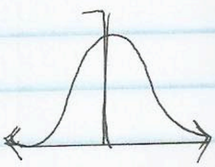
No close form expression for C.D.F (later in Exam. YES)

11/17

From 11/11 lecture:

	P.M.F	P.D.F	CDF	$E[X]$	$Var[X]$	$\lvert Supp[X] \rvert$
Discrete	$p(x) \in [0,1]$ $\sum_{x \in Supp[X]} p(x) = 1$	Not Exist	Exists	$\sum_{x \in Supp[X]} x\, p(x)$	$\sum (x-m)^2 p(x)$	$\leq \mathbb{N}$
Continuous	Not Exist	$f(x) \geq 0$ $\int_{Supp[X]} f(x)dx = 1$	Exists	$\int_{Supp[X]} x f(x) dx$	$\int (x-m)^2 f(x) dx$	$= \lvert \mathbb{R} \rvert$

	Quantile$[X, p]$
Discrete	$\min_x \{x : F(x) \geq p\}$
Continuous	x s.t. $F(x) = p$ $x = F^{-1}(p)$



- $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is this a P.D.F   Supp $= x \in \mathbb{R}$

(a) $f(x) \geq 0$ $\underbrace{\frac{1}{\sqrt{2\pi}}}_{+}$ $\underbrace{\frac{1}{e^{x^2/2}}}_{+}$ ✓

(b) $\int_{\mathbb{R}} f(x) dx = 1$. $\Rightarrow \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$ let $u = \frac{1}{\sqrt{2}} x \Rightarrow u = \frac{x^2}{2}$

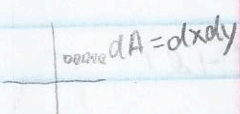
$du = \frac{1}{\sqrt{2}} dx \Rightarrow dx = \sqrt{2}\, du$

$\int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$

*Gamma Integral

$\int_{\mathbb{R}} e^{-u^2} \sqrt{u}\, du = \sqrt{2\pi} \Rightarrow \int_{\mathbb{R}} e^{-u^2} du = \sqrt{\pi}$

$(\int_{\mathbb{R}} e^{-u^2} du)^2 = \pi \Rightarrow \int e^{-u^2} du \int e^{-u^2} du = \pi$

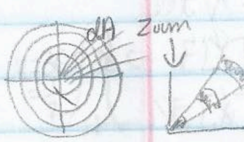


$\int e^{-x^2} dx \int e^{-y^2} dy = \pi$

$\iint e^{-(x^2+y^2)} \underbrace{dx\,dy}_{dA} = \pi$ The relationship firm $\Rightarrow r^2 = x^2 + y^2$

$dA = dx\,dy$

$x \in \mathbb{R}$, $y \in \mathbb{R}$ $\leftarrow A$



$\Rightarrow \int_{r \in [0,\infty)} \int_{\theta \in (0, 2\pi)} e^{-x^2}\, r\, dr\, d\theta$

$\rightarrow$ in terms of polar coordinate (r, θ).

$\Rightarrow \int_{r \in (0,\infty)} e^{-x^2} dr \int_{\theta \in (0, 2\pi)} d\theta = \int e^{-x^2} dr\, [\theta]_0^{2\pi} \Rightarrow \int_0^{\infty} e^{-x^2} dr = \pi = \frac{1}{2}$

$dA = r\,dr\,d\theta$.

$v = r^2$

$dv = 2r\,dr$

$\Rightarrow \frac{dr}{2} = r\,dr$

$\Rightarrow \int_0^{\infty} e^{-x} \frac{dr}{2} = \frac{1}{2} = 1 \quad [e^{-x}]_0^{\infty} = e^{-0} - \lim_{v \to \infty} \cancel{e^{-v}} = 1$ ✓ is a P.D.F.

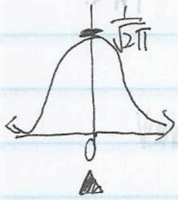
$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$ $Z \sim N(0,1)$

"normal", "gammsim"?

- Bell curve. - is a P.D.F.

- $\boxed{Z \sim N(0,1) = f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}}$ let $u = \frac{x^2}{2} \Rightarrow du = x\,dx$.

Proof: $-E[Z] = \int_{Supp[x]} x f(x)\,dx = \int_{x=-\infty}^{x=\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} e^{-u} du = \frac{1}{\sqrt{2\pi}}[-e^{-u}]_{x=-\infty}^{x=\infty} = \frac{1}{\sqrt{2\pi}}[-e^{-\frac{x^2}{2}}]_{-\infty}^{\infty}$



$= \frac{1}{\sqrt{2\pi}}\left(\underbrace{\lim_{x\to-\infty} e^{\frac{-x^2}{2}}}_{0} - \underbrace{\lim_{x\to\infty} e^{-\frac{x^2}{2}}}_{0}\right) = 0 \Rightarrow \boxed{\mu = 0}$ $\boxed{E[Z] = 0}$

$\frac{1}{e^{\frac{x^2}{2}}}$

$\boxed{Var[Z] = 1} = SE[Z]!$

→ use integration by parts

$-Var[Z] = E[Z^2] - \cancel{\mu^2}^{\,0} = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \underset{(math)}{\cdots} = 1.$

$-SE[Z] = 1$ $\boxed{\sigma^2 = \sigma = 1}$

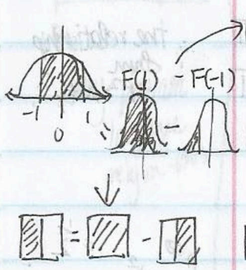
C.D.F $- F(x) = \int f(x)\,dx + C = \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}_{\text{not possible b/c. can't take the antiderivate}} + C$

$f(x) = \frac{d}{dx}[F(x)]$.

"Risd Alyoualim"

$F(0) = \frac{1}{2}$

*No close form for → so need computer to do the calculation. $F(x) = P(Z \le x) = \underbrace{\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dy}_{\text{done numerically. (need a computer)}}$.



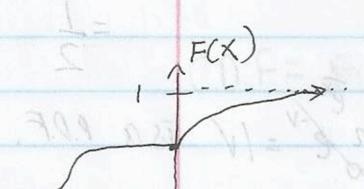
$\rightarrow P(Z \in [-1,1]) = F(1) - F(-1) \approx 0.68$.   68% will come between -1 & 1 realization

$P(Z \in [-2,2]) = F(2) - F(-2) \approx 0.95$ 95% " " " -2 & 2

$P(Z \in [-3,3]) = F(3) - F(-3) \approx 0.997$.   99.7% " " " " -3 & 3

"Empirical Rule", "3σ rule"

only make sense if the p.d.f is...



Review from 11/11 class.

start from discrete & jamming more & more

$-n \to \infty$

- $T \sim Geom(p)$, $E[T] = \frac{1}{p}$

$-\lim_{n \to \infty} p(t) = 0$ has no P.M.F.

$-F(x) = 1 - e^{-\lambda x}$

$f(x) = \lambda e^{-\lambda x}$

Exponential r.V

$X \sim Exp[\lambda] = f(x) = \lambda e^{-\lambda x}$

$E[X] = \frac{1}{\lambda}$

$\lambda = np$ (∞, 0)

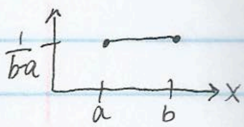
Uniform r.V

$F(X) = \frac{x-a}{b-a}$

See

$X \sim U(a,b) = f(x) = \frac{1}{b-a}$

$X \sim U(0,1)$ = standard Uniform.



- $X \sim Exp[\lambda]$

$Y = 2X \sim ?$

let $\lambda' = \frac{\lambda}{2}$

☆ Process not cover End (result) cover.

By def of C.D.F

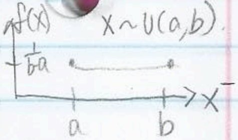
$-F_Y(x) = P(Y \le x) = P(2X \le x) = P(X \le \frac{x}{2}) = F_X(\frac{x}{2}) = 1 - e^{-\lambda \frac{x}{2}} = 1 - e^{-\lambda' x} = Exp[\lambda']$

$= Exp[\frac{\lambda}{2}]$

$\boxed{Y \sim Exp[\frac{\lambda}{2}]}$

$\Rightarrow Y \sim Exp[\frac{\lambda}{2}]$

$X \sim U(a,b)$.



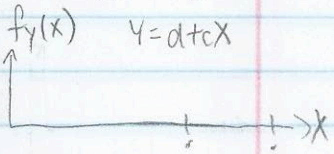
by def.

$-F_Y(x) = P(Y \le x) = P(cX + d \le x) = P(X \le \frac{x-d}{c}) = F_X(\frac{x-d}{c}) = \frac{\frac{x-d}{c} - a}{b-a} \cdot \frac{c}{c}$

$= \frac{x-(d+ca)}{cb-ca}$

$= \frac{x-(d+ca)}{(d+cb)-(d+ca)}$

$f_Y(x)$ $Y = d + cX$



let $a' = d + ca$

$b' = d + cb$

$= \frac{x - a'}{b' - a'}$ → matching it looks like the F(x) def.

$= U(a', b')$

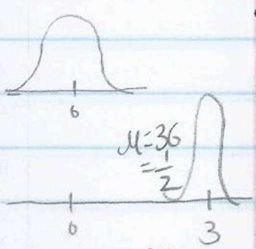
$= U(d+ca, d+cb)$

- $Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

let $X := \sigma Z + \mu$

$E[X] = \cancel{\sigma E[Z]}^{0} + \mu = \mu$

$SE[X] = SE[\sigma Z + \mu] = |\sigma| \cancel{SE[Z]}^{1} = |\sigma|$



shifting & skewing

$F_X(x) = P(X \le x) = P(\sigma Z + \mu \le x) = P(Z \le \frac{x-\mu}{\sigma}) = \underbrace{F_Z\left(\frac{x-\mu}{\sigma}\right)}_{\text{unknowable}}$

$f(x) = \frac{d}{dx} F_X[x] = \frac{d}{dx} F_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{du}{dx} \cdot \frac{d}{du}[F_Z(u)] = \frac{1}{\sigma}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\right) = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2}$

let $u = \frac{x-\mu}{\sigma} \Rightarrow \frac{du}{dx} = \frac{1}{\sigma}$

normal distribution

$\boxed{= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{\sqrt{2\pi}}(x-\mu)^2} \quad X \sim N(\mu, \sigma)}$

New f(x)

$Z \sim N(0,1)$ = standard normal r.V

$-\text{supp}[X] = \mathbb{R}$

$\mu \in \mathbb{R}$, $\sigma^2 \in (0, \infty)$

— parameter space