X_1, X_2 equal in distribution $X_1 \stackrel{d}{=} X_2$

if $p_1(x_1) = p_2(x_2)$ or $F(x_1) = F(x_2)$

if they exist

$p(x)$, $F(x)$ are "fingerprints" of the r.v. If they match, they are the same.

Lecture 10 3/12/15

$X \sim \text{Bernoulli}(p) := \begin{cases} 1 \text{ w.p. } p \\ 0 \text{ w.p. } 1-p \end{cases}$ ← nomenclature

$= \underbrace{p^x(1-p)^{1-x}}_{\text{PMF}} = p(x)$

10 cards 4R, 6B

$P(\text{getting 2R drawing of 3}) = \dfrac{\binom{4}{2}\binom{6}{1}}{\binom{10}{3}}$

$P(\text{getting } x \text{ R " "}) = \dfrac{\binom{4}{x}\binom{6}{3-x}}{\binom{10}{3}}$

$P(\text{getting } x \text{ R " } n) = \dfrac{\binom{4}{x}\binom{6}{n-x}}{\binom{10}{n}}$

N cards, K Red (success), N-K blue (failure)

$P(\text{" } x \text{ R " } n) = \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$ draws of success/failure without replacement

$F(x)$? Hard..

(Hyper)

$X \sim \text{Hypergeometric}(n, K, N) := \dfrac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$

equivalent parameterization $\Rightarrow R = N-K$, $N = K+R$

or $X \sim \text{Hypergeometric}(n, K, R) := \dfrac{\binom{K}{x}\binom{R}{n-x}}{\binom{K+R}{n}}$

we will see another parameterization in a little bit...

Parameter Space think... $K \not> N$, $n \not> N$

$N=0$?

$N=1$ what can K and n be?

K = 0/1, n = 1 ← can n be 0? No

$K=0$: $\dfrac{\binom{0}{x}\binom{1}{1-x}}{\binom{1}{1}}$ $K=1$: $\dfrac{\binom{1}{x}\binom{0}{1-x}}{\binom{1}{1}}$ $\text{Supp}[X] = \{0\}$ $X \sim \text{Deg}(0)$

$N=2$ what can K be? What can n be?

K = ~~0~~, 1, ~~2~~ n = ~~0~~, 1, ~~2~~

$\text{Supp}[X] = \{0, 1\}$

$X \sim \text{Hyper}(1, 1, 2) := \dfrac{\binom{1}{x}\binom{1}{1-x}}{\binom{2}{1}} = \dfrac{\frac{1!}{x!(1-x)!}\ \frac{1!}{x!(1-x)!}}{2} = \frac{1}{2} \stackrel{d}{=} \text{Bernoulli}\left(\frac{1}{2}\right)$

$$X \sim \text{Hyper}(1, 2, 2) := \frac{\binom{2}{x}\binom{0}{1-x}}{\binom{2}{1}} \quad \text{Supp}[X] = \{1\} \quad X \sim \text{Deg}(1)$$

all success

$$N = K \Rightarrow \text{Supp}[X] = \{n\} \Rightarrow X \sim \text{Deg}(n)$$

$$K = 0 \Rightarrow \text{Supp}[X] = \{0\} \Rightarrow X \sim \text{Deg}(0)$$

all failures

$$n = N \Rightarrow \text{Supp}[X] = \{K\} \Rightarrow X \sim \text{Deg}(K)$$

$$\left.\begin{array}{l} N \in \{2, 3, \ldots\} \\ K \in \{1, 2, \ldots, N-1\} \\ n \in \{1, 2, \ldots, N-1\} \end{array}\right\} \text{Parameter Space}$$

Special case

$n = 1$

$$X \sim \text{Hyper}(1, K, N) = \frac{\binom{K}{x}\binom{N-K}{1-x}}{\binom{N}{1}} = \frac{K!}{N\, x!\,(K-x)!} \frac{(N-K)!}{(N-K-1+x)!\,(1-x)!}$$

$$\text{Supp}[X] = \{0, 1\} \qquad P(X=1) = \frac{K!}{N\, 1!\,(K-1)!} \frac{\cancel{(N-K)!}}{\cancel{(N-K)!}\, 0!} = \frac{K}{N}$$

successes ← K; total ← N

$$P(X=0) = \frac{\cancel{K!}}{N(0!)\cancel{(K-0)!}} \frac{(N-K)!}{\cancel{(N-K-1)!}\, 1!} = \frac{N-K}{N}$$

failures ← N−K; total ← N

or complement rule $P(X=0) = 1 - P(X=1) = 1 - \frac{K}{N} = \frac{N-K}{N}$

$$\Rightarrow \text{Hyper}(1, K, N) \overset{d}{=} \text{Bernoulli}\left(\frac{K}{N}\right)$$

Support$[X]$?

$X \sim \text{Hypergeometric}(2, 4, 10)$ Supp$[X] = \{0, 1, 2\}$ $n < K,\ n < N-K$

$X \sim \text{Hypergeometric}(5, 4, 10)$ Supp$[X] = \{0, 1, 2, 3, 4\}$ $n > K,\ n < N-K$.

$X \sim \text{Hypergeometric}(8, 4, 10)$ Supp$[X] = \{2, 3, 4\}$ $n > K,\ n > N-K$

$X \sim \text{Hypergeometric}(5, 7, 10)$ Supp$[X] = \{2, 3, 4, 5\}$ $n < K,\ n > N-K$

	$n \le K$	$n > K$
$n \le N-K$	$\{0, \ldots, n\}$	$\{0, \ldots, K\}$
$n > N-K$	$\{n-(N-K), \ldots, n\}$	$\{n-(N-K), \ldots, K\}$

$$\text{Supp}[X] = \{\max\{0, n-(N-K)\}, \ldots, \min\{n, K\}\}$$

What if K is a proportion of total N? Remember this? 400R cards, 1000 cards

$p = 0.4$

"0.4n"

let $p := \frac{K}{N}$ the proportion of successes

$$X \sim \text{Hypergeometric}(n, p, N) = \frac{\binom{pN}{x}\binom{N-pN}{n-x}}{\binom{N}{n}} = p(x) \qquad p \in \left\{\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\right\}$$

otherwise fractional successes... NOT possible in discrete r.v.

What if $N \to \infty$

4RL 6B, $N=10$

40RL 60B, $N=100$

400RL, 600B, $N=1000$

⋮

$\uparrow$ $p = 0.4$ $\downarrow$

STOP

What is the limiting r.v.? X^*

$$p(x^*) := \lim_{N\to\infty} \frac{\binom{pN}{x}\binom{N-pN}{n-x}}{\binom{N}{n}} = \lim_{N\to\infty} \frac{\frac{(pN)!}{x!(pN-x)!}\frac{(N-pN)!}{(n-x)!(N-pN-(n-x))!}}{\frac{N!}{n!(N-n)!}} = \underbrace{\frac{n!}{x!(n-x)!}}_{\binom{n}{x}} \lim_{N\to\infty} \frac{\frac{(pN)!}{(pN-x)!}\frac{(N(1-p))!}{(N(1-p)-(n-x))!}}{\frac{N!}{(N-n)!}}$$