Lecture 6 2/19/15

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) \leq P(A) + P(B)$ Boole's Inequality

$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

$P$(At least one of you shares same bday)

$= \underbrace{P(\text{One pair}) + P(2\text{ pairs}) + \dots P(\tbinom{2?}{2}\text{ pairs})}_{\text{HARD}}$ ← compare shares same bday

$= 1 - P(\text{no one shares same bday})$ ← one of the most famous applications of the complement rule

3 people

$\dfrac{\frac{365 \cdot 364 \cdot 363}{3!}}{\frac{365^3}{3!}}$ ← sample w/out replacement ... otherwise you get a pair! ; all possible bdays $= .992 \Rightarrow P(\text{bday pair}) = .008$

general rule $\dfrac{{}_{365}P_n}{365^n} = 50\%$?

$\dfrac{\frac{365!}{(365-n)!}}{365^n} = 0.5$ Solve using computer $\Rightarrow n = 23$

$\dots = .99? \Rightarrow n \approx 60$

Odds(A) a way to express $P(A)$

$\text{Odds}(A) := \dfrac{P(A)}{1-P(A)}$ $P(\text{rolling } 6) = \frac{1}{6}$ $\text{Odds}(\text{rolling } 6) = 1:5$

More popular ... Odds Against (A)

$\text{OddsA}(A) := \dfrac{1-P(A)}{P(A)}$ $\text{OddsA}(\text{rolling } 6) = 5:1$ preview of expectation

Fair game for a bookie

If I bet \$1 ... the bookie owes me \$5 if I win

$\frac{1}{6}$: \$5 ; $\frac{5}{6}$: -\$1

$\frac{5}{6}(-1) + \frac{1}{6}(5) = 0$

Flip two coins. $P(H_2 \text{ knowing } H_1)? = \frac{1}{2}$

We say that knowing $H_1$ is "informationally irrelevant" to the prob. of $H_2$.

Knowing or being given is very important and has a special symbol.

$$P(A \mid B) = P(A) \quad \text{Knowing } B \text{ is informationally irrelevant}$$

definition of $A$ & $B$ "being independent" or $P(B|A) = P(B)$

$$P\left(\underset{\text{tomorrow}}{\text{IBM stock} \uparrow} \mid \text{Rains in Buenos Aires}\right) = P\left(\underset{\text{tomorrow}}{\text{IBM stock} \uparrow}\right)$$

$P(HH) = P(H, H) = P(H)P(H) = (\frac{1}{2})^2 = \frac{1}{4}$ seems reasonable and will be proven formally later when we get to Bayes Rule

$P(A, B) = P(A)\,P(B)$ ⟵ dense definition

"$P(A \text{ and } B)$"

"$P(A \cap B)$"

given $A_1, \ldots, A_n$ ind, $n \le \infty$

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)$$

$$P(\text{in 24 rolls of 2 you get at least one 66}) = P(\underset{\text{in 24}}{\text{one } 66}) + P(\underset{\text{in 24}}{\text{two } 66}) + \ldots + P(\underset{\text{in 24}}{24\ 66})$$

$$= 1 - P(\underset{\text{in 24x}}{\text{zero } 66})$$

$$= 1 - \underbrace{P(\sim 66, \sim 66, \ldots, \sim 66)}_{\text{ind?}}$$

$P(\sim 66) = 1 - P(66) = \frac{35}{36}$

$P(66) = P(6)^2 = \frac{1}{36}$

$$= 1 - P(\sim 66)^{24}$$

$$= 1 - \left(\frac{35}{36}\right)^{24} = .4914 \quad \text{Chevalier de Mere}$$

Opposite of Independence : Dependence

$P(A, B) \ne P(A)P(B)$ or $P(A|B) \ne P(A)$

$P(\text{given couple}) \ldots$

Trivial: A, B non-empty mutually exclusive $\Rightarrow$ independent? No

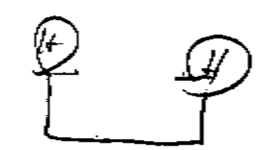
$P(A \cap B) \neq P(A)P(B)$

$0 = P(\phi) \neq P(A)P(B) \geq 0$ ✓

$P(H|T) = 0 \neq P(H) = \frac{1}{2}$

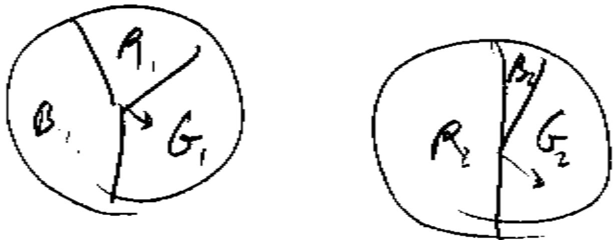
Flip coin connected by magic

Dependent?




$\frac{1}{2} = P(H,H) \overset{?}{=} P(H)P(H) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$ ✓

Sometimes independence doesn't mean "independent" in English e.g.



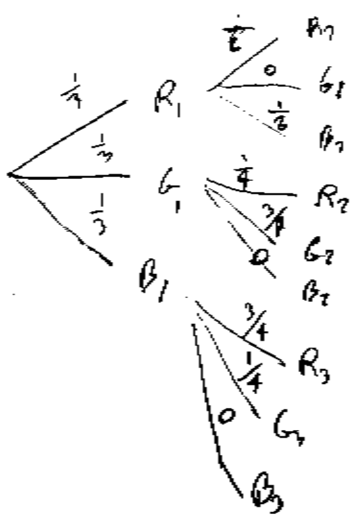

spinner 1

spinner 2

$P(R_1) = \frac{1}{3}$

$P(R_2) = \frac{1}{2}$

If ind. $P(R_1, R_2) = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$

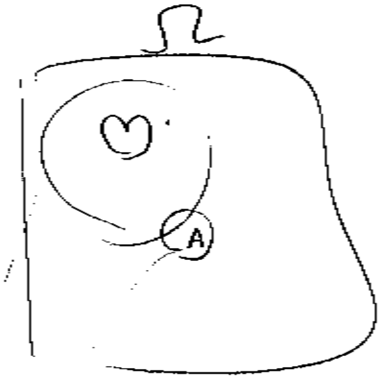



Clearly "dependent" in English.

But "dependent" probabilistically? $\Uparrow$ independence really means "is statistically irrelevant"

$$P(R_1)P(R_2) = \frac{1}{3}\left(\underbrace{\frac{1}{3}\cdot\frac{1}{2}}_{\frac{1}{6}} + \underbrace{\frac{1}{3}\cdot\frac{1}{4}}_{\frac{1}{12}} + \underbrace{\frac{1}{3}\cdot\frac{3}{4}}_{\frac{1}{4}}\right) = \frac{1}{6}$$

$\frac{1}{2}$




$$P(A) = \frac{\cancel{4}}{52} = \frac{1}{13}$$

$$\| \longrightarrow \text{Thus } A, \heartsuit \text{ are ind.}$$

$$P(A \mid \heartsuit) = \frac{1}{13}$$

But you can see there's subtlety here.
"Reducing fractions" is indeed an operation: you are doing something

Still informally irrelevant

What does "given" really mean?

It means we no longer operate in Ω, we operate in a new Ω' where Ω' ⊆ Ω. Here Ω' = ♡ ⊂ Ω

$P(T \mid BA) = 0.5$
$P(T) = 0.5,\ P(BA) = 0.4$


"Conditional Probability"


STOP

1c & smoke

this is what CP looks like for ind.

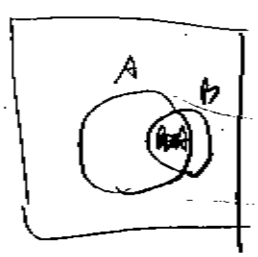
$n = 1{,}000 = |\Omega|$, 200 smokers (A), 60 lung cancer (B), 36 (A, B)

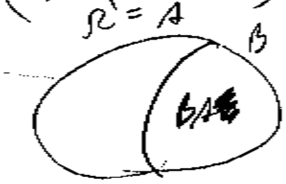
Assume lim. freq def to estimate prob

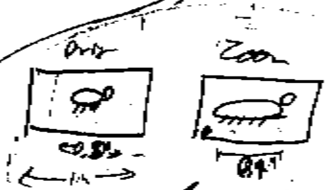
$$P(A) = \frac{200}{1000} = 0.2, \quad P(B) = \frac{60}{1000} = .06 \quad P(B,A) = \frac{36}{1000} = 0.036$$

because?

$$P(\text{lung cancer} \mid \text{smoking}) = P(B \mid A) = \frac{.036}{.2} = 0.18 \approx \frac{1}{5}$$




$$\text{Zoom} = \frac{1}{0.5} = 2$$

$$P(B \mid A) := P(BA) \cdot \underbrace{\frac{P(\Omega)}{P(\Omega')}}_{\frac{1}{P(A)}} = \frac{P(B,A)}{P(A)}$$

but bigger

also...

(def of cond prob)

If A, B ind

$$\Rightarrow P(A \mid B) = \frac{P(A,B)}{P(B)} \Rightarrow P(A,B) = P(A \mid B)P(B) = P(A)P(B) \blacksquare$$

Also $P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)}$ (Bayes Rule / Thm)