MATH 241 Spring 2014 Homework #3

Professor Adam Kapelner

Due 5PM outside my office, Tuesday, Feb 24, 2015

(this document last updated Thursday 19^{th} February, 2015 at 11:33am)

Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out". Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, read the section about balls/urns in Chapter 1 and the axioms of probability in Chapter 2 and the section on even independence in Chapter 3 (but not conditional probability). Chapter references are from the 7th edition of Ross. You also may need to review the first six pages of Chapter 1 of Donald Gillies "Philosophical Theories of Probability".

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]", purple problems are extra credit. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems.

This homework is worth 100 points but the point distribution will not be determined until after the due date. Late homework will be penalized 10 points per day.

15 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com` (make sure you upload both the hwxx.tex and the preamble.tex file). If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. **Handing it in without the printout incurs a penalty of 20 points.** Keep this page printed for your records. Write your name and section below where section A is if you're registered for the 9:15AM–10:30AM lecture and section B is if you're in the 12:15PM-1:30PM lecture.

NAME: ______________________________ SECTION (A or B): ______

Problem 1

A little bit more philosophy.

(a) [easy] What is the difference between probabilities that are objective and probabilities that are epistemic?

(b) [easy] Why would you call an objective probability "random" but not an epistemic probability? Explain.

(c) [easy] If all information was known, would there still be epistemic probabilities? Yes/no is fine.

(d) [easy] According to Laplace (and his interpretation of Newton), if all information was known about physical systems including all laws and all initial conditions, would there be randomness? Yes/no is fine.

(e) [difficult] Knowing what we known in the 21st century, if all information was known about physical systems including all laws and all initial conditions, would there be randomness? If so, what theory has demonstrated evidence for this?

(f) [easy] What upset Einstein in 1926 to say "God does not play dice with the universe?"

Problem 2

This problem involves using the multinomial coefficient to solve problems.

(a) [easy] Imagine you have 12 flowers: 4 red and 3 blue and 5 white. How many ways are there to arrange them in 12 flower pots if flowers of the same color are indistinct.

(b) [easy] We add 2 orange flowers to collection in part (a). How many ways to arrange the flowers now if flowers of the same color are indistinct?

(c) [easy] Imagine we have 5 flowers: one white, one blue, one red, one orange and one purple. How many ways to arrange them if flowers of the same color are indistinct? Use the multinomial coefficient and show that it is equal the number you arrive at using the permutation concept from lecture 2.

(d) [harder] Recall the binomial theorem:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

We wish to now extend this expansion to house many more intial terms such as $(a_1 + a_2 + \ldots + a_K)^n$. Explain in words that each term in the expansions should be of the form:

$$a_1^{x_1} a_2^{x_2} \ldots a_K^{x_K}$$

Look in your notes when we did the binomial expansion for a hint.

(e) [harder] Why should $x_1 + x_2 + \ldots + x_K = n$?

(f) [difficult] Given that $n = 7$ and $K = 4$, how many terms of the form $a_1^2 a_3^3 a_4^2$ will appear in the expansion? For a hint see next question.

(g) [difficult] It turns out the general formula looks like:

$$(a_1 + a_2 + \ldots + a_K)^n = \sum_{\text{something}} \binom{n}{x_1, x_2, \ldots, x_K} a_1^{x_1} a_2^{x_2} \ldots a_K^{x_K}$$

What is the sum indexed over?

Problem 3

We will get our feet wet with basic "axioms" and theorems. Assume all capital letters are sets. If the problem asks you to prove a fact, you may only use your knowledge of set theory and the definition of $\mathbb{P}(\cdot)$ given in the book / lecture. Some of the answers are in the book. Try to do them yourself and only use the book if you are having trouble.

(a) [easy] List (1) all assumptions prior to and (2) the three conditions that make $\mathbb{P}(\cdot)$, the set function that returns a probability. These three conditions are also known as the "axioms of probability."

(b) [easy] Prove that if A_1 and A_2 are disjoint (mutually exclusive), $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2)$.

(c) [easy] Prove that $\mathbb{P}(\varnothing) = 0$.

(d) [harder] Prove that $\mathbb{P}(A) \leq 1$.

(e) [easy] Assuming the previous theorem that $\mathbb{P}(A) \leq 1$, prove that $\mathbb{P}(A) \in [0, 1]$.

(f) [difficult] Prove that if $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

(g) [difficult] Prove that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A, B)$ (in the notes).

(h) [E.C.] Let $A_1 \subseteq A_2 \subseteq A_3, \subseteq \ldots$ (this is called a sequence of "increasing events.") Prove (on another sheet of paper) that:

$$\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}\left(\lim_{n \to \infty} A_n\right)$$

Problem 4

You are playing billiards. There are 15 balls on the table (save the cue ball which we are ignoring) numered 1, 2, ..., 15 and 6 pockets the balls can go into (4 corner pockets and two side pockets, displayed below). The goal of the game you are playing is to get all the 15 balls into any of the pockets.

(a) [easy] How many ways is there to sink all balls if you assume at least one ball goes into each hole?

(b) [easy] How many ways is there to sink all balls if you do not assume at least one ball goes into each hole (i.e. pockets can be empty now)?

(c) [difficult] How many ways to sink all balls if pockets can be empty and the balls are distinct?

(d) [E.C.] How many ways to sink all balls if pockets can be empty with the restriction that each pocket must have at least one odd ball and at least one even ball?

(e) [E.C.] On the game in (b), let's say you win a special prize if the holes have three holes have 4 balls and the other three holes have one ball (4+4+4+1+1+1=12). You don't care which pockets have which balls but assume that each pocket has at least one ball. What is the probability you win this special prize upon sinking all balls? Assume that balls go into random pockets. Kind of silly... but it makes you think.

Problem 5

Probability is rooted in gambling and thus we will be analyzing some gambling games. We will introduce the game of Roulette here. Basically, there's a ball that is dropped onto a spinning wheel with pockets for the ball to fall once the wheel and ball run out of momentum. There are 18 red pockets and 18 black pockets. There are two flavors of the game:

- European: There is one additional pocket colored green and labeled 0 (for a total of 18+18+1=37 pockets). An example of this wheel is pictured below on the left.
- American: There are two additional pockets colored green labeled 0 and 00 (for a total of 18+18+2=38 pockets). An example of this wheel is pictured below on the right.

The gambler can make bets on any of the spaces as well as red (R), black (B), green (G), an odd number, an even number and a slew of other exotic type bets which we won't enumerate. We will be analyzing payoffs when we get to random variables next week but we will not be discussing them now.


(a) [easy] What is the probability of the ball landing in a black pocket? Calculate for both European and American roulette.

(b) [easy] What is the probability of the ball landing in a green pocket? Calculate for both European and American roulette.

(c) [easy] What is the probability you see RRBBBRGRBB in 10 spins?

(d) [easy] In the 18 red pockets there are 9 even numbered pockets and 9 odd numbered pockets. What is the probability of getting a pocket wihich is both Red and Odd?

(e) [easy] What is the probability you see a spin that is both Red and Green?

(f) [easy] What is the probability you see a spin that is Red or Green?

(g) [easy] In America, you play the game 10 times and always bet on black. What is the probability you win all 10 times?

(h) [difficult] In America, you play the game 10 times and always bet on black. What is the probability you win at least once?