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MATH 241

Lecture 11 3/17/15

why?

$$X \sim \text{Hyper}(n, K, N) := \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} = p(x)$$

let $p = \frac{K}{N}$

$K \in \{1, \ldots, N-1\}$

$p \in \left\{\frac{1}{N}, \ldots, \frac{N-1}{N}\right\}$

$$X \sim \text{Hyper}(n, p, N) := \frac{\binom{pN}{x}\binom{N-pN}{n-x}}{\binom{N}{n}} = p(x)$$

10 cards, 4 Red

100 cards, 40 Red

1000 cards, 400 Red

⋮

eventually not replacing means nothing!

$$p(x) = \lim_{N\to\infty} \frac{\binom{pN}{x}\binom{N(1-p)}{n-x}}{\binom{N}{n}} = \lim_{N\to\infty} \frac{\frac{(pN)!}{x!(pN-x)!}\cdot\frac{(N(1-p))!}{(n-x)!(N(1-p)-n+x)!}}{\frac{N!}{n!(N-n)!}}$$

$$= \frac{n!}{x!(n-x)!} \lim_{N\to\infty} \frac{(pN)!}{(pN-x)!}\,\frac{(N(1-p))!}{(N(1-p)-n+x)!}\,\frac{(N-n)!}{N!}$$

$\frac{7!}{3!} = \overbrace{7\cdot6\cdot5\cdot4}^{} = 7\cdot\overset{(7-1)\cdots}{\cdots}\cdot(7-3)$ $\quad \frac{7!}{(7-4)!} = \underbrace{7\cdot6\cdot5\cdot4}_{4 \text{ times}} = 7\cdot\overset{(7-1)\cdots}{\cdots}\cdot(7-4+1)$

$\frac{(pN)!}{(pN-x)!} = \overbrace{(pN)(pN-1)\cdots(pN-x+1)}^{x \text{ times}}$ $\quad \frac{(N(1-p))!}{(N(1-p)-(n-x))!} = \overbrace{N(1-p)(N(1-p)-1)\cdots(N(1-p)-(n-x)+1)}^{n-x \text{ times}}$

$\frac{3!}{7!} = \frac{1}{\underbrace{7\cdot6\cdot5\cdot4}_{4 \text{ times}}} =$

$\frac{N!}{(N-n)!} = \underbrace{N(N-1)\cdots(N-n+1)}_{n \text{ times}}$

$\frac{(7-4)!}{7!} = \frac{1}{7\cdot6\cdot5\cdot4}$

What does this look like?

$$\binom{n}{x}\lim_{N\to\infty} \frac{pN}{N}\frac{pN-1}{N-1}\cdots\frac{pN-x+1}{N-x+1}\,\frac{N(1-p)}{N-x}\cdot\frac{N(1-p)-1}{N-x-1}\cdots\frac{N(1-p)-(n-x)+1}{N-n+1}$$

$\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x)\lim_{x\to c} g(x)$ (product rule)

$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$ if f, g diff. near c.

$\lim_{N\to\infty}\frac{pN}{N} = p$, $\lim\frac{pN-1}{N-1} = \lim p = p$, $\lim\frac{N(1-p)}{N-x} = \lim(1-p) = 1-p$, $\lim\frac{N(1-p)-(n-x)+1}{N-n+1} = \lim 1-p = 1-p$

Collect terms $\overbrace{p\cdots p}^{x \text{ times}} \cdots \overbrace{(1-p)\cdots(1-p)}^{n-x \text{ times}}$

$\Rightarrow p(x^*) = \binom{n}{x}p^x(1-p)^{n-x}$ $\quad X^*$, the limiting r.v. is called a "binomial r.v."

we will see why...

$X \sim \text{Binomial}(n, p) \overset{d}{=} \lim_{N\to\infty} \text{Hypergeometric}(n, p, N)$

No longer need N since $N\to\infty$ and now $\Rightarrow$ sample with replacement

~~Binomial powerful!, Sample with replacement.~~

~~Flip 10 coins $P(5H) = \frac{\frac{10!}{5!5!}}{2^{10}}$ Hd~~

~~$X \sim \text{Bin}(10, \frac{1}{2})$, $P(X=5) = \binom{10}{5}(\frac{1}{2})^5(\frac{1}{2})^{10-5} = \frac{\frac{10!}{5!5!}}{2^{10}}$~~

"You're going inside a bag, and pulling out a success or failure each time with equal prob."

Review... Independence of r.v.'s

$P(X_1 = x_1, X_2 = x_2) = P(X = x_1)\, P(X = x_2) \quad \forall x_1 \in Supp(X_1),\ \forall x_2 \in Supp(X_2)$

or $P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$

or $P(X_2 = x_2 \mid X_1 = x_1) = P(X_2 = x_2)$

If you tell me value of x_2, it doesn't affect the pmf of X_1

$P(A, B) = P(A)P(B)$ — remember these are events at the end of the day underneath the hood..

$X_1 \sim$ Bernoulli $\left(\frac{1}{3}\right)$ from unfair coin A

$X_2 \sim$ Bernoulli $\left(\frac{1}{3}\right)$ " B

$X_1 \overset{d}{=} X_2$ (since $F_{X_1}(x) = F_{X_2}(x)$) but are X_1, X_2 ind?

$P(X_1 = 1 \mid X_2 = 0) = P(H_1 \mid T_2) = P(H_1) = \frac{1}{3}$

$P(X_1 = 1 \mid X_2 = 1) = P(H_1 \mid H_2) = P(H_1) = \frac{1}{3}$

$P(X_1 = 0 \mid X_2 = 0) = P(T_1 \mid T_2) = P(T_1) = \frac{2}{3}$

$P(X_1 = 0 \mid X_2 = 1) = P(T_1 \mid H_2) = P(T_1) = \frac{2}{3}$

all cases $\Rightarrow$ ind.

independent and identically distributed. iid. "iid"

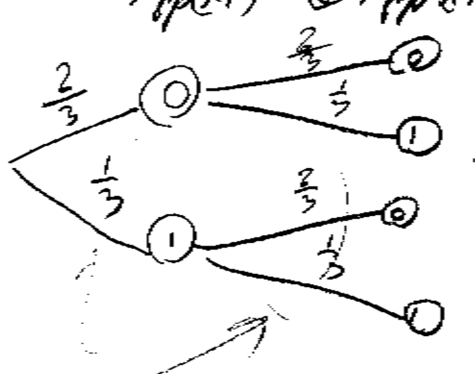
$X_1, X_2 \overset{iid}{\sim} Bern\left(\frac{1}{3}\right)$

"both X_1 and X_2 are ind. and iden. distributed as Bernoulli$\left(\frac{1}{3}\right)$'s"

$T_2 = X_1 + X_2$ — a new r.v.! What's its pmf.

due to ind of X_1, X_2

Supp(X_1)   Supp(X_2)



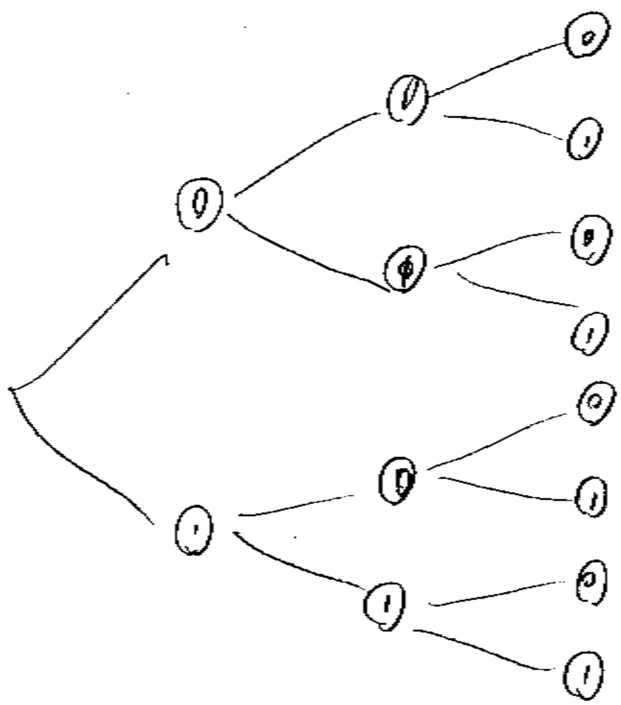
iden. dist.

$P(X_1 = x_1\ \&\ X_2 = x_2)$	T
$\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$	0
$\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)$	1
$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$	1
$\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$	2

$\Rightarrow T_2 \sim \begin{cases} 0 & \text{w.p. } 4/9 \\ 1 & \text{w.p. } \frac{2}{9} + \frac{2}{9} = \frac{4}{9} \\ 2 & \text{w.p. } \frac{1}{9} \end{cases}$

not Unif($\{0,1,2\}$)!!

$T_3 = X_1 + X_2 + X_3$ where $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$



p	T
$\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^3$	0
$\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2$	1
$\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2$	1
$\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1$	2
$\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2$	1
$\left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1$	2
$\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2$	2
$\left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0$	3

$$T \sim \begin{cases} 0 & \text{w.p. } 1 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^3 \\ 1 & \text{w.p. } 3 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^2 \\ 2 & \text{w.p. } 3 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1 \\ 3 & \text{w.p. } 1 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0 \end{cases}$$

How many ways to get 0-0-0? 0? $1 = \binom{3}{0}$

How many ways to get 1-0-0? 1? 0-1-0, 0-0-1 ~ indistinct $3 = \binom{3}{1}$

How many ways to get 2? 1-1-0, 1-0-1, 0-1-1 $3 = \binom{3}{2}$

How many ways to get 3? 1-1-1 $1 = \binom{3}{3}$

_ _ _ in order to get one success...

$T_n = X_1 + \dots + X_n$, $X_1, \dots, X_n \overset{iid}{\sim} \text{Bern}\left(\frac{1}{3}\right)$

$$T_n \sim \begin{cases} 0 & \text{w.p. } \binom{n}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^n \\ 1 & \text{w.p. } \binom{n}{1} \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{n-1} \\ \vdots & \\ n-1 & \text{w.p. } \binom{n}{n-1} \left(\frac{1}{3}\right)^{n-1} \left(\frac{2}{3}\right)^1 \\ n & \text{w.p. } \binom{n}{n} \left(\frac{1}{3}\right)^n \left(\frac{2}{3}\right)^0 \end{cases}$$

$\Rightarrow p(x) = \binom{n}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{n-x}$

If Bern(p)

$\Rightarrow p(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$\Rightarrow T_n \sim \text{Binom}(n, p)$

Amazing: two ways to think about the Binomial. Ind, iden dist, Bernoulli

do coin flip example

do lottery example

with Binomial CDF + complement rule

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Chevalier de Mere P(at least one 6,6 in 24 rolls of 2 die) $= 1 - P$(zero 6's in 24 rolls)

$X \sim \text{Binomial}(24, \frac{1}{36})$

$\Downarrow$

$1 - P(X=0)$

$= 1 - \binom{24}{0}\left(\frac{1}{36}\right)^0\left(\frac{35}{36}\right)^{24}$

$= 1 - \left(\frac{35}{36}\right)^{24} = .4914$ again ...

$X \sim \text{Binomial}(n, p)$

$Supp(X) = \{0, 1, \ldots, n\}$

$p(x) = \underbrace{\underbrace{\binom{n}{x}}_{\geq 1} \underbrace{p^x}_{\substack{\in(0,1) \\ >0}} \underbrace{(1-p)^{n-x}}_{\substack{\in(0,1) \\ >0}}}_{>0}$ is $p(x) > 0 \; \forall x \in Supp(X)$?

Need parameter space ...

$n=0$? No... Just like $N \in \{2, 3, \ldots\}$, $K \in \{1, 2, \ldots, N-1\}$, $\frac{K}{N} \in \{\frac{1}{N}, \ldots, \frac{N-1}{N}\} \Rightarrow (0,1)$

$n \in \mathbb{N}, \; p \in (0,1)$

If $n = 0 \Rightarrow$ doesn't make sense

If $p = 0 \Rightarrow p(x) = \binom{n}{x} 0^x (1-0)^{n-x}$ $\quad 0^x = 0 \; \forall x$ except $x=0$

$\lim_{x \to 0^+} x^x = 1 \Rightarrow 0^0 := 1$

$\Rightarrow X \sim \text{Bin}(n, 0) \stackrel{d}{=} \text{Deg}(0)$ Why?

If $p = 1 \Rightarrow p(x) = \binom{n}{x} 1^x 0^{n-x}$ 0 always except $x = n \Rightarrow X \sim \text{Bin}(n, 1) \stackrel{d}{=} \text{Deg}(n)$ Why?

STEP

Does $\sum_{x \in Supp(X)} p(x) \stackrel{?}{=} 1$ $\quad (a+b)^n = \sum$ $\quad \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = ((1-p) + (p))^n = 1^n = 1$ ✓

Remember $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$

Cool Trick... this is why it's called the binomial r.v.!

Special case $X \sim \text{Binomial}(1, p) = \binom{1}{x} p^x (1-p)^{1-x} = \frac{1}{x!(1-x)!}$

$\Rightarrow Supp(X) = \{0, 1\}$ $\quad \|$ Bernoulli(p) $\quad P(X=0) = 1-p, \; P(X=1) = p$