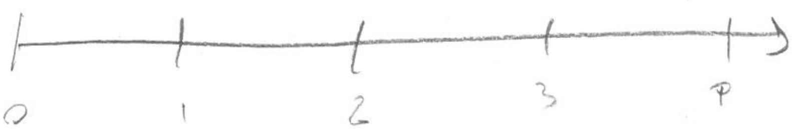
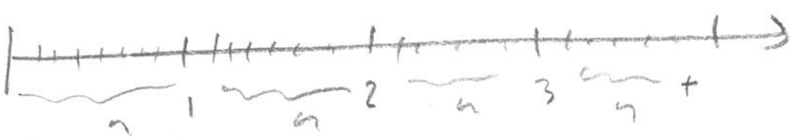
# Lecture 17 11/11/16 Math 201

$T \sim \text{Geometric}(p) :: p(t) = (1-p)^{t-1} p$ , $F(t) = 1-(1-p)^{t}$ $\quad E(T) = \frac{1}{p} \text{ exp} \cdot 1 \frac{\text{sec}}{\text{exp}}$

every second one iid Bernoulli...

Now... every second has n Bernoullis




$p(t) = (1-p)^{nt-1} p$ , $F(t) = 1-(1-p)^{nt}$

$E[T] = \frac{1}{p} \text{ exp} \cdot \frac{1 \text{ sec}}{n \text{ exp}} = \frac{1}{np}$

If n is large... immediately stops...

But what if $p \approx 0$ really small

let $\lambda = np$ , n large, p small but the product of the two is constant

$\Rightarrow p = \frac{\lambda}{n}$ Using this substitution...

$\Rightarrow p(t) = \left(1-\frac{\lambda}{n}\right)^{nt-1} \frac{\lambda}{n}$ , $F(t) = 1-\left(1-\frac{\lambda}{n}\right)^{nt}$

Now let $n \to \infty$ so that in every second there's infinite experiments; experiments occur continuously. What is the PMF?

$$\lim_{n\to\infty} p(t) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt-1} \frac{\lambda}{n} = \underbrace{\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt-1}}_{?} \underbrace{\lim_{n\to\infty} \frac{\lambda}{n}}_{0} = 0$$

No valid PMF! $\sum_t p(t) = 0! \neq 1$

What about CDF?

$$\lim_{n\to\infty} F(t) = \lim 1-\left(1-\frac{\lambda}{n}\right)^{nt} = 1 - \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nt} = 1 - \left(\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}\right)^{t}$$

$$\lim f(x)^a = \left(\lim f(x)\right)^a$$

famous limit!

Consider

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

| $n$ | $f(x)$ |
|---|---|
| 10 | 2.594 |
| 100 | 2.705 |
| 1000 | 2.717 |
| 10000 | 2.718 |
| ⋮ | ⋮ |

Converges... $\Rightarrow e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

or $e := \sum_{i=0}^{\infty} \frac{1}{i!}$ or $\int_1^e \frac{1}{x}\,dx = 1$

How about $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$ s.t. $a \in \mathbb{R}$ constant

let $\frac{1}{m} = \frac{a}{n} \Rightarrow n = ma$ if $n \to \infty \Rightarrow m \to \infty$

$$\Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{m}\right)^{ma} = \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^a = e^a$$

$$\Rightarrow \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \quad \Rightarrow \lim F(t) = 1 - e^{-\lambda t}$$

Is this a valid CDF?

$F(t) \in [0,1]$ ?

$F(t) \geq 0$ $\quad 1 - e^{-\lambda t} \geq 0$ ?

$1 \geq e^{-\lambda t}$ ?

$0 \geq -\lambda t$

$t \geq 0$ ? Yes

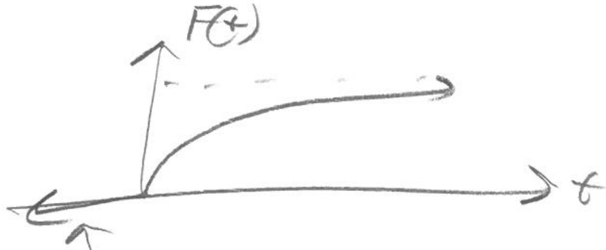
$F(t) \leq 1$ ?

$1 - e^{-\lambda t} \leq 1$

$-e^{-\lambda t} \leq 0$

$e^{-\lambda t} \geq 0$

$\frac{1}{e^{\lambda t}} \geq 0$ Yes




why

Comes from geometry

$\lim_{t \to -\infty} F(t) = 0$

$\lim_{t \to \infty} F(t) = \lim_{t \to \infty} 1 - e^{-\lambda t} = 1 - \lim_{t \to \infty} e^{-\lambda t} = 1 - \lim_{t \to \infty} \underbrace{\frac{1}{e^{\lambda t}}}_{\to 0} = 1$ ✓

Is $F(y) \geq F(x)$ if $y > x$?

Check derivative ... ensure always $\geq 0$

$\frac{d}{dt}[F(t)] = \frac{d}{dt}[1 - e^{-\lambda t}] = \lambda e^{-\lambda t} = \frac{\overbrace{\lambda}^{+}}{\underbrace{e^{\lambda t}}_{+}} \geq 0$ ✓

Since there is a CDF ⇒ T is a r.v. Discrete? No... no PMF. So what is it?

$Supp[T] = (0, \infty)$ $\quad |Supp(T)| = |\mathbb{R}| > |\mathbb{N}|$

Size of the Continuum (uncountable $\infty$)

⇒ T is a cont. r.v.

Why did this happen? ← $\infty$ means no "spaces" in between #'s. Infinite division means no missing #'s.

Is continuous time real? Quantum Theory again.

Planck length $1.62 \times 10^{-35}$ m, not possible to tell difference in location

$d_1 = d_2$! Crazy

speed of light

time it takes to cross Planck length: $5.3 \times 10^{-44}$ s

Time may be discrete! We don't know! Until we understand quantum gravity..

$p(3) = 0$ why? ← infinite resolution

$p(3.0000000\ldots) = 0$

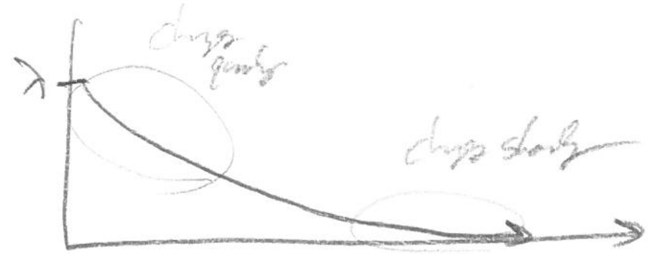
but $p(3.000) = P(T \in [2.995\bar{0}, 3.004\bar{9}])$ $\quad F(3.005) - F(2.995) > 0$

$= P(T \leq 3.0049) - P(T \leq 2.9950)$

How "fast" does the CDF change?

$f(t) := F'(t) = \lambda e^{-\lambda t}$

probability density function (PDF)
$\neq$ probability mass function (PMF)




$$P(T \in [a,b]) = \int_a^b f(t)\,dt = F(b) - F(a)$$ Fund. Thm. Calculus..

PDF: answers the question: How dense is the probability in a certain region?

let $\lambda = 2$

$f(1) = 0.27 \neq p(1) = 0$

$p(1) = P(T=1) = \int_1^1 f(t)\,dt = 0$

$f(0.1) = 1.63 \neq p(0.1) = 0$

But $f(0.1) > 1$! Possible? Yes ... PDF is not a prob.

PDF is completely abstract! It is good for

(a) Integration to find probs of regions

(b) Comparison $\quad \frac{f(0.1)}{f(1)} = \frac{1.63}{0.27} \approx 6 \Rightarrow$ realizations near 0.1 are 6x more likely than realizations near 1

Why $\lim_{\varepsilon \to 0} \frac{P(T \in [0.1, 0.1+\varepsilon])}{P(T \in [1, 1+\varepsilon])} = \lim_{\varepsilon \to 0} \frac{F(0.1+\varepsilon) - F(0.1)}{F(1+\varepsilon) - F(1)}$

def of deriv. $F'(t) = f(t)$

$= \lim_{\varepsilon \to 0} \frac{\frac{F(0.1+\varepsilon) - F(0.1)}{\varepsilon}}{\frac{F(1+\varepsilon) - F(1)}{\varepsilon}} = \frac{f(0.1)}{f(1)}$

$P(T \in (-\infty, \infty)) = 1$ Why? T has to realize something!

$= \int_{-\infty}^{\infty} f(t)\,dt$ by F.T.C. Analogous to $\sum_{x \in Supp(X)} p(x) = 1$ for the discrete case

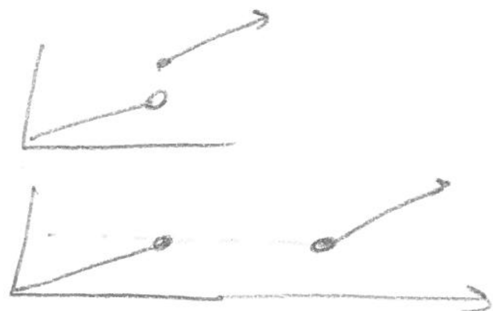
Def Cont r.v. X

(1) $|Supp(X)| = |\mathbb{R}|$

(2) $F(x)$ is a valid CDF with no "jumps". kinks allowed

(3) PMF does not exist

(4) $f(x) \geq 0$ and $\int_{Supp(X)} f(x)\,dx = 1$

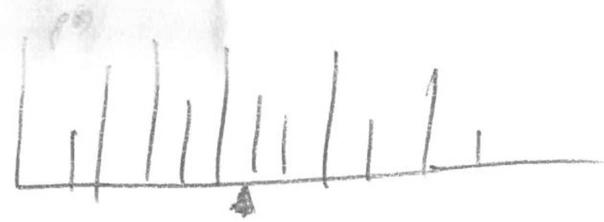


Def of Identical Distr for Cont. r.v.'s $X_1 \overset{d}{=} X_2$ if $f_1(x) = f_2(x)$

Def of iden. distr of both discrete & cont r.v.'s $X_1 \overset{d}{=} X_2$ if $F_1(x) = F_2(x)$

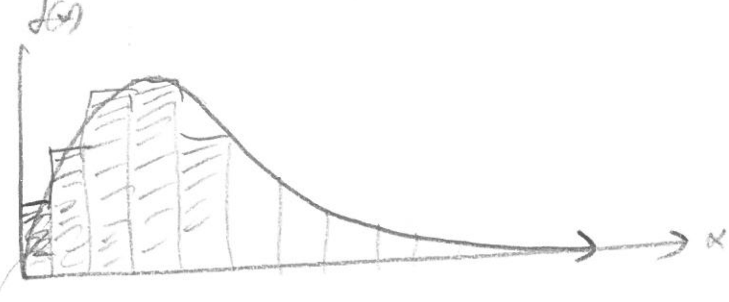
CDF's always exist

What is $E[X]$? If X is discrete

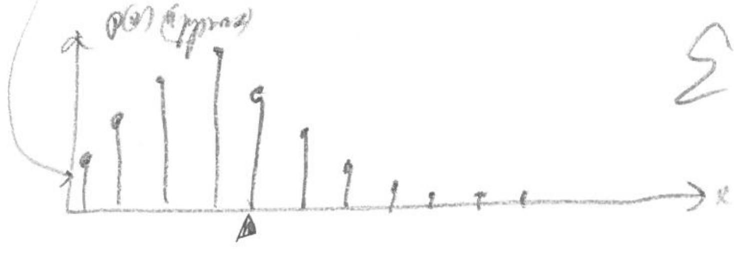



$\sum_{x \in Supp(X)} x\,p(x)$

Now..




Imagine




$\sum x\,p(x)$

Make the rectangles thinner & thinner...

midpoint   area of rectangle

$$\sum_{Supp(X)} x\, p(x) \longrightarrow \boxed{\mu = E(X) = \int_{Supp(X)} x\, f(x)\, dx}$$

$$E[g(X)] = \int_{Supp(X)} g(x) f(x)\, dx$$

Not proven... need Lebesgue integration to demonstrate

$$\Rightarrow Var(X) := E\left[(X-\mu)^2\right] = \int_{Supp(X)} (x-\mu)^2 f(x)\, dx$$

All rules from before apply now...

$$E[aX+c] = a\mu + c$$

$$Var(aX+c) = a^2\sigma^2 \Rightarrow SE(aX+c) = |a|\sigma$$

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = n\mu$$

If iden. distr

$$Var\left(\sum X_i\right) = \sum_{i=1}^{n} Var(X_i) = n\sigma^2$$

if indep   if iid

Who was this scrambled geometric? 8

$P(X \le x) =$ $P(X > x)$

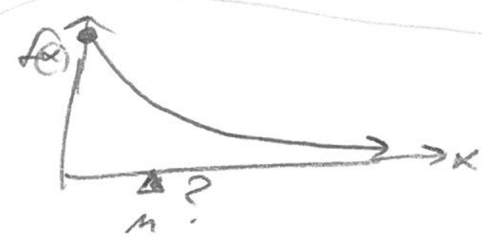
$F(x) = 1 - e^{-\lambda x} \Rightarrow 1 - F(x) = e^{-\lambda x}$

$X \sim Exp(\lambda) := \underbrace{\lambda e^{-\lambda x}}_{f(x) \text{ not } p(x)}$

$Supp(X) = (0, \infty)$ or $[0, \infty)$ but not $[0, \infty]$ why?

Param space $\lambda = np$ $n \to \infty$, $p \in (0,1)$

$\lambda \in (0, \infty)$




$$E(X) = \int_{Supp(X)} x f(x) dx$$

$$= \int_0^\infty x \lambda e^{-\lambda x} dx = \lambda \int_0^\infty x e^{-\lambda x} dx$$

recall $\int u dv = uv - \int v du$

let $u = x$, $dv = e^{-\lambda x} dx$

$\Rightarrow du = dx \Rightarrow v = -\frac{1}{\lambda} e^{-\lambda x} \Rightarrow \int v du = \int -\frac{1}{\lambda} e^{-\lambda x} dx = \frac{1}{\lambda^2} e^{-\lambda x}$

$$\Rightarrow E(X) = \lambda \left[ -\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} \right]_0^\infty$$

$$= -\left( \left( \lim_{x \to \infty} x e^{-\lambda x} + \lim_{x \to \infty} \frac{1}{\lambda} e^{-\lambda x} \right) - \left( 0 e^{-\lambda(0)} + \frac{1}{\lambda} e^{-\lambda(0)} \right) \right)$$

$X \sim Geom(p) \Rightarrow E(X) = \frac{1}{p}$ if $n = 1 \Rightarrow p = \lambda$

$$= -\left( (0 + 0) - (0 + \tfrac{1}{\lambda}) \right) = \boxed{\frac{1}{\lambda}}$$

So makes sense!

| | Single step | Multiple steps |
|---|---|---|
| Discrete | Geom | Neg Bin |
| Cont | Exponential | Erlang |

Next 2 & 2 (type of gamma)

Geometric was called "memoryless" ... Is exponential also memoryless?

$$P(X > a+b \mid X > b) = \frac{P(X > a+b \ \& \ X > b)}{P(X > b)} = \frac{P(X > a+b)}{P(X > b)}$$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda b}} = \frac{e^{-\lambda a}\cancel{e^{-\lambda b}}}{\cancel{e^{-\lambda b}}} = e^{-\lambda a}$$

$$= P(X > a) \Rightarrow \text{Yes!}$$

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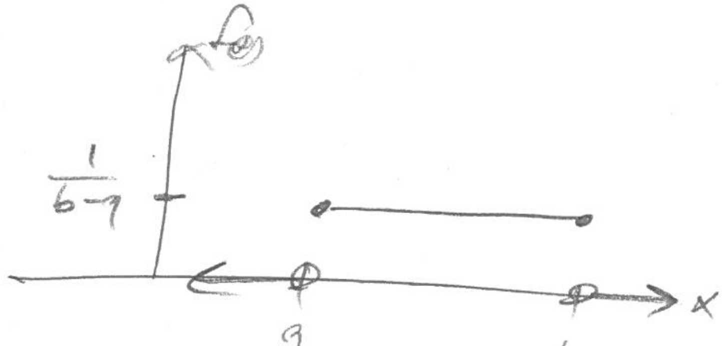
Recall $X \sim \text{Unif}(\{1, 7, 19\}) = \begin{cases} 1 & \text{wp } \frac{1}{3} \\ 7 & \text{wp } \frac{1}{3} \\ 19 & \text{wp } \frac{1}{3} \end{cases}$

discrete unif.

Let $X \sim \text{Unif}(a, b) := f(x) = \frac{1}{b-a}$

is the cont. uniform




$\text{Supp}(X) := [a, b]$

Param space

$a \in \mathbb{R}, b \in \mathbb{R}$ but $a < b$

Is this a PDF? Is $f(x) \geq 0$ always? Yes

Does $\int_{\text{Supp}(X)} f(x)dx = 1$? $\int_a^b \frac{1}{b-a} dx = \frac{1}{b-a} (x)_a^b = \frac{b-a}{b-a} = 1$ ✓

Find CDF ... the antiderivative

$$F(x) = \int f(x)\,dx + C$$

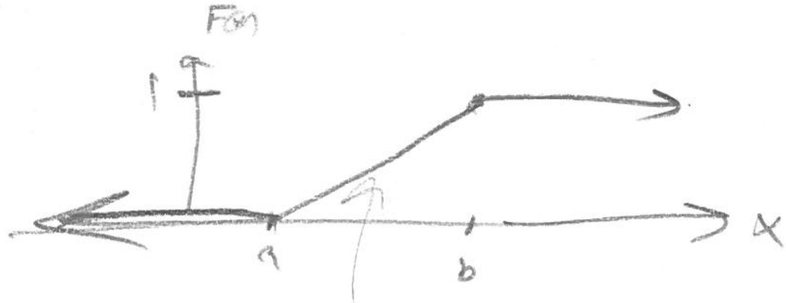
$$= \int \frac{1}{b-a}\,dx + c$$

$$= \frac{x}{b-a} + C \qquad \text{value of } C?$$

$F(a) = 0$ if $F(a) = 0 \Rightarrow \frac{a}{b-a} + C = 0 \Rightarrow C = \frac{-a}{b-a}$

$F(b) = 1$

$$\Rightarrow F(x) = \frac{x-a}{b-a}$$




slope? $\frac{1}{b-a}$ of course

so $a=0$, $b=1$

$\Rightarrow X \sim \text{Unif}(0,1)$ AKA the standard Uniform

$f(x) = 1$

$F(x) = x$

$\text{Supp}(X) = [0,1]$ CS people... this is the rand function!

$\mu$? $E(X) = \int_{\text{Supp}(X)} x\, f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \boxed{\frac{a+b}{2}}$

$\text{Med}(X) = \text{argmin}\{x : F(x) = 0.5\} \Rightarrow \frac{x-a}{b-a} = \frac{1}{2} \Rightarrow 2x - 2a = b - a \Rightarrow x = \frac{b+a}{2}$ makes sense!!

$$\sigma^2 = Var(X) = \int_{Supp(X)} (x-\mu)^2 f(x)\, dx = E(X^2) - \mu^2$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b - \frac{(a+b)^2}{4}$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{(a+b)^2}{4}$$

$$\begin{array}{r l}
 & b^2 + ab + a^2 \\
b-a \,\big) & b^3 - a^3 \\
 & -(b^3 - ab^2) \\
\hline
 & ab^2 - a^3 \\
 & -(ab^2 - a^2 b) \\
\hline
 & a^2 b - a^3 \\
 & -(a^2 b - a^3) \\
\hline
 & 0
\end{array}$$

$$\frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4} = \frac{(4b^2 + 4ab + 4a^2) - (3a^2 + 6ab + 3b^2)}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \boxed{\frac{(b-a)^2}{12}}$$

$$SE(X) = \sqrt{\sigma^2} = \frac{b-a}{\sqrt{12}}$$