Lecture 5 2/17/15

Subjective prob - epistemic - but different people can have different prob's given the same evidence

P(OJ Simpson guilty), P(my favorite color purple)

epistemic differs from l.r.f. or propensity.

It answers the question: How "certain" are you?

P(F=ma is true) → what degree of corroboration do we have?

In epistemic definition, you wouldn't use the term "random" because if you knew all information, you would be certain.

Prob < l.r.f. / prop. / degree of certainty / corroboration

→ epistemic prob is only a result of not having complete knowledge

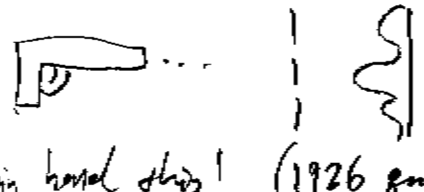
Do you get certainty in the objective world? Coin...

Newton's Laws in Principia (1687) ⇒ determinism.. fixed set of rules

If (a) know rules, (b) know initial conditions ⇒ predict future with 100% accuracy (no randomness)

<LAPLACE> coin flip

Was so until 1920's with wave/particle duality



double slit experiments.. the electron "chooses" a place to land. $\omega \in \Omega$.

Einstein hated this! (1926 quote)

2015: randomness built into the universe (on a small scale)
but small effects can snowball into big effects

Laplace writes:

> We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it – an intelligence sufficiently vast to submit these data to analysis – it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past would be present to its eyes.
>
> (1814: 4)

The vast intelligence here described has come to be known as Laplace's demon. The idea is obviously founded on that of a human scientist (perhaps Laplace himself) using Newtonian mechanics to calculate the future paths of planets and comets. Extrapolating from this success, it was natural to suppose that a sufficiently vast intelligence could calculate the entire future course of the universe. Laplace himself relates his vast intelligence to human successes in astronomy. As he says:

> The human mind offers, in the perfection which it has been able to give to astronomy, a feeble idea of this intelligence. Its discoveries in mechanics and geometry, added to that of universal gravity, have enabled it to comprehend in the same analytical expressions the past and future states of the system of the world.
>
> (Laplace 1814: 4)

> The regularity which astronomy shows us in the movements of the comets doubtless exists also in all phenomena.
>
> The curve described by a simple molecule of air or vapor is regulated in a manner just as certain as the planetary orbits; …
>
> (Laplace 1814: 6)

Regardless of the definition of what prob really is, mathematicians have their own def: P is a set function on all sets $A_i \subseteq \Omega$ — technically wrong... but okay for this class — when Ω is unique.

(a) $P(A_i) \geq 0$ (b) $P(\Omega) = 1$ (c) If A_1, A_2, A_3 disjoint $\Rightarrow P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \sum P(A_i)$

Equivalently $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ for countable sets $A_1, A_2, \ldots$

Thm 1 $P(A) = 1 - P(A^c)$ "Complement Rule"

$P(\Omega) = 1 \Rightarrow P(A \cup A^c) = 1 \Rightarrow P(A) + P(A^c) = 1 \Rightarrow \ldots$
(b) (set thry) by (c) algebra

Thm 2 $P(\phi) = 0$

$P(\phi) = 1 - P(\phi^c) = 1 - P(\Omega) = 1 - 1 = 0$
(Thm) set thy (b)

Thm 3 $P(A) = 0 \Rightarrow A = \phi$

Imagine not: $A \neq \phi$ but $P(A) = 0$

$P(A \cup A^c) = 1$

$P(A) + P(A^c) = 1$

$0 + P(A^c) = 1 \Rightarrow P(A^c) = 1$

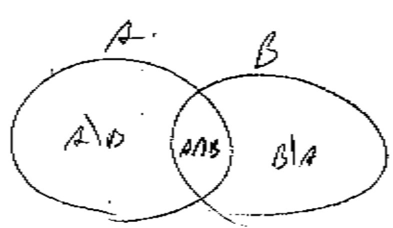
$\Rightarrow A^c = \Omega \Rightarrow A = (A^c)^c = \Omega^c = \phi$ by uniqueness of Ω $\perp$

Thm 4 If $A \subset B \Rightarrow P(A) < P(B)$

If $A \subset B \Rightarrow \exists\, C = B \setminus A \neq \phi$ where $C \cap A = \phi$ and $A \cup C = B$ $\Rightarrow P(C) > 0$

$\Rightarrow P(A \cup C) = P(B) \Rightarrow P(A) + P(C) = P(B) \Rightarrow P(B) - P(A) = P(C) \Rightarrow P(B) - P(A) > 0$
(c)

Thm 5 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$P(A \cup B) = P(A \setminus B \cup A \cap B \cup B \setminus A)$

$= P(A \setminus B) + P(A \cap B) + P(B \setminus A)$

$A = A \setminus B \cup A \cap B \Rightarrow P(A) = P(A \setminus B) + P(A \cap B)$

$B = B \setminus A \cup A \cap B \quad P(B) = P(B \setminus A) + P(A \cap B)$

$= \underbrace{P(A \setminus B) + P(A \cap B)}_{P(A)} + \underbrace{P(B \setminus A) + P(A \cap B)}_{P(B)} - P(A \cap B)$

Law of Inclusion-Exclusion

$\ldots P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{\text{all sets} \\ \{i,j\} \\ \text{order doesn't matter}}} P(A_i \cap A_j) + \sum_{''} P(A_i \cap A_j \cap A_k) - + \ldots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right)$

More Coming...

10 flowers 3 Red, 4 Green, 3 Blue

How many ways? $\frac{10!}{3!4!3!} := \binom{10}{3,4,3}$ ← multichoose notation

How many ways to pick sets of 3, 4, 3 w/out respect to order? Choosing batch of locations 1, 2, ..., 10.

Quicker way to think about it.

3 Red 7 not Red

$\underline{\sim R}$ $\underline{\sim R}$ $\underline{R}$ $\underline{\sim R}$ $\underline{R}$ $\underline{\sim R}$ $\underline{\sim R}$ $\underline{\sim R}$ $\underline{R}$ $\underline{\sim R}$

$\frac{10!}{7!3!} = \binom{10}{3}$

Now 7 not red, 4 Green, 3 Blue.

$\frac{7!}{4!3!} = \binom{7}{4}$

Now 3 not red or green, 3 blue

$\binom{3}{3}$

$$\binom{10}{3,4,3} = \binom{10}{3}\binom{7}{4}\binom{3}{3} = \binom{10}{4}\binom{6}{3}\binom{3}{3} = \binom{10}{3}\binom{7}{3}\binom{4}{4}$$

Identities...

Balls & Urns...

n balls, r urns, Balls are not distinguishable, Urns must have ≥ 1 balls

$\underbrace{\_\ \_\ \_\ \cdots\ \_}_{r}$ $r = 2$, $n = 4$

Urn 1 Urn 2

$\overset{\text{Urn 1}}{|000}\overset{\text{Urn 2}}{|0|}$

↑ place one divider in between the 4 balls

$r=3,\ n=5$

| O | O O | O O |

place two dividers in between 5 balls

How many dividers? $r-1$
How many places to put them $n-1$ $\Rightarrow \binom{n-1}{r-1}$

$X_1+X_2+\dots+X_r=n$ where $X_1,X_2,\dots,X_r\in\mathbb{N}$

How many solutions? Application of this type of class thinking

n items, m defective, m<n
How many ways to order them so no two defective items next to each other?
(p14)

⊗ O ⊗ O ⊗ O ⊗ O

m

Same as placing the $n-m$ good items in m urns

$X_1+\dots X_{m+1}=n-m$

$X_1\geq 0,\ X_{m+1}\geq 0$

$X_2\dots X_m\in\mathbb{N}$

...

Now, allow urns to be empty

|| O | O || $n=2,\ r=2$

place divider in one of these places

$n=3,\ r=3$

| || O || O || O ||

3 balls, 2 dividers → balls and dividers could be arranged

$\binom{5}{2}$

$\binom{n+r-1}{r-1}=\binom{n+r-1}{n}$

geometry

let $x_1'=x_1+1,\ \dots\ x_r'=x_r+1$

$x_1'+\dots+x_r'=n+r$

$\Rightarrow\binom{n+r-1}{r-1}$

Multi exp.

$(x_1+\dots+x_r)^n$

$=\sum\binom{n}{n_1\dots n_r}x_1^{n_1}\dots x_r^{n_r}$

How many?

$\binom{n+r-1}{r-1}$

RISK...

$X_1+X_2+\dots X_r=n$ s.t. $X_1,\dots X_r\in\mathbb{N}_0$ How many solutions?

What if balls are non distinguishable?