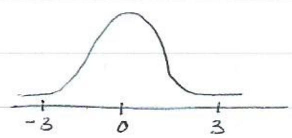
Math 241   11/23/16   A. Kapelner   Lec #19

answers q 3a HW7

$$\begin{cases} Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ Supp[Z] = \mathbb{R} \\ E[Z] = 0 \\ SE[Z] = 1 \end{cases}$$



$$X \sim N(\mu, \sigma^2) = \frac{1}{2\pi} e^{\frac{1}{}(x-\mu)^2}$$

1)

- X is the random variable model for male height.
- X is norm. distributed with mean 70" and s.e. 4".
- What is the probability of male is 78" + taller.

i) $X \sim N(70'', 4''^2)$

ii) $p(x \geq 78'') = p\left(\frac{x - 70''}{4''} \geq \frac{78''-70''}{4''}\right)$ → probability statement.

$= p(Z > 2) = 2.5\%$

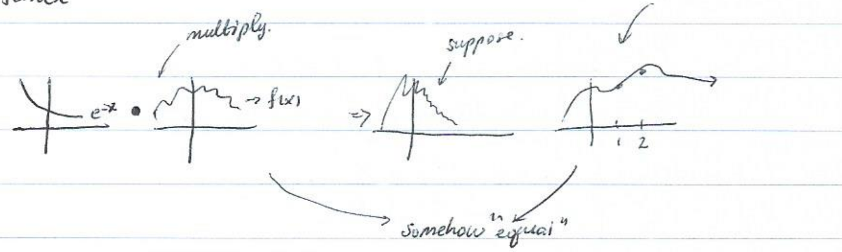
Bilateral Laplace Transform.   Note: [correction] $L(-t)$

for $f(x)$ let $L(t) = \int_{\mathbb{R}} e^{-tx} f(x) dx$

for $t=1$

for $t=2$

...



Theorem: If $L(t)$ exists, then $L(t)$ and $f(x)$ are one-to-one.

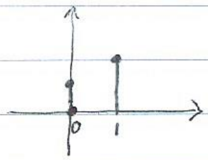
$E[X] = \int_{\mathbb{R}} x f(x) dx$ $E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f(x) dx$ if continuous

$\sum_{x \in Supp[X]} e^{tx} p(x)$ if discrete

Mass Generating Function : $M_X(t) := E[e^{tX}]$

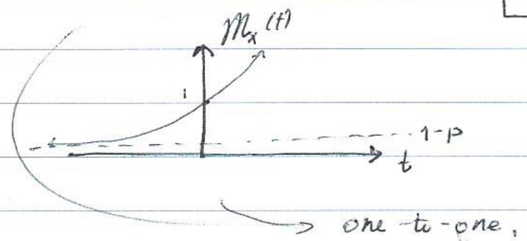
Theorem:

$X \sim Bern(p)$



$$M_X(t) = E[e^{tX}]$$
$$= \sum_{x \in Supp[X]} e^{tx} p(x)$$
$$= e^{t(0)} p(0) + e^{t(1)} p(1)$$
$$= \boxed{1-p-pe^t}$$

// X got "integrated out".



Consider: X is binomial (n, p)

Find: $E[X^{17}] = \sum_{x=0}^{n} x^{17} \binom{n}{x} p^x (1-p)^{n-x}$ // algebraically impossible to solve.

Recall the Taylor Series of $f(x)$

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots$$

$$f(x) \approx f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 \Rightarrow \text{2nd degree approx.}$$

Let $f(x) = e^x$ $\quad x=0 \Rightarrow c=0$

$$e^x = e^0 + \frac{e^0}{1!}x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \ldots$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Taylor expansion for a r.v. (tX):

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \ldots$$

$$M_X(t) = E(e^{tX}) = E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \ldots\right]$$

$$M'_X(t) = \frac{d}{dt}[M_X(t)] = \frac{d}{dt}E[\ ] = E\left[\frac{d}{dt}[\ ]\right] = E\left[X + \frac{tX^2}{1!} + \frac{t^2X^3}{2!} + \frac{t^3X^4}{3!} + \ldots\right]$$

$$M'_X(0) = E[X]$$

$$M''_X(t) = E\left[X^2 + tX^3 + \frac{t^2X^4}{2!} + \ldots\right]$$

$$M''_X(0) = E[X^2] \quad \ldots \quad M'''_X(0) = E[X^3] \ldots \quad \boxed{M_X^{(k)}(0) = E[X^k]}$$

"returns k^{th} moment"

let $Y = aX + c$ Linear Transformation of R.V.

$$M_Y(t) = E[e^{tY}] = E[e^{t(aX+c)}] = E[e^{taX} e^{tc}] = e^{tc} E[e^{taX}] =$$

let $t' := at$

$$= e^{tc} E[e^{t'X}] = e^{tc} M_X(t') = \boxed{e^{tc} M_X(at) = M_Y(t)}$$

Suppose

X_1, X_2 are independent random variables.

Let $Y = X_1 + X_2$

$$M_Y(t) = E[e^{tY}] = E[e^{t(X_1 + X_2)}] = E[e^{tX_1} \cdot e^{tX_2}] = E[e^{tX_1}] E[e^{tX_2}] =$$

$$= \boxed{M_{X_1}(t) M_{X_2}(t) = M_Y(t)}$$

$$\left\{ \begin{array}{l} \text{If } X_1, X_2 \text{ are independent and identically distr. (iid)} \\ \text{let } Y = X_1 + X_2 \\ \text{Then } M_Y(t) = M_{X_1}(t) M_{X_2}(t) = (M_X(t))^2 \end{array} \right.$$

I) Suppose $X_1, \dots, X_n \overset{iid}{\sim} Bern(p)$

Let $T = X_1 + \dots + X_n \sim Binom(n, p)$

Then $M_T(t) = (M_Y(t))^n = (1 - p + pe^t)^n$

II) Suppose $X \sim Geom(p)$

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} (1-p)^{x-1} p = p \sum_{x=0}^{\infty} e^{tx} (1-p)^{x-1} \frac{1-p}{1-p} = \frac{p}{1-p} \sum_{x=0}^{\infty} \underbrace{e^{tx}}_{(e^t)^x} (1-p)^x$$

$$= \frac{p}{1-p} \sum_{x=1}^{\infty} (e^t(1-p))^x = \frac{p}{1-p} \left(\sum_{x=0}^{\infty} (e^t(1-p))^x - 1 \right) = \frac{p}{1-p} \left(\frac{1}{1 - e^t(1-p)} - 1 \right)$$

$$= \frac{p}{1-p} \frac{e^t(1-p)}{1 - e^t(1-p)} = \boxed{\frac{p}{1 - e^t(1-p)} \text{ if } t < \ln\left(\frac{1}{1-p}\right)}$$

III) Suppose $X \sim Exp(\lambda)$;

$$M_X(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} \underbrace{e^{(t-\lambda)x}}_{e^{(t-\lambda)x}} dx = \frac{\lambda}{t-\lambda} \left[e^{(t-\lambda)x} \right]_0^{\infty} = \frac{\lambda}{t-\lambda} \left(\lim_{x \to \infty} e^{(t-\lambda)x} - 1 \right) =$$

$$= \frac{\lambda}{t-\lambda}(0 - 1) = \boxed{\frac{\lambda}{\lambda - t} \text{ if } t < \lambda} \quad \text{if } t - \lambda < 0 \Rightarrow t < \lambda$$

$X \sim Exp(\lambda)$

$Y = aX \sim ?$

$$\mathcal{M}_Y(t) = e^{tc} \mathcal{M}_X(at) = \frac{\lambda}{\lambda - at} \cdot \frac{\frac{1}{a}}{\frac{1}{a}}$$

$$= \frac{\lambda/a}{\frac{\lambda}{a} - t} = \frac{\lambda'}{\lambda' - t} \Rightarrow Y \sim Exp(\lambda')$$

let $\lambda' = \frac{\lambda}{a}$ $\Rightarrow Y \sim Exp\left(\frac{\lambda}{a}\right)$

$Z \sim N(0,1)$,

$$\mathcal{M}_Z(t) = E[e^{tZ}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2 + tx} dx = \text{[algebraic manipulation]}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-t)^2 + \frac{t^2}{2}} dx = e^{t^2/2} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dt}_{1} \longrightarrow \text{density for } N(t, 1^2)$$

$$= \boxed{e^{\frac{t^2}{2}}}$$

$X \sim N(\mu, \sigma^2)$

$X = \mu + \sigma Z$

$\mu \to t$, $\sigma \to a$

$$\mathcal{M}_X(t) = e^{t\mu} \mathcal{M}_Z(\sigma t) = e^{t\mu} e^{\frac{\sigma^2 t^2}{2}} := \boxed{e^{t\mu + \frac{\sigma^2 t^2}{2}}}$$

Preview:

LLN.

$\bar{X} \to \mu$.