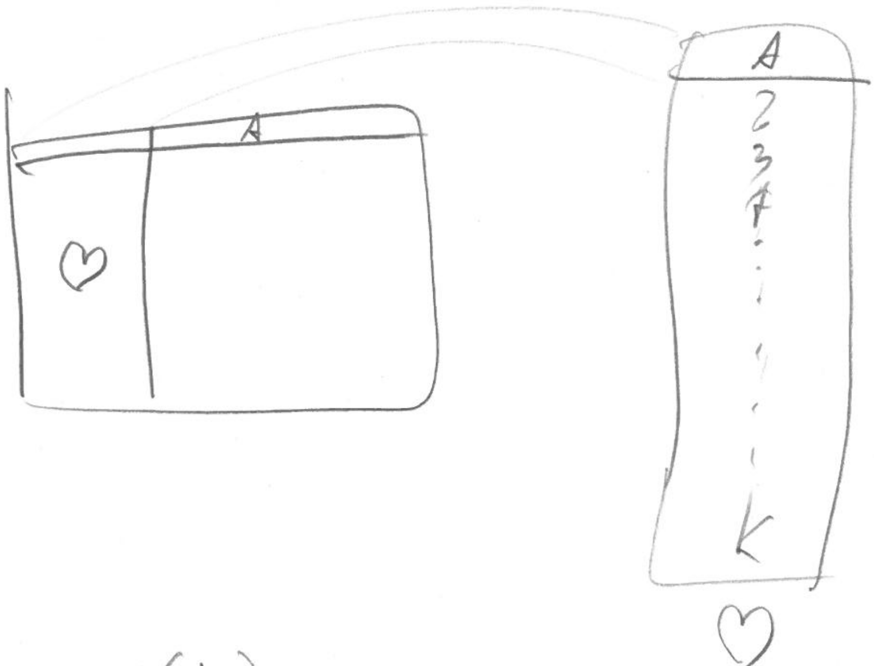
Lec 7 9/15/16 Math 241

Monty Hall Demo

"Conditional on"

"information you know"

$$P(A) = \frac{4}{52} = \frac{1}{13} \qquad P(A \mid \heartsuit) = \frac{1}{13}$$



$P(A) = P(A \mid \heartsuit)$ Since this didn't change... did the "information" of $\heartsuit$ matter in this prob calc?

$$P(\text{IBM stock} \uparrow \mid \text{rains in Buenos Aires}) = P(\text{IBM stock} \uparrow)$$

"not informationally relevant"

Def: A, B are independent events if

$P(A \mid B) = P(A)$

OR

$P(B \mid A) = P(B)$

$$\Rightarrow P(A \mid B) = P(A) \Rightarrow \frac{P(AB)}{P(B)} = P(A)$$

$$\Rightarrow P(AB) = P(A)P(B) \quad \text{(Mult. Rule)}$$

Let $A_1, A_2, \ldots$ be independent events

$$P(A_1 A_2 \ldots) = P\left(\bigcap_{i=1}^{\infty} A_i\right) = \prod_{i=1}^{\infty} P(A_i)$$

Flip two coins ..

$$P(H_1 \mid H_2) = P(H_1) = 0.5$$

$$P(H_1 H_2 H_3 H_4 H_5) = P(H)^5 = \left(\frac{1}{2}\right)^5 \quad \text{p enumerate} \quad \frac{1}{2^5} = |\Omega^5|$$

Chevalier de Mere

$$P(\geq 1 \text{ double } 6 \text{ in } 24 \text{ rolls}) = P(1\ 6\text{-}6 \text{ in } 24) + P(2\ 6\text{-}6 \text{ in } 24) + \ldots + P(24\ 6\text{-}6 \text{ in } 24)$$

HARD

EASY

$$= 1 - P(\text{Zero } 6\text{-}6 \text{ in } 24) \quad \text{compl rule}$$

$$= 1 - P(\underbrace{\text{Not } 6\text{-}6, \text{Not } 6\text{-}6, \text{Not } 6\text{-}6, \ldots \text{Not } 6\text{-}6}_{24 \text{ times}})$$

$$= 1 - P(\text{Not } 6\text{-}6)^{24} \quad \text{ind. mult. rule}$$

$$= 1 - \left(\frac{35}{36}\right)^{24} = .4914039$$

$$P(\text{Not } 6\text{-}6) = 1 - P(6,6)$$
$$= 1 - P(6)^2$$
$$= 1 - \left(\frac{1}{6}\right)^2$$
$$= \frac{35}{36}$$

3

If $P(B|A) \neq P(B)$ or $P(A|B) \neq P(A)$ or $P(AB) \neq P(A)P(B)$

$\Rightarrow$ A, B are not independent (dependent)

$P(\text{Q6A late})$ $P(\text{Q6A late} \mid \text{snowstorm})$

$P(\text{Q6A late})$ $P(\text{Q6A late} \mid \text{no traffic})$

$P(\text{lung cancer} \mid \text{smoke})$ $P(\text{lung cancer})$

A, B disjoint events w/ non-zero prob. $\Rightarrow$ independent? Test it

$P(A|B) \stackrel{?}{=} P(A)$

$\quad \| \qquad \neq$

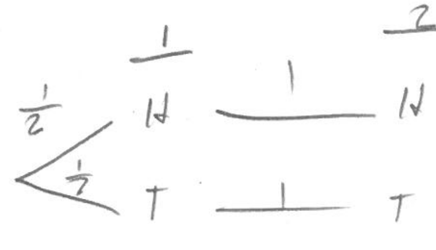
$\quad 0$

$0 = P(H|T) \neq P(H) = \frac{1}{2}$

Consider the Dependent Coin

Regular Coin

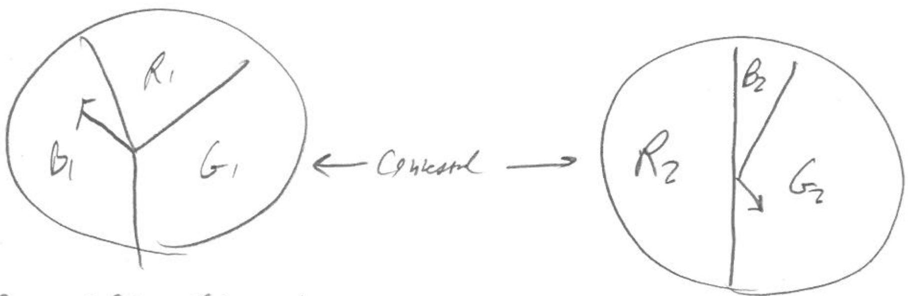
Magic Coin



$P(HH) = \frac{1}{2}$

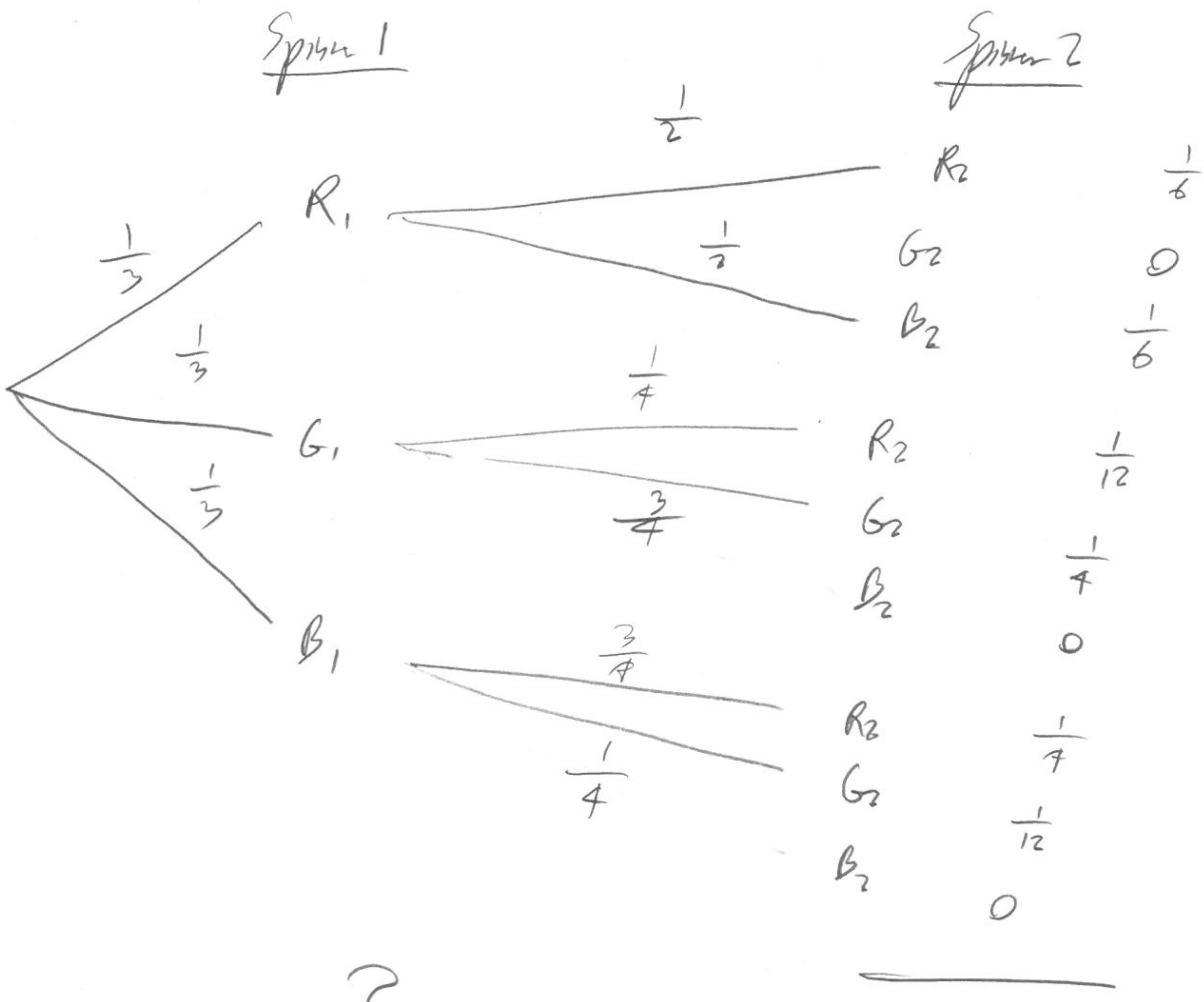
$P(TT) = \frac{1}{2}$

Sometimes "independent" doesn't mean "independent"



$P(R_1) = P(G_1) = P(B_1) = \frac{1}{3}$

$P(R_2) = \frac{1}{2}$, $P(G_2) = \frac{1}{3}$, $P(B_2) = \frac{1}{6}$



R_1, R_2 independent?

$P(R_1, R_2) \stackrel{?}{=} P(R_1)\,P(R_2) \Rightarrow \frac{1}{6} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ ✓

they are!!

What about R_1 & G_2

$$P(R_1 G_2) \stackrel{?}{=} P(R_1)\,P(G_2)$$

$$0 \stackrel{?}{=} \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

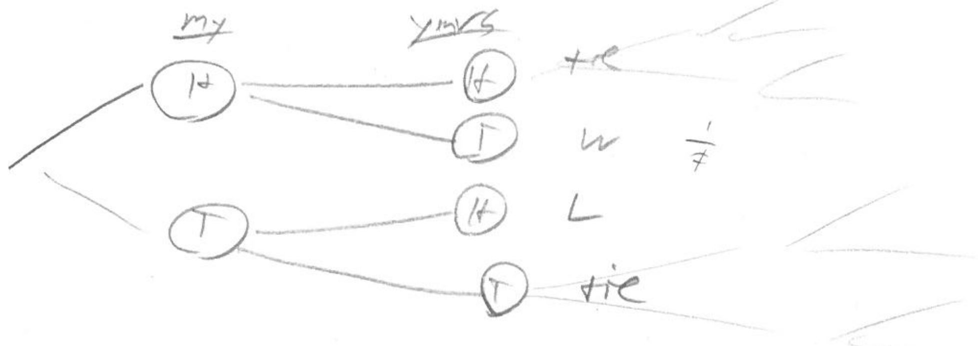
No... R_1, G_2 dependent!

It's just R_1 & R_2 ... informational irrelevance

Play game... H/T flip two coins H/T I win, T/H you win

HH or TT → tie.. play again $p := P(I \text{ win}) = ?$

SKIP



$$p = \frac{1}{4} + \frac{1}{4} p(\text{win}/\text{tie}) + \frac{1}{4} p(\text{win}/\text{tie})$$

indep.

$$p = \frac{1}{4} + \frac{1}{2} p \Rightarrow \boxed{p = \frac{1}{2}}$$

Question in office Hrs

1000 coin flips

$$P(600H, 400T) = \frac{\frac{1000!}{600!\,400!}}{2^{1000}} \quad // \quad \binom{1000}{600} = \binom{1000}{400}$$

who are you choosing

1 2 3 4 5 6 ... 1000

choosing places for the H (or T).

Birthday Problem

P(at least one pair of you share the same bday)

Review

$= P$(one pair same bday)
$+ P$(2 pairs same bday)
$+ P$(3 pairs same bday)
$+$
$\vdots$
$P\left(\binom{60}{2} \text{ pairs same bday}\right)$

lots to do!!

Is there an Easier way??

$= 1 - P(A^c) = 1 - P$(no one shares same bday)

Assume bdays equally likely
wrong but $\approx$ true

$$\frac{365 \cdot 364 \cdot 363 \cdots}{365^{60}} = \frac{{}_{365}P_{60}}{365^{60}} = ?\ 0.005877$$

all possible bdays

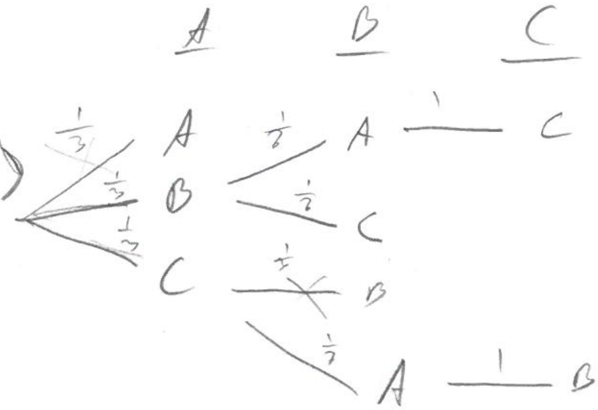
$\Rightarrow P(A) = 1 - \approx 99.4\%$

Hat Problem

n people walk into a room and put their hats on the table.
The hats are then randomly given out to everyone $p := P(\text{zero people get hat})$

$$1-p = P(\text{at least one person gets hat}) = \underbrace{P(1\,p\text{ gets hat}) + P(2\,p\text{ gets hat}) + \dots + P(n\text{ people get hat})}_{\text{Looks HARD}}$$

$n=3$



$p = 2\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot 1 = \frac{1}{3}$

But difficult... as n goes up!

let A_i: event in which i^{th} person gets their hat

$1-p = P(\geq 1 \text{ person gets their hat})$

... someone gets their hat! or rather.. gets hat

OR
$$= P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$P(A\cup B) = P(A)+P(B) - P(A\cap B) = \sum_{i=1}^{2} P(A_i) - P\left(\bigcap_{i=1}^{2} A_i\right)$$

$$= \sum P(A_i) - \sum_{i\neq j} P(A_i\cap A_j) + \sum_{i\neq j\neq k} P(A_i\cap A_j\cap A_k) - + - + \dots (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

general rule... proven w/ induction

$$P(A_1) = \frac{1\cdot (n-1)\cdot (n-2)\cdots 1}{n!} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$P(A_2) = \frac{(n-1)\cdot 1\cdot (n-2)\cdots}{n!} = \quad = \frac{1}{n}$$

$\vdots$

$$\Rightarrow \sum P(A_i) = 1$$

$$P(A_1 \cap A_2) = \frac{\underline{1}\ \underline{1}\ \underline{n-2}\ \underline{n-3} \cdots}{n!} = \frac{(n-2)!}{n!}$$

$$P(A_1 \cap A_3) = \frac{\underline{1}\ \underline{n-2}\ \underline{1}\ \underline{n-3}\ \underline{n-4} \cdots}{n!} = \frac{(n-2)!}{n!}$$

$\vdots$

How many?

$\binom{n}{2}$

$$\Rightarrow \sum_{i \neq j} P(A_i \cap A_j) = \binom{n}{2}\frac{(n-2)!}{n!} = \frac{\cancel{n!}}{2!\cancel{(n-2)!}}\frac{\cancel{(n-2)!}}{\cancel{n!}} = \frac{1}{2!}$$

$$P(A_1 \cap A_2 \cap A_3) = \frac{\underline{1}\ \underline{1}\ \underline{1}\ \underline{n-3}\ \underline{n-4} \cdots \underline{1}}{n!} = \frac{(n-3)!}{n!}$$

$\vdots$

How many?

$\binom{n}{3}$

$$\Rightarrow \sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k) = \binom{n}{3}\frac{(n-3)!}{n!} = \frac{n!}{(n-3)!\,3!}\frac{(n-3)!}{n!} = \frac{1}{3!}$$

- - - -

$$\Rightarrow p = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$

Recall

the center

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!}(x-c)^i \quad \forall c \in \mathbb{R} \quad \text{(Taylor Series)}$$

Usually if you want $f(x)$ where $x \approx 7$ $f(x) \approx f(7) + \frac{f'(7)}{1!}(x-7) + \frac{f''(7)}{2!}(x-7)^2$

Stop at 2 terms

If $c=0$

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i$$

$$e^x = e^0 + \frac{e^0}{1!}x + \frac{e^0}{2!}x^2 = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$\Rightarrow e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!}$$

$$\Rightarrow 1-e^{-1} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots -$$

$$\Rightarrow 1-p = 1-e^{-1} \Rightarrow p = e^{-1} \approx 0.368 \approx \frac{1}{3}$$

Balls & Urns if there's time...