L1

Lecture 2 Math 271 Feb 3, 2015

$C = \{9{:}15,\ 12{:}15\}$

$2^C = \{\phi,\ \{9{:}15\},\ \{12{:}15\},\ \{9{:}15,\ 12{:}15\}\}$ ← all groups of combinations of elements of every cardinality

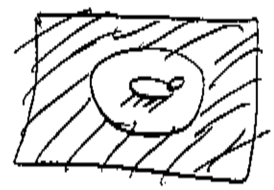
card 0   card 1   card 2

Any set A we have $a \in A$ or $a \notin A$ for all elements

What do we mean by "all elements"? There needs to be a "universe of discourse" AKA "Universe" (AKA "sample space" in prob & stats) and we denote it "Ω" "omega". Omega. Elements are ω, $\omega \in \Omega$ or "S"

It is all elements our current "scope" is limited to

Scope in CS? You decide what that scope is.



Let $\Omega = F \cup M$ (only 2 sexes)

Now $F \subseteq \Omega$, $M \subseteq \Omega$. All sets must be subsets of Ω.

Same as $\sum_{i=1}^{n} i = 1+2+\dots+n$

If n sets are unioned together and $\bigcup_{i=1}^{n} A_i = \Omega$

then the "set of sets" $\{A_1, A_2, \dots, A_n\}$ is called "collectively exhaustive" (Collectively, they exhaust or have, all elements in the Universe)

Ω

$A_1, \dots, A_n$ are collectively exhaustive

If $A_1 \cap A_2 = \emptyset \Rightarrow A_1, A_2$ "mutually exclusive"

If $A_i \cap A_j = \emptyset \ \forall i,j,\ i \neq j,\ i \in \{1,\dots,n\},\ j \in \{1,\dots,n\}$

all sets A_i are mutually exclusive $\Rightarrow \{A_1, \dots, A_n\}$ are mutually exclusive

mutually exclusive and collectively exhaustive looks like:

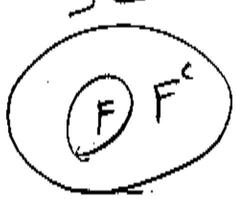
Ω

One more set operation: Complement

F^c all elements not in F $\quad F^c := \Omega \setminus F$

$= \{Bob, Joe, max\}$

Ω

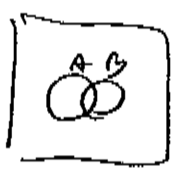


$F \cup F^c = ?$ $\quad F \cap F^c = ?$, $\Omega^c = ?$, $\phi^c = ?$

Where is this going? what's the chance if you pull out a name, it's a female?

$$\frac{|F|}{|\Omega|} = \frac{4}{7} \approx .57$$ be precise...

$\Omega \setminus A^c$, $A \cap A^c$, $A^c \cap \Omega$, $A \cup \Omega$, $A \cup A^c$



$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

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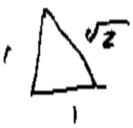
more vocabulary

$\mathbb{N} = \{1, 2, \dots\}$

$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$, $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\}$ set builder notation "such that"

Problem $\mathbb{Q}$ has holes  $\sqrt{2} \notin \mathbb{Q}$

$\mathbb{R} = \mathbb{Q} \cup \{\text{irrationals}\}$

e.g. $\sqrt{2}, \pi, e, \dots$

$[1,2] := \{x : x \geq 1 \;\&\; x \leq 2 \;\&\; x \in \mathbb{R}\}$ $\quad \min([1,2]) = ?$

$[a,b] := \{x \dots$ $\quad \min((1,2)) = ?$

$(1,2) := \{x : x > 1 \;\&\; x < 2 \;\&\; x \in \mathbb{R}\}$ $\quad (-\infty, \infty) = \mathbb{R}$

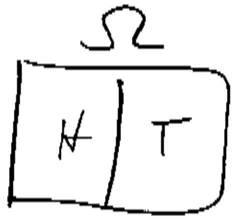
$(a,b) := \dots$

~~Set Theory~~

From now on, we will only be ~~interested~~ in sets whose elements are "outcomes". Outcomes are things that can "occur".

Ω is the "space" of all things that can occur.

For instance coin toss Heads or Tails $\omega_1 = H$, $\omega_2 = T$



$\{H\},\{T\}$ must exist? all elements?

$$2^{\Omega} = \{\phi\ \{H\}\ \{T\},\ \{H,T\}\}$$

elements of 2^{Ω} are subsets of Ω. They are called "events"

It is more convenient to build probability around "events" instead of outcomes.

Events are "measured" w.r.t. the size of Ω to determine probability.

$|\Omega| = 2$ does $|H|$ make sense? No! But $|\{H\}|$ makes sense

Outcomes are only important because they belong to events. abuse of notation

$P(\{H\}) = \frac{|\{H\}|}{|\Omega|} = \frac{1}{2}$ $P(H)$ is meaningless! P's domain is 2^{Ω} not Ω.

Our natural definition of $P(A) = \frac{|A|}{|\Omega|}$ ← the proportion of the sample space taken up by A.

$P(H \text{ or } T)$ $P(\phi) = \frac{|\phi|}{|\Omega|} = 0$ What is this event? the prob of "nothing" happening is 0.

$P(\{H\} \cup \{T\}) = P(\{H,T\}) = P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$ the prob of "something" happening is 1

How many events can we ask "what's the prob of ... happening"?

$$\Rightarrow |2^{\Omega}| = 4$$

More interesting: two coin tosses ← ordered pairs. Each outcome is distinct $HT \neq TH$

$$\Omega \times \Omega = \Omega^2 = \{HH, HT, TH, TT\}$$

← Cartesian product (Not on HW or exams)

$\Omega \times \Omega$

HH	HT
TH	TT

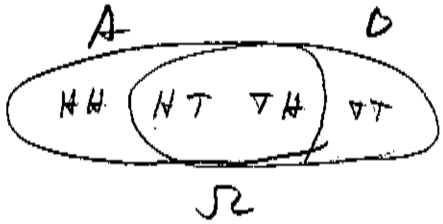
$|2^\Omega| = ?$ How many questions: what's the prob of ... ?

$$P(HH) = \frac{|\{HH\}|}{|\Omega|} = \frac{1}{4}$$

let $A = \{\omega : \omega \text{ has at least one } H\} = \{HH, HT, TH\}$

$P(A) = \frac{3}{4}$

let $B = \{\omega : \omega \text{ has at least one } T\} = \{TT, TH, HT\}$ $\quad P(B) = \frac{3}{4}$



$P(A \cup B) = 1$

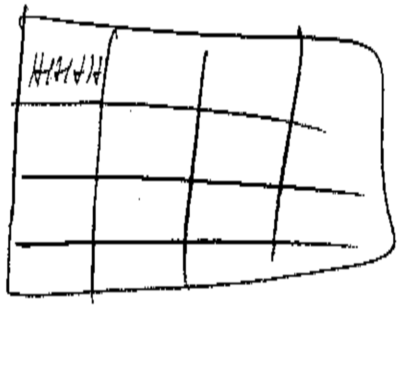
$P(A \cap B) = \frac{1}{2}$

$P(A|B) = \frac{1}{4}$

A, B mutually excl?

A, B coll. exhaustive?

Ω^4 (4 tosses) $\quad |\Omega^4| = 16 \quad |2^{\Omega^4}| = ?$ $\quad 2^{10} \approx 1000$, $2^6 = 64$ $\approx 64{,}000$



Does $P(HHHH) = P(HTHT)$?

Seems like $P(4H)$ should be less than $P(2H, 2T)$. why?

Ω^5 tosses $\quad |\Omega^5| = 32 \quad |2^{\Omega^5}| \approx 4$ billion $\quad$ powersets get big!!

Counting

Counting size of Ω. If $\Omega = \Omega_1 \times \Omega_2$, $|\Omega| = |\Omega_1||\Omega_2|$

e.g. flip coin, roll die

Ω

1H	2H	3H	4H	5H	6H
1T	2T	3T	4T	5T	6T

$$|\Omega| = \underset{(2)}{|\Omega_1|}\,\underset{(6)}{|\Omega_2|} = 12$$

What if $\Omega \neq \Omega_1 \times \Omega_2$? Need to learn how to count carefully!

Let F|M be the "pure" sample space $= \{J, M, S\}$

How many ways to arrange them in front of you?

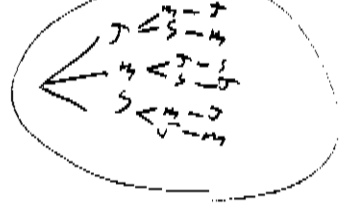
$\Omega = \{JMS, JSM, MJS, MSJ, SMJ, SJM\}$, $|\Omega| = 6 \neq 3^3 = 27$

Why?

$$|\Omega| = \underset{\text{\# of first possible}}{\underline{3}} \cdot \underset{\text{\# of second possible}}{\underline{2}} \cdot \underset{\text{\# of third possible}}{\underline{1}}$$

#'s like these come up a lot

$$3! = 3 \cdot 2 \cdot 1, \quad n! := \prod_{i=1}^{n} i = n(n-1)\cdots(1) \quad \text{factorial}$$



How about 5 people? $|\Omega| = 5! = 120$

10 people? $= 3.6\ m$

20 people? $= 2.7 \cdot 10^{32} =$ diam (universe) in ft.

This type of sample space's outcomes are called "permutations"

Given a set of objects A, how many ways to arrange them?

$|A|!$ Notation $_{10}P_{10}$

$$\underbrace{10}_{\text{10 people}} P \underbrace{10}_{\text{10 chairs}}$$

$\#$people $\geq$ $\#$chairs in the above notation

What if $\#$ chairs < $\#$ people?

Let's say 10 people, 1 chair $_{10}P_1$

$$\frac{10}{\#\text{ of 1st chair possibilities}} = \frac{10!}{9!}$$

" " 2 chairs

$$\underset{\#\,1^{st}}{\underline{10}} \cdot \underset{\#\,2^{nd}}{\underline{9}} = \frac{10!}{8!} \qquad {}_{10}P_2$$

" " 5 chairs

$$\underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} = \frac{10!}{5!} \qquad {}_{10}P_5$$

Permutation $_nP_k = \dfrac{n!}{(n-k)!}$

First example $_nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!$

For this to make sense, we must define $0! := 1$

5 Boys, 5 Girls together as couples alternating $(5!5!2)$

— Decks of cards Scotts / nude