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a function of two variables tha spits out 1 number.

$T = X_1 + X_2 \rightarrow g(X_1, X_2)$ $E[T] = \sum_{t \in supp[T]} t \; p(t)$

$f(x_1, x_2) = x_1^2 - \sqrt{x_2}$
$\mathbb{R}^2 \rightarrow \mathbb{R}$

$E[g(x_1, x_2)] = \sum_{\vec{x} \in Supp[\vec{x}]} (g(\vec{x}) \, p(\vec{x})$

$\sum_{(x_1, x_2)} g(x_1, x_2) \, p(x_1, x_2) \rightarrow$ joint mass function

$E[x_1 + x_2] = \sum_{\langle x_1, x_2 \rangle} (x_1 + x_2) \, p(x, x_2) =$

$\hookrightarrow \sum_{x_1 \in supp[x_1]} \sum_{x_2 \in supp[x_2]} x_1 \, p(x_1, x_2) + \sum_{x_2 \in supp[x_2]} \sum_{x_1 \in supp[x_1]} x_2 \, p(x_1, x_2)$

Assume X_1, X_2 indep.
$\Rightarrow p(x_1, x_2) = p(x_1) p(x_2)$

$= \sum_{x_1} x_1 \sum_{x_2} p(x_1, x_2) + \sum_{x_2} x_2 \sum_{x_1} p(x_1, x_2)$
$\hookrightarrow = E[x_1] + E[x_2]$

$= \sum_{x_1} x_1 \cdot \sum_{x_2} p(x_1) \, p(x_2) + \sum_{x_2} x_2 \sum_{x_1} p(x_1) p(x_2) =$

$= \sum_{x_1} \underbrace{x_1 p(x_1)}_{E[X_1]} \sum_{x_2} \underbrace{p(x_2)}_{1} + \sum_{x_2} \underbrace{x_2 p(x_2)}_{E[X_2]} \sum_{x_1} \underbrace{p(x_1)}_{1}$

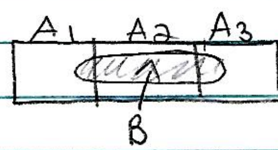
ex.

		X_1: 1	7	19	$p(x_2)$
X_2	5	$\frac{1}{15}$	$\frac{1}{3}$	$\frac{2}{15}$	$\frac{16}{30}$
	23	$\frac{1}{30}$	$\frac{1}{10}$	$\frac{1}{30}$	$\frac{5}{30}$
	88	$\frac{1}{30}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{9}{30}$
	$p(x_1)$	$\frac{4}{30}$	$\frac{19}{30}$	$\frac{7}{30}$	1

$P(x_1, x_2)$

$p(x_1)$ marginal

$Supp[X_1] = \{1, 7, 19\}$
$Supp[X_2] = \{5, 23, 88\}$
$P(B) = P(B, A_1) + P(B, A_2) + P(B, A_3)$



$P(X_1 = 1) = \frac{4}{30}$

$P(X_1 = 1, X_2 = 88) = P(X_1 = 1) P(X_2 = 88)$
$\frac{1}{30} \neq \frac{4}{30} \cdot \frac{9}{30}$

for any r.v.'s $X_1, \dots X_n$

$E[X_1 + \dots X_n] = E[X_1] + \dots E[X_n]$

$E[T] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$

If $X_1, \dots X_n$ identically distributed .. w/ exp. μ
(not necessarily independent)

$\boxed{E[T] = n\mu}$ Further $E[\bar{X}] = E\left[\frac{T}{n}\right] = \frac{1}{n} E[T] = \frac{1}{\cancel{n}} \cancel{n} \mu = \boxed{\mu}$

Recall $X_1, X_2, \dots X_r \overset{iid}{\sim} \text{Geom}(p)$
$T \sim \text{Neg Binom}(r, p)$

$E[T] \sum_{x=r}^{\infty} x \binom{x-1}{r-1} (1-p)^{x-r} p^r$ $\boxed{E[T] = \frac{r}{p}}$ $\rightarrow n\mu$

$X \sim \text{Hyper}(n, K, N)$ $X = X_1 + X_2 \dots + X_n$

$E[X] = \sum_{x \in supp[X]} \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$ $X_1, X_2 \dots X_n$ inden. distri

$\text{Bern}\left(\frac{K}{N}\right)$
$\downarrow$
μ

$\boxed{E[X] = n\frac{K}{N}}$

$Var[X] := E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2]$

$E[X^2] + E[-2\mu X] + E[\mu^2] = E[X^2] - 2\mu E[X] + \mu^2$

$= E[X^2] - 2\mu^2 + \mu^2 \rightarrow E[X^2] - \mu^2 = \sigma^2$ $\quad E[X] \rightarrow \mu$

$E[X^2] = \sigma^2 + \mu^2$

$E[X]$ 1^{st} moment $\quad E[|X-\mu|]$ 1^{st} centered moment

$E[X^2]$ 2^{nd} moment $\quad E[|X-\mu|^2]$ 2^{nd} centered moment $\rightarrow$ variance

$\vdots$ $\quad\quad \vdots$

$E[X^k]$ k^{th} moment $\quad E[|X-\mu|^k]$ k^{th} centered moment

$\left.\begin{array}{l} Var[X] \\ Var[X+c] \end{array}\right\}$ the same

$c \in \mathbb{R}$

$Var[X+c] = E[((X+\cancel{c}) - (M+\cancel{c}))^2] = E[(X-M)^2] = Var[X]$

$Var[X]$

$Var[aX]$ s.t $a \in \mathbb{R}$ $\rightarrow Var[aX] = E[(\underbrace{aX - aM}_{a(X-M)})^2] = E[a^2(X-M)^2]$

$= a^2 E[(X-M)^2] = a^2 Var[X]$

$SE[aX] = \sqrt{Var[aX]} = \sqrt{a^2 Var[X]} = |a| SE[X]$

$\boxed{\begin{array}{l} Var[aX+c] = a^2 \sigma^2 \\ SE[aX+c] = |a| SE[X] \end{array}}$

Two r.v's X_1, X_2

$Var[X_1+X_2] = E[((X_1+X_2) - (M_1+M_2))^2]$

$= E[X_1^2 + X_2^2 + M_1^2 + M_2^2 + 2X_1X_2 - 2X_2M_2 - 2X_1M_2 - 2X_2M_1 + 2M_1M_2]$

$= E(X_1^2) + E(X_2^2) + M_1^2 + M_2^2 + 2E[X_1, X_2] - 2M_1^2 - 2M_2^2 - 2M_1M_2$

$- 2\cancel{M_1M_2} + 2\cancel{M_1M_2}$

$= E[X_1^2] + E[X_2^2] - M_1^2 - M_2^2 + 2(E[X_1X_2] - M_1M_2)$

$\boxed{Var[X_1+X_2] = \sigma_1^2 + \sigma_2^2 + 2Cov[X_1, X_2]}$

Two r.v's X_1, X_2 indep. $\Rightarrow Var[X_1+X_2] = Var[X_1] + Var[X_2]$

$E[X_1X_2] \sum_{X_1}\sum_{X_2} X_1X_2\, p(X_1, X_2) \rightarrow \sum_{X_1}\sum_{X_2} X_1X_2\, p(X_1)\, p(X_2)$

$\sum_{X_1} X_1 p(X_1) \sum_{X_2} X_2 p(X_2) = E[X_1] \cdot E[X_2] = M_1 M_2$

$Cov[X_1, X_2] = M_1M_2 - M_1 - M_2 = 0$

* If $X_1, X_2, \ldots X_n$ indep

(not necessarily identically dist.)

$Var[T] \Rightarrow Var\left[\sum_{i=1}^{n}\right] = Var[X_1 + \ldots X_n] = Var[X_1] + \ldots + Var[X_n] = \sum_{i=1}^{n} \sigma_i^2$

$X_1, X_2, \cdots, X_n \overset{iid}{\sim}$

$$\boxed{\begin{aligned} Var[T] &= n\sigma^2 \\ SE[T] &= \sqrt{n}\,\sigma \end{aligned}}$$

If $X_1 \ldots X_n$ iid $E[\bar{X}] = \mu$

$$Var[\bar{X}] = Var\left[\frac{T}{n}\right] = \frac{1}{n^2} Var[T] = \frac{1}{\cancel{n^2}} \cancel{n}\sigma^2 = \boxed{\frac{\sigma^2}{n}} = \boxed{SE[\bar{X}] = \frac{\sigma}{\sqrt{n}}}$$

$p(x) = PMF$

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$E[ax+c] = a\mu + c$

$E[T_n] = \sum_{i=1}^{n} E[X_i] = n\mu$ → if identical distribution

$Var[ax+c]\ a^2\sigma^2 \rightarrow SE[ax+c] = |a| = \sigma$

If $X_1 \ldots X_n$ indep

$Var[T_n] = \sum_{i=1}^{n} Var[X_i] = n\sigma^2$ → if iid

$E[\bar{X}] = \mu$ if ident. distrib.

$Var[\bar{X}] = \frac{\sigma^2}{n}$ if iid

→ $SE[\bar{X}] = \frac{\sigma}{\sqrt{n}}$ if iid

$Y \sim Geom(p) = (1-p)^{Y-1}$

$Var[Y] = E[(Y-\mu)^2]$

$= E[(Y-\frac{1}{p})^2]$

$= E[Y^2] - (\frac{1}{p})^2 \rightarrow = \boxed{\frac{1-p}{p^2}}$ → variance for Geom.

$E[g(x)] = \sum_{x \in supp(x)} g(x)\, p(x)$

$E[Y^2] = \sum_{Y=1}^{\infty} Y^2 (1-p)^{Y-1} p$

let $Z = Y-1 \rightarrow Y = Z+1$

$Y = 1 \ldots \infty \rightarrow Z = 0 \ldots \infty$

$p\left(\sum_{Z=0}^{\infty} (Z+1)^2 (1-p)^2 = \right.$ $\rightarrow Z^2 + 2Z + 1$

$(1-p)\underbrace{\sum_{Z=1}^{\infty} Z^2 (1-p)^{Z-1} p}_{E[Y^2]} + (1-p)2\underbrace{\sum_{Z=1}^{\infty} Z (1-p)^{Z-1} p}_{\frac{1}{p}} + \overbrace{p\sum_{Z=0}^{\infty} (1-p)^Z}^{\frac{1}{p}}$

$\boxed{\text{CDF Geom} = 1-(1-p)^x}$

$X \sim \text{NegBinom}(r, p)$

$\text{Var}[X] = \boxed{r \frac{1-p}{p^2}}$

The geometric r.v has the memorylessness property

$P(X=7) \stackrel{?}{=} P(X=17 \mid X>10)$

$= \frac{P(X=17 \,\&\, P>10)}{P(X>10)}$ → $= 1-P(X \leq 0$

$= 1 - F(10)$

$= 1-(1-(1-p)^x) = 1-p^{10}$

$= \frac{(1-p)^{16} p}{(1-p)^{10}}$

$= (1-p)^6 p = P(X=7)$ ✓

$P(X=a) = P(X = a+b \mid X>b)$ → $P(X=a+b) = (1-p)^{a+b-1} p$

$= (1-p)^{a-1} p$

$\frac{P(X=a+b \text{ and } X>b)}{P(X>b)}$

$(1-p)^b$

$= \frac{(1-p)^{a+b-1} p}{(1-p)^b}$

$= (1-p)^{a-1} p$

Review

- Expectation and variance have no meaning unless number is large

Exam: HW, 4, 5, 6