11/17/16

Definition: Moment Generating Function

$$M_X(t) := E[e^{tX}], \quad e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \dots$$

Taylor series expansion

- I. $M_X(t) = M_Y(t) \Rightarrow X \overset{d}{=} Y$
- II. $E[X^k] = M_X^{(k)}(0)$
- III. ~~mixing~~ $Y = aX + c \Rightarrow M_Y = e^{tc} M_X(at)$
- IV. If X, Y independent, $M_{X+Y}(t) = M_X(t) M_Y(t) \overset{\text{if iid}}{=} (M_Y(t))^2$

$X \sim Bern(p) \Rightarrow M_X(t) = 1 - p + pe^t$

$X \sim Binom(n, p) \Rightarrow M_X(t) = (1 - p + pe^t)^n$

$X \sim Geom(p) \Rightarrow M_X(t) = \frac{pe^t}{1 - e^t(1-p)}$ if $t < \ln\left(\frac{1}{1-p}\right)$

$X \sim Exp(\lambda) \Rightarrow M_X(t) = \frac{\lambda}{\lambda - t}$ if $t < \lambda$

$Z \sim N(0,1) \Rightarrow M_Z(t) = e^{t^2/2}$

$X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$X \sim Deg(c) \Rightarrow M_X(t) = e^{tc}$

V. Levy's Continuity Theorem

$X_1, X_2, \dots$ is a sequence of r.v.

$\lim_{n \to \infty} M_{X_n}(t) = M_Y(t) \iff X_n \to Y$ converges

If n large, $M_{X_n}(t) \approx M_Y(t) \Rightarrow X_n \overset{d}{\approx} Y$

Law of Large Numbers

$X_1, \dots, X_n \overset{iid}{\sim}$ with mean μ

$\Rightarrow \bar{X}_n \to \mu$

$\sim Deg(\mu)$

$M_{\bar{X}_n}(t) = M_{\frac{T}{n}}(t) \overset{\text{III}}{=} M_T\left(\frac{t}{n}\right) = M_{X_1 + \dots + X_n}\left(\frac{t}{n}\right) \overset{\text{IV}}{=} \left(M_X\left(\frac{t}{n}\right)\right)^n$

$M_X(t) := E[e^{tX}]$

by def. of m.g.f $= \left(E\left[e^{\frac{t}{n}X}\right]\right)^n \overset{\text{Taylor series}}{=} \left(E\left[1 + \frac{t}{n}X + \frac{t^2X^2}{2!\,n^2} + \frac{t^3X^3}{3!\,n^3} + \dots\right]\right)$

- We say $f(n) = o(g(n))$ ("little 'o'") if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$

  $g(n)$ process "faster" than $f(n)$.

Ex: is $n^2 = o(n^3)$?

$\lim \frac{n^2}{n^3} = \lim_{n \to \infty} \frac{1}{n} = 0$ ✓

$$\frac{t^2x^2}{2n^2}+\frac{t^3x^3}{3!n^3}+\cdots = o\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty}\frac{\{}{\frac{1}{n}} = \lim_{n\to\infty}\frac{t^2x^2}{2n}+\frac{t^3x^3}{3!n^3}+\cdots = 0$$

$$=\left(E\left[1+\frac{t}{n}x+\frac{t^2x^2}{2!n^2}+\frac{t^3x^3}{3!n^3}+\cdots\right]\right)^n=\left(E\left[1+\frac{t}{n}x+o\left(\frac{1}{n}\right)\right]\right)^n=\left(1+\frac{t\mu}{n}+E\left[o\left(\frac{1}{n}\right)\right]\right)^n$$

$$=\left(1+\frac{t\mu}{n}+o\left(\frac{1}{n}\right)\right)^n$$

ignore expectation

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e \qquad \lim_{n\to\infty}\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n=e$$

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n=e^a$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}+o\left(\frac{1}{n}\right)\right)^n=e \quad \text{AND} \quad \lim_{n\to\infty}\left(1+\frac{a}{n}+o\left(\frac{1}{n}\right)\right)^n=e^a}$$

$$\lim_{n\to\infty}M_{\bar{X}_n}(t)=\lim_{n\to\infty}\left(1+\frac{t\mu}{n}+o\left(\frac{1}{n}\right)\right)^n=e^{t\mu}$$

$$\Rightarrow \boxed{\bar{X}\to\mu}$$

If $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ with mean μ and s.e. σ

What does $G_n = \dfrac{\bar{X}_n-\mu}{\frac{\sigma}{\sqrt{n}}}$ look like as n gets large? ← $\bar{X}$ standardized to have mean 0 and S.E 1

$$G_n=\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}=\frac{\sqrt{n}\left(\frac{X_1+X_2+\cdots+X_n}{n}-\mu\right)}{\sigma}=\frac{\frac{\sqrt{n}\,X_1+\cdots+X_n-\mu n}{n\sqrt{n}}}{\sigma}=\frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}}$$

$$=\frac{(X_1+\cdots+X_n)-(\mu+\cdots+\mu)}{\sigma\sqrt{n}}=\frac{(X_1-\mu)+(X_2-\mu)+\cdots+(X_n-\mu)}{\sigma\sqrt{n}}=\frac{1}{\sqrt{n}}\left(\frac{X_1-\mu}{\sigma}+\frac{X_2-\mu}{\sigma}+\cdots+\frac{X_n-\mu}{\sigma}\right)$$

$$\text{let } Z_i=\frac{X_i-\mu}{\sigma} \quad =\frac{1}{\sqrt{n}}(Z_1+\cdots+Z_n)$$

$E[Z]=0$

$\sigma[Z]=1$

$E[X^2]=1$

$Var[X]=E[X^2]-\mu^2$

if $\mu=0$

$\Rightarrow \sigma^2=E[X^2]$

if $\sigma=1$

$\Rightarrow E[X^2]=1$

$$M_{G_n}(t)=M_{\frac{1}{\sqrt{n}}(Z_1+\cdots+Z_n)}(t)=M_{Z_1+\cdots+Z_n}\left(\frac{t}{\sqrt{n}}\right)=\underset{\text{if iid}}{\left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)^n}=\left(E\left[e^{\frac{t}{\sqrt{n}}Z}\right]\right)^n$$

$$E\left[\left(1+\frac{t}{\sqrt{n}}Z+\frac{t^2Z^2}{2!n}+\underbrace{\frac{t^3Z^3}{3!n^{3/2}}+\frac{t^4Z^4}{4!n^2}}_{o(\frac{1}{n})}\right)\right]^n$$

$$E\left[\left(1+\frac{tZ}{\sqrt{n}}+\frac{t^2Z^2}{2n}+o\left(\frac{1}{n}\right)\right)\right]^n=\left(1+\frac{t^2}{2n}+o\left(\frac{1}{n}\right)\right)^n$$

$$E\left[\frac{tZ}{\sqrt{n}}\right]=\frac{t}{\sqrt{n}}E[Z]=0, \quad E\left[\frac{t^2Z^2}{2n}\right]=\frac{t^2}{2n}=E[Z^2]=\frac{t^2}{n}$$

$$\lim_{n\to\infty} M_{C_n}(t) = \lim_{n\to\infty}\left(1+\frac{t^2}{2n}+o\left(\frac{1}{n}\right)\right)^n = e^{\frac{t^2}{2}}$$

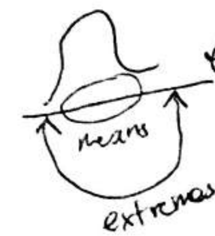
$$\Rightarrow \boxed{C_n \to Z \sim N(0,1)}$$

Central Limit Theorem CLT

So... if $X_1, X_2, \dots, X_n \overset{iid}{\sim}$ with mean μ and S.E. σ,

$$C_n = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \longrightarrow Z \sim N(0,1)$$

If n is large, $C_n \overset{d}{\approx} Z \sim N(0,1)$ $\quad \mu + \sigma Z \sim N(\mu, \sigma^2)$

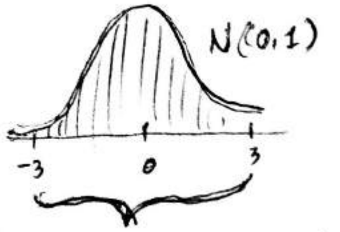


e^{x^2}

I. $C_n \overset{d}{\approx} Z \sim N(0,1)$

$\rightarrow$ II. $\bar{X} \overset{d}{\approx} N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

$\rightarrow$ III. $T \overset{d}{\approx} N\left(n\mu, (\sqrt{n}\sigma)^2\right)$



Normal curve — Defines what normal is inside the curve

EX: Take 100 random equally likely L or R steps. What is the probability you're more than 10 steps away from where you began?

Let $T = X_1 + X_2 + \dots + X_{100} \overset{\text{by CLT}}{\approx} N\left(n\mu, (\sqrt{n}\sigma)^2\right)$

$X_1, X_2, \dots, X_{100} \overset{iid}{\sim} \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$ $\quad N(0, 10^2)$

$\mu = 0$

$\sigma = \sqrt{\sigma^2} = \sqrt{1} = 1$

$$P(|T| > 10) = P(T < -10) + P(T > 10)$$

$$= P\left(\frac{T-0}{10} < \frac{-10-0}{10}\right) + P\left(\frac{T-0}{10} > \frac{10-0}{10}\right)$$

$$\approx P(Z < -1) + P(Z > 1) = P(Z \notin [-1, 1]) = 32,$$

EX2: $X_1, \dots, X_{30} \overset{iid}{\sim} \text{Geom}(p = \frac{1}{2})$

What is the prob. the average stopping time is more than 2.75?

$\bar{X}_{30} \approx N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$ by the CLT

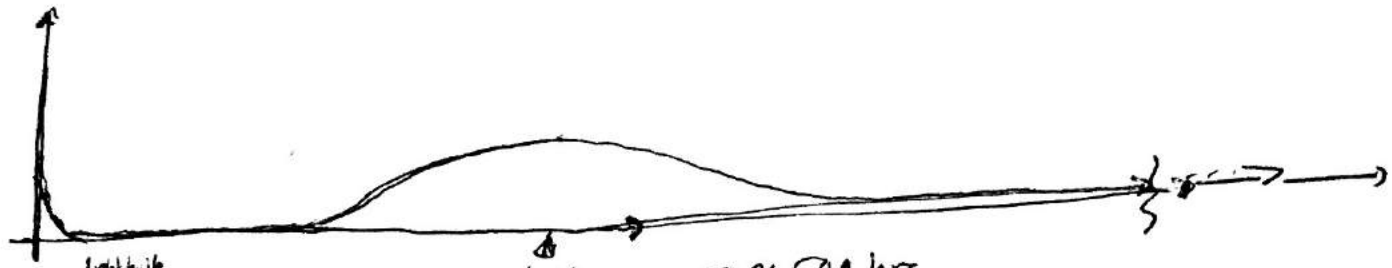
$\mu = \frac{1}{p} = \frac{1}{\frac{1}{2}} = 2$

$\sigma = \sqrt{\frac{1-p}{p^2}} \approx 1.414$

$\frac{\sigma}{\sqrt{n}} \approx \frac{1.414}{\sqrt{30}} \approx .258$

$$P(\bar{X} > 2.75) = P\left(\frac{\bar{X}-2}{.258} > \frac{2.75-2}{.258}\right) \overset{\text{by CLT}}{\approx} P(Z > 3) = .0015$$

ex 3: <u>Light Bulb failure times</u>



$\mu = 1000$ hrs $\quad \sigma \approx 500$ hrs.

$f_{\text{lightbulb}}(t)$ not NORMAL

You buy 50 lightbulbs, What is probability they last on average more than 1300 hrs?

$$\bar{X} \approx N\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) = N\left(1000, \left(\frac{500}{\sqrt{50}}\right)^2\right)$$

by CLT

$$\frac{500}{\sqrt{50}} \Downarrow 70.7$$

$$P(\bar{X} > 1300) = P\left(\frac{\bar{X} - 1000}{70.7} > \frac{1300-1000}{70.7}\right)$$

$$\approx P(Z > 4.24) \approx 0$$