Lecture 6 - September 13, 2016

- P is a set function
  $\exists\, \Omega$

(a) $P(\Omega) = 1$

(b) $P(A) \geq 0 \ \forall A$

(c) If $A_1, A_2, A_3 \ldots$ disjoint
$\rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

- <u>Theorem</u> $P(A) = 1 - P(A^c)$

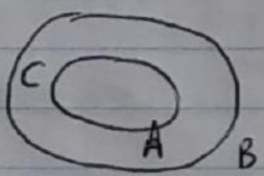
$\Omega = A \cup A^c$

$P(\Omega) = P(A \cup A^c)$

$P(\Omega) = P(A) + P(A^c)$ by condition (c) above

$1 = P(A) + P(A^c)$ by condition (a) above.

- <u>Theorem</u>: $A \subseteq B \Rightarrow P(A) \leq P(B)$



$C := B \setminus A$

$B = A \cup C \quad A \cap C = \phi$

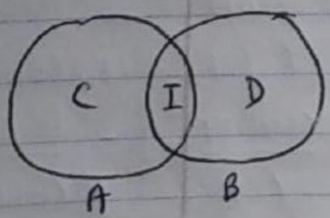
$P(B) = P(A \cup C)$

$P(B) = P(A) + P(C)$ by condition (c) above

$P(B) - P(A) = P(C) \geq 0$ by condition (b) above.

$P(B) \geq P(A)$

- $P(A \cup B) = P(A) + P(B) - P(AB)$



$C = A \setminus B$

$D = B \setminus A$

$I = BA$

* $P(A) = P(C) + P(I) \Rightarrow P(C) = P(A) - P(I)$
* $P(B) = P(D) + P(I) \Rightarrow P(D) = P(B) - P(I)$
* $P(A \cup B) = P(C) + P(D) + P(I)$

$\Downarrow$

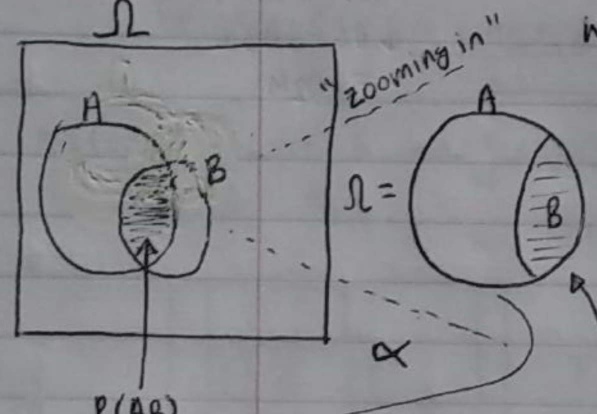
$P(A \cup B) = \left(P(A) - P(I)\right) + \left(P(B) - \cancel{P(I)}\right) + \cancel{P(I)}$

$\Downarrow$

$\underline{P(A \cup B) = P(A) + P(B) - P(AB)}$

(Example) $n = 1000$ people
200 smokers (A)
60 lung cancer (B)
36 smoke & lung cancer (C)

$P(A) = 0.2$
$P(B) = 0.06$
$P(AB) = 0.036$



what is probability of lung cancer among smokers
given smoking
conditional smoking?

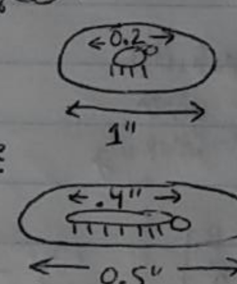
$P(B|A) \cdot \text{zoom} = P(BA) \cdot \frac{P(\Omega)}{P(A)} \Rightarrow \boxed{P(B|A) = \frac{P(BA)}{P(A)}}$

"given"
Probability of B given A.

Definition of conditional probability

Side note
$\Delta \propto$ (triangle)
$x \propto y$
$x = cy$ s.t. $c \in \mathbb{R}$
∝ means proportional

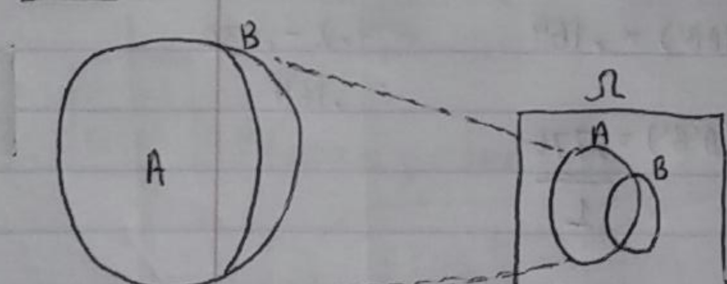
Side note
Zoom = $\frac{\text{begin size}}{\text{end size}}$



$= \frac{1''}{0.5''} = 2$

* $P(B|A) = \frac{P(BA)}{P(A)} = \frac{.036}{.2} = \boxed{.18}$ ↝ Says that about a $\frac{1}{5}$ chance of lung cancer.
low

* $P(A|B) =$



$P(A|B) = \frac{P(AB)}{P(B)}$
$= \frac{0.36}{.06} = \boxed{.6}$ ← high

Definition of conditional Probability

$\boxed{P(B|A) = \frac{P(BA)}{P(A)}}$ $\quad P(BA) = \frac{P(\Omega)}{P(A)} \Rightarrow P(A|B) = \frac{P(AB)}{P(B)}$

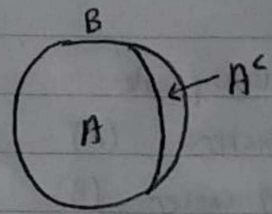
$P(AB) = P(A|B)(PB)$

Baye's Rule (1763)
$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}$

* P (lung cancer given didn't smoke)

$= P(B|A^c) = \frac{P(BA^c)}{\underbrace{P(A^c)}_{1-P(A)}} = \frac{.024}{0.8} = \boxed{.03}$

high b/c among people no smoke. Getting lung cancer 3% of time, never smoke... which means other things giving lung cancer



$P(B) = P(BA) + P(BA^c)$

$P(BA^c) = P(B) - P(BA)$

$.06 - .036$

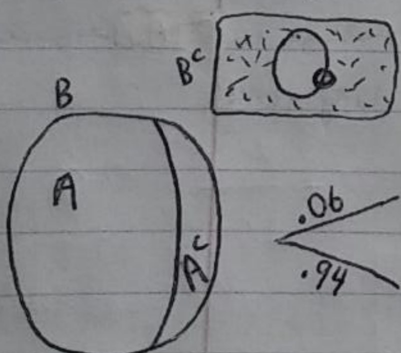
$= .024$

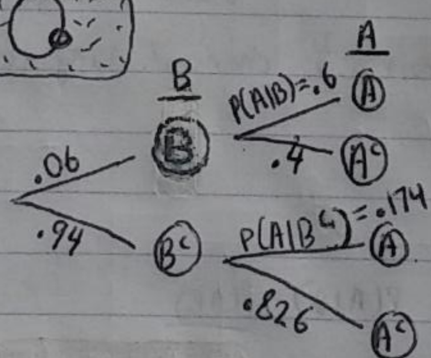
Probability l.c. given smoking

* $\frac{P(B|A)}{P(B|A^c)} = \frac{.18}{.03} = \boxed{6}$ } Risk ratio:
How much more likely to get lung cancer given smoking than non-smoking? → 6 times more likely

probability l.c. given non-smoking

* $P\left(\frac{A \ B}{A^c \ B^c} \middle| \underline{\quad} \right)$ → 8 probabilities that can be asked in this example.





$P(AB) = .036$

$P(A^cB) = .024$

$P(AB^c) = .164$

$P(A^cB^c) = .776$

$\overline{\quad 1 \quad}$

$P(A) = P(AB) + P(AB^c)$

$P(AB^c) = P(A) - P(AB)$

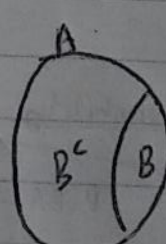
$= .2 - .036$

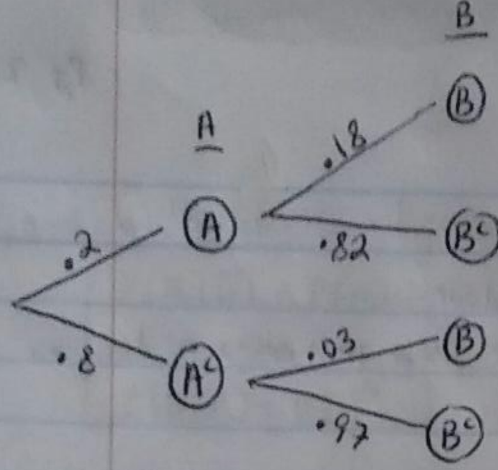
$= .164$

* $P(A|B) = 1 - P(A^c|B)$

$\frac{P(AB)}{P(B)} = 1 - \frac{P(A^cB)}{P(B)}$

$P(AB) = P(B) - P(A^cB) = P(B) = P(AB) + P(A^cB)$





$P(AB) = .036$

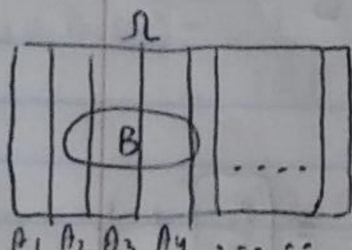
$P(AB^c) = .164$

$P(A^cB) = .024$

$P(A^cB^c) = .776$

Concludes lung cancer & smoking example

- Consider event B and disjoint and collectively exhaustive events. $A_1, A_2, A_3 \ldots$



$$
\begin{aligned}
P(B) &= P(B \cap \Omega) \quad T \\
&= P(B \cap (A_1 \cup A_2 \cup A_3 \cup \ldots)) \\
T \rightarrow &= P((B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3) \cup \ldots) \\
&= P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) + \ldots
\end{aligned}
$$

(T b/c the A's are collectively exhaustive)

$\forall i \neq j \ (B \cap A_i) \cap (B \cap A_j) = \phi$

$B \cap B \cap A_i \cap A_j$

$B \cap (A_i \cap A_j)$

$B \cap \phi$

$= \phi$

$$\boxed{P(B) = \sum_{i=1}^{\infty} P(BA_i)} \leftarrow \text{Law of total probability}$$

$$\boxed{P(A_\ell \mid B) = \frac{P(B \mid A_\ell)P(A_\ell)}{\sum_{i=1}^{\infty} P(B \mid A_i)P(A_i)}} \quad \text{Bayes Theorem}$$

$$\boxed{P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}} \quad \text{Bayes Rule}$$

- Ω

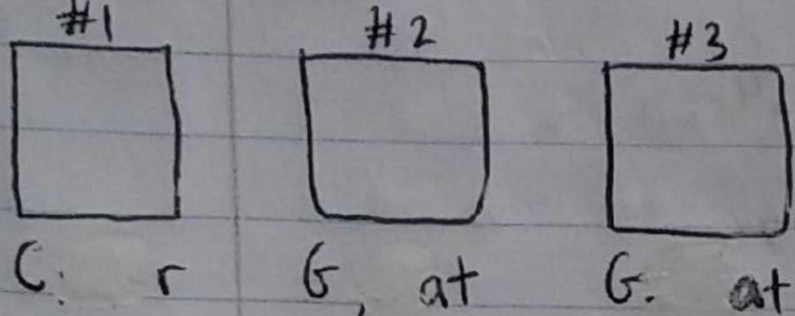
GG	GB
BG	BB

$P(\text{other is girl} \mid \text{one is girl}) = P(\{GG\} \mid \{GG, GB, BG\})$

$$\frac{P(\cap)}{P(\ )}$$

$$= \frac{P(GG)}{P(GG, GB, BG)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \boxed{\frac{1}{3}}$$

- **Monte Hall Game**

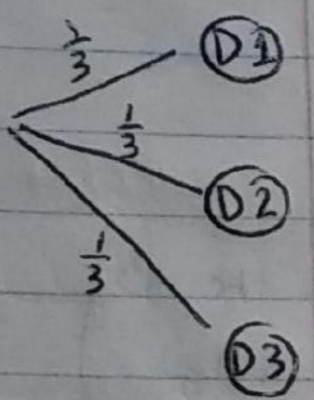


① Pick door

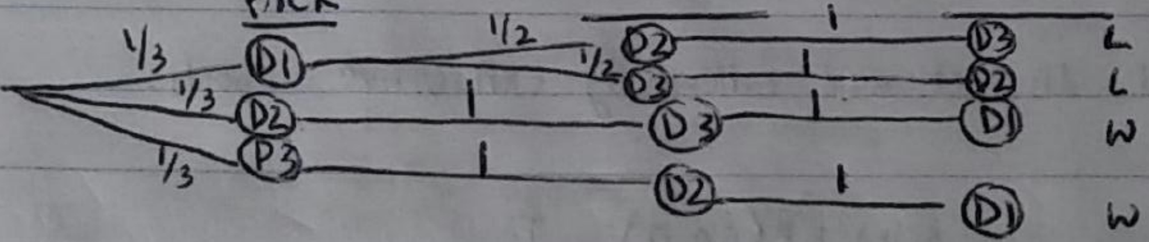
② Game show that opens a door to reveal ______

③ You ____________

Car In



Pick



$$P(w \mid \text{Car is } D1) = \frac{\frac{1}{3}\cdot\frac{1}{3}\cdot 1\cdot 1 + \frac{1}{3}\cdot\frac{1}{3}\cdot 1\cdot 1}{\frac{1}{3}} = \boxed{\frac{2}{3}} \neq \frac{1}{2}$$