Lec 4 9/6/16 Math 241

$\Omega = \{B, J, R, S, M, A\}$ $\quad P(\text{alternating gender}) = \dfrac{3\cdot 3\cdot 2\cdot 2\cdot 1\cdot 1 + 3\cdot 3\cdot 2\cdot 2\cdot 1\cdot 1}{|\Omega|} = \dfrac{2(3!)^2}{6!} = \dfrac{72}{6!} = \dfrac{1}{10}$

5 flowers 3 o's, 2 x's

If all distinct # orderings $= 5!$

but if orchids indistinguishable, $\dfrac{5!}{3!}$

" " " " chrys " , $\dfrac{5!}{2!}$

If orchids and chrys indistinguishable $\dfrac{5!}{3!\,2!} = \dfrac{5\cdot\cancel{4}^{12}\cdot 3\cdot 2}{\underbrace{3\cdot 2\cdot 2}_{12}} = 10$

OOOXX XOXOO
OOXOX XXOOO
OXOOX OXXOO
XOOOX OOXXO
XOOXO OXOXO

orderings

Does $\dfrac{5!}{12}\cdot 12 = 5!$? Does each of these orderings become 12 if you "begin to care" about their distinctness?

$XOOXO$ — X_1OOX_2O — $X_1O_1O_2X_2O_3$, $X_1O_1O_3X_2O_2$, $X_1O_2O_1X_2O_3$, $X_1O_2O_3X_2O_1$, $X_1O_3O_2X_2O_1$, $X_1O_3O_1X_2O_2$

$XOOXO$ — X_2OOX_1O — ⋮

12 units are collapsed together into 1

10 coin flips — 5H's, 5T's s.t. H,T's indistinguishable

$P(5H, 5T) = \dfrac{\frac{10!}{5!\,5!}}{2^{10}} = .2460938$

1000 coin flips

calculator bombs!

$P(600H, 400T) = \dfrac{\frac{1000!}{600!\,400!}}{2^{1000}} \in [0,1]$ easily handled #!

Solution to overflow... use logs

$$\ln(p) = \ln(1000!) - \ln(600!) - \ln(400!) - 1000\ln(2)$$

since $n! = \prod_{i=1}^{n} i \Rightarrow \ln(n!) = \sum_{i=1}^{n} \ln(i)$

no more blowup but tedious!!!

$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ Stirling Approx.

$$\Rightarrow \ln(n!) \approx \frac{1}{2}\ln(2\pi) + \left(n+\frac{1}{2}\right)\ln(n) - n$$

$$\Rightarrow \ln(p) = \left(\frac{1}{2}\ln(2\pi) + 1000.5\ln(1000) - \cancel{1000}\right) - \left(\frac{1}{2}\ln(2\pi) + 600.5\ln(600) - \cancel{600}\right)$$
$$- \left(\frac{1}{2}\ln 2\pi + 400.5\ln(400) - \cancel{400}\right)$$
$$- 1000\ln(2)$$

$$= 1000.5\ln(1000) - 600.5\ln(600) - 400.5\ln(400) - \frac{1}{2}\ln(2\pi) - 1000\ln(2)$$

$$= -23.79$$

$$\Rightarrow p = e^{\,} = 4.6 \times 10^{-11} \text{ (rare)}$$

$\frac{1000!}{600!\,400!} \leftrightarrows \frac{n!}{k!(n-k)!}$ s.t. $k \le n,\ n \in \mathbb{N}$ is a quantity that comes up a lot!

Imagine $\{J, B, S, R, M, A\}$ 6 chairs $\Rightarrow 6!$

But 4 chairs $\Rightarrow {}^{6}P_{4} = \frac{6!}{(6-4)!} = \frac{6!}{2!}$ $\quad \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}$

What if... I didn't care about the order of the 4 "people". How my orderings?

JBSRMA   Sample without replacement (but order doesn't matter)

{J, B, S, R}	{B, S, R, M}
{J, B, S, M}	{B, S, R, A}
{J, B, S, A}	{B, S, M, A}
{J, B, R, M}	{B, R, M, A}
{J, B, R, A}	{S, R, M, A}
{J, B, M, A}	
{J, S, R, M}	
{J, S, R, A}	
{J, S, M, A}	
{J, R, M, A}	

$\Rightarrow$ 15 unique s.t. order doesn't matter

$15 \cdot 4! =$ all orders

$6 \cdot 5 \cdot 4 \cdot 3$

"6 choose 4"

$$\Rightarrow \frac{{}_6P_4}{4!} = \frac{\frac{6!}{2!}}{4!} = \frac{6!}{4!\,2!} = {}_6C_4 \text{ or } \binom{6}{4}$$

4-combination of 6

$${}_nC_k := \binom{n}{k} := \frac{n!}{(n-k)!\,k!}$$

Identities

1. $\binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n$
2. $\binom{n}{n-1} = \frac{n!}{(n-(n-1))!\,(n-1)!} = \frac{n!}{1!\,(n-1)!} = n$
3. $\binom{n}{n} = \frac{n!}{n!\,0!} = 1$
4. $\binom{n}{0} = \frac{n!}{0!\,n!} = 1$ weird...
5. $\binom{n}{n-k} = \frac{n!}{(n-k)!\,\underbrace{(n-(n-k))!}_{k!}} = \binom{n}{k}$

each choice of k leaves n-k so there must be exact same #!

Consider order matters

$$P(\text{4 spots, I get Jane}) = \frac{\frac{1}{J}\frac{5}{2}\frac{4}{3}\frac{3}{4} + \frac{5}{1}\frac{1}{J}\frac{4}{3}\frac{3}{4}}{{}_6P_4} = \frac{4(\cancel{5\cdot4\cdot3})}{6\cdot\cancel{5\cdot4\cdot3}} = \frac{2}{3}$$

Consider order doesn't matter $\{J, \_, \_, \_\}$

$$= \frac{\binom{5}{3}}{\binom{6}{4}} = \frac{\frac{5\cdot\cancel{4}\cdot\cancel{3!}}{\cancel{2!}\cancel{3!}}}{\frac{\cancel{6}\cdot5\cdot\cancel{4!}}{\cancel{2}\cdot\cancel{4!}}} = \frac{10}{15} = \frac{2}{3}$$

all possible ways to choose the "other" 3 people

Recall set A, let $|A| = n$, $n \in \mathbb{N}$

and $|2^A| = 2^{|A|} = 2^n$

subsets of A (order doesn't matter)

$$2^A := \{B : B \subseteq A\}$$

$$= \{B : B \subseteq A \;\&\; |A| = 1\} \cup \{B : B \subseteq A \;\&\; |A| = 2\} \cup \ldots$$

why?
mutually exclusive?
collectively exhaustive?

$\Rightarrow \ldots$

$$2^A = \bigcup_{i=0}^{n} \{B : B \subseteq A \;\&\; |A| = i\}$$

$$2^n = |2^A| = \sum_{i=0}^{n} \left| \{B : B \subseteq A \;\&\; |A| = i\} \right|$$

How many subsets of n with size i s.t. order doesn't matter

$$\Rightarrow 2^n = \sum_{i=0}^{n} \binom{n}{i}$$

Cool...

$$(a+b)^2 = (a+b)(a+b) = \overbrace{a^2+ab+ba+b^2}^{4\text{ terms}} = a^2+2ab+b^2$$

$= 2\times 2$ terms

8 terms

$$(a+b)^3 = (a+b)(a+b)(a+b) = a^3 + \dots = a^3+3a^2b+3ab^2+b^3$$

4 a's → 3 a's

$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b) = \binom{4}{4}a^4 + \binom{4}{3}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{1}ab^3 + \binom{4}{0}b^4$$

a^0

take 4 of them

take 3 of them

$$(a+b)^n = \binom{n}{n}a^n + \binom{n}{n-1}a^{n-1}b + \binom{n}{n-2}a^{n-2}b^2 + \dots + \binom{n}{2}a^2b^{n-2} + \binom{n}{1}ab^{n-1} + \binom{n}{0}b^n$$

$$= \sum_{i=0}^{n}\binom{n}{i}a^i b^{n-i} = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$$

Binomial Thm.

$$(1+x)^n = \sum_{i=0}^{n}\binom{n}{i}1^{n-i}x^i = \sum_{i=0}^{n}\binom{n}{i}x^i = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

$$= 1 + \sum_{i=1}^{n-1}\binom{n}{i}x_i + x^n$$

$$= (1+x)(1+x)^{n-1} = (1+x)\sum_{i=0}^{n-1}\binom{n-1}{i}x^i = \sum_{i=0}^{n-1}\binom{n-1}{i}x^i + \sum_{i=0}^{n-1}\binom{n-1}{i}x^{i+1}$$

reindexing trick

$$= \sum_{i=0}^{n-1}\binom{n-1}{i}x^i + \sum_{i=1}^{n}\binom{n-1}{i-1}x^i$$

$$= \underbrace{\binom{n-1}{0}x^0}_{1} + \sum_{i=1}^{n-1}\left(\binom{n-1}{i}+\binom{n-1}{i-1}\right)x^i + \underbrace{\binom{n-1}{n-1}x^n}_{x^n}$$

$$\Rightarrow \binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$$

"Pascal's Rule"

1

1 1

1 2 1

1 3 3 1

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \qquad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \qquad \binom{5}{3} \quad \binom{5}{4}$$

$$\binom{6}{0} \quad \cdots \quad \binom{6}{4} \quad \cdots \quad \binom{6}{6}$$

$$\binom{6}{4} = \binom{5}{3} + \binom{4}{3} + \binom{4}{4}$$

$$= 10 + 4 + 1 = 15$$

10 {J . . . .} ⋮ {J . . .} + 4 {B . . . ⋮ {B . . . + 1 {{R, S, B, A

5 other people 3 seats

4 other people, 3 seats

4 people, 4 seats

ranks

$D = R \times S$, $R = \{A, 2, 3, \ldots 10, J, Q, K\}$ $|R| = 13$

suits

Sometimes $D = R \times S \cup \{Jok1, Jok2\}$ $S = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ $|S| = 4$

but we don't care... $|D| = 52$

$$= \{\langle 2, \diamondsuit \rangle \langle A, \clubsuit \rangle, \ldots\}$$