1

$X_1, \ldots, X_n \overset{iid}{\sim} \text{Bern}(p)$

Lecture 12 March 19, 2015 $T = X_1 + \ldots + X_n$

$\Rightarrow T \sim \text{Binomial}(n,p) := \binom{n}{x} p^x (1-p)^{n-x} = p(x)$, $p \in (0,1)$, $n \in \mathbb{N}$ $F(x) = ?$ regularized incomplete beta function (not closed form)

Does $\sum_{x \in Supp[X]} p(x) = 1$? → where the r.v. gets its name or from the coeff.

$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$ → binomial expansion

$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = ((1-p)+(p))^n = 1^n = 1$ ✓ easy!

Now ask a different question. What if $\underbrace{X_1, X_2, \ldots}_{\text{possibly infinite}} \overset{iid}{\sim} \text{Bern}(p)$?

let $T = \min_t \{X_t = 1\}$ i.e. the trial # of the "first success"

$P(T=1) = p$ $\quad$ $\underline{1}$

$P(T=2) = (1-p)p$ $\quad$ $\underline{0}\ \underline{1}$

$P(T=3) = (1-p)^2 p$ $\quad$ $\underline{0}\ \underline{0}\ \underline{1}$ $\quad$ Why not $\binom{3}{1}$? Only one way allowed...

$P(T=4) = (1-p)^3 p$ $\quad$ $\underline{0}\ \underline{0}\ \underline{0}\ \underline{1}$

$\vdots$

$P(T=x) = (1-p)^{x-1} p$ $\quad$ $\underbrace{\underline{0}\ \underline{0} \ldots \underline{0}}_{x-1}\ \underline{1}$ "if at first you don't succeed..."

$T \sim \text{Geometric}(p) := (1-p)^{x-1} p = p(x)$

Parameter Space

$p \in (0,1)$ $\quad$ $Supp[T] = \mathbb{N}$ $\quad$ Why $0 \notin Supp[T]$?

$p = 1 \Rightarrow T \sim \text{Deg}(1)$ $\quad$ $p = 0$ T d.n.e. (not possible!) not even Deg!!

Does $\sum_{x \in supp(X)} p(x) = 1$?

$q = 1-p \in (0,1)$

$$\sum_{x=1}^{\infty} (1-p)^{x-1} p \stackrel{?}{=} 1 \Rightarrow \sum_{x=1}^{\infty} (1-p)^{x-1} \stackrel{?}{=} \frac{1}{p} \Rightarrow \sum_{x=0}^{\infty} (1-p)^{x} = \frac{1}{1-(1-p)} = \frac{1}{p} \checkmark$$

the Geometric series, the reason it's called a "Geometric" r.v.

Reason... what does $\sum_{r=0}^{\infty} q^r$ when $q \in (0,1)$ equal? → converges since

Integral test for conv. of a series...

$$\int_0^{\infty} q^r \, dr = \frac{q^r}{\ln(q)}\Big]_0^{\infty} = \frac{1}{\ln(q)}\left(\lim_{r\to\infty} q^r - 1\right) = -\frac{1}{\ln(q)} < \infty \checkmark$$

$1 + q + q^2 + q^3 + \dots = T$

$\Rightarrow 1 + q\left(1 + q + q^2 + q^3 + \dots\right) = T$

$1 + qT = T \Rightarrow 1 = T(1-q) \Rightarrow T = \frac{1}{1-q}$

$\cancel{\sum_{r=0}^{\infty} q^r = 1 + \sum_{r=1}^{\infty} q^r \Rightarrow \frac{1}{1-q} = 1 + \sum_{r=1}^{\infty} q^r \Rightarrow \sum_{r=1}^{\infty} q^r = \frac{1}{1-q} - 1 = \frac{q}{1-q}}$

F(x) for Hypergeometric & binomial ⇒ Hard

Can we get F(x) for geometric

$F(x) = P(T \le x) = \sum_{y=1}^{x} (1-p)^{y-1} p$ ... not a good strategy... but possible...

$= 1 - P(X > x) = 1 - (1-p)^x \checkmark$

X occurs after x
so we have x failures

Examples. Flip coin until H. What is the prob you wait until 10th flip?

$T \sim Geom\left(\frac{1}{2}\right)$ $P(T=10) = \left(1-\frac{1}{2}\right)^9 \frac{1}{2} = \boxed{\frac{1}{2^{10}}}$

Defect rate of computer chips is $\frac{1}{1000}$, what is prob you see a defect at 500?

$T \sim Geom\left(\frac{1}{1000}\right)$ $P(T=500) = \boxed{\left(\frac{999}{1000}\right)^{499} \frac{1}{1000}} = .0006$

"success" is a "defect" ⇒ why??

What is prob you see a defect before or at 500 chips? $P(T \le 500) = F(500) = 1 - \left(\frac{999}{1000}\right)^{500} = .39$

Now ask a different question... $X_1, X_2, \ldots \overset{iid}{\sim} \text{Bern}(p)$ again

$$T = \min_t \left\{ \sum_{x=1}^{t} X_t = r \right\}$$ i.e. the # of trials until "r" successes

let $r = 3$ just for example's sake.

Can $T=1$? $T=2$? $T=3$?

$P(T=3) = p^3$ $\underline{1}\ \underline{1}\ \underline{1}$

succeed always on last trial

$P(T=4) = 3(1-p)^1 p^2 p$

$\underline{1}\ \underline{1}\ \underline{0}\ \underline{1}$

$\underline{1}\ \underline{0}\ \underline{1}\ \underline{1}$

$\underline{0}\ \underline{1}\ \underline{1}\ \underline{1}$

fails, success, last succ

need $r-1 = 2$ among the other trials any which way! $\binom{3}{2}$

$P(T=5) = 6(1-p)^2 p^2 p$

$\underline{1}\ \underline{1}\ \underline{0}\ \underline{0}\ \underline{1}$

$\underline{1}\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{1}$

$\underline{1}\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{1}$

$\underline{0}\ \underline{1}\ \underline{1}\ \underline{0}\ \underline{1}$

$\underline{0}\ \underline{1}\ \underline{0}\ \underline{1}\ \underline{1}$

$\underline{0}\ \underline{0}\ \underline{1}\ \underline{1}\ \underline{1}$ $\to \binom{4}{2}$

ways

$\vdots$

$$P(T=x) = \binom{x-1}{r-1}(1-p)^{x-r} p^{r-1} p = \binom{x-1}{r-1}(1-p)^{x-r} p^r = p(x)$$

$\underbrace{\_\ \_\ \ldots\ \_}_{x-1}\ \underline{1}$

Negative Binomial, need $r-1$ successes

does not start at 1

$$T \sim \text{NegBin}(r, p) := \binom{x-1}{r-1}(1-p)^{x-r} p^r \qquad \text{Supp}[T] = \{r, r+1, \ldots\}$$

<u>Parameter Space</u>

$p \in (0,1)$ if $p=1 \Rightarrow T \sim \text{Deg}(r)$, $p=0 \Rightarrow T$ d.n.e. $\quad r \in \mathbb{N}$

Special case

$$T \sim \text{NegBin}(1,p) = \binom{x-1}{1-1}(1-p)^{x-1}p^1 = (1-p)^{x-1}p = \text{Geom}(p)$$

$$\binom{x-1}{1-1} = \binom{x-1}{0} = 1$$

of course... by def!

analagously the same as $\text{Binom}(1,p) = \text{Bern}(p)$

Another way to think about this

$X_1, \ldots, X_r \overset{iid}{\sim} \text{Geom}(p)$

$$T = \underbrace{X_1}_{\text{wait for first}} + \underbrace{X_2}_{\text{wait for second}} + \ldots + \underbrace{X_r}_{\text{wait for }r\text{th}} \sim \text{NegBin}(\underbrace{r}_{\text{wait for }r!}, p)$$

Does $\sum_{x \in \text{Supp}[X]} p(x) = 1$? $\sum_{x=r}^{\infty} \binom{x-1}{r-1}(1-p)^{x-r}p^r \overset{?}{=} 1$

Hard... E.C. the trick used to prove that gives this r.v. its name.

$F(x) = ?$ HARD real incomplete regularized beta function.

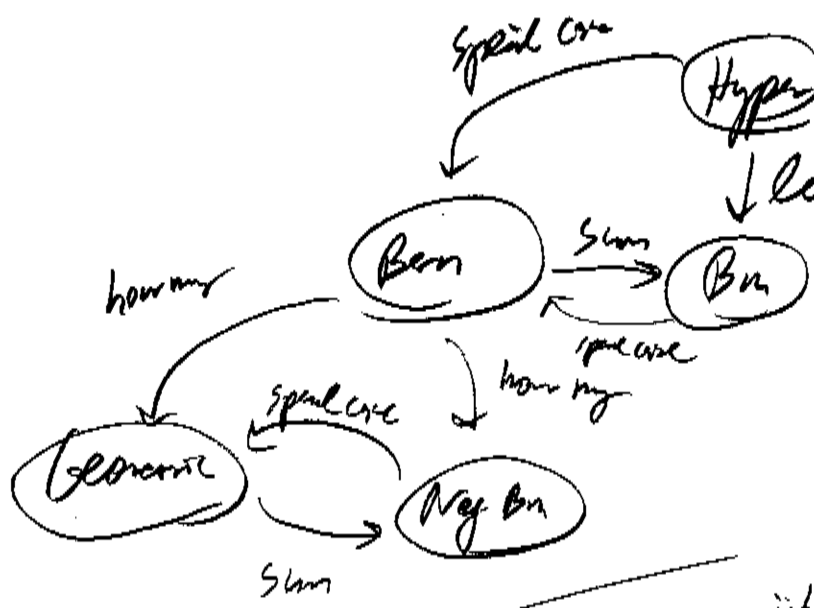
Roll dice until you get 17 6's

P(takes 100 rolls to get 17 6's)

$T \sim \text{NegBin}(17, \frac{1}{6})$

$$P(T=100) = \binom{100-1}{17-1}\left(\frac{5}{6}\right)^{100-17}\left(\frac{1}{6}\right)^{17} = \binom{99}{16}\left(\frac{5}{6}\right)^{83}\left(\frac{1}{6}\right)^{17} \approx 1.8\%$$

Done with discrete r.v.'s



all connected

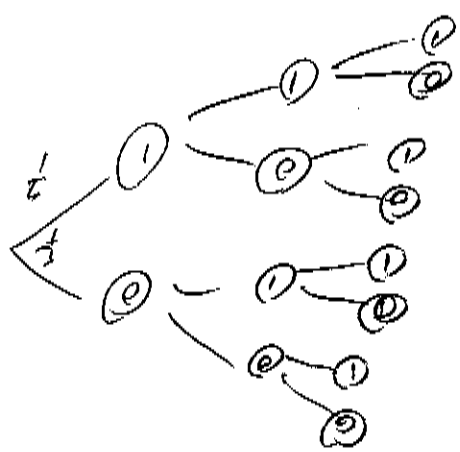
= STOP

let $X_1, \dots, X_n \overset{iid}{\sim} p(x)$ somehow

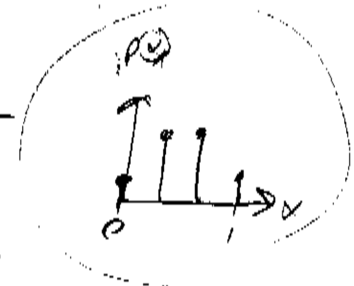
Define $\bar{X}_n = \frac{T_n}{n}$ ← the total r.v. $:= \sum_{i=1}^{n} X_i$

← the "avg. r.v."

let $X_1, X_2, X_3 \overset{iid}{\sim} \text{Bern}\left(\frac{1}{2}\right)$



$p(X_1,X_2,X_3)$	T	$\bar{X}$
$\frac{1}{8}$	3	1
	2	2/3
	2	2/3
	1	1/3
	2	2/3
	1	1/3
	1	1/3
	0	0



$$\bar{X}_3 \sim \begin{cases} 1 & \text{wp } \frac{1}{8} \\ \frac{2}{3} & \text{wp } \frac{3}{8} \\ \frac{1}{3} & \text{wp } \frac{3}{8} \\ 0 & \text{wp } \frac{1}{8} \end{cases}$$

$\bar{X}_n \to$? maybe $\frac{1}{2}$?

Philosophical jump (r.v.'s generate "data")

$X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ is a r.v. but also a "data generating process" (dgp)

"X" is the model but "x" is the "realization" → became real