Lecture #18 Math 241 A. Kapelner 11/17/16

	PMF	PDF	CDF	E[X]	Var(X)
Discrete r.v.	$p(x) \in [0,1]$ $\sum_{x \in Supp[X]} p(x) = 1$	NO PDF	Yes	$\sum_{x \in Supp[X]} x\,p(x)$	$\sum_{x \in Supp[X]} (x-\mu)^2 p(x)$
Continuous r.v	NO PMF	$f(x) \geq 0$ $\int_{Supp[X]} f(x)dx = 1$ $f(x) = \frac{d}{dx}(F(x))$	Yes	$\int_{Supp[X]} x f(x) dx$	$\lvert\mathbb{R}\rvert \leq \lvert\mathbb{N}\rvert$ $\int_{Supp[X]} (x-\mu)^2 f(x) dx$

Quantil(x, p)
$\min\{x : F(x) \geq p\}$
$x = F^{-1}(p)$ x st $F(x) = p$

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \frac{1}{e^{x^2/2}}$ where is it 0?

(a) $f(x) \geq 0$ $x \in \mathbb{R}$

(b) $\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$? No need to know.

The Proof:

$\frac{1}{\sqrt{2\pi}} \int e^{-\frac{x^2}{2}} dx = 1 \Rightarrow \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \Rightarrow \int_{\mathbb{R}} e^{-u^2} du \sqrt{2} = \sqrt{2\pi} \Rightarrow \int_{\mathbb{R}} e^{-u^2} du = \sqrt{\pi}$

$= \left(\int_{\mathbb{R}} e^{-u^2} du\right)^2 = \pi \Rightarrow \int e^{-u^2} du \int e^{-u^2} du = \pi \Rightarrow \int e^{-x^2} dx \int e^{-x^2} dy = \pi$

$\Rightarrow \underbrace{\iint_{x \in \mathbb{R}\ y \in \mathbb{R}}}_{\text{"Area"}} e^{-(x^2+y)} dx dy = \iint_{r \in (0,\infty)\ \theta \in (0,2\pi)} e^{-r^2} r dr d\theta = \int_{r \in (0,\infty)} e^{-r^2} r dr \int_{\theta \in (0,2\pi)} d\theta$

Recap
$r dr d\theta = dA = dx dy$

$\Rightarrow 2\pi \int e^{-r^2} r dr \stackrel{?}{=} \pi \Rightarrow \int_0^\infty e^{-r^2} r dr = \frac{1}{2}$

let $u = r^2$
$\frac{du}{dr} = 2r \Rightarrow du = 2r dr$
$\Rightarrow r dr = \frac{du}{2}$

$\Rightarrow \int_0^\infty e^{-u} \frac{du}{\cancel{2}} = \frac{1}{\cancel{2}} = \left[-e^{-u}\right]_0^\infty = \left(e^{-(0)} - \lim_{x \to \infty} e^{-x}\right) = 1 - 0 = 1$ ∎

$\Rightarrow$ qtg.

$Z \sim N(0,1) \cdot f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ → Bell Curve.

↳ Standard Normal Random Variable. aka "Gaussian"

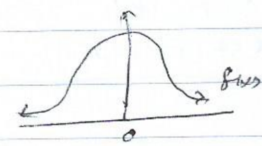
$$E[Z] = \int x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x e^{-x^2/2} \, dx = \frac{1}{2\sqrt{\pi}} \int_{x=-\infty}^{x=\infty} e^{-u} \, du = \frac{1}{\sqrt{2\pi}} \left[-e^{-u} \right]_{x=-\infty}^{x=\infty}$$

let $u = \frac{x^2}{2} \Rightarrow \frac{du}{dx} = x$;

$du = x\,dx$;

$$= \frac{1}{\sqrt{2\pi}} \left(\lim_{x \to -\infty} \underbrace{e^{-\frac{x^2}{2}}}^{\frac{1}{e^{x^2/2}}} - \lim_{x \to \infty} e^{-x^2/2} \right) = \frac{1}{\sqrt{2\pi}} (0 - 0) = 0$$

$\Rightarrow \boxed{\mu = 0}$



$$Var[Z] = E(Z^2) - \cancel{\mu^2}^{\,0} = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = \underbrace{\Longrightarrow}_{\text{done by Integ. by parts method.}} 1. \Rightarrow SE[Z] = 1$$

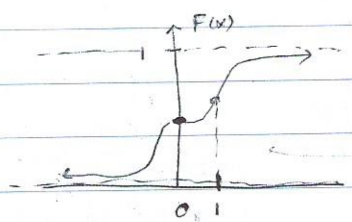
$\boxed{\sigma^2 = \sigma = 1}$

CDF:

$$F(x) = \int f(x)\,dx + C = \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + C}$$

Not possible. "use computer to solve"

"Risch Algorithm" → Wiki.



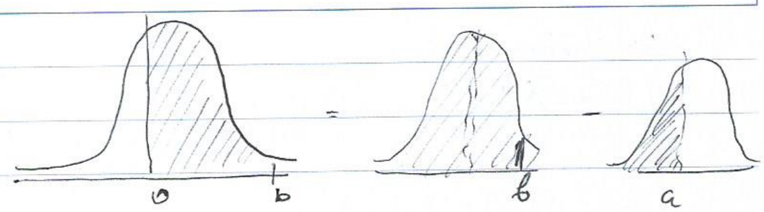
$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

Approximation.

F(0). "half of bell" "full ⇒ 1"

Express ... in terms of CDF

$P(Z \in [a,b]) =$



$F(b) - F(a)$

"between the given intervals"

"68-95-99.7 rule" "empirical rule"

$$\begin{cases} P(Z \in [-1,1]) = F(1) - F(-1) \approx .68 \\ P(Z \in [-2,2]) = F(2) - F(-2) \approx .95 \\ P(Z \in [-3,3]) = F(3) - F(-3) \approx .997 \end{cases}$$

$SE[Z] = 1$.

Exponential and uniform R.V. [Review]

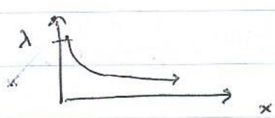
$X \sim Exp(\lambda) := \lambda e^{-\lambda x}$

$F(x) = 1 - e^{-\lambda x}$

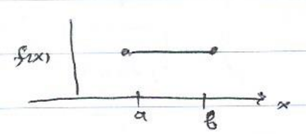
$X \sim$ geometric (p) $\quad Exp[X] = \frac{1}{p} \quad \lambda = np$ ($n \to \infty$, $p \to 0$) wtf.

Continue;

$F(x) = 1 - e^{-\lambda x} \quad E(x) = \frac{1}{\lambda}$

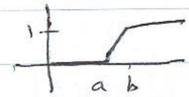


$X \sim U(a,b) = \underbrace{\frac{1}{b-a}}_{f(x)}$



f≡s.

$F(x) = \frac{x-a}{b-a}$



$X \sim Exp(\lambda)$

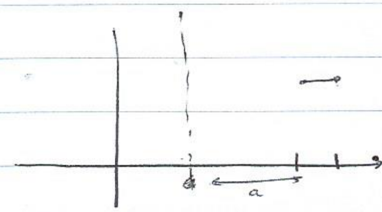
$Y = 2X \sim ?$ "How's this distributed?" $\quad \lambda' := \lambda/2$

$F_Y(x) = P(Y \le x) = P(2X \le x) = P(X \le \frac{x}{2}) = F_X(\frac{x}{2}) = 1 - e^{-\lambda x/2} = 1 - e^{-\lambda' x} \Rightarrow Y \sim Exp(\lambda') \Rightarrow$

$\Rightarrow \boxed{Y \sim Exp(\frac{\lambda}{2})}$

$X \sim \text{Uniform}(a, b)$ 

$Y = d + cX$?



$F_Y(x) = P(Y \le x) = P(d + cX \le x) = P\left(X \le \frac{x-d}{c}\right) = F_X\left(\frac{x-d}{c}\right) = \frac{\frac{x-d}{c} - a}{b-a} \cdot \frac{c}{c} =$

$= \frac{x - (d + ac)}{bc - ac} = \frac{x - (d + ac)}{(d + bc) - (d + ac)} = \frac{x - a'}{b'' - a'}$ $\rightarrow$ $Y \sim \text{Unif}(a', b')$

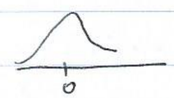
let $a' := d + ac$

$b' := d + bc$

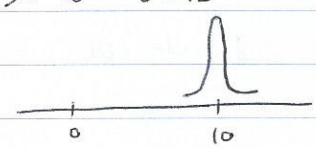
$\rightarrow$ $Y \sim U(d + ac, d + bc)$

Remember: $E[Z] = 0$ $SE[Z] = 1$

let, $X = \mu + \sigma Z$

Start with 

$\mu = 10$ $\sigma = 1/2$



"Still a bell shape"

$F_X(x) = P(X \le x) = P(\mu + \sigma Z \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right) \Rightarrow$ Stop.

$= F_Z\left(\frac{x-\mu}{\sigma}\right)$

$f(x) = \frac{d}{dx} F(x)$ $\longrightarrow$ $\frac{d}{dx}\left[F_Z\left(\frac{x-\mu}{\sigma}\right)\right]$ let $u = \frac{x-\mu}{\sigma} \Rightarrow \frac{du}{dx} = \frac{1}{\sigma}$

PDF

Thus, $\frac{du}{dx} \frac{d}{du}\left[F_Z(u)\right] = \frac{1}{\sigma} f_Z(u) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right) =$

$= \frac{1}{\sigma}\left(\frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2 / 2}\right) = \boxed{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}$ $\longrightarrow$ Normal PDF.

$\frac{(x-\mu)^2}{\sigma^2} / 2$

$E[X] = E[\mu + \sigma Z] = \mu$

$SE[X] = SE[\mu + \sigma Z] = |\sigma|$

parameters.

Param. space: $\mu \in \mathbb{R}$, $\sigma^2 \in (0, \infty)$

Supp $[X] = \mathbb{R}$

$X \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$11 - 4 = 7 - 1 = 6$

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MIQUELRIUS