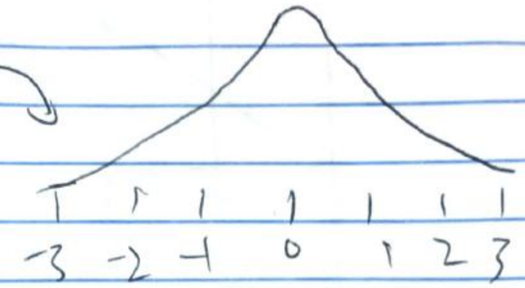
Lec #19.

11/22/2016

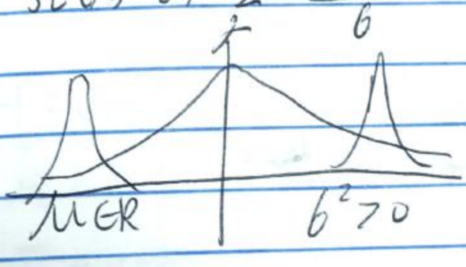
$Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x)$



$Supp(Z) = \mathbb{R}$, $E(Z) = 0$, $SE(Z) = 1$

$X \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$E(X) = \mu$, $SE(X) = \sigma$, $Z = \frac{X-\mu}{\sigma}$ ⇒ standardization



ex: X is male height, norm. distr w/ mean 70'', standard error 4'', Find prob. a male is more than 78'' tall.

$X \sim N(70'', 4''^2)$

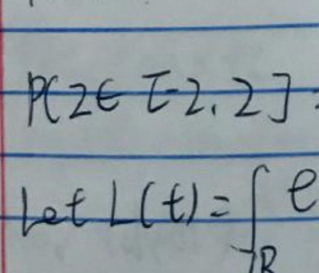
$P(X > 78) = P\left(\frac{X-70}{4} > \frac{78-70}{4}\right) = P(Z > 2) = 2.5\%$

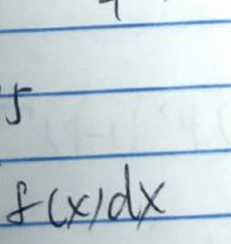
$P(Z \in [-2, 2]) = 0.95$

Let $L(t) = \int_{\mathbb{R}} e^{-tx} f(x)\,dx$ if $t = 1$

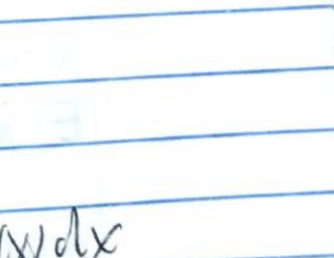
$L(1) = \int_{\mathbb{R}} e^{-x} f(x)\,dx$

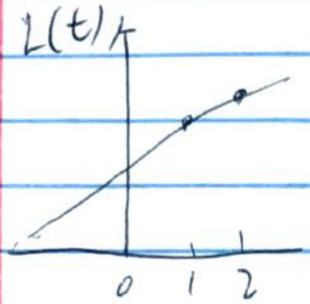
"Bilateral Laplace Transformation"





$f(x)$ $L(1) = $ Area →





Thm: if $L(t)$ exist, $L(t)$ & $f(x)$ are 1:1

If $f(x)$ is a PDF

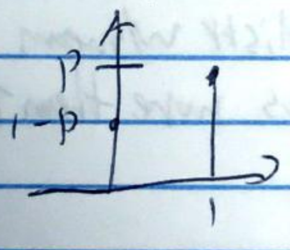
$M_X(t) = E[e^{tX}] := \int_{\mathbb{R}} e^{-tx} f(x)dx$, for cont.

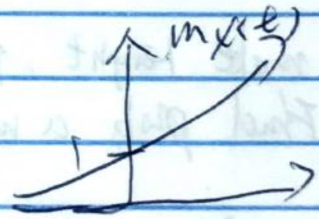
moment generating function (MGF) for r.v. X

$M_X(t) = E(e^{tX}) = e^{t(0)} \cdot p(0) + e^{t(1)} \cdot p(1) = 1-p+pe^t$

~~① If $M_X(t)$ $M_Y(t) \Rightarrow X \overset{d}{=} Y$~~

Recall ex: $X \sim Bern(p)$





if ① $m_X(t) = \cancel{E[e^{tX}] = e^{t(0)}}$ $m_Y(t) \Rightarrow X \overset{d}{=} Y$

$X \sim Binom(n, p)$

$E[X^{17}] = \sum_{x=0}^{n} x^{17} \binom{n}{x} p^x (1-p)^{n-x}$

Recall $\forall c$, $f(x)$ cont (Taylor series)

$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!} \cdot (x-c)^2 + \cdots$

$x \approx c$

$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2, \cdots \rightarrow 2^{nd}$ order

Let $f(x) = e^x \quad x \approx 0 \Rightarrow c = 0$

$$e^x = e^0 + \frac{e^0 x}{1!} + \frac{e^0 x^2}{2!} \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots$$

$$M_x(t) = E[e^{tx}] = E[1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots] \quad \frac{d}{dt}[M_x(t)]$$

$$= \frac{d}{dt}[E(\cdots)] = E[\frac{d}{dt}[\cdots]]$$

$$M'_x(t) = E[x + \frac{tx^2}{1!} + \frac{t^2x^3}{2!} + \frac{t^3x^4}{3!} \cdots]$$

$$M'x(0) = E[x] = \mu$$

$$M''x(t) = E[x^2 + \frac{tx^3}{1!} + \frac{t^2x^4}{2!} + \cdots]$$

$$M''x(0) = E[x^2]$$

$$M'''x(t) = E[x^3 + \frac{tx^4}{\cancel{1!}} + \frac{t^2x^5}{1!} + \cdots]$$

$$M'''x(t) = E[x^3]$$

② $M_x^{(k)}(0) = E[x^k]$

Let $Y = ax + c$

$$m_Y(t) = m_{ax+c}(t) = E[e^{t(ax+c)}] = E[e^{tax} e^{tc}]$$

$$= e^{tc} E[e^{tax}] \Downarrow$$

$L(t) = \int e^{tx} f(x)\,dx \quad$ Let $t' : at$ $= e^{tc} E[e^{t'x}] = e^{tc} M_x(t$

③ If $Y = ax + c \Rightarrow M_Y(t) = e^{tc} M_x(at)$

Δ Consider X_1, X_2 indepent r.v.'s Let $Y = X_1 + X_2$

$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}] = E[e^{tX_1} \cdot e^{tX_2}]$

$= E[e^{tX_1}] \cdot E[e^{tX_2}] = M_{X_1}(t) M_{X_2}(t)$

if $X_1 X_2 \sim$ iid.

$M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t) = (M_X(t))^2$

(4) If X_1, X_2 indepent $Y = X_1 + X_2 \Rightarrow M_Y(t) = M_{X_1}(t) \cdot M_2(t)$

Ex: $X \sim \text{Bern}(p) \Rightarrow M_X(t) = 1 - p + pe^t$

Recall: $X_1 X_2 \dots X_n \overset{iid}{\sim} \text{Bern}(p)$

$T = X_1 + X_2 + \dots X_n \sim \text{Bino}(n, p)$

$M_T(t) = E[e^{tT}] = E[e^{t(X_1+X_2 \dots X_n)}] = M_{X_1}(t) \dots M_{X_n}(t)$

$= (M_X(t))^n = (1 - p + pe^t)^n$

Ex: $X \sim \text{Geom}(p)$

$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} (1-p)^{x-1} p \cdot \frac{(1-p)}{(1-p)} = \frac{p}{1-p} \sum_{x=1}^{\infty} e^{tx} (1-p)^x$

$= \frac{p}{1-p} \sum_{x=1}^{\infty} (e^t (1-p))^x = \frac{p}{1-p} \left(\sum_{x=0}^{\infty} (e^t (1-p))^x - 1 \right)$

$= \frac{p}{1-p} \left(\frac{1}{1 - e^t (1-p)} - 1 \right) = \frac{p}{1-p} \cdot \frac{e^t (1-p)}{1 - e^t (1-p)}$

$M_X(t) = \frac{pe^t}{1 - e^t (1-p)}$ if $t < \ln\left(\frac{1}{1-p}\right)$

if $e^t (1-p) < 1$

$\Rightarrow e^t < \frac{1}{1-p}$

$\Rightarrow t < \ln \frac{1}{1-p}$

$X \sim Exp(\lambda)$

$Y = aX,\ a \in \mathbb{R}$

$m_Y(t) = e^{tc} m_X(at) = m_X(at) = \frac{\lambda}{\lambda - at} \frac{\frac{1}{a}}{\frac{1}{a}} = \frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t} = \frac{\lambda'}{\lambda' - t}$

$= \frac{\lambda'}{\lambda' - t} \Rightarrow Y \sim Exp(\lambda') = Exp(\frac{\lambda}{a})$ let $\lambda' = \frac{\lambda}{a}$

$Z \sim N(0,1) \Rightarrow M_Z(t) = E[e^{tZ}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}x^2 + tx} dx$

$= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x-t)^2} \cdot e^{\frac{t^2}{2}} dx = e^{\frac{t^2}{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx$

$= e^{\frac{t^2}{2}}$

$\mu = M'_Z(0) = t e^{\frac{t^2}{2}} \big|_{t=0} = 0$

$\sigma^2 = E[Z^2] - \mu^2 = E[Z^2] = M_Z''(0) = e^{\frac{t^2}{2}} + t \cdot t e^{\frac{t^2}{2}} \big|_{t=0} = 1 \Rightarrow \sigma = 1$

$\triangleright X_1, X_2, \ldots, X_n$ iid

$\bar{X} \to \mu$ law of large #'s

- Levy's continuity Thm
  $X_n \to X$ if $\lim_{n \to \infty} M_{X_n}(t) = M_X(t)$

$\mu \sim Deg(\mu)$, $X \sim Deg(c) \Rightarrow M_X(t) = E[e^{tX}] = e^{tc}$

$M_\mu(t) = e^{t\mu}$

$\lim_{n \to \infty} M_{\bar{X}}(t) = e^{t\mu}$

$M_{\bar{X}}(t) = E[e^{t\bar{X}}] = E[e^{x(\frac{X_1 + \ldots X_n}{n})}] = M_{X_1}$