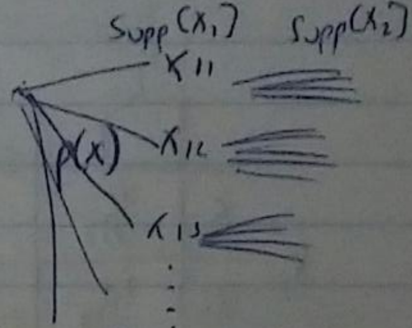
***Next unit**

$T_3 = X_1 + X_2$

$E(T) \stackrel{?}{=} f(E(X_1), E(X_2))$

$= \sum_{t \in Supp(T)} t \, \underline{p(t)}$



$\boxed{\text{Nov 8, 2016: Lecture 15}}$

HW due Monday

DO NOT STRESS
- * 4PM - 6PM → Do 10.9, 11.1, 11.2, 11.3
- * 8PM - 10PM → Do 11.4, 11.5, 11.6, 11.7

IGNORE THIS ⟷ SOME MIND NOTES ~~THOUGTH~~

- $T = X_1 + X_2$ [X_1 & X_2 are r.v]

$$E(T) = \underbrace{\sum_{t \in Supp[T]}}_{\boxed{\text{Also hard}}} t \, p(t) \quad \boxed{\text{HARD}}$$

$$E[X_1 + X_2] = \sum_{\langle X_1, X_2 \rangle \text{ pairs}} (x_1 + x_2) \, \underbrace{p(x_1, x_2)}_{\boxed{\text{Joint mass function}}}$$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad E[g(\vec{X})] = \sum_{\vec{x} \in Supp} g(\vec{x}) \, p(\vec{x})$$

$$\sum_{x_1 \in Supp[X_1]} \sum_{x_2 \in Supp(X_2)} x_1 p(x_1, x_2) + \sum_{x_2 \in Supp(X_2)} \sum_{x_1 \in Supp(X_1)} x_2 p(x_1, x_2) \Leftarrow \sum_{x_1 \in Supp[X_1]} \sum_{x_2 \in Supp(X_2)} (x_1 + x_2) p(x_1, x_2)$$

$$\Downarrow$$

$$\sum_{x_1} x_1 \underbrace{\sum_{x_2} p(x_1, x_2)} + \sum_{x_2} x_2 \sum_{x_1} p(x_1, x_2)$$

- X_1 & X_2 are independent r.v.

$T = X_1 + X_2$

$\Rightarrow p(x_1, x_2) = p(x_1) \, p(x_2)$

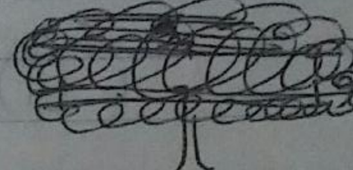
$$E(T) = \sum_{x_1} x_1 \sum_{x_2} p(x_1) p(x_2) + \sum_{x_2} x_2 \sum_{x_1} p(x_1) p(x_2)$$

constant — constant

$$\Downarrow$$

$$\boxed{E[X_1] + E[X_2]} \Leftarrow \underbrace{\sum_{x_1} x_1 p(x_1)}_{E[X_1]} \underbrace{\sum_{x_2} p(x_2)}_{1} + \underbrace{\sum_{x_2} x_2 p(x_2)}_{E(X_2)} \underbrace{\sum_{x_1} p(x_1)}_{1}$$

* $Supp[X_1] = \{1, 7, 19\}$
$Supp[X_2] = \{5, 23, 88\}$



		X_1			
		1	7	19	
X_2	5	$1/15$	$1/3$	$2/15$	$\frac{16}{30}$
	23	$1/30$	$1/10$	$1/30$	$\frac{5}{30}$
	88	$1/30$	$1/5$	$1/15$	$\frac{9}{30}$
		$\frac{4}{30}$	$\frac{19}{30}$	$\frac{7}{30}$	1

$p(X_1, X_2)$

$p(X_2)$

← all values must add up to 1.

$p(X_1) \rightarrow$ MARGINAL

* There's a relationship between joint mass fn & marginal function

$$\sum_{X_1} \sum_{X_2} p(X_1, X_2) = 1$$

$\rightarrow P(X_1 = 1, X_2 = 5) = \frac{1}{15}$

$\rightarrow p(X_1 = 1) = 4/30$

1 w/ a 5
1 w/ a 23
1 w/ a 88

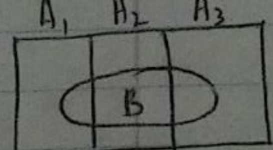
sums up to $4/30$

$P(X_1 = 1, X_2 = 5) +$
$P(X_1 = 1, X_2 = 23) +$
$P(X_1 = 1, X_2 = 88)$

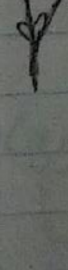
$\Downarrow$

$$\sum_{X_2} p(X_1 = 1, X_2)$$

$\Downarrow$



$P(B) = P(B, A_1) + P(B, A_2) + P(B, A_3)$



$\rightarrow p(X_1) = \sum_{X_2} p(X_1, X_2)$

"X^2 is margined out"

Calc 201 reference

$g(x) = \underbrace{\int_{\mathbb{R}} f(x, y)\, dy}$

Y is ~~margined~~ integrated out of integral

$$\overbrace{\sum_{x_1} x_1 \sum_{x_2} p(x_1, x_2)}^{p(x_1)} \Longrightarrow \sum_{x_1} x_1 p(x_1) + \sum_{x_2} x_2 p(x_2)$$

$$E[X_1] + E[X_2]$$

$$\rightarrow p(1, 88) \overset{?}{=} p(1)p(88)$$

$$30\ \frac{\cancel{1}}{\cancel{30}} \quad \frac{4}{30} \cdot \frac{9}{30} = \underline{36}$$

THESE TWO ARE DEPENDENT

- $X_1, X_2, \ldots, X_7$ r.v.'s

$$\Rightarrow E(T) = E[X_1 + \ldots X_7] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = E[X_1] + \ldots E[X_n]$$

$$= \mu_1 + \ldots + \mu_n$$

- let's say $X_1, X_2, \ldots X_n$ are identically distributed. (not necessarily independent).

$$E[T] = E[X_1 + X_2 + \ldots + X_n] = n\mu$$

$X \sim Binom(n, p)$

$X = X_1 + \ldots + X_n$

$X_1, \ldots X_n \overset{iid}{\sim} Bern(p)$

$$E[X] = np$$

$E[X] = n\mu$

$E[X] = \frac{r}{p}$

*$X \sim$ Neg Binomial(r, p)

$$E[X] = \sum_{x=r}^{\infty} x \binom{x-1}{r-1} (1-p)^{x-r} (p)^r$$

$X_1, X_2, \ldots X_r \overset{iid}{\sim} Geom(p) \Rightarrow X = X_1 + \ldots X_r \sim$ Neg Binomial(r, p)

- $X \sim Hyper(n, K, N)$

$$E[X] = \sum_{x \in supp(X)} x \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

Remember:

$$\left[X \sim Hyper(1, K, N) = Bern\left(\frac{K}{N}\right) \right]$$

$X = X_1 + X_2 + \dots + X_n$

s.t. $X_1, X_2, \dots, X_n$ $\underbrace{}_{}$ indent. dists. $Bern\left(\frac{K}{N}\right)$ ← not independent

$\rightarrow$ $E[X] = n\frac{K}{N}$

- $Var[X] := E[(X-\mu)^3]$

$$= E[X^2 - 2\mu X + \mu^3]$$

$$= E[X^2] + E[-2\mu X] + E[\mu^3]$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$\sigma^2 = E[X^2] - \mu^2$$

$$\Downarrow$$

$$\boxed{E[X^2] = \sigma^2 + \mu^2}$$

- $E[X]$ first moment
- $[X^2]$ second moment

⋮

- $E[X^k]$ k^{th} moment

- $E[|X-\mu|]$ 1^{st} centered moment
- $E[\underbrace{(X-\mu)^2}_{Var[X]}]$ 2^{nd} centered moment

⋮

- $E[|X-\mu|^k]$ 3^{rd} centered ~~centered~~ moment.

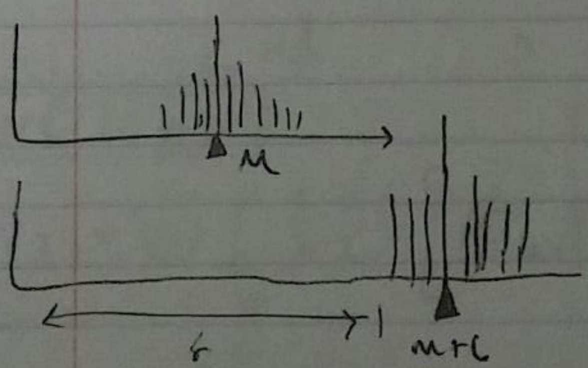
- $\frac{E[|X-\mu|]}{\sigma}$ 1st standardized moment

- $1 = \frac{E[(X-\mu)^2]}{\sigma^2}$ 2nd standardized moment

- $Skew[X] = \frac{E[|X-\mu|^3]}{\sigma^3}$ 3rd standardized moment

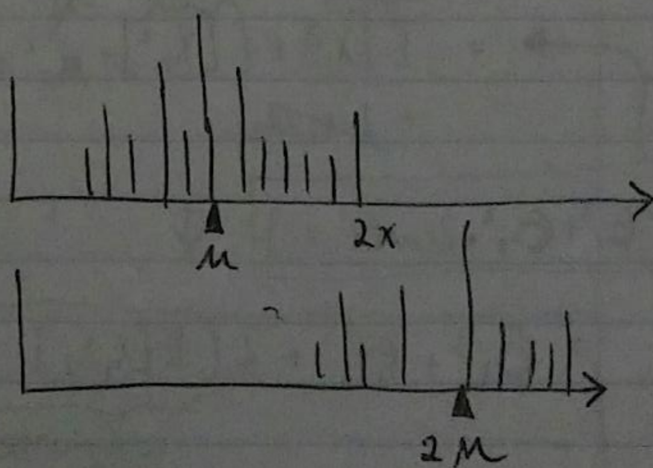
- $Kurt[X] = \frac{E[(X-\mu)^4]}{\sigma^4}$ 4th standardized moment

- $Var[X]$
  $= Var[X+c]$ such that $c \in \mathbb{R}$



- $Var[X+C] = E[(X+C)-(\mu+c)^2]$
  $= E[(X-\mu)^2]$
  $= Var[X]$

- $Var[X]$
  $Var[a-X]$ s.t. $a \in \mathbb{R}$



$$\mathrm{Var}[aX] = E[(aX - a\mu)^2]$$

$$= E[(a(x-\mu))^2]$$

$$= E[a^2(x-\mu)^2]$$

$$= a^2 E[(X-\mu)^2]$$

$$= a^2\,\mathrm{Var}[X] \Rightarrow \boxed{\mathrm{Var}[aX] = \rho^2\sigma^2}$$

• $SE[aX] = \sqrt{\mathrm{Var}(aX)}$

$$= \sqrt{a^2\sigma^2}$$

$$\boxed{SE[aX] = |a|\sigma}$$

$$\boxed{\begin{aligned}\mathrm{Var}[aX+c] &= a^2\sigma^2\\ SE[aX+c] &= |a|\sigma\end{aligned}}$$

• X_1, X_2 r.v.'s

$$\mathrm{Var}[X_1+X_2] = E\left[\left((X_1+X_2)-(\mu_1+\mu_2)\right)^2\right]$$

$$\downarrow$$

$$= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 + 2X_1X_2 - 2X_1\mu_1 - 2X_2\mu_2 - 2X_1\mu_2 - 2X_2\mu_1 + 2\mu_1\mu_2]$$

$$\downarrow$$

$$\sigma_1^2+\mu_1^2 \quad \sigma_2^2+\cancel{\mu_2^2} \quad -\mu_1^2 \quad \cancel{-\mu_2^2}$$

$$= E[X_1^2] + E[X_2^2] + \cancel{\mu_1^2} + \cancel{\mu_2^2} + 2E[X_1X_2] - 2\mu_1^2 - \cancel{2\mu_2^2} - 2\mu_1\mu_2 - \cancel{2\mu_1\mu_2} + \cancel{2\mu_1\mu_2}$$

$$\boxed{\mathrm{Var}[X_1+X_2] = \sigma_1^2 + \sigma_2^2 + 2\,\mathrm{cov}[X_1, X_2]}$$

$$\downarrow$$

$$= \sigma_1^2 + \sigma_2^2 + 2(\underbrace{E[X_1X_2] - \mu_1\mu_2}_{\substack{\text{covariance}\\ \mathrm{cov}[X_1, X_2]}})$$

$$Var[X_1+X_2] = \sigma_1^2 + \sigma_2^2$$

$\Rightarrow$

- If X_1, X_2 are independent (but not necessarily iden. distr.)

$$E[X_1 X_2] = \sum_{X_1}\sum_{X_2} x_1 x_2\, p(x_1, x_2) = \sum_{X_1}\sum_{X_2} x_1 x_2\, p(x_1)p(x_2)$$

$$= \underbrace{\sum_{X_1} x_1 p(x_1)}_{E[X_1]} \underbrace{\sum_{X_2} x_2 p(x_2)}_{E[X_2]} \Rightarrow \underline{\mu_1 \mu_2}$$

$$\boxed{\begin{array}{l}\text{if } X_1, X_2 \text{ independent} \\ cov[X_1, X_2] = \mu_1\mu_2 - \mu_1\mu_2 = 0\end{array}}$$

$X_1, \ldots, X_n$ independent

$$Var[T] = Var[X_1 + \ldots + X_n] = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \sigma_i^2$$

* $X_1, X_2, \ldots, X_n$ iid

$$\boxed{Var[T] = n\sigma^2}$$

- $Var[\bar{X}] = Var\left[\frac{T}{n}\right] = \frac{1}{n^2} Var[T] = \frac{1}{\cancel{n^2}} n\sigma^2 = \frac{\sigma^2}{n} = \boxed{\begin{array}{c}X_1, \ldots X_n \text{ iid} \\ Var[\bar{X}] = \\ \frac{\sigma^2}{n}\end{array}}$

$\frac{T}{n} = \underbrace{\frac{1}{n}}_{\text{constant}} T$

$$\Rightarrow \boxed{SE[\bar{X}] = \frac{\sigma}{\sqrt{n}}} \ast\ast\ast\ast$$