Lecture 11

Thursday, October 20, 2016 12:16 PM

$$X \sim \text{Geom}(p) := (1-p)^{x-1} p$$

$$\text{supp}[x] = \mathbb{N}$$

(ex) Play poker till Royal Flush

$$P\left(\begin{smallmatrix}\text{Royal}\\\text{Flush}\end{smallmatrix}\right) = \frac{4}{\binom{52}{5}} = 1.53/\text{million} = 0.00000153$$

PMF when you need 1 state

$$X \sim \text{Geometric}(0.00000153)$$

$$P(\underline{X = 1000000}) = (.9999985)^{999999} \cdot 0.00000153$$

(ex) Millionth time or sooner ← CDF when 'x' time or sooner

$$P(X \leq 1000000) = F(1000000)$$

$$= 1 - (1-p)^{x}$$

$$= 1 - .9999985^{1000000} \approx 77\%$$

$$X_1, X_2, \dots \overset{iid}{\sim} \text{Bern}(p)$$

$$X = \min\{t : X_t = 1\}$$

$$T = \min\left\{t : \sum_{i=1}^{t} X_i = r\right\}$$

$r = 2$

	0	0	0	1	0	0	0	0	1
	1	2	3	4	5	6	7	8	9

each coin doesnt know what happened previously

$r = 3$

	0	0	0	1	0	0	0	0	1	0	1
	1	2	3	4	5	6	7	8	9	10	11

(*) Succeed 3 times

multiplying because of independence

$P(T=1) = 0$

$P(T=2) = 0$

$P(T=3) = p^3$

$P(T=4) = 3(1-p)p^3$

1	1	0	1
0	1	1	1
1	0	1	1

$P(T=5) = \binom{4}{2}(1-p)^2 p^3$

"6"

1	0	1	0	1
1	1	0	0	1
1	0	0	1	1
0	0	1	1	1
0	1	0	1	1
0	1	1	0	1

$\dots t \dots = X$

$$P(T=x) = \binom{x-1}{2}(1-p)^{\boxed{x-3}}\, p^{\boxed{3}}$$

$$X \sim \text{NegBin}(r, p) = \binom{x-1}{r-1}(1-p)^{x-r} p^{r}$$

"Negative Binomial"

$$\text{Supp}[x] = \{r, r+1, r+2, \ldots\ldots\}$$

← underlying bernoulli(p)

$$\text{Param Space: } p \in (0,1), \; r \in \mathbb{N}$$

$$X \sim (1, p) = \binom{x-1}{\underbrace{1-1}_{\text{"0"}}}(1-p)^{x-1} p^{1}$$ ← Geometric PMF

$$= (1-p)^{x-1} \; p = \text{Geom}(p)$$

$\boxed{\text{Proof}}$

$$1 = \sum_{X \in supp[X]} p(x) \qquad 1 \overset{?}{=} \sum_{x=r}^{\infty} \binom{x-1}{r-1} (1-p)^{x-r} p^{r}$$

$a := 1-p$

$$\sum_{i=0}^{\infty} a^{i} = \frac{1}{1-a} \text{ if } a \in (0,1)$$

$$\sum_{x=0}^{\infty} (1-p)^{x} = \frac{1}{p} = p^{-1}$$

$$\sum_{x=1}^{\infty} (1-p)^{x-1} = p^{-1}$$

Take $\frac{d}{dx}\left[\quad\right]$ both sides

$$\sum_{x=2}^{\infty} (-1)(x-1)(1-p)^{x-2} = (-1)\, p^{-2}$$

← take $\frac{d}{dx}\left[\quad\right]$ both sides

↑ starts at 2 because with 1, it becomes 0

$$\sum_{x=3}^{\infty} (-1)^{2} (x-1)(x-2)(1-p)^{x-3} = (-1)(-2)\, p^{-3}$$

starts at 3 because at 2 it becomes 0

$$\sum_{x=4}^{\infty} \cancel{(-1)^{2}} \overset{d-2}{(x-1)} \overset{d-1}{(x-2)} \overset{d}{(x-3)} \overset{d+1}{(1-p)^{x-3}} = \overset{d!}{(-1)(-2)(-3)}\, \overset{d+1}{p^{-4}}$$

$$\sum_{x=r}^{\infty} \underbrace{(x-1)(x-2)\ldots\ldots(x-(x-1))}_{\frac{(x-1)!}{(x-r)!}}(1-p)^{x-r} = (r-1)!\,p^{-r}$$

$$\binom{x-1}{r-1} \searrow$$

$$\downarrow$$

$$\sum_{x=r}^{\infty} \frac{\underline{(x-1)!}}{\underline{(x-r)!}\,\underline{(r-1)!}}(1-p)^{x-r}p^{r} = \cancel{p^{-r}p^{r}} = 1$$

$$\swarrow$$

$$((x-1)-(x-r))!$$

$X \sim \text{NegBin}(r,p)$

but X_i = # of failures

$$i: \binom{x+r-1}{r-1}(1-p)^{x}p^{r}$$

$$\downarrow$$

$$\boxed{\frac{(x+r-1)!}{x!\,(r-1)!}} = \frac{(x+r-1)(x+r-2)\ldots\ldots r}{x!}$$

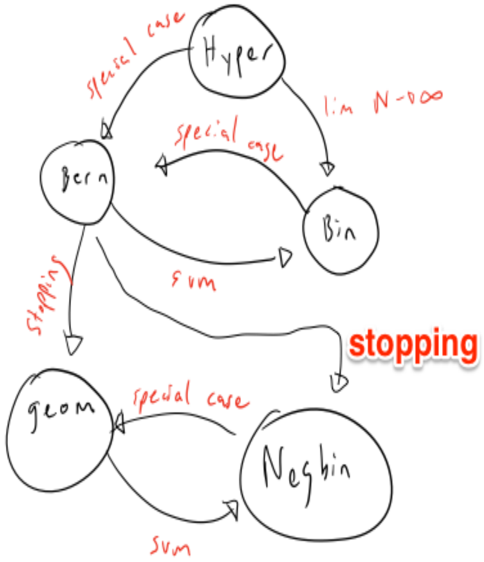
$$= (-1)^x \underbrace{\frac{(-r)(-r-1)\dots\dots(-r-x+1)}{x!}}$$

$$= (-1)^x \binom{-r}{x}$$

$$X_1, X_2, \dots, X_r \overset{iid}{\sim} Geom(p)$$

$$X_1 + X_2 + \dots\dots X_r \sim NegBin(r, p)$$

we will prove this in 6 lectures



$X \sim \mathrm{Bern}\left(\frac{1}{2}\right) \begin{cases} 1 & \text{wp } \frac{1}{2} \\ 0 & \text{wp } \frac{1}{2} \end{cases}$

$\overset{\text{small}}{X} \in \mathrm{supp}\,\overset{\text{big}}{[x]}$

$\underset{\text{R.V model}}{X \neq 1}$ $\underset{\text{realization}}{x = 1}$

$P(X=1)$ $x=1$

R.V becomes realization (ex 1 flip, 1 head)

Datum: Realization of a r.V

Data: Realization of r.v.'s

iid Data:

Listen to Lecture for coin example

ⓔⓧ $X_1, \ldots X_7 \overset{iid}{\sim} \mathrm{Hyper}(\underset{n}{\underset{\shortparallel}{3}}, \underset{k}{\underset{\shortparallel}{4}}, \underset{N}{\underset{\shortparallel}{8}})$ ← without replacement

$x_1 = 2, x_2 = 2, x_3 = 2, x_4 = 1, x_5 = 1, x_6 = 0, x_7 = 1$

Israels simulation $X \sim \mathrm{Binom}\left(8, \frac{1}{2}\right)$ ← with replacement

7 people simulation

$X_1, \ldots X_7 \overset{iid}{\sim} \mathrm{Binom}\left(8, \frac{1}{2}\right)$

$X_1 = 4, X_2 = 4, X_3 = 0, X_4 = 4, X_5 = 4, X_6 = 6, X_7 = 4$