Lecture 19 4/28/15 Math 241

$$Z \sim N(0,1) := \overbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}^{f(x)}$$ Std normal

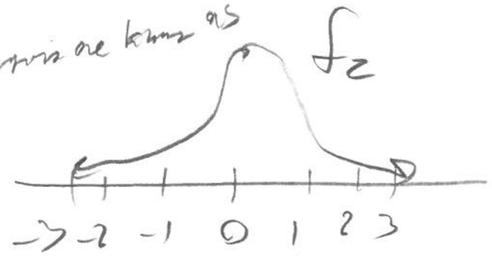
$$X = \mu Z + \sigma \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$ genl normal density/r.v./distr. "bell curve"

$E(X) = \mu$, $Var(X) = \sigma^2$, $SE(X) = \sigma$

Parameters

$\mu \in \mathbb{R}$, $\sigma \in (0, \infty)$ (since SE must be +)

realizations are known as



"Z scores"

-1: realization is 1 SE below mean $P(Z < -1) = 16\%$ $-1 \Rightarrow 16\%$ile

$+2$: 2SE's ~~below~~ above mean $P(Z \geq 2) = 0.025 \Rightarrow 97.5\%$ile

Z-scores are 1:1 with percentiles

Male Height

$$X \sim N(\underset{\uparrow}{70''}, \underset{\underset{SE}{\uparrow}}{3''}{}^2)$$ if Bob is 73" tall what is his percentile?

$x = 73'' \Rightarrow z = \frac{x-\mu}{\sigma} = \frac{73-70}{3} = \frac{3}{3} = 1 \Rightarrow 84\%$ile of heights

Many things in nature are normally dist but why???

Note:

$M_X(t) = L(-t)$ ← reflection of Bilateral Laplace transform through x-axis.

1:1 still applies. Thus...

PMF $p(x) \Leftrightarrow M_X(t)$ are 1:1

PDF $f(x) \Leftrightarrow M_X(t)$ are 1:1

Why do we care??

① $T = X_1 + X_2$. PMF / PDF difficult to compute!

if X_1, X_2 independent $E[g(X_1)h(X_2)] = E[g(X_1)]E[h(X_2)]$

But... $M_{X_1+X_2}(t) = E[e^{t(X_1+X_2)}] = E[e^{tX_1}]\,E[e^{tX_2}] = M_{X_1}(t)\,M_{X_2}(t)$

If indep...

the mgf of the sum is the product of the mgf's!

② Recall $X \sim \text{Binom}(n,p)$ $\quad \text{Var}[X] = E[X^2] - \mu^2$ difficult to compute! E.C. on HW 8?

$$\sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$$

What about $E[X^{17}]$? i.e. the 17^{th} moment.

$\sum_{x=0}^{n} x^{17} \binom{n}{x} p^x (1-p)^{n-x}$ forget it! TOO HARD!

Need easier way...

What is e^x? Taylor series

x close to 0

$$f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$e^x \approx e^0 + \frac{e^0}{1!}x + \frac{e^0}{2!}x^2 + \ldots = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$\frac{d}{dx} e^x = e^x$ for all derivatives

$$e := e^1 = \sum_{i=0}^{\infty} \frac{1}{i!} \quad (\text{another def of } e)$$

$$\Rightarrow e^{tX} \approx 1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \ldots$$

$$M_X(t) := E[e^{tX}] \approx E[1 + \ldots + \ldots]$$

$$= 1 + \frac{t\,E(X)}{1!} + \frac{t^2\,E(X^2)}{2!} + \frac{t^3\,E(X^3)}{3!} + \ldots$$

What about $M_X'(t)$?

$$\frac{d}{dt} M_X(t) = 0 + \frac{E(X)}{1!} + \frac{\cancel{2}t\,E(X^2)}{\cancel{2}\cdot 1} + \frac{\cancel{3}t^2\,E(X^3)}{\cancel{3}\cdot 2!} + \ldots$$

$$M_X'(0) = 0 + \frac{E(X)}{1!} + 0 + 0 + \ldots$$

$$\Rightarrow M_X'(0) = E(X) \;!!!$$

What about $M_X''(t)$?

$$\frac{d^2}{dt^2} M_X(t) = \frac{d}{dt} M_X'(t) = 0 + 0 + \frac{E(X^2)}{1} + \frac{\cancel{2}t\,E(X^3)}{\cancel{2}\cdot 1} + \ldots$$

$$M_X''(0) = E(X^2)\,!$$

What about $M_X'''(t)$?

$$\frac{d^3}{dt^3} M_X(t) = \frac{d}{dt} M_X''(t) = 0+0+0+\frac{E[X^3]}{1} + t(\cdots)$$

$$M_X'''(0) = E(X^3)$$

moments!

$$\Rightarrow M_X^{(n)}(0) = E[X^n] \quad \text{mgf!!}$$

$Y = aX$. What is $M_Y(t)$?

let $t' = at$

$$M_Y(t) = E(e^{tY}) = E[e^{atX}] = E(e^{t'X}) = M_X(t') = M_X(at)$$

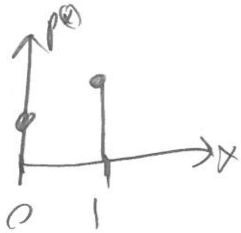
$Y = X + c$ What is $M_Y(t)$?

$$M_Y(t) = E(e^{tY}) = E[e^{t(X+c)}) = E(e^{tX}e^{tc}) = e^{tc}E(e^{tX}) = e^{tc}M_X(t)$$

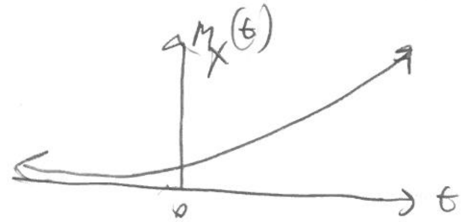
$$Y = aX + c \Rightarrow M_Y(t) = e^{tc} M_X(at)$$

Do some problems? $X \sim \text{Bern}(p)$

$$M_X(t) = E(e^{tX}) = \sum_{x \in \text{supp}(X)} e^{tx} p(x) = e^{t(0)}(1-p) + e^{t(1)}(p) = 1-p+pe^t$$



$\Longleftrightarrow$



SAME!

and if $X \overset{d}{=} Y \Leftrightarrow M_X(t) = M_Y(t)$

$X \sim Bin(n,p)$

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} \overbrace{e^{tx}}^{(e^t)^x} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} \underbrace{(pe^t)^x}_{b} \underbrace{(1-p)^{n-x}}_{a} = (1-p+pe^t)^n$$ by bin. thm again

$(a+b)^n =$

$X_1, \dots, X_n \overset{iid}{\sim} Bern(p)$

$T = X_1 + \dots + X_n \sim Bin(n,p)$ Prove it!

by iden. dist

$$M_{X_1+\dots+X_n}(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_n}(t) = (M_X(t))^n = (1-p+pe^t)^n \checkmark$$

by mgf of sum is product of mgfs for ind. r.v.'s

mgf of $Bin(n,p)$

DONE

$X \sim Exp(\lambda)$

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{tx-\lambda x}\,dx = \lambda \int_0^\infty e^{x(t-\lambda)}\,dx$$

$$= \lambda \left[\frac{1}{t-\lambda} e^{x(t-\lambda)}\right]_0^\infty = \frac{\lambda}{t-\lambda}\left[\frac{1}{e^{xc}}\right]_0^\infty = \frac{\lambda}{t-\lambda}(1-0) = \frac{\lambda}{t-\lambda}$$ only for $t < \lambda$ o/w d.n.e.!

$\lambda \in (0,\infty)$ for $t<\lambda$ $t-\lambda<0$

$Y = aX \sim Exp\left(\frac{\lambda}{a}\right)$ Let's see it now

$$M_Y(aX) = M_X(at) = \frac{\lambda}{at-\lambda} = \frac{\frac{1}{a}(\lambda)}{\frac{1}{a}(at-\lambda)} = \frac{\frac{\lambda}{a}}{t-\frac{\lambda}{a}} = \frac{\lambda'}{t-\lambda'}$$ which is the mgf for $Exp(\lambda' = \frac{\lambda}{a})$!

More cool stuff on HW...

$$Z \sim N(0,1) \quad M_Z(t) = E(e^{tZ}) = \int_{-\infty}^{\infty} e^{t\cancel{Z}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + tx} \, dx$$

Remember...

$$X \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \checkmark \, dx = 1 \text{ since it's a PDF!}$$

Can we make ~~this~~ look like that?

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx)}$$

$$x^2 - 2tx = (x-t)^2 - t^2 = x^2 - 2tx + t^2 - t^2$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left((x-t)^2 - t^2\right)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t^2)} e^{+\frac{t^2}{2}}$$

$$\Rightarrow e^{\frac{t^2}{2}} \underbrace{\int \frac{1}{\sqrt{2\pi(1)}} e^{-\frac{1}{2(1)}(x-t)^2} \, dx}_{\text{PDF for } X \sim N(t, 1^2)} = \boxed{e^{+t^2/2} = m_Z(t)}$$

$E[Z] = 0$? $m'(t) = te^{t^2/2}$, $E[Z] = m'(0) = 0$ ✓

$E[Z^2] = 1$? $m''(t) = \underbrace{+t(+te^{-t^2/2})}_{u \, dv/dt} \underbrace{+ e^{+t^2/2}(+1)}_{+ v \, du/dt} = 1$ ✓ $\quad u = +t, v = e^{-t^2/2}$

} HW

Again... what's so special about the normal density???

Review...

exist?

	PMF	PDF	CDF	MGF
discrete r.v.	Y	N	Y	≈ always
cont.	N	Y	Y	≈ always

Let $X_1, X_2, \ldots X_n \overset{iid}{\sim}$ something with mean μ, std error σ

$\bar{X} := \frac{X_1 + \ldots + X_n}{n}$ ind. & iden. dist!

Recall... $E[\bar{X}] = \mu$, $Var[\bar{X}] = \frac{\sigma^2}{n}$ $SE[\bar{X}] = \frac{\sigma}{\sqrt{n}}$

What above in 10.4, we said $\frac{X - E(X)}{SE(X)}$ has mean 0, SE 1

standardization!

$C_n := \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$ $\Rightarrow$ has mean 0, SE 1

but we don't know its distribution!

Goal: find C_n's distr as n gets BIG!

$$C_n = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\frac{X_1 + \ldots + X_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\frac{X_1 + \ldots + X_n}{n} - \frac{n\mu}{n}}{\frac{\sigma}{\sqrt{n}}} = \frac{X_1 + \ldots + X_n - (\mu + \ldots + \mu)}{\sigma\sqrt{n}}$$

$$= \frac{1}{\sqrt{n}}\left(\frac{(X_1 - \mu) + \ldots + (X_n - \mu)}{\sigma}\right) = \frac{1}{\sqrt{n}}\left(\frac{X_1 - \mu}{\sigma} + \ldots + \frac{X_n - \mu}{\sigma}\right)$$

n stdized r.v.'s let $Z_i = \frac{X_i - \mu}{\sigma}$

$$= \frac{1}{\sqrt{n}}(Z_1 + \ldots + Z_n) = \frac{Z_1}{\sqrt{n}} + \ldots + \frac{Z_n}{\sqrt{n}}$$

We know $E[Z] = 0$, $SE[Z] = 1 \Rightarrow Var(Z) = 1 \Rightarrow E[Z^2] = 1$ since $\mu = 0$

Keep these in mind!!!

Do we know $M_Z(t)$? No!! Since we don't know dist of X, we don't know dist of Z...

Do we know mgf of $\frac{Z}{\sqrt{n}}$? Yes, as a function of $M_Z(t)$

$Y = aX \quad M_Y(t) = M_X(at)$

$$\Rightarrow M_{\frac{Z}{\sqrt{n}}}(t) = M_Z\left(\frac{t}{\sqrt{n}}\right)$$

How about mgf of C_n?

by identical distribution of $X_1, \ldots, X_n$

$$M_{C_n}(t) = M_{\frac{Z_1}{\sqrt{n}} + \cdots + \frac{Z_n}{\sqrt{n}}}(t) = M_{\frac{Z_1}{\sqrt{n}}}(t) \cdot \ldots \cdot M_{\frac{Z_n}{\sqrt{n}}}(t) = \left(M_{\frac{Z_i}{\sqrt{n}}}(t)\right)^n$$

$$= \left(M_Z\left(\frac{t}{\sqrt{n}}\right)\right)^n \quad // \quad \text{STOP}$$

Recall a mgf is $M_X(t) = E(e^{tX}) = 1 + \frac{t E(X)}{1!} + \frac{t^2 E(X^2)}{2!} + \frac{t^3 E(X^3)}{3!} + \ldots$

$$\left(1 + \frac{\frac{t}{\sqrt{n}} E(Z)}{1!} + \frac{\frac{t^2}{n} E(Z^2)}{2!} + \frac{\frac{t^3}{n^{3/2}} E(Z^3)}{3!} + \frac{\frac{t^4}{n^2} E(Z^4)}{4!} + \ldots\right)^n$$

Recall... $E(Z) = 0$, $E(Z^2) = 1$ since Z is a standardized r.v.

$$\left(1 + \frac{\frac{t^2}{2}}{n} + \frac{\frac{t^3}{6} E(Z^3)}{n^{3/2}} + \frac{\frac{t^4}{24} E(Z^4)}{n^2} + \ldots\right)^n$$

call this $e(n)$

$\underline{e(n) \in o\left(\frac{1}{n}\right)}$ which means $\lim_{n \to \infty} \frac{e(n)}{\frac{1}{n}} = 0$ true?

goes to 0 faster than $\frac{1}{n}$ goes to 0.

$$\frac{\frac{\frac{t^3}{6} E(Z^3)}{n^{3/2}}}{\frac{1}{n}} + \frac{\frac{\frac{t^4}{24} E(Z^4)}{n^2}}{\frac{1}{n}} + \ldots = \underbrace{\frac{\frac{t^3}{6} E(Z^3)}{\sqrt{n}}}_{\lim = 0} + \underbrace{\frac{\frac{t^4}{24} E(Z^4)}{n}}_{\lim = 0} + \ldots \quad \lim = 0$$