Lecture 3 — September 1, 2016

- Working definition of Probability

$$P(A) = \frac{|A|}{|\Omega|} \quad \text{if } \forall \omega \quad P(\{\omega\}) = \frac{1}{|\Omega|}$$

"every outcome is equally likely to happen".

- Two coin flips

$|2^{\Omega}| \Rightarrow$ # of events possible

$$\Omega' = \Omega^2$$
$$\Omega = \{H, T\}$$

Ω'

<H,H>	<H,T>
<T,H>	<T,T>

$|\Omega'| = 4$

*What is probability of getting two heads?

$$P(\{\langle H,H \rangle\}) = P(H,H)$$
$$\downarrow$$
$$= \frac{1}{4}$$

*A: at least one H

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{4} = \frac{|\{\langle H,T\rangle, \langle T,H\rangle, \langle H,H\rangle\}|}{4} = \frac{3}{4}$$

*B: at least one T

$$P(B) = \frac{|B|}{|\Omega|} = \frac{|B|}{4} = \frac{|\{\langle H,T\rangle, \langle T,H\rangle, \langle T,T\rangle\}|}{4} = \frac{3}{4}$$

* $P(A \cup B) = \frac{4}{4} = 1$ (because have to get head or tail when flip coin)

"getting at least one heads or at least one tails"

* $P(A \cap B) = \frac{|\{\langle T, H\rangle, \langle H, T\rangle\}|}{4} = \frac{1}{2}$

"at least one head and at least one tail"

P(A and B) P(A, B)
P(A & B) P(AB)

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* 4 coin tosses

$\Omega' = \Omega^4$

$\Omega = \{H, T\}$

$|\Omega'| = 16$

Ω'

$\langle H,H,H,H\rangle$	$\langle T,T,T,T\rangle$	$\langle T,H,H,H\rangle$	

$|2^{\Omega'}|$ = # of events possible

* $P(HHHH) = \frac{1}{16}$

Not same thing

* $P(HTHT) = \frac{1}{16}$

* $P(2H, 2T) = \frac{|\{\overset{1}{\langle H,H,T,T\rangle}, \overset{2}{\langle T,T,H,H\rangle}, \overset{3}{\langle HTTH\rangle}, \overset{4}{\langle THHT\rangle}, \overset{5}{\langle HTHT\rangle}, \overset{6}{\langle THTH\rangle}\}|}{16} = \frac{6}{16}$

* A: at least one H

$P(A) = \frac{|A|}{16} = \frac{|\{\langle H,T,T,T\rangle \langle HTTH\rangle \ldots}{16}$ theres a lot of them.

Find probability of ~~there the one~~ getting no heads.

Use complement Rule $\leftarrow A^c = \{< 1\ H\} = \{\text{zero } H\}$

$1 - P(A^c)$

$\rightarrow 1 - \frac{1}{16} = \frac{15}{16}$

$\frac{|\{\langle T,T,T,T\rangle\}|}{16}$

* 10 coin flips

$\Omega' = \Omega^{10}$ ($\Omega = \{H,T\}$)

$|\Omega'| = 2^{10} \approx 1000$

$|2^{\Omega'}| = 2^{1000}$ = a big #. (# of events possible)

* $P(5H, 5T) = \dfrac{\text{a large \#??}}{2^{10}} = .2460938$ Answer (was given)

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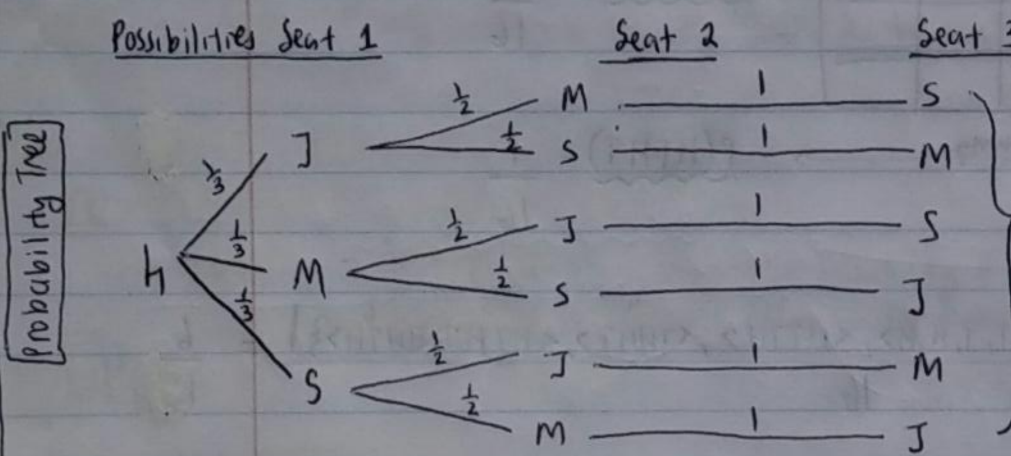
* $\Omega = \{\text{Jane, Mary, Susan}\}$ $(\Omega = \{J\,M\,S\})$

π, π, π → How many ways to order them?
3 chairs

$\{\underset{1}{\langle JMS \rangle}, \underset{2}{\langle MJS \rangle}, \underset{3}{\langle SJM \rangle}, \underset{4}{\langle S,M,J \rangle}, \underset{5}{\langle JSM \rangle}, \underset{6}{\langle MSJ \rangle}\}$



$|\Omega| = 6 \neq |\Omega^3| = 27$

sampling without replacement

sampling with replacement

* Another way

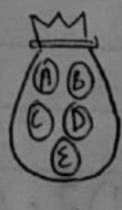
$\underset{\text{seat \#1}}{\underline{3}} \quad \underset{\text{seat \#2}}{\underline{2}} \quad \underset{\text{seat \#1}}{\underline{1}} = 3! = 6$

3 factorial

$\boxed{\cdot\ n! := \prod_{i=1}^{n} i}$

means product, big Π.

*



Factorials get big really fast:
$5! = 120$
$10! = 3.6$ M
$20! = 2.7 \times 10^{32}$

• 10 people, 3 chairs

$\#$ of orders $= 10 \cdot 9 \cdot 8 = \frac{10!}{7!} = \boxed{\frac{10!}{(10-3)!} = 10P_3} \rightarrow \boxed{nP_K := \frac{n!}{(n-k)!}}$ *

$\underset{\text{seat \#1}}{\underline{10}}$ $\underset{\text{seat \#2}}{\underline{9}}$ $\underset{\text{seat \#1}}{\underline{8}}$

permutation

$= 10! \stackrel{?}{=} 10P_{10} = \frac{10!}{(10-10)!} = \frac{10!}{0!} = \boxed{0! := 1}$ ← This a definition

• Example

Bob – Jane
Richard – Susan
Max – Alice

P(couples sit together) $= \frac{|A|}{|\Omega|} = \frac{48}{6!}$
↳ (P(A))

(Not just 3! b/c couples can switch on bench.)

$\underset{\text{seats 1\&2}}{\underline{3^{\,JB,\,BJ}}}$ $\underset{\text{seats 3\&4}}{\underline{2^{\,SR,\,RS}}}$ $\underset{\text{seats 5\&6}}{\underline{1^{\,AM,\,MA}}}$ $\stackrel{?}{=} 3! \cdot 2 \cdot 2 \cdot 2$

$\underbrace{3!} \cdot 2^3$

$6 \cdot 8 = 48$

another way

$\underset{1}{\underline{}}$ $\underset{2}{\underline{}}$ $\underset{3}{\underline{}}$ $\underset{4}{\underline{}}$ $\underset{5}{\underline{}}$ $\underset{6}{\underline{}}$

(white out)

*Another way to count sets

$\underbrace{\frac{6}{1}\ \frac{1}{2}}_{\text{couple}}\ \underbrace{\frac{4}{3}\ \frac{1}{4}}_{\text{couple}}\ \underbrace{\frac{2}{5}\ \frac{1}{6}}_{\text{couple}} = 6 \cdot 4 \cdot 2 = 48$

• $P\left(\text{alternating gender}\right) = \frac{}{6!}$

Bob Jane
Richard Susan
Max Alice

$\frac{3}{1}\ \frac{3}{2}\ \frac{2}{3}\ \frac{2}{4}\ \frac{1}{5}\ \frac{1}{6} = \underbrace{(3!)^2} * 2 = 72$

B

(*2 because can start off with a G)

$P(A) = P(A_1) + P(A_2) = P(A_1 \cup A_2)$ if A_1 & A_2 mutually exclusive

$P\left(\begin{smallmatrix}\text{alternating}\\ \text{gender}\end{smallmatrix}\right) = P(BGBGBG) + P(GBGBGB)$

→ $\underbrace{P(\text{BJ sit together})}_{P(A)} = \dfrac{}{6!}$

Answer: $\dfrac{10(4!)}{6!}$

$\underset{\text{seat 1\&2}}{\dfrac{1}{}}\ \dfrac{4}{3}\ \dfrac{3}{4}\ \dfrac{2}{5}\ \dfrac{1}{6} = \dfrac{(4!)2\cdot 5}{}$

BJ
JB — BJ can switch to JB

_ 1 _ 2&3 _ 4 _ 5 _ 6

_ 1 _ 2 _ 3&4 _ 5 _ 6

can sit in different bench order

- 100 ball select 3 without replacement : $\dfrac{100}{}\ \dfrac{99}{}\ \dfrac{98}{}$
- 100 ball select 3 with replacement : $\dfrac{}{100}\ \dfrac{}{100}\ \dfrac{}{100}$ $= \text{—} = .97$

- 10000 ball select 3 without replacement : $\dfrac{10000}{}\ \dfrac{9{,}999}{}\ \dfrac{9998}{}$
- " " w/ replacement : $\dfrac{}{10000}\ \dfrac{}{10000}\ \dfrac{}{10000}$ $= \text{—} = .9997$

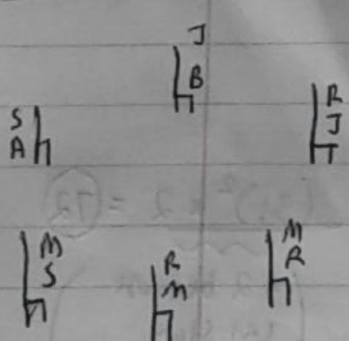
★ want to show

$$\lim_{n\to\infty} \frac{{}_nP_k}{n^k} = \frac{\frac{n!}{(n-k)!}}{n^k} = \frac{n(n-1)(n-2)\ldots(n-k+1)\}k}{\underbrace{n\cdot n\cdot n\cdot n \ldots \cdot n}_{k}}$$

$$= \lim \underset{1}{\frac{n}{n}} \cdot \lim \underset{1}{\frac{n-1}{n}} \cdot \lim \underset{1}{\frac{n-2}{n}} \cdots \lim \underset{1}{\frac{n-k+1}{n}} = \underline{\underline{1}}$$

$$\boxed{\lim f(x)\, g(x) = \lim f(x) \lim g(x) \text{ if } f, g \text{ cont.}}$$

- Back to couples example



can be rotated 6 times → $\dfrac{6!}{6}$

principle of dividing out invariance →

"same"
indistinguishable
nondistinct
non-unique

- Permutation: (order matters) → $_nP_r = \frac{n!}{(n-r)!}$ Choice of r things from set of n w/o replacement where order matters.

- Combination: order no matter. Choice of r things from set n w/o replacement → $_nC_r = \left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right] = \frac{_nP_r}{r!} = \frac{n!}{r!(n-r)!}$

- **Example: flowers**

5 flowers
3 orchids (O)
2 chrysantimums (X)

$O_1 O_2 O_3 X_1 X_2$
$O_1 O_3 O_2 X_1 X_2$
$O_2 O_1 O_3 X_1 X_2$
$O_2 O_3 O_1 X_1 X_2$
$O_3 O_2 O_1 X_1 X_2$
$O_3 O_1 O_2 X_1 X_2$

# of way to arrange 5 flowers is:

$\frac{5!}{3!}$ ← 5 permutations. 120 ways to arrange 3 distinct flowers.
↖ # of ways to arrange 3 orchids.

$\frac{5!}{2!}$ ← 5 permutations. 120 ways to arrange 5 distinct flowers.
↖ # of ways to arrange 2 chrysantimums

- **EX: 5 People, 3 Chairs.**

A, B, C, D, E, F

$\frac{6}{1}\ \frac{5}{2}\ \frac{4}{3} = 6 \times 5 \times 4 = 120$ permutations (120 ways to seat 6 people in 3 seats)

(care about whose sitting where. care about arrangement)

ABC	BAC	CAB
ACB	BCA	CBA

FBC	BFC	CFB
FCB	BCF	CBF

12 of 120 permutations

set of ways to arrange same 3 people

set of ways to arrange same 3 people

→ If I have 6 people sitting in 3 chairs, how many ways can I choose three people out of the 6 where dont care which chair sit on?

* 6 ways of arranging 3 people, certain group of 3 people.

→ How many different ways to choose 3 from the six?

* $\frac{120}{6}$ ← # of permutation / ← # of ways to arrange 3 people (3 letters) $= 20$

The # of ways to choose 3 people out of 6.