Lecture 15 11/8/16 Math 241

$T = X_1 + X_2$ $\quad E(T) = \sum_{t \in Supp(T)} t\, p(t)$ very complicated since structure...

Note $T = g(X_1, X_2) = g(\vec{X})$ s.t. $g: \mathbb{R}^2 \to \mathbb{R}$

$$E[g(\vec{X})] = \sum_{\vec{X} \in Supp(\vec{X})} g(\vec{X})\, p(\vec{X}) = \sum_{\langle x_1, x_2\rangle \in Supp(X_1) \times Supp(X_2)} g(x_1, x_2) \overbrace{p(x_1, x_2)}^{\text{joint mass function (jmf)}}$$

$$= \sum_{x_1 \in Supp(X_1)} \sum_{x_2 \in Supp(X_2)} g(x_1, x_2)\, p(x_1, x_2)$$

$$E(T) = E(X_1 + X_2) = \sum_{x_1} \sum_{x_2} (x_1 + x_2)\, p(x_1, x_2) = \sum_{x_1}\sum_{x_2} x_1\, p(x_1, x_2) + \sum_{x_2}\sum_{x_1} x_2\, p(x_1, x_2)$$

$$= \sum_{x_1} x_1 \sum_{x_2} p(x_1, x_2) + \sum_{x_2} x_2 \sum_{x_1} p(x_1, x_2)$$

Note if X_1, X_2 ind. $\Rightarrow p(x_1, x_2) = p(x_1)\, p(x_2)$

$$= \overbrace{\sum_{x_1} x_1\, p(x_1)}^{E(X_1)} \underbrace{\sum_{x_2} p(x_2)}_{1} + \overbrace{\sum_{x_2} x_2\, p(x_2)}^{E(X_2)} \underbrace{\sum_{x_1} p(x_1)}_{1}$$

$$= E(X_1) + E(X_2)$$

If not... we need to figure out

$$\sum_{x_2} p(x_1, x_2) \quad \& \quad \sum_{x_1} p(x_1, x_2)$$

Consider X_1, X_2 s.t. $Supp(X_1) = \{1, 7, 19\}$, $Supp(X_2) = \{5, 23, 88\}$

$p(X_1, X_2)$

X_2 \ X_1	1	7	19	$p(X_2)$ "marginal"
5	$\frac{1}{15}$	$\frac{1}{3}$	$\frac{2}{15}$	$\frac{16}{30}$
23	$\frac{1}{30}$	$\frac{1}{10}$	$\frac{1}{30}$	$\frac{5}{30}$
88	$\frac{1}{30}$	$\frac{1}{5}$	$\frac{1}{15}$	$\frac{9}{30}$
$p(X_1)$ "marginal"	$\frac{4}{30}$	$\frac{19}{30}$	$\frac{7}{30}$	1

$\sum\sum p(x_1, x_2) = ?$ 1 why?

Is X_1 and X_2 ?

$$\frac{4}{30} = P(X_1 = 1) \overset{?}{\neq} P(X_1 = 1 \mid X_2 = 23) = \frac{1}{30} \Rightarrow \text{No...}$$

What do we see here? $\sum_{x_1} p(x_1, x_2) = p(x_2)$, $\sum_{x_2} p(x_1, x_2) = p(x_1)$

Similar to $g(x) = \int f(x, y)\, dy$

where'd y go?? "Integrated out"

$$E(T) = \sum_{x_1} x_1 \underbrace{\sum_{x_2} p(x_1, x_2)} + \sum_{x_2} x_2 \underbrace{\sum_{x_1} p(x_1, x_2)}$$

$$= \sum_{x_1} x_1\, p(x_1) + \sum_{x_2} x_2\, p(x_2) = E(X_1) + E(X_2)$$

for any r.v.'s $X_1, X_2, \ldots, X_n$,

$$E(T) = E(\Sigma X_i) = \Sigma E(X_i) = E(X_1) + E(X_2) + \ldots + E(X_n)$$

for any r.v.'s $X_1, X_2, \ldots, X_n$ iden. distr (not necessarily indep.)

$$E(T) = \Sigma E[X_i] = nE(X_i) = n\mu$$

$$\Rightarrow E(\bar{X}) = E\left(\frac{T}{n}\right) = \frac{1}{n}E(T) = \frac{1}{n}n\mu = \mu$$

Recall $X_1, X_2, \ldots, X_r \overset{iid}{\sim} \text{Geom}(p)$

$$T = X_1 + X_2 + \ldots + X_r \sim \text{NegBin}(r, p) \quad \text{(by def.)}$$

$$E[T] = r\mu = r\frac{1}{p} = \boxed{\frac{r}{p}}$$

$X \sim \text{Hyper}(n, K, N)$

Imagine $X_1, X_2, \ldots, X_n$ are the r.v.'s for a single draw without replacement

$$X = X_1 + X_2 + \ldots + X_n$$

$X_1 \sim \text{Bern}\left(\frac{K}{N}\right)$

$X_2 \sim \text{Bern}\left(\frac{K}{N}\right)$

$\vdots$

$X_n \sim \text{Bern}\left(\frac{K}{N}\right)$

but not indep.

$$\boxed{E(X) = n\mu = n\frac{K}{N}}$$

Back to variance ...

$$Var(X) := E[(X-\mu)^2]$$

$$= E[X^2 - 2\mu X + \mu^2]$$

$$= E(X^2) + E[-2\mu X] + \underbrace{E(\mu^2)}$$ (just proven)

$$= E[X^2] - 2\mu \underbrace{E(X)}_{\mu} + \mu^2$$

$$\boxed{Var(X) = E(X^2) - \mu^2} \Rightarrow E(X^2) = \sigma^2 + \mu^2$$

$E(X^2)$ second moment $E(X^k)$

$E(X-\mu)$ second central moment $E[(X-\mu)^k]$

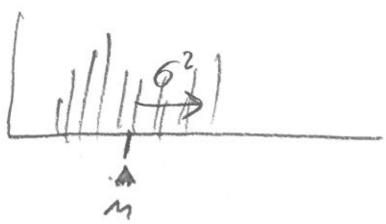
$\frac{E(X-\mu)^3}{\sigma^3}$ third std moment skewness

$\frac{E[(X-\mu)^4]}{\sigma^4}$ kurtosis

Recall linear transformation $Y = aX + c$ s.t. $a \in \mathbb{R}$, $c \in \mathbb{R}$

$Y = X + c$, $Var(X) = \sigma^2$

What is $Var(Y)$



σ^2 shouldn't change..

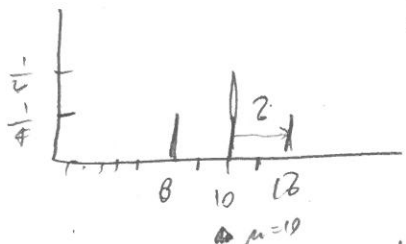
$$Var(X+c) = E[((X+c)-(\mu+c))^2] = E[(X-\mu)^2] = Var(X)$$

$$E(X+c) = \mu + c$$

$Y = aX$ $Var(X) = \sigma^2$

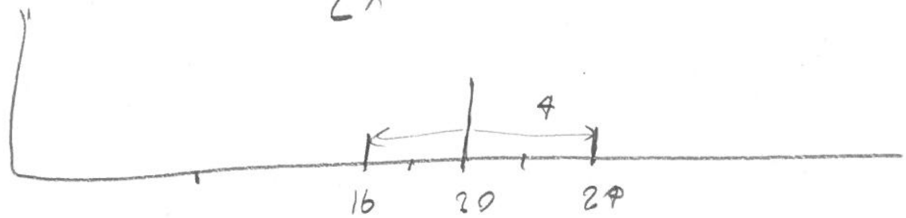
What is $Var(Y)$?

X



$$Var(X) = 2^2 \cdot \tfrac{1}{2} + 2^2 \cdot \tfrac{1}{2} = 4$$

$2X$



$\mu = 2\mu = 20$

$$Var(2X) = 4^2 \cdot \tfrac{1}{2} + 4^2 \cdot \tfrac{1}{2}$$
$$= 16 = 4\,Var(X) = 2^2\,Var(X)$$

Why? Variance is a prob weighted square error

$$Var(aX) = E[(aX - a\mu)^2] = E[(a(X-\mu))^2] = E[a^2 (X-\mu)^2] = a^2 E[(X-\mu)^2]$$
$$= a^2\,Var(X)$$

$E(aX) = a\mu$

$$Var[aX + c] = a^2 \sigma^2$$

$$\Rightarrow SD[aX + c] = \sqrt{Var[aX+c]} = |a|\sigma$$

X_1, X_2 two r.v.'s $\quad Var(T)$?

$Cov(X_1, X_2) := E[X_1 X_2] - \mu_1 \mu_2$

$$Var(X_1 + X_2) = E\left[\left((X_1 + X_2) - (\mu_1 + \mu_2)\right)^2\right]$$

$$= E[X_1^2 + X_2^2 + \mu_1^2 + \mu_2^2 + 2X_1X_2 - 2X_1\mu_1 - 2X_2\mu_2 - 2X_1\mu_2 - 2X_2\mu_1 + 2\mu_1\mu_2]$$

$$= \underbrace{E(X_1^2)}_{\sigma_1^2 + \mu_1^2} + \underbrace{E(X_2^2)}_{\sigma_2^2 + \mu_2^2} + \cancel{\mu_1^2} + \cancel{\mu_2^2} + 2E[X_1X_2] - \cancel{2\mu_1^2} - \cancel{2\mu_2^2} - \cancel{2\mu_1\mu_2} - 2\mu_2\mu_1 + \cancel{2\mu_1\mu_2} = \sigma_1^2 + \sigma_2^2 + 2(E[X_1X_2] - \mu_1\mu_2)$$

Recall $E[g(X_1, X_2)]$

If indep..

$$E[X_1 X_2] = \sum_{x_1}\sum_{x_2} x_1 x_2 \, p(x_1, x_2) = \sum_{x_1}\sum_{x_2} x_1 x_2 \, p(x_1) p(x_2) = \sum_{x_1} x_1 p(x_1) \sum_{x_2} x_2 p(x_2)$$

$$= E(X_1) E(X_2) = \mu_1 \mu_2$$

$$Cov[X_1, X_2] = E[X_1 X_2] - \mu_1 \mu_2 = 0$$

$$\Rightarrow Var[X_1 + X_2] = Var(X_1) + Var(X_2)$$

If $X_1, X_2, \ldots, X_n$ indep (but not necessarily ident. distr.)

$Var(T) =$

$$\Rightarrow Var[X_1 + \ldots + X_n] = Var(X_1) + \ldots + Var(X_n) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \sigma_i^2$$

If $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$

$$\Rightarrow Var[T] = n\sigma^2$$

$$\Rightarrow Var(\bar{X}) = Var\left(\frac{T}{n}\right) = \frac{1}{n^2} Var(T) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$$

$$\Rightarrow SE(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

Lec15 end

$X \sim Binom(n, p)$ $\quad Var(X) = n \, Var(X_i) = np(1-p)$

$\Rightarrow X = X_1 + \ldots + X_n$

when $X_1, \ldots, X_n \overset{iid}{\sim} Bern(p)$