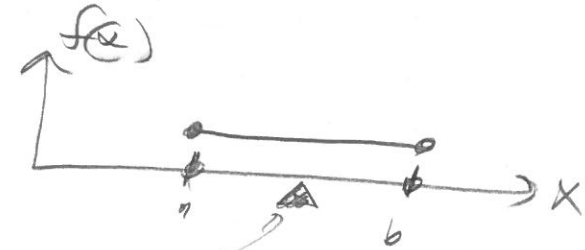
Next "Brand Name" cont. r.v.

$$X \sim \underbrace{U}_{\text{uniform}}(a,b) := \overbrace{\frac{1}{b-a}}^{\text{PDF}} \quad Supp[X] = [a,b]$$

Param space: $a<b$ $a,b \in \mathbb{R}$

Cont. analogue of discrete uniform
all #'s betw a,b are equally likely



Is it a r.v.? $\int_{x \in Supp(X)} f(x)\,dx = 1$?

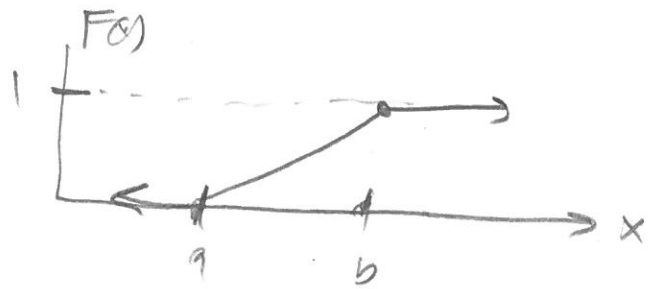
$$\int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[x\right]_a^b = \frac{b-a}{b-a} = 1 \checkmark$$

$$F(x) = \int f(x)\,dx = \frac{x}{b-a} + C$$

Solve for C... $F(a) = 0$

$$\Rightarrow \frac{a}{b-a} + C = 0 \Rightarrow C = -\frac{a}{b-a}$$

$$\Rightarrow F(x) = \frac{x}{b-a} - \frac{a}{b-a} = \frac{x-a}{b-a} \checkmark$$



$X \sim U(0,1)$ "the std uniform"

gives me a random # betw 0,1. Most important to CS. (Math.Random(); Java)

$f(x) = 1$, $F(x) = x$ Super simple!

$$\mu := E[X] = \int_{Supp(X)} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \boxed{\frac{b+a}{2}} \checkmark$$

$$Var[X] = \int (x-\mu)^2 f(x)\,dx = E[X^2] - \mu^2 = \int_a^b x^2 f(x)\,dx - \left(\frac{b+a}{2}\right)^2 = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{b^2+2ab+a^2}{4} =$$

$\frac{b^3-a^3}{b-a} = ?$ polynomial long division procedure:

good skill to know!!

$$\begin{array}{r} b^2 + a^2 + ab \\ b-a \,\big)\, b^3 - a^3 \\ (ab^2) \quad b^3 - ab^2 \\ \hline -a^3 + ab^2 \\ (aa^2) \quad -a^3 + a^2 b \\ \hline ab^2 - a^2 b \\ (ab) \quad ab^2 - a^2 b \\ \hline 0 \end{array}$$

Check this...

$(b-a)(b^2+a^2+ab) = b^3 - \cancel{ab^2} + \cancel{a^2b} - a^3 + \cancel{ab^2} - \cancel{a^2b}$ ✓

$$\frac{b^2+a^2+ab}{3} - \frac{b^2+2ab+a^2}{4}$$

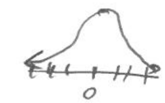
$$= \frac{4b^2+4a^2+4ab-3b^2-6ab-3a^2}{12}$$

$$= \frac{b^2+a^2-2ab}{12} = \boxed{\frac{(b-a)^2}{12}} \Rightarrow SE[X] = \frac{b-a}{\sqrt{12}}$$

Last brand new r.v. "the std. normal"

$Z \sim N(0,1) := \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{>0 \ \forall x}$ ✓ $\quad Supp[Z] = \mathbb{R}$

everywhere!!!



Is this a PDF?

$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1 \,?$ $\quad \int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx \stackrel{?}{=} \sqrt{2\pi}$

$u = \frac{x}{\sqrt{2}}$

let $u^2 = \frac{x^2}{2}$ $\quad \frac{du}{dx} = \frac{1}{\sqrt{2}}$

$\int_{\mathbb{R}} e^{-u^2} \sqrt{2}\, du \stackrel{?}{=} \sqrt{2\pi}$ $\quad \int_{\mathbb{R}} e^{-u^2} du \stackrel{?}{=} \sqrt{\pi}$

NOT ON EXAM

Imagine $x^2+y^2 = u^2$

$\Rightarrow \iint_{x \in \mathbb{R}, y \in \mathbb{R}} e^{-(x^2+y^2)} dx\,dy = \int_x e^{-x^2} dx \int_y e^{-y^2} dy \stackrel{?}{=} \sqrt{\pi} \cdot \sqrt{\pi} = \pi$

Polar coords...

$dA = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$

$= \iint_{\theta \in [0,2\pi],\, r \in [0,\infty)} e^{-r^2}\, r\,dr\,d\theta$

$q = r^2 \quad \frac{dq}{dr} = 2r \Rightarrow dr = \frac{1}{2}\frac{dq}{r}$

$= 2\pi \int_0^\infty r e^{-r^2} dr$

$= \cancel{2}\pi \int_0^\infty \cancel{r} e^{-q} \frac{1}{\cancel{2}} \frac{dq}{\cancel{r}}$

$= \pi \int_0^\infty e^{-q} dq = \pi \left[-e^{-q}\right]_0^\infty = \pi$ ✓

let $u = -\frac{x^2}{2} \Rightarrow \frac{du}{dx} = -x \Rightarrow dx = -\frac{du}{x}$

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$$\mu = E(Z) = \int_{\mathbb{R}} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x e^{-\frac{x^2}{2}} dx = -\frac{1}{\sqrt{2\pi}} \int_{u_1}^{u_2} e^u du = -\frac{1}{\sqrt{2\pi}} \left(e^u\right)_{u_1}^{u_2} = -\frac{1}{\sqrt{2\pi}} \left[e^{-\frac{x^2}{2}}\right]_{-\infty}^{\infty}$$

$$= -\frac{1}{\sqrt{2\pi}}(0-0) = \boxed{0}$$

$x = u \quad dv = xe^{-\frac{u^2}{2}}$

$\Rightarrow dx = du \Rightarrow v$

$$Var(Z) = E(X^2) - \cancel{\mu^2}^{0} = E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} dx = \ldots = 1$$

(integration by parts) $= \sigma^2 \Rightarrow SE(Z) = 1 = \sigma$

Z is called the std. normal and it is a stdized r.v.

$F_Z(x) = ?$ Not solvable in closed form!

$F(0) = ?$ 0.5 due to symmetry

A few to remember..

$P(Z \in [-1,1]) = F(1) - F(-1) \approx 0.68$

$P(Z \in [-2,2]) = F(2) - F(-2) \approx 0.95$

$P(Z \in [-3,3]) = F(3) - F(-3) \approx 0.997$

"68-95-99.7 Rule" or "3σ rule" "empirical rule"

Recin $X \sim Exp(\lambda) := \lambda e^{-\lambda x}$

let $Y = 2X$ how is Y distr? If $X \sim Bin(n,p)$ $2X$ was not binomial!

$$F_Y(x) := P(Y \le x) = P(2X \le x) = P\left(X \le \frac{x}{2}\right) = F_X\left(\frac{x}{2}\right)$$

let $u = \frac{x}{2} \Rightarrow \frac{du}{dx} = \frac{1}{2} \Rightarrow dx = 2du$

$$f_Y(x) \quad \frac{dF_Y(x)}{dx} = \frac{d}{dx} F_X\left(\frac{x}{2}\right) = \frac{1}{2}\frac{d}{du} F_X(u) = \frac{1}{2} f_X(u) = \frac{\lambda}{2} e^{-\lambda u} = \frac{\lambda}{2} e^{-\lambda \frac{x}{2}} = Exp\left(\frac{\lambda}{2}\right)$$

So if $Y = aX$, $X \sim Exp(\lambda)$, $Y \sim Exp\left(\frac{\lambda}{a}\right)$

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let $X \sim U(a,b)$, $Y = cX + d \sim U(ca+d, cb+d)$

$$F_Y(x) = P(Y \le x) = P(cX + d \le x) = P\left(X \le \frac{x-d}{c}\right) = F_X\left(\frac{x-d}{c}\right)$$

$$f_Y(x) = \frac{d}{dx} F_Y(x) = \frac{1}{c}\frac{d}{du} F_X(u) = \frac{1}{c} f_X(u) = \frac{1}{c}\frac{1}{b-a}$$

$Z \sim N(0,1)$ From HW 8 recall this...

$X = \sigma Z + \mu$ $\quad E[X] = \sigma \underbrace{E[Z]}_{0} + \mu = \mu$

$Var(X) = \sigma^2 \underbrace{Var(Z)}_{1} = \sigma^2 \Rightarrow SE[X] = \sigma$

How is X distr? $Supp[X] = \mathbb{R}$ still...

$$F_X(x) = P(X \le x) = P(\sigma Z + \mu \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right) = F_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$f_X(x) = \frac{d}{dx}\left[F_X(x)\right] = \frac{d}{dx}\left[F_Z\left(\frac{x-\mu}{\sigma}\right)\right]$$ let $u = \frac{x-\mu}{\sigma}$ $\quad \frac{du}{dx} = \frac{1}{\sigma} \Rightarrow dx = \sigma\, du$

~~log~~ density has a sqd error loss

$$= \frac{1}{\sigma}\frac{d}{du} F_Z(u) = \frac{1}{\sigma} f_Z(u) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

density of genl Normal r.v.

$X \sim N(\mu, \sigma^2)$

mean $\quad$ var.

But why is the normal density so important????

Wait two lectures.... We need to build some tools to understand the normal density

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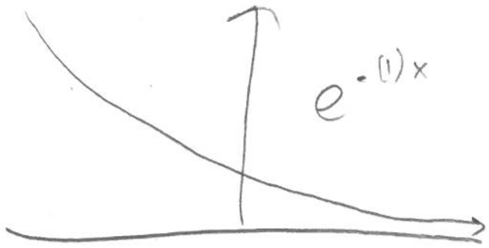
let $L(t) := \int_{\mathbb{R}} e^{-tx} f(x)\, dx$ "Bilateral Laplace Transform"

demon

"integrate x out" to be left with a function of t. It's an "operator" on function f.

if I want $L(1) = \int_{\mathbb{R}} e^{-x} f(x)\, dx$



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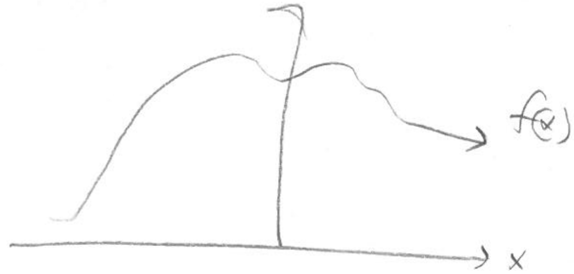

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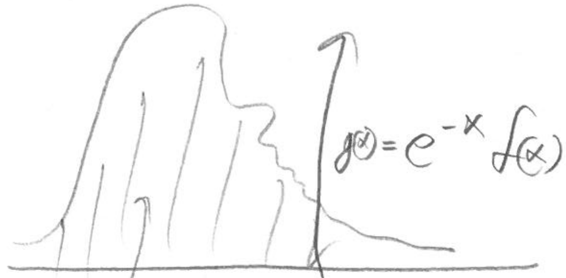
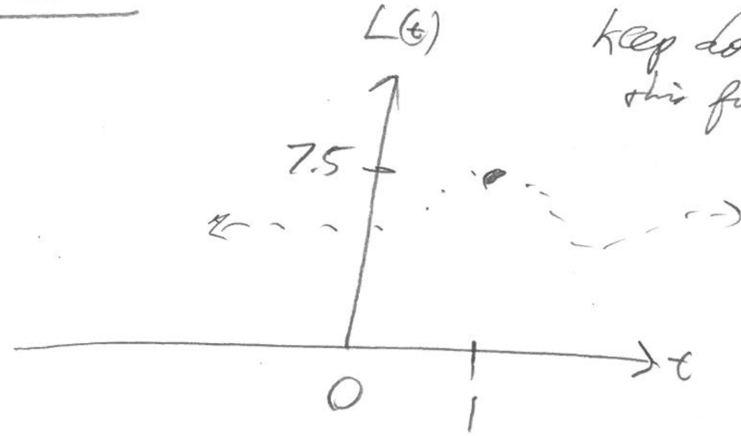


$\int e^{-x} f(x) = 7.5$



keep doing this for every t

turns out $L(t) \iff f(x)$ are $1{:}1$ (if $L(t)$ exists)

sometimes $\int = \infty$

If you give me $f(x)$, I give you $L(t)$ which is not shared by any other $f(x)$. " $L(t)$ " " $f(x)$ " " " $L(t)$

Thus $f(x)$ and $L(t)$ are two sides of the same coin. Looking at the same thing a different way.

Define $M_X(t) := E[e^{tX}] = \int_{x \in supp(X)} e^{tx} f(x)\,dx$ if cont. X r.v.

$= \sum_{x \in supp(X)} e^{tx} p(x)$ if discrete X r.v.

Moment generating function (mgf)