October 27, 2016

Let $T_n = X_1 + X_2 + \dots + X_n$ ⇒ "Total r.v. | Sum r.v."

n denotes how many things you're summing up.

$\overline{X_n} = \frac{T_n}{n} = \frac{X_1 + X_2 + \dots + X_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$

"Average r.v."

> $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(0.1)$

$T_3 \sim \text{Binom}(3, 0.1)$

$\text{Supp}[X] = \{0, 1, 2, 3\}$

$$T_3 \sim \begin{cases} 0 & \text{wp} \quad 0.729 \\ 1 & \text{wp} \quad 0.243 \\ 2 & \text{wp} \quad 0.027 \\ 3 & \text{wp} \quad 0.001 \end{cases}$$

$\text{Supp}[\overline{X}] = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$

$$\overline{X}_3 \sim \begin{cases} 0 & \text{wp} \quad 0.729 \\ \frac{1}{3} & \text{wp} \quad 0.243 \\ \frac{2}{3} & \text{wp} \quad 0.027 \\ 1 & \text{wp} \quad 0.001 \end{cases}$$

Def: Sample average $\bar{x}$ is a realization from $\overline{X}$

$\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$

$x = 5'11''$

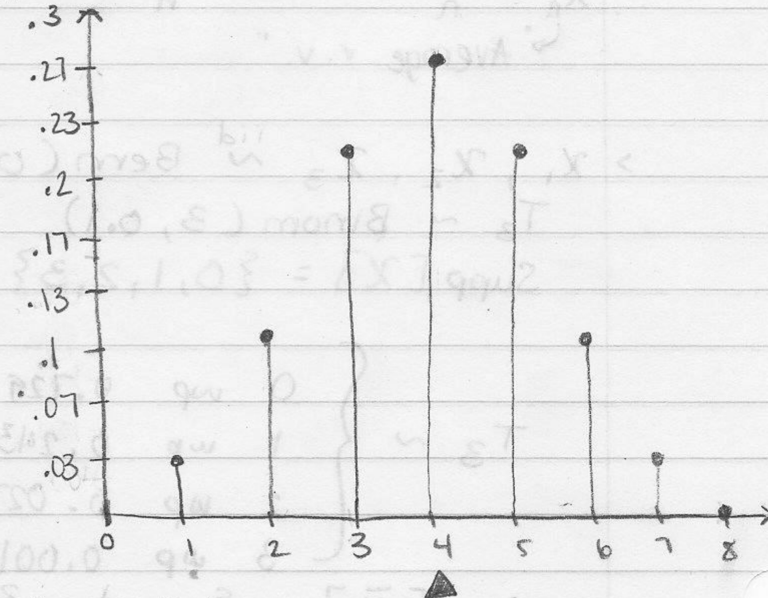
X is a r.v. for the adult height.

$P(X = 5'11'')$ ➡ Probability that you get a realization of X to be $5'11''$.

We need a model ➡ PMF

> $X \sim \text{Binom}(8, \frac{1}{2}) = \binom{8}{x}(\frac{1}{2})^x (1-\frac{1}{2})^{8-x}$
$= \frac{\binom{8}{x}}{2^8}$

x	p(x)	F(x)
0	0.004	0.004
1	0.031	0.035
2	0.109	0.145
3	0.219	0.363
4	0.273	0.637
5	0.219	0.855
6	0.109	0.965
7	0.031	0.996
8	0.004	1



$= \text{"}1\text{"}$

B/c $\sum_{x \in \text{Supp}[X]} p(x) = 1$ otherwise it "ain't" no PMF.

b/c it is the prob that x is less than 8, which is the entire support! So x has to be ≤ 8.

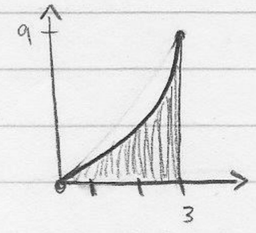
> "How many heads?"

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
x	2	5	4	4	3	5	4	5	6	5	7	4	5	3	5	5	1	4	3	3	4
$\bar{x}$	2	3.5	3.67				3.857														

$\frac{\text{Total Successes}}{n}$

$\bar{x} \longrightarrow 4$

Why? It's the balancing point. ▲



$f(x) = x^2$ where $x \in [0, 3]$

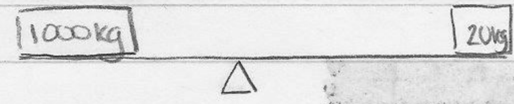
$G[f] = \int_{\mathbb{R}} f(x)\,dx = 9$

function function

"A function of a function"

$G : F \longrightarrow \mathbb{R}$

functions one number



Where is the balancing point?

$$\sum_i w_i (d_i - d^*) = 0 \Rightarrow \sum_i w_i d_0 - \sum_i w_i d^* = 0$$
$$\Rightarrow \frac{\sum_i w_i d_i}{\sum_i w_i d^*} = d^*$$

"Expectation of X"
It's not just a function - it's a function of a function.

$E[X] :=$

$$\mathcal{M} := \frac{\sum_{x \in Supp[X]} p(x)\, x}{\underbrace{\sum_{x \in Supp[X]} p(x)}_{=1}}$$ so $\mathcal{M} := \sum_{x \in supp[X]} x\, p(x)$

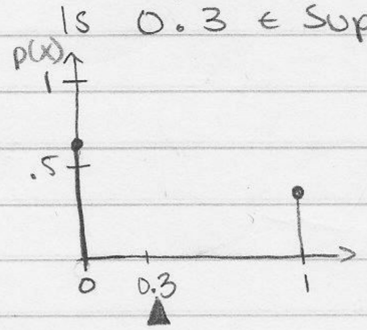
{"Mean", "Expected Value", "Expectation", "First Moment"!}

> Law of Large Numbers
$\bar{X} \to \mathcal{M}$

"The average is eventually going to get closer and closer to the balance point. If you take the average of a lot of different things, it becomes the expected value."

> $X \sim Bern(0.3)$
$E[X] = (0 \times p(0)) + (1 \times p(1)) = p(1) = 0.3$
Is $0.3 \in Supp[X]$? No, $\mathcal{M} \notin Supp[X]$ *

$E[X] := p$



> In the example where $X \sim Binom(8, \frac{1}{2})$, prove that the expected value is 4...

$$\mathcal{M} := \underbrace{0 \cdot p(0)}_{} + 1 \cdot p(1) + \ldots + 8 \cdot p(8)$$
$$= (0) + (.031) + (2 \cdot .109) + (3 \cdot .219) + (4 \cdot .273)$$
$$+ (5 \cdot .219) + (6 \cdot .109) + (7 \cdot .031) + (8 \cdot .004) = \boxed{4}$$

> For $X \sim \text{Binom}(n,p)$,

$$M := \sum_{x=0}^{n} x \left[\binom{n}{x} p^x (1-p)^{n-x} \right]$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= n \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,\underbrace{(n-x)!}_{((n-1)-(x-1))!}} p^x (1-p)^{n-x}$$

$$= n \sum_{x=1}^{n} \binom{n-1}{x-1} p^x (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{((n-1)-(x-1))}$$

Let $m = n-1$, $y = x-1$

$x = 1, \ldots, n \quad y = 0, \ldots, n-1 = 0, \ldots, m$

$$= np \underbrace{\left(\sum_{y=0}^{m} \binom{m}{y} p^y (1-p)^{m-y} \right)}_{=1} = np$$

* $M := np$

> $X \sim \text{Hyper}(n, K, N)$

$$M := \sum_{x \in \text{Supp}[X]} x \left[\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \right]$$

iid b/c the PMF is the same for all the cups

> Take out 3 nickels out of a cup of 7 nickels, 3 of which are spotted. How many spotted ones did you get?

$\Rightarrow X \overset{iid}{\sim} \text{Hyper}(3, 4, 7)$

lower case. Realizations

n	1	2	3	4	5	6	7							14							21
x	1	2	1	2	2	2	2	2	3	2	1	2	1	2	2	3	1	2	2	1	0
$\bar{x}$							1.857							1.857							1.762

For Hyper,

$$M := \sum_{x \in \text{Supp}[X]} x \left[\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \right] = \mu = n \cdot \underbrace{\frac{K}{N}}_{= p} \Rightarrow 3 \cdot \frac{4}{7} = 1.714$$

close to 1.762.

> Take out 1 nickel out of a cup of 6 nickels, 2 of which are spotted. Is your nickel spotted?

$\Rightarrow X \sim \text{Bern}\left(\frac{1}{3}\right)$

> Take out 1 nickel out of a cup of 6 nickels, 2 of which are spotted. Keep going until you get a spotted nickel, sampling with replacement.

$\Rightarrow X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right) \Rightarrow X \sim \text{Geom}\left(\frac{1}{3}\right)$

n	1	2	3	4	5
x	3	2	1	3	3
$\bar{x}$					2.4

$\bar{x} \to 3$ (The average is going to approach 3)

> Take out 1 nickel out of a cup of 6 nickels, 4 of which are spotted. Is your nickel spotted?

$\Rightarrow X \sim \text{Bern}\left(\frac{2}{3}\right)$

> Go until we get 3 spotted coins

$\Rightarrow X_1, \ldots, X_3 \overset{iid}{\sim} \text{Geom}\left(\frac{2}{3}\right) \Rightarrow X_1 + \ldots + X_3 \sim \text{NegBin}\left(3, \frac{2}{3}\right)$

n	1	2	3	4	5
x	5	9			
$\bar{x}$		4.5			