Average r.v.

Let $T_n = X_1 + X_2 + \dots + X_n$
total r.v. aka **sum r.v.**

upper case X

$\bar{X}_n = \frac{T_n}{n} = \frac{X_1 + \dots + X_n}{n} = \frac{1}{n}\sum_{i=0}^{n} X_i$
average r.v.

n – sample size as in total number of conducted trials

e.g. $X_1\ X_2\ X_3 \sim \text{Bernoulli}(0.1) = T_3 \sim \text{Binomial}(3, 0.1)$

$$T_3 \sim \begin{cases} 0 & \text{w.p. } \binom{3}{0}(.1)^0(1-p)^3 = 0.729 \\ 1 & \text{w.p. } \binom{3}{1}(.1)^1(1-p)^{3-1} = 0.243 \\ 2 & \text{w.p. } \binom{3}{2}(.1)^2(1-p)^{3-2} = 0.027 \\ 3 & \text{w.p. } \binom{3}{3}(.1)^3(1-p)^{3-3} = 0.001 \end{cases}$$

e.g. HTH 2 Heads

Σ=1

zero 1s, one 1s, two 1s, three 1s

$\text{Supp}[T_3] = \{0,1,2,3\}$

note: the distribution of an average r.v. $-\bar{X}$ is different that of a total r.v. $-T_n$

$$\bar{X}_3 \sim \begin{cases} 0 & \text{w.p. } \binom{3}{0}(.1)^0(1-p)^3 = 0.729 \\ 1/3 & \text{w.p. } \binom{3}{1}(.1)^1(1-p)^{3-1} = 0.243 \\ 2/3 & \text{w.p. } \binom{3}{2}(.1)^2(1-p)^{3-2} = 0.027 \\ 1 & \text{w.p. } \binom{3}{3}(.1)^3(1-p)^{3-3} = 0.001 \end{cases}$$

e.g. HTH 2 out of 3 $\frac{1+0+1}{3}$

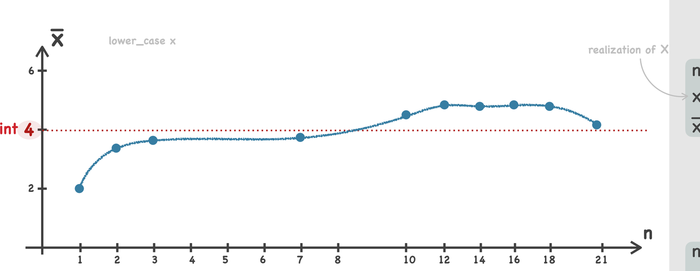
Σ=1

Def.

sample average $\bar{x}$ is a realization of $\bar{X}$

$\bar{x} := \frac{1}{n}\sum_{i=0}^{n} x_i$

lower case x



x , X , probability statement , PMF

prob. that you make a realization of this X and get x=5'11"

P(X=5'11") = ?

r.v. for human adult height

We have to specify a model for X.
If no model, this is meaningless.

p(5'11") = P(X=5'11") = PMF

you need a function that links an element of support to the probabilities that come out of it, without such function you can not answer this question

$X \sim \text{Binomial}(n,p) := \binom{n}{x} p^x (1-p)^{n-x}$

sample size; prob. of success; 1-p = prob of failure; n-x = # of failures; x = # of successes; p = prob. of success

1 cup: 8 coins , n=sample_size=8
shake the cup, quarry 8 coins, how many HEADS ?

$X \sim \text{Binomial}(8, 1/2) := \binom{8}{x}(1/2)^x(1-1/2)^{8-x} = \binom{8}{x}(1/2)^8 = \frac{\binom{8}{x}}{2^8} = \frac{8!}{2^8}\ \frac{1}{x!(8-x)!}$

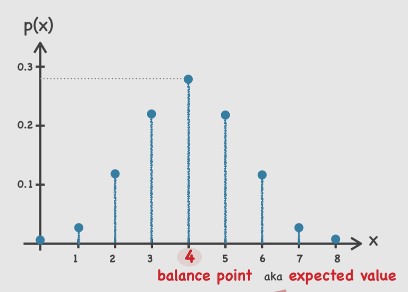
$(1/2)^8$, $\frac{8!}{x!(8-x)!}$

x	p(x)	F(x)
0	.004	.004
1	.031	.035 = .004 + .031
2	.109	.144
3	.219	.363
4	.273	.636
5	.219	.855
6	.109	.964
7	.031	.995
8	.004	1
	Σ=1	

$\binom{8}{x}(1/2)^8$

why equal ? $\binom{8}{3} = \binom{8}{5}$ 56 = 56

the sum over the support of the PMF is 1, otherwise it ain't no PMF :-)



∞ cups: 8 coins , n=sample_size=8
shake the cup , quarry 8 coins, how many HEADS ? – repeat to ∞

$X_1, X_2 \dots \sim \text{Binomial}(8, 1/2)$

realization build an empirical PMF and an empirical CDF. Show that they are about the same.

realization of X

n :	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x :	2	5	4	4	3	5	4	5	6	5	7	4	5	3
$\bar{x}$:	2	3.5	3.67				3.86			4.3		4.5		4.429

$\frac{x_1+x_2+x_3}{n} = \frac{2+5+4}{3}$

n :	15	16	17	18	19	20	21	...
x :	5	5	1	4	3	3	4	Supp[X]={0,...,8}
$\bar{x}$:		4.5		4.3			4.14	≈ 4 balance point

$\bar{X}$ sample average lower_case x

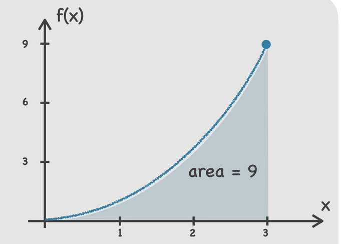
Mean

calculus

$f(x) = x^2 \quad x \in [0,3]$

function that takes a function $G[f] = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big]_0^3\ 9 - 0 = 9$

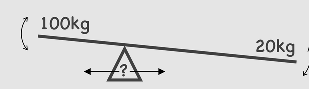
an integral is a function of a function



G: function → $\mathcal{R}$ (a single # e.g. 9)

it appears $\bar{x}$ → balance point or pivot point

physics w = weight
d = distance



where is the balance point ?

$\sum_i w_i(d_i - d^*) = 0$

$\sum_i w_i d_i - \sum_i w_i d^* = 0$

$\sum_i w_i d_i = d^* \sum_i w_i$

analog

$\frac{\sum_i w_i d_i}{\sum_i w_i} = d^*$ weights = probabilities PMF; distance = x $\frac{\sum_i p_i x_i}{\sum_i p_i} = x^*$ p(x), =1

$\frac{0\cdot100 + 1\cdot20}{100+20} = 0.17$

E[X] is another notation for M – function of a function
interchangeable

$E[X] := \mu := \sum_{x\in\text{Supp}[X]} x\ p(x)$

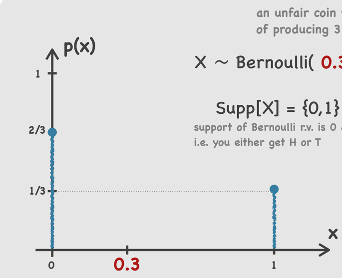
E for expression; mu

E: X → $\mathcal{R}$
space of r.v.'s; a single #

mean , expected value , expectancy , first moment

Law of Large #'s $\bar{X} \to \mu$ — you take the average of a lot of things and you get the expected value

Mean – Bernoulli



an unfair coin with expectancy of producing 3 Heads in 10 flips

X ~ Bernoulli(0.3)

Supp[X] = {0,1}
support of Bernoulli r.v. is 0 and 1
i.e. you either get H or T

0.3 ∈ Supp[X] ? – NO, 0.3 is neither 0 nor 1
in general $\mu \notin \text{Supp}[X]$

X (r.v. function) taken through E (mean function) returns p (prob.)

$E[X] = 0\cdot p(0) + 1\cdot p(1) = p(1) = 0.3$
or p in general Bernoulli

in the case of a single Bernoulli experiment E[X] aka μ = p
X (r.v. function) taken through E (mean function) returned p (prob.)

$$E[X \sim \text{Bernoulli}(p) := p^x(1-p)^{1-x}] = \mu = p$$

how about a 'mean' of a multiple Bernoulli experiments ?

$X_1, X_2, \dots, X_8 \sim \text{Bernoulli}(0.3) = X \sim \text{Binomial}(8, 0.3)$

E[X] ≈ 2.4 8 · 0.3 = 2.4

$E[X] = \sum_{x=0}^{8} x\,p(x) = 0\cdot p(0) + 1\cdot p(1) + 2\cdot p(2) + 3\cdot p(3) + 4\cdot p(4) + 5\cdot p(5) + 6\cdot p(6) + 7\cdot p(7) + 8\cdot p(8)$

x∈Supp[X]={0,...,8}

$\binom{8}{5}(0.3)^5(0.7)^3$ 56 · .00243 · .343 $\binom{8}{8}(0.3)^8(0.7)^0$ 1· 0.000066 ·1

= 0·0 + 1·.198 + 2·.2965 + 3·.2541 + 4·.1361 + 5·.04668 + 6·.01 + 7·.0012 + 8·.000066
= 0 + .198 + .593 + .7624 + .5445 + .2333 + .06 + .00857 + .0005
≈ 2.4

how do we get the 'balance point' ? Imagine Binomial(1000,0.3). Do we really need to perform 1000 summations ?
Or is there a general formula which will allow us to skip this laborious process ?

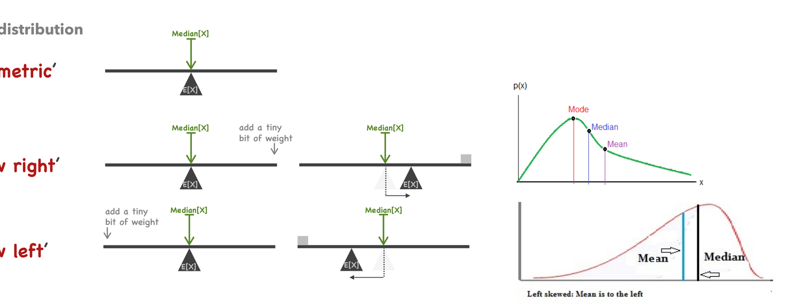
Warning!
Median and the Mean are NOT the same.
Median[X] ≠ E[X]

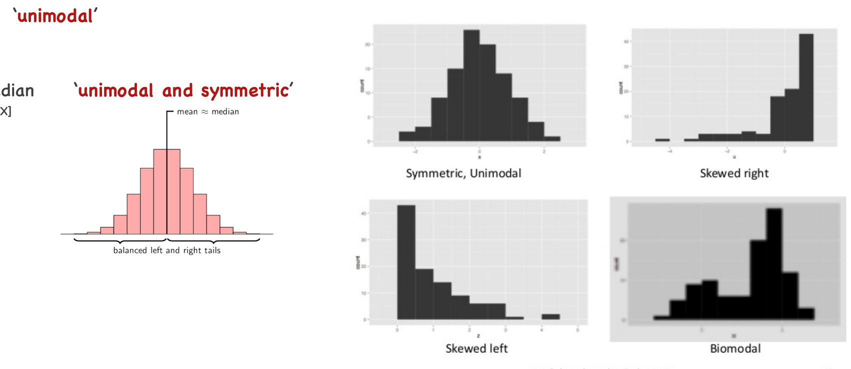
Median and the Mean are both measures of 'central tendency'

Median – says that there is 50% of the support below 4 and 50% above 4

Mean – says that if I balance a thing, where is the 'balancing point'. In our example it's 5

condition	types of distribution
Mean = Median (E[X] = Median[X])	'symmetric'
Mean > Median (E[X] > Median[X])	'skew right'
Mean < Median (E[X] > Median[X])	'skew left'
| Mode[X] | = 1 (one mode)	'unimodal'
Mode[X] = Mean = Median (Mode[X] = E[X] = Median[X])	'unimodal and symmetric'





Custom r.v.s

Ever wondered how to build custom r.v.s?

Bet on Black. Payout is 1:1 (bet \$1 win \$1) — Roulette in U.S. total of 38 pockets: 18 black, 18 red, 2 green 0 , 00

$X \sim \begin{cases} \$1 & \text{w.p. } 18/38 \\ -\$1 & \text{w.p. } 20/38 \end{cases}$ r.v. model of the payout of this bet

what is the expectation ?

$E[X] = \sum_{x\in\text{Supp}[X]} x\,p(x) = \$1\cdot p(\$1) + (-\$1)\cdot p(-\$1) = \$1\cdot 18/38 + (-\$1)\cdot 20/38 = -\$0.053$

meaning if you play many times on average you are loosing ≈5.3 cents (per play)

the more you play the more you loose in a long run you'll loose everything – you can't win

$X_1, X_2, \dots, X_n \overset{iid}{\sim} \begin{cases} \$1 & \text{w.p. } 18/38 \\ -\$1 & \text{w.p. } 20/38 \end{cases}$

$\lim_{n\to\infty} T_n = -\infty$

Bet on 'lucky' # 7. Payout is 35:1 (bet \$1 win \$35) — Roulette in U.S. total of 38 pockets: 18 black, 18 red, 2 green 0 , 00

$X \sim \begin{cases} \$35 & \text{w.p. } 1/38 \\ -\$1 & \text{w.p. } 37/38 \end{cases}$ r.v. model of the payout of this bet

what is the expectation ?

$E[X] = \sum_{x\in\text{Supp}[X]} x\,p(x) = \$35\cdot p(\$35) + (-\$1)\cdot p(-\$1) = \$35\cdot 1/38 + (-\$1)\cdot 37/38 = -\$0.053$

Expectation is a 'long run' property and it means very little when you have only few random variables.

hint: – play video draw poker duces wild
– black jack you loose 0.5% on average

Bet on 'dozen' # 1...12 Payout is 2:1 (bet \$1 win \$2) — Roulette in U.S. total of 38 pockets: 18 black, 18 red, 2 green 0 , 00

$X \sim \begin{cases} \$2 & \text{w.p. } 12/38 \\ -\$1 & \text{w.p. } 26/38 \end{cases}$ r.v. model of the payout of this bet

what is the expectation ?

$E[X] = \sum_{x\in\text{Supp}[X]} x\,p(x) = \$2\cdot p(\$2) + (-\$1)\cdot p(-\$1) = \$2\cdot 12/38 + (-\$1)\cdot 26/38 = -\$0.053$

Bet on 'dozen' # 1...12 Payout is 2:1 (bet \$1 win \$2) — Roulette in Europe total of 37 pockets: 18 black, 18 red, 1 green 0

$X \sim \begin{cases} \$2 & \text{w.p. } 12/37 \\ -\$1 & \text{w.p. } 25/37 \end{cases}$ r.v. model of the payout of this bet

what is the expectation ?

$E[X] = \sum_{x\in\text{Supp}[X]} x\,p(x) = \$2\cdot p(\$2) + (-\$1)\cdot p(-\$1) = \$2\cdot 12/37 + (-\$1)\cdot 25/37 = -\$0.027$

in a long run you will loose half the money playing the European Roulette over the US version.

if X models a payout of a game, "Fair Game" is if E[X] = 0

Basic r.v. transportation – Uber example

P(traffic) = 0.3

$W \sim \begin{cases} 7\text{min} & \text{w.p. } 0.7 \\ 12\text{min} & \text{w.p. } 0.3 \end{cases}$ not a Bernoulli

$E[W] = \sum_{x\in\text{Supp}[X]} x\,p(x) = 7\text{min}\cdot p(7\text{min}) + 12\text{min}\cdot p(12\text{min}) = 7\text{min}\cdot 0.7 + 12\text{min}\cdot 0.3 = 8.5$

on average in all my trips the time spend in the taxi is ≈8.5 min

$X_1, X_2, \dots, X_8 \sim \text{Bernoulli}(1/2) = X \sim \text{Binomial}(8, 1/2)$

E[X] ≈ 4 8 · 0.5 = 4

$E[X] = \sum_{x=0}^{8} x\,p(x) = 0\cdot p(0) + 1\cdot p(1) + 2\cdot p(2) + 3\cdot p(3) + 4\cdot p(4) + 5\cdot p(5) + 6\cdot p(6) + 7\cdot p(7) + 8\cdot p(8)$

$\binom{8}{5}(0.5)^8$ 56 · .0039 $\binom{8}{8}(0.5)^8$ 1· 0.0039

= 0·0.004 + 1· .031 + 2·.109 + 3· .219 + 4· .273 + 5· .219 + 6·.109 + 7· .031 + 8· .004
= 0 + .031 + .218 + .657 + 1.092 + 1.095 + .654 + .217 + .032
≈ 4
balance point

self-note:
Conducting a single Bernoulli experiment (e.g. flip of a coin) means converting a theoretical r.v. X~Bernoulli into its practical 'realization' x equal to either 0 or 1. Shaking a cup of 8 coins is analogous to performing 8 simultaneous Bernoulli experiments which can also be expressed by X~Binomial. Every time we shake this cup of 8 coins, we can expect the number of Heads to be somewhere between 0 and 8, and since we know that there is 50% chance of getting a Head it is reasonable to expect about 4 Heads. This expectation of 4 Heads is known as the 'balance point' and we deduced it to be the product of 8·50%. Statistics gives us a great tool known as the "Mean" which allows us to calculate the 'balance point'.

n · p = 8 · 0.5 = 4

end of class experiments

1 cup: 7-coins , 4-marked , n=sample_size=3
shake the cup, quarry 3 coins, how many Marked?

$n\frac{K}{N} = 3\ \frac{4}{7} = 1.714$

X ~ Hypergeometric(n,K,N) (3,4,7)

p ⟶ n p = Binomial

n :	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x :	1	2	1	2	2	2	2	2	3	2	1	2	1	2
$\bar{x}$:	1	1.5	1.33				1.71							1.786

n :	15	16	17	18	19	20	21	...
x :	2	3	1	2	2	1	0	Supp[X]={0,...,3}
$\bar{x}$:							1.714	≈ 1.714 balance point

1 cup: 6-coins , 2-marked , n=sample_size=1
shake the cup, quarry 1 coin, is it Marked?

X ~ Bernoulli(2/6) – balance point 2/6

1 cup: 6-coins , 4-marked , n=sample_size=1
shake the cup, quarry 1 coin, is it Marked?

X ~ Bernoulli(4/6) – balance point 4/6

∞ cups – in each cup: 6-coins , 2-marked , n=sample_size=1
shake the cups, stop on 1st Marked:
quarry 1 coin, from 1st cup , is it Marked? – NO
quarry 1 coin, from 2nd cup , is it Marked? – NO
quarry 1 coin, from 3rd cup , is it Marked? – Yes , STOP
realization x=3

X ~ Geometric(2/6)

n :	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x :	3	2	1	3	3	2	5	8	3	2	1	9	1	5
$\bar{x}$:		2.5	2				2.71							3.43

n :	15	16	17	18	19	20	21	...
x :	2	3	1	2	2	1	5	Supp[X]={1,...}
$\bar{x}$:							3.05	≈ 3 balance point

∞ cups – in each cup: 6-coins , 4-marked , n=sample_size=1
shake the cups, stop on the 3rd Marked:
quarry 1 coin, from 1st cup , is it Marked? – Yes
quarry 1 coin, from 2nd cup , is it Marked? – NO
quarry 1 coin, from 3rd cup , is it Marked? – Yes
quarry 1 coin, from 4th cup , is it Marked? – NO
quarry 1 coin, from 5th cup , is it Marked? – Yes , STOP
realization x=5

X ~ Negative Binomial(3,4/6)

n :	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x :	5	9	6	3	7	6	5	4	6	7	5	5	3	6
$\bar{x}$:		7	6.66				5.86							5.5

n :	15	16	17	18	19	20	21	...
x :	3	5	4	5	4	5	3	Supp[X]={3,...}
$\bar{x}$:							5.05	≈ 4.5 balance point

Mean – Binomial

based on the above revelation:

if E[X~Bernoulli(p)] = p , E[X~Binomial(n,p)] = np ????

Lets put the Binomial through the Mean function and see what comes out.

$E[X] = \sum_{x\in\text{Supp}[X]} x\,p(x) = \mu = \sum_{x=0}^{n} x\binom{n}{x}p^x(1-p)^{n-x}$ | expand the choose notation

$= \sum_{x=0}^{n} x\,\frac{n!}{x!(n-x)!}\,p^x(1-p)^{n-x}$ | $x! = x\cdot(x-1)!$

x∈Supp[0,...,n]

$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!}\,p^x(1-p)^{n-x}$ | $(n-x)! = [(n-1)-(x-1)]!$

0 · smthg = 0

$= n\sum_{x=1}^{n}\frac{(n-1)!}{(x-1)![(n-1)-(x-1)]!}\,p^x(1-p)^{((n-1)-(x-1))}$ | express in choose notation

$n = n-1$, $k = x-1$; $\frac{n!}{k!(n-k)!} = {}_nC_k = \binom{n}{k}$

$= n\sum_{x=1}^{n}\binom{n-1}{x-1}\,p^x(1-p)^{((n-1)-(x-1))}$ | create homogeneous terms

$= n\,p\sum_{x=1}^{n}\binom{n-1}{x-1}\,p^{x-1}(1-p)^{((n-1)-(x-1))}$

let: $m = n-1$, $y = x-1$; $x = 1\dots n$, $y = 0\dots n-1 = 0\dots m$

$= n\,p\sum_{y=0}^{m}\binom{m}{y}\,p^y(1-p)^{m-y} = 1$ (as previously proved)

$= n\,p$

$$E[X \sim \text{Binomial}(p) := \binom{n}{x}p^x(1-p)^{n-x}] = \mu = n\,p$$

Mean – Geometric

$X \sim \text{Geometric}(0.2) := (1-0.2)^{x-1}\ 0.2$

Warning!
Mode and Min are NOT the same

Min[X] = min{ Supp[X] } = 1
the smallest value in the support

look at all the probabilities find the largest one and tell me who is responsible

it the most likely value you will get

Mode[X] = arg_max{ p(x) } = 1
x∈Supp[X]

CDF; special case of Quantile

Quantile[X,p] := arg_min{ F(x)≥p } ⟶ **Median**[X,0.5] = 4
aka
Percentile if p is a percent — 70th Percentile = 6
e.g. 0.2·100 = 20%

what is a 95th percentile ?
Lets say 95% on the exam was 17. It means that at the most 5% students scored higher than 17, and at the minimum 95% of the students scored below 17.

90th Percentile = 11

Tertiles	Quartiles	Quintiles	Deciles
Quantile[X,0.33]	Quantile[X,0.25]	Quantile[X,0.20]	Quantile[X,0.10]
Quantile[X,0.66]	Quantile[X,0.50]	Quantile[X,0.40]	Quantile[X,0.20]
	Quantile[X,0.75]	Quantile[X,0.60]	...
		Quantile[X,0.80]	Quantile[X,0.90]

75th percentile – 25th percentile
IQR = Q[X,0.75] – Q[X,0.25] = 7 – 2 = 5
represents middle 50% of data

Max[X] = max{ Supp[X] } = d.n.e.
the largest value in the support

Range[X] = Max[X] – Min[X] = d.n.e.

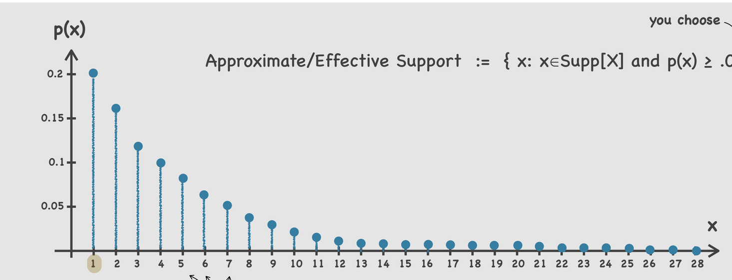
$0.8^{x-1}\ 0.2$

x	p(x)	F(x)
1	.200	.200
2	.160	.360 = .200 + .160
3	.120	.480
4	.102	.582
5	.082	.664
6	.066	.730
7	.052	.782
8	.042	.824
9	.037	.861
10	.027	.888
11	.021	.909
12	.017	.926
13	.014	.940
14	.011	.951
15	.009	.960
16	.007	.967
17	.006	.973
18	.005	.978
19	.004	.982
20	.003	.985
21	.002	.987
22	.001	.988
23	.001	.989
24	.001	.990
25	.001	.991
26	.001	.992
27	.000	.992
28	.000	.993
...		
	$\Sigma\approx1$	

Supp[X] = {1,...} ⟶ Why ≠ 1 ? In theory we could keep trying to ∞ and never get our success which is that necessary stopping factor.

you choose

Approximate/Effective Support := { x: x∈Supp[X] and p(x) ≥ .001 }



where is the **balance point** – E[X] – μ

Geometric PMF – p(x)

$\mu = \sum_{x\in\text{Supp}[X]} x\,p(x) \to \sum_{x=1}^{\infty} x\,(1-p)^{x-1}p$ | pull the const p

x∈Supp = [1,...]

$= p\sum_{x=1}^{\infty} x\,(1-p)^{x-1}$

reindexing trick
let $y = x-1 \to x = y+1$
$x = 1\dots\infty \to y = 0\dots\infty$

$= p\sum_{y=0}^{\infty}(y+1)\,(1-p)^y$ | split Σ

$= p\left[\sum_{y=0}^{\infty} y(1-p)^y + \sum_{y=0}^{\infty} 1\,(1-p)^y\right]$

geometric series $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$

$= p\left[\sum_{y=0}^{\infty} y(1-p)^y + \sum_{y=0}^{\infty} 1\,\frac{1}{1-(1-p)}\right]$

$= p\sum_{y=0}^{\infty} y(1-p)^y + \cancel{p}\sum_{y=0}^{\infty}\frac{1}{\cancel{p}}$ | ps cancel

$= \sum_{y=0}^{\infty} y\,(1-p)^y\,p + 1$ | $x^3 = x^2\cdot x^1 = x^{3-1}\cdot x^1$

$= \sum_{y=0}^{\infty} y\,(1-p)^{y-1}(1-p)\,p + 1$ | pull the constant (1-p)

$= (1-p)^1\sum_{y=0}^{\infty} y\,(1-p)^{y-1}\,p + 1$

$= (1-p)^1\ \mu + 1$

$\cancel{\mu} = 1\cancel{\mu} - p\mu + 1$

$= -p\mu + 1$

$p\mu = 1$

thus our **balance point** is = 5

$$E[X \sim \text{Geometric}(p) := (1-p)^{x-1}p] = \mu = \frac{1}{p}$$

$\frac{1}{0.2} = 5$