Lec 18 Math 241 11/17/16

Lec 17 Review

$T \sim \text{Geom}(p)$ by the second $E[T] = \frac{1}{p}$

put n experiments into each second but let $\lambda = np$

let $n \to \infty$

$p(t) = 0 \quad \forall t$ No PMF!

but...

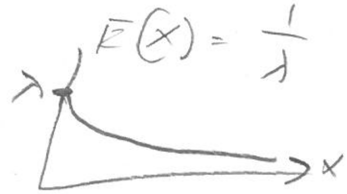
$F(t) = 1 - e^{-\lambda t}$

T is a cont. r.v.

$f(t) = \frac{d}{dt} F(t)$ expresses quickness of CDF piling up prob or "prob. density"

prob. dens. fxn (PDF)

$X \sim Exp(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{f(x) \text{ not } p(x)}$ $E(X) = \frac{1}{\lambda}$ memoryless as well



	PMF	PDF	CDF exists	$E(X)$	$Var(X)$	$\lvert Supp(X) \rvert$	Quantile[p]
Discrete	✓ $\sum p(x) = 1$, $p(x) \in [0,1]\ \forall x$	$= 0$	✓	$\sum x p(x)$	$\sum (x-\mu)^2 p(x)$	$\leq \lvert \mathbb{N} \rvert$	min $\{x : F(x) \geq p\}$
Cont	$= 0$	✓ $\int f(x)dx = 1$, $f(x) \geq 0\ \forall x$	✓	$\int x f(x) dx$	$\int (x-\mu)^2 f(x) dx$	$= \lvert \mathbb{R} \rvert > \lvert \mathbb{N} \rvert$	x s.t. $F(x) = p$

$f(x) = 100$? Yes!

does not mean it exists in closed form

since continuous... there will be an exact p^*. No need for $\geq$.

$X \sim Unif(a, b)$

$E(X) = \frac{a+b}{2}$ $SE(X) = \frac{b-a}{\sqrt{12}}$ Hard proof!

$X \sim \text{Unif}(0,1)$ std. uniform

$$Z \sim N(0,1) := \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{f(x) \text{ PDF}}$$

normal r.v.,
gaussian r.v.
"bell curve"

$\text{Supp}(Z) := \{x : f(x) > 0\}$

Note... always > 0

$= \mathbb{R}$

"Anything could happen"

Is this a r.v.?

(a) $f(x) \geq 0$ ✓

(b) $\int_{\text{Supp}(X)} f(x)\,dx = 1 \Rightarrow \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = 1$ proof...

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$$

let $u = \frac{1}{\sqrt{2}} x \Rightarrow \frac{x^2}{2} = u^2$

$du = \frac{1}{\sqrt{2}}\,dx \Rightarrow dx = \sqrt{2}\,du$

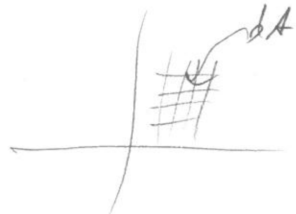
$$\Rightarrow \int_{\mathbb{R}} e^{-u^2} \sqrt{2}\,du = \sqrt{2\pi} \Rightarrow \overbrace{\int_{\mathbb{R}} e^{-u^2}\,du}^{\text{Gaussian Integral}} = \sqrt{\pi}$$

still $(-\infty, \infty)$

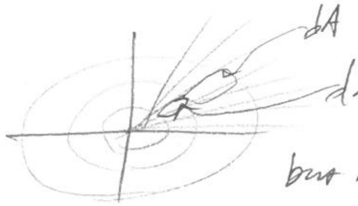
$$\left(\int_{\mathbb{R}} e^{-u^2}\,du\right)^2 = \pi \Rightarrow \int_{\mathbb{R}} e^{-u^2}\,du \int_{\mathbb{R}} e^{-u^2}\,du = \pi \Rightarrow \int_{\mathbb{R}} e^{-x^2}\,dx \int_{\mathbb{R}} e^{-y^2}\,dy = \pi$$

$$\Rightarrow \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x^2+y^2)}\,dx\,dy = \pi$$

height of as rect. prisms (parallelepipeds)



$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \pi$$

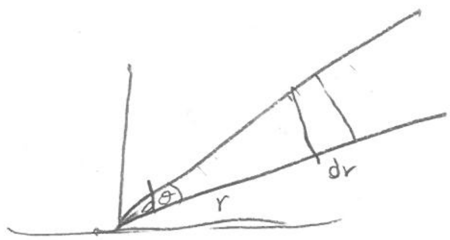


$dA(r)$

polar coord. switch

but not same size !

$x^2+y^2=r^2$



$$dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta = r\,dr\,d\theta$$

$$\left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right) dr\,d\theta$$

$$\Rightarrow \underset{(r)}{\int_0^\infty}\underset{(\theta)}{\int_0^{2\pi}} e^{-r^2}\, r\,dr\,d\theta = \pi = \int_0^\infty e^{-r^2} r\,dr \int_0^{2\pi} d\theta = \pi = 2\pi \int_0^\infty e^{-r^2} r\,dr = \cancel{\pi}\,\frac{1}{2}$$

(A)

$u = r^2 \Rightarrow du = 2r\,dr \Rightarrow \frac{du}{2} = r\,dr$

$$\int_0^\infty e^{-u}\frac{du}{\cancel{2}} = \cancel{\tfrac{1}{2}}\ 1 \Rightarrow \int_0^\infty e^{-u}\,du = 1 \ :\ -\left[e^{-u}\right]_0^\infty = 1 \quad e^{-0} - \lim_{x\to\infty} e^{-u} = 1$$

$u = \frac{x^2}{2} \quad du = x\,dx$

$$E[Z] = \int_{\mathrm{Supp}(X)} x f(x)\,dx = \int_{\mathbb{R}} x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{x=-\infty}^{x=\infty} e^{-u}\,du = \frac{1}{\sqrt{2\pi}}\left[-e^{-u}\right]_{x=-\infty}^{x=\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left[-e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty} = \frac{1}{\sqrt{2\pi}}\left(\lim_{x\to-\infty} e^{-\frac{x^2}{2}} - \lim_{x\to\infty} e^{-\frac{x^2}{2}}\right) = 0 \Rightarrow \boxed{\mu = 0}$$

$$Var(Z) = E(Z^2) - \cancel{(E(Z))^2} = \int_{\mathbb{R}} x^2\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx = \ldots = 1 \Rightarrow \boxed{\sigma^2 = \sigma = 1}$$

int. by parts

$$F(x) = \underbrace{\int \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx}_{\text{antiderivative impossible (Risch algorithm)}} + C$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, y\, e^{-\frac{y^2}{2}}\,dy \quad \text{(numerically)}$$

$$F(0) = \frac{1}{2}$$

$$P(Z \in [-1,1]) = F(1) - F(-1) \approx 0.68$$

$$P(Z \in [-2,2]) = F(2) - F(-2) \approx 0.95$$

$$P(Z \in [-3,3]) = F(3) - F(-3) \approx 0.997$$

"3σ rule", "empirical rule", "68-95-99.7 rule" let $X \sim Exp(\lambda)$ $f := \lambda e^{-\lambda x}$ $F(x) = 1 - e^{-\lambda x}$, $Y = 2X$ Can we get distr?

$$F_Y(x) = P(Y \le x) = P(2X \le x) = P\left(X \le \frac{x}{2}\right) = F_X\left(\frac{x}{2}\right)$$

$$= 1 - e^{-\lambda \frac{x}{2}}$$

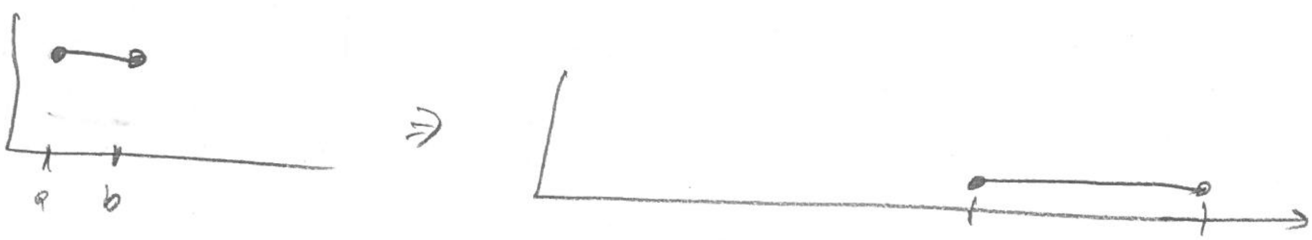
$$= 1 - e^{-\frac{\lambda}{2} x}$$

$$= 1 - e^{-\lambda' x} \quad \text{if } \lambda' = \frac{\lambda}{2}$$

$$= F_Y(x) \quad \text{s.t. } Y \sim Exp(\lambda') = Exp\left(\frac{\lambda}{2}\right)$$

Wow...

let $X \sim U(a,b)$ $f = \frac{1}{b-a}$ $F_X(x) = \frac{x-a}{b-a}$ $Y = cX + d$ lin transf



$d = 10$ $c = 2$

How is Y distr?

$$F_Y(x) = P(Y \le x) = P(cX + d \le x) = P(cX \le x - d) = P\left(X \le \frac{x-d}{c}\right) = F_X\left(\frac{x-d}{c}\right)$$

$$= \frac{\left(\frac{x-d}{c}\right) - a}{b-a} = \frac{x - d - ac}{c(b-a)} = \frac{x - (ca + d)}{cb - ca}$$

$$= \frac{x - (d + ac)}{(bc + d) - (ac + d)} = \frac{x - a'}{b' - a'}$$

let $a' = d + ac$, $b' = d + bc$

$$Y \sim Unif(a', b') = Unif(d + ac, d + bc)$$

why?

$Z \sim N(0,1)$ let $X = \sigma Z + \mu \Rightarrow E(X) = \mu,\ SE(X) = \sigma$

$F_X(x) = P(X \le x) = P(\sigma Z + \mu \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right) = F_Z\left(\frac{x-\mu}{\sigma}\right)$ unknown since $F_Z(x)$ unknown!

but... $f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} F_Z\left(\frac{x-\mu}{\sigma}\right) = \frac{d}{du} \frac{1}{\sigma} F_Z(u) = \frac{1}{\sigma} f_Z(u)$

let $u = \frac{x-\mu}{\sigma}$ $\frac{du}{dx} = \frac{1}{\sigma} \Rightarrow dx = \sigma\, du$

$= \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$

$\frac{d}{du}\ \frac{du}{dx}$

$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}$

$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

general normal distr.

$X \sim N(\mu, \sigma^2)$ $Var(X)$ why?

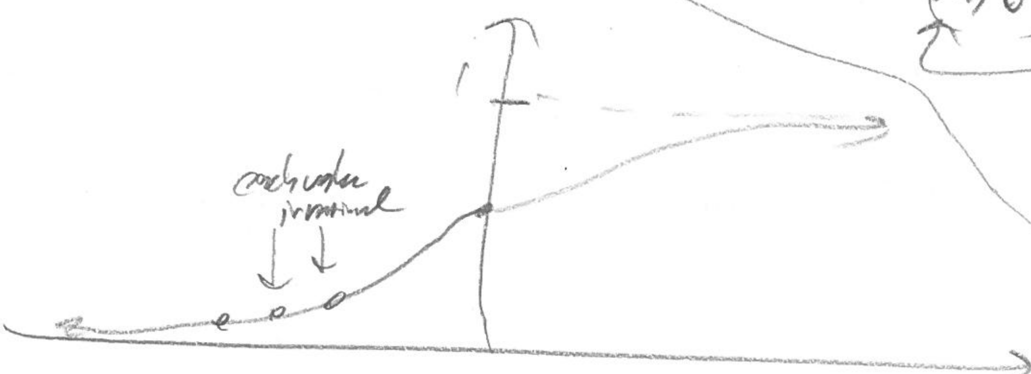
$E(X)$

Param Space

$\mu \in \mathbb{R}$

$\sigma^2 \in (0, \infty)$

$Supp(X) = \mathbb{R}$



$P(Z \le -1) = .16$

$P(Z \le -2) = .025$

$P(Z \le -3) = .0015$

But why so important?

"integrate x out"

let $L(t) = \int_{\mathbb{R}} e^{-tx} f(x)\, dx$ Bilateral Laplace Transform