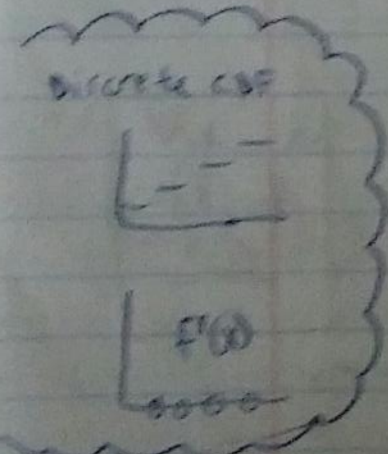
Lecture 18: November 17, 2016

	PMF	PDF
Discrete r.v.	$p(x)\in[0,1]$ $\sum_{x\in Supp(X)} p(x)=1$	no PDF
Continuous r.v.	no PMF	$f(x)\geq 0$ $\int_{Supp(X)} f(x)\,dx = 1$ $f(x)=\frac{d}{dx}\left(F(x)\right)$



	CDF	E[X]
Discrete r.v.	Yes	$\sum_{x\in Supp(X)} x\,p(x)$
Ctn. r.v.	Yes	$\int_{Supp(X)} x\,f(x)\,dx$

	Var[X]	|Supp[X]|	Quantile(X, p)
Discrete r.v.	$\sum_{x\in Supp(X)} (x-\mu)^2\,p(x)$	$\leq \lvert\mathbb{N}\rvert$	$\min\{x : F(x)\geq p\}$
Ctn. r.v.	$\int_{Supp(X)} (x-\mu)^2 f(x)\,dx$	$\lvert\mathbb{R}\rvert$	$x = F^{-1}(p)$ [x s.t. F(x) = p]

$\cdot\ f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$= \dfrac{1}{\sqrt{2\pi}} \cdot \dfrac{1}{e^{x^2/2}}$

(a) $f(x) \geq 0 \quad x \in \mathbb{R}$ ✓

(b) $\displaystyle\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = 1$?

Prove

Proof

$\dfrac{1}{\sqrt{2\pi}} \displaystyle\int e^{-\frac{x^2}{2}} dx = 1 \Rightarrow \int e^{-\frac{x^2}{2}} dx = \sqrt{2}\pi$

let $u = \dfrac{1}{\sqrt{2}} x \Rightarrow u^2 = \dfrac{x^2}{2}$

$\Rightarrow du = \dfrac{1}{\sqrt{2}} dx$

$dx = du\sqrt{2}$

$\displaystyle\int e^{-u^2} du = \sqrt{\pi}$

$\left(\displaystyle\int e^{-u^2} du\right)^2 = \pi \Rightarrow \int e^{-u^2} du \int e^{-u^2} = \pi$

$\Downarrow$

$(r^2 = x^2 + y^2)$

$\displaystyle\underbrace{\int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}}}_{A} \overbrace{e^{-(x^2+y^2)}}^{f(x,y)} \underbrace{dx\,dy}_{dA} \longleftarrow \int e^{-x^2} dx \int e^{-y^2} dy = \pi$

$\displaystyle\int_{r \in (0,\infty)} \int_{\theta \in (0,2\pi)} e^{-r^2} r\,dr\,d\theta = \int_{r \in (0,\infty)} e^{-r^2} r\,dr \int_{\theta \in (0,2\pi)} d\theta = 2\pi \int e^{-r^2} r\,dr \overset{?}{=} \underline{\pi}$

$\downarrow$

next page $\Leftarrow$ $\dfrac{1}{2} \Leftarrow \displaystyle\int_0^\infty e^{-u} \frac{du}{2} \Leftarrow$ let $u = r^2$ $\quad \dfrac{du}{dr} = 2r \Rightarrow du = 2r\,dr \Rightarrow r\,dr = \dfrac{du}{2}$ $\Leftarrow \displaystyle\int_0^\infty e^{-r^2} r\,dr = \frac{1}{2}$

$$= -e^{-u}\Big]_0^{\infty} = \left(e^{-(0)} - \lim_{x\to\infty} e^{-x}\right) = 1 - 0 = 1 \checkmark$$

- $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ PDF? $\Rightarrow$ Through proof, yes is a PDF.

$$\boxed{Z \sim N(0,1) :- f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}$$

"bell curve"

Standard normal r.v. "gassium"

- $E[Z] = \int x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x e^{-\frac{x^2}{2}} dx$

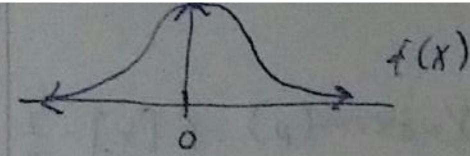
$\left[\text{let } u = \frac{x^2}{2} \Rightarrow \frac{du}{dx} = x \Rightarrow du = x\,dx\right]$

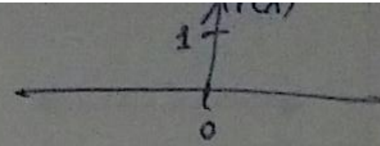
$$\frac{1}{\sqrt{2\pi}} \left[-e^{-u}\right]_{x=-\infty}^{x=\infty} \Leftarrow \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} e^{-u} du$$

$$\frac{1}{\sqrt{2\pi}} \left(\underbrace{\lim_{x\to-\infty} e^{-\frac{x^2}{2}}}_{\frac{1}{e^{x^2/2}}} - \lim_{x\to\infty} e^{-\frac{x^2}{2}}\right) = \frac{1}{\sqrt{2\pi}}(0 - 0) = 0$$

$\Downarrow$

$$\boxed{\mu = 0}$$





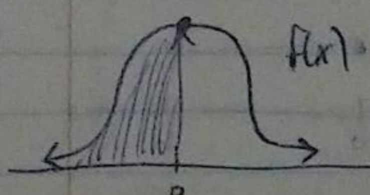
- $Var[Z] = E[Z^2] - \cancel{\mu^2}^{\,\nearrow 0}$ $\rightarrow$ $F(x) = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \ldots\ldots$

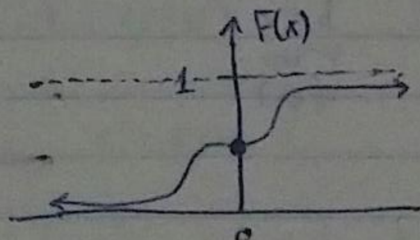
integration by parts

$\Downarrow$

$\boxed{\sigma^2 = \sigma = 1} \Leftarrow SE[Z] = 1 \Leftarrow = 1$

- $F(x) = \int f(x)\,dx + C = \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}_{\text{not possible}} + C$

- 

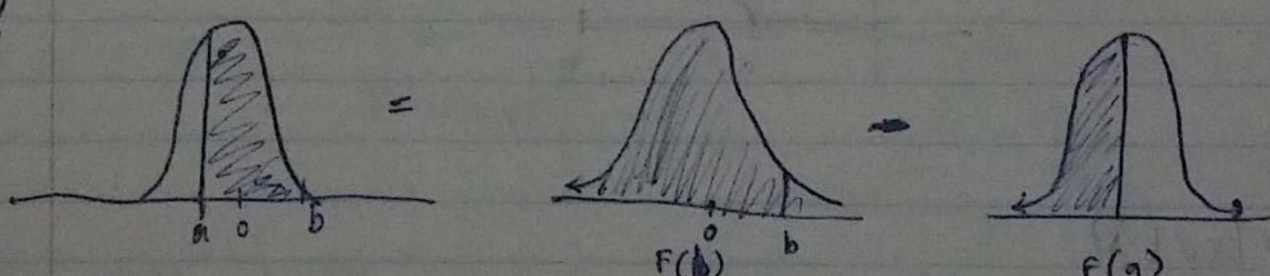


F(0)

F(1)

The bell is one, integrates to 1

- $P(Z \in [a, b]) = F(b) - F(a)$



between -1, 1 SE's

- $P(Z \in [-1, 1]) = F(1) - F(-1) \approx .68$
- $P(Z \in [-2, 2]) = F(2) - F(-2) \approx .95$
- $P(Z \in [-3, 3]) = F(3) - F(-3) \approx .997$

~~[illegible] rule~~

"66 - 95 - 99.7" rule

"3σ rule"

"Empirical rule"

$X \sim Geom(p) \cdot E[X] = \frac{1}{p}$

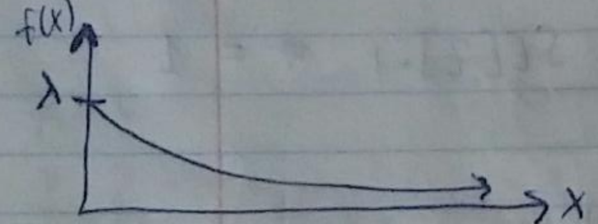
$\lambda = np$

$\downarrow \infty \quad \downarrow 0$

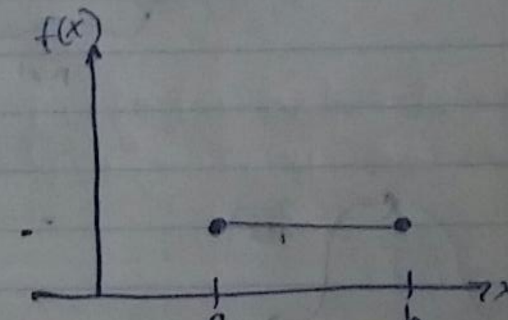
• $X \sim Exp(\lambda) := \lambda e^{-\lambda x}$

$F(x) = 1 - e^{-\lambda x}$

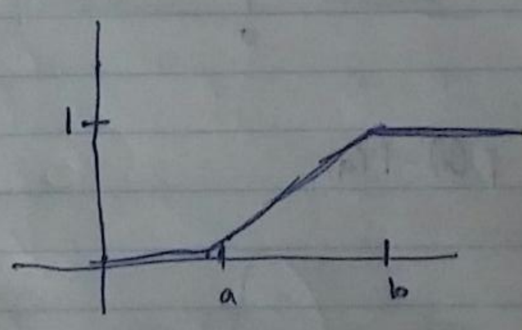
$E[X] = \frac{1}{\lambda}$



• $X \sim Uniform(a,b) = \underbrace{\frac{1}{b-a}}_{f(x)}$



$F(x) = \frac{x-a}{b-a}$



• $X \sim Exp(\lambda)$

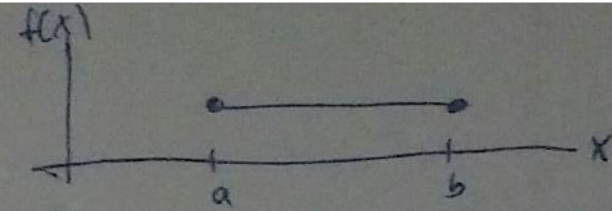
$Y = 2X \sim ?$

$\lambda' = \frac{\lambda}{2}$

$F_Y(x) = P(Y \le x) = P(2X \le x) = P(X \le \frac{x}{2}) = F_X(\frac{x}{2}) = 1 - e^{-\lambda \frac{x}{2}}$

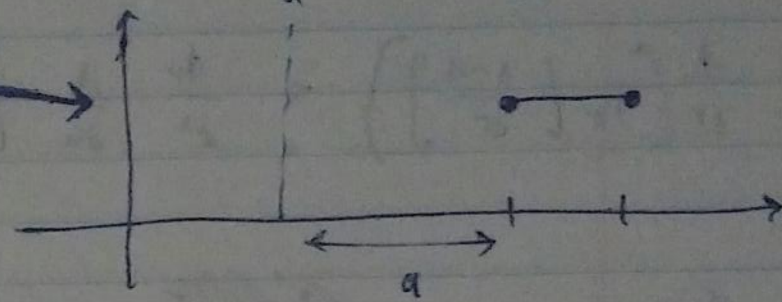
$\Downarrow$

$\boxed{Y \sim Exp(\frac{\lambda}{2})} \Leftarrow Y \sim Exp(\lambda') \Leftarrow \underbrace{1 - e^{-\lambda' x}}_{\text{CDF for exponential}}$



- $X \sim \text{Uniform}(a,b)$

  $Y = d + cX \sim ?$



$F_Y(x) = P(Y \le x)$

$= P(d + cX \le x) = P\left(X \le \frac{x-d}{c}\right) = F_X\left(\frac{x-d}{c}\right) = \frac{\frac{x-d}{c} - a}{b-a} \cdot \frac{c}{c}$

$\Downarrow$

$Y \sim \text{Uniform}(a', b')$

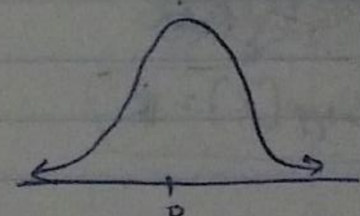
$\frac{x-a'}{b'-a'} = \frac{x-(d+ac)}{(d+bc)-(d+ac)} \Leftarrow \frac{x-(d+ac)}{bc-ac}$

(let $a' = d+ac$
$b' = d+bc$)

$\boxed{Y \sim \text{Uniform}(d+ac,\ d+bc)}$

$\boxed{Z \sim N(0,1) := f(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}$ $E[Z] = 0,\ SE[Z] = 1$

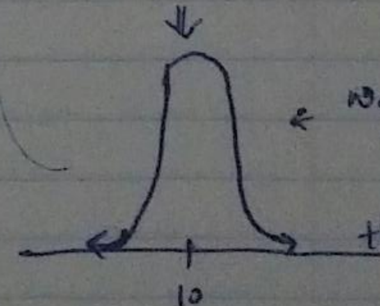
$X = \mu + \sigma Z$



$F_X(x) = P(X \le x) = P(\mu + \sigma Z \le x)$ $\mu = 10 \quad \sigma = \frac{1}{2}$

$\Downarrow$

STOP $F_Z\left(\frac{x-\mu}{c}\right) \Leftarrow P\left(Z \le \frac{x-\mu}{\sigma}\right)$



$\Leftarrow$ want PDF of this thing!

$f(x) = \frac{d}{dx} F(x) \longrightarrow \frac{d}{dx}\left[F_Z\left(\frac{x-\mu}{\sigma}\right)\right]$

PDF

let $u = \frac{x-\mu}{\sigma} \Rightarrow \frac{du}{dx} = \frac{1}{\sigma} \longrightarrow$ Next Page

$$\frac{d}{dx}\left(F_Z\left[\frac{x-\mu}{\sigma}\right]\right) = \frac{du}{dx}\,\frac{d}{du}\left[F_Z(u)\right] = \frac{1}{\sigma}\, f_Z(u)$$

$$\Downarrow$$

$$\frac{1}{\sigma}\left(\frac{1}{\sqrt{2\pi}}\, e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}\right) \Leftarrow \frac{1}{\sigma}\, f_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$\frac{(x-\mu)^2}{2\sigma^2}$$

NORMAL PDF

$$\boxed{\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}$$

$$E[X] = (\mu + \sigma Z) = \mu$$
$$SE[X] = SE[\mu + \sigma Z] = |\sigma|$$

- Normal PDF

$$\boxed{X \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}$$

$\boxed{\star\ \text{Param Space}: \mu \in R,\ \sigma^2 \in (0, \infty)}$

$\boxed{\text{Supp}(X) = \mathbb{R}}$