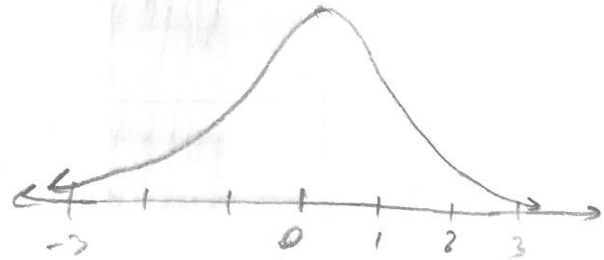
Lec 19 Math 241 11/22/16

$Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$



$Supp(Z) = \mathbb{R}$

$\underbrace{E(Z) = 0,\ SE(Z) = 1}_{\text{standard}}$

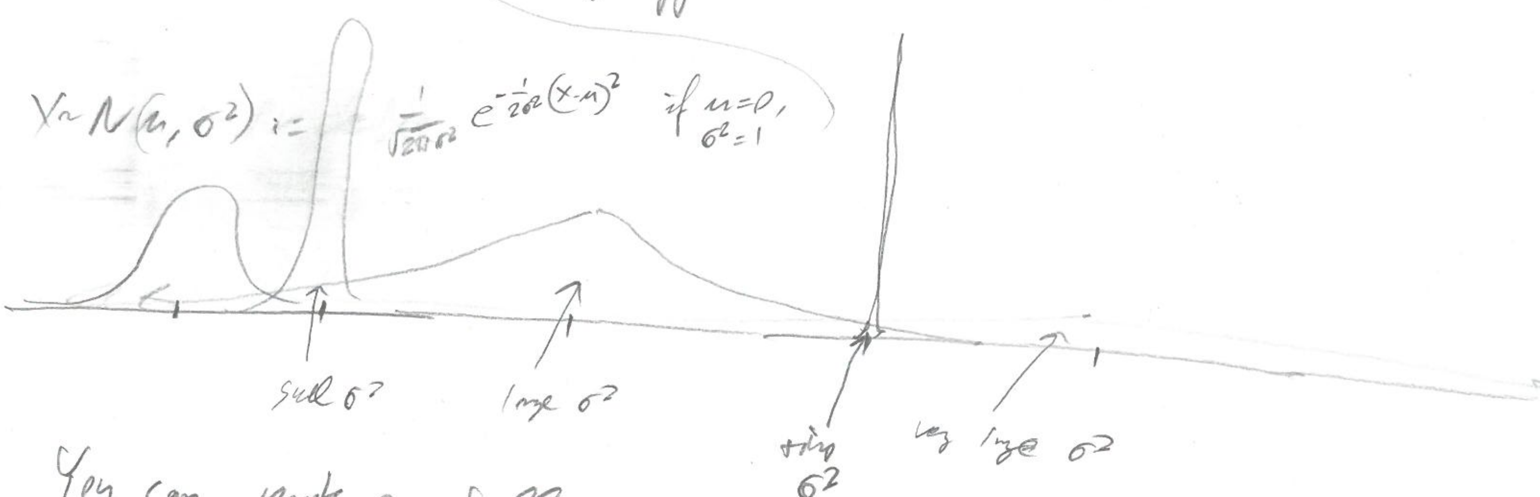
why draw this way?

$P(Z \in [-3,3]) \approx 99.7\%$
$P(Z \in [-2,2]) \approx 95\%$
$P(Z \in [-1,1]) \approx 68\%$ } Empirical Rule

Z-scores
-2? $+1$? $+3$?
meaning?

why approx? Exact answers unknown...

$X \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ if $\mu = 0$, $\sigma^2 = 1$



You can make any bell curve you want all $Supp(X) = \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma^2 \in (0, \infty)$

$\sigma^2 = 0$? Not possible. $\sigma^2 < 0$... $Var(X) \geq 0$ so not possible

Recall: $X = \mu + \sigma Z \Rightarrow Z = \frac{X - \mu}{\sigma}$

I can take any normal rv do this linear transform and get std normal

e.g. X is Male Height. Height is normally distr, $\mu = 70''$, $\sigma = 4''$

What is the prob someone is more than $78''$ tall?

$$X \sim N(70'', 4''^2)$$

$$\Rightarrow Z = \frac{X - 70''}{4''} \sim N(0,1)$$

78'' = z-score = 2

$$P(X \geq 78'') = P\left(\frac{X-70''}{4''} \geq \frac{78-70''}{4''}\right) = P(Z \geq 2) \approx 2.5\%$$

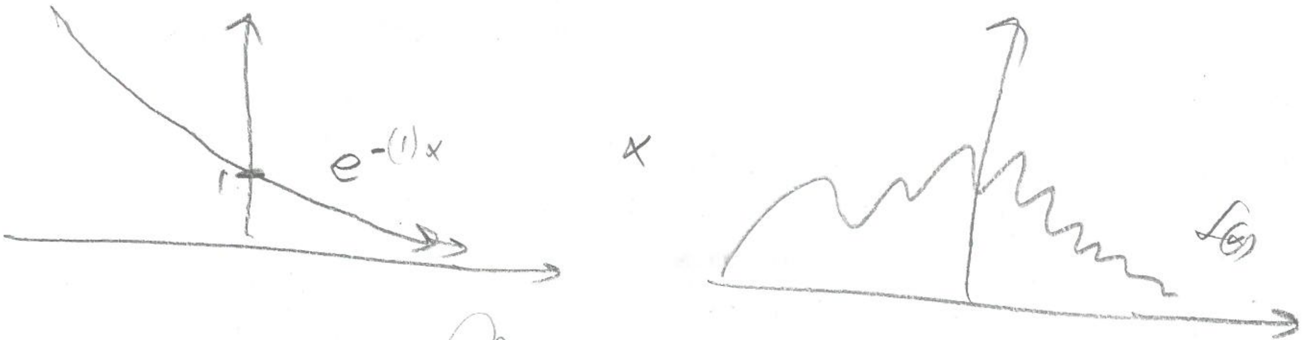
Lots of things are normally distr... why?... Patience...

$$\underbrace{\text{let } L(t) = \int_{\mathbb{R}} e^{-tx} f(x)\, dx}$$

L is called the "Bilateral Laplace Transform" of f

What does this look like?

$$L(1) = \int e^{-(1)x} f(x)\, dx$$



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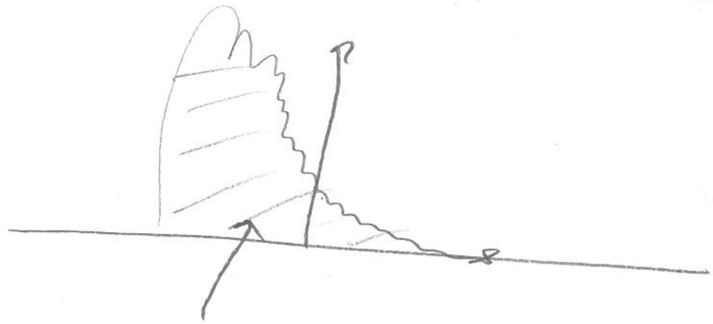
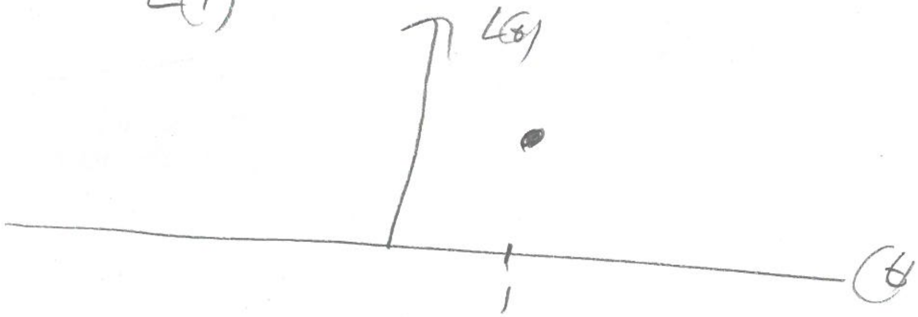


$L(1) =$



Do this for all values of t..

Thm: if $L(t)$ exists... $L(t)$ & $f(x)$ are 1:1.

$L(t) \Rightarrow f(x)$, $f(x) \Rightarrow L(t)$

Note: if $f(x)$ is PDF then

since $E g(x) = \int g(x) f(x)\, dx$

$$\int_{\mathbb{R}} e^{-tx} f(x)\, dx = E\left[e^{tX}\right]$$

Define $\underbrace{M_X(t)}$ $:= E[e^{tX}] = \int e^{tx} f(x)\,dx$ cont.

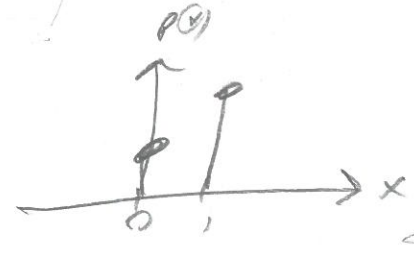
$= \sum_{x \in S_p(x)} e^{tx} p(x)$ discrete.

moment generating function (MGF) of r.v. X

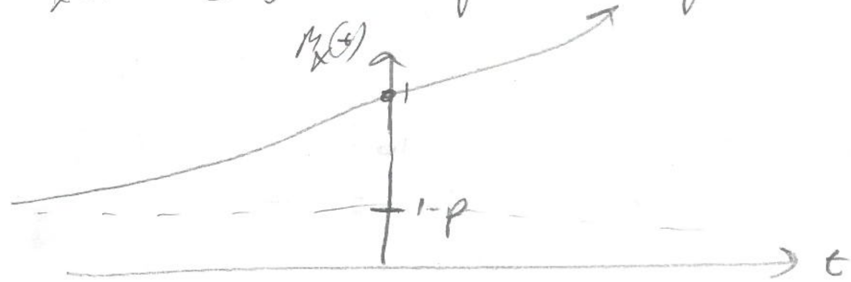
$\boxed{\text{I} \quad M_X(t) \text{ is 1:1 with } f(x) \text{ or } p(x)}$

identically distr. r.v.'s have same mgf!

e.g. $X \sim Bern(p) := p^x (1-p)^{1-x}$



$M_X(t) = E[e^{tX}] = e^{t(0)} p(0) + e^{t(1)} p(1) = 1-p + pe^t$



are the same!!

Only imply each other..

genotype vs phenotype

Why is $M_X(t)$ useful..

Consider $X \sim Binomial(n,p)$ $E[X^{17}] = \underbrace{\sum_{x=0}^{n} x^{17} \binom{n}{x} p^x (1-p)^{n-x}}_{\text{not possible to figure out...}}$

Recall $f(x)$ when $x \approx c$ can be approximated by...

$$f(x) \approx f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3$$ 3rd degree approx.

$$\approx \sum_{i=0}^{\infty} \frac{f^{(i)}(c)}{i!}(x-c)^i$$

let $f(x) = e^x$ to see what happens when we do a Taylor expansion $x \approx 0 \Rightarrow c = 0$

$$e^x \approx e^0 + \frac{e^0 x}{1!} + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \dots$$

$$\Rightarrow e^x = \sum_{i=0}^{\infty} \frac{1}{i!} x^i$$

$$e^{tX} = 1 + \frac{tX}{1!} + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \frac{t^4 X^4}{4!} + \dots$$

$$M_X(t) = E(e^{tX}) = E[\quad]$$

$$= 1 + tE(X) + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E(X^3) + \frac{t^4}{4!}E[X^4] + \dots$$

$$\frac{d}{dt} M_X(t) = M'_X(t) = E(X) + \frac{t}{1!}E(X^2) + \frac{t^2}{2!}E(X^3) + \frac{t^3}{3!}E(X^4) + \dots$$

let $t = 0$

$$M_X'(0) = E(X)$$

$$M''_X(t) = E[X^2] + \frac{t}{1!}E[X^3] + \frac{t^2}{2!}E(X^4) + \dots$$

$$M_X''(0) = E(X^2)$$

$$M'''_X(t) = E(X^3) + \frac{t}{1!}E(X^4) + \dots$$

this is why it gets its name!!! Generates moments...

$$M_X'''(0) = E(X^3)$$

II $\boxed{M_X^{(k)}(0) = E[X^k]}$

Other cool properties

$Y = aX + c$

III

$$M_Y(t) = M_{aX+c}(t) = E\left[e^{t(aX+c)}\right] = E\left[e^{atX} e^{tc}\right] = e^{tc} E\left[e^{t(aX)}\right] = e^{tc} M_X(at)$$

$Y = X_1 + X_2$ and X_1, X_2 indep

IV

$$M_Y(t) = M_{X_1+X_2}(t) = E\left[e^{t(X_1+X_2)}\right] = E\left[e^{tX_1} e^{tX_2}\right] = E\left[e^{tX_1}\right] E\left[e^{tX_2}\right] = M_{X_1}(t) M_{X_2}(t)$$

the mgf of a sum of indep. r.v.'s is the product of the mgfs of the constituent r.v.'s

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$X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(p)$

$T = X_1 + \ldots + X_n \sim \text{Binom}(n,p)$ what is mgf of binomial?

why? I

$$M_T(t) = M_{X_1+\ldots+X_n}(t) = M_{X_1}(t) \cdot \ldots \cdot M_{X_n}(t) = \left(M_X(t)\right)^n = \boxed{\left(1-p+pe^t\right)^n}$$

$X \sim \text{Geom}(p)$

$$M_X(t) = E\left[e^{tX}\right] = \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1}p = p \sum e^{tx}(1-p)^{x-1} = \frac{p}{1-p} \sum_{x=1}^{\infty} (e^t)^x (1-p)^x = \frac{p}{1-p} \sum_{x=1}^{\infty} \left(e^t(1-p)\right)^x$$

$$= \frac{p}{1-p}\left(\sum_{x=0}^{\infty}\left(e^t(1-p)\right)^x - 1\right)$$

if $e^t(1-p) < 1 \Rightarrow$

$\Rightarrow e^t < \frac{1}{1-p}$

$\Rightarrow t < \ln\left(\frac{1}{1-p}\right)$

$$= \frac{p}{1-p}\left(\frac{1}{1-e^t(1-p)} - 1\right) = \frac{p}{\cancel{1-p}} \frac{e^t\cancel{(1-p)}}{1-e^t(1-p)} = \frac{pe^t}{1-e^t(1-p)} \quad \text{if } t < \ln\left(\frac{1}{1-p}\right)$$

$X \sim Exp(\lambda)$

$$m_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx = \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)x}\right]_0^\infty = \frac{\lambda}{t-\lambda}(0-1) = \frac{\lambda}{\lambda - t} \mathbb{1}_{t<\lambda}$$

if $t-\lambda < 0 \Rightarrow t < \lambda$

$Y = a X$

$$m_Y(t) = m_{aX}(t) = m_X(at) = \frac{\lambda}{\lambda - at} = \frac{\lambda' a}{\lambda' a - ta} = \frac{\lambda'}{\lambda' - t} \Rightarrow Y \sim Exp\left(\overset{\lambda'}{\overbrace{\frac{\lambda}{a}}}\right)$$

$\lambda' = \frac{\lambda}{a}$

$X \sim Deg(c) \Rightarrow m_X(t) = E[e^{tX}] = e^{tc}$

LLN

$Z \sim N(0,1)$

$$m_Z(t) = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2 + tx} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-t)^2} e^{\frac{1}{2}t^2} dx$$

$$-\frac{1}{2}x^2 + tx = -\frac{1}{2}(x^2 - 2tx) = -\frac{1}{2}\left((x-t)^2 - t^2\right) = -\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2$$

complete the square

$$e^{\frac{t^2}{2}} \overbrace{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx}^{1} = e^{\frac{t^2}{2}}$$

$X \sim N(t,1) \Rightarrow f(x) =$

$X \sim N(\mu, \sigma^2) \Rightarrow X = \mu + \sigma Z$

$$\Rightarrow m_X(t) = e^{t\mu} e^{\frac{(\sigma t)^2}{2}} = e^{t\mu + \frac{\sigma^2 t^2}{2}}$$

Prove:

$$SE(Z) = 1 \Rightarrow SE(Z) = \sqrt{Var(Z)} = \sqrt{E(Z^2) - (m_Z'(0))^2} = \sqrt{E(Z^2)} = \sqrt{m_Z''(0)} = 1$$

$m_Z'(t) = t e^{\frac{t^2}{2}}$ $\quad m_Z'(0) = 0$

$m_Z''(t) = e^{t^2/2} + t^2 e^{t^2/2}$, $\quad m_Z''(0) = 1$