November 17, 2016

Review >

	PMF	PDF	CDF	$E[X]$	$Var[X]$	Quantile(p)
Discrete r.v.	$p(x) \in [0,1]$ $\sum_{x \in Supp[X]} p(x) = 1$	No	Yes	$\sum_{x \in Supp[X]} x p(x)$	$\leq \lvert \mathbb{N} \rvert$	$\min \{x : F(x) \geq p\}$
Continuous r.v.	No	$f(x) \geq 0$ $\int_{Supp[X]} f(x) = 1$ $f(x) = \frac{d}{dx}(F(x))$	Yes Does not mean that it appears in closed form	$\int_{Supp[X]} x f(x) dx$	$= \lvert \mathbb{R} \rvert > \lvert \mathbb{N} \rvert$	x s.t. $F(x) = p$

> $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Is this a PDF?

(a) $f(x) \geq 0$?

Yes b/c $e^{-\frac{x^2}{2}}$ never goes to 0.

(b) $\int_{supp[X]} f(x)\,dx = 1$?

Yes, $\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$.

> Proof:

$$\frac{1}{\sqrt{2\pi}} \int e^{-\frac{x^2}{2}} dx = 1 \Rightarrow \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

Let... $u = \frac{1}{\sqrt{2}} x \qquad du = \frac{1}{\sqrt{2}} dx$

$u^2 = \frac{x^2}{2} \qquad dx = \sqrt{2}\, du$

$$\Rightarrow \int_{\mathbb{R}} e^{-u^2} du \sqrt{2} = \sqrt{2\pi} \Rightarrow \int_{\mathbb{R}} e^{-u^2} du = \sqrt{\pi} \Rightarrow \left(\int_{\mathbb{R}} e^{-u^2} du\right)^2 = (\sqrt{\pi})^2$$

$$\Rightarrow \left(\int_{\mathbb{R}} e^{-u^2} du\right)^2 = \pi \Rightarrow \int_{\mathbb{R}} e^{-u^2} du \int_{\mathbb{R}} e^{u^2} du = \pi \Rightarrow \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy = \pi$$

$$\Rightarrow \int_{x\in\mathbb{R}}\int_{y\in\mathbb{R}} e^{-(x^2+y^2)}\,dx\,dy \Rightarrow \int_{r\in(0,\infty)}\int_{\theta\in(0,2\pi)} e^{-r^2}\,r\,dr\,d\theta \Rightarrow \int_{r\in(0,\infty)} e^{-r^2}\,r\,dr \int_{\theta\in(0,2\pi)} d\theta$$

$$\Rightarrow 2\pi \int e^{-r^2}\,r\,dr \stackrel{?}{=} \pi$$

Let $u = r^2$ $\quad du/dr = 2r \quad du = 2r\,dr \quad r\,dr = \frac{du}{2}$

$$\Rightarrow \int_0^\infty e^{-u}\frac{du}{2} = \frac{1}{2} \Rightarrow [-e^{-u}]_0^\infty \; (e^{-(0)} - \lim_{x\to\infty} e^{-x}) = 1 - 0 = 1$$

> $Z \sim N(0,1) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

"standard normal r.v." "bell curve" "Gaussian r.v."

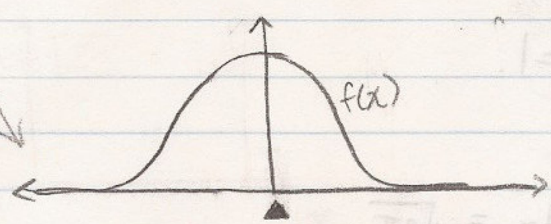
This is one case of the general normal dist.

> $E[Z] = \int_{supp[X]} x f(x)\,dx = \int_{supp[X]} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$

Let $u = \frac{x^2}{2}$ $\quad \frac{du}{dx} = x \quad du = x\,dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} e^{-u}\,du = \frac{1}{\sqrt{2\pi}}[-e^{-u}]_{x=-\infty}^{x=\infty} = \frac{1}{\sqrt{2\pi}}\left[-e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\lim_{x\to-\infty} e^{-\frac{x^2}{2}} - \lim_{x\to\infty} e^{-\frac{x^2}{2}}\right) = 0 \Rightarrow \mu = 0$$



by integration by parts.

> $Var[Z] = E[Z^2] - \cancel{\mu^2} = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \stackrel{\downarrow}{=} 1$

$Var[Z] = 1$

$SE[Z] = 1$

so $\sigma^2 = \sigma = 1$

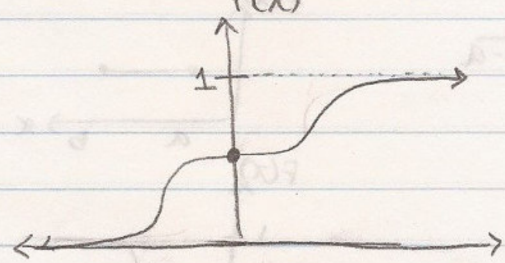
> CDF

$$F(x) = \int f(x)\,dx + c = \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx + c}$$

antiderivative

This has no closed form expression. We need a computer to do this.

$F(x)$, (big F), will be from a computer.

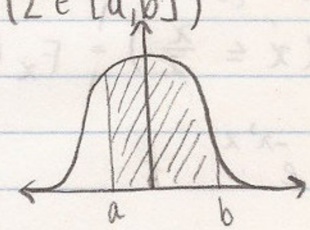


What is $F(0)$? The amount of probability up until 0?

$F(x) = \int_{-\infty}^{x} \frac{1}{2\pi} e^{-\frac{y^2}{4}}\,dy$

$\frac{1}{2}$ because we proved that the bell $=1$. Since the bell is symmetrical, at $x=0$, the CDF $= \frac{1}{2}$.

> $P(Z \in [a,b])$



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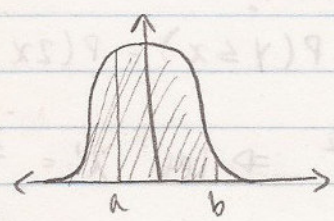

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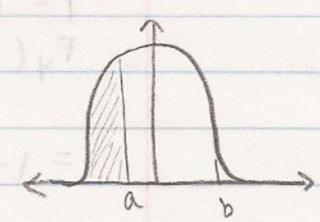


$P(Z \in [a,b]) \quad = \quad F(b) \quad - \quad F(a)$

$P(Z \in [-1,1]) = F(1) - F(-1) \approx 0.68$
$P(Z \in [-2,2]) = F(2) - F(-2) \approx 0.95$
$P(Z \in [-3,3]) = F(3) - F(-3) \approx 0.997$

Collectively together is called the "68-95-997 rule" or "3σ rule" or "empirical rule"

"The amount of prob btwn -1 standard errors and 1 standard error is 0.68"

> Review:

$X \sim \text{Geom}(p) \quad \text{Exp}[X] = \frac{1}{p} \quad \lambda = \overset{\infty \searrow \swarrow 0}{np}$

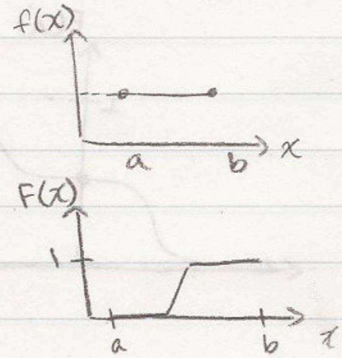
$X \sim \text{Exp}(\lambda) := \lambda e^{-\lambda x}$

$F(x) = 1 - e^{-\lambda x}$

$E[X] = \frac{1}{\lambda}$

$X \sim \text{Unif}(a,b) := f(x) = \frac{1}{b-a}$

$F(x) = \frac{x-a}{b-a}$



> $X \sim \text{Exp}(\lambda)$

$Y = 2X \sim ?$

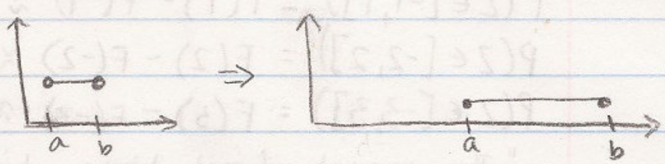
$F_Y(x) = P(Y \leq x) = P(2X \leq x) \overset{\text{divide by 2}}{=} P(\overbrace{X \leq \tfrac{x}{2}}^{\text{CDF of } X}) = F_X\left(\frac{x}{2}\right)$

$= 1 - e^{-\lambda \frac{x}{2}} \Rightarrow \text{Let } \lambda' = \frac{\lambda}{2} \Rightarrow 1 - e^{-\lambda' x}$

$= Y \sim \text{Exp}(\lambda') = Y \sim \text{Exp}\left(\frac{\lambda}{2}\right)$

> $X \sim \text{Unif}(a,b)$

$Y = d + cX \sim ?$



$F_Y(x) = P(Y \leq x) = P(d + cX \leq x) = P\left(X \leq \frac{x-d}{c}\right) = F_X\left(\frac{x-d}{c}\right)$

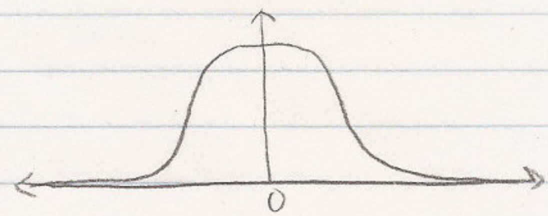
$= \frac{\frac{x-d}{c} - a}{b-a} \times \frac{c}{c} = \frac{x-d-ac}{c(b-a)} = \frac{x-(d+ac)}{bc-ac} = \frac{x-(d+ac)}{(d+bc)-(d+ac)}$

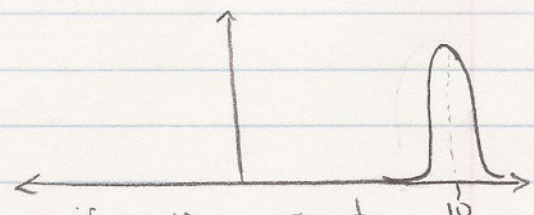
$\Rightarrow \text{Let } a' = d + ac \Rightarrow \frac{x-a'}{b'-a'} \Rightarrow Y \sim \text{Unif}(a', b')$

$b' = d + bc \qquad \Rightarrow Y \sim \text{Unif}(d+ac, d+b)$

> Returning to the normal rv,
remember that $Exp[Z] = 0$, $SE[Z] = 1$

Let $X = \mu + \sigma Z$





if $\mu = 10$, $\sigma = \frac{1}{2}$
It remains a bell shape.

$F_X(x) = P(X \le x) = P(\mu + \sigma Z \le x) = P(Z \le \frac{x-\mu}{\sigma}) = F_Z(\frac{x-\mu}{\sigma})$ STOP

But F_Z is antidifferentiable... so we find the PDF.

$f(x) = \frac{d}{dx} F(x) \longrightarrow \frac{d}{dx}\left[F_Z\left(\frac{x-\mu}{\sigma}\right)\right]$

Let $u = \frac{x-\mu}{\sigma}$ $\quad \frac{du}{dx} = \frac{1}{\sigma}$

$$\frac{du}{dx}\frac{d}{dx}\left[F_Z(u)\right] = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sigma}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$$

$$X \sim \text{Normal}(\overset{\text{center}}{\mu}, \overset{\text{how much to squish}}{\sigma^2}) := e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

"General Normal Distribution"

$E[X] = E[\mu + \sigma^2] = \mu$
$SE[X] = SE[\mu + \sigma^2] = |\sigma|$ — skinnier / fatter bellcurve
Param Space: $\mu \in \mathbb{R}$ $\quad \sigma^2 \in (0, \infty)$
$Supp[X] = \mathbb{R}$