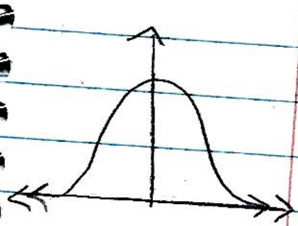
11/17/16

| | PMF | PDF | CDF | E[X] | Var[X] | |
|---|---|---|---|---|---|---|
| Discrete | $p(x) \in [0,1]$, $\sum_{x \in supp[x]} p(x) = 1$ | not exist | exist | $\sum_{x \in supp(x)} x\,p(x)$ | $\sum (x-\mu)^2 p(x)$ | $\lvert supp[x] \rvert \leq \lvert \mathbb{N} \rvert$ |
| Continous | does not exist | $\int_{supp[x]} f(x)dx = 1$, $f(x) \geq 0$ | exist | $\int_{supp[x]} x\, f(x)dx$ | $\int (x-\mu)^2 f(x)dx$ | $= \lvert \mathbb{R} \rvert$ |

| | Quantile[X, p] |
|---|---|
| Discrete | $\min x \{X : F(x) \geq p\}$ |
| Continous | $x$ s t $f(x) = p$, $x = F^{-1}(p)$ |



$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad x \in \mathbb{R}$$ is this a PDF?

a) $f(x) \geq 0$ $\quad \underbrace{\frac{1}{\sqrt{2\pi}}\ \frac{1}{e^{x^2/2}}}_{\text{Both positive}}$ ✓

b) $\int_{\mathbb{R}} f(x)dx = 1 \longrightarrow \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$

let $n = \frac{1}{\sqrt{2}} x \Rightarrow n^2 = \frac{x^2}{2}$

$dn = \frac{1}{\sqrt{2}} dx \Rightarrow dx = \sqrt{2}\, dn$

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = 2\sqrt{\pi}$$

$$\int_{\mathbb{R}} e^{-n^2} \sqrt{2} dn = 2\pi \rightarrow \int_{\mathbb{R}} e^{-n^2} dn = \sqrt{\pi} \Rightarrow$$

$$\Rightarrow \left(\int_{\mathbb{R}} e^{-n^2} dn\right)^2 = \pi \rightarrow \int e^{-n^2} dn \int e^{-n^2} dn = \pi$$

$$\Rightarrow \int e^{-x^2} dx \int e^{-y^2} dy - \pi \rightarrow \iint_A e^{-(x^2+y^2)} \underbrace{dxdy}_{dA} = \pi$$

$$\overbrace{f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}^{\text{Bell curve}}$$ is a PDF

$Z \sim N(0,1)$ "normal", "gassium r.v"

$$E[Z] = \int_{supp[x]} x\, f(x)dx = \int_{\mathbb{R}} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

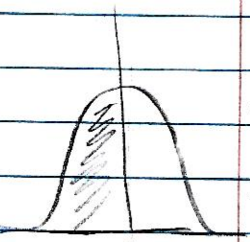
let $u = \frac{x^2}{2}$

$du = x dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=\infty} e^{-u} du = \frac{1}{\sqrt{2\pi}} - \left[e^{-u}\right]_{x=-\infty}^{x=\infty} = \frac{1}{\sqrt{2\pi}} \left[-e^{-x^2/2}\right]_{-\infty}^{\infty}$$

$$\frac{1}{\sqrt{2\pi}}\left(\overbrace{\lim_{x\to-\infty} e^{-x^2/2}}^{\frac{1}{e^{x^2/2}}=0} - \overbrace{\lim_{x\to\infty} e^{-x^2/2}}^{\frac{1}{e^{x^2/2}}=0}\right) = 0 \Rightarrow \boxed{\mu = 0}$$

$$\boxed{\begin{matrix} SE[Z]=1 \\ \sigma^2=\sigma=1 \end{matrix}}$$



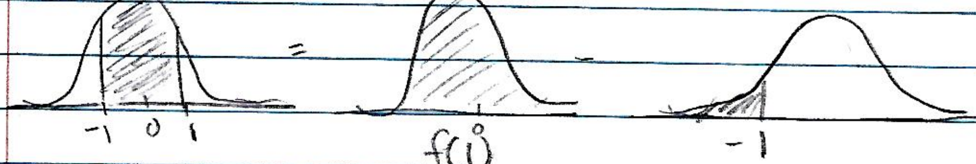
$f(0) = \frac{1}{2}$

$$Var[X] = E[Z^2] - \cancel{\mu^2}\, 0 = \int x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \; \ldots \ldots = 1$$

(work)

$$F(x) = \int f(x)\,dx + C \quad \int \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx + C$$ → not possible ("risch algorithm)

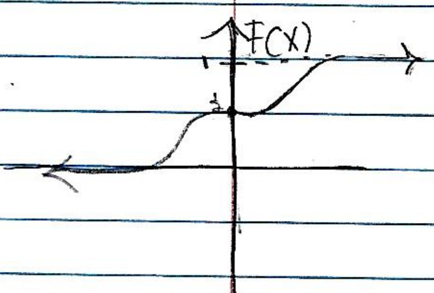
$P(Z \in [-1,1]) = f(1) - f(-1) \approx 0.68$




$P(Z \in [-2,2]) = F(2) - F(-2) \approx 0.95$

$P(Z \in [-3,3]) = F(3) - F(-3) \approx 0.997$

"Emperical rule", "3σ rule"




$\leftarrow f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$T \sim Geom(p)$

$\lim_{n\to\infty} p(t) = 0$ has no pmf

$F(X) = 1 - e^{-\lambda x}$ → CDF

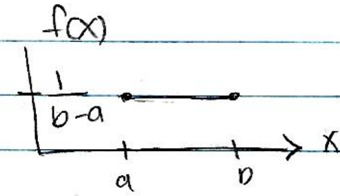
PDF $\boxed{X \sim Exp(\lambda) = f(x) = \lambda e^{-\lambda x}}$ exponential r.v

$\lambda = \overset{\infty}{n}\underset{0}{p}$ $\quad E[X] = \frac{1}{\lambda}$

$X \sim \text{Uniform}(a,b)$

$:= \frac{1}{b-a} \rightarrow f(x)$

$\hookrightarrow F(x) = \frac{x-a}{b-a}$




$X \sim U(0,1)$

$Y = aX$

$= \frac{\lambda}{a}$

$X \sim \text{Exp}(\lambda)$

$Y = 2X \sim ?$

let $\lambda' = \frac{\lambda}{2}$

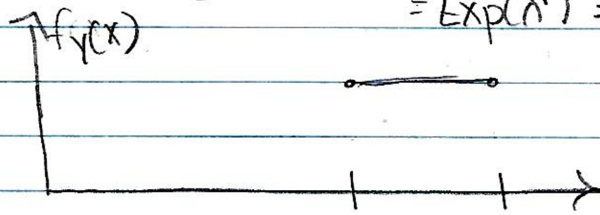
$F(y) = P(Y \le x) = P(2X \le x)$

$= P(X \le \frac{x}{2}) = F_X(\frac{x}{2})$

$\rightarrow F_X(\frac{x}{2}) = 1 - e^{-\lambda \frac{x}{2}} = 1 - e^{-\lambda' x}$

$= \text{Exp}(\lambda') = \text{Exp}(\frac{\lambda}{2}) = Y \sim \text{Exp}(\frac{\lambda}{2})$

$Y = d + cX$




$X \sim U(a,b)$

$Y = cX + d$

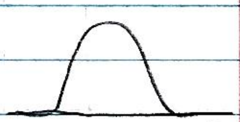
$F_Y(x) = P(Y \le x) = P(cX + d \le x) = P(X \le \frac{x-d}{c}) = F_X(\frac{x-d}{c})$

$F_X\left(\frac{x-d}{c}\right) = \frac{\frac{x-d}{c} - a}{b-a} \cdot \frac{c}{c} = \frac{x-(d+ca)}{cb-ca} = \frac{x-(d+ca)}{(d+cb)-(d+ca)}$

let $a' = d + ca$

$b' = d + cb$

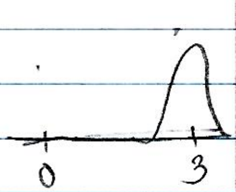
$\rightarrow \frac{x-a'}{b'-a'} = U(a', b') = U(d+ca, d+cb)$




$Z \sim \text{Norm}(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Let $X := \sigma Z + \mu$

$E[X] = \sigma E[\underbrace{Z}_{0}] + \mu = \mu$




$SE[X] = SE[\sigma Z + \mu] = |\sigma| \, SE[\overset{1}{Z}] = |\sigma|$

$F_X(x) = P(X \le x) = P(\sigma Z + \mu \le x) = P(Z \le \frac{x-\mu}{\sigma})$

$\underbrace{F_Z\left(\frac{x-\mu}{\sigma}\right)}_{\text{unknowable}}$ $\quad f(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} F_Z\left(\frac{x-\mu}{\sigma}\right)$

let $u = \frac{x-\mu}{\sigma} \rightarrow \frac{du}{dx} = \frac{1}{\sigma}$

$$\frac{du}{dx}\frac{d}{du}\left[\frac{F(u)}{2}\right] = \frac{1}{\sigma}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}$$

$$\boxed{= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-2\sigma^2(x-\mu)^2} = X \sim N(\mu, \sigma)}$$ PDF

Param space  Supp $[X] = \mathbb{R}$

$\mu \in \mathbb{R}$

$\sigma^2 \in (0, \infty)$