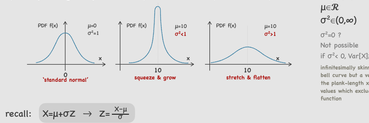
General Normal r.v. cont...

$X \sim \text{Normal}(\mu, \sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

by manipulating the parameters you can make any bell curve you want

$\text{Supp}[X] = \mathbb{R}$
$\mu \in \mathbb{R}$
$\sigma^2 \in (0, \infty)$
$\sigma^2 = 0$?
Not possible
if $\sigma^2 < 0$, $\text{Var}[X] \geq 0$ so not possible
infinitesimaly skinny bell curve is NOT a a bell curve but a vertical line , thus under the plant-length x would have multiple values which excludes a possibility of a function

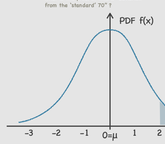


recall: $X = \mu + \sigma Z \rightarrow Z = \frac{X-\mu}{\sigma}$

I can take any normal r.v. and do this linear transformation and get standard normal
e.g.
X is a rand. var model for male height. X is normally distributed with mean-μ=70" and SE-σ= 4". What is the probability a male is more than 78" tall ?

typical test question

model: $X \sim \text{Normal}(70", 4"^2)$ (μ, σ^2)
$X_{new} = 70" + 4"Z \rightarrow Z = \frac{X-70"}{4} \rightarrow Z \sim \text{Normal}(0, 1)$
standardization
4 is far more important than 16
probability statement: $P(X \geq 78") = P(\frac{X-70"}{4} \geq \frac{78"-70"}{4}) = P(Z \geq 2) \approx 2.5\%$
what is the prob. of this 8" deviation from the 'standard' 70" ?
subtract the mean and divide by S.E.
why is Height 'normally' distributed ? - next lecture



MGF - Moment Generating Function

$M_X(t) := E[e^{tX}] = \begin{cases} \sum_{x \in \text{supp}[X]} e^{tx} p(x) & \text{if discrete r.v.} \\ \int_{x \in \text{Supp}[X]} e^{tx} f(x)\, dx & \text{if continuous r.v.} \end{cases}$

g(x)

MGF - Moment Generating Function of r.v. X
which is an Expectation of g(X)

property 1

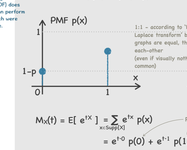
if $M_X(t) = M_Y(t) \rightarrow X \overset{d}{=} Y$ identically distributed r.v.s have the same MGF
$M_X(t)$ is 1:1 with PDF or PMF

if two MGFs are equal, then X & Y are 'equal in distribution' (their PMFs or PDFs are equal)

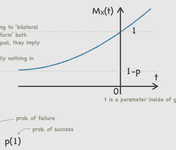
self-note:
its like having two different pointers to the same memory location but one pointer (MGF) allows you to do things that other one (PMF or PDF) does not. Thus now you can perform transformations which were previously impossible.

MGF - Bernoulli

e.g.
$X \sim \text{Bernoulli}(p) := p^x (1-p)^{1-x}$



1:1 - according to 'bilateral Laplace transform' both graphs are equal, they imply each-other
(even if visually nothing in common)



t is a parameter inside of g(x)

$M_X(t) = E[e^{tX}] = \sum_{x \in \text{supp}[X]} e^{tx} p(x)$
prob. of failure
prob. of success
$= e^{t \cdot 0} p(0) + e^{t \cdot 1} p(1)$
$= e^0 \cdot 1 - p + e^t \cdot p$
$M_X(t) = 1 - p + pe^t$

MGF - Binomial

$X \sim \text{Binomial}(n, p) = X \sim \text{Bernoulli}(p) + \dots + X \sim \text{Bernoulli}(p) = n$
We have a cup with 1 coin. We draw the coin out. Record whether it ar T and put the coin back in the cup. We repeat the experiment n-times then see # of Hs.
We have a cup with n coins. We draw them ALL out and record how many Hs.

$T = X_1 + X_2 + \dots + X_n$
property 1
property 2
$M_T(t) = M_{X_1 + X_2 + \dots + X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \dots \cdot M_{X_n}(t) = (M_X(t))^n$
$= (1 - p + pe^t)^n$
and we already know the MGF of Bernoulli
$M_X(t) = (1 - p + pe^t)^n$

if $X_1, X_2 \overset{iid}{\sim} Y = X_1 + X_2$
$M_Y(t) = M_{X_1}(t) M_{X_2}(t)$ property 4
$= (M_X(t))^2$ property 1

MGF - Moment

$Y = aX + c$
$E[e^{t(aX+c)}]$ & $x^{m+n} = x^m x^n$
$M_Y(t) = M_{aX+c}(t) = E[e^{t(aX+c)}] = E[e^{taX} e^{tc}] = e^{tc} E[e^{taX}]$
let $t' = ta$
$= e^{tc} E[e^{tX}]$
$= e^{tc} M_X(t')$
$= e^{tc} M_X(ta)$
Thus as soon as you say that X is linearly transformed you immediately know its an MGF
$M_{aX+c}(t) = e^{tc} M_X(ta)$
$M_Y(t) = e^{tc} M_X(ta)$
property 3

X_1 , X_2 are independent r.v.s
let $Y = X_1 + X_2$
$E[e^{(X_1+X_2)}]$ & $x^{m+n} = x^m x^n$
$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}] = E[e^{tX_1} \cdot e^{tX_2}]$
$= E[e^{tX_1}] \cdot E[e^{tX_2}]$
$= M_{X_1}(t) \cdot M_{X_2}(t)$
property 4
$M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t)$
the MGF of a sum of independent r.v.s is the product of MGF's of the constituent r.v.s
if iid : $M_Y(t) = (M_X(t))^2$
previously you couldn't just add two PMFs , here you just multiply two MGFs very convenient since PMF and its MGF are the same 1:1

MGF - Geometric

$X \sim \text{Geometric}(p) := (1-p)^{x-1} p$

$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} (1-p)^{x-1} p$ | pull cont. p
$= p \sum_{x=0}^{\infty} e^{tx} (1-p)^{x-1}$ | $\cdot \frac{1-p}{1-p}$
$= \frac{p}{1-p} \sum_{x=1}^{\infty} (e^t)^x (1-p)^x$ | update index
$= \frac{p}{1-p} \sum_{x=1}^{\infty} (e^t (1-p))^x$ | $a^m b^m = (ab)^m$
$= \frac{p}{1-p} \sum_{x=0}^{\infty} (e^t (1-p))^x - 1$ | update index
Geometric Series $a = e^t(1-p)$ $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$
$= \frac{p}{1-p} [\frac{1}{1-e^t(1-p)} - 1]$ | adjust denominator & -
$= \frac{p}{1-p} \frac{e^t(1-p)}{1-e^t(1-p)}$ | cancel (1-p)
$= \frac{p e^t}{1-e^t(1-p)}$
$M_X(t) = \frac{p e^t}{1-e^t(1-p)}$; $1 \; t < (1/1-p)$
true if (indicator function)
if $e^t(1-p) < 1$
$e^t < 1/1-p$
$t < \ln(1/1-p)$
in both sides

MGF - Exponential

$X \sim \text{Exponential}(\lambda) := \lambda e^{-\lambda x}$

$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$ | pull cont. λ
$= \lambda \int_0^\infty e^{tx} e^{-\lambda x} dx$ | $a^m b^m = (ab)^m$
$= \lambda \int_0^\infty (e^{t-\lambda})^x dx = \frac{\lambda}{t-\lambda} [(e^{t-\lambda})^x]_0^\infty = \frac{\lambda}{t-\lambda} (\lim_{x \to \infty} e^{t-\lambda x} - 1)$
$= \frac{\lambda}{t-\lambda} (0 - 1) = \frac{\lambda}{\lambda - t}$
if $t - \lambda < 0 \rightarrow t < \lambda$
$M_X(t) = \frac{\lambda}{\lambda - t}$; $t < \lambda$

$\lim_{x \to \infty} e^{-x} = 0$

$Y = aX$
property 3
$M_Y(t) = M_{aX}(t) = M_X(at) = \frac{\lambda}{\lambda - at} = \frac{\lambda' a}{\lambda' a - at} = \frac{\lambda'}{\lambda' - t} \rightarrow Y \sim \text{Exponential}(\lambda')$
let $\lambda' = \frac{\lambda}{a}$
then $\lambda = \lambda' a$
or ·
$\frac{\frac{a}{1}}{\frac{1}{a}} = \frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t}$
$Y \sim \text{Exponential}(\frac{\lambda}{a})$

using MGF
remember ?
using CDF

let $X \sim \text{Exponential}(\lambda)$, $Y = 2X$ how is this distributed ?
a clever trick - look at the CDF of Y
$F_Y(x) = P(Y \leq x) = P(2X \leq x) = P(X \leq \frac{x}{2}) = F_X(\frac{x}{2})$ now we can use the CDF of X
$= 1 - e^{-\lambda \frac{x}{2}}$
$= 1 - e^{-\frac{\lambda}{2} x}$
$Y \sim \text{Exponential}(\frac{\lambda}{2})$ if $\lambda' = \frac{\lambda}{2}$
$= 1 - e^{-\lambda' x}$
Exponential CDF $F(x) = 1 - e^{-\lambda x}$
$Y \sim \text{Exponential}(\lambda')$ s.t. $= F_Y(x)$

MGF - usefulness

Why is $M_X(t)$ useful?
consider $X \sim \text{Binomial}(n, p) := \binom{n}{x} p^x (1-p)^{n-x}$
find $E[X^{17}] = \sum_{x=0}^{n} x^{17} \binom{n}{x} p^x (1-p)^{n-x}$
not possible to figure out

recall Taylor Series of f(x) near x=c
$f(x) = \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots = \sum_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!}(x-c)^i$
$f(x) = \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3$ 3rd degree approximation
$f(x) = \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2$ 2nd degree approximation

let $f(x) = e^x$ near x=0 → c=0
$e^x \approx \frac{e^0}{0!} x^0 + \frac{e^0}{1!} x^1 + \frac{e^0}{2!} x^2 + \frac{e^0}{3!} x^3 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = \sum_{i=0}^{\infty} \frac{x^i}{i!}$
$e^{tX} \approx 1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \frac{t^5X^5}{5!} + \dots$
$M_X(t) = E[e^{tX}] = E[1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \frac{t^5X^5}{5!} + \dots]$ E[X+Y] = E[X]+E[Y]
$= E[1] + \frac{t}{1!} E[X] + \frac{t^2}{2!} E[X^2] + \frac{t^3}{3!} E[X^3] + \frac{t^4}{4!} E[X^4] + \dots$
$\frac{d}{dt} M_X(t) = \frac{d}{dt} E[e^{tX}] = E[\frac{d}{dt}[e^{tX}]]$ anything that's a function of the r.v. X is a constant when differentiating with respect to t
$M'_X(t) = 0 + \frac{1}{1!} E[X] + \frac{t}{2!} E[X^2] + \frac{t^2}{3!} E[X^3] + \frac{t^3}{4!} E[X^4] + \dots$
let t=0 → $M'_X(0) = E[X] = \mu$ '1st moment'
$M''_X(t) = 0 + 0 \cdot E[X] + \frac{1}{1!} E[X^2] + \frac{t}{2!} E[X^3] + \frac{t^2}{3!} E[X^4] + \dots$
let t=0 → $M''_X(0) = E[X^2]$ '2nd moment'
$M'''_X(t) = 0 + 0 + 0 \cdot E[X^2] + \frac{1}{1!} E[X^3] + \frac{t}{2!} E[X^4] + \dots$
let t=0 → $M'''_X(0) = E[X^3]$ '3rd moment'
property 2
$M^{(k)}_X(0) = E[X^k]$ this is how MGF gets its name ! it generates moments ... you can just crank out moments by taking derivatives
the kth derivative of MGF at t=0 gives you the kth moment'

MGF - Normal

$M_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{(2\pi)}} e^{-\frac{x^2}{2}} dx$ | pull cont. through
$= \frac{1}{\sqrt{(2\pi)}} \int_{\mathbb{R}} e^{tx} e^{-\frac{x^2}{2}} dx$ | $a^m b^m = (ab)^m$
$= \frac{1}{\sqrt{(2\pi)}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2 + tx} dx$
$-^1/_2 x^2 + tx = -^1/_2(x^2 - 2tx) = -^1/_2((x-t)^2 - t^2) = -^1/_2(x-t)^2 + {}^1/_2 t^2$
complete the square
$(x-t)^2 = x^2 - 2tx + t^2$
$(x-t)^2 - t^2 = x^2 - 2tx$
$= \frac{1}{\sqrt{(2\pi)}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-t)^2} e^{\frac{1}{2}t^2} dx$
$= e^{\frac{1}{2}t^2} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}(x-t)^2} dx}_{=1}$ if $X \sim \text{Normal}(1, 1) \rightarrow f(x)$

Standard Normal $M_Z(t) = e^{\frac{t^2}{2}}$

if $X \sim \text{Normal}(\mu, \sigma^2) \rightarrow X = \mu + \sigma Z$
$M_X(t) = e^{t\mu} e^{\frac{\sigma^2 t^2}{2}}$
$= e^{t\mu + \frac{\sigma^2 t^2}{2}}$
General Normal
$M_X(t) = e^{t\mu + \frac{\sigma^2 t^2}{2}}$

property 3
$M_{aX+c}(t) = e^{tc} M_X(ta)$
$M_{Z \sim \mu}(t) = e^{t\mu} M_Z(t\sigma)$
$= e^{t\mu} e^{\frac{\sigma^2 t^2}{2}}$

if $X \sim \text{Deg}(c) \rightarrow M_X(t) = e^{tc}$

Lévy's Continuity Theorem

if $X_1, \dots, X_n$ is a sequence of r.v.'s
$\lim_{n \to \infty} M_{X_n}(t) = M_Y(t) = e^{t\mu} \iff X_n \rightarrow Y$
if the Limit of LHS-MGF approaches RHS-MGF, then X_n and Y are the same
if n large, $M_{X_n}(t) \approx M_Y(t) \rightarrow X_n \overset{d}{\approx} Y$ approx. equality distributed
if the MGFs are equal then you have the same r.v.

Law of Large Numbers LLN

if $X_1, \dots, X_n \overset{iid}{\sim}$ something s.t. $E[X] < \infty$, $\bar{X}_n \rightarrow \mu$ r.v. sequence converges to μ
thus if $Y \sim \text{Deg}(\mu) \rightarrow \mu$, we are done
Lec-12
$\lim_{n \to \infty} M_{\bar{X}_n}(t) = M_Y(t) = e^{t\mu}$
$M_{\bar{X}_n}(t) = M_{\frac{X_1 + \dots + X_n}{n}}(t) = M_{\frac{1}{n}\Sigma}(t) = M_\Sigma(\frac{t}{n}) = (M_X(\frac{t}{n}))^n = (E[e^{\frac{t}{n}X}])^n = (E[1 + \frac{tX}{n} + \frac{t^2X^2}{2!n^2} + \frac{t^3X^3}{3!n^3} + \dots])^n$
by Def of $\bar{X}_n$; T total r.v. ; by property 3 ; by property 4 ; by Def of MGF ; by Taylor Series
We say f(n) is 'little o' of g(n) if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$ g(n) 'grows' faster then f(n)
e.g. $n^2 = o(n^3)$
'little o' similar to CS 'Big O'
$\lim_{n \to \infty} \frac{n^2}{n^3} = \lim_{n \to \infty} \frac{1}{n} = 0$ ✔
n^3 / n^2 ; past this point n^2 is $o(n^3)$
We can say $\frac{t^2X^2}{2!n^2} + \frac{t^3X^3}{3!n^3} + \dots \geq o(\frac{1}{n})$ b/c $\lim_{n \to \infty} \frac{\frac{t^2X^2}{2!n^2} + \frac{t^3X^3}{3!n^3} + \dots}{\frac{1}{n}} = \lim_{n \to \infty} \frac{t^2X^2}{2!n} + \frac{t^3X^3}{3!n^2} + \dots = 0$
$= (E[1 + \frac{tX}{n} + o(\frac{1}{n})])^n$
$= (E[1] + E[\frac{tX}{n}] + E[o(\frac{1}{n})])^n$
$= (1 + \frac{t\mu}{n} + E[o(\frac{1}{n})])^n$
this E can be ignored
$= (1 + \frac{t\mu}{n} + o(\frac{1}{n}))^n$
recall $\lim_{n \to \infty} (1 + \frac{1}{n})^n = e$
$\lim_{n \to \infty} (1 + \frac{n}{n})^n = e^n$
$\lim_{n \to \infty} (1 + \frac{1}{n})^n = e$
fact
anything that grows faster than $^1/_n$ does not matter e.g $^1/_{n^{1000}}$
but if n=billion , $f(n)=3.08 \approx 2.718$
n=trillion, $f(n)=2.90 \approx 2.718$
slow !!
$\lim_{n \to \infty} (1 + \frac{n}{n} + \frac{1}{n})^n = e^n$
$\lim_{n \to \infty} M_{\bar{X}_n}(t) = \lim_{n \to \infty} (1 + \frac{t\mu}{n} + o(\frac{1}{n}))^n = e^{t\mu} \longrightarrow \bar{X}_n \rightarrow \mu$
Deg MGF
$\bar{X}$ MGF
b/c the long run behavior of the MGF of $\bar{X}$ gets closer and closer to MGF of Deg(μ) , $\bar{X}$ converges to μ
if $X \sim \text{Deg}(c) \rightarrow M_X(t) = e^{tc}$
if $X \sim \text{Deg}(\mu) \rightarrow M_X(t) = e^{t\mu}$

Central Limit Theorem CLT

Crown Jewel of Math 241

if $X_1, \dots, X_n \overset{iid}{\sim}$ mean-μ and SE-σ
$C_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$ C_n is the standardized $\bar{X}_n$ $\mu = 0$ $\sigma = 1$
subtract the mean and divide by S.E. S.E. of $\bar{X} = \frac{\sigma}{\sqrt{n}}$
how does C_n look like as n gets large ?
let's examine the long-run behavior of C_n MGF and see if it matches anything we know.
$= \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{\sqrt{n}(\frac{X_1 + \dots + X_n}{n} - \mu)}{\sigma} = \frac{\sqrt{n}(\frac{X_1 + \dots + X_n}{n} - \frac{n\mu}{n})}{\sigma} = \frac{X_1 + \dots + X_n - n\mu}{\sigma \cdot \sqrt{n}} = \frac{(X_1 - \mu) + \dots + (X_n - \mu)}{\sigma \cdot \sqrt{n}}$
by Def of $\bar{X}_n$; $\frac{a+b}{c} = a/c + b/c$; $\frac{a}{b/c} = \frac{a \cdot c}{b}$
standardization subtract the mean and divide by S.E.
$= \frac{1}{\sqrt{n}} [\frac{(X_1 - \mu)}{\sigma} + \dots + \frac{(X_n - \mu)}{\sigma}]$
$= \frac{1}{\sqrt{n}} [Z_1 + \dots + Z_n]$
$\text{Var}[X] = E[X^2] - \mu^2$; if μ=0 , $\text{Var}[X] = E[X^2]$
if σ=1 , $\sigma^2 = 1$ and $E[X^2] = 1$
we know that $E[Z] = 0$ $SE[Z] = 1$ $E[Z^2] = 1$
let $Z_i = \frac{(X_i - \mu)}{\sigma}$
MGF of C_n
$M_{C_n}(t) = M_{\frac{1}{\sqrt{n}}(Z_1 + \dots + Z_n)}(t) = (M_{Z_i}(\frac{t}{\sqrt{n}}))^n = (E[e^{\frac{tZ_i}{\sqrt{n}}}])^n = (E[1 + \frac{tZ_i}{\sqrt{n}} + \frac{t^2Z_i^2}{2!n} + \frac{t^3Z_i^3}{3!n^{3/2}} + \frac{t^4Z_i^4}{4!n^2} + \dots])^n$
by property 3 ; by property 4 ; by Def of MGF ; by Taylor Series ; just as before $= o(\frac{1}{n})$
remember μ standard normal = 0
$E[\frac{tZ_i}{\sqrt{n}}] = E[\frac{t}{\sqrt{n}} Z] = \frac{t}{\sqrt{n}} E[Z] = \frac{t}{\sqrt{n}} \mu = 0$
$E[\frac{t^2Z_i^2}{2!n}] = E[\frac{t^2}{2!n} Z^2] = \frac{t^2}{2!n} E[Z^2] = \frac{t^2}{2!n} 1$
$= (E[1 + \frac{tZ_i}{\sqrt{n}} + \frac{t^2Z_i^2}{2!n} + o(\frac{1}{n})])^n$
$= (E[1] + E[\frac{tZ_i}{\sqrt{n}}] + E[\frac{t^2Z_i^2}{2!n}] + E[o(\frac{1}{n})])^n$
this E can be ignored
$= (1 + 0 + \frac{t^2}{2!n} + o(\frac{1}{n}))^n$
What happens as n→∞
$\lim_{n \to \infty} M_{C_n}(t) = \lim_{n \to \infty} (1 + \frac{t^2}{2!n} + o(\frac{1}{n}))^n = e^{\frac{t^2}{2}} = M_Z(t) \longrightarrow C_n \rightarrow Z \sim \text{Normal}(0, 1)$
our C_n converges to the 'standard normal'
by the same 'e' transformation as we did in LLN

Central Limit Theorem - CLT

if $X_1, \dots, X_n \overset{iid}{\sim}$ with mean-μ , SE-σ , and large n
$C_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \rightarrow Z \sim \text{Normal}(0, 1)$

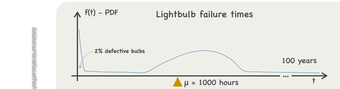
1 the standardized avg. of n r.v.'s looks like the 'standard normal'
or
the standardized average r.v. is approximately distributed as 'standard normal'

corollaries:

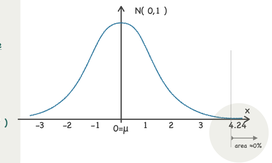
2 if μ large $C_n \overset{d}{\approx} Z \sim \text{Normal}(0, 1)$ approx. equally distributed
if the mean is large, C_n and 'standard normal' are approx. equally distributed

3 $C_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \approx Z \implies \bar{X}_n \approx \frac{\sigma}{\sqrt{n}} Z + \mu \implies \bar{X} \overset{d}{\approx} \text{Normal}(\mu, (\frac{\sigma}{\sqrt{n}})^2)$ 'average'
S.E. of $\bar{X}$
the 'avg. r.v.' is approx. equally distributed as 'general normal' with $\sigma = {}^\sigma/_{\sqrt{n}}$ which is the S.E. of $\bar{X}$

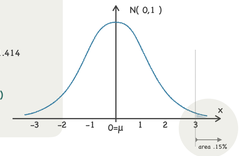
4 $n\bar{X} \approx \sigma\sqrt{n} Z + n\mu \implies T \approx \sigma\sqrt{n} Z + n\mu \implies T \overset{d}{\approx} \text{Normal}(n\mu, (\sigma\sqrt{n})^2)$ 'how far away'
total r.v.
from where you begin
the 'total r.v.' is approx. equally distributed as 'general normal' with μ=nμ and σ=σ√n



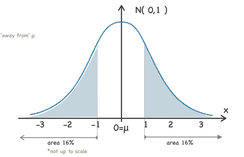
You buy 50 lightbulbs. What is the probability they last on average more than 1300 hours?
keyword
$X_1 + \dots + X_{50} \overset{iid}{\sim} F_X(\mu=1000, \sigma \approx 500)$
$\bar{X} \approx \text{Normal}(\mu, (\frac{\sigma}{\sqrt{n}})^2)$ S.E. = $\frac{\sigma}{\sqrt{n}} = \frac{500}{\sqrt{50}} = 70.7$
standardization subtract the mean and divide by S.E. μ=1000 σ=70.7
$\implies \bar{X} \approx \text{Normal}(1000, (70.7)^2)$
$P(\bar{X} \geq 1300) = P(\frac{\bar{X} - 1000}{70.7} \geq \frac{1300 - 1000}{70.7}) \approx P(Z \geq 4.24) \approx 0$



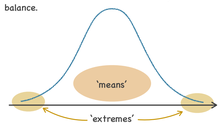
You are to flip a fair coin 30 times. You stop on the 1st Head. What is the probability that the average stopping time is more than 2.75 ?
keyword
$X_1 + X_2 + \dots + X_{30} \overset{iid}{\sim}$ Geometric $(^1/_2)$ $\mu = \frac{1}{p} = \frac{1}{1/2} = 2$
$\sigma^2 = \frac{1-p}{p^2} \implies \sigma = \sqrt{\frac{1-p}{p^2}} \approx 1.414$
$\bar{X} \approx \text{Normal}(\mu, (\frac{\sigma}{\sqrt{n}})^2)$ S.E. = $\frac{\sigma}{\sqrt{n}} = \frac{1.414}{\sqrt{30}} = .258$
standardization subtract the mean and divide by S.E. μ=2 σ=.258
$\implies \bar{X} \approx \text{Normal}(2, (.258)^2)$
by CLT
$P(\bar{X} \geq 2.75) = P(\frac{\bar{X} - 2}{.258} \geq \frac{2.75 - 2}{.258}) \approx P(Z \geq \approx 3) = .0015$

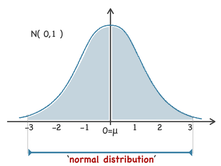


Take 100 random equally likely Left/Right steps. What is the probability you are more than 10 steps away from where you began ?
keyword
$X_1 + X_2 + \dots + X_{100} \overset{iid}{\sim}$ Rademacher $\begin{cases} 1 \text{ w.p. } 1/2 \\ -1 \text{ w.p. } 1/2 \end{cases}$
let $T = X_1 + X_2 + \dots + X_{100} \overset{\text{by CLT}}{\approx} \text{N}(n\mu, (\sqrt{n}\sigma)^2) = \text{N}(0, 10^2)$
long-run asymptotic distribution ; where we begin - mean ; standard deviation 'away from' μ
μ , σ²
10 to the Left ; 10 to the Right
$P(|T| > 10) = P(T < -10) + P(T > 10)$
$= P(\frac{T-0}{10} < \frac{-10-0}{10}) + P(\frac{T-0}{10} > \frac{10-0}{10})$
$= P(Z < -1) + P(Z > 1)$ symmetric bell curve
$= P(Z \in [-1, 1])$ remember the quantiles '68-95-997' rule ?
$= 100 - 68 = 32\%$
standardization subtract the mean and divide by S.E. μ=0 σ=10



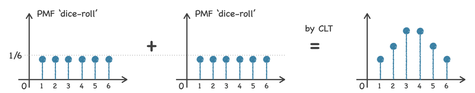
There is something very special about this shape. This function is very good at making the harmony between the 'means' and the 'extremes'. It creates a mathematically natural balance.





it defines what 'normalness' means, anything outside of this range is considered abnormal

anything creates a 'bell' 😱!



already starting to look like a bell