Basic Set Theory

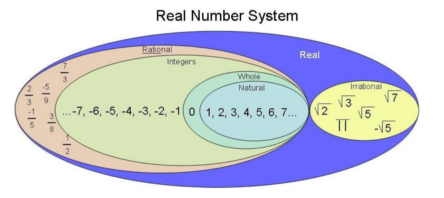
CS aphorism: the hardest things in CS are two things:
1 - name things correctly
2 - cash & validation

Set theory is the branch of mathematical logic that studies sets, which informally are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. The language of set theory can be used in the definitions of nearly all mathematical objects.
The modern study of set theory was initiated by Georg Cantor and Richard Dedekind in the 1870s. After the discovery of paradoxes in naive set theory, numerous axiom systems were proposed in the early twentieth century, of which the Zermelo–Fraenkel axioms, with the axiom of choice, are the best-known.

Set theory is commonly employed as a foundational system for mathematics, particularly in the form of Zermelo–Fraenkel set theory with the axiom of choice. Beyond its foundational role, set theory is a branch of mathematics in its own right, with an active research community. Contemporary research into set theory includes a diverse collection of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals.

- wikipedia.org

Real Number System



important sets

some mathematicians do not consider 0 a natural number

$\mathcal{N}$ = natural numbers = {0,1,2,3....}
$\mathcal{Z}$ = integers = {....,-3,-2,-1,0,1,2,3,...}
$\mathcal{Z}^+$ = positive integers = {1,2,3,.....}
$\mathcal{R}$ = set of real numbers
$\mathcal{R}^+$ = set of positive real numbers
$\mathcal{Q}$ = set of rational numbers $\mathcal{Q} = \{p/q \mid p \in \mathcal{Z}, q \in \mathcal{Z}, \text{and } q \neq 0\}$

$\mathcal{C}$ = set of complex numbers $\mathcal{C} = \{a+bi \mid a,b \in \mathcal{R}\}$
a,b - real numbers | variables
$i = \sqrt{-1}$ - not a real number | constant

boldface is being used

Basic Set Theory

Set Name — assignment aka definition — elements aka members — set braces denote the beginning / end of enumeration

F := { Jeane , Mary , Susan , <u>Dana</u> }

- order is irrelevant
- duplicates are irrelevant

In the above example Set Name:F stands for Female first names. and Set Name:M stands for Male first names.

M := { Bob , Joe , Max , <u>Dana</u> }

Venn Diagram

Jeane , Mary , Susan
Dana
Bob , Joe , Max

Sets can contain an <u>infinite</u> number of elements:

$\mathcal{N}$:= { 1 , 2 , 3 , 4 , ... }

ellipses . . . are used when the general pattern of the elements is <u>obvious</u> and reads "and so forth"

$\mathcal{Z}$:= { ... , -2 , -1 , 0 , 1 , 2 , ... }

Operations on Sets

MNEMONIC: element ∈/∉ set

set inclusion - Jeane is a member of F

Jeane ∈ F

set exclusion - Jeane is NOT a member of F

Jeane ∉ F

Set Equality / Inequality

equality - LHS & RHS have the same elements

{ Jeane , Mary , <u>Dana</u> , Susan } = F   (order irrelevant)

inequality - LHS & RHS do <u>NOT</u> have the same elements

{ Jeane , Mary } ≠ F   M ≠ F

Subsets

subset - <u>ALL</u> elements of LHS are in RHS

{ Jeane , Mary } ⊆ F

proper subset - <u>ALL</u> elements of LHS are in RHS but NOT vice versa

{ Jeane , Mary } ⊂ F

⊆ same as ⊂ or =

Predicate is a sentence that <u>includes a variable</u> thus is <u>neither True nor False</u>

thus a **predicate** is NOT a **proposition** (statement that is T or F) however it can be made into one by assigning some value to it's variable or using a **quantifier**

Predicates

∈ ∉ = ≠ ⊂ ⊆ are predicate functions which return T/F

≠ (M , F) → True   are sets M and F not equal? → True

≠ : set → { T , F }   if ≠ is a set, T & F are its members

= (M , F) → False   are sets M and F equal? → True

= (param1:Set1 , param2:Set2) -> Bool T/F
≠ (param1:Set1 , param2:Set2) -> Bool T/F
⊂ (param1:Set1 , param2:Set2) -> Bool T/F
⊆ (param1:Set1 , param2:Set2) -> Bool T/F

∈ (param1:SetMember , param2:Set) -> Bool T/F
∉ (param1:SetMember , param2:Set) -> Bool T/F

ex

singleton set - set with only 1 member

{ Jeane } ⊂ F ?   T - since there are elements in RHS that are not in LHS

Jeane ⊂ F ?   F - in order to be a proper subset, member Jeane would have to be a set. ONLY a Set can be a Subset of another Set, this this notation is ILLEGAL

{ Jeane } ∈ F ?   F - LHS is a set and RHS does NOT contain LHS among its members

Jeane ∈ F ?   T - Jeane is a member of RHS

Jeane ⊆ F ?   F - Jeane is NOT a set - notation illegal

{ Jeane } ⊆ F ?   T - LHS is a subset of RHS because element Jeane is contained by both LHS and RHS

set_Union (param1:Set1 , param2:Set2) -> Set3

Set Union - combines the elements of both sets

MNEMONIC: set ∪ set = set

set union - combines the elements of both sets

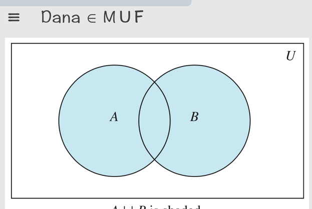
F ∪ M

{ { Jeane , Mary , Susan , <u>Dana</u> } { Bob , Joe , Max , <u>Dana</u> } }

collapse or flatten (Ruby)

{ Jeane , Mary , Susan , Bob , Joe , Max , <u>Dana</u> }

x ∈ F ∨ x ∈ M → x ∈ M ∪ F
Dana ∈ F ≡ Dana ∈ M ∪ F



A ∪ B is shaded.

a member is in M or in F

"inclusive or" aka "non-exclusive or"

ex

{ Jeane } ∪ { Jeane } = { Jeane } ?

T - since the duplicate elements count as one , and as a consequence the union of LHS && RHS flattens into a singleton set equivalent to either of the sides.

F ∪ M → <u>Dana</u> ∈ F ∪ M ?   T - since ∪ is inclusive

Let's use a definition of $\mathcal{N}$ which does NOT include zero. We want to include zero among the elements of $\mathcal{N}$

$\mathcal{N} \cup \{0\} \rightarrow \mathcal{N}_0 := \{0, 1, 2, 3, \cdots\}$

set_Intersect (param1:Set1 , param2:Set2) -> Set3

Set Intersection - returns only the elements in-common

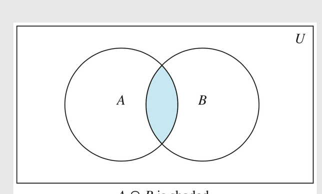
MNEMONIC: set ∩ set = set

F ∩ M

{ { Jeane , Mary , Susan , <u>Dana</u> } { Bob , Joe , Max , <u>Dana</u> } }

{ <u>Dana</u> }

x ∈ F ∧ x ∈ M → x ∈ M ∩ F



A ∩ B is shaded.

a member is in M and in F

ex

F ∩ { Bob , Joe } = { }   "empty set" aka "null set"   ∅ := { }

T - since LHS && RHS have NO elements in common (they are mutually exclusive)

odds ∩ evens = ∅

ex

∅ ⊂ F   T - empty set is a proper subset of F since F also has elements which ∅ does not

∅ ⊆ F   T - vacuously true

∅ ∈ F   F - ∅ is a not a member but a set

∅ ∉ F   T - look above

Empty Set

the Empty Set or Null Set is the set with <u>no elements</u>

symbolized ∅ or {}

∅ is a subset of any set :   ∅ ⊆ A

∅ ≠ { ∅ }

singleton set : a set with <u>one element</u>

{ a } , { ∅ }

set_Diff (param1:Set1 , param2:Set2) -> Set3

Set Difference / Subtraction - LHS wipes away RHS

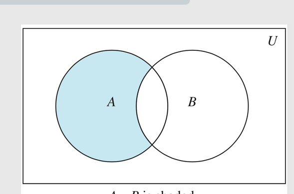
MNEMONIC: set \ set = set

F \ M

{ { Jeane , Mary , Susan , <u>Dana</u> } { Bob , Joe , Max , <u>Dana</u> } }

{ Jeane , Mary , Susan }   All elements in F that are NOT in M

x ∈ F ∧ x ∉ M → x ∈ F\M



A − B is shaded.

a member is in F and NOT in M

ex

if A ∩ B = ∅ → A\B = A   wipes away B
B\A = B   wipes away A

A \ ~~B~~
B \ ~~A~~

if A = B , A ∩ B = A or B   (A , B)

if A ⊆ B → A\B = ∅

wipe away B so everything cancels

if A\B = ∅ , A ∩ B = A

Def: A & B are mutually exclusive if A ∩ B = ∅

mutual exclusivity:

"Mutually exclusive" is a statistical term describing two or more events that cannot occur simultaneously. For example, it is impossible to roll a five and a three on a single die at the same time.
In logic, two mutually exclusive propositions are propositions that logically cannot be true in the same sense at the same time.   - wikipedia.org

two sets are called disjoint if their intersection is the <u>empty set</u> ∅

Def:
every non-empty set S has at least 2 subsets: ∅ and itself
1 b/c a ∈ ∅ is always false   ∅ ⊆ S for every set S
2 b/c a ∈ S → a ∈ S is true   S ⊆ S for every set S

set_Comp (param1:Set1 , param2:Set2) -> Set3

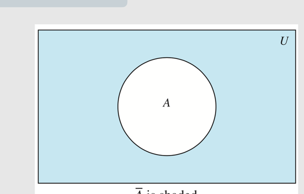
Set Complement - negation

F^C or $\overline{F}$

{ { Jeane , Mary , Susan , <u>Dana</u> } { Bob , Joe , Max , <u>Dana</u> } }

{ Bob , Joe , Max }   everything that is NOT in F

x ∉ F → x ∈ F^C



$\overline{A}$ is shaded.

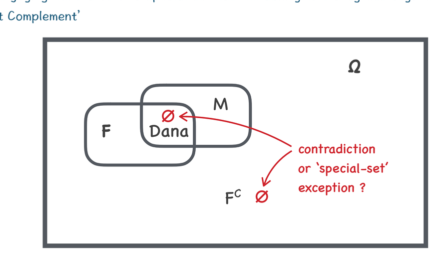
everything NOT in F

self-note: Can a set resulting from a 'Set Complement' still contain an 'empty set' ?

Def: every non-empty set has at least 2 subsets (∅ & itself)

By this definition both F & M must have an ∅ among their elements. Also the ∅ must be among 2^Ω. Lets place ∅ at F ∩ M.

- If 'Set Complement' does return a set, does that resulting F^C also contain by definition an ∅ among its members?
- If this is true, then how can F^C contain a member which is simultaneously present by definition in F ?
- Can the application of 'Set Complement' on F bereave F of its ∅ ?
- OR ∅ enjoying the status of a special-set has the ability to transgress beyond the scope of 'Set Complement'



Set Builder Notation - use when it is <u>impossible</u> to list all elements of a set

E := { 2n : n ∈ $\mathcal{Z}$ }

"all elements 2n (even)" "such that" "n is an element of the integers"

E := { ... , -4 , -2 , 0 , 2 , 4 , ... } - even numbers

Power Set - set of subsets

MNEMONIC: $2^{\#\text{ of elements}}$ = 2^3 = 8 subsets

A := { 1 , 2 , 3 }

2^A := { B : B ⊆ A } = { ∅ , A , {1} , {2} , {3} , {1,2} , {1,3} , {2,3} }

power set of A - <u>ALL</u> subsets of A

note: in textbooks we see this as P(A)

self-note:
We have a Set Named 'A' . Set 'A' is defined as: A:={1,2,3} therefore our set A has 3 members: 1, 2, and 3. Since Set A is NOT and 'Empty Set' it can have multiple subsets. If we take ALL of the possible subsets of A we can create another set called the 'Power Set' and denoted as 2^A. This Power Set contains as members all of the set A's possible subsets.

<u>parent-set</u>   <u>all possible subsets</u>
A:={1,2,3}   B:={1} C:={2} D:={3} E:={1,2} F:={1,3} G:={2,3}
A
{}

Def: every non-empty set has at least 2 subsets: - itself - empty set

power set 2^A := { A, {}, B, C, D, E, F, G }

CS binary representation:

FFF   TFF   FFT   TFT
{ ∅ , {1,2,3} , {1} , {2} , {3} , {1,2} , {1,3} , {2,3} }
TTT   FTF   TTF   FTT

3 variables: 1,2,3
2^3 = 8
2 choices: - present T - absent F
8 possible combinations TTT TFT etc···

Set Cardinality - set size (number of elements)

overloaded |abs val| symbol aka 'pipe' character

|A| = 3

|F ∪ M| ≠ |F| + |M|
7   4   4

Heuristic Method of Counting :

for any set S ... $|2^S| = 2^{|S|}$

compare

ex

$\lvert 2^A\rvert = 8$	$\lvert 2^{F\cup M}\rvert = 7$	$\lvert\varnothing\rvert = 0$
$\lvert F\cap M\rvert = 1$	$\lvert F\setminus M\rvert = 3$	$\lvert F\rvert - \lvert M\rvert = 0$

heuristic: enabling a person to discover or learn something for themselves ; "a hands on" or interactive heuristic approach to learning
- an approach to problem solving, practical method, not guaranteed to be optimal but sufficient to get the job done···

Universal Set U / domain

the Universal Set U is the set containing <u>everything currently under consideration</u>
- sometimes implicit
- sometimes explicitly stated
- contents depend on the context

Special Set Ω

Special Set Ω

universe of discourse , sample space , CS - scope

- all elements we are limited to ; you define it

Ω := FUM = { ... (names) }
note:
F ⊆ Ω , M ⊆ Ω , $2^F \subseteq \Omega$

Dice Roll Ω := { 1,2,3,4,5,6 }
Coin Flip Ω := { Heads , Tales }

Probability

$P(A) = \frac{|A|}{|\Omega|}$

Ω := FUM   working def for probability of A

What is the probability a "random" name is a female ?

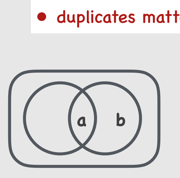
$P(F) = \frac{|F|}{|\Omega|} = \frac{4}{7}$

Ordered Pair - Tuples ()

- order matters
- duplicates matter

(a , b) := { {a} , {a,b} }

(a,b) ≠ (b,a)
(a,b) ≠ { a,b }



Cartesian Product

A × B := { (a,b) : a∈A , b∈B }

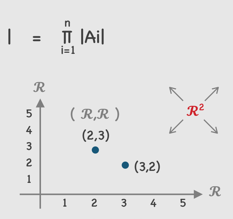
A = { 1,2 }   B = { 3,4 }

A × B = { (1,3) , (1,4) , (2,3) , (2,4) }

| A×B | = |A| · |B| = 4   if finite

A^2 = A × A = { (1,1) , (1,2) , (2,1) , (2,2) }
A^n = A × A × A × ... × A^n n∈$\mathcal{N}$

$|A^n| = |A|^n$ $\left|\prod_{i=1}^{n} A_i\right| = \prod_{i=1}^{n} |A_i|$



Cartesian Plane

set identities

propositional calculus (a.k.a. propositional logic) and set theory are both instances of an algebraic system called a *Boolean Algebra*

- the <u>operators</u> in set theory are <u>analogous</u> to the corresponding operators in propositional calculus
- as always there must be a universal set U
- all sets are assumed to be subsets of U

propositional calculus

TABLE 6 Logical Equivalences.

Equivalence	Name
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

∧ ∩
∨ ∪
¬¬ $\overline{\overline{\ }}$
T U
F ∅

set theory

TABLE 1 Set Identities.

Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$ not not A	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

ex

F ∩ Ω = F
F ∪ Ω = Ω
∅ ∩ Ω = ∅   higher order of triviality
∅ ∪ Ω = Ω
$\varnothing^C$ = Ω
F ∈ 2^Ω = Ω
F \ Ω = ∅   wipe away Ω
Ω \ F = F^C everything NOT in F
$(F^C)^C$ = F   double negation
A ∪ A^C = Ω
A ∩ A^C = ∅
A ⊆ A^C - False

$|A| + |A^C| = |\Omega|$ for |Ω| finite

$|A| = |\Omega| - |A^C|$ useful fact

{ A , A^C } where A^C := Ω\A

collectively exhaustive   mutually exclusive

$A_1, A_2, A_3, \cdots, A_n\}$: n≤∞ $A_1, A_2, A_3, \cdots, A_n\}$: n≤∞

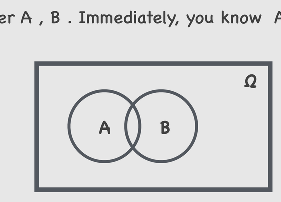
$\bigcup_{i=1}^{n} A_i = \Omega$ $A_i \cap A_j = \varnothing$ for all i≠j

all possible Ω-subsets are included   they got nothing in-common

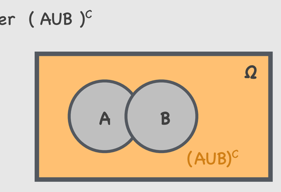
collectively exhaustive:

In probability theory and logic, a set of events is jointly or collectively exhaustive if at least one of the events must occur. For example, when rolling a six-sided die, the outcomes 1, 2, 3, 4, 5, and 6 are collectively exhaustive, because they encompass the entire range of possible outcomes.
- wikipedia.org

consider A , B . Immediately, you know A,B ⊆ Ω



consider (A∪B)^C



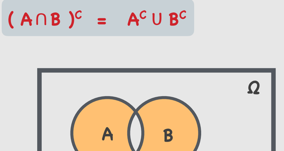
can I express this differently ?

algebra $(3 \cdot 4)^7 = 3^7 \cdot 4^7$

(A∪B)^C = ? $A^C \cup B^C$ False
$A^C \cup B^C$ = A ∩ B   True

(A∪B)^C ≠ A ∩ B

$(A \cup B)^C = A^C \cap B^C$
$(A \cap B)^C = A^C \cup B^C$ de Morgan Laws



$\mathcal{N}$:= { 1 , 2 , 3 , ··· }

aleph - countable infinity
| $\mathcal{N}$ | = $\aleph_0$ infinity that you can count to

$\mathcal{Z}$:= {···,-2,-1,0,1,2,···}
| $\mathcal{Z}$ | = $2\aleph_0 + 1$?? seams reasonable but FALSE
| $\mathcal{Z}$ | = $\aleph_0$

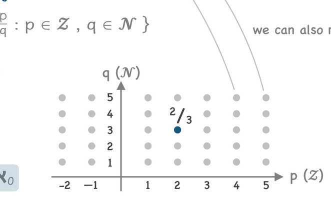
back to $\mathcal{N}$

why ? Lets map {···,-2,-1,0,1,2,···}
{1 , 2 , 3 , 4 , 5 , ···}

rational numbers
$\mathcal{Q}$:= { $\frac{p}{q}$: p ∈ $\mathcal{Z}$, q ∈ $\mathcal{N}$ }

we can also map them to $\mathcal{N}$

| $\mathcal{Q}$ | = ?



| $\mathcal{Q}$ | = $\aleph_0$

are all numbers ∈ $\mathcal{Q}$? No! e.g. √2 , π ∉ $\mathcal{Q}$

real numbers

where is √2 , π , etc... ?

rationals   irrationals
$\mathcal{R}$:= $\mathcal{Q}$ ∪ { all holes }

no holes - the continuum

[3,7] := { x: x≥3 , x≤7 } ⊂ $\mathcal{R}$
(10,17] := { x: x>10 , x≤17 } ⊂ $\mathcal{R}$

| $\mathcal{R}$ | = ?
assume | $\mathcal{R}$ | = $\aleph_0$

consider x ∈ (0,1) , b(x) = x in base 2
0.0101101010···
0.1101001110···
0.0111100000···
0.0000101111···

counting enumeration — we keep listing till infinity

lets flip the diagonal
0.1101101010···
0.1001001110···
0.0101100000···
0.0001101111···

(0,1) ⊂ $\mathcal{R}$ | (0,1) | ≤ | $\mathcal{R}$ |

0.1001 ··· we get a real number that is not equal to any of the above numbers meaning the above list is <u>incomplete</u> and we can not enumerate all of the numbers

| $\mathcal{R}$ | ≠ $\aleph_0$ | $\mathcal{R}$ | > $\aleph_0$

$\mathcal{C}$:= | $\mathcal{R}$ | uncountable infinity = the cardinality of the continuum

$\aleph_0 < \mathcal{C}$ uncountable infinity is larger than countable infinity

self-note: there are infinitely many numbers between 0 and 1. Those infinitely many numbers already lie on an infinitely long continuum. Diving into this space between 0 and 1 is like opening large pocket which contains infinitely many numbers. This could be akin to opening a portal into a different dimension. This dimension itself can have its own continuum which could mirror the original exit dimension or be akin to a nano-sized space with its own physics. Can we map those back to $\mathcal{N}$ (just like before) or are they impossible to be counted ? Having this infinitely large subspaces within the continuum makes the already infinitely large continuum even larger. This alludes to the fact that it might be impossible to count those ever increasing packets of sub-infinity. Therefore we might be looking at a different form of infinity - the uncountable infinity (which could be infinitely times larger than the countable infinity)

Probability

Probability

Ω - sample space

outcomes all possible things that can happen in a trial
Ω = { ω_1 , ω_2 , ω_3 , ... }

outcomes contained in events

events
A := {ω_1 , ω_2}
B := {ω_1 , ω_3}

e.g.

coin flip Ω = { H,T }   2 possible outcomes: heads , tails
|Ω| = 2   cardinality of Ω = number of possible outcomes

In a coin flip only Heads or Tails could happen and nothing else. Heads and Tails are mutually exclusive - (be careful outcomes are NOT sets)

{ H } , { T }   H , T
events   outcomes

{ H } ⊆ Ω   H ∈ Ω
events are subsets   outcomes are members

Sets in which all elements are outcomes are called 'events' . An event is defined by:
A ⊆ Ω → A ∈ 2^Ω

set of outcomes - event   event space i.e. all possible events

here, 2^Ω = { ∅ , { H } , { T } , { H,T } }
all must boil down to these 4 things

$P(A) = \frac{|A|}{|\Omega|}$ if Ω is finite

working def for probability of A

$P(H) = \frac{|H|}{|\Omega|}$ this does NOT compute ILLEGAL !! | member | ???

$P(\{H\}) = \frac{|\{H\}|}{|\Omega|} = 1/2$ probability of an event
or | {H,T} |

$\forall \omega_i \; P(\omega_i) = \frac{1}{|\Omega|}$ probability of an outcome

ω : lower case Omega

In probability theory, the <u>sample space</u> of an experiment or random trial is the set of all possible outcomes or results of that experiment. A sample space is usually denoted using set notation, and the possible outcomes are listed as elements in the set. It is common to refer to a sample space by the labels S, Ω, or U (for "universal set").
- wikipedia.org

In probability theory, an <u>event</u> is a set of outcomes of an experiment (a subset of the sample space) to which a probability is assigned. A single outcome may be an element of many different events, and different events in an experiment are usually not equally likely, since they may include very different groups of outcomes.
Typically, when the sample space is finite, any subset of the sample space is an event (i.e. all elements of the power set of the sample space are defined as events). However, this approach does not work well in cases where the sample space is uncountably infinite, most notably when the outcome is a real number. So, when defining a probability space it is possible, and often necessary, to exclude certain subsets of the sample space from being events.
- wikipedia.org

for the time being
Def: P: 2^Ω → [0,1]

P({H,T}) = 1
P({H} ∪ {T}) = 1
P({H} ∩ {T}) = 0   absurd - coin edge?

P(Ω) = 1   P(∅) = 0

$P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$

$P(\varnothing) = \frac{|\varnothing|}{|\Omega|} = 0$ trivial events

Complement Rule - useful triviality

$P(A^C) = \frac{|A^C|}{|\Omega|} = \frac{|\Omega| - |A|}{|\Omega|} = 1 - \frac{|A|}{|\Omega|} = 1 - P(A)$

$P(A) = 1 - P(A^C)$

$|2^\Omega| = 2^{|\Omega|}$ numb of possible events
≠ |Ω|

for Ω = { H,T }
P({H}) = 1/2   both events are equally likely
P({T}) = 1/2