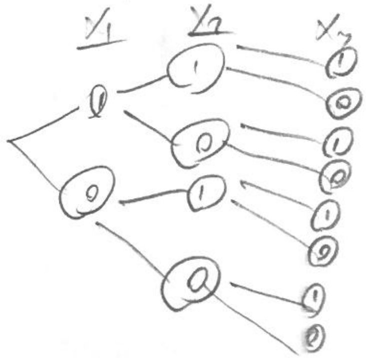
Lecture 13 3/24/14

$X_1, \ldots, X_n \overset{iid}{\sim} p(x)$ ← some discrete r.v. (general now)

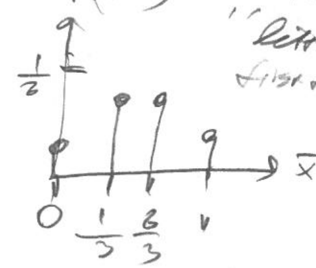
$T_n := X_1 + \ldots + X_n \sim p(t)$ the distr. of the sum

$\bar{X}_n := \frac{X_1 + \ldots + X_n}{n} \sim p(\bar{x})$ the distr. of the arithmetic avg. Why is adding up #'s and dividing by the # of #'s a good thing to do?

e.g. $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}(\frac{1}{2})$



$P(X_1, X_2, X_3)$	T	$\bar{X}$
$\frac{1}{8}$ ↓	3	1
	2	2/3
	2	2/3
	2	1/3
	1	2/3
	1	1/3
	0	1/3
		0



free var. I'll call "little $\bar{x}$" (first time not "x"!)

Where is $\bar{X}$ going?

$\bar{X} \to \text{Deg}(\frac{1}{2})$?

Philosophical Jump

$X \sim \text{Bern}(\frac{1}{2})$ is a r.v. but also a (d.g.p.) "data ~~flowing~~ generating process"

"X" is the process and "x" is the realization. Realizing means to "make real"

$X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ (coin flipping)

$y = 0$, $x = 1$ (coin flipped)

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}\left(\frac{1}{2}\right)$

$x_1 = 0 \leftarrow$ datum

$\left.\begin{array}{l} x_1 = 0 \\ x_2 = 1 \\ x_3 = 1 \\ \vdots \end{array}\right\}$ data

"datum" is the realization of a r.v.

"data" are " realizations of r.v.'s

iid data " " " iid "

A datum x must be in $\text{Supp}[X]$ ← possible datum values!

$P(X = x)$: what is the prob of getting this datum?

Ex height 5'8" what is the prob of 5'8"

if you have a prob. model X, $P(X = 5'8")$ can be computed...

Sometimes models hard to build... ~~but~~ this is what we actually believe...

Def: the sample avg $\bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$ is a realization for $\bar{X}_n$.

In class demos

$X_1, \ldots, X_6 \overset{iid}{\sim} \text{Hyper}(3, 9, 8)$ $\quad \bar{x}_6 = \{\ldots\}$

$\bar{x}_6 = \{\ldots\} \approx 4$?

$X_1, \ldots, X_6 \overset{iid}{\sim} \text{Binomial}\left(8, \frac{1}{2}\right)$

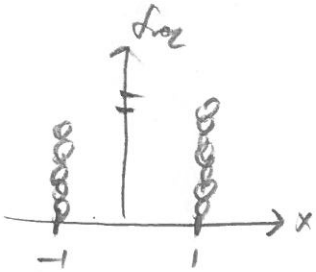
$X_1, \ldots, X_6 \overset{iid}{\sim} \text{Geom}\left(\frac{1}{2}\right)$ $\quad X_1, \ldots, X_6 \overset{iid}{\sim} \text{NegBin}\left(3, \frac{1}{2}\right)$ $\quad \bar{x}_6 = \{\ldots\}$

$\bar{x}_6 = \{\ldots\}$

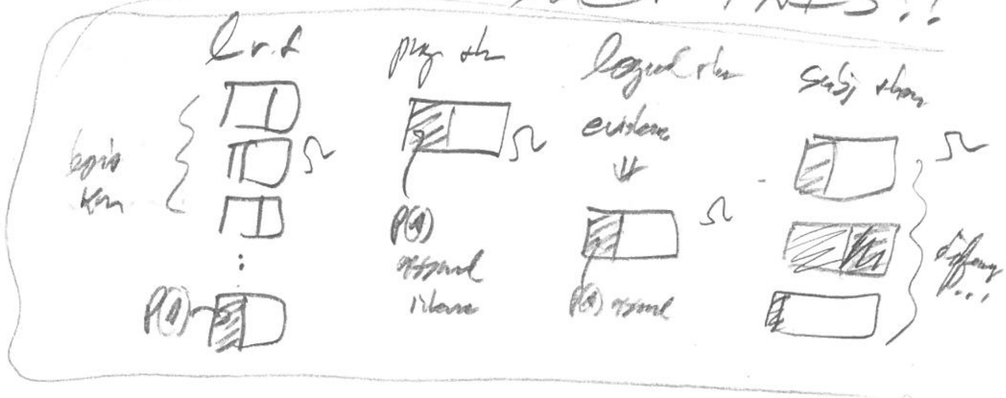
$= \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$ "random walk"

Last demo

$X_1, \ldots, X_{80} \sim$ Rademacher $\quad \bar{X}_{80} = \frac{-\sum \mathbb{1}_{X_i=-1} + \sum \mathbb{1}_{X_i=1}}{80}$



SKIP THIS!!



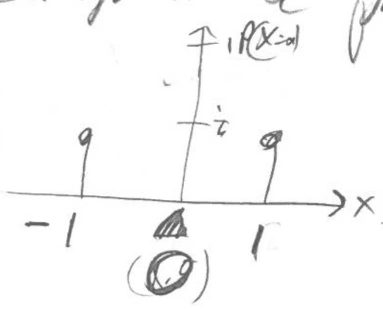
In the limit as $n \to \infty$, what should the pivot be?

The pivot is a very special property of the r.v. It is called the "expected value" or "expectation" or "mean". It is denoted "μ" or "$E[X]$". How to define?

I could define it as $\bar{X}_n \xrightarrow{n \to \infty} E(X)$. The idea being as $n \to \infty$ $\bar{X}$ gets close to Deg(μ). Similar to $P(A) := \lim_{n \to \infty} \frac{1}{n} \sum \mathbb{1}_{\omega_i \in A}$.

But we don't have to! If we know the model, i.e. the PMF/CDF, we know what $\bar{X}$ converges to because we can compute it. $\bar{X}$ converges to the "prob weighted avg of the outcome values"

def. of exp.



$E[X] = (\frac{1}{2})(-1) + \frac{1}{2}(1) = 0$ ✓

Thus, ... $\mu := E[X] := \sum_{x \in Supp(X)} x \cdot p(x)$

$\Rightarrow \bar{X}_n \to E(X)$ is a thm. Called the Law of Large #'s (LLN). Proof beyond scope of course but you need to know this fact!!

$X \sim Unif(\{1, 3, 7\}) \Rightarrow E(X) = \ldots$

Lets say $X \sim Bern(\frac{1}{3})$, $E(X) = \sum_{x \in Supp[X]} x\,p(x) = \ldots$

$X \sim Bern(p)$, $E(X) = \ldots$ genl Bernoulli

$$X \sim Binomial\left(8, \tfrac{1}{2}\right) \Rightarrow E(X) = \ldots = \sum_{x=0}^{8} x \binom{8}{x} \overbrace{\left(\tfrac{1}{2}\right)^{x}\left(\tfrac{1}{2}\right)^{8-x}}^{\frac{1}{2^8}}$$

$$= \frac{1}{2^8}\left(0\binom{8}{0} + 1\binom{8}{1} + 2\binom{8}{2} + 3\binom{8}{3} + 4\binom{8}{4} + 5\binom{8}{5} + 6\binom{8}{6} + 7\binom{8}{7} + 8\binom{8}{8}\right)$$

$$= \frac{1}{256}\left(8 + 2\cdot 28 + 3\cdot 56 + 4\cdot 70 + 5\cdot 56 + 6\cdot 28 + 7\cdot 8 + 8\right)$$

$$= \frac{1}{256}\left(8 + 56 + 168 + 280 + 168 + 56 + 8\right)$$

$$= \frac{1}{256}\,1024 = \boxed{4}$$

How many H do you expect in 8 coin flips?

(y=0 free) $\binom{n-1}{x-1} = \frac{(n-1)!}{(n-1-x+1)!(x-1)!}$

$$X \sim Binomial(n, p) \quad E(X) = \sum x\,p(x) = \sum_{x=0}^{n} x\binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x\binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x} = \sum_{x=1}^{n} \frac{n!}{(n-x)!(x-1)!} p^x (1-p)^{n-x} = np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!(x-1)!} p^{x-1} (1-p)^{n-x}$$

let $y = x - 1 \Rightarrow x = y + 1$, $m = n - 1 \Rightarrow n = m + 1$

$$= np \sum_{y=0}^{n-1=m} \binom{m}{y} p^y (1-p)^{n-(y+1)} \qquad n - y - 1 = m - y$$

$$= np \underbrace{\sum_{y=0}^{m} \binom{m}{y} p^y (1-p)^{m-y}}_{=\sum p(x) = 1} = \boxed{np}$$

STOP

$X \sim Geom(p)$

let $y = x - 1 \Rightarrow x = y + 1$

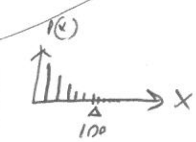
$$\mu = E(X) = \sum_{x=0}^{\infty} x\,p(x) = \sum_{x=1}^{\infty} x(1-p)^{x-1} p = p\sum_{x=1}^{\infty} x(1-p)^{x-1} = p\sum_{x=1}^{\infty} x(1-p)^{x-1} = p\sum_{y=0}^{\infty} (y+1)(1-p)^{y}$$

$$= p\left(\sum_{y=0}^{\infty} y(1-p)^y + \underbrace{\sum_{y=0}^{\infty} (1-p)^y}_{\frac{1}{1-(1-p)} = \frac{1}{p}}\right) = 1 + p\sum_{y=0}^{\infty} y(1-p)^y = \boxed{\frac{1}{p}}$$

$$1 + (1-p)\underbrace{\sum_{y=1}^{\infty} y(1-p)^{y-1} p}_{\mu}$$

@ $y = 0$, sum $= 0$

$$\mu = 1 + (1-p)\mu \Rightarrow \mu = 1 + \mu - \mu p \Rightarrow 0 = 1 - \mu p \Rightarrow \boxed{\mu = \frac{1}{p}}$$



$p = \frac{1}{100} \Rightarrow$ you expect 100 experiments