Lecture 17 4/21/15 Math 241

Recall $T \sim \text{Geometric}(p)$ think of free variable as time

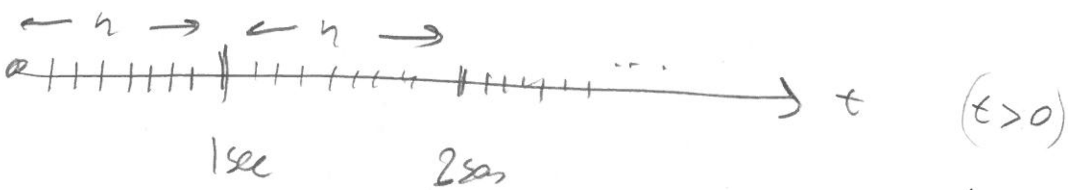
$\underline{0}$ $\underline{0}$ $\underline{1}$

Bern(p) Bern(p) Bern(p)

$t=1$ $t=2$ $t=3$

$$p(t) = (1-p)^{t-1} \cdot p$$

What if in every second, we run n separate experiments?



As a function of time, $p(t) = (1-p)^{nt-1} p$

If p is high, this doesn't make too much sense $E(T) = \frac{1}{p}$ ← # of experiments

$E(T) = \frac{1}{np} \approx 0$ ← # of seconds, if n is large

Here's what we'll do (Poisson), let $p \to 0$ and $n \to \infty$

but $\lambda = np$ as $n\uparrow$ $p\downarrow$ and the product is constant!

$\Rightarrow p = \frac{\lambda}{n}$

$$\Rightarrow p(t) = \left(1 - \frac{\lambda}{n}\right)^{nt-1} \frac{\lambda}{n}$$

Pretend experiments are happening continuously $n \to \infty$ $T \sim \lim_{n \to \infty} \text{Geometric}\left(\frac{\lambda}{n}\right)$

$$p(t) = \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^{nt-1} \frac{\lambda}{n} = \lim \left(1 - \frac{\lambda}{n}\right)^{nt-1} \lim \cancel{\frac{\lambda}{n}}^{\to 0} = 0$$

We have a Geometric r.v. with $p(t) = 0$ everywhere!

$\Rightarrow p(t)$ is <u>NOT</u> a PMF since it is zero above 0 and hence cannot sum to 1.

$1 = \int_1^e \frac{1}{x}\,dx$ or $e := \sum_{i=0}^{\infty} \frac{1}{i!}$

$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$n=10 \to 2.5937$
$n=100 \to 2.7048$
$n=1000 \to 2.7169$
$n=1e6 \to 2.7183$

What about $F(t)$?

Recall $X \sim \text{Geometric}(p)$, $F(x) = 1-(1-p)^x$

Now $F(t) = \lim_{n\to\infty} 1-\left(1-\frac{\lambda}{n}\right)^{nt} = 1 - \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{nt} = 1 - \left(\overbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}^{e^{-\lambda}}\right)^t = 1-e^{-\lambda t}$

let $a \in \mathbb{R}$

$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$ let $\frac{a}{n} = \frac{1}{m} \Rightarrow n = ma$

$n\to\infty$ $ma\to\infty$ a const so $m\to\infty$

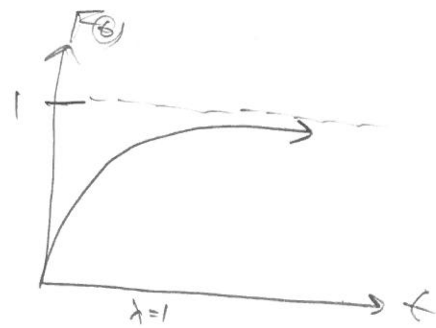
$= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{ma} = \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^a = e^a$

Is this a CDF?

$\lim_{t\to-\infty} F(t) = F(0) = 1-e^{-\lambda(0)} = 1-1=0$ ✓ (since $t>0$)

$\lim_{t\to\infty} F(t) = 1 - \lim_{t\to\infty}\frac{1}{e^{\lambda t}} = 1-0=1$ ✓

$\frac{dF}{dt} > 0$? $\frac{dF}{dt} = \lambda e^{-\lambda t} = \frac{\overset{>0}{\lambda}}{\underset{>0}{e^{\lambda t}}} > 0$ ✓



$\lambda=1$

$F(1) = 1-e^{-1} = 0.63$

$F(6) = 1-e^{-6} = 0.998$

$P(T\in(3,4)) = F(4)-F(3) = (1-e^{-4})-(1-e^{-3}) = .031$

This thing has a CDF but no PMF. It is a r.v. but not discrete.

$\text{Supp}[T]$? = any time > 0 $= (0,\infty)$

uncountable ∞

$|\text{Supp}[T]| = |\mathbb{R}| > |\mathbb{N}|$ How did this happen? due to limit

$n\to\infty$ infinite divisions $\Rightarrow$ no missing numbers

But is time actually continuous?

Ancient Greeks thought so. And they thought space was continuous as well.
But the atomic theory / quantum contradicts this!

|——| Planck length $\simeq 1.62 \times 10^{-35}$ m

not possible to tell the difference between 2 locations < 1 PL apart.

speed of light

$\longrightarrow$

|——| Planck time $\simeq 5.3 \times 10^{-44}$ s

time cannot be distinguished < 1 PT.

thus it seems time is discrete. So T as a r.v. is technically "fake" but a good model for a geometric with high n, low p.

$p(3) = 0$. Why is this

$t = 3.0000\ldots$ s   I can't write this down!

If I stop at 6 0's, I'm really saying it could have been between:

$$t \in [2.999950,\ 3.000049]$$

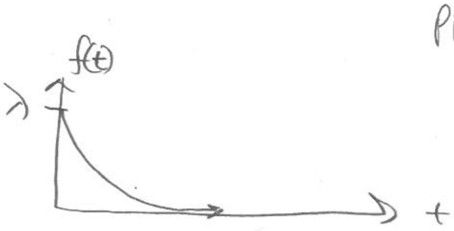
so $p(3.000000) = F(\ ) - F(\ ) > 0$!

New question: how does the CDF change?

$f(t) := F'(t) = \frac{dF}{dt} = \lambda e^{-\lambda t}$

$P(T \in (a,b)) = F(b) - F(a) = \int_a^b f(t)\,dt$ F. T. of C.

probability density function (PDF)
(how dense is the probability packed around t?)



$\lambda = 1$ $f(3) = 0.05 \neq p(3) = 0$

PDF $\neq$ PMF $\Rightarrow$ PDF does not measure $P(T = ?)$.

The PDF is completely abstract. PDF is good for comparing two r.v.'s at a certain element in the support (see more later).

Def: X is a continuous r.v. if $|Supp(X)| = |\mathbb{R}| > |\mathbb{N}|$.

Shared with discrete

① $Supp(X) \subseteq \mathbb{R}$

② $F(x)$ is a CDF

Not Shared

① PMF $p(x)$ d.n.e. Why?

② $f(x) := F'(x)$ exists and

a) $\int_{x \in Supp(X)} f(x)\,dx = 1$

$\rightarrow$ $f(x)$ is now a fingerprint: $X_1 \overset{d}{=} X_2 \iff f_1 = f_2$ if cont.

by F.T.C. $\Rightarrow P(X \in [a,b]) = \int_a^b f(x)\,dx \quad a<b,\ a,b \in Supp(X)$

$\Rightarrow P(X = a) = P(X \in [a,a]) = \int_a^a f(x)\,dx = 0$

b) $f(x) > 0 \quad \forall x \in Supp(X)$

Note $f(x)$ can be > 1

It is not a probability!!!!

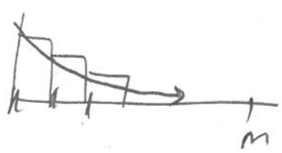
Express $P(X \le x)$ in terms of $f(x)$

$$F(x) = \int_{\min\{Supp(X)\}}^{x} f(y)\,dy$$

(or $-\infty$ if no min)

What is $E(X)$? Previously, $E(X) = \sum_{x \in Supp(X)} x\, p(x)$

Now..

$E(X) =$ 

$S = Supp(X) = [0, m]$ split into rectangles. Let rectangles shrink to 0 width

$\mu := E(X) = \int_{x \in Supp(X)} x f(x)\,dx \quad \Rightarrow \quad Var(X) := \int_{x \in Supp(X)} (x-\mu)^2 f(x)\,dx$

and

$E(g(X)) = \int_{x \in Supp(X)} g(x) f(x)\,dx \quad \Downarrow \quad Var(g(X)) = \int_{x \in Supp(X)} \left(g(x) - E(g(X))\right)^2 f(x)\,dx$

All rules apply:

$E[aX+c] = a\mu + c$

$Var[aX+c] = a^2\sigma^2$

$SE[aX+c] = |a|\sigma$

$E[\sum X_i] = \sum E(X_i)$ for all $X_1, X_2, \ldots$

$Var[\sum X_i] = \sum Var(X_i)$ for all $X_1, X_2, \ldots$ independent

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What r.v.'s have we been studying?

$X \sim \underbrace{Exp}(\lambda) := \underbrace{\lambda e^{-\lambda x}} \leftarrow f(x)$ not $p(x)$

exponential w/ param λ

for cont. r.v.'s we put $f(x)$ here, not $p(x)$.

Exponential used to model waiting times. $Supp(X) = (0,\infty)$ or $[0,\infty)$

$$E(X) = \int_0^\infty x\lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty \underset{u}{x}\, \underbrace{e^{-\lambda x}dx}_{dv} =$$

$\lambda \in (0,\infty)$

$\lambda = np$, $p \in (0,1)$ but $n \to \infty$

$\int u\,dv = uv - \int v\,du$

$dv = e^{-\lambda x}dx$

$v = -\frac{1}{\lambda}e^{-\lambda x}$ $\quad \int v\,du = \int -\frac{1}{\lambda}e^{-\lambda x}dx$

$= +\frac{1}{\lambda^2}e^{-\lambda x}$

Degenerate case $\lambda = 0$ now illegal!

$f(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & o/w \end{cases}$ $\quad \int f(x)\,dx \neq 1$

$$= \lambda\left[-\frac{1}{\lambda}xe^{-\lambda x} - +\frac{1}{\lambda^2}e^{-\lambda x}\right]_0^\infty$$

$$= \lambda\left[-xe^{-\lambda x}\right]_0^\infty - \frac{1}{\lambda}\left[e^{-\lambda x}\right]_0^\infty$$

$$= (0-0) - \frac{1}{\lambda}(0-1) = \boxed{\frac{1}{\lambda}}$$

$Var[X]$? HW...

$X \sim Geom(p)$ $\quad E(X) = \frac{1}{p}$

Now $p = \frac{\lambda}{n}$ but now rescaled in $X = nt$

$E[X] = \frac{n}{\lambda}$ exp's now in seconds $\cdot \frac{1}{n}\frac{sec}{exp} = \frac{1}{\lambda}$ makes sense!

Geometric : discrete waiting time for one event

Exponential : cont " "

Neg Bin : discrete " multiple events

Erlang / Gamma : cont " " (Math 272)

If Geometric memoryless $\Leftrightarrow$ Exponential memoryless as well:

$$P(X > x + x_0 \mid X > x_0) = \frac{P(X > x + x_0 \ \& \ X > x_0)}{P(X > x_0)} = \frac{P(X > x + x_0)}{P(X > x_0)}$$

$$X \sim Exp(\lambda) \Rightarrow F(x) = 1 - e^{-\lambda x} \Rightarrow P(X > x) = 1 - F(x) = + e^{-\lambda x}$$

$$\Rightarrow \frac{e^{-\lambda(x + x_0)}}{e^{-\lambda x_0}} = e^{-\lambda x} = P(X > x) \checkmark$$

See lecture before midterm 2

Given $f(x)$, solve for $F(x)$

$$\Rightarrow F(x) = \int_{x \in supp(X)} f(x)\, dx + C$$

How to solve for C?

You know $F(\min(supp(X))) = 0$

$X \sim Exp(\lambda)$

$$\int f(x)\, dx = -e^{-\lambda x} + C \qquad F(0) = 0$$

$$-e^{-(0)\lambda} + C = 0 \Rightarrow -1 + C = 0 \Rightarrow C = 1$$

$$\Rightarrow F(x) = 1 - e^{-\lambda x}$$