10/13 – $X \sim$ Bernouilli (p)
$X \sim$ Binomial (n, p)
$X \sim$ Hyper (n, K, N)

☆ $\lim_{N \to \infty}$ Hyper (n, p, N) = Binom (n, p). b/c pull things from a big # ...

- $\sum_{x \in Supp[X]} p(x) = 1$ $\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = 1.$

binomial thm: Recall $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

let $a = p$
$b = 1-p$
$c = X$

$(p + (1-p))^n = \sum_{i=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$
$= n$
$= 1$

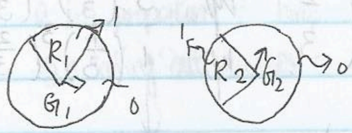
- X_1 and X_2 are independent $X_1, X_2 \overset{ind}{\sim}$

$P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$ for all possible value in $\forall x_1 \in Supp[X_1]$,
–or– $P(X_2 = x_2 \mid X_1 = x_1) = P(X_2 = x_2)$ $\forall x_1 \in Supp[X_2]$,
–or– $\underbrace{P(X_1 = x_1, X_2 = x_2)}_{\text{Joint Mass Function (JMF)}} = P(X_1 = x_1) P(X_2 = x_2)$

- Def: if X_1, X_2 are independent and identically distr. (iid) means
$X_1 \overset{d}{=} X_2$ and $X_1, X_2 \overset{ind}{\sim}$ denotes $X_1, X_2 \overset{iid}{\sim}$

distributed

$X_1, X_2 \overset{iid}{\sim}$ Bern $\left(\frac{1}{3}\right)$



3 ways • Note that •

$T=1$: $\underline{1}\ \underline{0}\ \underline{0}$, $\underline{0}\ \underline{1}\ \underline{0}$, $\underline{0}\ \underline{0}\ \underline{1}$ $\Big\}\binom{3}{1}$ $T=2$: $\underline{1}\ \underline{1}\ \underline{0}$, $\underline{1}\ \underline{0}\ \underline{1}$, $\underline{0}\ \underline{1}\ \underline{1}$ $\Big\}\binom{3}{2}$ $T=0$: $\underline{0}\ \underline{0}\ \underline{0}\ \}\binom{3}{0}$

$T=3$: $\underline{1}\ \underline{1}\ \underline{1}\ \}\binom{3}{3}$

• Let $T_2 = X_1 + X_2 = g(X_1, X_2)$ $\qquad X_1, X_2 \overset{iid}{\sim} Bern(\frac{1}{3})$

$Supp[X] = \{0, 1\}$

– $Supp[T_2] = \{0, 1, 2\}$

$T_2 = X_1 + X_2$

Supp $[X_1]$ Supp $[X_2]$ T

$\frac{1}{3}$: 1 — $\frac{1}{3}$: 1 $\quad P(X_1=1, X_2=1) = \frac{1}{9}$ $\quad 2$

$\frac{1}{3}$: 1 — $\frac{2}{3}$: 0 $\quad P(X_1=1, X_2=0) = \frac{2}{9}$ $\quad 1$

$\frac{2}{3}$: 0 — $\frac{1}{3}$: 1 $\quad P(X_1=0, X_2=1) = \frac{2}{9}$ $\quad 1$

$\frac{2}{3}$: 0 — $\frac{2}{3}$: 0 $\quad P(X_1=0, X_2=0) = \frac{4}{9}$ $\quad 0$

$\qquad 1$

$$T_2 \sim \begin{cases} 0 & wp\ \frac{4}{9} \\ 1 & wp\ \frac{4}{9} \\ 2 & wp\ \frac{1}{9} \end{cases}$$

• Let $T_3 = X_1 + X_2 + X_3$

Supp $[X_1]$	Supp $[X_2]$	Supp $[X_3]$	JMF	T
$\frac{1}{3}$: 1	$\frac{1}{3}$: 1	1	$(\frac{1}{3})^3(\frac{2}{3})^0$	3
		0	$(\frac{1}{3})^2(\frac{2}{3})^1$	2
	$\frac{2}{3}$: 0	1	$(\frac{1}{3})^2(\frac{2}{3})^1$	2
		0	$(\frac{1}{3})^1(\frac{2}{3})^2$	1
$\frac{2}{3}$: 0	$\frac{1}{3}$: 1	1	$(\frac{1}{3})^2(\frac{2}{3})^1$	2
		0	$(\frac{1}{3})^1(\frac{2}{3})^2$	1
	$\frac{2}{3}$: 0	1	$(\frac{1}{3})^1(\frac{2}{3})^2$	1
		0	$(\frac{1}{3})^0(\frac{2}{3})^3$	0

$$T_3 \sim \begin{cases} 0 & wp\ 1\,(\frac{1}{3})^0(\frac{2}{3})^3 \\ 1 & wp\ 3\,(\frac{1}{3})^1(\frac{2}{3})^2 \\ 2 & wp\ 3\,(\frac{1}{3})^2(\frac{2}{3})^1 \\ 3 & wp\ 1\,(\frac{1}{3})^3(\frac{2}{3})^0 \end{cases} \qquad \begin{matrix} 1 = \binom{3}{0} \\ 3 = \binom{3}{1} \\ 3 = \binom{3}{2} \\ 1 = \binom{3}{3} \end{matrix}$$

☆ It's the 3rd row of the Pascal Δ. ☆

The pattern → $T_3 \sim \binom{3}{x}(\frac{1}{3})^x(\frac{2}{3})^{3-x} = Binom(3, \frac{1}{3})$

- $X_1, X_2, \dots, X_n \overset{iid}{\sim} \text{Bern}(\frac{1}{3})$

$$T_n \sim \text{Binom}(n, \tfrac{1}{3}) \begin{cases} 0 \text{ up} & \binom{n}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^n \\ 1 \text{ up} & \binom{n}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{n-1} \\ 2 \text{ up} & \binom{n}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{n-2} \\ \vdots & \\ n-2 \text{ up} & \binom{n}{n-2}\left(\frac{1}{3}\right)^{n-2}\left(\frac{2}{3}\right)^2 \\ n-1 \text{ up} & \binom{n}{n-1}\left(\frac{1}{3}\right)^{n-1}\left(\frac{2}{3}\right)^1 \\ n \text{ up} & \binom{n}{n}\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^0 \end{cases}$$

- $X_1, X_2, \dots, X_n \overset{idd}{\sim} \text{Bern}(p)$

$$T_n \sim \text{Binom}(n, p) \begin{cases} 0 \text{ up} & \binom{n}{1}(p)\quad(1-p) \\ 1 \text{ up} & (\quad)(p)\quad(1-p) \\ 2 \text{ up} & (\quad)(p)\quad(1-p) \\ \vdots & \\ n-2 \text{ up} & (\quad)(p)\quad(1-p) \\ n-1 \text{ up} & (\quad)(p)\quad(1-p) \\ n \text{ up} & (\quad)(p)\quad(1-p) \end{cases}$$

$\frac{1}{3}$ became p when Bern(p) not $(\frac{1}{3})$.
$\frac{2}{3}$ became $1-p$

Both are the same thing mathematically

Think in this way

2 concepts of Binomial:

Binomial (n, p) is $\lim_{N \to \infty}$ Hyper(n, p, N) or $X_1, X_2, \dots X_n \overset{iid}{\sim}$ Bern(p)

$T = X_1 + X_2 + \dots + X_n$

→ sampling with or without replacement is the same as long as the bag is big just like what we did in lec 1 & 2 (random variable).

a really big big bag with a big proportion of things you taking out successes

→ taking proportion of red balls of 38% pull 1 out another one still 38% caz the bag is so big

or

doing a whole bunch of bern of iid and you ask at the end "how many successes did I get"

or

☆ sampling with or without is the same → a really big bag wit a small bag 38% take a ball out (Is it a red?) put it back in (repeat)

CDF of the binomial ~~PMF~~

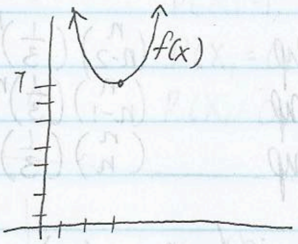
- $F(x) = P(X \le x)$

$$\sum_{i=0}^{x} \binom{n}{i} p^i (1-p)^{n-i} = \underbrace{I_{1-p}(n-k, 1+k)}_{\text{regularized binomial gamma function}} := (n-k)\binom{n}{k} \int_0^{1-p} t^{n-k-1} (1-t)^k \, dt$$

-End of Binom-

- $f(x) = 7 + (x-3)^3$

positive parabola, to the right 3



min of function = range ← $\min\{f(x)\} = 7$

how you get to the min of range = min of X ← $\arg\min\{f(x)\} = 3$

← $\max\{f(x)\} =$ undefined

$\arg\max\{f(x)\} =$ undefined.

- $X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$.

= infinite sequence of iid Bern(p)'s.

you get a 1 you stop

$T := \min\{t : X_t = 1\} =$ $\underset{1}{0}$ $\underset{2}{0}$ $\underset{3}{1}$ $\underset{4}{0}$ $\underset{5}{0}$ $\underset{6}{1}$

↓

first success also known as "stepping time"

$\frac{1}{1}$ − $p(1) = p(T=1) = p$

$\underset{1}{0}$ $\underset{2}{1}$ − $p(2) = p(T=2) = p(X_1=0, X_2=1) = P(X_1=0)P(X_2=1) = (1-p)p$

$\underset{1}{0}$ $\underset{2}{0}$ $\underset{3}{1}$ − $p(3) = p(T=3) = p(X_1=0, X_2=0, X_3=1) = P(X_1=0)P(X_2=0)P(X_3=1) = (1-p)^2 p$

− p ⋮ ⋮

− $p(t) = p(T=t) = (1-p)^{t-1} p$

$$\overbrace{\underbrace{0 \ 0 \ \ldots \ 0}_{t-1} \ \underbrace{1}_{1}}^{t}$$

when is the first success we get: $X \sim \text{Geometric}(p) := (1-p)^{x-1} p$

• $X \sim Geometric(p) := (1-p)^{x-1} p$

- Parm Space

$p \in (0, 1)$

- can't include 0,1 under lives bem...

unbounded

- There's no maximum, it could be anything...

- $Supp[X] = \mathbb{N}$

(all possible things that could happen) (all natural #)

If $p = 1 = 1$.

$p = 0$ = invalid (illegal).

$$\sum_{x \in Supp[X]} p(x) = 1 \qquad \sum_{x=1}^{\infty} (1-p)^{x-1} p \overset{?}{=} 1$$

→ now proving this is the case.

$q := 1-p$.

$$\sum_{x=1}^{\infty} (1-p)^{x-1} = \frac{1}{p}$$

$$\sum_{x=1}^{\infty} q^{x-1} = \frac{1}{1-q}$$

reindexing $x=1 = x=0$

$$\underbrace{\sum_{x=0}^{\infty} q^{x}}_{S} = \frac{1}{1-q}$$

$S = q^0 + q^1 + q^2 + q^3 + \dots$

$S = 1 + q + q^2 + q^3 + \dots$

$S = 1 + q(1 + q + q^2 + \dots)$.

$S = 1 + qS$

$\Rightarrow S - qS = 1$

$(1-q)S = 1$

$S = \frac{1}{1-q} = \sum_{x=0}^{\infty} q^x$

- CDF

difficult to do.

$$F(x) = P(X \leq x) = \sum_{i=1}^{x} (1-p)^{i-1} p$$

$= 1 - p(X > x)$ → looks like

$= 1 - (1-p)^x = F(x)$.

Mathematical Proof:

$\frac{0}{1}$ $\frac{0}{2}$ $\frac{0}{3}$ ..... $\frac{0}{x}$ | $\overline{x+1}$ $\overline{x+2}$ — — —

if $X > x$

1 must be large than x.

other proof.

from what we proved $=1$

$$\bullet\ P(X > x) = P(X = x+1) + P(X = x+2) + \dots = \sum_{i=x+1}^{\infty} (1-p)^{i-1} p = \sum_{i=1}^{\infty} (1-p)^{x+i-1} p = (1-p)^x \boxed{\sum_{i=1}^{\infty} (1-p)^{i-1} p}$$