October 20th, 2016

$X \sim \text{Geom}(p) := (1-p)^{x-1} p$

$\text{Supp}[X] = \mathbb{N}$ Parameter Space: $p \in (0,1)$

→ The stopping time of $X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$

→ Questions on Midterm 2.

You play poker until you get a royal flush.

$P(\text{Royal Flush}) = \dfrac{4}{\binom{52}{5}} = 1.53/\text{Million} = .00000153$

a) Build a random variable model for the hand and number in which you get a royal flush. (Playing until you get a royal flush).

$X \sim \text{Geom}(.00000153)$

b) What's the probability you get a royal flush on the millionth hand?

$P(X = .00000153) = (1 - .00000153)^{1000000-1} \cdot .00000153$

$= .9999985^{999999} \cdot .00000153$

c) What's the probability you get a royal flush on the millionth hand or sooner?

$P(X \leq 1000000) = \underset{\text{CDF}}{F(1000000)} = 1 - (1 - .00000153)^{1000000}$

$= 1 - (.9999985)^{1000000}$

$= .777 \approx 77\%$

→ Geometric looks like: $X = \min\{t : X_t = 1\}$

$T = \min\{t : \sum_{i=1}^{t} X_i = r\}$

You are doing this experiment with Bernouillis until you get r successes.

So if $r = 3$,

$\underset{1}{0}\ \underset{2}{0}\ \underset{3}{0}\ \underset{4}{1}\ \underset{5}{0}\ \underset{6}{0}\ \underset{7}{1}\ \underset{8}{0}\ \underset{9}{1}$ } You got $r=3$ in $X=9$

→ For $r = 3$ ...

$P(X=0) = 0$
$P(X=1) = 0$
$P(X=2) = 0$

} Because you cannot get 3 successes in 0, 1, or 2 experiments.

$P(X=3) = p^3$ } $\underset{1}{1}\ \underset{2}{1}\ \underset{3}{1}$ (The prob of success)^3

$P(X=4) = \binom{3}{2}(1-p)^1 p^3$
$\binom{3}{2} = 3$

0	1	1	1
1	0	1	1
1	1	0	1
1	2	3	4

You have to get a 1 in the end. What's left is 3 spaces here and 4 spaces here to decide where the last two 1's go. Hence why there is a $\binom{3}{2}$ and a $\binom{4}{2}$.

$P(X=5) = \binom{4}{2}(1-p)^2 p^3$
$\binom{4}{2} = 6$

0	0	1	1	1
0	1	0	1	1
1	0	0	1	1
1	0	1	0	1
1	1	0	0	1
0	1	1	0	1
1	2	3	4	5

$P(X=x) = \binom{x-1}{2}(1-p)^{x-3}\, p^3$ } $\underbrace{0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,1\,0}_{x-1}\ \underset{x^{th}}{1}$

↑
PMF

→ $X \sim NegBin(r, p) := \binom{x-1}{r-1}(1-p)^{x-r}\, p^r$
Negative binomial

$Supp[X] = \{r, r+1, r+2, \ldots\}$ → Has to be greater or equal to r b/c you can't get r successes if you try less than r times. So you can't try to get $r=3$ successes in 1 try.

Parameter Space

$r \in \mathbb{N}$
$p \in (0,1)$ → B/c it came from Bernouilli.

waiting for first success.

$$\rightarrow X \sim \text{NegBin}(1, p) = \binom{x-1}{1-1}(1-p)^{x-1}p^1$$
$$= 1(1-p)^{x-1}p$$
$$= (1-p)^{x-1}p = \text{Geom}(p)$$

Imagine the following scenarios:

$$X_1, X_2, \ldots, X_r \overset{iid}{\sim} \text{Geom}(p)$$
$$X_1 + X_2 + \ldots + X_r \sim \text{NegBin}(r, p)$$

These two are really the same.

$$\sum_{x \in Supp[X]} p(x) = 1$$

This must be true for all PMFs. If it isn't true, then it isn't a PMF.

$$\sum_{x=r}^{\infty} \binom{x-1}{r-1}(1-p)^{x-r}p^r \overset{?}{=} 1$$

Proof:

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a} \text{ s.t. } a \in (0,1)$$ Geometric Series

$$\sum_{i=0}^{\infty} (1-p)^i = \frac{1}{p}$$

$$\sum_{x=1}^{\infty} (1-p)^{x-1} = p^{-1}$$ Reindexing

$$\sum_{x=2}^{\infty} (-1)(x-1)(1-p)^{x-2} = (-1)p^{-2}$$ First derivative, $\frac{d}{dp}$

$$\sum_{x=3}^{\infty} \underset{d}{(-1)^2}\,\underset{d-1}{(x-1)}\,\underset{d}{(x-2)}(1-p)^{\underset{d+1}{x-3}} = \underbrace{(-1)(-2)p^{-3}}_{(-1)^d d!\, p^{-d}}$$ Second Deriv.

$$\sum_{x=r}^{\infty} (x-1)(x-2)\ldots(x-(r-1))(1-p)^{x-r} = (r-1)!\,p^{-r}$$

$$\sum_{x=r}^{\infty} \frac{(x-1)!\,(1-p)^{x-r}}{(x-r)!\,(r-1)!} = p^{-r}\,(p^r)$$

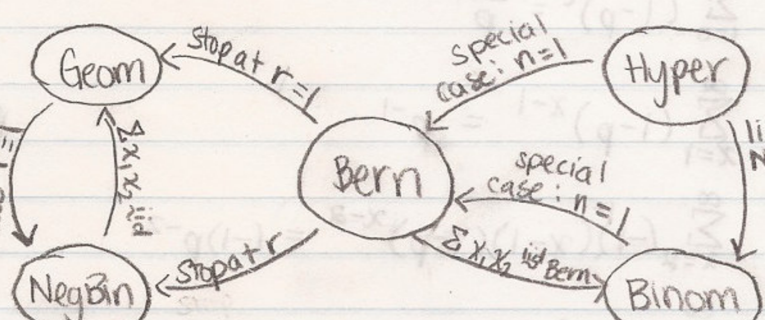
$$\sum_{x=r}^{\infty} \binom{x-1}{r-1}(1-p)^{x-r}p^r = 1$$

$X \sim \text{NegBin}(r, p) = \binom{x+r-1}{r-1}(1-p)^x \underbrace{p^r}$ } Equivalent Parameterization

↑ the # of failures

$$\binom{x+r-1}{r-1}$$
$$= \frac{(x+r-1)!}{(r-1)!\,(x)!}$$
$$= \frac{(x+r-1)(x+r-2)\ldots r}{x!}$$
$$= \frac{(-1)^x(-r)(-r-1)(-r-2)\ldots(-r-x+1)}{x!}$$
$$= (-1)^x\binom{-r}{x}$$
$$= (-1)^x\binom{-r}{x}(1-p)^x p^r$$

This is why it's called a negative binomial.



$X \sim \text{Bern}(\frac{1}{2}) = \begin{cases} 1 \text{ wp } \frac{1}{2} \\ 0 \text{ wp } \frac{1}{2} \end{cases}$

<u>Big X</u>	vs. <u>Small x</u>
• Big X refers to the <u>model</u>.	• Small x is something that comes out of the model.
• Coin is big X	• Flipped coin that landed on heads is small x.
• Random variable.	

• Allowed to have value of 1. It could have been 0 but it wasn't. It's become set in stone. ← Realization of r.v.

• $x \in \text{Supp}[X]$

Coin Pick Experiments.

- 8 coins , 4 spotted

a) One person randomizes the coins and picks 3. 2 are spotted. What just happened?

big $\rightarrow X \sim \text{Hyper}(\overset{n}{3}, \overset{K}{4}, \overset{N}{8})$

small $\rightarrow x = 2$

b) 7 people shake the cups and each pick 3. Each person gives their individual results of spotted coins. What happened?

big $\rightarrow X_1, X_2, \dots, X_7 \overset{iid}{\sim} \text{Hyper}(3,4,8)$

small $\rightarrow x_1 = 1$ $\quad x_4 = 3$ $\quad x_7 = 2$

$x_2 = 1$ $\quad x_5 = 1$

$x_3 = 3$ $\quad x_6 = 2$

$\text{Supp}[X] = \{0,1,2,3\}$

c) One person pours out all 8 coins. He tells us how many <u>heads</u> he got. What just happened?

8 Bernouillis , summed up so it is a...

$\Rightarrow X \sim \text{Binom}(8, \frac{1}{2})$

d) 7 people shake the cups and pour out all coins. They each tell us how many heads they got. What happened?

$X_1, X_2, \dots, X_7 \overset{iid}{\sim} \text{Binom}(8, \frac{1}{2})$

$\text{Supp}[X] = \{\underset{\text{no heads}}{0}, 1, 2, \dots, \underset{\text{all heads}}{8}\}$

$x_1 = 3$ $\quad x_4 = 4$ $\quad x_7 = 5$

$x_2 = 3$ $\quad x_5 = 5$

$x_3 = 4$ $\quad x_6 = 6$