11/29/16

<u>Def</u> moment generating function for r.v X:
$m_X(t) = E[e^{tX}]$ recall $e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \cdots$

Taylor series expanssion

<u>Proposition</u>

I $m_X(t) = m_Y(t) \Rightarrow X \overset{d}{=} Y$

II $m^{(k)}_X(0) = E[X^k]$

III $Y = aX + c \Rightarrow m_Y(t) = e^{tc} m_X(at)$

IV If X, Y are ind.
$m_{X+Y}(t) = m_X(t)\, m_Y(t) = (m_X(t))^2$

V $\lim_{n\to\infty} m_{X_n}(t) = m_Y(t) \Longleftrightarrow X \to Y$

$X_1, X_2 \ldots X_n$ sequence of r.v's
$\Rightarrow$ if n large $\to X_n \overset{d}{\approx} Y$
X_n is approx equally distributed as Y

$X \sim Bern(p) \to m_X(t) = 1 - p + pe^t$ (def)

$X \sim Binom(n,p) \Rightarrow m_X(t) = (1 - p + pe^t)^n$

$X \sim Geom(p) \Rightarrow m_X(t) = \frac{pe^t}{1 - e^t(1-p)}$ if $t < \ln\left(\frac{1}{1-p}\right)$

$X \sim Exp(\lambda) = m_X(t) = \frac{\lambda}{\lambda - t}$ if $t < \lambda$.

$Z \sim N(0,1) \Rightarrow m_X(t) = e^{t^2/2}$

$X \sim N(\mu, \sigma^2) \Rightarrow m_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$X \sim Deg(c) \Rightarrow m_X(t) = e^{tc}$

If $X_1, \ldots X_n$ iid some dist. with mean μ
$\Rightarrow \overline{X}_n \to \mu$
$\mu \sim Deg(\mu)$
$\lim_{n\to\infty} m_{\overline{X}}(t) = e^{t\mu}$

$$M_{\bar{X}}(t) = M_{\frac{T}{n}}(t) \overset{\text{Fact III}}{=} M_T\left(\frac{t}{n}\right)$$

$$= \underbrace{\prod_{i=1}^{n} M_X\left(\frac{t}{n}\right) = \left(M_X\left(\frac{t}{n}\right)\right)^n}_{\text{Fact IV}} \underset{\text{def mgF}}{=} E\left[e^{\frac{t}{n}X}\right]$$

Taylor Series exp e^{tX}

$$= \left(E\left[1 + \frac{tX}{n} + \frac{t^2X^2}{2!n^2} + \frac{t^3X^3}{3!n^3} + \cdots\right]\right)^n$$

We Say $f(n) = o(g(n))$ "little o"

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ $g(n)$ goes larger "faster" than $f(n)$

$n^2 = o(n^3)$

$\lim \frac{n^2}{n^3} = \lim \frac{1}{n} = 0$ ✓

$$\frac{t^2X^2}{2!n^2} + \frac{t^3X^3}{3!n^3} + \cdots = o\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \frac{\frac{t^2X^2}{2!n^2} + \frac{t^3X^3}{3!n^3} + \cdots}{1/n} = \lim_{n\to\infty} \frac{t^2X^2}{2n} + \frac{t^3X^3}{3!n^2} + \cdots = 0 \checkmark$$

$$= \left(E\left[1 + \frac{tX}{n} + o\left(\frac{1}{n}\right)\right]\right)^2 = \left(1 + \frac{tM}{n} + E\left[o\left(\frac{1}{n}\right)\right]\right)^n$$

$$= \left(1 + \frac{tM}{n} + o\left(\frac{1}{n}\right)\right)^n$$

Ignore Expectation

What that Shows $= \lim_{n\to\infty}\left(1 + \frac{tM}{n} + o\left(\frac{1}{n}\right)\right)^n = e^{tM}$ Reminder $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Does $\lim\left(1 + \frac{a}{n} + o\left(\frac{1}{n}\right)\right)^n \overset{\text{still?}}{=} e^a$ → yes

$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$

note: - The average of random variables converges to the mean
↳ Law of large numbers → $\bar{X}_n \to M$

Let $X_1 \ldots X_n \overset{iid}{\sim}$, with mean M se σ

Consider $C_n = \dfrac{\bar{X}_n - M}{\frac{\sigma}{\sqrt{n}}}$ → $\bar{X}$ standerized to have mean 0, s.e 1

when you take and standerize it it always gives you mean 0 ; SE=1

$$\frac{\sqrt{n}(\bar{X}-M)}{\sigma} \to \frac{\sqrt{n}\left(\frac{X_1+X_2 \ldots X_n}{n} - M\right)}{\sigma}$$

$$= \frac{\sqrt{n}\left(\frac{X_1 \ldots X_n}{n}\right) - \frac{nM}{n}}{\sigma} \Rightarrow \frac{\sqrt{n}\,\frac{X_1-M+X_2-M \ldots X_n-M}{n}}{\sigma}$$

$$\frac{(X_1-M)+(X_2-M) \ldots + (X_n-M)}{\sigma\sqrt{n}} \to \frac{1}{\sqrt{n}}\left(\frac{X_1-M}{\sigma} + \frac{X_2-M}{\sigma} \ldots \frac{X_n-M}{\sigma}\right)$$

let $Z_i := \dfrac{X_i - M}{\sigma} = \dfrac{1}{\sqrt{n}}(Z_1 + Z_2 + \ldots Z_n)$ → $\boxed{Z_i \text{ is } X_i \text{ standerized} \quad E[Z_i]=0,\ SE[Z_i]=1}$

$$\lim_{n\to\infty} M_{C_n}(t) = M_{C_n}(t) = M_{\frac{1}{\sqrt{n}}(Z_1+\ldots Z_n)}(t)$$

$$= \underset{\text{by III}}{M_{Z_1+\ldots Z_n}}\left(\tfrac{t}{\sqrt{n}}\right) = \underset{\text{by IV}}{M_{Z_1}}\left(\tfrac{t}{\sqrt{n}}\right) \ldots M_{Z_n}\left(\tfrac{t}{\sqrt{n}}\right) = \left(M_Z\left(\tfrac{t}{\sqrt{n}}\right)\right)^n$$

$$= \left(E\left[e^{\frac{t}{\sqrt{n}}Z}\right]\right)^n = \left(E\left[1 + \frac{tZ}{\sqrt{n}} + \frac{t^2Z^2}{2!\,n} + \underbrace{\frac{t^3Z^3}{3!\,n^{3/2}} + \ldots}_{o\left(\frac{1}{n}\right)}\right]\right)^n$$

Var[X] = E[X^2] - M^2
If M^2 = 0
Var[X] = E[X^2]
If Var[X] = 1
E=[X^2] = 1

$$= \left(E\left[1 + \frac{tZ}{\sqrt{n}} + \frac{t^2Z^2}{2n} + \ldots o\left(\frac{1}{n}\right)\right]\right)^n$$

$$= \left(1 + E\left[\frac{t^2}{\sqrt{n}}\right] + E\left[\frac{t^2Z^2}{2n}\right] + E\left[o\left(\frac{1}{n}\right)\right]\right)^n$$

$$= \left(1 + \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n$$

$$\lim_{n\to\infty} M_{C_n}(t) = \lim_{n\to\infty}\left(1 + \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n = e^{\frac{t^2}{2}} \to C_n \to N(0,1)$$

The mgf of $Z \sim N(0,1)$

CLT → central limit theorem

If $X_1 \ldots X_n$ iid w/mean M and SE σ

$$\Rightarrow \frac{\bar{X}-M}{\frac{\sigma}{\sqrt{n}}} \to Z \sim N(0,1)$$

$\Rightarrow$ If n large . . .

$$\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \underset{\text{II}}{\overset{d}{\approx}} Z \sim (0,1) \Rightarrow \bar{X} \underset{\text{III}}{\overset{d}{\approx}} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) - T \underset{\text{IV}}{\overset{d}{\approx}} N\left(\mu, (\sqrt{n}\,\sigma)^2\right)$$

general $Z \sim N(0,1) := \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

$X = \mu + \sigma Z \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}}$

example $X_1, \ldots, X_{30}$ iid Geom(p)

$p = \frac{1}{2}$

What is the probability the avg realization is more than 2.75?

$P(\bar{X} > 2.75)$

$\bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

$\mu = \frac{1}{p} = 2$

$\sigma = \sqrt{\frac{1-p}{p^2}} = \sqrt{2} \approx 1.141$

$\frac{\sigma}{\sqrt{n}} = \frac{1.41}{\sqrt{30}} \approx .250$

$P(\bar{X} > 2.75) \approx P\left(\frac{\bar{X}-2}{.250} > \frac{2.75-2}{.250}\right) \approx P(Z > 3) = 0.15$

example Random Walk 100 steps

What is the probability you are more than 10 steps away from where you started?

Let $T = X_1 + X_2 \ldots + X_{100}$

where $X_1 \ldots X_{100} \overset{iid}{\sim} \begin{cases} 1 & \text{wp } \frac{1}{2} \\ -1 & \text{wp } \frac{1}{2} \end{cases}$

For CLT, we know $T \overset{d}{\approx} N(n\mu, (\sqrt{n}\,\sigma)^2)$

$\mu = 0, \; \sigma = 1 \;\rightarrow\; T \overset{d}{\approx} N(0, 10^2)$

$P(|T| > 10) = P(T < -10) + P(T > 10)$

$= P\left(\frac{T-0}{10} < \frac{-10-0}{10}\right) + P\left(\frac{T-0}{10} > \frac{10-0}{10}\right)$

$P(Z \in [-1, 1]) = 32\%$