$$M_{\bar{X}}(t) = E[e^{t\bar{X}}] = E[e^{x(\frac{X_1 + \dots X_n}{n})}] = M_{X_1}$$

→ Nov 29. 2016

Def. moment generating function for v.r. X:

$$M_X(t) = E(e^{tX})$$

Recall: $e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} \dots$ (Taylor series expansion)

Ⅰ $M_X(t) = M_Y(t) \Rightarrow X \overset{d}{=} Y$

Ⅱ $E[X^k] = M_X^{(k)}(0)$

Ⅲ $Y = aX + C \Rightarrow M_Y(t) = e^{tc} M_X(at)$

Ⅳ if X, Y independent, $M_{X+Y}(t) = M_X(t) \cdot M_Y(t) \overset{\text{if iid}}{\approx} (M_Y(t))^2$

$X \sim Bern(p) \Rightarrow M_X(t) = 1 - p + pe^t$

$X \sim Bino(n,p) \Rightarrow M_X(t) = (1-p+pe^t)^n$

$X \sim Geom(p) \Rightarrow M_X(t) = \frac{pe^t}{1 - e^t(1-p)}$ Ⅱ $t < \ln(\frac{1}{1-p})$

$X \sim Exp(\lambda) \Rightarrow M_X(t) = \frac{\lambda}{\lambda - t}$ Ⅱ $t < \lambda$

$Z \sim N(0,1) \Rightarrow M_Z(t) = e^{t^2/2}$

$X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$X \sim Deg(c) \Rightarrow M_X(t) = e^{tc}$

Levy's Continuity Theorem.

$X_1, X_2 \dots X_n$ is a sequence of r.v.

$\lim_{n \to \infty} M_{X_n}(t) = M_Y(t) \Leftrightarrow \underset{\text{Converges}}{X_n \to Y}$

if $n \to$ large

$M_{X_n}(t) \approx M_Y(t) \Rightarrow X_n \overset{d}{\approx} Y$

請记住此事只当n变大

Ex: if $X_1, \cdots X_n$ iid same distr mean μ. $\Rightarrow \bar{X}_n \to \mu$

$\mu \sim Deg(\mu)$

$$\lim_{n\to\infty} M_{\bar{X}_n}(t) = e^{t\mu}$$

$$M_{\bar{X}_n}(t) = M_{\frac{T}{n}}(t) = M_T\left(\tfrac{t}{n}\right) = \prod_{i=1}^{n} M_{X_i}\left(\tfrac{t}{n}\right) = \left(m_X\left(\tfrac{t}{n}\right)\right)^n$$

fact III

fact IV

$$= \left(E\left[e^{\frac{t}{n}X}\right]\right)^n = \left(E\left[1 + \frac{tX}{n} + \frac{t^2X^2}{2!\,n^2} + \frac{t^3X^3}{3!\,n^3} \cdots\right]\right)^n$$

Def MGF

Tayler series exp e^{tX}

$$= \left(E\left[1 + \frac{tX}{n} + o\left(\tfrac{1}{n}\right)\right]\right)^2 = \left(1 + \frac{t\mu}{n} + E\left[o\left(\tfrac{1}{n}\right)\right]\right)^n$$

$$= \left(1 + \frac{t\mu}{n} + o\left(\tfrac{1}{n}\right)\right)^n$$ Ignoe expectation

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$, $\lim_{n\to\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n = e$

$\lim_{n\to\infty}\left(1+\frac{1}{n}+o\left(\frac{1}{n}\right)\right)^n = e$, $\lim_{n\to\infty}\left(1+\frac{a}{n}+o\left(\frac{1}{n}\right)\right)^n = e^a$

$$\lim_{n\to\infty} M_{\bar{X}_n}(t)$$

$$= \lim_{n\to\infty}\left(1 + \frac{t\mu}{n} + o\left(\tfrac{1}{n}\right)\right)^n = e^{t\mu} \qquad \boxed{\bar{X} \to \mu}$$

Ex: let $X_1, \dots X_n \overset{iid}{\sim}$ with mean μ and s.e. σ,

$\bar{X}$ standard to has mean 0, s.e. 1

$$C_n := \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} = \frac{\sqrt{n}\left(\frac{X_1 + \dots X_n}{n} - \mu\right)}{\sigma}$$

$$= \frac{\sqrt{n}\left(\frac{X_1 + \dots X_n}{n} - \frac{n\mu}{n}\right)}{\sigma} = \frac{\sqrt{n}\,\frac{X_1 - \mu + X_2 - \mu + \dots X_n - \mu}{n}}{\sigma}$$

$$= \frac{(X_1 - \mu) + (X_2 - \mu) + \dots + (X_n - \mu)}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}}\left(\frac{X_1 - \mu}{\sigma} + \frac{X_2 - \mu}{\sigma} + \dots \frac{X_n - \mu}{\sigma}\right)$$

let $Z_i := \frac{X_i - \mu}{\sigma}$ (Z_i is X_i stdial)

$$= \frac{1}{\sqrt{n}}(Z_1 + Z_2 + \dots Z_i) \qquad E[Z_i] = 0,\ SE[Z_i] = 1$$

$$M_{C_n}(t) = M_{\frac{1}{\sqrt{n}}(Z_1 + \dots Z_i)}(t) \overset{\text{by III}}{=} M_{Z_1 + \dots Z_n}\left(\frac{t}{\sqrt{n}}\right) \overset{\text{by IV}}{=} M_{Z_1}\left(\frac{t}{\sqrt{n}}\right) + \dots M_{Z_i}\left(\frac{t}{\sqrt{n}}\right) = \left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

$$\lim_{n \to \infty} M_{C_n}(t) = \left(E\left[e^{\frac{t}{\sqrt{n}}Z}\right]\right)^n = \left(E\left[1 + \frac{tZ}{\sqrt{n}} + \frac{t^2Z^2}{2!\sqrt{n}} + \frac{t^3Z^3}{3!\sqrt{n}} + \dots\right]\right)^n$$

$$= \left(E\left[1 + \frac{tZ}{\sqrt{n}} + \frac{t^2Z^2}{n} + o\left(\frac{1}{n}\right)\right]\right)^n$$

$$= \left(1 + E\left[\frac{tZ}{\sqrt{2}}\right] + E\left[\frac{t^2Z^2}{n}\right] + E\left[o\left(\frac{1}{n}\right)\right]\right)^n$$

$$\frac{t}{\sqrt{n}}E[Z] = 0, \quad \frac{t^2}{\sqrt{n}}E[Z^2] = \frac{t^2}{\sqrt{n}}$$

$$= \left(1 + \frac{t^2}{n} + o\left(\frac{1}{n}\right)\right)^n$$

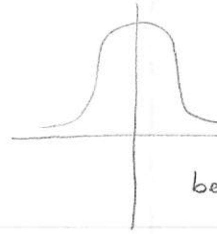
$$\lim_{n \to \infty} M_{C_n}(t) = \lim_{n \to \infty}\left(1 + \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n = e^{\frac{t^2}{2}} \to \text{the MGF of } Z \sim N(0,1)$$

$C_n \to N(0,1)$ "CLT central limit theorem"

(I) If $X_1, \dots X_n \overset{iid}{\sim}$ with mean μ, s.e. σ $\Rightarrow \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \to Z \sim N(0,1)$

if $n \to$ large. (II) $\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \overset{d}{\approx} Z \sim N(0,1)$ (III) $\Rightarrow \bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

(IV) $\Rightarrow T \overset{d}{\approx} N(\mu, (\sqrt{n}\,\sigma)^2)$



$e^{-\frac{x^2}{2}}$

bell curve.

$$Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

shift

$$X = \mu + \sigma Z \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

Why call normal? because all outcome is in the Normall range.

Ex: $X_1, \dots, X_{30} \overset{iid}{\sim} \text{Geom}(p)$, $p = \frac{1}{2}$

Mpx is prob. the avg. realization is more than 2.75?

$\frac{\sigma}{\sqrt{n}} = \frac{1.414}{\sqrt{30}} = 0.250$

$P(\bar{X} > 2.75)$ $\qquad \bar{X} \overset{d}{\approx} N(\mu, (\frac{\sigma}{\sqrt{n}})^2)$, $\quad \mu = \frac{1}{p} = \frac{1}{\frac{1}{2}} = 2$; $\sigma = \sqrt{\frac{1-p}{p^2}} = \sqrt{2} \approx 1.414$

$$= P\left(\frac{\bar{X}-2}{0.250} > \frac{2.75-2}{0.250}\right) \approx P(Z > 3) = 0.15\%$$

by the CLT

$\bar{X} \overset{d}{\approx} N(2, 0.250^2)$

Ex: Random Walk 100 setp. What's the prob. you are more than 10 step away from where you start?

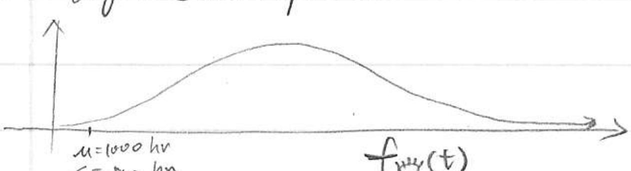
Let $T = X_1 + X_2 + X_3 + \dots X_{100}$; $X_1 \dots X_{100} \overset{iid}{\sim} \begin{cases} 1 & wp\ \frac{1}{2} \\ -1 & wp\ \frac{1}{2} \end{cases}$ $\quad \mu = 0$, $\sigma = 1$

From the CLT we know, $T \overset{d}{\approx} N(n\mu, (\sqrt{n}\sigma)^2) \Rightarrow T \overset{d}{\approx} N(0, 10^2)$

$$P(|T| > 10) = P(T < -10) + P(T > 10) = P\left(\frac{T-0}{10} < \frac{-10-0}{10}\right) + P\left(\frac{T-0}{10} > \frac{10-0}{10}\right)$$

$$= P(Z < -1) + P(Z > 1) = P(Z \notin [-1, 1]) = 32\%.$$

Ex: Light bulb lifetime.



$f_{💡}(t)$

you buy 50 bulbs, what is the prob the avg bulb lasts more than 1300 hr?

$\bar{X} \overset{d}{\approx} N(\mu, (\frac{\sigma}{\sqrt{n}})^2)$ by CLT.

$$P(\bar{X} > 1300) = P\left(\frac{\bar{X}-1000}{\frac{500}{\sqrt{50}}} > \frac{1300-1000}{\frac{500}{\sqrt{50}}}\right)$$