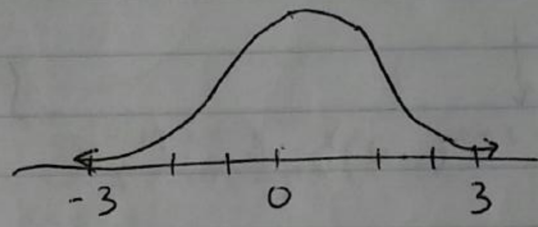
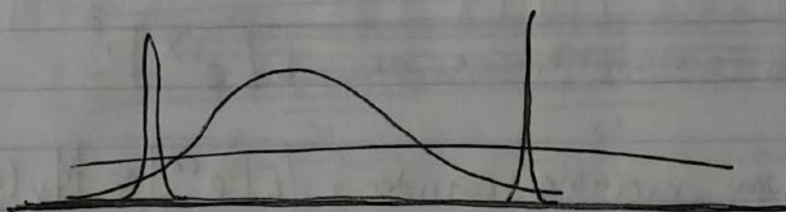
Lecture 19: November 22, 2016

- $Z \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ (Answer to 3a)

$Supp[Z] = \mathbb{R}$
$E[Z] = 0$
$SE[Z] = 1$



- $X \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\mu\sigma^2}} e^{\frac{1}{2\sigma^2}(x-\mu)^2}$ → Allows us to center curve wherever we want



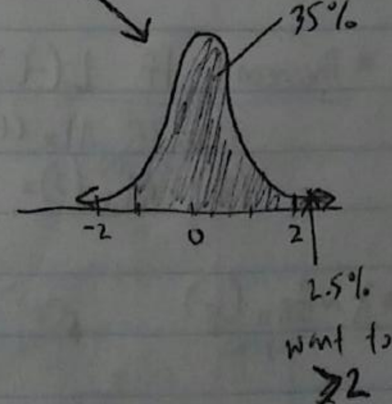
- X is random variable model for male height. X is normally distributed with mean 70 inches & SE 4 inches. What is probability that a male is more than 78 inches tall?

$\boxed{Z = \frac{X-\mu}{\sigma} \sim N(0,1)}$

$X \sim N(70, 4^2) := P(X \geq 78) := P\left(\frac{X-70}{4} \geq \frac{78-70}{4}\right)$

$\Downarrow$

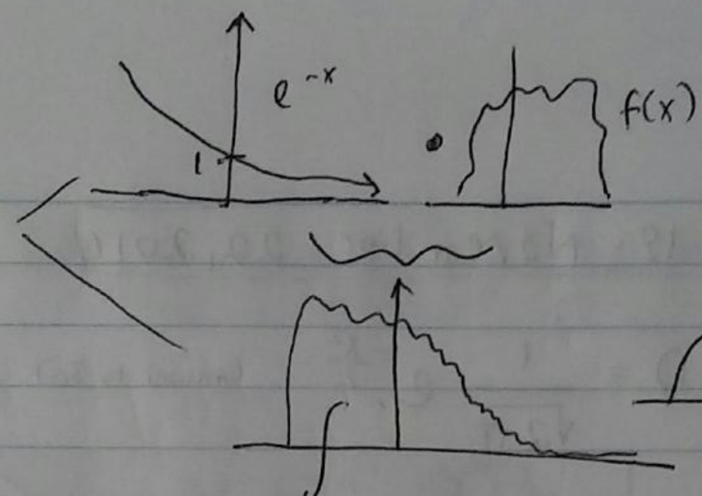
$\boxed{2.5\%} \Leftarrow P(Z > 2)$

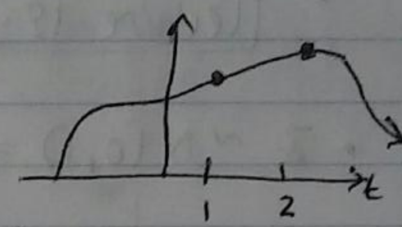


*Why should height be normally distributed?

- fn. $f(x)$, let $\underbrace{L(t) := \int_{\mathbb{R}} e^{+tx} f(x)\,dx}_{\text{Bilateral Laplace Transform}}$

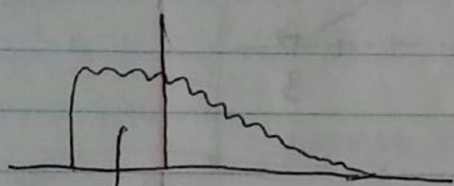
* $L(-1) = \int_{\mathbb{R}} e^{+x} f(x)\,dx$





* $L(-2) = \int_{\mathbb{R}} e^{+2x} f(x)\,dx$

$\int dx =: 7$



$\int dx := 7$

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* Theorem: If $L(-t)$ exists, $L(-t)$ & $f(x)$ are $\underbrace{\text{one-to-one}}$.

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• X is a random variable. Consider $\boxed{E[e^{tX}] \Leftarrow M_X(t)}$

$E(X) = \int_{\mathbb{R}} x f(x)\,dx$ (cts)

$E[X] = \sum_{x \in Supp[X]} x p(x)$ discrete

if continuous: $\int_{\mathbb{R}} e^{tX} f(x)\,dx$

if discrete: $\sum_{x \in Supp(X)} e^{tX} p(x)$

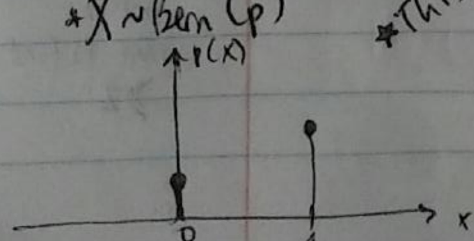
Moment Generating Function (mgf) of X

* Theorem: If $L(-t)$ exits, $L(-t)$ & $f(x)$ are one-to-one.

(I) If $M_X(t) = M_Y(t) \Rightarrow X' = Y$

(II) $M_X^{(k)}(0) = E[X^k]$

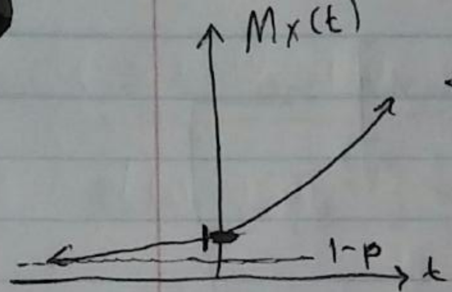
* $X \sim Bern(p)$ *This



$M_X(t) = E[e^{tX}] = \sum_{x \in Supp(X)} e^{tX} p(x)$

$= e^{t(0)} p(0) + e^{t(1)} p(1)$

$= (1-p) + pe^t = \boxed{1-p+pe^t}$



* & this are equal & are one-to-one.

*Consider $X \sim Binomial(n,p)$

~~[illegible]~~ Find $E[X^{17}] = \underbrace{\sum_{x=0}^{n} x^{17} \binom{n}{x} p^x (1-p)^{n-x}}_{\text{impossible, really hard to solve}}$

Recall the Taylor Series of $f(x)$ near $x=c$.

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots$$

$\Downarrow$

can say

$$\underbrace{f(x) \approx f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots}_{\text{called a 2nd order approximation}}$$

Let $f(x) = e^x$ near $x=0 \ni c=0$

$$e^x = e^0 + \frac{e^0}{1!}x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \dots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad e := \sum_{i=0}^{\infty} \frac{1}{i!}$$

$\Downarrow$

$$* \ e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \dots *$$

$$M_X(t) := E[e^{tX}] := E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \dots\right]$$

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going to mark this as ♥

$$* \frac{d}{dt}\left[M_X(t)\right] = \frac{d}{dt} E\Big[\overbrace{1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \dots}^{\text{going to mark this as } \heartsuit}\Big]$$

$$\frac{d}{dt}\int e^{tX} f(x)\,dx$$

$$E\left[X + \frac{tX^2}{1!} + \frac{t^2X^3}{2!} + \frac{t^3X^4}{3!} + \dots\right] \Leftarrow E\left[\frac{d}{dt}[\heartsuit]\right]$$

$\underline{M'_X(0) = E[X]}$

$$M''_X(t) = E\left[0 + X^2 + tX^3 + \frac{t^2X^4}{2!} + \dots\right]$$

$\underline{M''_X(0) = E[X^2]}$

$$M'''_X(t) = E\left[0 + X^3 + tX^4 + \dots\right]$$

$$M'''(0) = E[X^3]$$

leads us to rule # II

$$\boxed{M_X^{(k)}(0) = E[X^k]}$$

• let $Y = aX + c$ [move it over by c, expand or contract by a]

$$M_Y(t) = E[e^{tY}] = E[e^{t(aX+c)}] = E[e^{taX}e^{tc}]$$

$$\Downarrow$$

$$e^{tc}M_X(t') = \boxed{e^{tc}M_X(at)} = e^{tc}E[e^{t'X}] \Leftarrow e^{tc}E[e^{taX}]$$

let $t' = at$

$$e^{tc}M_X(at) \Downarrow M_Y(t)$$

Leads us to Rule # III

$$\boxed{M_Y(t) = e^{tc}M_X(t')}$$

• X_1 & X_2 are independent random variables. let $Y = X_1 + X_2$

$$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}] = E[e^{tX_1} + e^{tX_2}]$$

$$\Downarrow$$

$$\boxed{M_{X_1}(t)\, M_{X_2}(t)} \Leftarrow E[e^{tX_1}]\, E[e^{tX_2}]$$

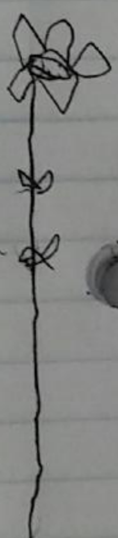
leads us to rule # IV

$$M_Y(t) = M_{X_1}(t)\, M_{X_2}(t)$$

If $X_1, X_2 \overset{iid}{\sim}$

$$Y = X_1 + X_2$$

$$M_Y(t) = M_{X_1}(t)\, M_{X_2}(t) = (M_X(t))^2$$



• $X_1, \dots, X_n \overset{iid}{\sim} Bern(p)$

$$T = X_1 + \dots + X_n \sim Binomial(n,p)$$

what is $M_T(t) = ?$

By rule #4, b/c iid & summing then:

$$M_T(t) = (M_Y(t))^n = \boxed{(1-p+pe^t)^n}$$

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a} \quad \text{if } a \in (0,1)$$

• $X \sim Geom(p)$

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx}(1-p)^{x-1}p \Rightarrow p\sum_{x=0}^{\infty} e^{tx}(1-p)^{x-1}\frac{1-p}{1-p}$$

$$\Downarrow$$

$$\frac{p}{1-p}\left(\sum_{x=0}^{\infty}(e^t(1-p))^x - 1\right) \Leftarrow \frac{p}{1-p}\sum_{x=1}^{\infty}(e^t(1-p))^x \Leftarrow \frac{p}{1-p}\sum_{x=0}^{\infty} \underbrace{e^{tx}}_{(e^t)^x}(1-p)^x$$

↳ if $e^t(1-p) < 1 \Rightarrow e^t < \frac{1}{1-p} < t < \ln\left(\frac{1}{1-p}\right)$

$$\Downarrow$$

$$\frac{p}{1-p}\left(\frac{1}{1-e^t(1-p)} - 1\right) \Rightarrow \frac{p}{\cancel{1-p}}\,\frac{e^t\cancel{(1-p)}}{1-e^t(1-p)} = \boxed{\frac{p}{1-e^t(1-p)} \quad \text{if } t < \ln\left(\frac{1}{1-p}\right)}$$

$X \sim Geom(p)$

- $\underline{X \sim Exp(\lambda)}$ [redacted]

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx \Rightarrow \lambda \int_0^\infty e^{(t-\lambda)x}\,dx$$

$$\frac{\lambda}{t-\lambda}(0-1) = \boxed{\frac{\lambda}{\lambda-t} \text{ if } t<\lambda} \Leftarrow \frac{\lambda}{t-\lambda}\left(\lim_{x\to\infty} e^{(t-\lambda)x} - 1\right) \Leftarrow \frac{\lambda}{t-\lambda}\left[e^{(t-\lambda)x}\right]_0^\infty$$

$$\lim_{x\to\infty} e^{-x} = 0$$

(if $t-\lambda \le 0 \Rightarrow t<\lambda$)

$$\lim_{x\to\infty} e^{-32x} = 0 \qquad \lim_{x\to\infty} e^{x} = \infty$$

$$\lim_{x\to\infty} e^{ax} \begin{cases} 0 & \text{if } a \text{ is } - \\ \infty & \text{if } a \text{ is } + \end{cases}$$

- $\underline{X \sim Exp(\lambda) \quad Y = aX \sim ?}$

$$M_Y(t) = e^{tc} M_X(at) = \frac{\lambda}{\lambda - at} \cdot \frac{\frac{1}{a}}{\frac{1}{a}} = \frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t} = \frac{\lambda'}{\lambda' - t}$$

let $\lambda' = \frac{\lambda}{a}$

$Y \sim Exp(\lambda')$

$Y \sim Exp(\frac{\lambda}{a})$

- $\underline{Z \sim N(0,1)}$, $M_Z(t) = E(e^{tZ}) = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$

$$\Downarrow$$

$$\cancel{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{1}{2}(x-t)^2}} \Leftarrow \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2 + tx}\,dx$$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-t)^2 + \frac{t^2}{2}}\,dx \Rightarrow \cancel{\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}}\ e^{\frac{t^2}{2}} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2}\,dt}_{f(x) - \text{density} - \text{for } N(t, 1^2)}$$

this integral is 1

So...

$$= \boxed{e^{\frac{t^2}{2}}}$$

$$\left(-\frac{1}{2}x^2 + tx = -\frac{1}{2}(x^2 - 2tx)\right)$$

Note:

$(x-t)^2 =$

$x^2 - 2xt + t^2$

$(x-t)^2 - t^2 =$

$x^2 - 2tx$

$\downarrow$

$\frac{1}{2}\left((x-t)^2 - t^2\right)$

$\downarrow$

$-\frac{1}{2}(x-t)^2 + \frac{t^2}{2}$

- $\underline{X \sim N(\mu, \sigma^2)}$

we know $X = \underbrace{\mu}_{c} + \underbrace{\sigma}_{a} Z$

$$M_X(t) = e^{t\mu} M_Z(\sigma t) = e^{t\mu} e^{\frac{\sigma^2 t^2}{2}} := \boxed{e^{t\mu + \frac{\sigma^2 t^2}{2}}}$$

$\boxed{\text{• Next class go over LLN } \bar{X} \to \mu}$