Math 241 Fall 2016
Midterm Examination One

Solutions

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Full Name ________________________________ Section (A or B) ____

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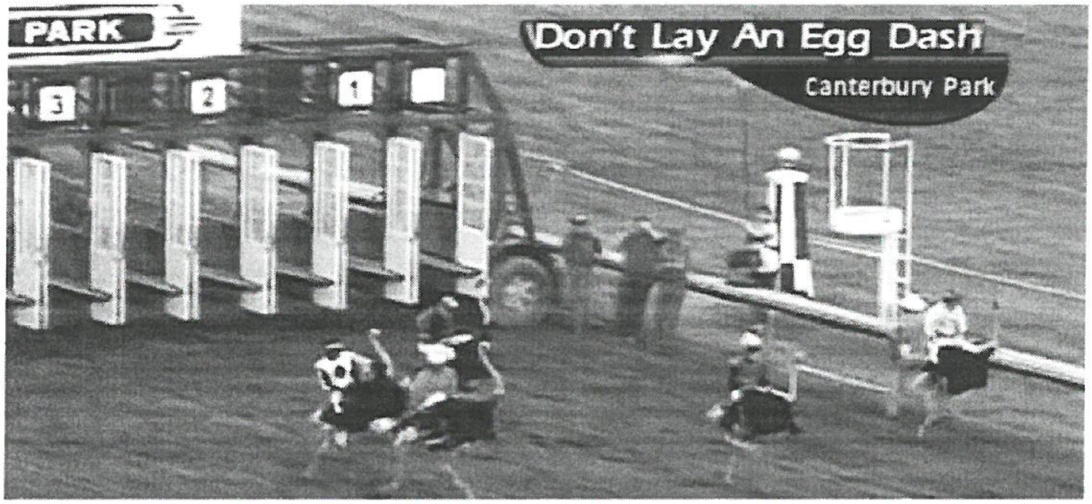
signature date

Instructions

This exam is seventy five minutes and closed-book. You are allowed one page (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If the question reads "compute," this means the solution will be a number otherwise you can leave the answer in choose, permutation, exponent, factorial or any other notation which could be resolved to a number with a computer. I advise you to skip problems marked "[Extra Credit]" until you have finished the other questions on the exam, then loop back and plug in all the holes. I also advise you to use pencil. The exam is 100 points total plus extra credit. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. Good luck!

Problem 1 The Canterbury Park racetrack in Shakopee, Minnesota usually is home to horse racing, but at times they host a whimsical Ostrich race called the "Don't Lay an Egg Dash". Supposedly, ostriches are about as fast as horses. You can train a horse, but ostriches are erratic and true wild cards.



In this race (see above) there are 5 ostriches named "Flightless Fred", "Longneck Ned", "Zippy", "Try-n-Fly" and "Birdman". One will win the race (come in first), one will come in second, etc.

(a) [3 pt / 3 pts] How many possible distinct outcomes are there in this race?

$5!$

(b) [4 pt / 7 pts] Would the assumption of each of these distinct outcomes being equally likely be a good assumption? Why or why not?

Yes, since ostriches are "erratic", previous data will likely be not useful in predicting the winners

(c) [2 pt / 9 pts] Despite what you wrote for (b), assume each outcome is equally likely for the remainder of the problem. What is the probability "Flightless Fred" wins?

$\frac{1}{5}$

(d) [4 pt / 13 pts] What is the probability "Flightless Fred" wins, "Birdman" comes in second and "Try-n-Fly" comes in third? Note: this is called a "trifecta" bet at the track.

$$\frac{1}{{}_5P_3}$$

(e) [5 pt / 18 pts] What is the probability "Flightless Fred", "Birdman" and "Try-n-Fly" all finish in the top 3 in an unspecified order? Note: this is called a "trifecta-box" bet at the track.

$$\frac{1}{\binom{5}{3}}$$

(f) [6 pt / 24 pts] What is the probability "Flightless Fred" finishes in the top 3 places? Note: this is called a show bet at the track.

union of disjoint events

$$P(1^{st}\text{ place}) + P(2^{nd}\text{ place}) + P(3^{rd}\text{ place})$$

$$= \frac{\underline{1}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}}{5!} + \frac{\underline{4}\ \underline{1}\ \underline{3}\ \underline{2}\ \underline{1}}{5!} + \frac{\underline{4}\ \underline{3}\ \underline{1}\ \underline{2}\ \underline{1}}{5!}$$

$$= \frac{3}{5}$$

(g) [5 pt / 29 pts] What is the probability "Flightless Fred" and "Birdman" finish neck-neck? They don't have to win, but they have to finish one after the other.

$$P(1^{st}/2^{nd}) + P(2^{nd}/3^{rd}) + P(3^{rd}/4^{th}) + P(4^{th}/5^{th})$$

$$= 2\frac{\underline{1}\ \underline{3}\ \underline{2}\ \underline{1}}{5!} + 2\frac{\underline{3}\ \underline{1}\ \underline{2}\ \underline{1}}{5!} + 2\frac{\underline{3}\ \underline{2}\ \underline{1}\ \underline{1}}{5!} + 2\frac{\underline{3}\ \underline{2}\ \underline{1}\ \underline{1}}{5!}$$

$$= \frac{2}{5!}(3!) = \frac{2}{5 \cdot 4} = \boxed{\frac{1}{10}}$$

(h) [5 pt / 34 pts] The jockeys (the person riding the bird) on "Flightless Fred", "Long-neck Ned" and "Zippy" are women and the jockeys on "Try-n-Fly" and "Birdman" are men. What is the probability first, second and third place feature 2 women and one man in no particular order?

$$\frac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}}$$

(i) [3 pt / 37 pts] What is the number of unique probability questions can I ask about the outcome of this race?

Sample space $|\Omega| = 5!$

event space $|2^{\Omega}| = 2^{|\Omega|} = 2^{5!}$

Problem 2 This problem is about the philsophical theory of probability.

(a) [2 pt / 39 pts] Let $p := \mathbb{P}$ (Trump wins the election in November). Posit a *numeric answer* for p.

any $x \in (0,1)$ e.g. 0.5

(b) [5 pt / 44 pts] Explain why there is no "correct" answer to (a). Make sure to mention in your answer which definition of probability you are invoking.

There is no objective definition of probability that can be employed to compute this probability. I used the subjective definition to answer (a).

(c) [2 pt / 46 pts] Would Laplace believe that the event "Trump wins the election in November" to be *truly* random? Yes/no only.

No (he believed the world is deterministic... including the presidential election)

Problem 3 This problem is about sets and the mathematical theory of probability.

(a) [3 pt / 49 pts] Simplify $\{\varnothing\} \cap \{\{\varnothing\}\}$.

ϕ

(b) [3 pt / 52 pts] Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. Simplify $2^A \cap 2^B$.

2^A

(c) [3 pt / 55 pts] Compute $\mathbb{P}(A \mid A^C)$.

0

(d) [6 pt / 61 pts] Consider events A and B. Prove that $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$. Use the "axioms" of the probability set function and your knowledge of set theory and *any* theorems from class that may be useful.

$P(A \cup B) = P(A) + P(B) - P(AB)$ inclusion-exclusion thm from class

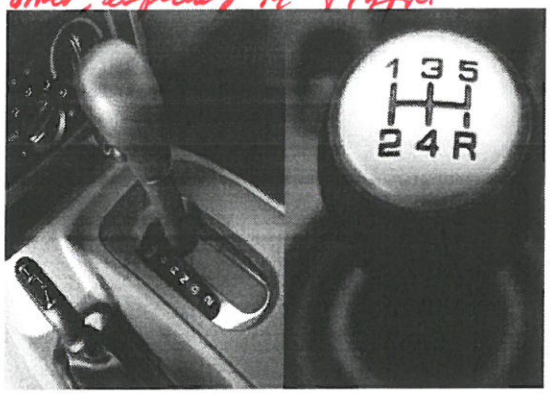
$P(AB) = P(A) + P(B) - P(A \cup B)$ algebra

$P(AB) \geq 0$ axiom (b)

$P(A) + P(B) - P(A \cup B) \geq 0$ subst.

$P(A \cup B) \leq P(A) + P(B)$ algebra

Problem 4 For the most part, motor vehicles are produced with one of two types of transmissions: manual and automatic. Manual is less common in cities because it's considered more burdensome for the driver, especially in traffic.



The percentage of new manual vehicles produced for 2017 is 46.2%. Americans consume 21.1% of worldwide vehicles. And 6.5% of new vehicles in American are estimated to be manual.

(a) [2 pt / 63 pts] Let A denote the event that a new car is in America and M denote the event that a new car has a manual transmission. What is $\mathbb{P}(A)$? Compute explicitly.

0.211

(b) [2 pt / 65 pts] Let A denote the event that a new car is in America and M denote the event that a new car has a manual transmission. What is $\mathbb{P}(M)$? Compute explicitly

0.462

(c) [2 pt / 67 pts] Let A denote the event that a new car is in America and M denote the event that a new car has a manual transmission. What is $\mathbb{P}(M \mid A)$? Compute explicitly

0.065

(d) [4 pt / 71 pts] What is the probability that the car is in America given that it is a manual transmission? Compute explicitly.

$$P(A|M) = \frac{P(M|A)\,P(A)}{P(M)} = \frac{0.065 \cdot 0.211}{0.462} = \boxed{.03}$$

(e) [7 pt / 78 pts] How much more likely is a car to be manual outside of America than within America? Compute Explicitly.

$$\frac{P(M|A^c)}{P(M|A)} = \frac{0.5681616}{0.065} = \boxed{8.74}$$

$$P(M|A^c) = \frac{P(M A^c)}{\underbrace{P(A^c)}_{1-P(A)}} = \frac{.448}{.789} = .568161$$

$$P(M) = P(M A) + P(M A^c)$$
$$= P(M|A)P(A) + P(M A^c)$$
$$\Rightarrow P(M A^c) = P(M) - P(M|A)P(A) = .462 - .065 \cdot .211 = .448$$

(f) [3 pt / 81 pts] [Extra Credit] Odds Against an event A is defined as $\mathbb{P}\left(A^C\right)/\mathbb{P}(A)$. What are the odds against a car being American given that its automatic?

(g) [4 pt / 85 pts] You see seven new cars go by in New York. What is the probability all of them were manual? Compute explicitly.

America

$$P(M_1 M_2 \ldots M_7 | A) = P(M|A)^7 = .065^7 = \boxed{4.9 \times 10^{-9}}$$

(h) [6 pt / 91 pts] You see seven new cars go by in New York. What is the probability at least one of them was manual? Compute explicitly.

$$P(\geq 1 \text{ M in } 7 \mid A) = 1 - P(0 \text{ M} \mid A) = 1 - P(M^c \mid A)^7 = 1 - (1 - P(M \mid A))^7$$

$$= 1 - (1 - 0.065)^7 = \boxed{.375}$$

(i) [~~4~~ 3 pt / 95 pts] You see seven new *manual* cars go by in New York. What is the probability the next one (the eigth car) is automatic? Compute explicitly.

$$1 - P(M \mid A) = 1 - 0.065 = \boxed{.935}$$

(j) [~~7~~ 6 pt / 100 pts] List all assumption(s) you use to answer (~~f~~ g), (~~g~~ h) and (~~h~~ i). There should be at least two. Are they reasonable?

1. Independence of the transmission type among cars on the street of NYC you are standing at. Yes, this is reasonable.
2. $P(M \mid A) = P(M \mid NYC)$. No this is not reasonable. Manual is less common in cities as the problem header suggests.

(k) [3 pt / 103 pts] You go to a party with 20 cars and everyone puts their keys in a bag and the keys are distributed out randomly. What is the approximate probability at least one person gets the keys to their car?

$$\frac{1}{e}$$