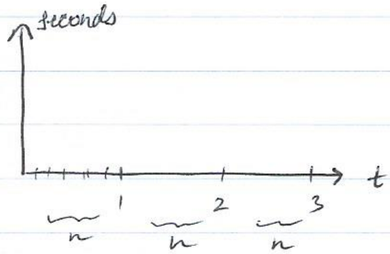
Lecture #17

Math 241 — A Kapelbaur — Nov 11, 2016

Let $T \sim \text{geometric}(p) = \underbrace{[(1-p)^{t-1}]\,p}_{p(t)}$ $\quad F(t) = 1-(1-p)^t \quad F(t) = \frac{1}{p} \cdot \text{exp} \cdot \text{sec}/\text{exp}.$

$1 - F(t) = (1-p)^t$

$\underbrace{}_{\text{[Kap. notes]}}$



$\Rightarrow \quad E(T) = \frac{1}{p}\,\text{exp} \cdot \frac{\text{sec}}{n \cdot \text{exp}} = \frac{1}{np}\,\text{sec.}$

Assume $n \uparrow$ but $p \downarrow$

Let $\lambda = np \Rightarrow p := \frac{\lambda}{n}$ // reparametrization

$p(t) = \left[\left(1-\frac{\lambda}{n}\right)^{nt-1}\right]\frac{\lambda}{n} \qquad F(t) = 1 - \left(1-\frac{\lambda}{n}\right)^{nt}$

Let $n \to \infty$ but λ is still λ

$\lim_{n\to\infty} p(t) = \lim_{n\to\infty}\left[\left(1-\frac{\lambda}{n}\right)^{nt}\right]\frac{\lambda}{n} = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nt} \cdot \lim_{n\to\infty}\frac{\lambda}{n} = 0$ forall t.

$\sum_{t \in supp[t]} p(t) = 0 \quad \rightarrow p(t)$ is not valid ; T is not a discrete r.v.

$\lim_{n\to\infty} F(t) = \lim_{n\to\infty} 1 - \left(1-\frac{\lambda}{n}\right)^{nt} = 1 - \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nt} = \underline{1-e^{-\lambda t}}$

Note: 1) $\lim f(x)^a = \lim f(x)^a$, $a \in \mathbb{R}$

2) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

CDF's: $\longrightarrow$ Cumulative Distr. Func. "collector"

① $F(t) \in [0,1]$

$1-e^{-\lambda t} \geq 0$

$1 \geq e^{-\lambda t}$

$0 \geq -\lambda t$

$-\lambda t \leq 0$ ✓ (pos. pos.)

$1-e^{-\lambda t} \leq 1$

$-e^{-\lambda t} \leq 0$ (pos.)

② $\lim_{t\to-\infty} F(t) = 0$ ✓

③ $\lim_{t\to-\infty} F(t) = 1$

$\lim_{t\to\infty} 1-e^{-\lambda t} = 1 - \lim e^{-\lambda t}$

$= 1 - \lim \frac{1}{e^{\lambda t}}$

$= 1 - 0 = 1$

probability density function (pdf)

$\tilde{f(x)} = \frac{d}{dt}[F(t)] = \frac{d}{dt}\left[1-e^{-\lambda t}\right] = \lambda e^{-\lambda t}$ is ≥ 0.

$\Rightarrow f(t) \rightarrow$ CDF

$\Rightarrow$ T is a r.v. but not discrete because $\Rightarrow$ No Valid PMF.

Note: For a discrete r.v. T, the support of $T = |\mathbb{R}| > |\mathbb{N}|$

PDF is an abstract [redacted] "thing" good for:

① Integrating to get probability. [Using Fund. Theorem of Calculus]

② Compare the relative likelihood of two points.

$$\sigma \approx \frac{f(0.1)}{f(1)} \quad \frac{P(T[0.1, 0.1+\varepsilon]) \cdot \frac{1}{\varepsilon}}{P(T\in[1, 1+\varepsilon] \cdot \frac{1}{\varepsilon}} \lim_{\varepsilon\to\infty} \frac{P(T[0.1, 0.1+\varepsilon])}{\varepsilon} \Rightarrow \frac{\overset{(\varepsilon)}{P(T[1, 1+\varepsilon)}}{\varepsilon} = \cancel{\quad}$$

$\longrightarrow F(1+\varepsilon) - F(\varepsilon)$

$P(T\in(-\infty,\infty)) = 1 = \int_{-\infty}^{\infty} f(t)dt = 1$ } PDF Property.

↳ Integrate over the Support.

ala Sum of $p(x) \Rightarrow 1$ for drv x.

<u>Properties of Cont. R.V. X,</u>

① $|Supp[X]| = |\mathbb{R}|$

② Has valid CDF $F(x)$: no jums, though gaps are allowed.

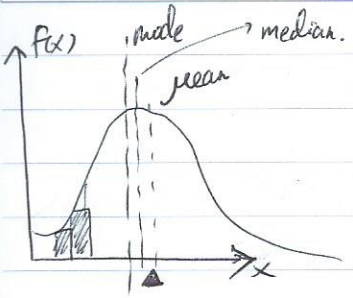
③ PMF $\Leftrightarrow$ DNE

④ PDF exists $f(x)$ $\left[\begin{array}{l}(a)\ f(x) \geq 0 \\ (b) \int_{Supp[X]} f(x)dx = 1\end{array}\right]$

(Let) x_1, x_2 be cont. rv' | $x_1 \overset{d}{=} x_2$ if

$f_1(x) = f_2(x)$ PDF's are equal or

$F_1(x) = F_2(x)$ CDF's are equal.



$$E[X] \approx \sum_{x \in Approx\,Supp[X]} x\,p(x)$$

$$\boxed{E[X] = \int_{Supp[X]} x f(x)\,dx}$$

$$E[g(x)] = \int_{supp[X]} g(x) f(x)\,dx$$

$$Var[X] = E[(x-\mu)^2] \Rightarrow \sigma^2 = \int_{Supp[X]} (x-\mu)^2 f(x)\,dx$$

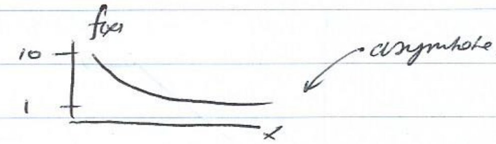
Note: $E[aX+b] = a\bar{X}+b = a\mu+b$

$Var[aX+b] = a^2\sigma^2 \Rightarrow SE[aX+b] = |a|\sigma$

$E[\sum x_i] = \sum E[X_i] = n\mu \Rightarrow$ if iid.

$Var[\sum x_i] = \sum Var[X_i] = n\sigma^2$

$\hookrightarrow$ if indep.



$X \sim Exp(\lambda) = \lambda e^{-\lambda x}$ $\qquad \lambda = np$ where $p \in [0,1]$; $n \in \mathbb{N}$;

$\hookrightarrow$ Exponential

$Supp[X] = (0, \infty)$

Parameter Space : $\lambda \in (0, \infty)$

$$E[X] = \int_0^\infty x \underbrace{\lambda e^{-\lambda x}}_{f(x)}\,dx$$

$$\Rightarrow \lambda \int_0^\infty x e^{-\lambda x}\,dx = \lambda \int_0^\infty x e^{-\lambda x}\,dx$$

$\to$ Long integration using Integration by Parts

This the result is

$$\boxed{= \frac{1}{\lambda}}$$

Thus, $X \sim$ geometric (p) $\quad E[X] = \frac{1}{p} = \frac{1}{np} = \frac{1}{\lambda}$

Note: Exponential has memoryless-ness property ;

$\Rightarrow e^{-\lambda a}$ [proof is skipped]

num.

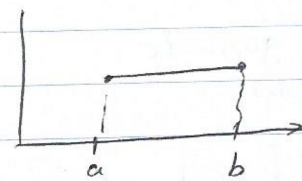
$$p(x > a+b \mid x > b)$$

$$\to \frac{p(x > a+b \;\&\; x > b)}{\underbrace{p(x > b)}_{e^{-\lambda b}}} = \frac{e^{-\lambda a}\cancel{e^{-\lambda b}}}{\cancel{e^{-\lambda b}}} = e^{-\lambda a}$$

	Single stop	Mult. Stop
Disc.	Geom	Neg. Bin.
Cont.	Exp	Erlag [gama] (?)

$X \sim$ Uniform $(\{1, 7, 283\})$

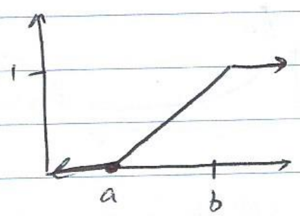
$X \sim (a,b) \rightarrow$ uniform $\overset{\text{density}}{\Rightarrow}$ $\frac{1}{b-a}$ $\sim f(x)$



P.S. $a \in \mathbb{R}$, $b \in \mathbb{R}$ but $a<b$.

CDF $F(x) = \int f(x)dx + C$

$$= \int \left(\frac{1}{b-a}\right) dx + C = \frac{x}{b-a} + C$$



$$F(a) = 0 \Rightarrow \frac{a}{b-a} + C = 0$$

$$\Rightarrow C = \frac{-a}{b-a}$$

$$\Rightarrow F(x) = \frac{x-a}{b-a}$$

$$E[x] = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b =$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{(\cancel{b-a})(b+a)}{2(\cancel{b-a})} = \frac{b+a}{2}$$

$$Var[x] = \sigma^2 = E[x^2] - \mu^2 = \int_a^b x^2 \frac{1}{b-a} d(x) - \left(\frac{a+b}{2}\right)^2 \Rightarrow \frac{1}{b+a}\left(\frac{x^3}{3}\right)_a^b = \frac{1}{3}\frac{b^3-a^3}{-a+b}$$

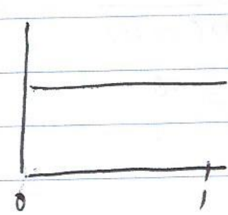
$$= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4} \Longrightarrow \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12}$$

$$\sigma = \frac{b-a}{\sqrt{12}}.$$

$a=0$, $b=1$

$X \sim$ Uniform$(0,1) = 1$

Standard Uniform.



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