September 29, 2016

> 10 cards : 4 Red, 6 Blue

$P(2 \text{ Red}, \text{Drawing } 3) = \dfrac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}}$

(without replacement, order doesn't matter)

$P(x \text{ Red}, \text{Drawing } 3) = \dfrac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$

$P(x \text{ Red}, \text{Drawing } n) = \dfrac{\binom{4}{x}\binom{6}{n-x}}{\binom{10}{n}}$

"successors" "Failures"

> 10 cards : K Red , 10 - K Blue

$P(x \text{ Red}, \text{Drawing } n) = \dfrac{\binom{K}{x}\binom{10-K}{n-x}}{\binom{10}{n}}$

> N Cards : K Red , N - K Blue

$P(x \text{ Red}, \text{Drawing } n) = \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$

$$\boxed{X \sim \text{Hypergeometric}(n, K, N) := p(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}}$$

Example: $N = 118$ $K = 37$ $n = 23$

$P(x = 16)$

$X \sim \text{Hyper}(23, 37, 118) := p(x=16) = \dfrac{\binom{37}{16}\binom{81}{7}}{\binom{118}{23}}$

what's the probability of getting 16 sucessors if you take 23 out?

How many you pick — Total number of successors — Successors + failures

successors — failures (23-16)

Parameter Space

$N = 0 ? \Rightarrow n = 0$

$N = 1 ? \Rightarrow n = 1, \; K = 0 \text{ or } 1.$

because if you only have 1 ball in a bag, you can only pick 1.

the one ball can either be a successor or a failure

$N = 2 ? \Rightarrow$ if $n = 1$:

$K = 0 \Rightarrow X \sim \text{Deg}(0) \rightarrow$ Not Interesting!

$K = 1$

$K = 2 \Rightarrow X \sim \text{Deg}(1) \rightarrow$ Not Interesting!

$n=1$ $\begin{cases} N=2, \quad K=1 \\ \hookrightarrow X \sim Hyper(1,1,2) = \dfrac{\binom{1}{x}\binom{1}{x}}{\binom{2}{1}} \Rightarrow \begin{matrix} P(X=0)=\frac{1}{2} \\ P(X=1)=\frac{1}{2} \end{matrix} = Bern(\tfrac{1}{2}) \end{cases}$

$n=2 \Rightarrow X \sim Deg(K) \rightarrow$ Not Interesting!

$$\boxed{\begin{aligned} \text{Parameter Space} \Rightarrow \quad & N \in \mathbb{N} \setminus \{1\} \\ & K \in \{1,2,\dots,N-1\} \\ & n \in \{1,2,\dots,N-1\} \end{aligned}}$$

K: → cannot $=N$ b/c then they are all success

n: → cannot $=N$ b/c you'll pick all

$X \sim Hyper(1,K,N) = Bern(\frac{K}{N})$

What is the support? $Supp[X] = \{0,1\}$

$$\Downarrow$$

$$= \frac{\binom{K}{N}\binom{N-K}{1-x}}{\binom{N}{1}} \begin{matrix} \Rightarrow P(X=1) = \frac{K}{N} = P \\ \Rightarrow P(X=0) = 1-\frac{K}{N} = 1-P \end{matrix}$$

$X \sim Hyper(2,4,10)$ 6 failures

$Supp[X] = \{0,1,2\} \Rightarrow$ $\left[\begin{matrix} n<K, \; n<N-K \\ Supp[X]=\{0,1,\dots,n\} \end{matrix}\right]$

what is possible? what can happen? You're picking from 2 balls. How many successes can you get?

2 fail, 0 success

1 fail, 1 success

0 fail, 2 success

$X \sim Hyper(5,4,10)$ 6 failures

$Supp[X] = \{0,1,2,3,4\} \Rightarrow$ $\left[\begin{matrix} n \geq K, \; n<N-K \\ Supp[X]=\{0,1,\dots K\} \end{matrix}\right]$

5 fail, 0 success

4 fail, 1 success

3 fail, 2 success

2 fail, 3 success

1 fail, 4 success

no more than $4=K$

$X \sim Hyper(8,4,10)$ 6 failures $\Rightarrow$ $\left[\begin{matrix} n \geq K, \; n \geq N-K \\ Supp[X]=\{n-(N-K),\dots,K\} \end{matrix}\right]$

$Supp[X] = \{2,3,4\}$

6 fail, 2 success

5 fail, 3 success

4 fail, 4 success

$X \sim Hyper(5,7,10) \Rightarrow$ $\left[\begin{matrix} n<K, \; n \geq N-K \\ Supp[X]=\{n-(N-K),\dots,n\} \end{matrix}\right]$

$Supp[X]$ $\{2,3,4,5\}$

	$n < K$	$n \geq K$
$n < N-K$	$\{0, \dots, n\}$	$\{0, \dots, K\}$
$n \geq N-K$	$\{n-(N-K), \dots, n\}$	$\{n-(N-K), \dots, K\}$

$$\text{Supp}[X] = \{\max(0, n-(N-K)), \dots, \min(n, K)\}$$

$$\sum_{x \in \text{Supp}[X]} P(x) = 1$$

n: Sample size ← how many you pick → n: Sample size
N: population size OR N: population size
K: # of successes K: proportion of successes

> Equivalent Parameterization

let $p = \frac{K}{N} \Rightarrow K = pN$ } one-to-one

$$X \sim \text{Hyper}(n, p, N) := p(x) = \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$$

> Consider a bag where 50% are successes

Ⓐ $p = 0.5$ $n = 6$ $N = 100$

$$p(X=3) = \frac{\binom{50}{3}\binom{50}{3}}{\binom{100}{6}} = .3223$$

Ⓑ $p = 0.5$ $n = 6$ $N = 1000$

$$p(X=3) = \frac{\binom{500}{3}\binom{500}{3}}{\binom{1000}{6}} = .3134$$

Ⓒ $p = 0.5$ $n = 6$ $N = 10,000$

$$p(X=3) = \frac{\binom{5000}{3}\binom{5000}{3}}{\binom{10000}{6}} = .3136$$

} Converges

If you increase the population size and keep the amount of successors the same, it will converge to one number. It doesn't matter if you take out and put it back in. Eventually, as you keep adding more balls to the population, the probability will be the same

The limiting random value (r.v.)

$\lim_{N\to\infty} P(x=3) = ? \Rightarrow X \sim Hyper(n, p, N)$ and $N \to \infty$

$$\lim_{N\to\infty} \frac{\frac{(pN)!}{x!(pN-x)!} \frac{((1-p)N)!}{(n-x)!((1-p)N-n+x)!}}{\frac{N!}{n!(N-n)!}}$$

factored out using $\lim af(x) = a \lim f(x)$

$$\boxed{\frac{\frac{1}{x!(n-x)!}}{\frac{1}{n!}} = \frac{n!}{x!(n-x)!} = \binom{n}{x}}$$

$$= \frac{\frac{1}{x!(n-x)!}}{\frac{1}{n!}} \lim_{N\to\infty} \frac{\frac{(pN)!}{(pN-x)!} \frac{((1-p)N)!}{((1-p)N-n+x)!}}{\frac{N!}{(N-n)!}}$$

$$= \binom{n}{x} \lim_{N\to\infty} \frac{\overbrace{\overbrace{(pN)(pN-1)\cdots(pN-x+1)}^{x} \overbrace{(1-pN)(1-pN-1)\cdots((1-p)N-n+x+1)}^{n-x}}^{n}}{\underbrace{(N)(N-1)\cdots\cdots(N-n+1)}_{n}}$$

$\lim f(x)g(x) = \lim f(x) \lim g(x)$

$$= \binom{n}{x} \underbrace{\underbrace{\lim_{N\to\infty} \frac{pN}{n}}_{p} \cdot \underbrace{\lim \frac{pN-1}{N-1}}_{p} \cdot \cdots \cdot \underbrace{\lim \frac{pN-x+1}{N-x}}_{p}}_{p^x} \cdot \underbrace{\underbrace{\lim \frac{(1-p)N}{N-x}}_{1-p} \cdot \underbrace{\lim \frac{(1-p)N-1}{N-x-1}}_{1-p} \cdot \cdots \cdot \underbrace{\lim \frac{(1-p)N-n+x+1}{N-n+1}}_{(1-p)}}_{(1-p)^{n-x}}$$

$= \binom{n}{x} p^x (1-p)^{n-x}$

$\Rightarrow X \sim Binomial(n,p) := p(x) := \binom{n}{x} p^x (1-p)^{n-x}$

$Supp[X] = \{0, 1, \ldots, n\}$ ← everything

In an indefinitely large bag, what are the possible values?

$= Deg(0)$

$X \sim Binom(n, 0) = \binom{n}{x} 0^x (1-0)^{n-x} \Rightarrow P(x=0) = \binom{n}{0} 0^0 1^n$ ← $\{0^0 := 1\}$ motivation force

$= Deg(n)$

$X \sim Binom(n, 1) = \binom{n}{x} 1^x 0^{n-x}$

$X \sim Binom(1, p) = \binom{1}{x} p^x (1-p)^{1-x} = p^x (1-p)^{1-x} = Bern(p)$

$Supp[X] = \{0, 1\}$

$\binom{1}{0} = 1$

$\binom{1}{1} = 1$