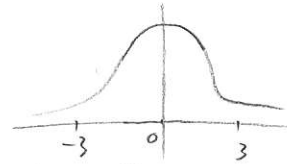
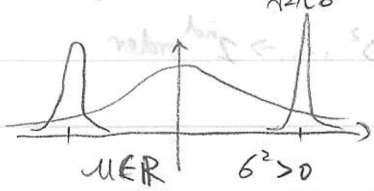
11/22/2016



△ $Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x)$, $Supp(Z) = \mathbb{R}$, $E(Z) = 0$, $SE(Z) = 1$

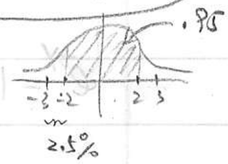
$X \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$, $E(X) = \mu$, $SE(X) = \sigma$, $Z = \frac{X-\mu}{\sigma}$

(Standardization)



Ex: X is male height, norm. distr w/mean 70", standard error 4",
Find prob. a male is more than 78" tall.

$P(Z \in [-2,2]) = .95$

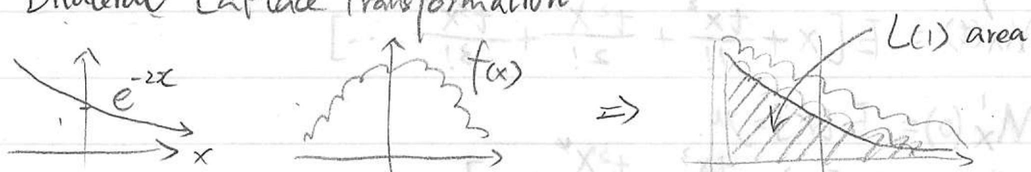


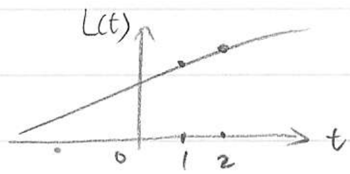
$X \sim N(70'', 4''^2)$

$P(X > 78) = P\left(\frac{X-70}{4} > \frac{78-70}{4}\right) = P(Z > 2) = 2.5\%$

△ $L(t) = \int_{\mathbb{R}} e^{-tx} f(x)\,dx$ if $t=1$ $L(1) \int_{\mathbb{R}} e^{-x} f(x)\,dx$

"Bilateral LaPlace Transformation"





Thm: if $L(t)$ exist, $L(t)$ & $f(x)$ are 1:1.

If $f(x)$ is a PDF.

$M_X(t) = E[e^{tX}] := \int_{\mathbb{R}} e^{tx} f(x)\,dx$. (for cont) $M_X(t) := E[e^{tX}] = \sum_{x \in Supp[x]} e^{tx} p(x)$

(for discrete)

⇑

MGF r.v. X (moment generating functions)

$M_X(t) = E(e^{tX}) = e^{t(0)} \cdot p(0) + e^{t(1)} \cdot p(1) = 1 - p + pe^t$

① If $M_X(t) = M_Y(t) \Rightarrow X \stackrel{d}{=} Y$

Recall $\forall c$, $f(x)$ cont.

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 \ldots\ldots$$ (Taylor series)

$x \approx c$

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 \ldots \Rightarrow 2^{nd} \text{ order}$$

Let $f(x) = e^x$ $\quad x \approx 0 \Rightarrow c = 0$

$$e^x = e^0 + \frac{e^0 x}{1!} + \frac{e^0 x^2}{2!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \ldots$$

$$M_X(t) = E[e^{tX}] = E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3x^3}{3!} + \ldots\right] \quad \frac{d}{dt}[M_X(t)]$$

$$= \frac{d}{dt}[E(\ldots)] = E\left[\frac{d}{dt}[\ldots]\right]$$

$$M_X'(t) = E\left[X + \frac{tX^2}{1!} + \frac{t^2X^3}{2!} + \frac{t^3X^4}{3!} \ldots\right]$$

$$M_X'(0) = E[X] = \mu$$

$$M_X''(t) = E\left[X^2 + \frac{tX^3}{1!} + \frac{t^2X^4}{2!} + \ldots\right]$$

$$M_X''(0) = E[X^2]$$

$$M_X'''(t) = E\left[X^3 + tX^4 + \frac{t^2X^5}{1!} + \ldots\right]$$

$$M_X'''(t) = E[X^3]$$

② $M_X^{(k)}(0) = E[X^k]$ The MGF ...

$$= \int_{\mathbb{R}} e^{tax} \cdot e^{tc} f(x)\,dx.$$

Let $Y = aX + c$; $M_Y(t) = M_{aX+c}(t) = E[e^{t(aX+c)}] = E[e^{tax} \cdot e^{tc}]$

$$= e^{tc} E[e^{taX}]$$

$$\Downarrow$$

$L(t) = \int e^{tx} f(x)\,dx$ Let $t' := at$. $= e^{tc} E[e^{t'X}] = e^{tc} M_X(t')$

$$= e^{tc} M_X(at)$$

③. If $Y = aX + c \Rightarrow M_Y(t) = e^{tc} M_X(at)$

△ Consider X_1, X_2 indep. r.v.'s Let $Y = X_1 + X_2$.

$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}] = E[e^{tX_1} \cdot e^{tX_2}] = E[e^{tX_1}] \cdot E[e^{tX_2}]$

$= M_{X_1}(t) M_{X_2}(t)$

if $X_1, X_2 \sim$ iid. $M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t) = (M_X(t))^2$ 同分布①

④ If X_1, X_2 indep. $Y = X_1 + X_2 \Rightarrow M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t)$

△Ex: $X \sim Bern(p) \Rightarrow M_X(t) = 1 - p + pe^t$

Recall: $X_1, X_2, \dots X_n \overset{iid}{\sim} Bern(p)$

$T = X_1 + X_2 + X \dots X_n \sim Bino(n, p)$

$M_T(t) = E[e^{tT}] = E[e^{t(X_1+X_2\dots X_n)}] \overset{\text{by indep ④}}{=} M_{X_1}(t) \dots M_{X_n}(t)$

$\overset{\text{by iden distr.}}{=} (M_X(t))^n = (1 - p + pe^t)^n$

Ex: $X \sim Geom(p)$

$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1} p \cdot \frac{(1-p)}{(1-p)} = \frac{p}{1-p} \sum_{x=1}^{\infty} \overbrace{e^{tx}}^{(e^t)^x}(1-p)^x$

$= \frac{p}{1-p} \sum_{x=1}^{\infty} (e^t(1-p))^x = \frac{p}{1-p}\left(\sum_{x=0}^{\infty} (e^t(1-p))^x - 1\right)$

$= \frac{p}{1-p}\left(\frac{1}{1-e^t(1-p)} - 1\right) = \frac{p}{1-p} \cdot \frac{e^t(1-p)}{1-e^t(1-p)}$

$$\boxed{\begin{aligned} &\text{if } e^t(1-p) < 1 \\ &\Rightarrow e^t < \tfrac{1}{1-p} \\ &\Rightarrow t < \ln\left(\tfrac{1}{1-p}\right) \end{aligned}}$$

$$\boxed{M_X(t) = \frac{pe^t}{1-e^t(1-p)} \quad \mathbb{1}_{t < \ln(\frac{1}{1-p})}}$$

$X \sim Exp(\lambda) \Rightarrow M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$

$= \lambda \int_0^\infty e^{(t-\lambda)x} dx = \frac{\lambda}{t-\lambda} [e^{(t-\lambda)x}]_0^\infty$

$= \frac{\lambda}{t-\lambda} \left(\lim_{x \to \infty} e^{(t-\lambda)x} - 1 \right) = \frac{x}{t-\lambda}(0-1) = \boxed{\frac{\lambda}{\lambda - t} \mathbb{1} t < \lambda} = M_X(t)$

if $t - \lambda < 0 \Rightarrow t < \lambda$.

$X \sim Exp(\lambda)$

$Y = aX,\ a \in \mathbb{R}$ Let $\lambda' = \frac{\lambda}{a}$

$M_Y(t) = e^{tc} M_X(at) = M_X(at) = \frac{\lambda}{\lambda - at} \cdot \frac{\frac{1}{a}}{\frac{1}{a}} = \frac{\lambda'}{\lambda' - t} \Rightarrow Y \sim Exp[\lambda'] = Exp[\frac{\lambda}{a}]$

$(x-t)^2 = x^2 - 2tx + t^2 \rightarrow -\frac{1}{2}(x^2 - 2tx) = -\frac{1}{2}((x-t)^2 - t^2) = -\frac{1}{2}(x-t)^2 + \frac{1}{2}t^2$

$Z \sim N(0,1) \Rightarrow M_Z(t) = E[e^{tZ}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}x^2 + tx} dx$

$= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x-t)^2} e^{\frac{t^2}{2}} dx = e^{\frac{t^2}{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dx = \boxed{e^{\frac{t^2}{2}}}$

(Another way to proof:)

$\mu = M'_Z(0) = t e^{\frac{t^2}{2}} \Big|_{t=0} = 0$

$\sigma^2 = E[Z^2] - \mu^2 = E[Z^2] = M_Z''(0) = e^{\frac{t^2}{2}} + t \cdot t e^{\frac{t^2}{2}} \Big|_{t=0} = 1 \Rightarrow \sigma = 1$

$\triangle$ $X_1, X_2, \ldots X_n \overset{iid}{\sim}$ $\bar{X} \to \mu$ Law of large #'s

conv i's distr.

$\boxed{\text{Levy's Continuing Thm. } X_n \to X \text{ if } \lim_{n \to \infty} M_{X_n}(t) = M_X(t)}$

$\mu \sim Deg(x)$. $X \sim Deg(c) \Rightarrow M_X(t) = E[e^{tX}] = e^{tc}$

$M_\mu(t) = e^{t\mu}$

$\lim_{n \to \infty} M_{\bar{X}}(t) = e^{t\mu}$ (next page)

$$M_{\bar{X}}(t) = E[e^{t\bar{x}}] = E[e^{x(\frac{x_1 + \dots x_n}{n})}] = M_{x_1}$$