Lec 18.

11/17/2016

	PMF	PDF	CDF	$E[X]$	$Var[X]$	$\|Supp[X]\|$
Discrete	$p(x) \in [0,1]$ $\sum p(x) = 1$ $x \in supp[X]$	not exist	exists	$\sum x p(x)$ $x \in supp[X]$	$\sum (x-\mu)^2 \cdot p(x)$	$\leq \|\mathbb{N}\|$
Continues	not exist	$f(x) \geq 0$ $\int f(x)\,dx = 1$ $supp[X]$	exist	$\int x f(x)\,dx$ $supp[X]$	$\int (x-\mu)^2 f(x)\,dx$	$\|\mathbb{R}\|$

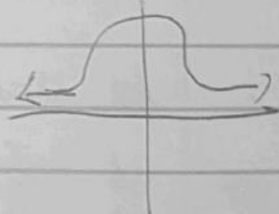
Discrete Quantile $[X, P]$

$\min_x \{x : F(x) \geq P\}$

Continues x s.t. $F(x) = P$

$x = F^{-1}(P)$

Ex: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is this a PDF



a) $f(x) \geq 0$ $\frac{1}{\sqrt{2\pi}}$ is $+$, $e^{-\frac{x^2}{2}}$ is $+$ ✓

b) $\int f(x)\,dx = 1 \Rightarrow \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = 1 \Rightarrow \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$

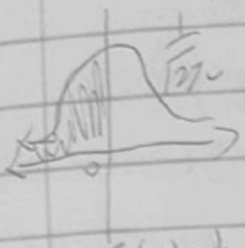
let $u = \frac{1}{\sqrt{2}} x \Rightarrow u^2 = \frac{x^2}{2}$, $du = \frac{1}{\sqrt{2}}\,dx \Rightarrow dx = \sqrt{2}\,du$

$\int_{\mathbb{R}} e^{-u^2} \sqrt{2}\,du = \sqrt{2\pi} \Rightarrow \int_{\mathbb{R}} e^{-u^2}\,du = \sqrt{\pi} \Rightarrow \left(\int_{\mathbb{R}} e^{-u^2}\,du\right)^2 = \pi \Rightarrow$

$\int e^{-u^2}\,du \int e^{-u^2}\,du = \pi \Rightarrow \int e^{-x^2}\,dx \int e^{-y^2}\,dy = \pi \Rightarrow \iint_{x \in \mathbb{R}, y \in \mathbb{R}} e^{-(x^2+y^2)}\,dx\,dy$ 先

$v = r^2$, $dv = 2r\,dr \Rightarrow \frac{dv}{2} = r\,dr \Rightarrow \int_0^\infty e^{-v} \frac{dv}{2} = \frac{1}{2} = 1$

$[e^{-v}]_0^{\infty} = e^{-0} - \lim_{v \to \infty} e^{-v} = 1$ ✓



This is PDF. $Z \sim N(0,1) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

"standard normal r.v", "bell curve" "Gaussian r.v"

$F(0) = \frac{1}{2}$

This is one case of the general normal dist

let $u = \frac{x^2}{2} \Rightarrow du = x\,dx$

$E[Z] = \int_{supp[X]} x f(x)\,dx : \int_{x=-\infty}^{x=\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} e^{-u}\,du = \frac{1}{\sqrt{2\pi}} [-e^{-u}]_{x=-\infty}^{x=\infty}$

$= \frac{1}{\sqrt{2\pi}} [-e^{-\frac{x^2}{2}}]_{-\infty}^{\infty}$

$= \frac{1}{\sqrt{2\pi}} (\lim_{x \to \infty} e^{-\frac{x^2}{2}} - \lim_{x \to \infty} e^{-\frac{x^2}{2}}) = 0 \Rightarrow \mu = 0.$

$Var[Z] = E[Z^2] = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = (work) = 1$

by integration by parts

$SE[Z] = 1 \quad \boxed{\sigma^2 = \sigma = 1}$

$\triangle\ F(X) = \int f(x)\,dx + C = \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + C}_{\text{not possible.}}$

$\triangle\ F(X) = P(Z \le X) = \int_{-\infty}^{x} \frac{1}{2\pi} e^{-\frac{x^2}{2}} dx$

what $F(0)$? the amount of probability up until 0. $\frac{1}{2}$ because we proved that bell = 1, since the bell is symmetrical, at $x=0$ the CDF $= \frac{1}{2}$.

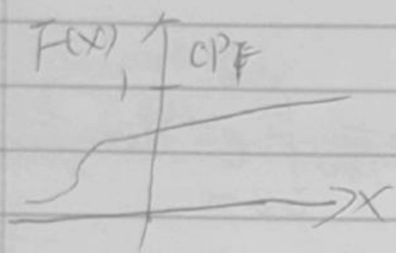
▷ $P(Z \in [-1,1]) = F(1) - F(-1) \simeq 0.68$

$P(Z \in [-2,2]) = F(2) - F(-2) \simeq 0.95$

$P(Z \in [-3,3]) = F(3) - F(-3) \simeq 0.997$

"3σ rule"
or "Empirical rule"
or "68-95-99.7 rule"

"The amount of prob. between -1 SE and 1 SE is 0.68"



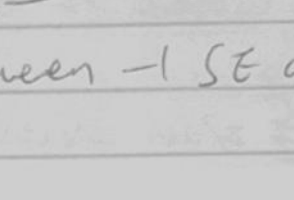
review midterm 2.

$T \sim Geom(p)$, $E(T) = \frac{1}{p}$. $\lim_{n \to \infty} P(t) = 0$ has no PMF

$F(x) = 1 - e^{-\lambda x}$

$X \sim Exp(\lambda) = f(x) = \lambda e^{-\lambda x}$ exponential r.v $E[X] = \frac{1}{\lambda}$

$X \sim U(a,b)$ $f(x) = \frac{1}{b-a}$

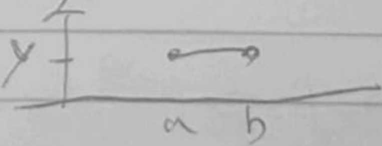


let $\lambda' = \frac{\lambda}{2}$

Ex: $X \sim Exp(\lambda)$ $Y = 2X \sim$?

$F_Y(x) = P(Y \le x) = P(2X \le x) = P(X \le \frac{x}{2}) = F_X(\frac{x}{2}) = 1 - e^{-\lambda \frac{x}{2}} \Rightarrow$

$1 - e^{-\lambda' x} = Exp(\lambda') \Rightarrow Exp(\frac{\lambda}{2}) \Rightarrow Y \sim Exp(\frac{\lambda}{2})$



Ex: $Y = d + cX$

$F_Y(x) = P(Y \le x) = P(cX + d \le x) = P(X \le \frac{x-d}{c}) = F_X(\frac{x-d}{c}) = \frac{\frac{x-d}{c} - a}{b-a} \cdot \frac{c}{c}$

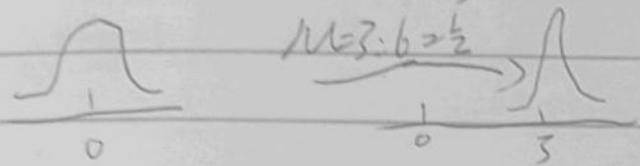
$\frac{x-(d+ca)}{cb-ca} = \frac{x-(d+ca)}{(d+cb)-(d+ca)} = \frac{x-a'}{b'-a'} = U(a', b') = U(d+ca, d+cb)$

(let $a' = d + ca$, $b' = d + cb$)

Ex: $Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ let $X := \sigma Z + \mu$

$E[X] = \sigma \cancel{E(Z)}^{0} + \mu = \mu$

$SE[X] = SE[\sigma Z + \mu] = |\sigma| \cancel{SE[Z]}^{1} = |\sigma|$



$F_X(x) = P(X \le x) = P(\sigma Z + \mu \le x) = P(Z \le \frac{x-\mu}{\sigma}) = F_Z(\frac{x-\mu}{\sigma})$

$f(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} F_Z(\frac{x-\mu}{\sigma}) = \frac{du}{dx} \cdot \frac{d}{du}[F_Z(u)] = \frac{1}{\sigma}(\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}})$

let $u = \frac{x-\mu}{\sigma} \Rightarrow \frac{du}{dx} = \frac{1}{\sigma}$

$X \sim N(\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma}(x-\mu)^2}$

$X \sim Bern(p)$