty} \frac{1}{i!}$

$\int \frac{1}{x}dx = 1$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$f(n)$

n	$f(n)$
10	2.514
100	2.705
1000	2.717
10000	2.78

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n, a\in\mathbb{R} \qquad \text{let } \frac{1}{m}=\frac{a}{n} \to n=ma$$

$$\text{if } n\to\infty \to m=\infty$$

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{ma} = \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^n\right)^a = e^a$$

CDF's

① $f(t)\in[0,1] \to$ probality $0 > x > 1$ $\qquad t\in(0,\infty)$

$$1-e^{-\lambda t} \geq 0$$
$$1 \geq e^{-\lambda t}$$
$$0 \geq -\lambda t$$
$$-\lambda t \leq 0 \checkmark$$

$$1-e^{-\lambda t} \leq 1$$
$$-e^{-\lambda t} \leq 0$$
$$\underbrace{}_{+}$$

λ is positive
t is positive
$-(+)(+) = -$

② $\lim_{t\to-\infty} F(t) = 0 \checkmark$

{ Taken geometric, modified it
PMF not valid, CDF still valid }

③ $\lim_{t\to-\infty} F(t) = 1 \checkmark$

$$\lim_{t\to\infty} 1-e^{-\lambda t} = 1-\lim_{t\to\infty} e^{-\lambda t}$$
$$1 - \lim \frac{1}{e^{\lambda t}} = 1-0 = 1$$

✳

$$\boxed{f(t) \frac{d}{dt}[F(t)]} = \frac{d}{dt}\left[1-e^{-\lambda t}\right] = \lambda e^{-\lambda t} \overset{?}{\geq} 0 \text{ Yes}$$
$$\underbrace{np}_{(+)} \quad (+)$$

need monotinicity
means slope
is positive
or ≥ 0.

T is random
Supp$[T] = (0,\infty)$
$|$supp$[t]| = \mathbb{R}$
↳ size $\mathbb{R}$ > size $\mathbb{N}$
$|$supp$[t]| = |\mathbb{R}| > |\mathbb{N}| \to$ for a discrete R.V
T is a continous r.v.

{ → so F(t) is a CDF.
T is a r.v but not
discrete because it has
no valid PMF. }

supp for Geometric
Supp$[x] = \{1\ldots\}$

If quantum gravity is real

$\vdash x \rule{2cm}{0.4pt} x \dashv$ plank lenght 1.6×10^{-35} m
↳cannot distinguish between here and here
→ light $3 E 8 m/s$, plank time: $5.3\times10^{-44} s$

$P(T=3) = p(3) = 0$

$p(T=3) = P(T=3.00000...) \to$ no interval contains infinite information

$P(T=3.000\overline{9}) = P(T \in [2.9995\overline{0}, 3.0004\overline{9}])$

stop

$\hookrightarrow F(3.0004\overline{9}) - F(2.9995\overline{0})$

$P(T \in [a,b]) = \int_a^b f(t)\,dt = F(b) - F(a)$

important

fundamental Theorem of Calculus.

$f(t) := \dfrac{d}{dt}(F(t)) = \dfrac{d}{dt}[1 - e^{-\lambda t}] = \lambda e^{-\lambda t} \geq 0$

Probability density function (PDF)

$f(t)$

$*$

exponential decay function

t

$F(t)$

density of stuff here is greater than density of stuff Here

$\lambda = 2$ $f(1) = 2e^{-2 \cdot 1} \approx 0.27 \neq P(1) = 0 \to$ does not measure prob. of 1

$f(0.1) = 2e^{2 \cdot 01} \approx 1.63 > 1$ It measure Prob. density of 1

$\hookrightarrow$ CDF moving very quickly
collecting prob. quickly
$\to$ not measuring prob.
measuring prob. density

PDF is an abstrac metric good for 2 things
① Integrity to get prob (region) via F.T.C
② copare two paints relative likelihood

$6 \approx \dfrac{f(0.1)}{f(1)} = \dfrac{P(T \in [0.1, 0.1 + \varepsilon])}{P(T \in [1, 1+\varepsilon])} \dfrac{\frac{1}{\varepsilon}}{\frac{1}{\varepsilon}} \to$ realization around .1
$\to$ realization around 1

consider $\lim\limits_{\varepsilon \to 0} \dfrac{P(T \in [0.1, 0.1 + \varepsilon])}{\varepsilon} \to F(0.1 + \varepsilon) - F(0.1)$

$\dfrac{P(T \in [1, 1 + \varepsilon])}{\varepsilon} \to F(1 + \varepsilon) - F(\varepsilon)$

$$1 = P(T \in (-\infty, \infty)) = \boxed{\int_{-\infty}^{\infty} F(t) \, dt = 1}$$

$\rightarrow$ PDF property

$\hookrightarrow$ If I sum up PDF have to get 1

$\rightarrow$ integrate over the support

$*\rightarrow$ the prob. that winds up in the supp, is the integral of supp.

$\rightarrow$ It has to realize something

$$a|a \rightarrow \sum_{X \in Supp[X]} p(x) = 1 \quad \text{for discrete r.v's } X.$$

Properties of continuous r.v X

① $|Supp[X]| \neq |\mathbb{R}|$ $\rightarrow$ continum.

② has valid CDF $F(x)$

with no jumps, but gaps ok

③ PMF does not exist

④ PDF exist $f(x)$ (a) $f(x) \geq 0$

⑥ $\int_{Supp[X]} f(x) \, dx = 1$

X_1, X_2 cont. r.v

and $X_1 \overset{d}{=} X_2$ if

$f_1(x) = f_2(x)$ PDFs the same or

$F_1(x) = F_2(x)$ CDFs the same

$$E[X] \approx \sum_{X \in Approx\, Supp[X]} X p(x)$$

$$\boxed{E[X] = \int_{Supp[X]} X f(x) \, dx}$$

$$E[g(x)] = \int_{\text{supp}[x]} g(x) f(x) \, dx$$

$$\text{Var}[x] = E[(x-M)^2]$$

$$\boxed{\sigma^2 = \int_{\text{supp}[x]} (x-M)^2 f(x) \, dx}$$

$$E[ax+c] = aM + c$$
$$\text{Var}[ax+c] = a^2 \sigma^2 \rightarrow SE[ax+c] = |a|\sigma$$
$$E[\Sigma x_i] = \Sigma E[x_i] = nM \quad \text{if iid}$$
$$\text{Var}[\Sigma x_i] = \Sigma \text{Var}[x_i] = n\sigma^2$$
↳ if indep. ↳ if iid

"Brand name r.v"

$$X \sim \text{Exp}(\lambda) = \lambda e^{-\lambda x} \qquad \in N$$
↳ exponential r.v $\qquad \lambda = np$
↳ (0,1)

$$\text{Supp}[X] = (0, \infty) \qquad \text{param space: } \lambda \in (0, \infty)$$
↳ rate

$$\int u \, dv = uv - \int v \, du$$
↳ integration by parts

$$E[X] = \int_0^\infty x \underbrace{\lambda e^{-\lambda x}}_{f(x)} \, dx$$

$$= \lambda \int_0^\infty x e^{-\lambda x} \, dx$$

let $u = \lambda \qquad$ let $dv = e^{-\lambda x}$

$du = dx \qquad v = -\frac{1}{\lambda} e^{-\lambda x}$

$$= \lambda \left[-\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} \right]_0^\infty$$

$$\left\{ \int v \, du = \int -\frac{1}{\lambda} e^{-\lambda x} \, dx = \frac{1}{\lambda^2} e^{-\lambda x} \right\}$$

$$- \left(\lim_{x \to \infty} \frac{x e^{-\lambda x}}{e^{\lambda x}} + \lim_{x \to \infty} \frac{1}{\lambda} e^{-\lambda x} \right) - \left(0 e^{-\lambda(0)} + \frac{1}{\lambda} e^{-\lambda(0)} \right)$$

$$- \left((0+0) - \left(0 + \frac{1}{\lambda} \right) \right) = \boxed{\frac{1}{\lambda}}$$

$X \sim \text{Geom}(p)$

$E[X] = \frac{1}{p} \rightarrow = \frac{1}{np} \Rightarrow \frac{1}{\lambda}$

Exponential has the memorylessness property

$P(X > a+b \mid X > b)$

$= P(X > a) = e^{-\lambda a}$

$\underbrace{P(X > a+b) \rightarrow e^{-\lambda(a+b)}}$

$\dfrac{\underbrace{P(X > a+b \ \& \ X > b)}}{\underbrace{P(X > b)}_{e^{-\lambda b}}} = \dfrac{e^{-\lambda a} e^{-\lambda b}}{e^{-\lambda b}} = e^{-\lambda a}$

$* P(X \leq x) = F(x) = 1 - e^{-\lambda x}$

$P(X > x) = 1 - F(x) = e^{-\lambda x}$

	Single stop	Mult. Stop
discrete	Geom	Neg Binom
contin.	Exponential	Erlang (gamma)

$X \sim \text{Uniform}(\{1, 7, 283\}) \rightarrow$ Discrete uniform

$X \sim (a, b) \rightarrow$ "uniform"

density $= \dfrac{1}{b-a}$ $\underbrace{\quad}_{f(x)}$

$supp[X] = [a, b]$

Param space
$a \in \mathbb{R}$
$b \in \mathbb{R}$ but $a < b$

CDF
constant levels
differentiate
to 0

$F(x) = \int f(x) \, dx + C$

$= \int \dfrac{1}{b-a} \, dx + C$

$= \dfrac{x}{b-a} + C$

$F(a) = 0 \Rightarrow \dfrac{a}{b-a} + C = 0$

$\Rightarrow C = \dfrac{-a}{b-a}$

$\boxed{\Rightarrow F(x) = \dfrac{x-a}{b-a}}$

$$\star \; E(X) = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)}$$

$$= \boxed{\frac{b+a}{2}} \; E[X]$$

$$\text{Var}[X] = \sigma^2 = E[X^2] - \mu^2$$

$$= \int_a^b x^2 \frac{1}{b-a} dx) - \left(\frac{a+b}{2}\right)^2 \to \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b = \frac{1}{3}\frac{b^3-a^3}{b-a}$$

$$\begin{array}{r} b^2 + ab + a^2 \\ b-a \overline{\big)\, b^3 - a^3} \\ -(b^3 - ab^2) \\ \hline ab^2 - a^3 \\ + ab^2 - a^2b \\ \hline a^2b - a^3 \\ -a^2b - a^3 \\ \hline 0 \end{array}$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{(4b^2 + 4ab + 4a^2) - (3a^2 + 6ab + 3b^2)}{12}$$

$$= \frac{a^2 - 2ab + b^2}{12} = \frac{(b-a)^2}{12}$$

$$\boxed{\sigma = \frac{b-a}{\sqrt{12}}}$$

$a = 0, \; b = 1$

$X \sim \text{unifor}(0,1) = 1$

Standard uniform