Math 241

Lecture 14 3/26/15

$X \sim \text{Geom}(p) := (1-p)^{x-1} p \quad , \quad \text{Supp}[X] = \mathbb{N} \quad , \quad p \in (0,1) \Rightarrow 1-p \in (0,1)$

$$\mu = E[X] := \sum_{x \in \text{Supp}[X]} x\, p(x) = \sum_{x=1}^{\infty} x (1-p)^{x-1} p = p \sum_{y=0}^{\infty} (y+1)(1-p)^{y}$$

$y = x-1 \Rightarrow x = y+1$

$$= p \left(\sum_{y=0}^{\infty} y(1-p)^y + \underbrace{\sum_{y=0}^{\infty} (1-p)^y}_{\frac{1}{1-(1-p)} = \frac{1}{p}} \right) = p \sum_{y=0}^{\infty} y(1-p)^y + 1 = 1 + p \sum_{y=1}^{\infty} y(1-p)^y$$

$$= 1 + p(1-p) \sum_{y=1}^{\infty} y(1-p)^{y-1} = 1 + (1-p)\mu \Rightarrow \mu = 1 + (1-p)\mu \Rightarrow \mu = 1 + \mu - p\mu$$

$$\Rightarrow \cancel{\mu} + p\mu = 1 + \cancel{\mu} \Rightarrow \boxed{\mu = \frac{1}{p}}$$

$X \sim \text{NegBin}(r, p)$ $E(X)$ ... you'll see later
(too tedious manually to calculate)

or "percentile" of a r.v.

New idea $\text{Quantile}[X, p] := \min_x \{F(x) \geq p\}$

it's the "first x" (in ascending order) that captures $p \geq P(X \leq x)$ i.e.
more than p prop. of the support is $\leq$ to it.

$X \sim \text{Bin}(10, 0.4)$

x	$p(x)$	$F(x)$
0	0.0060	0.0060
1	0.0404	0.0464
2	0.1209	0.1673
3	0.2150	0.3823
4	0.2508	0.6331
5	0.2007	0.8338
6	0.1115	0.945
7	0.0425	0.9877
8	0.0106	0.9983
9	0.0016	0.9999
10	0.0001	1.0000
sum	1	?

10%ile of X ? 2

20%ile ?

30%ile ?

40%ile ?

75%ile ?

95%ile ?

$\text{Median}(X) := \text{Quantile}[X, 0.5]$

s.t. $\approx \frac{1}{2}$ distr below, $\frac{1}{2}$ above

$\text{Median}(X) = E(X)$? NO

$\text{IQR}(X) := \text{Quantile}[X, 0.75] - \text{Quantile}[X, 0.25]$

How much ~~of the~~ support is in the middle 50% of the prob mass?

$\text{Mode}[X] := \underset{x \in \text{Supp}(X)}{\text{argmax}} \{p(x)\}$ here, 4

Roulette in America. Bet on Black. What is the expected winnings? Payout 1:1. \$1

$X \sim \begin{cases} \$1 & \text{wp } \frac{18}{38} \\ -\$1 & \text{wp } \frac{20}{38} \end{cases}$ $E(X) = (1)\frac{18}{38} + (-1)\frac{20}{38} = -\frac{2}{38} \approx \boxed{-\$0.053}$ interpret?

Bet on lucky #7. Payout 35:1

$$X \sim \begin{cases} \$35 & \text{w.p. } \frac{1}{38} \\ -\$1 & \text{w.p. } \frac{37}{38} \end{cases}$$

$$E(X) = (35)\frac{1}{38} + (-1)\frac{37}{38} = -\frac{2}{38} = \$-0.053$$

"Dozen Bet" $\{1, \dots, 12\}$ Payout 2:1

$$Y \sim \begin{cases} \$2 & \text{w.p. } \frac{12}{38} \\ -\$1 & \text{w.p. } \frac{26}{38} \end{cases} \qquad E(X) = 2\frac{12}{38} + (-1)\frac{26}{38} = -\$0.053$$

All bets in Roulette have same expectation!

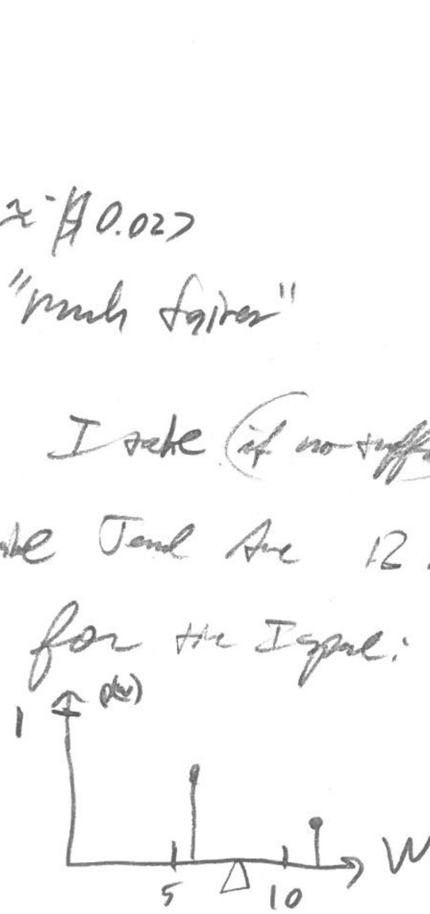
In Europe... bet on Red

$$X \sim \begin{cases} \$1 & \text{w.p. } 18/37 \\ -\$1 & \text{w.p. } 19/37 \end{cases} \qquad E(X) = \frac{18}{37} - \frac{19}{37} \approx -\$0.027$$

"much fairer"

New Question: I ride Uber. If I take (if no traffic) the Van Wyck, it's 7 min; if I take Jewel Ave 12 min. Prob(no traffic) = 70%. Create a r.v. for the [illegible]:

$$W \sim \begin{cases} 6^{\text{min}} & \text{w.p. } 0.7 \\ 12 & \text{w.p. } 0.3 \end{cases}$$



$$E[W] = 6 \cdot 0.7 + 12 \cdot 0.3 = \boxed{7.8 \text{ min}}$$ Explain...

How much do I pay for time? Uber charges \$0.40/min.

Let $B := \$0.40/\text{min} \cdot W$ $\quad B = f(W)$ "transformation theory"
really simple function...

What is $\text{Supp}[B]$?

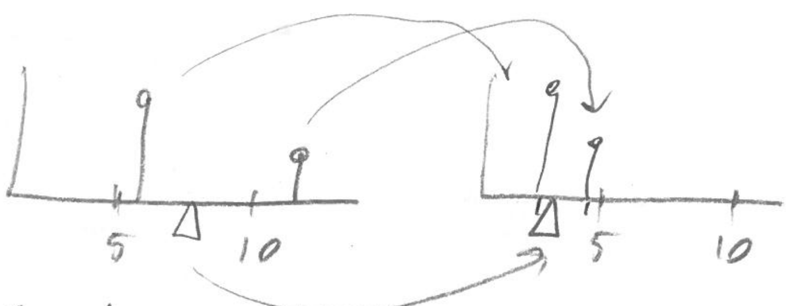
if $w = 6 \Rightarrow b = 2.4$
if $w = 12 \Rightarrow b = 4.8$ $\quad \Rightarrow \text{Supp}[B] = f(\text{Supp}[W]) = \{f(w) : w \in \text{Supp}[W]\} = f(\{6, 12\}) = \{2.4, 4.8\}$

What about $p_B(b)$? Is it related to $p_W(w)$?

Yes $p_B(b) = p_W\left(\frac{b}{0.4}\right)$ $\quad P(B = 2.4) = P(W = 6)$, $\quad P(B = 4.8) = P(W = 12)$

inverse function
(OK since 1:1)

$$E[B] = \sum_{b \in \text{Supp}[B]} b\, p_B(b) = \sum_{b \in \text{Supp}[B]} \overset{f(w)}{b}\, p_W\Big(\overset{w}{\tfrac{b}{0.4}}\Big) = \sum_{w \in \text{Supp}[W]} f(w)\, p(w) = \sum_{w \in \text{Supp}[W]} 0.40\, w\, p(w) = 0.4\, E[W]$$
$$= \$3.12$$



everything gets
multiplied by common value... so does pivot pt.

$Y = aX$, $\quad E[Y] = a\,E[X]$ (you'll see why later)

$$\sum_{y \in \text{Supp}[Y]} y\, p_Y(y) \overset{\text{same}}{=} \sum_{x \in \text{Supp}[X]} a x\, p_X(x) = a \sum x\, p_X(x) = a\, E[X]$$

But there's a base fare of \$3 as soon as you get in the taxi.

$T = \$3 + B$ $\quad$ What is $E[T]$?

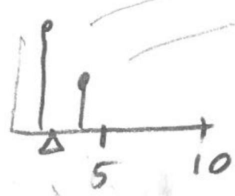
$$E[T] = \sum_{t \in \mathrm{Supp}(T)} t\, p_T(t) = \sum_{b \in \mathrm{Supp}(B)} (b+3)\, p_B(b) = \sum b\, p_B(b) + \sum 3\, p_B(b)$$

$\to 1$

$$= E(B) + 3$$

$$p_T(t) = p_B(t-3) = p_B(b)$$

In general $E[aX + c] = a\,E(X) + c$



Everything is shifted by 3 even the pivot point!

How much do we expect a datum to be away?

$e(x,\mu)$ error function

1. $e(x,y) = e(y,x)$ $\forall x,y \in \mathrm{Supp}$
2. $e(x,y) \geq 0$ $\forall x,y \in \mathrm{Supp}$
3. $e(x,x) = 0$ $\forall x$

$e(x,\mu) = |x-\mu|$

$e(x,\mu) = (x-\mu)^2$

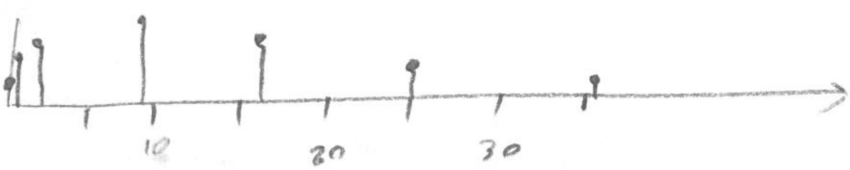
$e(x,\mu) = (x-\mu)^4$

$X = \mathrm{Bin}(6, \frac{1}{2})$

$E(X) = 6 \cdot \frac{1}{2} = 3$

$0 \to 0$
$1 \to 1$
$2 \to 4$
$\vdots$
$6 \to 36$

$Y = g(X) = X^2 \sim ?$ Need to go by def...



Y "varies" more from its expectation than X does.

Expectation is its "balancing" center.

what is $E(Y)$?

p doesn't change but the location in the support does

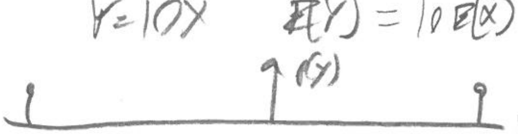
$$E[X^2] = \sum_{x \in \mathrm{Supp}(X)} x^2\, p(x)$$

$$= 0p(0) + 1p(1) + 4p(2) + 9p(3) + 16p(4) + 25p(5) + 36p(6) = \boxed{17.5}$$

No easier way!!

$$E(Y) = E[g(X)] = \sum_{x \in \mathrm{Supp}(X)} g(x)\, p(x)$$

$X \sim$ Rademacher $E(X) = 0$ $Y = 10X$ $E(Y) = 10\,E(X) = 0$ but are they the same??



$E(X) = E(Y) \not\Longrightarrow X \overset{d}{=} Y$

Y is more dispersed than X.