November 29, 2016

> Definition:

Moment Generating Function (MGF)

$M_X(t) := E[e^{tX}]$

$e^{tx} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots$ } Taylor Series Expansion.

> Facts (Properties):

I $M_X(t) = M_Y(t) \Rightarrow X \overset{d}{=} Y$ } The PMFs or the PDFs are equal.

II $E[X^k] = M_X^{(k)}(0)$

III If $Y = aX + c \Rightarrow M_Y(t) = e^{tc} M_X(at)$

IV If X, Y are independent,

$M_{X+Y}(t) = M_X(t) M_Y(t)$

If iid,

$M_{X+Y}(t) = M_X(t) M_Y(t) = (M_X(t))^2$

> $X \sim Bern(p) \Rightarrow M_X(t) = 1-p+pe^t$

$X \sim Binom(n,p) \Rightarrow M_X(t) = (1-p+pe^t)^n$

$X \sim Geom(p) \Rightarrow M_X(t) = \frac{pe^t}{1-e^t(1-p)}$ if $t < \ln\left(\frac{1}{1-p}\right)$

$X \sim Exp(\lambda) \Rightarrow M_X(t) = \frac{\lambda}{\lambda - t}$ if $t < \lambda$

$Z \sim N(0,1) \Rightarrow M_Z(t) = e^{\frac{t^2}{2}}$

$X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$X \sim Deg(c) \Rightarrow M_X(t) = e^{tc}$

V Levy's Continuity Theorem

$X_1, X_2, \ldots$ is a sequence of r.v.s.

$\lim_{n\to\infty} M_{X_n}(t) = M_Y(t) \iff X_n \xrightarrow{\text{converges}} Y$

If n large, $M_{X_n}(t) \approx M_Y(t) \Rightarrow X_n \approx Y$

> Law of Large Numbers (LLN)

$x_1, \dots, x_n \overset{iid}{\sim}$, with mean μ.

$\bar{x}_n \xrightarrow{\text{converges}} \mu$.

↑ it becomes μ with probability = 1.

↳ Kinda like $\sim Deg(\mu) \Rightarrow M_\mu(t) = e^{t\mu}$

$$M_{\bar{x}_n}(t) = \underset{\substack{\uparrow \\ \text{def. of } \bar{x}}}{M_{\frac{T}{n}}(t)} \underset{\substack{\uparrow \\ \text{III}}}{=} M_T\left(\tfrac{t}{n}\right) = M_{x_1+\dots+x_n}\left(\tfrac{t}{n}\right) \underset{\substack{\uparrow \\ \text{IV}}}{=} \left(M_x\left(\tfrac{t}{n}\right)\right)^n$$

$$\underset{\substack{\uparrow \\ \text{Def of MGF}}}{=} \left(E\left[e^{\frac{t}{n}x}\right]\right)^n \underset{\substack{\uparrow \\ \text{Taylor series}}}{=} \left(E\left[1 + \tfrac{t}{n}x + \tfrac{t^2x^2}{2!n^2} + \tfrac{t^3x^3}{3!n^3} + \dots\right]\right)^n$$

"little o"

[We say $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

↳ $g(n)$ grows faster than $f(n)$

Ex: Is $n^2 = o(n^3)$?

$\lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓]

Is $\frac{t^2x^2}{2!n^2} + \frac{t^3x^3}{3!n^3} + \dots = o\left(\frac{1}{n}\right)$?

$$\lim_{n\to\infty} \frac{\frac{t^2x^2}{2!n^2} + \frac{t^3x^3}{3!n^3} + \dots}{\frac{1}{n}} = \lim_{n\to\infty} \frac{t^2x^2}{2n} + \frac{t^3x^3}{3!n^2} + \dots = 0 \checkmark$$

$$M_{\bar{x}_n}(t) = \left(E\left[1 + \tfrac{tx}{n} + o\left(\tfrac{1}{n}\right)\right]\right)^n = \left(1 + \tfrac{t\mu}{n} + E\left[o\left(\tfrac{1}{n}\right)\right]\right)^n$$

$$= \left(1 + \tfrac{t\mu}{n} + o\left(\tfrac{1}{n}\right)\right)^n$$

$\lim_{n\to\infty} (1+\frac{1}{n})^n = e$

$\lim_{n\to\infty} (1+\frac{a}{n})^n = e^a$

$\lim_{n\to\infty} (1+\frac{a}{n} + \frac{1}{n^2})^n = e$

$n = 1$ billion $\Rightarrow f(n) = 3.08 \neq 2.718$

$n = 1$ trillion $\Rightarrow f(n) = 2.90 \neq 2.718$

"As $n \to \infty$, is the n going to matter? Anything that goes quicker than $\frac{1}{n} = e$." * fact.

$\lim_{n\to\infty} (1 + \frac{a}{n} + o(\frac{1}{n}))^n = e^a$

$\lim_{n\to\infty} M_{\bar{x}_n}(t) = \lim_{n\to\infty} ((1 + \frac{t\mu}{n} + o(\frac{1}{n}))^n = e^{t\mu}$

> If $x_1, x_2, \ldots, x_n \overset{iid}{\sim}$ with mean μ and S.E. σ, what does $C_n = \frac{\bar{x}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$ look like as n gets larger?

$E[\bar{x}] = \mu$

$SE[\bar{x}] = \frac{\sigma}{\sqrt{n}}$

$\bar{x}$ standardized to have mean 0 and SE 1.

$$C_n = \frac{\sqrt{n}(\bar{x}-\mu)}{\sigma} = \frac{\sqrt{n}\left(\frac{x_1+\ldots+x_n}{n}\right) - \mu}{\sigma} = \frac{\frac{\sqrt{n}\, x_1+\ldots+x_n - \mu n}{\cancel{n}\sqrt{n}}}{\sigma}$$

$$= \frac{x_1+\ldots+x_n - n\mu}{\sigma\sqrt{n}} = \frac{(x_1+\ldots+x_n) - (\mu+\ldots+\mu)}{\sigma\sqrt{n}}$$

$$= \frac{(x_1-\mu)+(x_2-\mu)+\ldots+(x_n-\mu)}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}}\left(\frac{x_1-\mu}{\sigma} + \frac{x_2-\mu}{\sigma} + \ldots + \frac{x_n-\mu}{\sigma}\right)$$

Let $Z_i = \frac{x_i-\mu}{\sigma} \Rightarrow \frac{1}{\sqrt{n}}(Z_1+\ldots+Z_n) = C_n$

$E[Z] = 0$ $\quad E[Z^2] = 1$

$SE[Z] = 1$

↳ $Var[X] = E[X^2] - \mu^2$. Since $\mu = 0$, $\sigma^2 = E[X^2]$. Since $\sigma = 1$, $E[X^2] = 1$.

$$M_{C_n}(t) = M_{\frac{1}{\sqrt{n}}(Z_1+\dots+Z_n)}(t) \underset{\text{III}}{=} M_{Z_1+\dots+Z_n}\left(\frac{t}{\sqrt{n}}\right) \underset{\text{IV}}{=} \left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

$$\underset{\text{def of MGF}}{=} \left(E\left[e^{\frac{t}{\sqrt{n}}\cdot Z}\right]\right)^n \underset{\text{Taylor series}}{=} \left(E\left[1+\frac{t}{\sqrt{n}}\cdot Z+\frac{t^2Z^2}{2!\,n}+\frac{t^3Z^3}{3!\,n^{\frac{3}{2}}}+\frac{t^4Z^4}{4!\,n^2}+\dots\right]\right)^n$$

$$= \left(E\left[1+\frac{tZ}{\sqrt{n}}+\frac{t^2Z^2}{2n}+o\left(\frac{1}{n}\right)\right]\right)^n = \left(1+\frac{t^2}{2n}+o\left(\frac{1}{n}\right)\right)^n$$

$$E\left[\frac{tZ}{\sqrt{n}}\right] = \frac{t}{\sqrt{n}}E[Z] = 0$$

$$E\left[\frac{t^2Z^2}{2n}\right] = \frac{t^2}{2n}E[Z^2] = \frac{t^2}{2n}$$

$$= \lim_{n\to\infty} M_{C_n}(t) = \lim_{n\to\infty}\left(1+\frac{t^2}{2n}+o\left(\frac{1}{n}\right)\right)^n = e^{\frac{t^2}{2}}$$

$\Downarrow$ MGF of the standard normal.

> Central Limit Theorem (CLT)

If $x_1, \dots, x_n \overset{iid}{\sim}$ with mean μ and SE σ

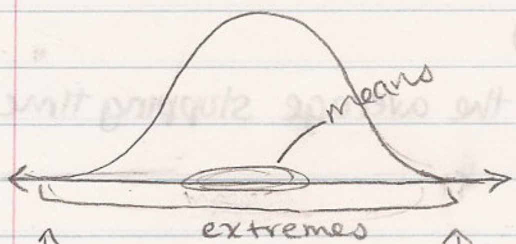
I $C_n = \dfrac{\bar{x}_n - \mu}{\frac{\sigma}{n}} \longrightarrow Z \sim N(0,1)$

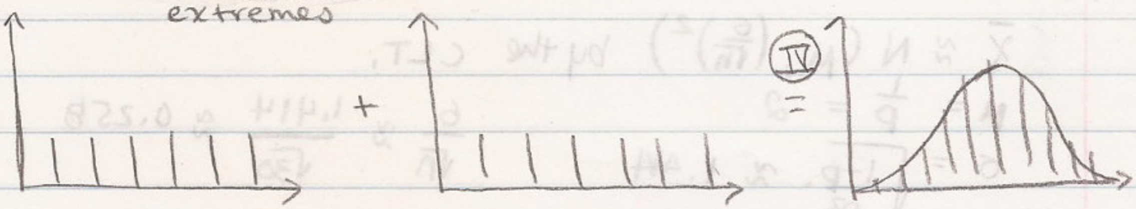
If n is large,

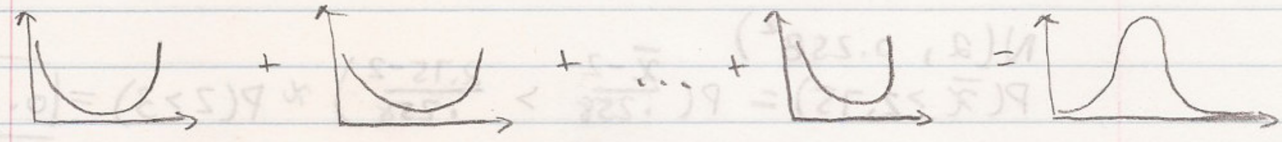
II $C_n \overset{d}{\approx} N(0,1)$

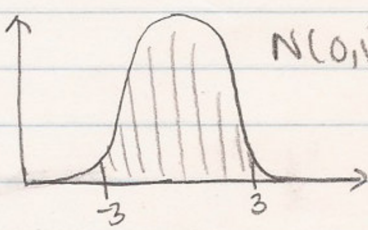
III $\bar{x} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) \rightarrow$ "what is the average...?"

IV $T \overset{d}{\approx} N\left(n\mu, (\sqrt{n}\sigma)^2\right) \rightarrow$ "How far away...?"









$N(0,1)$ Why is it called normal? Because the curve defines what is normal. If it's not within the 99.7%, it's not normal!

> Problem Solving

- Take 100 random equally likely left and right steps. What is the probability that you are more than 10 steps away from where you began?

Let $T = x_1 + \dots + x_{100} \overset{d}{\approx} \sim N(n\mu, (\sqrt{n}\sigma)^2)$

$\mu = 0$ b/c its average $= \frac{1+(-1)}{2} = 0$

$\sigma = \sqrt{\sigma^2} = \sqrt{1} = 1$

$x_1, x_2, \dots, x_n \overset{iid}{\sim} \begin{cases} 1 \text{ wp } \frac{1}{2} \\ -1 \text{ wp } \frac{1}{2} \end{cases}$

$N(0, 10^2) = P(|T| > 10) = P(T < -10) + P(T > 10)$

$n\mu = 100 \cdot 0 = 0$, $(\sqrt{n}\sigma)^2 = (\sqrt{100} \cdot 1)^2 = 10^2$

$= P\left(\frac{T-0}{10} < \frac{-10-0}{10}\right) + P\left(\frac{T-0}{10} > \frac{10-0}{10}\right)$

$\approx P(Z < -1) + P(Z > 1) = P(Z \notin [-1, 1])$

$= 1 - 68\% = \boxed{32\%}$

- $x_1, \dots, x_{30} \overset{iid}{\sim} \text{Geom}\left(\frac{1}{2}\right)$

What's the probability that the average stopping time is more than 2.75?

$\bar{x} \approx N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$ by the CLT.

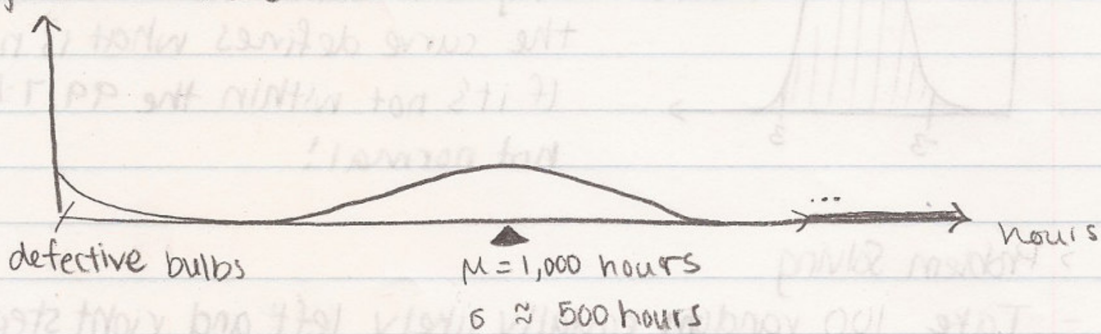
$\mu = \frac{1}{p} = 2$ $\qquad \frac{\sigma}{\sqrt{n}} \approx \frac{1.414}{\sqrt{30}} \approx 0.258$

$\sigma = \sqrt{\frac{1-p}{p^2}} \approx 1.414$

$N(2, 0.258^2)$

$P(\bar{x} > 2.75) = P\left(\frac{\bar{x}-2}{.258} > \frac{2.75-2}{.258}\right) \approx P(Z > 3) = \boxed{0.0015}$

- Lightbulb Failure Times



$f_{\bar{x}}(t)$ is not normal

You buy 50 light bulbs. What is the probability they last, on average, more than 1300 hours?

$\bar{x} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) = N\left(1000, \left(\frac{500}{\sqrt{50}}\right)^2\right) = N(1000, 70.7^2)$

$P(\bar{x} > 1300) = \left(\frac{\bar{x}-1000}{70.7} > \frac{1300-1000}{70.7}\right) = P(Z > 4.24) \approx 0$