Lecture 18 10/27/16 R.V. vs. realization review

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Let $T_n = X_1 + X_2 + \dots + X_n$ sum r.v.

$$\overline{X}_n = \frac{T_n}{n} = \frac{X_1 + \dots + X_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$ average r.v.

e.g. $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(0.1)$

$T_3 \sim \text{Bin}(3, 0.1)$

$$T \sim \begin{cases} 0 & \text{wp } .729 \\ 1 & \text{wp } .243 \\ 2 & \text{wp } .027 \\ 3 & \text{wp } .001 \end{cases} \qquad \overline{X} \sim \begin{cases} 0 \\ 1/3 \\ 2/3 \\ 1 \end{cases}$$

Def: Sample average $\bar{x}$ is a realization from $\overline{X}$

$$\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$P(X = 5'8'') = ?$

we need to specify model for X... no model... meaningless question...

Note $\bar{x}$'s for experiments

Lecture 11 Review...

Recall $X \sim Binom(n,p): \binom{n}{x} p^x (1-p)^{n-x}$

$X \sim Binom(8, \frac{1}{2}) = \binom{8}{x} (\frac{1}{2})^x (1-\frac{1}{2})^{8-x} = \binom{8}{x} (\frac{1}{2})^8 = \frac{8!}{2^8} \frac{1}{x!(8-x)!}$

(3 digits)

| $x$ | $p(x)$ | $F(x)$ |
|---|---|---|
| 0 | 0.004 | 0.004 |
| 1 | 0.031 | 0.035 |
| 2 | 0.109 | 0.145 |
| 3 | 0.217 | 0.363 |
| 4 | 0.273 | 0.637 |
| 5 | 0.219 | 0.855 |
| 6 | 0.109 | 0.965 |
| 7 | 0.031 | 0.996 |
| 8 | 0.004 | 1 |

Supp(X)

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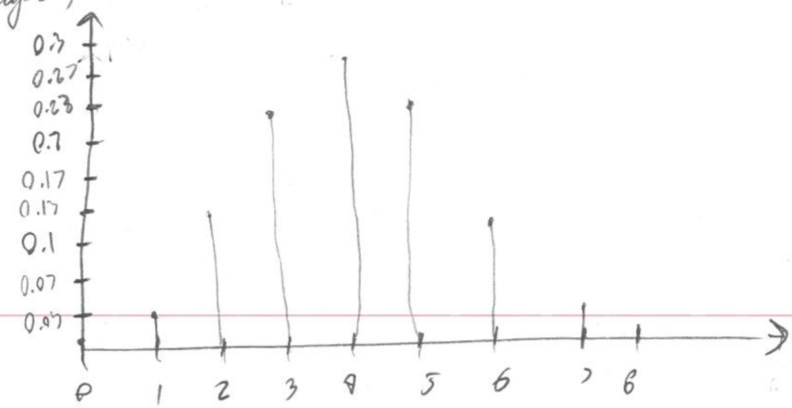
Why is ?




Now do many $X_1, \dots \overset{iid}{\sim} Binom(8, \frac{1}{2})$ realizations and build an empirical PMF and an empirical CDF. Show they are about the same

| $n$ | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|
| $X$ | | | | | | |
| $\bar{X}_n$ | | | | | | |

$X \sim Hyper(3, 4, 8)$

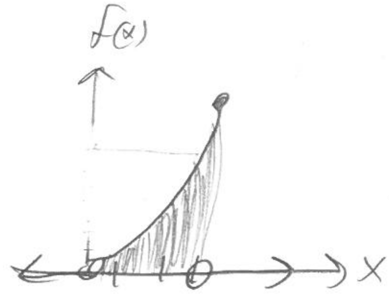
$\lim_{n \to \infty} \bar{X}_n = 4$? Yes. Hold on...

$X \sim Geom(\frac{1}{2})$

$X \sim NegBin(3, \frac{1}{2})$

| $n$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $X$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
| $\bar{X}$ | | | | | | | | | |

Recall calculus...




$f(x) = x^2$ where $x \in [0, 3]$

$\int_0^3 f(x)\,dx = 9$ What is an integral? A function of a function...

$G[f] = \int_0^3 f(x)\,dx$ $G$: functions $\longrightarrow \mathbb{R}$ (a single #)

functional function A "function of a function".

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It appears $\bar{X} \longrightarrow$ the "pivot pt" or the "balance pt"!

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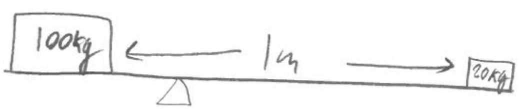
HS physics




Where is the balance pt? Net Angular momentum $= 0$

$$\sum_i w_i(d_i - d^*) = 0 \Rightarrow \sum w_i d_i - \sum w_i d^* = 0 \Rightarrow \frac{\sum w_i d_i}{\sum w_i} = d^*$$

$$= \frac{0 \cdot 100 + 1 \cdot 20}{100+20} = 0.17m$$

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Let's use same principle here... Weights = probs distance = X $\frac{\sum_i p_i x_i}{\sum p_i} = x^*$

$p(x)$

$1$

interchangeable

$E[X] := \mu := \sum_{x \in supp[X]} x\,p(x)$ "E" for expectation $E: \mathcal{X} \longrightarrow \mathbb{R}$

space of r.v.'s a single #

$X \sim \text{Bern}(0.3)$ $\quad E(X) = 0 \cdot p(0) + 1 \cdot p(1) = \overset{0 \cdot 0.7 +}{1 \cdot 0.3} = 0.3$

$X \sim \text{Bern}(p)$

$E(X) = 0 \cdot p(0) + 1 \cdot p(1) = p(1) = \boxed{p}$

For general Bernoulli

Why should $\bar{X} \to \mu$ ? proof later...

$X \sim \text{Binom}(8, 0.5) \Rightarrow$

We showed empirically (via observed data) that $E(X) \approx 4$

Let's prove it...

$$E(X) = \sum_{x \in Supp(X)} x_i \, p(x_i) = 0 \cdot p(0) + 1 \cdot p(1) + 2\,p(2) + 3\,p(3) + 4\,p(4) + 5\,p(5) + 6\,p(6) + 7\,p(7) + 8\,p(8)$$

$$= 0 + 0.031 + 2 \cdot 0.109 + 3 \cdot 0.219 + 4 \cdot 0.273 + 5 \cdot 0.219 + 6 \cdot 0.109 + 7 \cdot 0.031 + 8 \cdot 0.004$$

$$= \boxed{4}$$

What about $X \sim \text{Binom}(n,p)$ $\quad E[X] = f(n,p)$ ? $= np$ ?

$$\sum_{x \in Supp(X)} x\,p(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} \cdots = \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} \cdots$$

$= 0 + \sum_{x=1}^{n} \cdots$

$$= n \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\underbrace{(n-x)!}_{((n-1)-(x-1))}} \cdots = n \sum_{x=1}^{n} \binom{n-1}{x-1} p^x (1-p)^{n-x} = np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{(n-1)-(x-1)}$$

let $m = n-1$

let $y = x-1$

$x = 1 \ldots n$

$y = 0 \ldots n-1 = 0 \ldots m$

$$= np \underbrace{\left( \sum_{y=0}^{m} \binom{m}{y} p^y (1-p)^{m-y} \right)}_{1} = \boxed{np}$$

HARD:

$X \sim \text{Hyper}(n, K, N)$ $\quad E(X) = \sum_{x \in Supp(X)} x \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$ ? Wait... easier way...

$X \sim \text{Unif}(1, 10, 100)$ $\quad E(X) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 10 + \frac{1}{3} \cdot 100 = \frac{1}{3}(1 + 10 + 100) = \frac{111}{3}$

$X \sim \text{Unif}(A)$ $\quad E(X) = \sum_{x \in A} x\,p(x) = \frac{1}{|A|} \sum_{x \in A} x$