Lecture # 16

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Let $\underline{Y \sim \text{geometric}(p)}$,

Then, $Var[Y] = E[(Y-\mu)^2] = E[Y^2] - \mu^2 = E[Y^2] - (1/p)^2$

Note: $E[Y^2] = \sum_{y=1}^{\infty} y^2(1-p)^{y-1}p$

$= \sum_{z=0}^{\infty} (z+1)^2(1-p)^z p$ let $z = y-1 \Rightarrow y = z+1$

$= p\left(\sum_{z=0}^{\infty} (z+1)^2(1-p)^z p\right.$ $y = 1 \ldots \infty$ $z = 0 \ldots \infty$

$\bullet \underbrace{\sum_{z=0}^{\infty} z^2(1-p)^z p}_{\infty} + \underbrace{2p\sum_{z=0}^{\infty} z(1-p)^z}_{\infty} + p\underbrace{\sum_{z=0}^{\infty} (1-p)^z}_{1}$

$= \underbrace{(1-p)\sum_{z=1}^{\infty} z^2(1-p)^{z-1}p}_{E[Y^2]} + \underbrace{2(1-p)\sum_{z=1}^{\infty} z(1-p)^{z-1}}_{E[X] = 1/p} + 1$

$\to$ Geometric.

$\Rightarrow E[Y^2] = (1-p)E[Y^2] + \frac{2(1-p)}{p} + 1$

$1-(1-p) = p$

$\Rightarrow pE[Y^2] = \frac{2(1-p)}{p} + 1$

$\Rightarrow E[Y^2] = \frac{2(1-p)}{p^2} + \frac{1}{p}$

$\Rightarrow Var[Y] = \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2-2p-1+p}{p^2} \to \underline{Var[Y] = \frac{1-p}{p^2}}$,

$\sigma^2 = \frac{1-p}{p^2}$ $\to \underline{Var[X] = r\frac{(1-p)}{p^2}}$

Let $X \sim$ Negative Binomial (r, p)

$Var[X] = \sum_{x=r}^{\infty} (x - \frac{r}{p})^2 \binom{x-1}{r-1}(1-p)^{x-r}p^r$ $\Rightarrow Var[X] = r\frac{1-p}{p^2}$ (?)

$E[X] = r/p$

Note: $x_1, \ldots, x_r \overset{iid}{\sim} Geom(p)$

Since: $X = \underbrace{x_1 + x_2 + \ldots + x_r}_{r \text{ terms}}$ Recall: $E[X] = r/p$

Thus $Var[X] = r \cdot \frac{1-p}{p^2}$

Note: Geometric Distribution has memoryless-ness property, that is the probability of a success after n number of failures is the same as the p of success at "any point of time" (in this case, at any trial)

General Case of Memoryless-ness:

$$p(x = a+b \mid x > a) = \frac{p(x = a+b \ \& \ x > a)}{p(x>a)} = \frac{p(x=a+b)}{1-F(a)}$$

$$= \underbrace{\frac{(1-p)^{a+b-1}}{(1-p)^a} = (1-p)^{b-1}p = p(x=b)}_{\text{Due to iid Beroullis.}}$$

Nate Silver said $P(\text{Clinton wins the race}) = 0.75$ $\leftrightarrow$ Well.

Model: $X \sim \text{Benoulli}(0.75)$

$E[X] = 0.75$

Average realization is close to $E[X]$ due to Law of Large Numbers (LLN)

$E[X]$ however is meaningless as the same election would not happen over & over again.