Lecture 3 Math 241 Feb 5, 2014

9:15

Set of objects A, how many ways to arrange them? $|A|!$

$\{J, M, S\}$ $\quad {}_3P_3$ ← # spots to arrange them; # of objects $= 3! = \underline{3} \cdot \underline{2} \cdot \underline{1} = 6$

What if less spots than objects?

$\frac{10!}{9!} = \frac{10!}{(10-1)!}$

10 people, one chair $\quad {}_{10}P_1 \quad \underset{1^{st}\text{ chair}}{\underline{10}} = 10$

$= \frac{10!}{8!} = \frac{10!}{(10-2)!}$

10 people two chairs $\quad {}_{10}P_2 \quad \underline{10} \cdot \underline{9} = 90$

n people k chairs s.t. $k \leq n$ $\quad {}_nP_k = \frac{n!}{(n-k)!}$

${}_{10}P_{10} = \frac{10!}{(10-10)!} = \frac{10!}{0!} = 10! \Rightarrow 0! := 1$ to make our def consistent.

5 B, 5 G as couples. How many ways to arrange them in 10 chairs w/ couples sitting next to each other? While alternating B-G-B-G-... or G-B-G-B

$\underline{5}\ \underline{5}\ \underline{4}\ \underline{4}\ \underline{3}\ \underline{3}\ \underline{2}\ \underline{2}\ \underline{1}\ \underline{1} = (5!)^2 (2)$

How about couples next to each other but not necessarily alternating?

$(5!)^2\, 2^5$

10 people, 10 chairs Bob & Joe ~~sit~~ together

$\langle B, J \rangle$ $\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{{}_8P_8 = 8!}$ $\quad 9!(2) \quad 9 \cdot 8!(2)$

10 people 5 chairs Bob must sit $\quad \underline{B}\ \underbrace{\_\ \_\ \_\ \_}_{{}_9P_4} \quad 5 \cdot {}_9P_4$

10 people 5 chairs B, J must sit next to each other $\quad \langle B, J \rangle\ \underbrace{\_\ \_\ \_}_{{}_8P_3} \quad 4 \cdot {}_8P_3 \cdot 2$

5 men 4 women M/W ranked separately, How many rankings?

m|F $\{J, M, S\}$

Ω ← permutations $= \{ \ldots\ldots \}$ 6 of them

this can be thought of as "Choose one from $\{J, M, S\}$, seat her. Then choose one of the remaining and seat her etc. This is "sampling without replacement""

$m|F \times m|F \times m|F = \{JJJ, \ldots, SSS\} \to 27$ elements

J re goes back — "Sampling with replacement" — returns home

n objects, k draws without replacement $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!} = {}_nP_k$

n objects, k draws with replacement $n(n)\cdots(n) = n^k$

If n is large and k is fixed $n^k \approx {}_nP_k$

1000 people 5 seats $\frac{{}_{1000}P_5}{1000^5} = \frac{1000 \cdot 999 \cdot 998 \cdot 997 \cdot 996}{1000^5} \approx 99\%$.

objects that can be reused (coins, spinners, dice...)

objects can't be reused — people

In the limit it's the same

$$\lim_{n\to\infty} \frac{{}_nP_k}{n^k} = \frac{n(n-1)\cdots(n-k+1)}{n(n)\cdots(n)}$$

Remember $\lim_{x\to a} f(x)g(x) = \lim f(x) \lim g(x)$

$$= \left(\lim \frac{n}{n}\right)\left(\lim \frac{n-1}{n}\right)\cdots\left(\lim \frac{n-k+1}{n}\right) = 1$$

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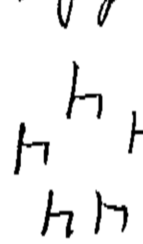
5 people 5 chairs (in a ring/circle)



How many ways to seat people if the circle is rotationally invariant?

$\frac{{}_5P_5}{5}$ "dividing out invariance" principle

9 flowers 5 B, 4 R

How many ways to arrange flowers if each flower distinct? ${}_9P_9$

" " blues were all completely the same and reds " .. ? $\dfrac{{}_9P_9}{{}_5P_5\,{}_4P_5} = \dfrac{9!}{5!4!}$

Ω^6: 6 coin flips

HTHTTH ≠ HTHTHT ← these are different flips!

How many ways to have 3H 3T in 6 flips?

$\dfrac{{}_6P_6}{{}_3P_3\,{}_3P_3} = \dfrac{6!}{3!3!}$ } "Binomial coefficient"

$P(HHHTTT) = \dfrac{1}{2^6} = \dfrac{1}{64}$

$P(3H, 3T) = \dfrac{\frac{6!}{3!3!}}{2^6} = \dfrac{\frac{6\cdot5\cdot4}{3\cdot2}}{2^6} = \dfrac{20}{64}$

1437 flowers, 1000 Blue, 400 Red, 30 Green, 7 Yellow

How many ways to arrange?

$\dfrac{1437!}{1000!\,400!\,30!\,7!}$ } "Multinomial coefficient"

10 people 4 chairs. How many permutations?

${}_{10}P_4$

What if we don't care about the order of the four people?

$\dfrac{{}_{10}P_4}{4!}$ ← all the combinations of 4 people out of 10.

How many combinations of k objects out of n total?

$\dfrac{{}_nP_k}{k!} = \dfrac{n!}{(n-k)!\,k!} := {}_nC_k = \binom{n}{k}$ ← notation I'll use...

("choose" notation)

52 cards in a deck

$R = \{A, 2, 3, \ldots, 10, J, Q, K\}$ $\Omega = R \times S$

$S = \{S, H, C, D\}$

5-card draw poker is a game where you are dealt five cards, without replacement. Order doesn't matter — comb (not perm). How many possible hands?

$|\Omega| = \binom{52}{5}$

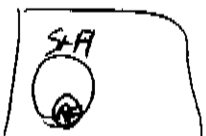
Many ways to win! There's a hierarchy of winning hands.

(1) Royal Flush 10 J Q K A in any suit

$P(\text{Royal Flush}) = \frac{|\text{Royal Flush}|}{|\Omega|}$ ← How many are there? $= \frac{\binom{4}{1}}{\binom{52}{5}} \approx \frac{1}{2.6m}$ ← 4 suits ← choose 1

(2) Straight Flush A 2 3 4 5 6 7 8 9 10 J Q K A ← order

$P(\text{St Flush}) = \frac{\binom{10}{1} - \binom{4}{1}}{\binom{52}{5}}$



(3) 4 of a kind AAAA 7 $\frac{\binom{13}{1}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}}$

More next time

1000 coin flips $P(500H, 500T) = \frac{\frac{1000!}{500!\,500!}}{2^{1000}}$ — $\binom{1000}{500}$ → Choose 500 places out of 1000 for the H (others automatically T)

Computational problem! #'s too big!

Use logs

$\ln(P) = \underbrace{\ln 1000! - 2\ln 500!}_{} - \underbrace{1000\ln 2}_{OK}$ → $\ln 500! = \sum_{i=1}^{500} \ln(i)$ Can be done but 500 operations! Slow

⇒ Need approximation for ! operation!

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$ still too big!

$$1000! \approx \sqrt{2\pi(1000)}\left(\left(\frac{1000}{2.718}\right)^{1000}\right)$$

$$\ln(n!) \approx \frac{1}{2}\ln(2\pi) + \left(n+\frac{1}{2}\right)\ln(n) - n$$

$$\ln(p) = \frac{1}{2}\ln 2\pi + 1000.5\ln(1000) - 1000 - 2\left(\frac{1}{2}\ln 2\pi + 500.5\ln(500) - 500\right) - 2\ln(1000)$$
$$= -3.6797$$

$$e^{\ln(p)} = \boxed{p = .0252}$$

~~Roll a die 6 times. Define a win as having at least one roll # = roll value.~~

~~e.g. If you get a 4 on the 4th roll, ⇒ you win~~

~~1 6 6 6 6 6 P(win on first) $\frac{6^5}{6^6} = \frac{1}{6}$~~

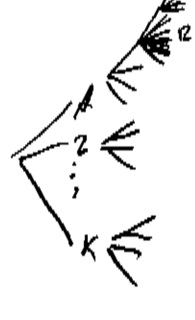
④ Full House 99933

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$$

Why not $\binom{13}{2}\binom{4}{3}\binom{4}{2}$? 99933 = 99333 No...

picking no. since 9, 3's

A2
A3
⋮
AK



$13\cdot12 = \binom{13}{1}\binom{12}{1} \neq \binom{13}{2} = \frac{13\cdot12}{2}$ ✓ Half

HARD!