10/27. • $T_n = X_1 + X_2 + \ldots + X_n$ "sum r.v." / "total r.v"

Sample Average Random Variable $\rightarrow \bar{X}_n := \frac{T_n}{n} = \frac{X_1 + X_2 + \ldots + X_n}{n}$

n ↓ "sample size"

• $X_1, X_2, X_3 \overset{iid}{\sim} Bern(0.1)$

$T_3 \sim Binomal(3, 0.1) = \begin{cases} 0 & up & 0.729 \\ 1 & up & 0.243 \\ 2 & up & 0.027 \\ 3 & up & 0.001 \end{cases}$

Supp — Find the prob from BinPMF

$\frac{T_n}{n}$

$\bar{X}_3 \sim \begin{cases} 0 & up & 0.729 \\ \frac{1}{3} & up & 0.243 \\ \frac{2}{3} & up & 0.027 \\ 1 & up & 0.001. \end{cases}$

$0, \frac{1}{3}, \frac{2}{3}, 1$ — realization

$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$

Realization from $\bar{X}$

– Big X is the model for r.v
Small x is the realization of X (outcome).
$P(X = 5'8")$ need model of $p(x)$. P.M.F.

• $X \sim Binom(8, \frac{1}{2}) = \binom{8}{x}(\frac{1}{2})^x(1-\frac{1}{2})^{8-x}$ (P.M.F)

$= \frac{\binom{8}{x}}{2^8}$

May on the test

→ ☆ sum this up ↓ got 1. b/c of the formula of PMF

☆ $F(8) = 1$ b/c it collects everything. That's what C.D.F do.

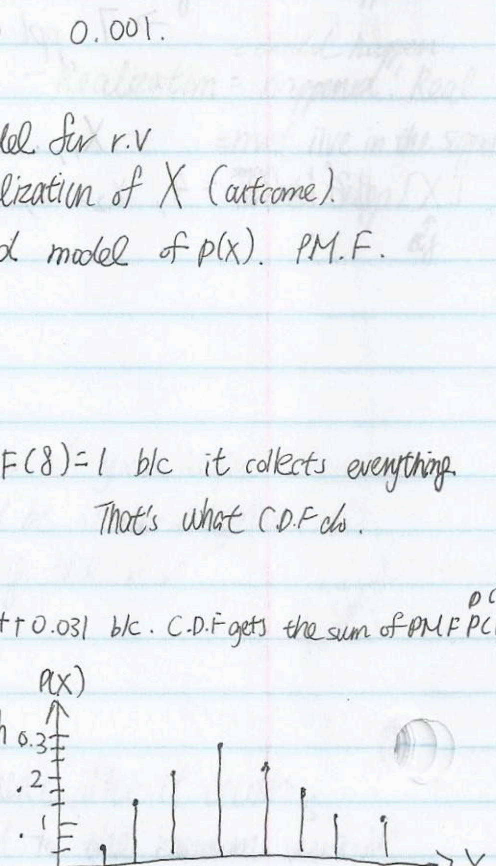
• Why $F(8) = 1$.
• Graph

Supp of X

X	P(X)	F(x)
0	0.004	0.004
1	0.031	0.035
2	0.109	0.145
3	0.219	0.363
4	0.273	0.637
5	0.219	0.855
6	0.109	0.965
7	0.031	0.996.
8	0.009.	1

← = 0.004 + 0.031 b/c. C.D.F gets the sum of PMF $P(0)$ $P(1)$

Graph



balance pt → The thing we came.

• $X_1, \dots, X_n \overset{iid}{\sim} Binom(8, \frac{1}{2}) = \binom{8}{x}(\frac{1}{2})^x(1-\frac{1}{2})^{8-x} = \frac{\binom{8}{x}}{2^8}$

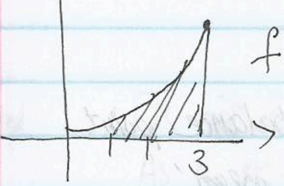
n	1 ... 7	 14	 21
X	5 4 4 3 4 3	3 4 4 5 5 2 5	4 1 6 4 4 4 8
$\bar{X}_n$	3.714	3.857	4.048

Getting closer & closer to 4.

$\bar{X}$ as n goes big. $\lim_{n \to \infty} \bar{X}_n$.

* Balance pt → 4. (Taking a function & finding the important thing).

✳. A function of a function



$f(x) = x^2 \quad x \in [0, 3]$. $G[f] := \int_{\mathbb{R}} f(x)dx = 9$.

$G: \tilde{F} \to \mathbb{R}$

$\tilde{F}$: space of functions; $\mathbb{R}$: (#).

• How to find the balance point?

$$\underset{p(x)'s}{\sum_i} w_i(d_i - d^*) = 0 \Rightarrow \sum w_i d_i = \sum w_i d^* \Rightarrow d^* = \frac{\sum w_i d_i}{\sum w_i}$$

$i \to x \in Supp[X]$, $d_i \to X$

$$\frac{100 \cdot 0 + 20 \cdot 1}{100 + 20} = 0.17.$$

Four Names

• mean

• expectation

• expected value

• first moment

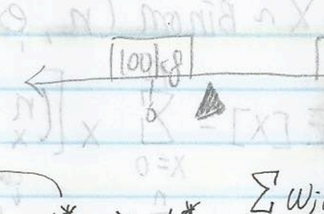
$$E[X] := \mu := \frac{\sum_{x \in Supp[X]} x\, p(x)}{\underbrace{\sum_{x \in Supp[X]} p(x)}_{1}} = \underbrace{\sum_{x \in Supp[X]} x\, p(x)}_{\text{the expected value}}.$$

E : all r.v's $\to \mathbb{R}$.

Ex: $X \sim Bern(0.3)$

Supp of Bern → (0, 1)

$$E[X] = 0 \cdot p(0) + 1 \cdot p(1) = 0.3$$



– Is $E[X] \in Supp[X]$? No. Either 0 or 1. Never a 0.3.

The expected value is unexpected. You need to do many experiments to find it!

General Case.

- $X \sim Bern(p)$

$$\underset{\text{1st value of supp}}{E[X] = 0 \cdot p(0)} + \overset{\text{2nd}}{1 \cdot p(1)} \ldots$$

$$= p$$

- $X \sim Binom(8, \frac{1}{2})$

$$\sum_{X \in Supp(X)} x\, p(x) \leftarrow E[X] = 0 \cdot p(0) + 1 \cdot p(1) + \ldots + 8 \cdot p(8).$$

According to the table we did

$$= 0 + 1 \cdot 0.031 + 2 \cdot 0.109 + 3 \cdot 0.219 + 4 \cdot 0.273 + \ldots + 8 \cdot 0.004$$

$$= \boxed{4}$$

4 is the expected value; the balance point (we see 4 most of the time) the mean!

General Case.

- $X \sim Binom(n, p)$

Not on Exam.

Binom P.M.F

$$\text{Proof} \quad E[X] = \sum_{X=0}^{n} x \left[\binom{n}{x} p^x (1-p)^{n-x}\right]$$

$$= \sum_{X=1}^{n} x \frac{\overset{\text{constant (pull it out)}}{n!}}{x!(n-x)!} \ldots = n \sum \underbrace{\frac{(n-1)!}{(x-1)!(n-x)!}}_{\binom{n-1}{x-1}} p^{x-1} (1-p)^{((n-1)-(x-1))}$$

$$(n-x) = (n-1)-(x-1)$$

let $m := n-1$

let $y := x-1$

$x = 1 \ldots n$

$y = 0 \ldots n-1 = 0 \cdot m$

$$= np \underbrace{\sum_{y=0}^{m} \binom{m}{y} p^y (1-p)^{m-y}}_{=1} = np.$$

That is the binom P.M.F sum over ...?

i.i.d experiments.

takes out · success · sample size.

- $X \sim Hyper(3, 4, 7)$

n	1 7	... 14	... 21
X	1 1 0 0 1 2 3	1 3 2 3 2 2 2	3 1 2 1 2 2 2
$\bar{X}$	1.143	1.643	1.714

↑ expected value.

* 4 spotted, 2 unspotted. 1 out = Bern $(\frac{4}{6})$

- $X \sim$ Geometric

* 2 spotted, 4 unspotted. = Bern $(\frac{2}{6})$ Bern $(\frac{1}{3})$.

- $\bar{X} \to E[X]$

As n gets bigger → i.i.d experiments in a long run → expected value.

average value

= "Law of Large #'s"

* 4 spotted 2 unspotted. = Bern $(\frac{4}{6})$ = Bern $(\frac{2}{3})$.

taking 3 out

= Neg Bin $(3, \frac{2}{3})$

↓

waited until you get 3 success.