MATH 241 Spring 2015 Homework #9

Professor Adam Kapelner

Due 5PM, Tuesday, Apr 28, 2015

(this document last updated Thursday 23rd April, 2015 at 3:14pm)

Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out". Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. But for this homework set, read about continous r.v.'s: their properties (definition, expectation, variance, etc), the exponential r.v., the uniform r.v. and most importantly the normal / Gaussian r.v.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]", purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems. If the problem asks you for a computation, **round to two or three decimals (do not answer in an exact fraction).**

This homework is worth 100 points but the point distribution will not be determined until after the due date. Late homework will be penalized **10 points per day** up to a maximum of 50 points. Read more about this policy in the syllabus.

Between 1–15 points are arbitrarily given as a bonus (conditional on quality) if the homework is typed using LaTeX (15 points only if it is perfect LaTeX). Links to instaling LaTeX and program for compiling LaTeX are found on the syllabus (please read carefully).

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. **Handing it in without this printout is NOT ACCEPTABLE.** There is also a redo policy which you can read about in the syllabus.

Keep this page printed for your records (if using LaTeX, this page will not show but a shortened header appears). Write your name and section below where section A is if you're registered for the 9:15AM–10:30AM lecture and section B is if you're in the 12:15PM-1:30PM lecture.

NAME: ______________________________ SECTION (A or B): ______

Fundamentals of Continuous r.v.'s We will learn about this other type of r.v.

Problem 1

This problem will focus on the continuous exponential r.v. and you will see how it's built from the discrete geometric r.v.

(a) [easy] Let $X \sim \text{Geometric}\,(p)$ and use t to indicate the free variable. In each unit of time, we have n experiments now. Write the PMF for this r.v.

(b) [easy] Let $n \to \infty$ and $p \to 0$ but keep thier product pinned at the constant $\lambda = np$. Show that the PMF of this new r.v. T is zero everywhere.

(c) [harder] Find the CDF of T by taking the same limit as the last problem.

(d) [easy] Let $\lambda = 2.92$. What is $\mathbb{P}\,(T = 2)$?

(e) [easy] Let $\lambda = 3.12$. What is $\mathbb{P}\,(T \leq 2)$?

(f) [easy] Let $\lambda = 4.56$. What is $\mathbb{P}\,(T \in [2, 2.7])$?

(g) [easy] What is $\text{Supp}[T]$?

(h) [easy] What is $|\text{Supp}[T]|$? That is, what is the size of this set?

(i) [easy] What is the parameter space of T?

(j) [easy] Run the following in `R`. It will generate 5 realizations from $T_1, \ldots, T_t \overset{iid}{\sim} \text{Exp}(\lambda = 6.56)$: `rexp(5, 6.56)` and write them below.

(k) [easy] Look at the first draw. Is this number really a draw? Or is it rounded?

(l) [difficult] Assume it's rounded from the decimal after the last decimal you see. Find the probability the computer spits out that number when realizing the r.v.

(m) [easy] Derive the PDF $f(t)$ from the CDF.

(n) [easy] Let $\lambda = 4.56$. Compute $p(0.1)$ using the PMF and $f(0.1)$ using the PDF.

(o) [harder] Interpret the PDF at 0.1, $f(0.1)$. What does that number mean?

(p) [harder] In the last problem you got an answer greater than 1. Why should it be possible that the PDF can yield numbers greater than 1?

(q) [difficult] Derive the CDF from the PDF. This will involve anti-differentiation. And you have to worry about the constant of integration and solve for it (somehow). Justify how you solve for the constant to arrive at the same CDF you found in (c).

(r) [easy] Run the following lines in `R` *one at a time* which will plot the PDF for $T \sim \text{Exp}(0.1)$, $T \sim \text{Exp}(1)$, $T \sim \text{Exp}(10)$ and $T \sim \text{Exp}(100)$. Print out this image and attach it to your homework.

```
par(mfrow = c(2, 2))
ts = seq(0, 4, 0.01)
plot(ts, dexp(ts, 0.1), type = "l", ylim = c(0, 1))
plot(ts, dexp(ts, 1), type = "l", ylim = c(0, 1))
plot(ts, dexp(ts, 10), type = "l", ylim = c(0, 1))
plot(ts, dexp(ts, 100), type = "l", ylim = c(0, 1))
#last line placeholder
```

How do you design an exponential r.v. to give large numbers — should λ be big or small?

(s) [easy] Let $\lambda = 4.56$, compute $\mathbb{P}(T \in [0,1])$ using integration on the PDF.

(t) [easy] Let $\lambda = 4.56$, compute $\mathbb{P}(T \in [0,1])$ using a difference of CDF values.

(u) [easy] What theorem describes why the last two problems should be equal? Write the statement of this theorem below.

(v) [easy] Let $\lambda = 4.56$, what is $\mathbb{P}(T = 1)$ using the integral definition of probability?

(w) [difficult] Let $T \sim \text{Exp}(\lambda)$. Show that $\mathbb{E}[T] = 1/\lambda$ using the definition of expectation for continuous r.v.'s.

(x) [easy] Let's say $\lambda = 2$, what is $\mathbb{E}[T]$? Pretend the unit of time is seconds.

(y) [difficult] Pretend you are approximating the exponential with a geometric r.v.. Let $n = 1000$ and $p = 0.002$ to have $\lambda = 2$. Show that the expectation of that geometric r.v. is the same as $\mathbb{E}[T]$ in the previous problem.

(z) [harder] For discrete r.v.'s, $\mathbb{E}[X] := \sum_{x \in \text{Supp}[X]} xp(x)$. For continuous r.v.'s, $\mathbb{E}[X] := \int_{x \in \text{Supp}[X]} xf(x)dx$. Why is the formula different?

(aa) [difficult] Let $T \sim \text{Exp}(\lambda)$. Show that $\mathbb{V}\text{ar}[T] = 1/\lambda^2$ using the definition of expectation for continuous r.v.'s. Very googlable.

(bb) [easy] Let $T \sim \text{Exp}(\lambda)$. Show that $\mathbb{SE}[T] = 1/\lambda$ *by assuming the answer of the last question*. I marked it easy for this reason.

(cc) [easy] Assume $T \sim \text{Exp}(\lambda)$ and $Y = 10sec + 5T$. What is $\mathbb{E}[Y]$ and $\mathbb{SE}[Y]$?

(dd) [difficult] For a discrete r.v., we defined the mode as:

$$\text{Mode}[X] := \underset{x \in \text{Supp}[X]}{\arg\max} \{p(x)\}$$

where $p(x)$ was the PMF. For continuous r.v.'s, keep the definition the same but replace the PMF with the PDF. Using this definition, find the mode of $T \sim \text{Exp}(\lambda)$. Does this make sense and why?

(ee) [difficult] Let $T \sim \text{Exp}(\lambda)$. Show that the median $[T] = \ln(2)/\lambda$.

(ff) [harder] Prove the memorylessness property of the exponential r.v. (see notes)

(gg) [difficult] Are exponential r.v.'s *real*? Do they exist in the real world? Asked another way, is time really continuous? Discuss. Consult your notes.

(hh) [E.C.] The convolution of two continuous r.v.'s occurs when you are trying to find the density of $T = X_1 + X_2$. It is denoted by the $*$ symbol and its formula is below:

$$f_T(x) = f_{X_1}(x) * f_{X_2}(x) := \int_{\mathbb{R}} f_{X_1}(s) f_{X_2}(x - s) ds$$

For X_1 and X_2 only supported on $[0, \infty)$ such as the exponential r.v., the formula becomes:

$$f_T(x) = f_{X_1}(x) * f_{X_2}(x) = \int_0^x f_{X_1}(s) f_{X_2}(x - s) ds$$

We know that when adding waiting time in the discrete case, the sums of $r \overset{iid}{\sim}$ Geometric (p) r.v.'s are distributed as a NegBin $(r,\, p)$.

prove that the sum of $\overset{iid}{\sim}$ Exp (λ) r.v.'s are distributed as an Erlang(r, λ) which is the continuous analogue of the discrete negative binomial distribution.

That is, prove the convolution of $r \overset{iid}{\sim} \text{Exp}(\lambda)$ has the Erlang "footprint," defined by its density:

$$f_T(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}$$

You should use induction. Do this on a separate piece of paper.

Problem 2

We will now get our feet wet with the Uniform r.v.

(a) [easy] Let $X \sim U(0,1)$, the standard uniform r.v. What is the support of X?

(b) [easy] Let $X \sim U(0,1)$. What is the PDF of X?

(c) [harder] Let $X \sim U(0,1)$. Draw the CDF of X.

(d) [easy] Let $X \sim U(a,b)$, the general uniform r.v. What is the PDF of X?

(e) [harder] Let $X \sim U(a, b)$. Solve for the CDF of X by finding the correct antiderivative of $f(x)$, the PDF from the last problem (see notes).

(f) [easy] Let $X \sim U(a, b)$. Find the median $[X]$ using the CDF from the previous problem.

(g) [difficult] Let $X \sim U(a, b)$. Find the arbitrary Quantile$[X, p]$ where $p \in [0, 1]$.

(h) [E.C.] Let $X \sim U(a, b)$. Compute the arbitrary nth moment, $\mathbb{E}[X^n]$.

(i) [difficult] Let $X \sim U(a, b)$. I bungled the variance calculation in class because I mis-factored $b^3 - a^3$. Find $\mathbb{V}\text{ar}[X]$ using the formula $\mathbb{V}\text{ar}[X] = \mathbb{E}[X^2] - \mu^2$ where you assume that we know $\mu = \frac{1}{2}(a + b)$ which was proved in class.

Problem 3

This is an introduction to the normal r.v. We will do more with it for next homework assignment.

(a) [easy] Let $Z \sim \mathcal{N}(0, 1)$. Write the density $f_Z(x)$ and verify that its value is positive for all real numbers.

(b) [easy] What is the $\text{Supp}[Z]$?

(c) [easy] Use the $\Phi(x) := F_Z(x)$ notation to denote $\int_{-\infty}^{2} f_Z(x)dx$.

(d) [easy] What is $\mathbb{P}(Z \in [-1, 1])$?

(e) [easy] What is $\mathbb{P}(Z \in [-2, 2])$?

(f) [easy] What is $\mathbb{P}(Z \in [-3, 3])$?

(g) [harder] Draw the density $f_Z(x)$ and be careful to denote the x-axis like we did in class. Then illustrate the empirical rule just like we did in class. I want to see the 68, 95, 99.7%'s denoted as areas.

(h) [harder] Draw $\Phi(x) := F_Z(x)$, i.e. the CDF of the standard normal r.v. Use the same scale as your drawing of the PDF of Z.

(i) [easy] What is $\mathbb{P}(Z \notin [-1, 1])$?

(j) [easy] What is $\mathbb{P}(Z \notin [-2, 2])$?

(k) [easy] What is $\mathbb{P}(Z \notin [-3, 3])$?

(l) [easy] What is $\mathbb{P}(Z > 1)$?

(m) [easy] What is $\mathbb{P}(Z > -1)$?

(n) [easy] What is $\mathbb{P}(Z < -1)$?

(o) [easy] What is $\mathbb{P}(Z < -2)$?

(p) [easy] What is $\mathbb{P}(Z > 3)$?

(q) [harder] What is $\mathbb{P}(Z \in (-3, 1))$?

(r) [E.C.] In class we proved that $\int f(x)dx = 1$ for the standard normal. Go through the proof line by line and explain why each line is correct and follows from the previous line in English.

(s) [E.C.] Prove that

$$\int_{-\infty}^{\infty} e^{-a(x+b)^2}\, dx = \sqrt{\frac{\pi}{a}}.$$

(t) [E.C.] Prove that

$$\int_{0}^{\infty} e^{-ax^b} dx = \frac{\Gamma\left(\frac{1}{b}\right)}{ba^{\frac{1}{b}}}.$$