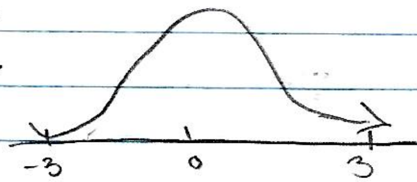
11/22/16

$Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x)$ ↳



$Supp[Z] = \mathbb{R}$

$E[Z] = 0$

$SE[Z] = 1$

$X \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$E[X] = \mu$

$SE[X] = \sigma$

$\mu \in \mathbb{R}$ $\quad \sigma^2 > 0$

$Z = \frac{X - \mu}{\sigma}$ → standardization

* ex X is male height, norm. distr w/ mean 70", standard error 4". Find prob. a male is is more than 78" tall.

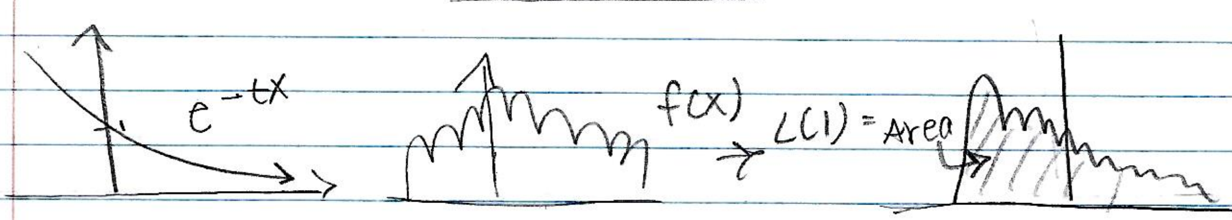
$X \sim N(70'', 4''^2)$

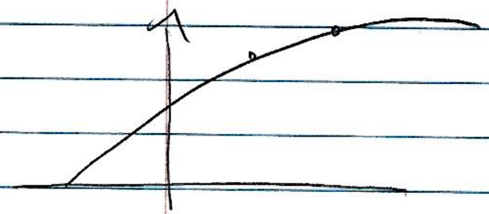
$P(X > 78) = P\left(\frac{X-70}{4} > \overbrace{\frac{78-70}{4}}^{8}\right) = P(Z > 2) = 2.5\%$

$P(Z \in [-2, 2]) = .95$

Let $L(t) := \int_{\mathbb{R}} e^{-tx} f(x)\,dx$ $\quad$ If $t=1$ $\quad L(1) = \int_{\mathbb{R}} e^{-x} f(x)\,dx$

"Bilateral LaPlace Transformation"





Thus if $L(t)$ exists, $L(t)$ & $f(x)$ are 1:1

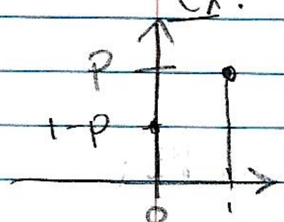
If $f(x)$ is a PDF

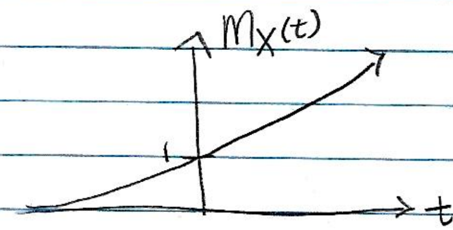
$M_X(t) := E[e^{tX}] := \int_{\mathbb{R}} e^{-tx} f(x)\,dx$ for cont.

moment generating function (MGF) for r.v X

$M_X(t) := E[e^{tx}] = \sum_{x \in Supp[X]} e^{tx} p(x)$ → for discrete.

ex: $X \sim Bern(p)$





$M_X(t) = E[e^{tx}] = e^{t(0)}p(0) + e^{t(1)}p(1) = 1-p+pe^t$

if (T) $M_X(t) = M_Y(t) \Rightarrow X \overset{d}{=} Y$

$X \sim Binom(n,p)$

$E[X^{17}] = \sum_{x=0}^{n} x^{17} \binom{n}{x} p^x (1-p)^{n-x}$

Recall $\forall c$, $f(x)$ cont.

$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}\cdot(x-c)^2 + \dots$

Taylor series

$x \approx c$

$f(x) \approx f(c) + f'(c) + \frac{f''(c)}{2}(x-c)^2$ → 2^{nd} order approx

let $f(x) = e^x$ $x \approx 0 \Rightarrow c = 0$

$e^x = e^0 + \frac{e^0 x}{1!} + \frac{e^0 x^2}{2!} + \dots$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \dots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \dots$$

$$M_X(t) = E[e^{tX}] = E\left[1 + tX + \frac{t^2X^2}{2!} + \dots\right]$$

Consider $\frac{d}{dt}\left[M_X(t)\right] = \frac{d}{dt}\left[E[\dots]\right] = E\left[\frac{d}{dt}[\dots]\right]$

$$\frac{d}{dt}\int f(t,x)\,dx$$

$$= E\left[X + \frac{tX^2}{1!} + \frac{t^2X^3}{2!} + \frac{t^3X^4}{3!} + \dots\right] = M_X'(t)$$

$$M_X'(0) = E[X] = \mu$$

$$M_X''(t) = E\left[X^2 + tX^3 + \frac{t^2X^4}{2!} + \dots\right]$$

$$M_X''(0) = E[X^2]$$

$$M_X'''(t) = E\left[X^3 + tX^4 + \dots\right]$$

$$M_X'''(0) = E[X^3]$$

II $M_X^{(k)}(0) = E[X^k]$

let $Y = aX + c$

$$M_Y(t) = M_{aX+c}(t) = E\left[e^{t(aX+c)}\right] = E\left[e^{taX}e^{tc}\right]$$

$$= e^{tc}E\left[e^{taX}\right]$$

Let $t' = at \rightarrow e^{tc} E[e^{t'X}] = e^{tc} M_X(t')$

III If $Y = aX + c$

$M_Y(t) = e^{tc} M_X(at)$

Consider X_1, X_2 independent r.v's

let $Y = X_1 + X_2$

$M_Y(t) = E[e^{tY}] = E[e^{tX_1} e^{tX_2}] = E[e^{tX_1}] E[e^{tX_2}]$

$= M_{X_1}(t) M_{X_2}(t)$

If X_1, X_2 iid

$M_Y(t) = M_{X_1}(t) M_{X_2}(t) = (M_X(t))^2$

IV If X_1, X_2 indp, $Y = X_1 + X_2 \rightarrow M_Y(t) = M_{X_1}(t) M_{X_2}(t)$

~~ex~~: $X \sim Bern(p) \rightarrow M_X(t) = 1 - p + pe^t$

Recall: $X_1, X_2 \ldots X_n$ iid $Bern(p)$ {By ind IV}

$T = X_1 + X_2 \ldots + X_n \sim Binom(n, p)$

$M_T(t) = E[e^{tT}] = E[e^{t(X_1 + \ldots X_n)}] = M_{X_1}(t) \ldots M_{X_n}(t)$

$= (M_X(t))^n := (1 - p + pe^t)^n$

By iden. distr.

$X \sim Geom(p)$

$M_X(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} (1-p)^{x-1} p \cdot \frac{(1-p)}{(1-p)} = \frac{p}{1-p} \sum_{x=1}^{\infty} \underbrace{e^{tx}}_{(e^t)^x} (1-p)^x$

$= \frac{p}{1-p} \left(\sum_{x=0}^{\infty} (e^t(1-p))^x - 1 \right)$ if $e^t(1-p) < 1$

$\Rightarrow e^t < \frac{1}{1-p}$

$\Rightarrow t < \ln\left(\frac{1}{1-p}\right)$

$= \frac{p}{1-p} \left(\frac{1}{1 - e^t(1-p)} - 1 \right) = \frac{p}{\cancel{1-p}} \cdot \frac{e^t \cancel{(1-p)}}{1 - e^t(1-p)} = M_X(t) = \frac{pe^t}{1 - e^t(1-p)}$ $\forall t < \ln\left(\frac{1}{1-p}\right)$

$X \sim Exp(\lambda) \Rightarrow M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$

$= \lambda \int_0^\infty e^{(t-\lambda)x} dx = \frac{\lambda}{t-\lambda} \left[e^{(t-\lambda)x} \right]_0^\infty$

$\frac{\lambda}{t-\lambda} \left(\lim_{x \to \infty} e^{(t-\lambda)x} - 1 \right) \underset{g}{=} \frac{\lambda}{t-\lambda}(0-1) = \boxed{M_X(t) = \frac{\lambda}{\lambda - t} \quad \text{if } t < \lambda}$

If $t - \lambda < 0 \Rightarrow t < \lambda$

$X \sim Exp(\lambda)$

$Y = aX,\ a \in \mathbb{R}$

let $\lambda' = \frac{\lambda}{a}$

$M_Y(t) = e^{tc} M_X(at) = M_X(at) = \frac{\lambda}{\lambda - at} \cdot \frac{\frac{1}{a}}{\frac{1}{a}} = \frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t} = \frac{\lambda'}{\lambda' - t}$

$= \frac{\lambda'}{\lambda' - t} \Rightarrow Y \sim Exp(\lambda') = Exp\left(\frac{\lambda}{a}\right)$

$Z \sim N(0,1) \Rightarrow M_Z(t) = E[e^{tZ}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}x^2 + tx}$

$\mu = M_Z'(0) = Z t e^{\frac{t^2}{2}} \Big|_{t=0} = 0$

$\sigma^2 = E[Z^2] - \cancel{\mu^2}^{0} = E[Z^2] = M_Z''(0) = e^{\frac{t^2}{2}} + t \cdot t e^{\frac{t^2}{2}} \Big|_{t=0} = 1$

$\boxed{\sigma = 1}$ <=