10/13/16

- $X \sim$ Bernoulli (p)
- $X \sim$ Binomial (n,p)
- $X \sim$ Hypergeometric (n,K,N)

- $\sum_{x \in Supp[X]} p(x) = 1$
- $\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$

$$\lim_{N \to \infty} Hyper(n,p,N) = Binom(n,p)$$

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \leftarrow \text{Binomial Theorem}$$

let $a=p$, $b=1-p$, $i=x$

$$= (p+(1-p))^n = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

$$\to (1)^n = 1$$

X_1 & X_2 are independent r.v's if

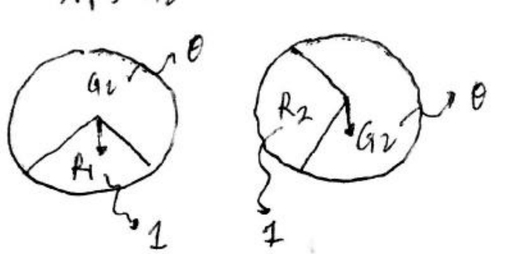
- $P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$
- $P(X_2 = x_2 \mid X_1 = x_1) = P(X_2 = x_2)$
- $P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1) P(X_2 = x_2)$

$\forall x_1 \in Supp[X_1]$, $\forall x_2 \in Supp[X_2]$

- X_1 & X_2 are independent and identically distributed if X_1, X_2 are indep. and $X_1 \overset{d}{=} X_2$ and $X_1, X_2 \overset{iid}{\sim}$ → independent & identically distributed

istributed

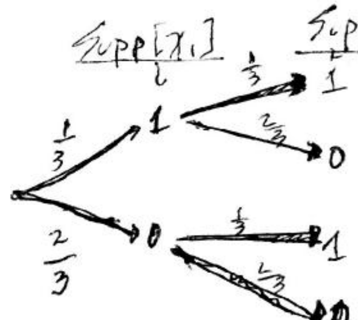
$X_1, X_2 \overset{iid}{\sim}$ Bern $(\frac{1}{3})$



$Supp[T] = \{0, 1, 2\}$

$T = X_1 + X_2 = g(X_1, X_2)$

$$T \sim \begin{cases} 0 & \text{wp } \frac{4}{9} \\ 1 & \text{wp } \frac{4}{9} \\ 2 & \text{wp } \frac{1}{9} \end{cases}$$



$P(X_1 = 1, X_2 = 1) = \frac{1}{9} = 2$

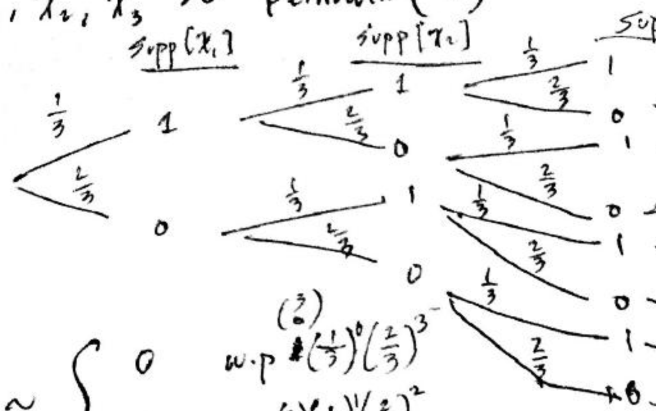
$P(X_1 = 1, X_2 = 0) = \frac{2}{9} = 1$

$P(X_1 = 0, X_2 = 1) = \frac{2}{9} = 1$

$P(X_1 = 0, X_2 = 0) = \frac{4}{9} = 0$

n choices

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bernoulli}\left(\frac{1}{3}\right)$



	T
$\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 =$	3
$\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 =$	2
$\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 =$	2
$\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 =$	1
$\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 =$	2
$\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 =$	1
$\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 =$	1
$\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 =$	0

$$T \sim \begin{cases} 0 & \text{w.p } \binom{3}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{3} \\ 1 & \text{w.p} \binom{3}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 \\ 2 & \text{w.p} \binom{3}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 \\ 3 & \text{w.p} \binom{3}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0 \end{cases}$$

0	0	0	$\binom{3}{0}$
1	0	0	$\binom{3}{1}$
0	1	0	
0	0	1	
1	1	0	$\binom{3}{2}$
1	0	1	
0	1	1	
1	1	1	$\binom{3}{3}$

$T_n = \sum_{i=1}^{n} X_i$

$\text{Supp}[T_n] = \{0, 1, 2, \dots, n\}$

$$T \sim \begin{cases} 0 & \text{w.p } \binom{n}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{n} \\ 1 & \text{w.p. } \binom{n}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{n-1} \\ 2 & \text{w.p. } \binom{n}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{n-2} \\ \vdots & \\ n-2 & \text{w.p. } \binom{n}{n-2}\left(\frac{1}{3}\right)^{n-2}\left(\frac{2}{3}\right)^{2} \\ n-1 & \text{w.p. } \binom{n}{n-1}\left(\frac{1}{3}\right)^{n-1}\left(\frac{2}{3}\right)^{1} \\ n & \text{w.p. } \binom{n}{n}\left(\frac{1}{3}\right)^{n}\left(\frac{2}{3}\right)^{0} \end{cases}$$

$\sim \text{Binom}(n, \frac{1}{3})$

$X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(p)$?

All the $\left(\frac{1}{3}\right)$ becomes (p) and all the $\left(\frac{2}{3}\right)$ becomes $(1-p)$

2 Concepts for Binomial

$$\lim_{N \to \infty} \text{Hyper}(n, p, N) = \text{Binom}(n, p)$$

OR

$$X_1, X_2, \dots X_n \overset{iid}{\sim} \text{Bern}(p) ; X_1 + X_2 + \dots X_n \sim \text{Binom}(n, p)$$

- 1000 coin flips
- $P(600\,H) = P(X=600) = \binom{1000}{600}\left(\frac{1}{2}\right)^{600}\left(\frac{1}{2}\right)^{400}$

$X \sim Binom\left(1000, \frac{1}{2}\right)$

$F(x) = P(X \leq x)$

CDF of Binomial

$$F(x) = \sum_{i=0}^{x} p(i) = \sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i}$$

Incomplete regularized gamma function

$$I_{1-p}(n-k, 1+k) := (n-k)\binom{n}{k} \underbrace{\int_0^{1-p} t^{n-k-1}(1+t)^k \, dt}_{\text{no closed form}}$$

$\boxed{X_1, X_2, \ldots \overset{iid}{\sim} Bern(p)}$

$\boxed{\text{* Infinite Sequence of iid r.v's}}$

$\underline{0}\,\underline{0}\,\underline{1}\,\underline{0}\,\underline{1}\,\underline{0}\,\underline{0}\,\underline{1}\ldots$

$\{3, 5, 8, \ldots\}$

$T = \min\{t : X_t = 1\} \to$ "First Success" "Stopping time"

$p(1) = p(T=1) = P(X_1 = 1) = p$

$p(2) = p(T=2) = P(X_1 = 0, X_2 = 1) = P(X_1 = 0) \cdot P(X_2 = 1) = (1-p)p$

$p(3) = p(T=3) = P(X_1 = 0, X_2 = 0, X_3 = 1) = \ldots (1-p)^2 p$

$\vdots$

$p(x) = p(T=x) = \ldots = (1-p)^{x-1} p$

$X \sim Geometric(p) := (1-p)^{x-1} \cdot p$

parameter space

- $p \in (0,1)$

$Supp[X] = \{1, 2, 3, \ldots\}$ infinite

$$\sum_{x \in Supp[X]} p(x) = 1$$

$$\sum_{x=1}^{\infty} \frac{(1-p)^{x-1}\cancel{p}}{\cancel{p}} \overset{?}{=} \frac{1}{p} = \sum_{x=1}^{\infty} (1-p)^{x-1} \overset{?}{=} \frac{1}{p}$$

$\boxed{\text{let } q := 1-p}$

$$\sum_{x=1}^{\infty} (q)^{x-1} = \frac{1}{p} \longrightarrow S = \sum_{x=1}^{\infty} q^{x-1} = q^0 + q^1 + q^2 + \ldots$$

$= 1 + q + q^2 + q^3 + \ldots$

$= 1 + q(1 + q + q^2 + \ldots)$

$1 + qS$

$S - qS = 1$

$(1-q)S = 1$

$S = \frac{1}{1-q}$

Geometric Series

$$\boxed{\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}}$$

- $F(x) = P(X \leq x)$

$$\sum_{i=1}^{x} (1-p)^{x-1} p$$

$1 - P(\underset{1}{0}\ \underset{2}{0}\ \underset{3}{0}\ \underset{4}{0} \ldots \underset{x}{0}) \mid \underset{x+1}{\_}\ \underset{x+2}{\_}\ \underset{x+3}{\_}\ \underset{x+4}{\_} \ldots$

$= (1-p)^x$