Lecture 16 Math 241 4/2/15 (See p5 Lec 15 first)

If $X_1, \ldots, X_n$ ind... not necessarily iden distr

In general $Var[X_1 + \ldots + X_n] = Var[X_1] + \ldots + Var[X_n]$

or $Var\left(\sum X_i\right) = \sum Var(X_i)$

$$Var[\bar{X}_n] = Var\left(\frac{X_1 + \ldots + X_n}{n}\right) = Var\left[\frac{1}{n}(X_1 + \ldots + X_n)\right] = \frac{1}{n^2} Var(X_1 + \ldots + X_n) =$$

$$= \frac{1}{n^2}(\underbrace{\sigma^2 + \ldots \sigma^2}_{n}) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

$$\Rightarrow SD[\bar{X}_n] = \frac{\sigma}{\sqrt{n}} \qquad \lim_{n \to \infty} SD[\bar{X}_n] = 0$$

$$\Leftrightarrow \lim_{n \to \infty} Var[\bar{X}] = 0$$

general r.v X

If $SD(X) = 0 \Rightarrow X \sim Deg$

$SD[X] = 0 \Rightarrow Var(X) = 0$

$\Rightarrow \sum_{x \in supp(X)} (x - \mu)^2 p(x)$, $(x-\mu)^2 \to$ only $= 0$ if $x = \mu$, $p(x) > 0$

$\Rightarrow supp(X) = \mu \Rightarrow X \sim Deg$

"proof" of LLN $\bar{X} \to \mu$

Back to Uber Example...

$T = 3 + 0.40W$, $E(T) = 3 + 0.40E(W) = ?$

$Var(T) = (0.40)^2 \sigma_W^2$, $SD(T) = 0.40\sigma_W = ?$

Calculate

$X_1, \ldots, X_n \overset{iid}{\sim} Bern(p)$

$T_n = X_1 + \ldots + X_n$ $\quad E[T_n] = E[X_1 + \ldots + X_n] = \overbrace{p + \ldots + p}^{n} = \boxed{np}$ much easier!

$Var(T_n) = Var[X_1 + \ldots + X_n] = \overbrace{p(1-p) + \ldots + p(1-p)}^{n} = \boxed{np(1-p)}$ easier than...

$Var(T_n) = E(T_n^2) - (E(T_n))^2 = E(T_n^2) - n^2p^2 = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} = \boxed{np(1-p)}$ ~~HARDER~~!!!

$$E(T_n^2) = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \cdot x \frac{n(n-1)!}{x \cdot (x-1)!(n-x)!} p \cdot p^{x-1} (1-p)^{n-x}$$

why?

$$= np \sum_{x=1}^{n} x \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x} = np \sum_{y=0}^{n-1} (y+1) \binom{n-1}{y} p^y (1-p)^{n-y-1} = np \sum_{y=0}^{m} (y+1) \binom{m}{y} p^y (1-p)^{m-y}$$

let $y = x-1 \Rightarrow x = y+1$ $\qquad$ let $m = n-1 \Rightarrow n = m+1$

$$= np \left(\underbrace{\sum_{y=0}^{m} y \binom{m}{y} p^y (1-p)^{m-y}}_{Y \sim Bin(m,p) \Rightarrow E(Y) = mp = (n-1)p} + \underbrace{\sum_{y=0}^{m} \binom{m}{y} p^y (1-p)^{m-y}}_{1} \right) = np((n-1)p + 1)$$

$$= np(np - p + 1) = n^2p^2 - np^2 + np$$

$X \sim \text{Geom}(p)$ $\quad E(X) = \frac{1}{p}$, $Var(X) = \frac{1-p}{p^2}$ (Annoying proof)

$:= (1-p)^{x-1} p$

What about $P(X = 17 \mid X > 10) = \dfrac{\overbrace{P(X=17 \,\&\, X \geq 10)}^{P(X=17)}}{P(X>10)}$

$A \cap B = B$

$$= \frac{(1-p)^{16} p}{\underbrace{(1-p)^{10}}_{1-F(10)}} = (1-p)^{6} p = P(X=7)$$

$$P(X = b+x \mid X > b) = \frac{(1-p)^{b+x-1} p}{(1-p)^{b}} = (1-p)^{x-1} p = P(X=x)$$

"memorylessness" comes from independence

Midterm 2 ↑

Final ↓

$\boxed{\text{REVIEW}}$