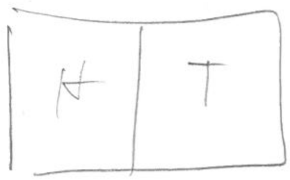
Lecture 8 9/27/16 Math 241

$n=3$ flips

$\Omega = \{H, T\}$



$\omega_1 = H$
$\omega_2 = T$
$\omega_3 = H$

What is the average of the 3 flips? How "randomly spread" are the 3 flips?

You can't perform computations easily on arbitrary sets

What if I created a function e.g.

$$\mathbb{1}_{\omega = H} = \begin{cases} 1 & \text{if } \omega = H \\ 0 & o/t \end{cases}$$

$n=3$ $\quad 1, 1, 0 \quad \Rightarrow \quad$ Avg $\bar{x} = \frac{1+1+0}{3} = \frac{2}{3}$

What did we do?

difficult to do computations

easy to do computations (functions with a quant like \$, in, etc)

Generally, $X : \Omega \to \mathbb{R}$

X: experimental outcome $\to$ values you care ~~about~~ in your random model

X, is a function called a random variable ("r.v."), $X(\omega)$

$X(H) = 1$
$X(T) = 0$

What is $P(X = 1)$?

technically illegal since $P : 2^{\Omega} \to [0,1]$

since $X(H)=1$

$$P(\{\omega : X(\omega)=1\}) = P(\{H\}) = \frac{|\{H\}|}{|\Omega|} = \frac{1}{2} \Rightarrow \begin{array}{l} P(X=1)=\frac{1}{2} \\ P(X=0)=\frac{1}{2} \end{array}$$

we will use the abuse of notation

<u>Support</u>: r.v.'s have a $\underbrace{\text{"range"}}_{\subseteq \mathbb{R}}$ of $\underbrace{\text{possible values}}_{x:\ P(X=x)>0}$ — the # of values is limited

big X (function)

$$\text{Supp}(X) := \{x : P(X=x) > 0\} \subseteq \mathbb{R}$$

little x: an arbitrary value in the range of the function

why not ≥ 0? Impossible realizations are not interesting

$P(X=17)=0$

Def: discrete r.v. is one s.t $|\text{Supp}(X)| \leq |\mathbb{N}|$

Problem 3 — C+ble ∞

Assume $|\Omega|$ c+ble. $\Rightarrow \Omega = \{\omega_1, \omega_2, \ldots\}$ s.t: $P(\{\omega_i\})>0$

$$\sum_{x \in \text{Supp}(X)} P(X=x) = 1 \quad \text{Intuitively... Why?}$$

$$\bigcup_{x \in \text{Supp}(X)} \{\omega : X(\omega)=x\} \stackrel{?}{=} \Omega$$

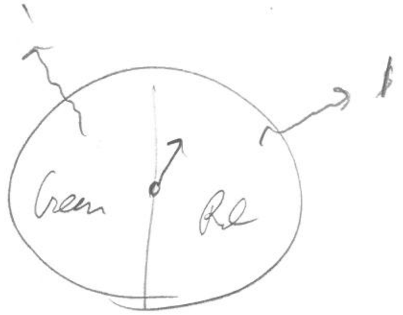
If not, $\exists\, \omega$ s.t $X(\omega) \notin \text{Supp}(X) \Rightarrow P(\{\omega\})=0 \Rightarrow$ Contradiction.

$$\{\omega : X(\omega)=x_1\} \cap \{\omega : X(\omega)=x_2\} \stackrel{?}{=} \phi$$

Yes b/c $X(\omega_i) = x_1$ and x_2 which violates the def. of function — must excl.

$$\Rightarrow P(\Omega) = P(\{\omega : X(\omega)=x_1\}) + P(\{\omega : X(\omega)=x_2\}) + \ldots$$

$$1 = \sum_{x \in \text{Supp}(X)} P(X=x) \quad \checkmark$$



$$X = \begin{cases} 1 & \text{if } \omega = \text{Red} \\ 0 & \text{if } \omega = \text{Green} \end{cases}$$

$P(X=1) = \frac{1}{2}$

$P(X=0) = \frac{1}{2}$

is X the "same" as previously. Technically, NO since

$X : \{H, T\} \to \{1, 0\}$, $X : \{R, G\} \to \{1, 0\}$

But when looking at the values and their prob's only, it doesn't matter

"with probability"

$$\Rightarrow \quad X \sim \begin{cases} 1 & \text{wp } \frac{1}{2} \\ 0 & \text{wp } \frac{1}{2} \end{cases}$$

"distributed as"

There are many Ω's that can produce this r.v.

$\Rightarrow$ We don't care about Ω anymore. We know it's there, we know there's some underlying experiment and sample space, but we don't need to know what it is

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This r.v. is very special. It is the first of the "brand name" r.v.'s we'll discuss

definition

$$X \sim \text{Bernoulli}\left(\frac{1}{2}\right) := \begin{cases} 1 & \text{wp } \frac{1}{2} \\ 0 & \text{wp } \frac{1}{2} \end{cases}$$

$\text{Supp}[X] = \{0, 1\}$

More generally,

$$X \sim \text{Bernoulli}(p) := \begin{cases} 1 & \text{wp } p \\ 0 & \text{wp } 1-p \end{cases} \leftarrow \text{Why?}$$

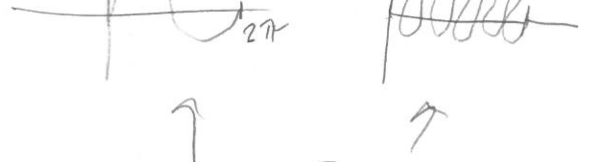
X is distributed Bernoulli w/ "parameter" p.

A parameter is a choice which defines a model.

For example

$f(x) = \sin(x)$ is a special case of $f(x) = \sin(ax)$ s.t. a is a constant

a defines the frequency/period of the wave



Same? No... but same family

Valid values of a here? $a \in \mathbb{R}$?

What if $a = 0$?  Is this a sine curve?

$a \in \mathbb{R} \setminus \{0\}$

Here, we would argue "no" but some say "yes".
This is the trivial case which is uninteresting.

$X \sim \text{Bernoulli}(p)$ Def: Parameter space: Knob you can turn on an oven e.g.

What are the possible values of p? p is a prob.

$p \stackrel{?}{\in} [0,1]$ What if $p = 1$? $X = \{1 \text{ wp } 1$ $\quad 1,1,1,1,1,1$

What if $p = 0$? $X = \{0 \text{ wp. } 0$ $\quad 0,0,0,0,0,0,0$

if $X \sim \text{Deg}(c) := \{ c \text{ w.p. } 1$ $\quad \text{Supp}[X] = \{c\}$

the "degenerate" r.v.

X is a r.v. by definition but it is trivial and uninteresting. Why? it just spits out the same value each time.

Just like $f(x) = \sin(0x) = 0$ is uninteresting and $a = 0$ was not included in set of values defining the sine curve family, so too $p = 0$, $p = 1$ are not included in the parameter space

$$\Rightarrow p \in (0, 1)$$

More notation:

defined anywhere

$p(x) := P(X = x)$ $\quad p: \mathbb{R} \to [0, 1]$

1-dim discrete r.v.

prob. mass function (PMF)

if $x \in \text{Supp}(X)$

$p(x) > 0$

if $x \notin \text{Supp}(X)$

$p(x) = 0$

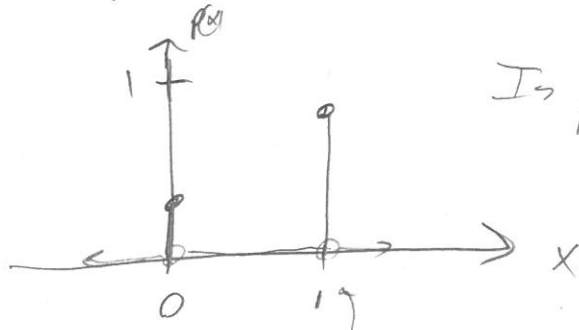
$$\sum_{x \in \text{Supp}(X)} p(x) = 1$$ (Same proof as before)

"Something has to happen"

"prob of anything happening is 1"

It is useful to plot PMF's.

$X \sim \text{Bern}\left(\frac{3}{4}\right)$



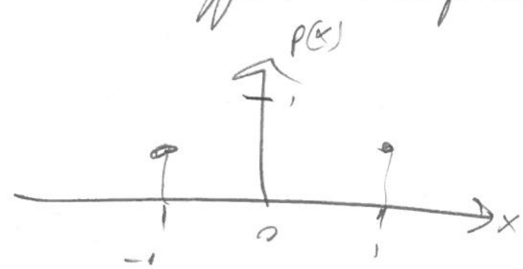
Is $p(x)$ a cont. function?

# of discontinuities?

$\text{Supp}[X]$

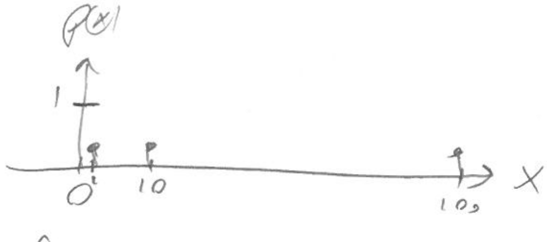
Usually no since it's implicitly understood that values not in the support have prob. 0.

$X \sim \text{Rademacher} := \begin{cases} -1 & \text{w.p. } \frac{1}{2} \\ 1 & \text{w.p. } \frac{1}{2} \end{cases}$



Random Walk in one dim.

$X \sim \text{Unif}(\{1, 10, 100\}) = \begin{cases} 1 & \text{w.p. } \frac{1}{3} \\ 10 & \text{w.p. } \frac{1}{3} \\ 100 & \text{w.p. } \frac{1}{3} \end{cases}$



Discrete Uniform, Generally $X \sim \text{Unif}(A)$ $\text{Supp}(X) = A$

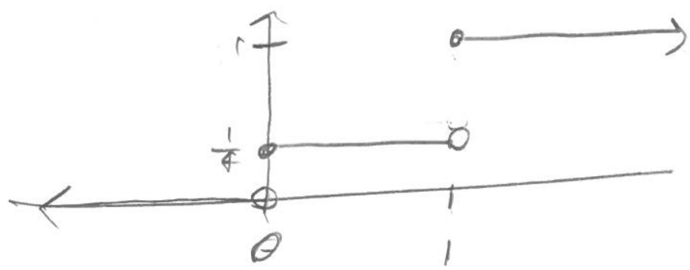
Parameter space: $A \subset \mathbb{R}$ s.t. $|A| \in \mathbb{N} \setminus \{1\}$

if $|A| = 1 \Rightarrow$ Deg.!

More notation

$F(x) := P(X \leq x)$ Cumulative Distribution Function (CDF)

$X \sim \text{Bernoulli}\left(\frac{3}{4}\right)$



$F(-37) = 0$

$F(1001) = 1$

Properties (derivable from def.)

① $\lim_{x \to \infty} F(x) = 1$

② $\lim_{x \to -\infty} F(x) = 0$

③ $x \le y \Rightarrow F(x) \le F(y)$ "monotonicity"

monotonically increasing

$\Rightarrow$ ④ $F(x) \in [0,1]$

$x < y \not\Rightarrow$ only $F(x) < F(y)$

"not strictly" monotonically increasing

$F(x)$ continuous if X is a discrete r.v.?

~~$\text{Supp}(X) = \{x_1, x_2, \ldots\}$ ordered smallest to largest~~

~~$F(x_2) = p(x_1) + p(x_2)$~~

$\forall \varepsilon > 0 \; F(x_2 - \varepsilon) = p(x_1) \Rightarrow$ discontinuous

# of discontinuities? $\text{Supp}(X)$

$X \sim \text{Bernoulli}(p) := \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$

piecewise function
(ugly, difficult)

$p(x) = p^x (1-p)^{1-x}$ nice!

$X_1 \sim \text{Bernoulli}(p)$ $\quad p_1(x) = p^x(1-p)^{1-x}$ (equal)

$X_2 \sim \text{Bernoulli}(p)$ $\quad p_2(x) = p^x(1-p)^{1-x}$ $=$

Def: $X_1 \overset{d}{=} X_2$ if $p_1(x) = p_2(x)$ or $F_1(x) = F_2(x)$

X_1 and X_2 are "equal in distribution"

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10 cards 4 R, 6 B $\qquad$ Draw cards <u>without</u> replacement

$$P(2R \text{ out of } 3 \text{ cards}) = \frac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}}$$

$$P(x\, R \text{ out of } 3 \text{ cards}) = \frac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$$

$$P(x\, R \text{ out of } n \text{ cards}) = \frac{\binom{4}{x}\binom{6}{n-x}}{\binom{10}{n}}$$

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10 cards K R, 10-K B

$$P(x\, R \text{ out of } n \text{ cards}) = \frac{\binom{K}{x}\binom{10-K}{n-x}}{\binom{10}{n}}$$

N cards K R, N-K Blue

$$P(\text{''}) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$