$\ln(p) = \ln(1000!) - 2\ln(500!) - 1000\ln(2)$

$\ln(n!) = \ln\left(\prod_{i=1}^{n} i\right) = \sum_{i=1}^{n} \ln(i)$

$\ln(1000!) =$ Still 1000 operations! Doable but

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$$p := P(500H, 500T) = \frac{\frac{1000!}{500!\,500!}}{2^{1000}}$$

Stirling's Approx

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad 1000! \approx \sqrt{2\pi(1000)}\left(\frac{1000}{e}\right)^{1000}$$

too big

$$\ln(n!) \approx \frac{1}{2}\ln(2\pi) + \left(n+\frac{1}{2}\right)\ln(n) - n$$

$$\ln(p) \approx \frac{1}{2}\ln(2\pi) + 1000.5\ln(1000) - 1000 - 2\left(\frac{1}{2}\ln(2\pi) + 500.5\ln(500) + 500\right) - 1000\ln 2 \approx -3.6797$$

9:15

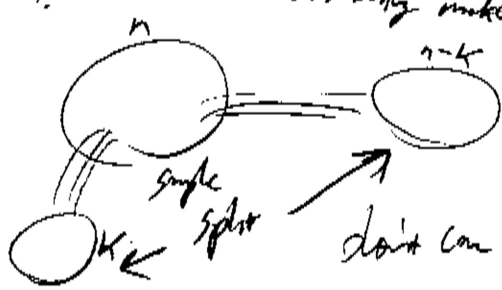
$$p \approx \exp(-3.6797) \approx \boxed{.0252}$$

Why? don't you expect half the coins to be heads? Is 2.5% ~~too low~~?

$$\binom{n}{k} = \frac{{}_nP_k}{k!} = \frac{n!}{(n-k)!\,k!}$$

Trivialities

1. $\binom{n}{1} = n$
2. $\binom{n}{n-1} = n$
3. $\binom{n}{n} = 1$
4. $\binom{n}{0} \overset{?}{=} \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!} = 1$ But doesn't really make sense
5. $\binom{n}{k} = \binom{n}{n-k}$



don't care about order...

$(a+b)^2 = 1a^2 + 2ab + 1b^2$

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3 = (a+b)(a+b)(a+b)$

$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 \qquad (a+b)(a+b)(a+b)(a+b)$

$\binom{4}{4} \quad \binom{4}{3} \quad \binom{4}{2} \quad \binom{4}{1} \quad \binom{4}{0}$ ← makes sense now?

$(a+b)^n = \binom{n}{n}a^n b^0 + \binom{n}{n-1}a^{n-1}b + \binom{n}{n-2}a^{n-2}b^2 + \dots + \binom{n}{1}ab^{n-1} + \binom{n}{0}a^0 b^n$

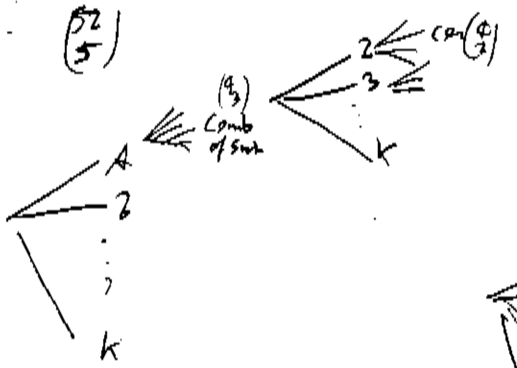
$= \sum_{i=0}^{n} \binom{n}{n-i} a^{n-i} b^i = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$ (Binomial Thm)

same

Poker!

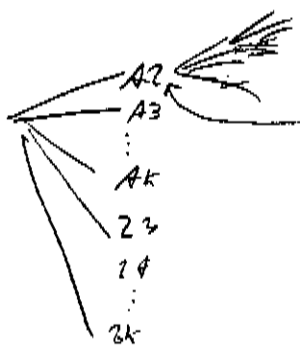
Full House

$\dfrac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$ Why not $\binom{13}{2}\binom{4}{3}\binom{4}{2}$



Why half

$\binom{13}{1}\binom{12}{1} = 13 \cdot 12 \neq \frac{13 \cdot 12}{2} = \binom{13}{2}$

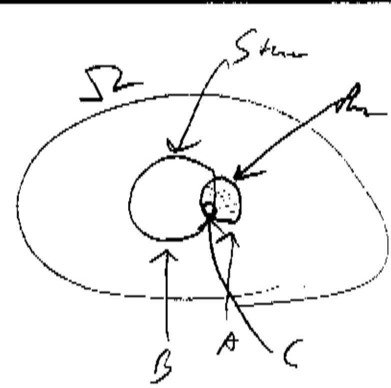


same allows for AAA22 and AA222

treats these as unique!

Flush

$$\frac{\binom{4}{1}\binom{13}{5} - \binom{4}{1}\binom{10}{1}}{\binom{52}{5}}$$



$$\left(A \setminus (A \cap B)\right) \cup C$$

P(no dupes)

$$\frac{3\cdot 2\cdot 1}{3^3} \qquad \frac{4!}{4^4}$$

Straight

$$\frac{\binom{10}{1}\binom{4}{1}^5}{\binom{52}{5}}$$

3-of-a-kind

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$$

Two-Pair

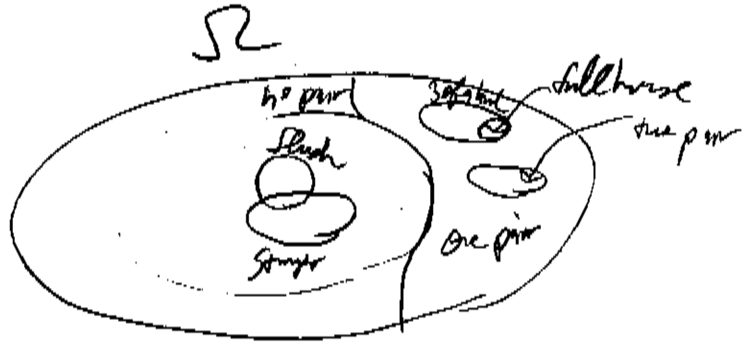
$$\frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}$$

1-pair — no str or flushes

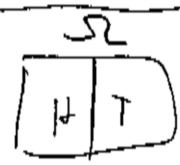
$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3$$

No pair hand ___

$$\binom{13}{5}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}^5 - \binom{4}{1}\binom{13}{5} + \binom{4}{1}\binom{10}{1}$$

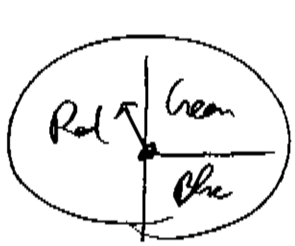


Previously...



$P(\{H\}) = \frac{|\{H\}|}{|\Omega|}$ hence $P(A) = \frac{|A|}{|\Omega|}$

But



$P(\{Red\}) = \frac{|\{Red\}|}{|\Omega|} = \frac{1}{3}$?

$\Longrightarrow P(A) = \frac{|A|}{|\Omega|}$ is not a general def of prob.

When does it apply? Only when $\omega_1, \omega_2, \ldots \in \Omega$ are equally likely outcomes,

i.e. $P(\{\omega_i\}) = \frac{1}{|\Omega|}$ for $i = 1, 2, \ldots$ (all ω's)

In the spinner case $P(\text{Red}) = \frac{1}{2}$ due to geometry but..

What if Ω's ω's are of unknown likelihood?

$\Omega = \{\text{sunny}, \text{cloudy}, \text{rainy}\}$ $\quad P(\text{sunny})$?

Further Ω $\boxed{H \mid T}$ How do you know I'm telling the truth? I could've made up this picture!

Need def. of probability! — the plane isn't real!

(I) Limiting Freq. (the book's approach)

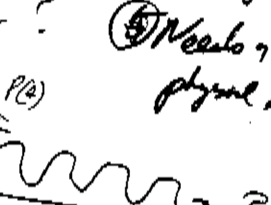
let $\mathbb{1}_{\omega \in A} = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$ the "indicator" function

$$P(A) := \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \mathbb{1}_{\omega_i \in A}}{n}$$

where $\omega_1 \in \Omega$ is the outcome of the first experiment
$\omega_2 \in \Omega$ is the outcome of the 2nd exp.
$\vdots$
$\omega_n \in \Omega$ " " nth exp.

What about single events?

Problems: ① who chooses $\omega \in \Omega$ and how and why?
② We never see $n = \infty$ only n large
③ Who's to say this limit converges?
④ Never give true probs!
⑤ Needs a physical system



$\rightarrow P(\infty \text{ is } 1440\text{cm})$, $P(\text{sunny tomorrow in } 11367)$, etc.

Historically, this is the first definition. Prob. began with the study of gambling games. If you spend hours gambling you see large n and you get an idea on $P(A)$.

e.g. M le Chevalier de Mere (1654) claimed that

$P(\{\text{at least one 6-6 in 24 rolls of 2-dice}\}) < \frac{1}{2}$

In truth $= .4914$ (calc. in a couple classes)

Long run freq is "objective" meaning a property of the physical / natural world. If humans didn't exist, $P(\{H\}) = \frac{1}{2}$ regardless.

(1950's)

II The propensity theory (also objective). A coin has an inherent property that induces a limiting freq of 50% Heads.

Calculating $P(A) \approx \frac{1}{n} \sum \mathbb{1}_{\omega_i \in A}$ is a way to estimate the propensity, not define it.

Canonical example. Uranium decomposition. U238 has $\frac{1}{2}$ life of 4.5 Byrs. P(one atom decaying after 4.5 Byr) $= \frac{1}{2}$.

This is hardwired in due to quantum mechanics. Possibly same as a coin?

Weakness: needs a physical object! Solves problem of ~~freq~~ theory.

Epistemic views: "degree of ~~belief~~" concerned with knowledge of humans

P(economy will go up) → something we believe (not property of physical world)

P(my favorite color is purple)

P(Aliens exist)

"degree of confirmation"

(III) Logical: given a set of evidence, anyone rational must have some probability value about an idea. (More on this later)

(IV) Subjective: given a set of evidence, differences of opinion are allowed.

P(OJ is guilty) $\boxed{\text{STOP}}$ No more $\omega \in \Omega$ setup?

If prob is determ...

All of these definitions never explain the phenomenon. What is randomness? Does it exist?

~~Determinism~~ Newton's principia (1687) 3 laws of motion, gravity

⇒ determinism: universe operates on a certain fixed ~~sets~~ set of rules.

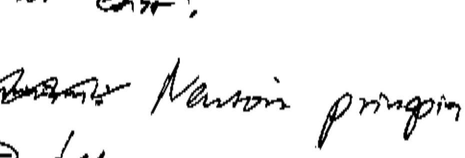
If you (a) know the rules (b) know the conditions exactly via measurement

⇒ you know the future with certainty! No room for 'randomness'!

Coin flip: exactly H or T. ω is an illusion due to imperfect a or b.

Laplace's 1814 philosophical essay on prob's (read)

Wasn't until the 1920's, wave-particle duality / quantum mechanics



double slit experiment
⇒ the electron chooses a final location!

Einstein hated this! ...

⇒ 2015: randomness is built into the fabric of the ~~[illegible]~~ universe but only on a very small scale. But small things can have huge snowball effects...