Lecture 9 Math 241 9/29/16

10 cards
4 R
6 B

$$P(2R \text{ in } 3 \text{ cards}) = \frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}}$$

$$P(xR \text{ in } 3 \text{ cards}) = \frac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$$

$$P(xR \text{ in } n \text{ cards}) = \frac{\binom{4}{x}\binom{6}{n-x}}{\binom{10}{n}}$$

K R, 10-K B

$$P(xR \text{ in } n \text{ cards}) = \frac{\binom{K}{x}\binom{10-K}{n-x}}{\binom{10}{n}}$$

N cards N-K B

$$P(xR \text{ in } n \text{ cards}) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} =: p(x) :=$$

$X \sim \text{Hyper}(n, K, N)$ → three parameters! Bern had just 1!

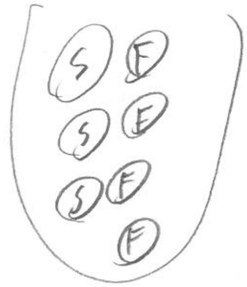
Review...

$X \sim \text{Bern}(p) := p^x(1-p)^{1-x}$

Verify $\sum_{x \in Supp(X)} p(x) = 1$

$$= \sum_{x \in \{0,1\}} p^x(1-p)^{1-x} = p^0(1-p)^{1-0} + p^1(1-p)^{1-1}$$

$$= 1-p+p = \boxed{1} \checkmark$$



N=7
K=3
n ... up to you

n: sample size
K: # successes
N: population size

3 knobs you can turn!

e.g. 100 students, 53 females
pick 8 at random, what's the prob of 6 being female?

Model $X \sim \text{Hyper}(8, 53, 100)$

Prob Query: $P(X=6) = \dfrac{\binom{53}{6}\binom{47}{2}}{\binom{100}{8}}$ → 6+2=8 Why?

use calculator... & Stirling's approx if needed

What can these knobs be?

$N=0$? Absurd $n=0$

$N=1$? $\Rightarrow K=0$ or 1 ~~so $P(X=0)$~~

$\Rightarrow n=1 \Rightarrow X \sim \text{Deg}(0)$ or (1)

$N=2$ if $n=2$ $X \sim \text{Deg}(K)$ you get everything!

if $n=1$

$K=0 \Rightarrow X \sim \text{Deg}(0)$

$K=2 \Rightarrow X \sim \text{Deg}(1)$

$K=1$

$X \sim \text{Hyper}(1,1,2) = \dfrac{\binom{1}{x}\binom{1}{1-x}}{\binom{2}{1}} \Rightarrow \begin{matrix} P(X=0)=\frac{1}{2} \\ P(X=1)=\frac{1}{2} \end{matrix} = \text{Bern}\left(\frac{1}{2}\right)$ Why?

$N=3$

$K=1,2$

$n=1,2$ Parameter space

$\Rightarrow \boxed{\begin{matrix} N \in \mathbb{N} \setminus \{1\} \\ K \in \{1,2,\dots,N-1\} \\ n \in \{1,2,\dots,N-1\} \end{matrix}}$

It must be this...

$X \sim \text{Hyper}(1,K,N) = \text{Bern}\Big(\underbrace{\tfrac{K}{N}}_{=p}\Big)$ $\text{Supp}(X) = \{0,1\}$

Proof

$= \dfrac{\binom{K}{x}\binom{N-K}{1-x}}{\binom{N}{1}} =$

$P(X=0) = \dfrac{\binom{K}{0}\binom{N-K}{1}}{N} = \dfrac{N-K}{N} = 1-\dfrac{K}{N}$ "$1-p$"

$P(X=1) = \dfrac{\binom{K}{1}\binom{N-K}{1-1}}{N} = \dfrac{K}{N}$ "p"

general model

$X \sim \text{Hyper}(n,K,N)$ $\text{Supp}(X)$?

(a) $X \sim \text{Hyper}(2, 4, 10) \Rightarrow \text{Supp}[X] = \{0, 1, 2\}$

(b) $X \sim \text{Hyper}(5, 4, 10) \Rightarrow \text{Supp}(X) = \{0, 1, 2, 3, 4\}$ why $5 \notin \text{Supp}$?

(c) $X \sim \text{Hyper}(8, 4, 10) \Rightarrow \text{Supp}(X) = \{2, 3, 4\}$ why $1 \notin \text{Supp}$?

(d) $X \sim \text{Hyper}(5, 7, 10) \Rightarrow \text{Supp}(X) = \{2, 3, 4, 5\}$ why $6 \notin \text{Supp}$?

4 cases of $X \sim \text{Hyper}(n, K, N)$

(a) $n \leq K,\ n < N-K$ choose less than # successes & $n <$ # failures

$\text{Supp}(X) = \{0, 1, \ldots, n\}$

(b) $n \geq K,\ n < N-K$

$\text{Supp}(X) = \{0, 1, \ldots, K\}$

(c) $n \geq K,\ n \geq N-K$

$\text{Supp}(X) = \{n-(N-K), \ldots, K\}$

(d) $n < K,\ n \geq N-K$

$\text{Supp}(X) = \{n-(N-K), \ldots, n\}$

	$n < K$	$n \geq K$
$n < N-K$	$\{0, \ldots, n\}$	$\{0, \ldots, K\}$
$n \geq N-K$	$\{n-(N-K), \ldots, n\}$	$\{n-(N-K), \ldots, K\}$

$$\text{Supp}[X] = \{\max(0, n-(N-K)), \ldots, \min(n, K)\}$$

$$\sum_{x \in \text{Supp}(X)} p(x) = 1\ ?$$ Hard... HW...

"Equivalent parameterization"

let $p = \frac{K}{N} \Rightarrow K = pN$

$$X \sim \text{Hyper}(n, p, N) := \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}}$$

$N = 9$, $p = 0.7$ not legal! why $K = 6.3$ makes no sense

param space

$N \in \mathbb{N} \setminus \{1\}$

$n \in \{1, \ldots, N-1\}$

$p \in \left\{\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\right\}$

Consider $p = 0.5$, $n = 6$, $N = 100$

$$P(X=3) = \frac{\binom{50}{3}\binom{50}{3}}{\binom{100}{6}} = .3223$$

Now $N = 1000$

$$P(X=3) = \frac{\binom{500}{3}\binom{500}{3}}{\binom{1000}{6}} = .3134$$

$N = 10{,}000$

$$P(X=3) = \frac{\binom{5000}{3}\binom{5000}{3}}{\binom{10000}{6}} = .3126$$

$\downarrow$

Converge?

$$\lim_{N\to\infty} P(X=3) = \,?$$

Generally, what is the limiting r.v.?

$X \sim \text{Hypergeom}(n, p, N)$ and $N \to \infty$

$$\lim_{N\to\infty} p(x) = \lim_{N\to\infty} \frac{\binom{pN}{x}\binom{(1-p)N}{n-x}}{\binom{N}{n}} = \lim_{N\to\infty} \frac{\frac{(pN)!}{x!\,(pN-x)!}\,\frac{((1-p)N)!}{(n-x)!\,((1-p)N-n+x)!}}{\frac{N!}{(N-n)!\,n!}} = \frac{\frac{1}{x!\,(n-x)!}}{\frac{1}{n!}} \lim_{N\to\infty} \frac{\frac{(pN)!}{(pN-x)!}\,\frac{((1-p)N)!}{((1-p)N-n+x)!}}{\frac{N!}{(N-n)!}}$$

$$\frac{\frac{1}{x!\,(n-x)!}}{\frac{1}{n!}} = \frac{n!}{x!\,(n-x)!} = \binom{n}{x}$$

factor out constants

$$\lim_{x\to\infty} a f(x) = a \lim_{x\to\infty} f(x)$$

$$= \binom{n}{x} \lim_{N\to\infty} \frac{\overbrace{(pN)(pN-1)(pN-2)\cdots(pN-x+1)}^{x \text{ terms}}\overbrace{((1-p)N)((1-p)N-1)((1-p)N-2)\cdots((1-p)N-n+x+1)}^{n-x \text{ terms}}}{\underbrace{(N)(N-1)(N-2)\cdots\cdots(N-n+1)}_{n \text{ terms}}}$$

Split up n terms with a limit $\quad \lim f(x)g(x) = \lim f(x) \lim g(x)$

$$= \binom{n}{x} \underbrace{\underbrace{\lim_{N\to\infty} \frac{pN}{N}}_{p} \underbrace{\lim \frac{pN-1}{N-1}}_{p} \cdots \underbrace{\lim \frac{pN-x+1}{N-x+1}}_{p}}_{x} \cdot \underbrace{\underbrace{\lim \frac{(1-p)N}{N-x}}_{1-p} \cdot \underbrace{\lim \frac{(1-p)N-1}{N-x-1}}_{1-p} \cdots \underbrace{\lim \frac{(1-p)N-n+x+1}{N-n+1}}_{1-p}}_{n-x}$$

$$= \binom{n}{x} p^x (1-p)^{n-x} =: p(x)$$

$X \sim \text{Binomial}(n, p)$

$$\text{Hypergeometric}(n, p, N) \longrightarrow \text{Binomial}(n, p)$$

Recall $\lim_{N\to\infty} \frac{N^{P_K}}{N^K} = 1$ sampling without replacement is the same as sampling with replacement if N large

$\text{Supp}[X] = \{0, 1, \dots, n\}$

Parameter Space

$n \in \mathbb{N}$

$p \in (0, 1)$

$\boxed{0^0 := 1}$?

this is one of the motivations

$\binom{n}{x} 0^x 1^{n-x}$ $\quad P(X=0) = \binom{n}{0} 0^0 1^n$

$X \sim \text{Binomial}(n, 0) = \text{Deg}(0)$

$X \sim \text{Binomial}(n, 1) = \text{Deg}(n)$

$X \sim \text{Binomial}(1, p) = \underbrace{\binom{1}{x}}_{1} p^x (1-p)^{1-x} = p^x (1-p)^{1-x} = \text{Bern}(p)$

$\text{Supp}(X) = \{0, 1\}$ $\quad \binom{1}{0} = 1, \binom{1}{1} = 1$ $\quad$ of course it is!