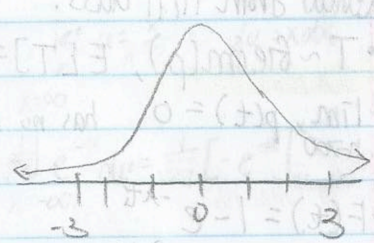
11/22 Standard Normal

- $Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x)$

  $Supp[Z] = \mathbb{R}$

  $E[Z] = 0$

  $SE[Z] = 1$



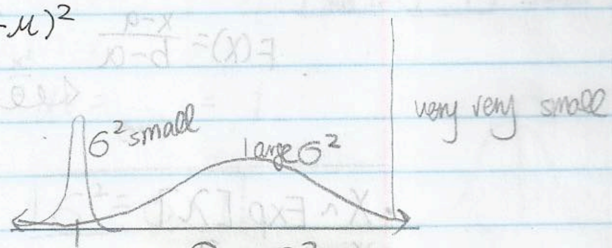
- $X \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

  $E[X] = \mu$

  $SE[X] = \sigma$

  $Z = \frac{X-\mu}{\sigma}$

$\sigma^2 > 0$ — one pt only not vaild.



$\mu \in \mathbb{R}, \sigma^2 > 0.$

* Typical Test Question

Question: X is male height
Normal distribution w/ replacement.
70", s.e. 4"

$X \sim N(70", 4"^2)$

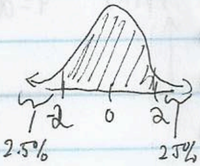
Find probability a male is more than 78" tall

$P(X > 78)$

$= P\left(\frac{X-70}{4} > \frac{78-70}{4}\right)$

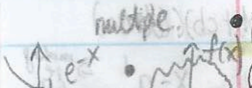
$= P(Z > 2) = 2.5\%.$

$P(Z \in [-2, 2]) = 95\%$

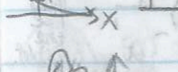


- "Bilalend LaPlace Transform". idea: (looking a function in a different way)

multiple



- let $L(t) = \int_{\mathbb{R}} e^{-tx} f(x)\,dx$ if $t=1$ $L(1) = \int_{\mathbb{R}} e^{-x} f(x)\,dx$.



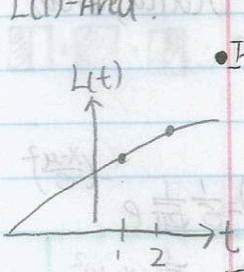
L(1) = Area.

Thm: if L(t) exists, L(t) & f(x) are 1:1.

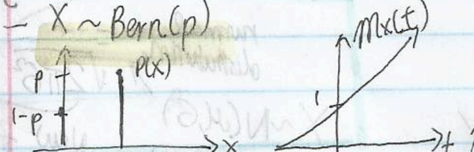
Moment Generating function (MGF) for r.v X.

- If f(x) is a PDF, then — $M_X(t) := E[e^{tx}] := \int_{\mathbb{R}} e^{-tx} f(x)\,dx$ for continuous.

  — $M_X(t) := E[e^{tx}] = \sum_{x \in Supp[X]} e^{tx} p(x)$ for discrete.



— $X \sim Bern(p)$



$E[e^{tx}] = e^{t(0)} p(0) + e^{t(1)} p(1) = 1 - p + pe^t$

Fact: If same MGF ⇒

$\boxed{\text{(I)}\ M_X(t) = M_Y(t) \Rightarrow X = Y}$

→ These 2 graphs represent the same info. smoothly. same (need to go through the process to check)

- $X \sim Binom(n, p)$

$$E[X^{17}] = \sum_{x=0}^{n} x^{17} \binom{n}{x} p^x (1-p)^{n-x}$$ not possible

Taylor Series

Recall $\forall c$ $f(x)$ continuous.

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots$$

$x \approx c$

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2$$ 2nd order approximation

Let $f(x) = e^x$ $x \approx 0 \Rightarrow c = 0$

$$e^x = e^0 + \frac{e^0 x}{1!} + \frac{e^0 x^2}{2!} + \dots$$ from Taylor Series.

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$e^{tX} = 1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \dots$$

* Moment Generating Function

$$M_X(t) = E[e^{tX}]$$ take whatever inside (expand it) by Taylor series.

$$= E\left[1 + tX + \frac{t^2 X^2}{2!} + \frac{t^3 X^3}{3!} + \dots\right]$$

Consider: $\frac{d}{dt}[M_X(t)] = \frac{d}{dt}[E[\dots]] = E[\frac{d}{dt}[\dots]]$

$$M'_X(t) = E[X + \frac{tX^2}{1!} + \frac{t^2 X^3}{2!} + \frac{t^3 X^4}{3!} + \dots]$$

$$M'_X(0) = E[X] = \mu$$ The first moment

$$M''_X(t) = E[X^2 + tX^3 + \frac{t^2 X^4}{2!} + \dots]$$

$$M''_X(0) = E[X^2]$$ The second moment.

$$M'''_X(t) = E[X^3 + tX^4 + \dots]$$

$$M'''_X(0) = E[X^3]$$

Fact

$$\boxed{\text{II} \; M_X^{(k)}(0) = E[X^k]}$$

The MGF generates moment

- let $Y = aX + c$

$$M_Y(t) = M_{aX+c}(t) = E[e^{\overset{taX+tc}{t(aX+c)}}] = E[e^{taX} e^{tc}] = e^{tc} E[e^{taX}]$$

$$= \int_{\mathbb{R}} e^{taX} e^{tc} f(x)\,dx$$

- let $t' := at$

$$= e^{tc} E[e^{t'X}] = e^{tc} M_X(t') = e^{tc} M_X(at)$$

Fact

$\boxed{\text{III} \text{ if } Y = aX + c \Rightarrow M_Y(t) = e^{tc} M_X(at)}$

- Consider X_1, X_2 indep. r.v's.
let $Y = X_1 + X_2$

$$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}] = E[e^{tX_1} \cdot e^{tX_2}] = E[e^{tX_1}]E[e^{tX_2}] = M_{X_1}(t) M_{X_2}(t)$$

If $X_1, X_2 \sim iid$

$$M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t) = (M_X(t))^2$$

$\boxed{\text{IV} \text{ If } X_1, X_2 \text{ indep. } Y = X_1 + X_2 \Rightarrow M_Y(t) = M_{X_1}(t) M_{X_2}(t)}$

- $X \sim Bern(p) \Rightarrow M_X(t) = 1 - p + pe^t$

Recall

$X_1, X_2, \dots X_n$ iid $Bern(p)$.

$T = X_1 + \dots + X_n \sim Binom(n,p)$

By Rule #4

$$- M_T(t) = E[e^{tT}] = E[e^{t(X_1+\dots+X_n)}] \overset{\text{by indep IV}}{=} M_{X_1}(t) \cdot \ldots \cdot M_{X_n}(t) \overset{\text{by identical distru}}{=} (M_X(t))^n = \boxed{(1-p+pe^t)^n}$$

- $X \sim Geom(p)$

*only responsible for the answer formulas

$$- M_X(t) = E[e^{tX}] = \sum_{X=1}^{\infty} e^{tX}(1-p)^{X-1} p \cdot \frac{(1-p)}{(1-p)} = \frac{p}{1-p} \sum_{X=1}^{\infty} \overset{(e^t)^X}{e^{tX}}(1-p)^X = \frac{p}{1-p} \sum_{X=1}^{\infty} (e^t(1-p))^X$$

$$= \frac{p}{1-p}\left(\sum_{X=0}^{\infty} (e^t(1-p))^X - 1\right)$$

$$= \frac{p}{1-p}\left(\frac{1}{1-e^t(1-p)} - 1\right) = \frac{p}{1-p} \frac{e^t(1-p)}{1-e^t(1-p)} = \overset{M_X(t)}{\boxed{\frac{pe^t}{1-e^t(1-p)} \overset{\text{indicator}}{\mathbb{1}}_{t < \ln(\frac{1}{1-p})}}}$$

(if $e^t(1-p) < 1 \Rightarrow e^t < \frac{1}{1-p}$
$\Rightarrow t < \ln(\frac{1}{1-p})$)

$- X \sim Exp(\lambda)$

$$\Rightarrow M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx = \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)x}\right]_0^\infty$$

$$= \frac{\lambda}{t-\lambda}\left(\lim_{x\to\infty} e^{(t-\lambda)x} - 1\right) \underset{\uparrow}{=} \frac{\lambda}{t-\lambda}(0-1) = \boxed{M_X(t) = \frac{\lambda}{\lambda - t} \ \mathbb{1}\ t<\lambda}$$

if $t-\lambda<0 \Rightarrow t<\lambda$

will see on next HW

$Y = aX,\ a \in \mathbb{R}$.

let $\lambda' = \frac{\lambda}{a}$

Proof:

$$M_X(t) = e^{tc} M_X(at) = \frac{\lambda}{\lambda - at} \cdot \frac{\frac{1}{a}}{\frac{1}{a}} = \frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t} = \frac{\lambda'}{\lambda' - t} \Rightarrow Y \sim Exp[\lambda'] = Exp\left[\frac{\lambda}{a}\right]$$

$- Z \sim N(0,1)$

$$\Rightarrow M_Z(t) = E[e^{tZ}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int e^{\underbrace{-\frac{1}{2}x^2 + tx}} dx$$

$$\to -\frac{1}{2}(x^2 - 2tx)$$

$$= -\frac{1}{2}((x-t)^2 - t^2)$$

$$= -\frac{1}{2}(x-t)^2 + \frac{1}{2}(t^2)$$

$(x-t)^2$
$= x^2 - 2tx + t^2$

complete by the square

$$= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x-t)^2} e^{\frac{t^2}{2}} dx$$

$$= e^{\frac{t^2}{2}} \int_{\mathbb{R}} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2}} dx$$

this piece is $f(x)$ for Normal $N(t,1) \to$ the whole thing is 1.

$$Z \sim N(0,1) \Rightarrow M_Z(t) = \boxed{e^{\frac{t^2}{2}}}$$

$$\mu = M_Z'(0) = te^{\frac{t^2}{2}}\Big|_{t=0} = 0.$$

$e^0 = 1$

$$\sigma^2 = E[Z^2] - \cancel{\mu^2}^{\,0} = E[Z^2] = M_Z''(0) = e^{\frac{t^2}{2}} + t \cdot te^{\frac{t^2}{2}}\Big|_{t=0} = 1 \Rightarrow \boxed{\sigma = 1}$$

$\boxed{\mu = 0}$
$\boxed{\sigma^2 = 1}$

"$\rightarrow$" conv. in distr. – $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$

Proof not cover.

$\bar{X} \rightarrow \mu$ is the Law of Large Numbers.

$$\boxed{\begin{array}{l}\text{Levy's Continuity Thm} \\ X_n \rightarrow X \text{ if } \lim_{n\to\infty} M_{X_n}(t) = M_X(t).\end{array}}$$

Proof it's true

– $M \sim Deg(\mu)$

$X \sim Deg(c) \Rightarrow M_X(t) = E[e^{tX}] = e^{tc}$

WTP

– $M_m(t) = e^{tm}$

$\lim_{n\to\infty} M_{\bar{X}}(t) = e^{tm} \rightarrow M_{\bar{X}}(t) = E[e^{t\bar{X}}] = E[e^{t(\frac{X_1+\ldots+X_n}{n})}]$

$= M_{X_1+\ldots+X_n}(\frac{t}{n})$

$= (M_X(\frac{t}{n}))^n$ expansion of MGF.

$(1 + \frac{t\mu}{n} + \frac{t^2E[X^2]}{2!\,n^2} + \frac{t^3E[X^3]}{3!\,n^3} + \ldots)^n.$