MATH 214 Spring 2015 Homework #1

Professor Adam Kapelner

Due 5PM outside my office, Tuesday, Feb 10, 2015

(this document last updated Thursday 19^{th} February, 2015 at 5:47pm)

Instructions and Philosophy

The path to success in this class is to do many problems. Unlike other courses, exclusively doing reading(s) will not help. Coming to lecture is akin to watching workout videos; thinking about and solving problems on your own is the actual "working out". Feel free to "work out" with others; **I want you to work on this in groups.**

Reading is still *required*. For this homework set, read the section about sample spaces in Chapter 2 and permutations in Chapter 1 in Ross. Chapter references are from the 7th edition.

The problems below are color coded: green problems are considered *easy* and marked "[easy]"; yellow problems are considered *intermediate* and marked "[harder]", red problems are considered *difficult* and marked "[difficult]". There are no E.C. problems on this homework. The *easy* problems are intended to be "giveaways" if you went to class. Do as much as you can of the others; I expect you to at least attempt the *difficult* problems.

This homework is worth 100 points but the point distribution will not be determined until after the due date. Late homework will be penalized 10 points per day.

15 points are given as a bonus if the homework is typed using LaTeX. Links to instaling LaTeX and program for compiling LaTeX is found on the syllabus. You are encouraged to use `overleaf.com`. If you are handing in homework this way, read the comments in the code; there are two lines to comment out and you should replace my name with yours and write your section. If you are asked to make drawings, you can take a picture of your handwritten drawing and insert them as figures or leave space using the "\vspace" command and draw them in after printing or attach them stapled.

The document is available with spaces for you to write your answers. If not using LaTeX, print this document and write in your answers. **Handing it in without the printout incurs a penalty of 20 points.** Keep this page printed for your records. Write your name and section below where section A is if you're registered for the 9:15AM–10:30AM lecture and section B is if you're in the 12:15PM-1:30PM lecture.

NAME: ______________________________ SECTION (A or B): ______

Set Theory Problems below are related to set theory. The sets we talk about in class are composed of outcomes in a universe that are events. Some of the problems below will be about abstract sets that are divorced from the sets used in probability.

Problem 1

These are questions on abstract set theory. Assume capital letters are arbitrary sets and Ω is the universe for all the following questions. Answer as succinctly as possible.

(a) [easy] Answer the following as best as possible.
$A \cup A =$
$A \cap A =$
$A \cap \varnothing =$
$A \cup \Omega =$
$A \cap \Omega =$
$A \cup A^C =$
$\varnothing^C =$
$\Omega^C =$
$A \backslash A =$
$A \backslash \Omega =$
$A \backslash \varnothing =$

(b) [easy] Are the following true (T) or false (F) for arbitrary sets A, B, C? The last one is extra credit and requires an explanation for bonus points.
$A \subseteq \Omega$
$A \subset \Omega$
$\varnothing \subseteq A$ and $A \subseteq \Omega$
$A \subseteq A \cup B$
$A \subseteq A \cap B$
$A \in A$

(c) [harder] Are the following true (T) or false (F) for the arbitrary set A?
$A \subseteq A$
$A \subset A$
$\varnothing \subseteq A$
$\varnothing \subset A$
$\varnothing \subseteq \varnothing$
$\varnothing \subset \varnothing$

(d) [harder] Are the following true (T) or false (F)? The symbol "$\Rightarrow$" denotes logical implication *i.e.* if the conditions on the l.h.s are met, the statement on the r.h.s is always true. Commas should be interpreted to mean "and."
$A \subseteq B \Rightarrow A \cap B = A$
$A \subseteq B, \; B \subseteq C \Rightarrow A \subseteq C$
$A \subseteq B, \; B \subseteq C \Rightarrow A \subset C$
$A \subseteq B, \; A \subseteq C \Rightarrow A \subset B \cap C$
$A \subset A \cup B$

(e) [harder] Express $A \cap B$ only in terms of set subtraction (by using the symbol "$\backslash$").

(f) [easy] Explain why $A \cup B = B \cup A$ *in English*.

(g) [harder] Draw three Venn diagrams illustrating the distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ one for each of three configurations of $A, B,\ C$ that you decide. You must draw A, B, C as circles. "Different configurations" means that the circles overlap differently.

Problem 2

Consider the sample space Ω where you flip a fair coin and roll a fair die.

(a) [easy] Draw this event space in a box similar to how we did in class.

(b) [easy] What is $|\Omega|$?

(c) [easy] Are singleton sets of the outcomes in Ω mutually exclusive? collectively exhaustive?

(d) [easy] Does it matter if the coin is flipped before the die, after the die, or simultaneously with the die? Explain *in English*.

(e) [easy] Consider the set T which represents all outcomes where the coin was flipped tails and E which represents the set of outcomes where the die rolled an even number. Draw a Venn diagram of T and E using circles.

(f) [harder] Describe fully the set $2^{(E \cup T)^C}$ *i.e.* list all its elements.

Problem 3

A "full deck of cards" has 52 cards where each card has two characteristics: (1) one of four suits $\spadesuit$, $\heartsuit$, $\clubsuit$ and $\diamondsuit$ and (2) one of 13 ranks: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K and each card is unique. The game Euchre (see `http://en.wikipedia.org/wiki/Euchre` for more information), 24 playing cards are used consisting of only aces, kings, queens, jacks, tens, and nines.

(a) [easy] Let Ω_E denote the sample space of a Euchre deck and Ω denote the sample space of a full deck. Is $\Omega_E \subset \Omega$ true?

(b) [harder] Construct Ω_E, the event space of a Euchre deck by using set notation and operations on Ω, the event space of a full deck of cards. Use the "..." notation used in class to specify your sets explicitly and use rank and suit such as 4$\clubsuit$ to denote the ω's $\in \Omega$. Hint: use the Cartesian product (denoted by $\times$) on two sets.

(c) [difficult] Let B be the set of black cards, F the set of face cards and $\spadesuit$ the set of spades. Describe the set on the r.h.s of:

$$\{A\spadesuit, 9\heartsuit, K\diamondsuit\} \cup \varnothing \;\subseteq\; \Big((B \cap F)^C \cup \spadesuit\Big)^C \setminus (\{10\spadesuit, 10\diamondsuit, 10\heartsuit\} \cap \Omega)$$

(d) [difficult] Do this problem after completing the last questions since it has to do with counting. Given 5 Euchre cards, how many ways is there to order them

(e) [difficult] You are dealt five Euchre cards out of the 24 total hands. How many ways is there to order all hands?

Problem 4

We will review the notation $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ as well as their subsets as introduced in Lecture 2.

(a) [easy] Draw a number line for x and shade in the area that represents the set $[1, 3] \cup (4, 9]$. If the set includes a number on the endpoint, draw a solid circle "$\bullet$" and if does not include the number, draw an open circle "$\circ$."

(b) [easy] Draw a number line for $Z \subset \mathbb{R}$ where $Z := \{x \in \mathbb{R} : |x| > 2\}$. This Z notation we'll be using in a couple months when we get to the normal distribution.

(c) [easy] Draw on the number line the set $[0, 1] \cap \left(0, \frac{1}{2}\right) \cap \left[0, \frac{1}{4}\right]$.

(d) [harder] Find the set $A := \bigcup_{i=1}^{\infty} \left[0, \frac{1}{i}\right]$.

(e) [difficult] Find the set $B := \bigcap_{i=1}^{\infty} \left[0, \frac{1}{i}\right]$.

(f) [easy] Find the set $\mathbb{Z} \backslash \mathbb{N}$.

(g) [harder] Describe the set $\mathbb{R} \backslash \mathbb{Q}$ as best as you can in English and give an example of an element of this set.

Counting Problems below are related to counting. We will review the methods learned in class and expand our horizons.

Problem 5

In this problem, we imagine rolling different sized-dice. Assume the outcomes (each face of each die) are equally likely for that die (see middle of page 9 in H, T & Z for a definition).

Let R be a standard 6-sided die, let S be an 10-sided die, let T be a 12-sided die, and let U be a 18-sided die. What is the sample size of Ω (*i.e.* $|\Omega|$) for the experiment where we...

(a) [easy] roll R 3 times?

(b) [harder] roll R then S then T then U?

(c) [harder] roll R 34 times, then roll S 45 times, then roll T 12 times, then roll U 76 times.

(d) [harder] Roll R and then roll S only if R rolled greater than or equal to 4. Construct the universe of discourse in this situation by enumerating each outcome of Ω below.

(e) [harder] Would each $\omega \in \Omega$ here be equally likely? If yes, explain why. If no, provide a counterexample.

Problem 6

Examine the following words and tell me how many *permutations* there are of the letters. We do not care about keeping track of the individual common letters. For example, in the word *dad*, there are two $d's$ and we want to treat the permutation $d_1 d_2 a$ the *same* as $d_2 d_1 a$.

(a) [easy] town

(b) [easy] tsktsk (yes, this is a real word!)

(c) [harder] mississippi

(d) [difficult] supercalifragilisticexpialidocious

Problem 7

Below is a standard chessboard. Rows one and eight have the following pieces: two rooks, two knights, two bishops, a king and a queen. Rows two and seven have 8 pawns. Rows one and two have all black pieces and rows seven and eight have all white pieces.

(a) [easy] How many ways are there to place the black queen on a white square?

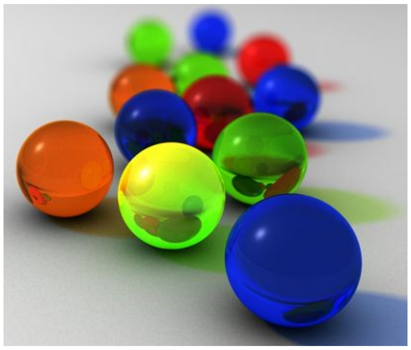
(b) [harder] How many ways are there to set up the pieces in the back ranks of both white and black *i.e.* arrange the two rooks, two knights, two bishops, king and queen on the first row of 8 squares. Note that this game is called "Fischer Random Chess" after the famous grandmaster Bobby Fischer who proposed the idea to make standard chess more exciting.

(c) [difficult] The game progresses and white takes two black pawns and black takes two white pawns. How many ways are there to arrange the pieces on the board? We don't care about pieces of a type being unique (*i.e.* all white pawns are the same, all black rooks are the same, etc)

(d) [E.C.] In the answer to (c) are all arrangements "equally likely" during an actual chess game? Explain why or why not.

Problem 8

We have 4 blue marbles, 4 green marbles, 2 orange marbles, and 2 red marbles. For the following questions, if you are using "choose notation", please write your choose notation, then write the formula using factorials, then write the actual number after you compute it.



(a) [easy] Viewing all the marbles as *unique*, how many ways are there to order the marbles? Note that "order" is another way of saying "permute."

(b) [harder] Viewing all marbles of the same color as *interchangeable*, how many ways is there to order the marbles?

(c) [E.C.] If I pick 4 marbles at random from the collection, how many ways are there to get two-of-a-kind *i.e.* two marbles of one color and two marbles of a different color.