Math 241 Spring 2015
Midterm Examination One

Solutions

Professor Adam Kapelner

March 5, 2015

Full Name ________________________________ Section (A or B) _____

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Instructions

This exam is seventy five minutes and closed-book. You are allowed one page (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If I say "compute," this means the solution will be a number. I advise you to skip problems marked "[Extra Credit]" until you have finished the other questions on the exam, then loop back and plug in all the holes. I also advise you to use pencil.

The exam is 100 points total. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. Good luck!

Problem 1 "Risk" is the game of world domination. Below is a typical board.

In this question, we will consider a variant of the game where there are 6 continents:

- North America
- South America
- Africa
- Europe
- North Asia
- South Asia / Oceania

Each continent is split into 5 countries. Thus, there is a total of $6 \times 5 = 30$ countries.

(a) [3 pt / 3 pts] The game comes with 30 illustrated cards where each card represents one of the 30 countries. How many ways is there of shuffling the cards if order matters? Answer using factorial notation, permutation or choose notation.

$$\boxed{30!}$$

(b) [5 pt / 8 pts] Estimate the value of your answer in (a). You may answer only in terms of a decimal (round to the nearest two decimals) and the exponentiation function.

$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$\ln(n!) \approx \frac{1}{2}\ln 2\pi + \frac{1}{2}\ln n + n\ln n - n = \frac{1}{2}\ln(2\pi) + \left(n + \frac{1}{2}\right)\ln(n) - n$

$\ln(30!) \approx \frac{1}{2}\ln(2\pi) + (30.5)\ln(30) - 30 = 74.66 \implies \boxed{30! \approx e^{74.66}}$

(c) [6 pt / 14 pts] You play the game with four other players (thus there is a total of five players). In the beginning of the game, cards are dealt out to each player so each player is dealt 6 random cards after a thorough shuffle (5 players × 6 cards = 30). What is the probability of being dealt China, Canada, Russia, Argentina, South Africa and Western United States *in that order*? Answer using factorial notation, permutation or choose notation.

$$\frac{1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1}{{}_{30}P_6} = \boxed{\frac{1}{{}_{30}P_6} = \frac{1}{30} \cdot \frac{1}{29} \cdot \frac{1}{28} \cdot \frac{1}{27} \cdot \frac{1}{26} \cdot \frac{1}{25}}$$

(d) [4 pt / 18 pts] Now assume order *does not matter* within a 6 card hand. What is the total number of unique 6 card hands that you can be dealt? Answer using factorial notation, permutation or choose notation.

$$\boxed{\binom{30}{6}}$$

(e) [4 pt / 22 pts] In a six card hand, what is the probability of being dealt all five countries in North America? Answer using factorial notation, permutation or choose notation.

North America has 5 countries, choose 5

pick another continent

$$\boxed{\frac{\binom{5}{5}\binom{5}{1}\binom{5}{1}}{\binom{30}{6}}}$$

pick one of the 5 countries in this continent

(f) [5 pt / 27 pts] In a six card hand, what is the probability of being dealt three pairs (of two countries) where each pair comes from a different continent (e.g. one possible hand is two African countries, two North Asian countries and two South American countries)? Remember, order still does not matter in your dealt hand. Answer using factorial notation, permutation or choose notation.

for one continent, choose 2 of the 5 countries

do that 3 times

pick 3 of the six continents to be the pairs

$$\boxed{\frac{\binom{6}{3}\binom{5}{2}^3}{\binom{30}{6}}}$$

(g) [4 pt / 31 pts] After cards are dealt, each player must now distribute army units among their five countries. Each player has 45 army unit pieces to distribute. All army unit pieces are equal in value and indistinguishable. Assume it is silly to leave a country completely empty (then an opponent can appropriate that territory without any battle). How many ways can you distribute your army units among the ~~five~~ six countries that were dealt to you at the beginning of the game? Countries are distinct because 10 armies on France and 8 armies on Mexico is a different strategic arrangement than 8 armies on France and 10 armies on Mexico.

$n = 45$ armies (which can be thought of as indistinct balls)
$r = 6$ countries (which can be thought of as distinct urns)
n balls to r urns without empty urns $\Rightarrow \binom{n-1}{r-1} = \binom{45-1}{6-1} = \boxed{\binom{44}{5}}$

(h) [4 pt / 35 pts] On second thought, it may sometimes be strategic to leave territories empty in order to have other countries be stronger (if those countries are heavy cross-roads). How many ways can you distribute the armies if you are allowed to leave your countries empty?

n balls in r urns with the possibility of empty urns $\Rightarrow \binom{n+r-1}{r-1} = \binom{45+6-1}{6-1} = \boxed{\binom{50}{5}}$

(i) [3 pt / 38 pts] [Extra Credit] Given that all players can leave countries empty, how many total initial board arrangements are there given one deal of the 30 cards?

All countries are given out so all five players have $\binom{50}{5}$ ways to distribute armies $\Rightarrow \boxed{\binom{50}{5}^5}$

(j) [3 pt / 41 pts] [Extra Credit] Given that all players can leave countries empty, how many total initial board arrangements are there? In this case, all possible arrangement of 30 cards to six players are possible.

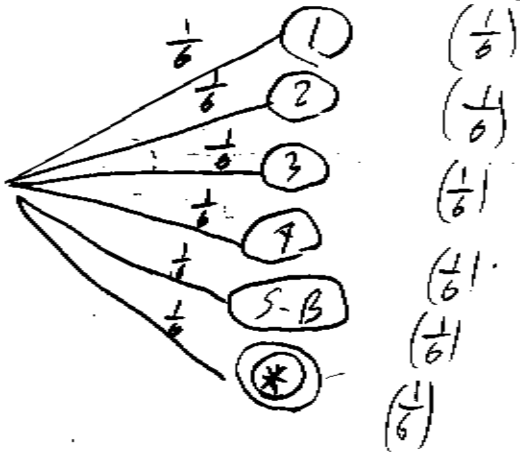
$$\boxed{\binom{30}{6}\binom{24}{6}\binom{18}{6}\binom{12}{6}\binom{6}{6}\binom{50}{5}^5}$$

hands for first player

hands for second player, etc

from part (i)

(k) [4 pt / 45 pts] Once the countries of the world are divvied up and the players apportion their armies, the gameplay starts. A player's turn consists of choosing adjoining countries to invade and going to "battle." Battle consists of the attacker first rolling a 6 sided fair die whose faces are labeled: 1, 2, 3, 4, ☠ (Skull-Bones) and a ⊛ (Bonus). Draw a probability tree for the outcomes of the attacker dice roll.



marginal probabilities not needed for full credit

(l) [4 pt / 49 pts] Rolling the die ten times yields the following outcome: 1, ☠, ☠, ☠, ☠, ☠, 4, 4, 4, 4. What is the probability of this event?

$$P(1, SB, SB, SB, SB, SB, 4, 4, 4, 4) = P(1)\,P(SB)\cdots P(4) = \boxed{\left(\tfrac{1}{6}\right)^{10}}$$

due to independence

fair die see (k)

(m) [5 pt / 54 pts] The attacker rolls first. If he rolls a ☠, he immediately loses. By "lose" we mean he loses one of the armies located in his attacking country. Assuming he wins on every other face of the die — the 1, 2, 3, 4 and bonus (as assumptiong we will modify in later questions), what is the probability he wins *at least once* in 13 rolls if he begins with 13 armies?

$P(\text{lose}) = P(SB) = \frac{1}{6}$ (see k)

$P(\text{win} \geq 1 \text{ times in 13 rolls})$

$= 1 - P(\text{win 0 times in 13 rolls})$ complement rule

$= 1 - P(\text{lose 13 times in a row})$ ″

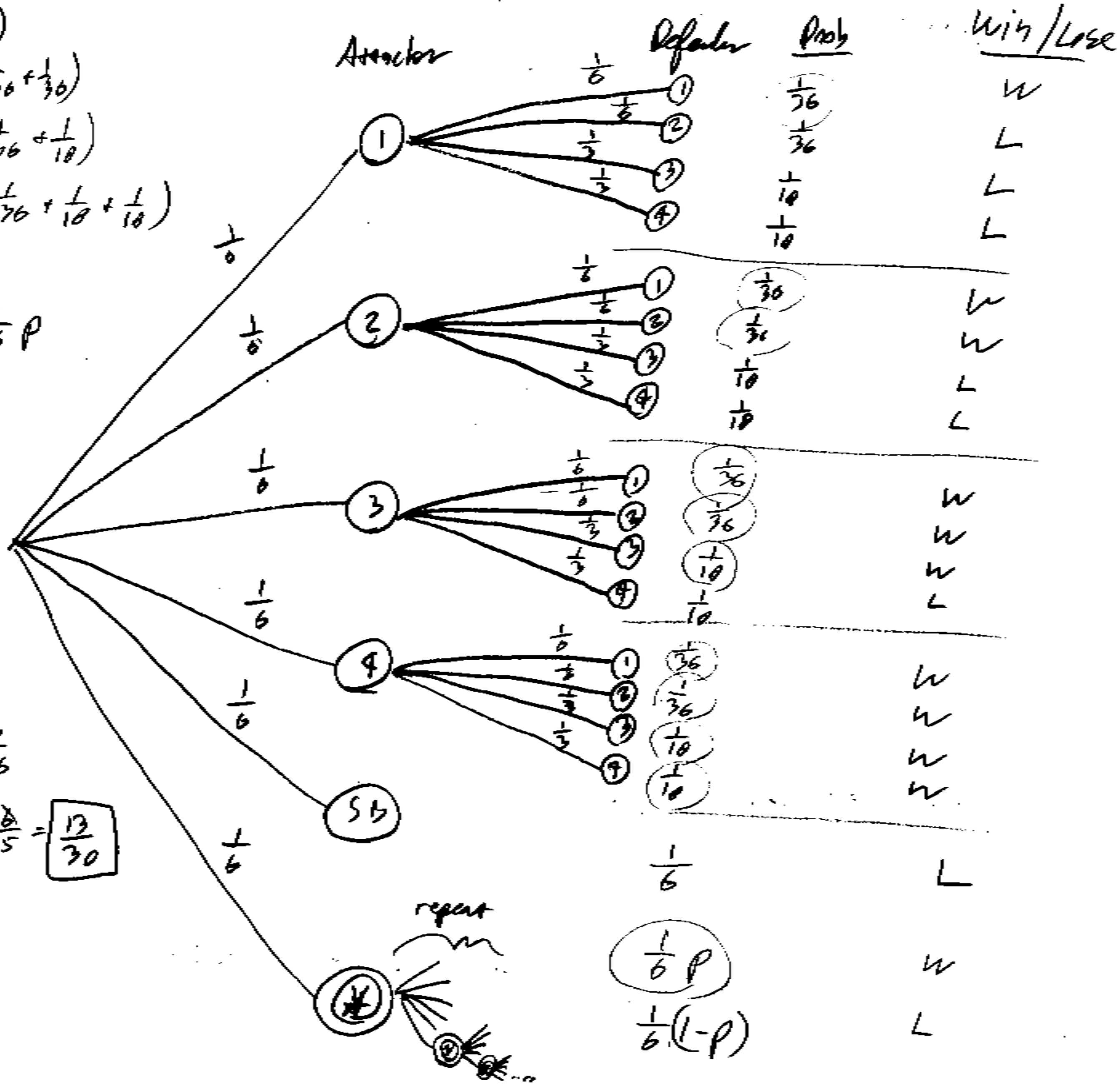
$= 1 - P(\text{lose})^{13}$ independence

$= \boxed{1 - \left(\frac{1}{6}\right)^{13}}$

(n) [9 pt / 63 pts] The defender die also has six sides with the following outcomes: 1, 2, 3, 3, 4, 4. This die is also fair. Now we can fully explain how one battle cycle works. The attacker goes first. If he gets a ☠, he automatically loses, if he gets a ⍟, he gets a bonus card and rolls again. If he gets a number, the defender then rolls. The attacker wins on a tie. What is the probability the attacker wins one battle cycle?

$$p := P(\text{win}) = \left(\frac{1}{36}\right) + \left(\frac{1}{36} + \frac{1}{36}\right) + \left(\frac{1}{36} + \frac{1}{36} + \frac{1}{18}\right) + \left(\frac{1}{36} + \frac{1}{36} + \frac{1}{18} + \frac{1}{18}\right) + \frac{1}{6}p$$

$$= \frac{13}{36} + \frac{1}{6}p$$



$$\Rightarrow \frac{5}{6}p = \frac{13}{36}$$

$$\Rightarrow p = \frac{13}{\cancel{36}_{6}} \cdot \frac{\cancel{6}}{5} = \boxed{\frac{13}{30}}$$

(o) [5 pt / 68 pts] What is the probability the attacker wins given that he rolls a 2?

$$P(W|2) = \frac{P(W,2)}{P(2)} = \frac{\frac{1}{36} + \frac{1}{36}}{\frac{1}{6}} = \frac{\frac{2}{36}}{\frac{1}{6}} = \boxed{\frac{1}{3}}$$

(p) [5 pt / 73 pts] What is the probability the attacker rolled a ☠ given that the defender won?

$$P(SB \mid L) = \frac{P(SB, L)}{P(L)} = \frac{\frac{1}{6}}{1-p} = \frac{\frac{1}{6}}{1-\frac{13}{30}} = \frac{\frac{1}{6}}{\frac{17}{30}} = \boxed{\frac{5}{17}}$$

P(win) from pb(n)

Problem 2 Some theoretical questions are below. The subparts are all independent unless otherwise indicated.

(a) [4 pt / 77 pts] Imagine a finite sample space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ of equally likely outcomes. Your friend tries to argue that all event elements in 2^Ω have equal probability. Show that your friend is wrong via a mathematical proof.

If equally likely, $P(\{\omega_i\}) = \frac{1}{|\Omega|} = \frac{1}{n}$ $\forall\, i = 1, \ldots, n$

But $\phi \in 2^\Omega$ and $P(\phi) = 0 \neq \frac{1}{n}$

(b) [4 pt / 81 pts] Consider sets C, D, E, F, G, H such that $C \subset D \subset E \subset F \subset G \subset H$. Prove that $\mathbb{P}(D) < \mathbb{P}(G)$. You can feel free to use theorems from class and your knowledge of set theory.

Since $D \subset E \subset F \subset G \Rightarrow D \subset G$ from set theory

we learned in class that $P(A) < P(B)$ when $A \subset B$, thus, invoking this theorem, we conclude $P(D) < P(G)$

(c) [4 pt / 85 pts] Imagine that the probability of A is greater than a half. Prove that the probability of $\Omega \backslash A$ is less than a half.

$P(\Omega) = 1$ (b1)

$\Rightarrow P(A \cup A^c) = 1$ (set theory)

$\Rightarrow P(A) + P(A^c) = 1$ (c)

$\Rightarrow 1 - P(A^c) = P(A)$ algebra

$1 - P(A^c) > \frac{1}{2}$ (given)

$\Rightarrow \frac{1}{2} > P(A^c)$ (algebra)

$\Rightarrow P(A^c) < \frac{1}{2}$ "

$\Rightarrow P(\Omega \setminus A) < \frac{1}{2}$ (def of complement of set)

(d) [3 pt / 88 pts] What is $\mathbb{Q} \cup \mathbb{Z} \cup \mathbb{N}$? $\boxed{\mathbb{Q}}$

Since $\mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{N} \subset \mathbb{Q}$

(e) [4 pt / 92 pts] It is known that $A_1, A_2, \ldots, A_n$ are collectively exhaustive but otherwise arbitrary. Fill in the circle: $\sum_{i=1}^{n} \mathbb{P}(A_i) \bigcirc 1$ with one of the following relation symbols: $<, >, \leq, \geq, =$ or $\neq$ and explain why this is the answer in one or two sentences.

$\boxed{\sum_{i=1}^{n} P(A_i) \geq 1}$; If $A_1, \ldots, A_n$ disjoint, $\sum_{i=1}^{n} P(A_i) = 1$ by (c)

If $A_1, \ldots, A_n$ are not disjoint $\sum_{i=1}^{n} P(A_i) > 1$

Since there will be some overlap that gets double-counted.

(f) [3 pt / 95 pts] [Extra Credit] Show that $\mathbb{P}(A^C \mid B^C) = 1 + \frac{\mathbf{P}(A,B) - \mathbf{P}(A)}{1 - \mathbf{P}(B)}$.

$$P(A^c \mid B^c) = \frac{P(A^c, B^c)}{P(B^c)} = \frac{\overbrace{P(A^c) - P(A^c, B)}^{\text{total prob.}}}{\underbrace{1 - P(B)}_{\text{complement}}} = \frac{\overbrace{(1 - P(A))}^{\text{complement}} - \overbrace{(P(B) - P(A,B))}^{\text{total prob.}}}{1 - P(B)} = \overbrace{\frac{(1 - P(B)) + (P(A,B) - P(A))}{1 - P(B)}}^{\text{algebra}} \overbrace{=}^{\text{algebra}} 1 + \frac{P(A,B) - P(A)}{1 - P(B)}$$

(g) [4 pt / 99 pts] In the expansion of $(w + x + y + z)^6$, how many terms are x^3y^3? Your final answer must be a number *only*. No other notation is allowed. Of course you can use other notation to arrive at your final result.

$$(w+x+y+z)^6 = \sum_{\substack{\text{all ways} \\ \text{of } x_1+\dots+x_4=6 \\ x_i \geq 0}} \binom{6}{x_1, x_2, x_3, x_4} \Longrightarrow x^3y^3 \text{ has coefficient } \binom{6}{0,3,3,0} = \frac{6!}{0!3!3!0!}$$

$$= \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 3 \cdot 2} = 5 \cdot 4 = \boxed{20}$$

(h) [5 pt / 104 pts] A coin of unknown fairness is flipped 1,000,000 times and you observe 499,958 heads. Explain in a few sentences or less if you believe the coin is fair by interpreting this experimental data according to the long run frequency definition of probability.

Acc'd to the long run freq. definition of probability, $P(H) = \lim_{n \to \infty} \frac{1}{n} \sum \mathbb{1}_{\omega_i \in H}$. n here is $1{,}000{,}000 \approx \infty$ so $P(H) \approx \frac{499{,}958}{1{,}000{,}000} = .49996 \approx 0.5$, the value of $P(H)$ indicating a "fair coin." So yes, this experimental data supports the coin being fair because it appears this limit is converging.

(i) [5 pt / 109 pts] Assume the same experimental data as in the previous question except now we know the coin is fair. Explain in a few sentences or less the interpretation of this experimental data according to the propensity definition of probability.

If the coin is fair, $P(H) = \frac{1}{2}$. By the propensity definition, the coin has an inherent propensity to flip to heads half the time. This propensity induces the long run frequency close to $\frac{1}{2}$ which is exactly what the experimental data shows.