12

x , X , probability statement , PMF

prob. that you make a realization of this X and get x=5'11"

$P(X=5'11") = ?$ — r.v. for human adult height

We have to specify a model for X.
If no model, this is meaningless.

$p(5'11") = P(X=5'11") = PMF$ — you need a function that links an element of support to the probabilities that come out of it, without such function you can not answer this question

$X \sim Binomial(n,p) := \binom{n}{x} p^x (1-p)^{n-x}$ — sample size; prob. of success; 1-p = prob of failure; n-x = # of failures; x = # of successes; p = prob. of success

Average r.v.

Let $T_n = X_1 + X_2 + ... + X_n$ — total r.v. aka sum r.v.

$\bar{X}_n = \frac{T_n}{n} = \frac{X_1 + ... + X_n}{n} = \frac{1}{n}\sum_{i=0}^{n} X_i$ — average r.v. (upper case X); n - sample size as in total number of conducted trials

Flip an unfair coin 3 times. Head=Success with prob. = 0.1. How many Heads?

e.g. $X_1, X_2, X_3 \overset{iid}{\sim} Bernoulli(0.1) = T_3 \sim Binomial(3, 0.1)$ — A cup with 3 (unfair) coins. Head=Success with prob. = 0.1. Shake then capsize the cup. How many Heads?

$T_3 \sim$
- 0 w.p. $\binom{3}{0}(.1)^0(1-p)^3 = 0.729$
- 1 w.p. $\binom{3}{1}(.1)^1(1-p)^{3-1} = 0.243$
- 2 w.p. $\binom{3}{2}(.1)^2(1-p)^{3-2} = 0.027$
- 3 w.p. $\binom{3}{3}(.1)^3(1-p)^{3-3} = 0.001$

Σ=1; e.g. HTH 2 Heads

$Supp[T_3] = \{0,1,2,3\}$ (zero 1s, one 1s, two 1s, three 1s); H → 1

note: the distribution of an average r.v. $\bar{X}$ is different than that of a total r.v. T_n

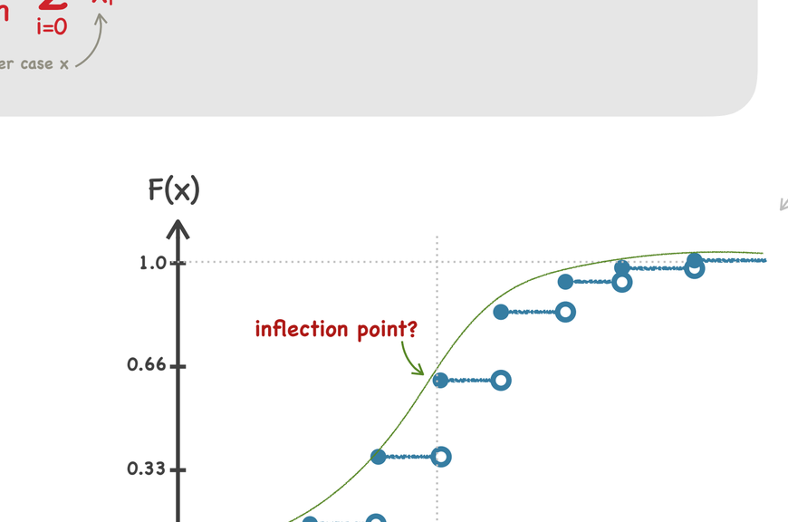
$\bar{X}_3 \sim$
- 0 w.p. $\binom{3}{0}(.1)^0(1-p)^3 = 0.729$
- 1/3 w.p. $\binom{3}{1}(.1)^1(1-p)^{3-1} = 0.243$
- 2/3 w.p. $\binom{3}{2}(.1)^2(1-p)^{3-2} = 0.027$
- 1 w.p. $\binom{3}{3}(.1)^3(1-p)^{3-3} = 0.001$

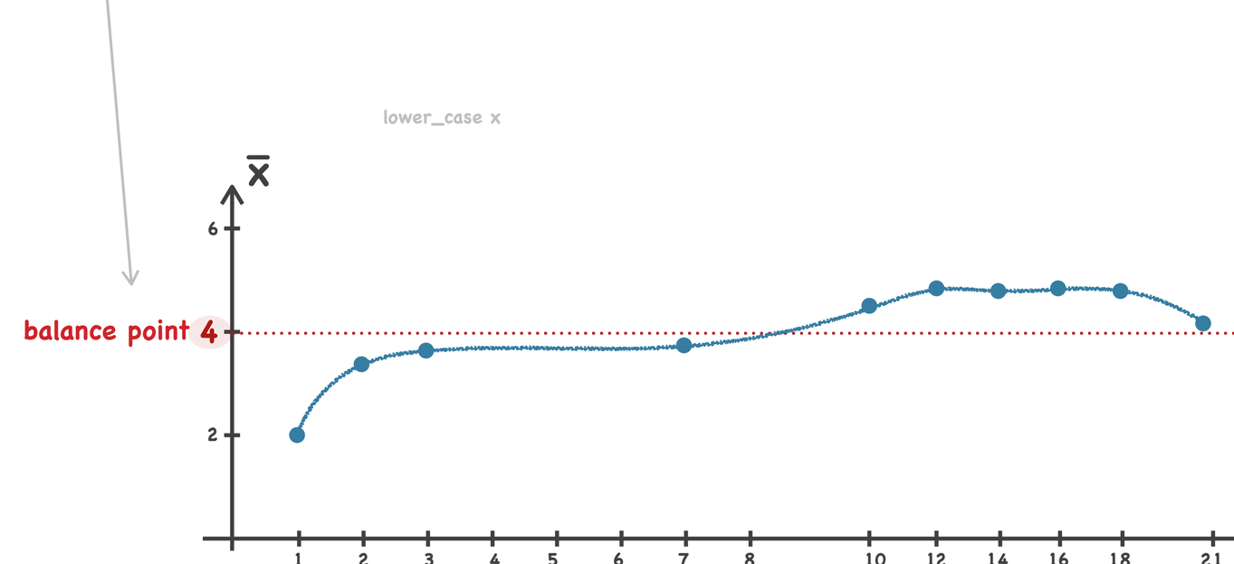
Σ=1; e.g. HTH 2 out of 3 $\frac{1+0+1}{3}$

$Supp[\bar{X}_3] = \{\frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}\}$

Def. **sample average $\bar{x}$ is a realization of $\bar{X}$** (lower case x; upper case X)

$\bar{x} := \frac{1}{n}\sum_{i=0}^{n} x_i$ (lower case x)





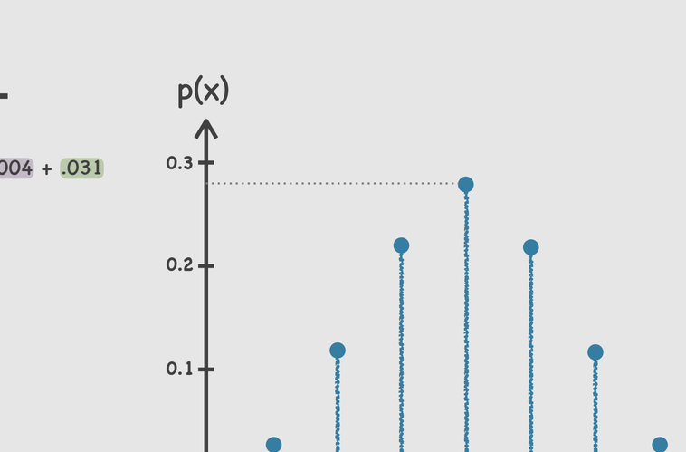
1 cup: 8 fair coins , n=sample_size=8
shake then capsize the cup, how many HEADS ?

$X \sim Binomial(8, 1/2) := \binom{8}{x}(1/2)^x(1-1/2)^{8-x} = \binom{8}{x}(1/2)^8 = \frac{\binom{8}{x}}{2^8} = \frac{8!}{2^8}\frac{1}{x!(8-x)!}$ — $(1/2)^8 \leftarrow x^n x^m = x^{n+m}$

why equal ? $\binom{8}{3} = \binom{8}{5}$ (56 = 56)

x	p(x) (PMF)	F(x) (CDF)
0	.004	.004
1	.031	.035 = .004 + .031
2	.109	.144
3	.219	.363
4	.273	.636
5	.219	.855
6	.109	.964
7	.031	.995
8	.004	1
	Σ=1	

the sum over the support of the PMF is 1, otherwise it ain't no PMF :-)



1 cup: 8 coins , n=sample_size=8
shake then capsize the cup, how many HEADS ? - repeat to ∞

$X_1, X_2 ... \sim Binomial(8, 1/2)$

realization build an empirical PMF and an empirical CDF. Show that they are about the same.

n : (# of trials)	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x : (realizations of X)	2	5	4	4	3	5	4	5	6	5	7	4	5	3
x̄ : (sample average)	2	3.5	3.67				3.86			4.3		4.5		4.429

$\frac{x_1+x_2+x_3}{n} = \frac{2+5+4}{3}$

n :	15	16	17	18	19	20	21	...
x :	5	5	1	4	3	3	4	Supp[X]={0,...,8} (no Hs, all Hs)
x̄ :		4.5		4.3			4.14	≈ 4 balance point

after infinitely many trials the average number of Heads for all of the trials is 4

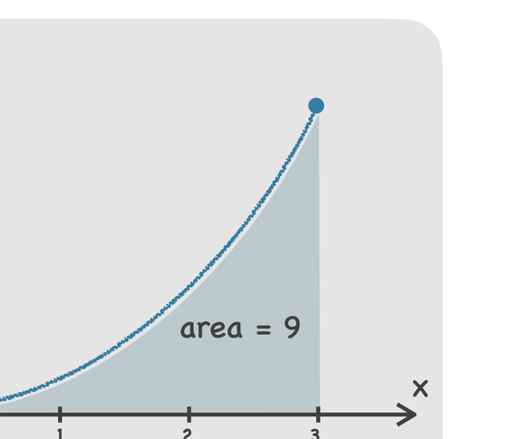
$\bar{X}$ sample average lower_case x

Mean

calculus

$f(x) = x^2 \quad x \in [0,3]$

function that takes a function $G[f] = \int_0^3 x^2 dx = \frac{x^3}{3}\Big]_0^3 \; 9 - 0 = 9$



an integral is a function of a function

G: function → ℛ (a single # e.g. 9)

it appears $\bar{x}$ (sample average) → balance point or pivot point

physics w = weight, d = distance

where is the balance point ?

$\sum_i w_i(d_i - d^*) = 0$

$\sum_i w_i d_i - \sum_i w_i d^* = 0$

$\sum_i w_i d_i = d^* \sum_i w_i$

$\frac{\sum_i w_i d_i}{\sum_i w_i} = d^*$ — analog: weights = probabilities PMF; distance = x — $\frac{\sum_i p_i x_i}{\sum_i p_i} = x^*$ ($p(x)$; denominator =1)

$\frac{0 \cdot 100 + 1 \cdot 20}{100+20} = 0.17$

summing all of the probabilities of the Support of a PMF = 1 — remember? $X \sim$ {1 w.p. 0.6; 0 w.p. 0.3; 2 w.p. 0.1}, Σ=1

E[X] is another notation for μ - function of a function (interchangeable)

$E[X] := \mu := \sum_{x \in Supp[X]} x\, p(x)$ — E for Expectation; mu

E: X → ℛ (space of r.v.'s → a single #)

mean , expected value , expectancy , first moment

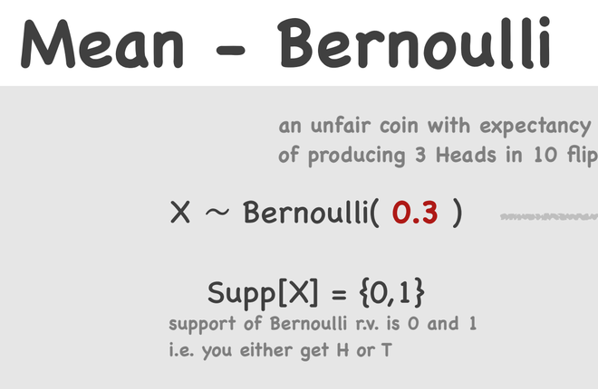
Law of Large #'s $\bar{X} \to \mu$ — you take the average of a lot of things and you get the expected value

Mean - Bernoulli

an unfair coin with expectancy of producing 3 Heads in 10 flips

$X \sim Bernoulli(0.3)$

$Supp[X] = \{0,1\}$ — support of Bernoulli r.v. is 0 and 1 i.e. you either get H or T



0.3 ∈ Supp[X] ? - NO, 0.3 is neither 0 nor 1
in general **μ ∉ Supp[X]**

X (r.v. function) taken through E (mean function) returns p (prob.)

E[X] = 0·p(0) + 1·p(1) = p(1) = 0.3 — or **p** for general Bernoulli

in the case of a single Bernoulli experiment E[X] aka μ = p
X (r.v. function) taken through E (mean function) returned p (prob.)

$E[X \sim Bernoulli(p) := p^x(1-p)^{1-x}] = \mu = p$

how about a 'mean' of a multiple iid Bernoulli experiments ?

$X_1, X_2, ..., X_8 \overset{iid}{\sim} Bernoulli(0.3) = X \sim Binomial(8, 0.3)$ — E[X] ≈ 2.4 (8 · 0.3 = 2.4)

$E[X] = \sum_{x=0}^{8} x\,p(x) = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 4 \cdot p(4) + 5 \cdot p(5) + 6 \cdot p(6) + 7 \cdot p(7) + 8 \cdot p(8)$ — x∈Supp[X]={0,...,8}

$\binom{8}{5}(0.3)^5(0.7)^3$ = 56 · .00243 · .343; $\binom{8}{8}(0.3)^8(0.7)^0$ = 1 · 0.000066 · 1

= 0·0 + 1·.198 + 2·.2965 + 3·.2541 + 4·.1361 + 5·.04668 + 6·.01 + 7·.0012 + 8·.000066
= 0 + .198 + .593 + .7624 + .5445 + .2333 + .06 + .00857 + .0005
≈ 2.4

How do we get the 'balance point'? We sum the products of all the values in the support of X and their corresponding probabilities. The above example had only 9 values in the support of X. However, the volume of calculations was quite large. Imagine Binomial(1000,0.3). Do we really need to perform 1000 product summations ?
Or is there a general formula which will allow us to skip this laborious process ?

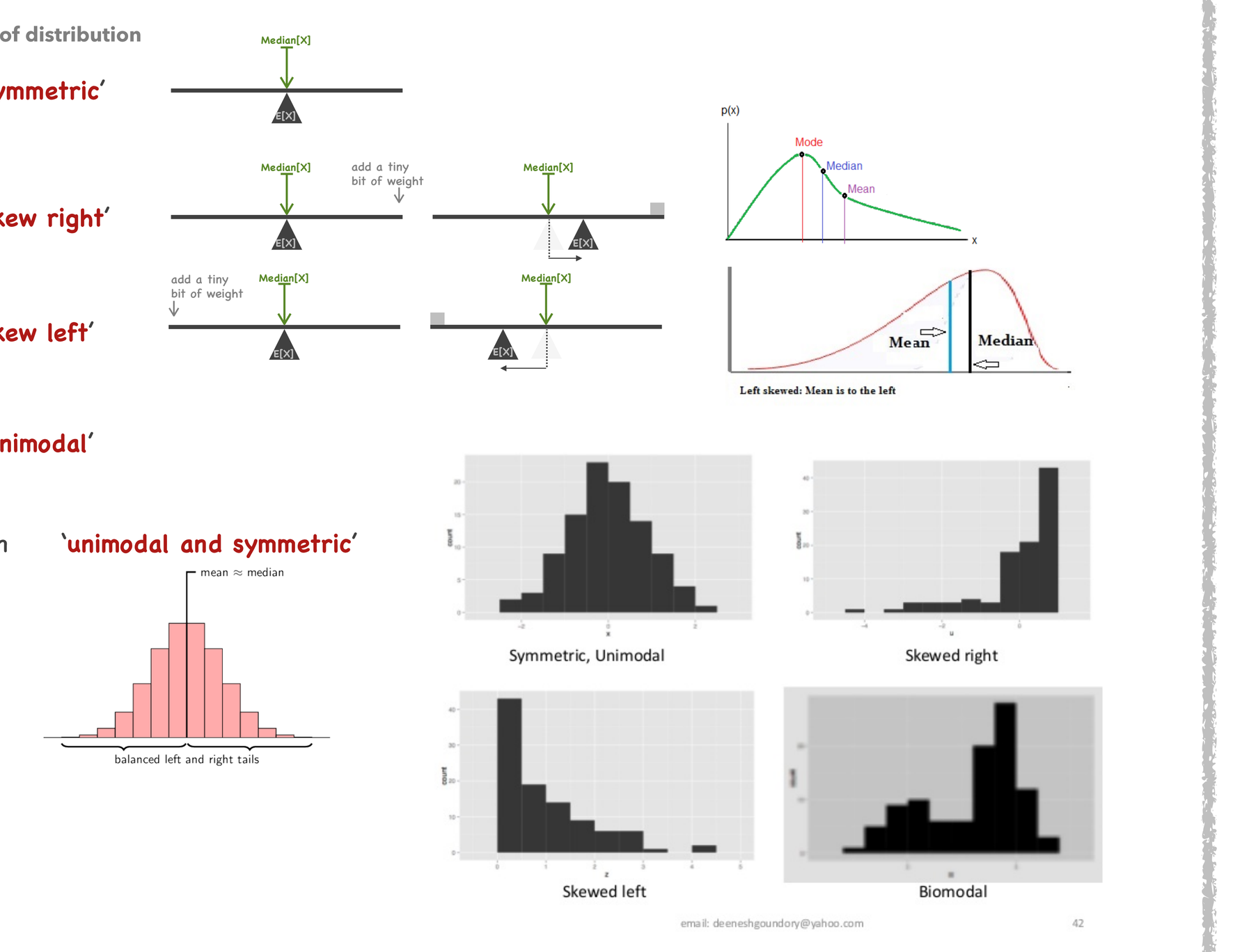
Warning!
Median and the Mean are NOT the same.
Median[X] ≠ E[X]

Median and the Mean are both measures of 'central tendency'

Median - says that there is 50% of the support below 4 and 50% above 4

Mean - says that if I balance a thing, where is the 'balancing point'. In our example it's 5

condition	types of distribution
Mean = Median (E[X] = Median[X])	'symmetric'
Mean > Median (E[X] > Median[X])	'skew right'
Mean < Median (E[X] > Median[X])	'skew left'
| Mode[X] | = 1 (one mode)	'unimodal'
Mode[X] = Mean = Median (Mode[X] = E[X] = Median[X])	'unimodal and symmetric'



Custom r.v.s

Ever wondered how to build custom r.v.s?

Bet on Black. Payout is 1:1 (bet $1 win $1) — Roulette in U.S. (total of 38 pockets: 18 black, 18 red, 2 green 0 , 00)

Rademacher $X \sim$ {\$1 w.p. 18/38; -\$1 w.p. 20/38} r.v. model of the payout of this bet

what is the expectation ?

$E[X] = \sum_{x \in Supp[X]} x\,p(x)$ = \$1·p(\$1) + (-\$1)·p(-\$1) = \$1·18/38 + (-\$1)·20/38 = - \$0.053

meaning if you play many times on average you are loosing ≈5.3 cents (per play)

the more you play the more you loose in a long run you'll loose everything - you can't win

$X_1, X_2, ..., X_n \overset{iid}{\sim}$ {\$1 w.p. 18/38; -\$1 w.p. 20/38}

$\lim_{n \to \infty} T_n = -\infty$

Bet on 'lucky' # 7. Payout is 35:1 (bet $1 win $35) — Roulette in U.S. (total of 38 pockets: 18 black, 18 red, 2 green 0 , 00)

$X \sim$ {\$35 w.p. 1/38; -\$1 w.p. 37/38} r.v. model of the payout of this bet

what is the expectation ?

$E[X] = \sum_{x \in Supp[X]} x\,p(x)$ = \$35·p(\$35) + (-\$1)·p(-\$1) = \$35·1/38 + (-\$1)·37/38 = - \$0.053

Expectation is a 'long term' property and it means very little when you have only few random variables.

hint: - play video draw poker duces wild
- black jack you loose 0.5% on average

Bet on 'dozen' # 1...12 Payout is 2:1 (bet $1 win $2) — Roulette in U.S. (total of 38 pockets: 18 black, 18 red, 2 green 0 , 00)

$X \sim$ {\$2 w.p. 12/38; -\$1 w.p. 26/38} r.v. model of the payout of this bet

what is the expectation ?

$E[X] = \sum_{x \in Supp[X]} x\,p(x)$ = \$2·p(\$2) + (-\$1)·p(-\$1) = \$2·12/38 + (-\$1)·26/38 = - \$0.053

Bet on 'dozen' # 1...12 Payout is 2:1 (bet $1 win $2) — Roulette in Europe (total of 37 pockets: 18 black, 18 red, 1 green 0)

$X \sim$ {\$2 w.p. 12/37; -\$1 w.p. 25/37} r.v. model of the payout of this bet

what is the expectation ?

$E[X] = \sum_{x \in Supp[X]} x\,p(x)$ = \$2·p(\$2) + (-\$1)·p(-\$1) = \$2·12/37 + (-\$1)·25/37 = - \$0.027

in a long run you will loose half the money playing the European Roulette over the US version.

if X models a payout of a game, "Fair Game" is if E[X] = 0

Basic r.v. transportation - Uber example

P(traffic on route A) = 0.3 — If there is traffic on route A, take route B.
- route A is 7min
- route B is 12min

$W \sim$ {7min w.p. 0.7; 12min w.p. 0.3} not a Bernoulli b/c Supp[X] ≠ {0,1}

$E[W] = \sum_{x \in Supp[X]} x\,p(x)$ = 7min · p(7min) + 12min · p(12min) = 7min · 0.7 + 12min · 0.3 = 8.5 min

- on average in all my trips the time spend in the taxi is ≈8.5 min
- this exact value pertains to an infinite amount of trips

$X_1, X_2, ..., X_8 \overset{iid}{\sim} Bernoulli(1/2) = X \sim Binomial(8, 1/2)$ — E[X] ≈ 4 (8 · 0.5 = 4)

self-note:
Conducting a single Bernoulli experiment (e.g. flip of a coin) means converting a theoretical r.v. X~Bernoulli into its practical 'realization' x equal to either 0 or 1. Shaking a cup of 8 coins is analogous to performing 8 simultaneous iid Bernoulli experiments which can also be expressed by X~Binomial. Every time we shake this cup of 8 coins, we can expect the number of Heads to be somewhere between 0 and 8. Since we know that there is 50% chance of getting a Head, it is reasonable to expect about 4 Heads per trial. This 'expectation' of 4 Heads is known under multiple names: 'mean' , 'expectation', 'first moment', and is denoted by the lower case greek letter μ or as E[X]. Performing the above 8 iid Bernoulli's can be mathematically expressed as the product of 8-flips and prob. of getting a Head. 8·0.5=4

$E[X] = \sum_{x=0}^{8} x\,p(x) = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 4 \cdot p(4) + 5 \cdot p(5) + 6 \cdot p(6) + 7 \cdot p(7) + 8 \cdot p(8)$

$\binom{8}{5}(0.5)^8$ = 56 · .0039; $\binom{8}{8}(0.5)^8$ = 1 · 0.0039

= 0·0.004 + 1·.031 + 2·.109 + 3·.219 + 4·.273 + 5·.219 + 6·.109 + 7·.031 + 8·.004
= 0 + .031 + .218 + .657 + 1.092 + 1.095 + .654 + .217 + .032
≈ 4

balance point aka mean aka expectation E[X] aka μ

n · p = 8 · 0.5 = 4

end of class experiments...

1 cup: 7-coins , 4-marked , n=sample_size=3
shake the cup, quarry 3 coins, how many Marked?

$n\frac{K}{N} = 3\,\frac{4}{7} = 1.714$ — P → n p = Binomial

X ~ Hypergeometric(n,K,N) (3,4,7)

n :	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x :	1	2	1	2	2	2	2	2	3	2	1	2	1	2
x̄ :	1	1.5	1.33				1.71							1.786

n :	15	16	17	18	19	20	21	...
x :	2	3	1	2	2	1	0	Supp[X]={0,...,3}
x̄ :							1.714	≈ 1.714 balance point

1 cup: 6-coins , 2-marked , n=sample_size=1
shake the cup, quarry 1 coin, is it Marked?
X ~ Bernoulli(2/6) - balance point 2/6

1 cup: 6-coins , 4-marked , n=sample_size=1
shake the cup, quarry 1 coin, is it Marked?
X ~ Bernoulli(4/6) - balance point 4/6

∞ cups - in each cup: 6-coins , 2-marked , n=sample_size=1
shake the cups, stop on the 1st Marked coin:
quarry 1 coin, from the 1st cup , is it Marked? - NO
quarry 1 coin, from the 2nd cup , is it Marked? - NO
quarry 1 coin, from the 3rd cup , is it Marked? - Yes , STOP
(realization x=3)

X ~ Geometric(2/6)

n :	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x :	3	2	1	3	3	2	5	8	3	2	1	9	1	5
x̄ :		2.5	2				2.71							3.43

n :	15	16	17	18	19	20	21	...
x :	2	3	1	2	2	1	5	Supp[X]={1,...}
x̄ :							3.05	≈ 3 balance point

the alternative way to conduct this experiment:
1 cup with 6 coins in it. 2 of the coins are marked.
- shake the cup then quarry 1 coin, is it Marked ? - NO
- shake the cup then quarry 1 coin, is it Marked ? - NO
- shake the cup then quarry 1 coin, is it Marked ? - NO
- shake the cup then quarry 1 coin, is it Marked ? - Yes → STOP
record the number of shakes, repeat the entire precess again

the alternative way of looking at this experiment:
The cup can be viewed as a single entity capable to deliver a marked coin with the probability of 1/3. It is as if we had a single 'unfair coin' with probability of Heads = 1/3. We keep flipping this 'unfair coin' until we get Heads. We record the number of flips. We repeat the experiment again.

$X \sim Bernoulli(1/3) :=$ {1 wp 1/3 (Heads = marked coin); 0 wp 2/3}

Thus a single trial means that we keep conducting iid Bernoulli experiments over and over until we get the first success which is the 'stopping time' for each trial.

∞ cups - in each cup: 6-coins , 4-marked , n=sample_size=1
shake the cups, stop on the 3rd Marked:
quarry 1 coin, from the 1st cup , is it Marked? - Yes
quarry 1 coin, from the 2nd cup , is it Marked? - NO
quarry 1 coin, from the 3rd cup , is it Marked? - Yes
quarry 1 coin, from the 4th cup , is it Marked? - NO
quarry 1 coin, from the 5th cup , is it Marked? - Yes , STOP
(realization x=5)

X ~ Negative Binomial(r,p)(3,4/6) (r: stopping time)

n :	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x :	5	9	6	3	7	6	5	4	6	7	5	5	3	6
x̄ :		7	6.66				5.86							5.5

n :	15	16	17	18	19	20	21	...
x :	3	5	4	5	4	5	3	Supp[X]={3,...}
x̄ :							5.05	≈ 4.5 balance point = r/p

In the above NegBinomial experiment, a single trial means that we keep conducting iid Bernoulli experiments over and over until we get the r^{th} success which is the 'stopping time' for each trial.

Mean - Binomial

based on the above revelation:

if E[X~Bernoulli(p)] = p , E[X~Binomial(n,p)] = np ????

Lets put the Binomial through the Mean function and see what comes out.

$E[X] = \sum_{x \in Supp[X]} x\,p(x) = \mu = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$ | expand the choose notation

$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$ | $x! = x \cdot (x-1)!$ — x∈Supp[0,...,n]

$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x}$ | $(n-x)! = [(n-1)-(x-1)]! = (n-1-x+1)!$ — 0 · smthg = 0

$= n \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)![(n-1)-(x-1)]!} p^x (1-p)^{(n-1)-(x-1)}$ | express in choose notation — 6! = 6·5!; n = n-1, k = x-1; $\frac{n!}{k!(n-k)!} = {}_nC_k = \binom{n}{k}$

$= n \sum_{x=1}^{n} \binom{n-1}{x-1} p^x (1-p)^{(n-1)-(x-1)}$ | create homogeneous terms

$= n\,p \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{(n-1)-(x-1)}$ — $p^x = p^1 \cdot p = p^{x-1} \cdot p$ — let: m = n-1, y = x-1; x = 1...n, y = 0...n-1 = 0...m

$= n\,p \sum_{y=0}^{m} \binom{m}{y} p^y (1-p)^{m-y} = 1$ (as previously proved)

$= n\,p$

$E[X \sim Binomial(p) := \binom{n}{x} p^x (1-p)^{n-x}] = \mu = n\,p$

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Warning! Mode and Min are NOT the same

Min[X] = min{ Supp[X] } = 1 — the smallest value in the support

look at all the probabilities find the largest one and tell me who is responsible — it's the most likely value you will get

Mode[X] = arg_max{ p(x) } = 1 ($x \in Supp[X]$)

Quantile[X,p] := arg_min{ F(x)≥p } → (special case of Quantile) **Median**[X,0.5] = 4
aka
Percentile if p is a percent — 70th Percentile = 6; 90th Percentile = 11
e.g. 0.2·100 = 20%

what is a 95th percentile ?
Lets say 95% on the exam was 17. It means that at the most 5% students scored higher than 17, and at the minimum 95% of the students scored below 17.

Tertiles	Quartiles	Quintiles	Deciles
Quantile[X,0.33]	Quantile[X,0.25]	Quantile[X,0.20]	Quantile[X,0.10]
Quantile[X,0.66]	Quantile[X,0.50]	Quantile[X,0.40]	Quantile[X,0.20]
	Quantile[X,0.75]	Quantile[X,0.60]	...
		Quantile[X,0.80]	Quantile[X,0.90]

75th percentile - 25th percentile
IQR = Q[X,0.75] - Q[X,0.25] = 7 - 2 = 5
represents middle 50% of data

Max[X] = max{ Supp[X] } = d.n.e. — the largest value in the support

Range[X] = Max[X] - Min[X] = d.n.e.

Mean - Geometric

$X \sim Geometric(0.2) := (1-0.2)^{x-1}\, 0.2$

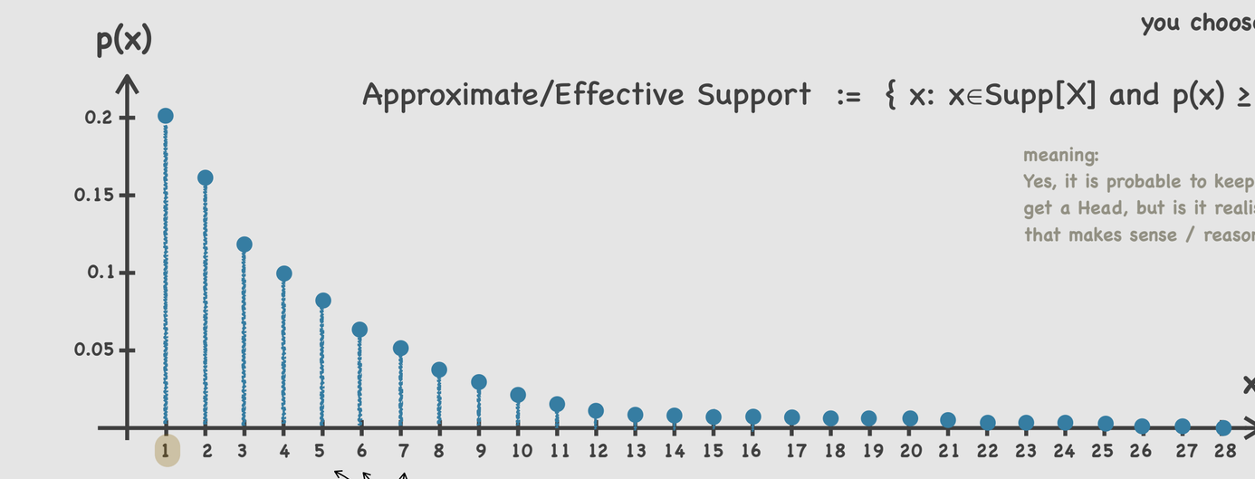
$0.8^{x-1}\,0.2$

x	p(x)	F(x)
1	.200	.200
2	.160	.360 = .200 + .160
3	.120	.480
4	.102	.582
5	.082	.664
6	.066	.730
7	.052	.782
8	.042	.824
9	.037	.861
10	.027	.888
11	.021	.909
12	.017	.926
13	.014	.940
14	.011	.951
15	.009	.960
16	.007	.967
17	.006	.973
18	.005	.978
19	.004	.982
20	.003	.985
21	.002	.987
22	.001	.988
23	.001	.989
24	.001	.990
25	.001	.991
26	.001	.992
27	.000	.992
28	.000	.993
...	Σ≈1	

Supp[X] = {1,...} — Why ≠ 1 ? In theory we could keep trying to ∞ and never get our success which is that necessary stopping factor.

Approximate/Effective Support := { x: x∈Supp[X] and p(x) ≥ .001 } (you choose)

meaning: Yes, it is probable to keep flipping that proverbial coin and never get a Head, but is it realistic/believable ? Choose a value for p(x) that makes sense / reasonably realistic.



where is the balance point - E[X] - μ

$\mu = \sum_{x \in Supp[X]} x\,p(x) \to \sum_{x=1}^{\infty} x\,(1-p)^{x-1}p$ | pull the const p — Geometric PMF - p(x); x∈Supp = [1,...]

$= p \sum_{x=1}^{\infty} x\,(1-p)^{x-1}$ — reindexing trick: let y=x-1 → x=y+1; x=1...∞ → y=0...∞

$= p \sum_{y=0}^{\infty} (y+1)(1-p)^y$ | split Σ

$= p\left[\sum_{y=0}^{\infty} y(1-p)^y + \sum_{y=0}^{\infty} 1\,(1-p)^y\right]$ — geometric series $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$

$= p\left[\sum_{y=0}^{\infty} y(1-p)^y + \sum_{y=0}^{\infty} 1\,\frac{1}{1-(1-p)}\right]$

$= p \sum_{y=0}^{\infty} y(1-p)^y + p \sum_{y=0}^{\infty} \frac{1}{p}$ | ps cancel

$= \sum_{y=0}^{\infty} y\,(1-p)^y\,p + 1$ | $x^3 = x^2 \cdot x^1 = x^{3-1} \cdot x^1$

$= \sum_{y=0}^{\infty} y\,(1-p)^{y-1}(1-p)^1\,p + 1$ | pull the constant $(1-p)^1$

$= (1-p)^1 \sum_{y=0}^{\infty} y\,(1-p)^{y-1}\,p + 1$

$= (1-p)^1\,\mu + 1$

$\mu = 1\mu - p\mu + 1$

$= -p\mu + 1$

$p\mu = 1$ — thus our balance point is = 5

$E[X \sim Geometric(p) := (1-p)^{x-1}p] = \mu = \frac{1}{p}$ — $\frac{1}{0.2} = 5$