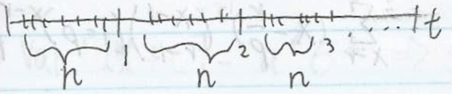
11/11

- $T \sim \text{Geometric}(p) := \underbrace{(1-p)^{t-1}p}_{p(t)}$

$F(t) = 1-(1-p)^t$

$1-F(t) = (1-p)^t$

$E(t) = \frac{1}{p}$ #exp.



$p(t) = (1-p)^{nt-1}p$, $F(t) = 1-(1-p)^{nt}$

$E[T] = \frac{1}{p}\text{ exp}\cdot\frac{1\text{ sec}}{n\text{ exp}} = \frac{1}{np}\text{ sec}$

- Imagine n large, but p small

Let $\lambda := np \Rightarrow p := \frac{\lambda}{n}$ (a reparametization.

$p(t) = (1-\frac{\lambda}{n})^{nt-1}\frac{\lambda}{n}$, $F(t) = 1-(1-\frac{\lambda}{n})^{nt}$

- Let $n \to \infty$ but λ remains λ

- $\lim_{n\to\infty} p(t) = \lim_{n\to\infty} (1-\frac{\lambda}{n})^{nt}\frac{\lambda}{n} = \lim_{n\to\infty}(1-\frac{\lambda}{n})^{nt}\cdot\underbrace{\lim_{n\to\infty}\frac{\lambda}{n}}_{0} = 0 \;\forall t$

$\Rightarrow p(t)$ is not valid

$\sum_{t\in Supp(T)} p(t) = 0 \Rightarrow T$ is not discrete r.v (valid the definition).

pops up everywhere.

pattern!

definition of e

$$= \lim_{n\to\infty}\underbrace{(1+\frac{1}{n})^n}_{f(n)} \Leftarrow$$

- $\lim_{n\to\infty} F(t) = \lim_{n\to\infty} 1-(1-\frac{\lambda}{n})^{nt} = 1-\lim_{n\to\infty}(1-\frac{\lambda}{n})^{nt} = 1-(\lim_{n\to\infty}(1-\frac{\lambda}{n})^n)^t$

- $\lim f(x)^k = (\lim f(x))^k$

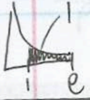
$k \in \mathbb{R}$

n	f(n)
10	2.594
100	2.705
1000	2.717
10000	2.718

$\not\to 1$

$e := \sum_{i=0}^{\infty}\frac{1}{i!}$

$\int_1^e \frac{1}{x}dx = 1$



- $\lim_{n\to\infty}(1+\frac{a}{n})^n$, $a \in \mathbb{R}$ $\curvearrowright$ $\lim_{n\to\infty}(1+\frac{1}{m})^{ma} = (\lim_{m\to\infty}(1+\frac{1}{m})^n)^a = e^a$

let $\frac{1}{m} = \frac{a}{n} \Rightarrow n = ma$ if $n \to \infty \Rightarrow m = \infty$

pattern matching

Thus, $\lim_{n\to\infty} F(t) = \lim_{n\to\infty} 1-(1-\frac{\lambda}{n})^{nt} = 1-\lim_{n\to\infty}(1-\frac{\lambda}{n})^{nt} = 1-(\underbrace{\lim_{n\to\infty}(1-\frac{\lambda}{n})^n}_{e^{-x}})^t$

$$= \boxed{1-e^{-\lambda t}}$$

- <u>CDF's</u>

are locked between 0 & 1 (if in 0 /1 = invalid)

① $F(t) \in [0, 1]$. $\qquad t \in (0, \infty)$ = positive #.

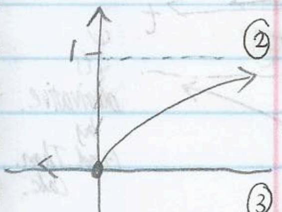
$1 - e^{-\lambda t} \geq 0 \qquad 1 - e^{-\lambda t} \leq 1$

$1 \geq e^{-\lambda t} \qquad -\underbrace{e^{-\lambda t}}_{(+)} \leq 0$

$0 \geq -\lambda t$

☆ Beginning the experiment as t=0.

$-\underbrace{\lambda}_{(+)}\underbrace{t}_{(+)} \leq 0$ ✓



② $\lim_{t\to-\infty} F(t) = 0$

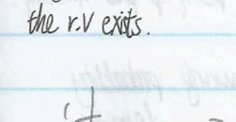
collecting $\overset{\text{supp.}}{\text{thing}} \to -\infty = 0$. $t \in [0,1]$

③ $\lim_{t\to\infty} F(t) = 1$

If you have CDF the r.V exits.

$\lim_{t\to\infty} 1 - e^{-\lambda t} = 1 - \lim e^{-\lambda t}$

$= 1 - \lim \frac{1}{e^{\lambda t}} = 1 - 0 = 1$ ✓.



Not valid. collected more support.

- $f(t) := \frac{d}{dt}[F(t)] = \frac{d}{dt}[1 - e^{-\lambda t}] = \underbrace{\lambda}_{\underbrace{n}_{+}}\underbrace{e^{-\lambda t}}_{+} \geq 0$? Yes. $\Rightarrow$ F(t) is a. C.D.F

$\Rightarrow$ T is a r.v. but not discrete because there's no valid P.M.F.

T is a r.v.

need monotonicity (slope ≥ 0). needs to go ↑ or stand constant.

$\underset{\text{everything that could pop out}}{\text{Supp}[T]} = (0, \infty) = [0, \infty) = [0, \infty\cancel{)}$ $\Rightarrow$ Turns out T is a continuous r.v.

$|\text{Supp}[T]| = \underset{\text{size of real \#s}}{|\mathbb{R}|} > |\mathbb{N}|$ ← for a discrete r.v

• Discrete V.S. continuous.

no spaces between # = discrete.

$P(T=3) = \underset{\downarrow}{p(3)} = 0$

P.M.F = 0 everywhere.

$= p(T = 3.\underbrace{00000000\ldots}_{\text{contains infinite info.}}) = 0$

never stops.

- we dont know if time = discrete/continuous

$P(T = 3.000) = P(T \in [2.9995\bar{0}, 3.0004\bar{9}])$

$\uparrow$ once you stop. everything rounded to 3.000.

$= F(3.0004\bar{9}) - F(2.9995\bar{0})$

same.

Plack length
1.62×10^{-35} m.

• The Fundemental Thm of Calc

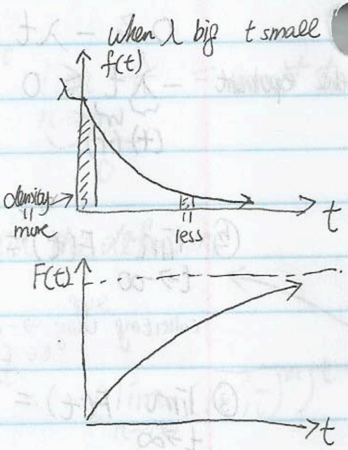
$$P(T \in [a, b]) = \int_a^b f(t)\,dt = F(b) - F(a)$$

$\uparrow$ important. $f(t) := \frac{d}{dt}[F(t)]$

$$f(t) := \frac{d}{dt}[F(t)] = \frac{d}{dt}[1 - e^{-\lambda t}] = \lambda e^{\lambda t} \geq 0$$

$\uparrow$ probability density function (P.D.F).

when λ big t small



$\downarrow$ its derivative. by Fund. Thm. Calc.

$\lambda = 2$

$f(1) = 2e^{-2 \cdot 1} \approx 0.27$

$f(0.1) = 2e^{2 \cdot 0.1} \approx 1.63 > 1$

greater than 1 is allowed
Density > 1 = C.D.F is moving quickly picking up prob quickly.

p of everything = 0 $\neq \ \hookrightarrow p(1) = 0 = \int_1^1 f(t)\,dt = 0$

(good for 2 things).

☆ P.D.F is an abstract metric ☆

① Integrating to get prob(region) via F.T.C

② Compare two points relative likelihoods.

It's measuring probability density NOT probability!

$6 = \frac{\text{relatively small regim}}{\text{small regim}}$

six

- $6 \approx \frac{f(0.1)}{f(1)}$ $\quad \frac{P(T \in [0.1, 0.1+\varepsilon])}{P(T \in [1, 1+\varepsilon])} \frac{\frac{1}{\varepsilon}}{\frac{1}{\varepsilon}}$ Consider $\lim_{\varepsilon \to 0} \frac{\overbrace{P(T \in [0.1, 0.1+\varepsilon])}^{F(0.1+\varepsilon)-F(0.1)} / \varepsilon}{\underbrace{P(T \in [1, 1+\varepsilon]}_{F(1+\varepsilon)-F(\varepsilon)} / \varepsilon} = \frac{f(0.1)}{f(1)}$

realization and 0.1

realization and 1

- $1 = P(T \in (-\infty, \infty])$

$= \int_{-\infty}^{\infty} f(t)\,dt = 1$

☆ P.D.F property

summing up the P.D.F = 1 got to be smth.

$1 = P[T \in Supp[T]]$

$= \int_{Supp[T]} f(t)\,dt = 1$ $\xrightarrow{\text{same as}}$ a/a $\sum_{x \in Supp[X]} p(x) = 1$ for discrete r.v's X.

Got to be realize somewhere

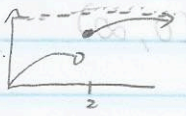
Properties of continuous r.v. X

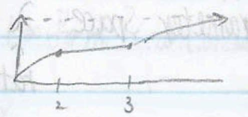
① $|Supp[X]| = |\mathbb{R}|$ continuum.

② Has valid ☆ C.D.F $F(X)$

(no jumps but gaps allowed)

can never pop up!





③ PMF does not exist. (b/c =0)

④ PDF exists $f(x)$ (a) $f(x) \geq 0$

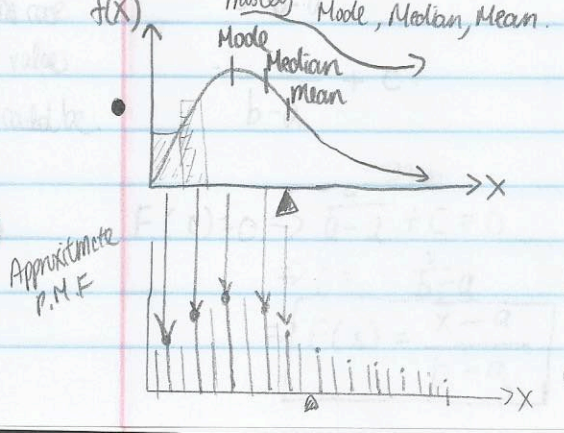
(b) $\int_{Supp[X]} f(x)\,dx = 1$

- X_1, X_2 continuous r.v.'s.

$X_1 \overset{d}{=} X_2$ if $f_1(X) = f_2(X)$ P.D.F's the same

OR.

$F_1(x) = F_2(x)$ C.D.F's the same.



No matter what you do = approximation.

area = probability

appr. until the little rectangles becomes a curve

$E[X] \approx \sum_{X \in \text{Appro } Supp[X]} x\,p(x)$

$\Downarrow$

$\boxed{E[X] = \int_{Supp[X]} x f(x)\,dx}$

- $E[X] = \int_{Supp[X]} x f(x)\,dx$
- $\sigma^2 = \int_{Supp[X]} (X-m)^2 f(x)\,dx.$

$E[g(X)] = \int_{Supp[X]} g(x) f(x)\,dx$

$Var[X] = E[(X-m)^2]$

- $E[aX+c] = am + c$
- $Var[aX+c] = a^2\sigma^2 \Rightarrow SE[aX+c] = |a|\sigma$
- $E[\sum X_i] = \sum E[X_i] \overset{\text{if identical dist}}{=} nm$
- $Var[\sum X_i] \underset{\text{if indep.}}{=} \sum Var[X_i] \underset{\text{if iid}}{=} n\sigma^2$

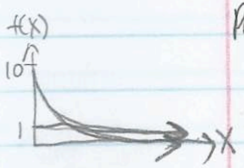
Brand Name r.v. "Exponential R.V"

$X \sim Exp[\lambda] := \lambda e^{-\lambda X}$ $\qquad \lambda = \overset{\mathbb{N}}{n}\underset{(0,1)}{p}$

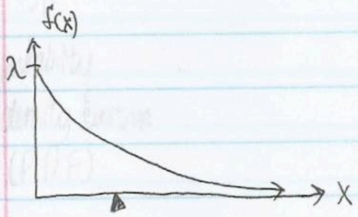
λ ‖ the higher the # the quicker it

$Supp[X] = (0, \infty)$

Parameter Space: $\underset{\text{rate}}{\lambda} \in (0, \infty)$



Properties:



$$E[X] = \int_0^\infty x \underbrace{\lambda e^{-\lambda x}}_{f(x)}\,dx$$

$$= \lambda \int_0^\infty x e^{-\lambda x}\,dx$$

$$= \lambda\left[-\frac{1}{\lambda}xe^{-\lambda x} - \frac{1}{\lambda^2}e^{-\lambda x}\right]_0^\infty =$$

Recall:

$\int u\,dv = uv - \int v\,du$

let $u = x$
$\Rightarrow du = dx$

let $dv = e^{-\lambda x}$
$\Rightarrow v = -\frac{1}{\lambda}e^{-\lambda x}$

$\int u\,du = \int -\frac{1}{\lambda}e^{-\lambda x}dx$
$= \frac{1}{\lambda^2}e^{-\lambda x}$

$$-\left(\left(\lim_{x\to\infty} \underbrace{xe^{-\lambda x}}_{\frac{x}{e^{\lambda x}}} + \lim_{x\to\infty}\frac{1}{\lambda}e^{-\lambda x}\right) - \left(0e^{-\lambda(0)} + \frac{1}{\lambda}e^{-\lambda(0)}\right)\right)$$

$$-((0+0)-(0+\frac{1}{\lambda})) = \boxed{\frac{1}{\lambda}}$$

- $X \sim Geom(p)$

$E[X] = \frac{1}{p}$
$= \frac{1}{np}$
$= \frac{1}{\lambda}$

$$\boxed{E[X] = \frac{1}{\lambda}}$$

continuous r.v.

- Expoential has the memoryesness property

$P(X \le x) = F(x) = 1 - e^{-\lambda x}$

$P(X > x) = 1 - F(x) = e^{-\lambda x}$

$$P(X>a+b \mid X>b) = P(X>a) = e^{-\lambda a}$$

$$\rightarrow = \frac{\overbrace{P(X>a+b \ \& \ X>b)}^{P(X>a+b)}}{\underbrace{P(X>b)}_{e^{-\lambda b}}} = \frac{e^{-\lambda a}\cancel{e^{-\lambda b}}}{\cancel{e^{-\lambda b}}} = e^{-\lambda a} \checkmark$$

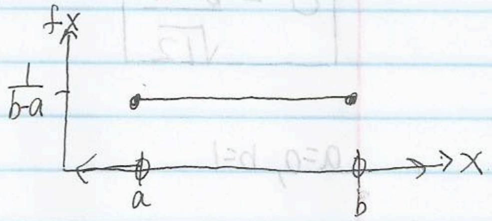
	Singe Stop	Multiple Stop
Discrete	Geom	Neg Bin
Cont	Expotential	Erlang (gamma) (not doing it).

- $X \sim Uniform(\{1, 7, 28\})$
=discrete uniform

$X \sim Uniform(a,b) = \underbrace{\frac{1}{b-a}}_{f(x)}$

= "Uniform" (continuous).

Uniform
= completely random
= everything has the same probability.
picks up any probability in the supp.



$Supp[X] = [a,b]$

Para Space = $a \in \mathbb{R}$, $b \in \mathbb{R}$ $-\infty$ ∞ but $a < b$

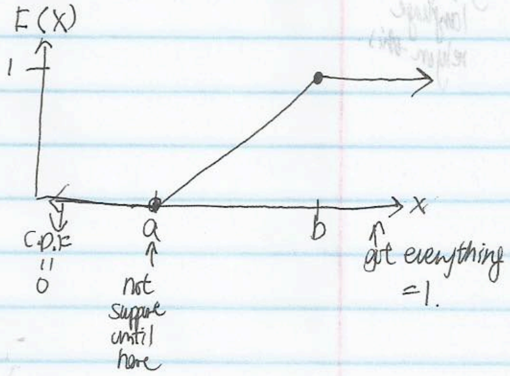
Classic antiderivate function

$F(x) = \int f(x)\,dx + C$

$= \int \frac{1}{b-a}\,dx + c$

$= \frac{x}{b-a} + c$

C.D.F in this case has strict stright values of what it could be.



$F(0) = 0 \Rightarrow \frac{a}{b-a} + C = 0$

$\Rightarrow C = -\frac{a}{b-a}$

$\Rightarrow \boxed{F(x) = \frac{x-a}{b-a}}$

$$-E[X] = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \boxed{\frac{b+a}{2}}$$

$$\boxed{E[X] = \frac{b+a}{2}}$$

- Variance $\sigma^2 = E[X^2] - m^2$

Proof $= \int_a^b x^2 \frac{1}{b-a}\,dx - \left(\frac{a+b}{2}\right)^2$

$$\boxed{\sigma^2 = \frac{(b-a)^2}{12}}$$

$$\begin{array}{r} b^2+ab+a^2 \\ b-a \overline{)\, b^3-a^3} \\ \underline{-\;b^3-ab^2} \\ ab^2-a^3 \\ \underline{-(ab^2-a^2b)} \\ a^2b-a^3 \\ \underline{-\;a^2b-a^3} \\ 0 \end{array}$$

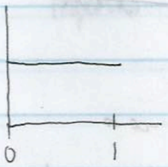
$$\frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b = \frac{1}{3}\,\frac{b^3-a^3}{b-a} = \frac{b^2+ab+a^2}{3}$$

$$= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{(4b^2+4ab+4a^2)-(3a^2+6ab+3b^2)}{12} = \frac{a^2-2ab+b^2}{12} = \boxed{\frac{(b-a)^2}{12}}$$

$$\boxed{\sigma = \frac{b-a}{\sqrt{12}}}$$

$a=0,\ b=1$



$X \sim \text{Uniform}(0,1) = \underline{\underline{1}}$

Standard Uniform

every program language relying on this.