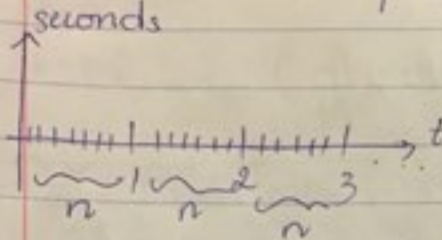
Nov 11, 2016

$T \sim \text{Geometric}(p) = \underbrace{(1-p)^{t-1} p}_{p(t)}$

$F(t) = 1-(1-p)^t$

$1 - F(t) = (1-p)^t$



$p(t) = (1-p)^{nt-1} p$

$F(t) = 1-(1-p)^{nt}$

$E(t) = \frac{1}{p}$ # exp

$\Rightarrow E(T) = \frac{1}{p} \text{exp} \cdot \frac{1 \text{sec}}{n \cdot \text{exp}} = \frac{1}{np} \text{sec}$

Imagine n large but p small

Let $\lambda = np \Rightarrow p := \frac{\lambda}{n}$ (Leparametonzation)

$p(t) = \left(1-\frac{\lambda}{n}\right)^{nt-1} \frac{\lambda}{n}$, $F(t) = 1 - \left(1-\frac{\lambda}{n}\right)^{nt}$

Let $n \to \infty$ but λ remains λ

$\lim_{n\to\infty} p(t) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt} \frac{\lambda}{n} = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt} \cdot \underbrace{\lim_{n\to\infty} \frac{\lambda}{n}}_{0}$

$\sum_{t \in \text{Supp}(T)} p(t) = 0 \Rightarrow \boxed{p(t) = 0}$ is not valid

$\Rightarrow T$ is not a discrete r.v

$\left(\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n\right)^t = e^{-\lambda t}$

$\lim_{n\to\infty} F(t) = \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nt} = 1 - \underbrace{\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt}} = \boxed{1-e^{-\lambda t}}$

$\lim f(x)^a = \lim f(x)^a$, $a \in \mathbb{R}$

$\lim_{n\to\infty} \underbrace{\left(1+\frac{1}{n}\right)^n}_{f(n)} =: e$

n	$f(n)$
10	2.514
100	2.705
1000	2.717
10000	2.718

$\not\to 1$

$$e := \sum_{i=0}^{\infty} \frac{1}{i!}$$

$$\int_1^e \frac{1}{x}\,dx = 1$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n \qquad \text{Let } \frac{1}{m} = \frac{a}{n} \Rightarrow n = ma$$

if $n \to \infty \Rightarrow m = \infty$

$a \in \mathbb{R}$

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{ma} = \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^n\right)^a = e^a$$

CDF's: $F(t) \in [0,1]$

(1) $1 - e^{-\lambda t} \geq 0$

$1 \geq e^{-\lambda t}$

$0 \geq -\lambda t$

$-\lambda t \leq 0$

$+ \quad +$

$1 - e^{-\lambda t} \leq 1$

$-e^{-\lambda} \leq 0$

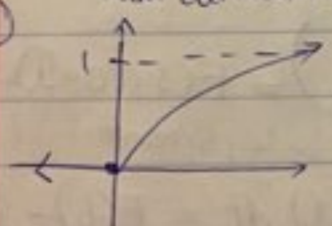
$+$

$t \in (0, \infty)$

Geom $t \to \infty$

Supp[X] $\in \mathbb{N}$

(2) take derivative



(CDF)

$$\lim_{t\to-\infty} F(t) = 0$$

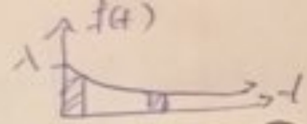
(3) $\lim_{t\to\infty} F(t) = 1$

$$\lim_{t\to\infty} 1 - e^{-\lambda t} = 1 - \lim e^{-\lambda t}$$

$$= 1 - \lim \frac{1}{e^{\lambda t}}$$

$$= 1 - 0 = 1$$

→ probability density function (PDF)



$f(t) := \frac{d}{dt}[F(t)] = \frac{d}{dt}\left[1 - \cancel{\phi} e^{-\lambda t}\right] = \lambda e^{-\lambda t} \geq 0$? ⇒ Yes

np ⊕ ⊕⊕

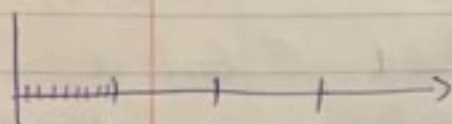
⇒ $F(t)$ is a CDF

⇒ T is a r.v but not discrete because it has no valid PMF

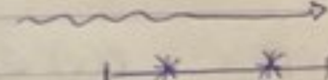
T is a r.v. $\text{Supp}[T] = (0, \infty) = [0, \infty)$

$|\text{Supp}[T]| = |\mathbb{R}| > |\mathbb{N}|$ ← for a discrete r.v

⇒ T is a continuous r.v



light $3E^{8}$ m/s



Planck length 1.62×10^{-35} m

Planck Time 5.3×10^{-44} s

$P(T=3) = p(3) = 0$

$P(T=3) = P(T = \underbrace{3.000000\cdots}_{\text{contain infinite information}}) = 0$

$P(T = 3.\underset{\uparrow \text{stop}}{000}) = P(T \in [2.9995\bar{0}, 3.0004\bar{9}]) = F(3.0004\bar{9}) - F(2.9995\bar{0})$

← important

$$\underbrace{P(T \in [a,b]) = \int_a^b f(t)\,dt = F(b) - F(a)}_{\text{fund. Thm. calc}}$$

$\lambda = 2$, $f(1) = 2e^{-2 \cdot 1} \simeq 0.27 \neq p(1)$

$p(1) = 0$ $\int_1^1 f(t)\,dt = 0$

$f(0.1) = 2e^{2 \cdot 0.1} \simeq 1.63 > 1$

PDF is an abstract magic good for 2 things

① Integrate to get prob (negiq) via F.T.C

② Compare two points ~~rel~~ relative likelihoods

$$6 \simeq \frac{f(0.1)}{f(1)} \qquad \frac{P(T[0.1, 0.1+\varepsilon])}{P(T \in [1, 1+\varepsilon]} \cdot \frac{\frac{1}{\varepsilon}}{\frac{1}{\varepsilon}} \lim_{\varepsilon \to \infty} \frac{\overbrace{\frac{P(T[0.1, 0.1+\varepsilon])}{\varepsilon}}^{F(0.1+\varepsilon)-F(0.1)}}{\underbrace{\frac{P(T \in [1, 1+\varepsilon])}{\varepsilon}}_{F(1+\varepsilon)-F(\varepsilon)}}$$

$$= \frac{f(0.1)}{f(1)}$$

$P(T \in (-\infty, \infty)) = 1$

$$\boxed{\int_{-\infty}^{\infty} f(t)\,dt = 1}$$ PDF property

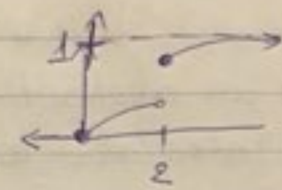
$\Rightarrow P(T \in \text{Supp}[T]) = \int_{\text{Supp}[T]} ft(dt) = 1$

ala: $\sum_{x \in \text{Supp}[X]} p(x) = 1$ for discrete r.v's X

Properties of continuous r.v X

① $|Supp[X]| = |\mathbb{R}|$
continuum

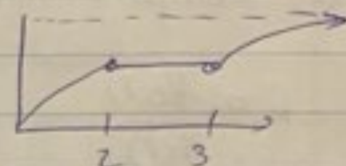
② Has valid CDF $F(x)$ no jumps
but gap allowed



③ PMF does not exist

④ PDF exists $f(x)$ (a) $f(x) \geq 0$

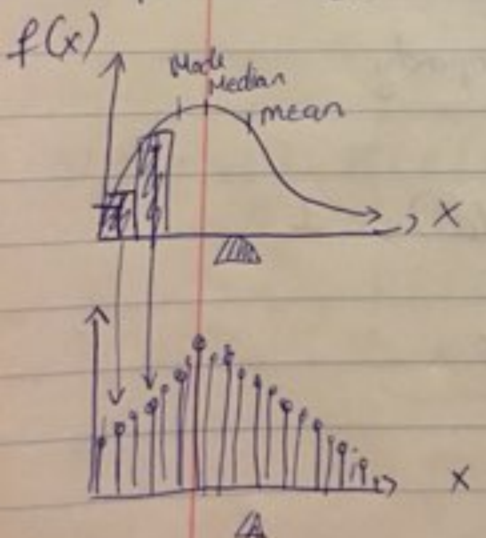
(b) $\displaystyle\int_{Supp[X]} f(x)\,dx = 1$



X_1, X_2 continuous r.v's $\quad X_1 \overset{d}{=} X_2$ if

$f_1(x) = f_2(x)$ PDF's the same or

$F_1(x) = F_2(X)$ CDF's same



$$E[X] \approx \sum_{x \in ApproSupp[X]} x\,p(x) \Rightarrow \boxed{E[X] = \int_{Supp[X]} x\,f(x)\,dx}$$

Approximate

$$E[g(X)] = \int_{Supp[X]} g(x) f(x)\,dx$$

$$Var[X] := E[(X-\mu)^2]$$

$$\boxed{\sigma^2 = \int_{Supp[X]} (x-\mu)^2 f(x)\,dx}$$

$E[aX+c] = a\mu + c$

$Var[aX+c] = a^2\sigma^2 \Rightarrow SE[aX+c] = |a|\sigma$

$E[\Sigma X_i] = \Sigma E[X_i] = n\mu$

if identical

$Var[\Sigma X_i] = \Sigma Var[X_i] = n\sigma^2$

if indep. if iid

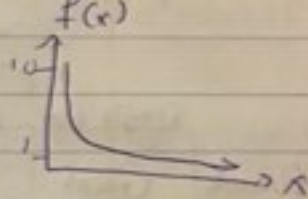
$X \sim Exp(\lambda) := \lambda e^{-\lambda x}$ exponential r.v

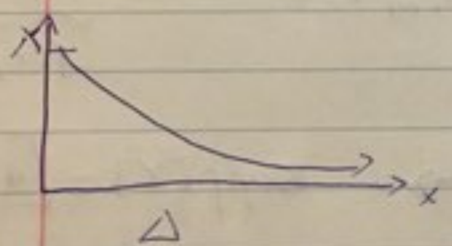
$Supp[X] = (0, \infty)$ $\quad \lambda = np \in \mathbb{N}$, $(0,1)$

Parameter space

$\lambda \in (0, \infty)$

rate





Δ

$$E[X] = \int_0^\infty \underbrace{x \lambda e^{-\lambda x}}_{f(x)} dx$$

$$\Rightarrow \lambda \int_0^\infty x e^{-\lambda x} dx = \lambda \int_0^\infty x e^{-\lambda x} dx$$

$$\int u\,dv = uv - \int v\,du$$

let $u = x \Rightarrow du = dx$

$dv = e^{-\lambda x}$

$\Rightarrow v = \frac{-1}{\lambda} e^{-\lambda x}$

$$= \lambda \left[-\frac{1}{\lambda} x e^{-\lambda x} - \frac{1}{\lambda^2} e^{-\lambda x} \right]_0^\infty$$

$$= -\left(\underbrace{\lim_{x \to \infty} x e^{-\lambda x}}_{\frac{x}{e^{x}}} + \lim_{x \to \infty} \frac{1}{\lambda} e^{-\lambda x} \right) - \left(0 \cdot e^{-\lambda(0)} + \frac{1}{\lambda} e^{-\lambda(0)} \right)$$

$$-\left((0 + 0) - \left(0 + \frac{1}{\lambda}\right) \right) = \boxed{\frac{1}{\lambda}}$$

$P(X \le x) = F(x) = 1 - e^{-\lambda x}$

$P(X > x) = 1 - F(x) = e^{-\lambda x}$

$X \sim \mathrm{Geom}(p) \quad E[X] = \frac{1}{p} = \frac{1}{np} = \frac{1}{\lambda}$

Exponential has the memorylessness property

$P(X > a+b \mid X > b) = P(X > a) = e^{-\lambda a}$

$$\frac{\overbrace{P(X > a+b \;\&\; a > b)}^{P(X>a+b) = e^{-\lambda(a+b)}}}{\underbrace{P(X > b)}_{e^{-\lambda b}}} = \frac{e^{-\lambda a} e^{-\lambda b}}{e^{-\lambda b}} = e^{-\lambda a}$$

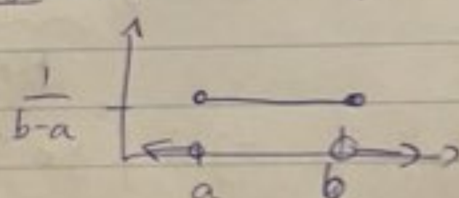
	Single Step	Mult Step
Discrete	Geom	NegBin
Cont	Exponential	~~Erlang~~ Erlang (gamma)

$X \sim \mathrm{Uniform}(\{1, 7, 28\})$

Discrete uniform

$X \sim \mathrm{Uniform}(a, b) := \underbrace{\frac{1}{b-a}}_{f(x)}$

"Uniform"



$\mathrm{supp}[X] = [a, b]$

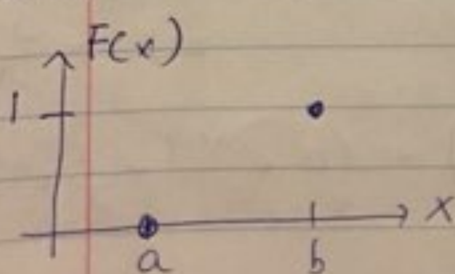
Parameter Space

$a \in \mathbb{R}$

$b \in \mathbb{R}$ but $a < b$

$F(x) = \int f(x)\,dx + C$

$= \int \frac{1}{b-a}\,dx + C = \frac{x}{b-a} + c$



$F(a) = 0 \Rightarrow \frac{a}{b-a} + C = 0$

$\Rightarrow c = \frac{a}{b-a}$

$\Rightarrow \boxed{F(x) = \frac{x-a}{b-a}}$

$$E[X] = \int_a^b x f(x)\,dx \quad \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \boxed{\frac{b+a}{2}}$$

$$\sigma^2 = E[X^2] - \mu^2 = \int_a^b x^2 \frac{1}{b-a}\,dx - \left(\frac{a+b}{2}\right)^2$$

$$\begin{array}{r} b^2+ab+a^2 \\ b-a\,\overline{)\,b^3-a^3} \\ -b^3-ab^2 \\ \hline ab^2-a^3 \end{array}$$

$$\frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b$$

$$= \frac{1}{3}\,\frac{b^3-a^3}{b-a}$$

$$= \frac{1}{3}\,\frac{(b-a)(b^2+ab+a^2)}{b-a}$$

$$= \frac{1}{3}(b^2+ab+a^2)$$

$$= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4}$$

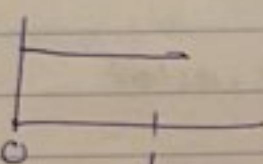
$$= \frac{4b^2+4ab+4a^2-(3a^2+6ab+3b^2)}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \boxed{\frac{(b-a)^2}{12}}$$

$$\boxed{\sigma = \frac{b-a}{\sqrt{12}}}$$

<u>Ex</u>: $a=0, b=1$



$X \sim \text{Uniform}(0,1) = 1$

Standard uniform.