Nate Silver

4/25/17

$X_1, \dots, X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$p(\theta, \sigma^2) \propto \frac{1}{\sigma^2}$

Both θ, σ^2 unknown

If σ^2 known

$p(\theta | X, \sigma^2) = N(\bar{x}, (\frac{\sigma}{\sqrt{n}})^2)$

If θ known

$p(\sigma^2 | X, \theta) = \text{InvGamma}(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2})$

If both unknown

$p(\theta, \sigma^2 | X) \propto p(X | \theta, \sigma^2) p(\theta, \sigma^2)$

$= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(X_i - \theta)^2}\right)\left(\frac{1}{\sigma^2}\right)$

$\propto (\sigma^2)^{\frac{-n}{2}-1} e^{\frac{-(n-1)s^2/2}{\sigma^2}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}$

$\propto \text{NormInvGamma}(\mu = \bar{x}, \lambda = n, \alpha = \frac{n}{2}, \beta = \frac{(n-1)s^2}{2})$

Sampling $\Rightarrow$ "pull", "draw" a realization. Make data.

$X \sim \text{Bern}(0.5)$? Flip a coin

$X \sim \text{Bin}(10, 0.5)$ Flip 10 coins

$X \sim \text{Bin}(10, 0.23859764 2)$?

$X \sim N(11.2, 3.7^2)$?

$F(x) := P(X \leq x)$ CDF

For a cont. R.V. X, what is the distribution of $Y := F_X(X)$?

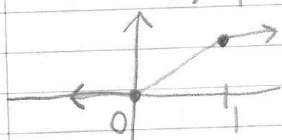
Never on test; just a proof

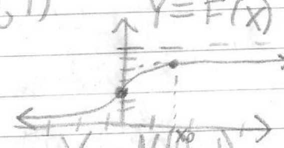
$f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right| = f_X(x) \frac{1}{\left|\frac{dy}{dx}\right|} = f_X(x) \cdot \frac{1}{\left|\frac{d}{dx}[F(x)]\right|}$

$= f_X(x) \frac{1}{|f_X(x)|} = 1$

$Supp(Y) = [0, 1]$ $\quad f_Y(y) = 1$

$\Rightarrow Y \sim U(0, 1)$ $\quad \Rightarrow X = F^{-1}(Y)$



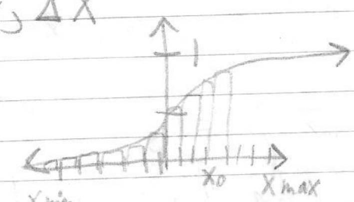


To sample x,

Important
① Sample y_0 from $U(0,1)$
② Compute $x_0 = F_x^{-1}(y_0)$
③ Retain x_0

If F^{-1} is not available, pick X_{min}, X_{max}, ΔX

Approximate $F(x)$ using numerical derivative



$G := \langle X_{min}, X_{min} + \Delta x, X_{min} + 2\Delta x, X_{min} + 3\Delta x, \ldots, X_{max} \rangle$

Compute $F(x) \quad \forall x \in G$

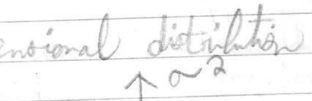
Approximate $x_0 \approx \min_{x_0 \in G} F(x_0) \geq y$

<u>What if X is discrete?</u>

$G := Supp(X)$

<u>This is sampling</u> from 1-dimensional distribution

$\theta, \sigma^2 | X \sim NInvG(\ldots)$



We know how to sample from

Write steps on test (r-norm, r-invgamma)

$f(x)$ but how to sample from $f(x,y)$?

$f(x,y)$ ("really $p(x,y)$" b/c discrete)

		Y: 2	4	6	
X	1	0.2	.05	.1	.35
	3	.1	.05	.1	.25
	5	.2	.1	.1	.4
		.5	.2	.3	

joint mass fn

$f(x)$

X and Y are not independent

$f(y)$

$f(y|x=1) \rightarrow \dfrac{\{.2, .05, .1\}}{.35}$

Recall Bayes Rule

$f(x,y) = f(y|x)\, f(x)$ conditional independence

Sampling

① Draw x_0 from $f(x)$

② Draw y_0 from $f(y|x=x_0)$

③ Return $\langle x_0, y_0 \rangle$

$P(\theta, \sigma^2|X) = P(\theta|X, \sigma^2)\, P(\sigma^2|X)$

$P(\theta|X, \sigma^2) = N(\bar{X}, (\frac{\sigma}{n})^2)$

$$\underline{P(\sigma^2|X)} = \frac{P(\theta, \sigma^2|X)}{P(\theta|X, \sigma^2)} \propto \frac{(\sigma^2)^{-\frac{n}{2}-1} e^{\frac{-(n-1)s^2/2}{\sigma^2}} e^{\frac{-1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}{\underbrace{\frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}}_{(\sigma^2)^{-\frac{1}{2}}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}$$

$$\propto (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}} e^{\frac{-(n-1)s^2/2}{\sigma^2}}$$

$$-\frac{n}{2}-\frac{1}{2}+1-1 = -(\frac{n}{2}+\frac{1}{2}-1)-1 = -(\frac{n}{2}-\frac{1}{2})-1 = -(\frac{n-1}{2})-1$$

$$\propto \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

Sampling from NInvGamma

① Sample σ_0^2 from $\text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

② Sample θ_0 from $N\left(\bar{X}, \left(\frac{\sigma_0}{\sqrt{n}}\right)^2\right)$

③ Return $\langle \theta_0, \sigma_0^2 \rangle$

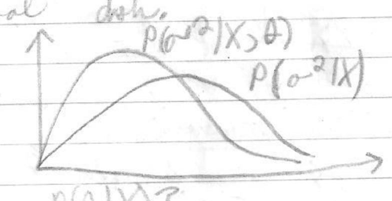
What is $P(\sigma^2|X)$? Is it $P(\sigma^2|X, \theta)$?

$$P(\sigma^2|X,\theta) = \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$$
$$\neq$$
$$P(\sigma^2|X) = \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

If $P(\theta, \sigma^2) \propto \frac{1}{\sigma^2}$

Recall $P(\sigma^2|X) = \int_{\mathbb{R}} P(\theta, \sigma^2|X)\,d\theta$

"marginal" dist.



What is $P(\theta|X)$?

$P(\theta|X) = \int_0^\infty P(\theta, \sigma^2|X)\,d\sigma^2 \longrightarrow$ Integral for HW

σ^2 is the "nuisance parameter" b/c don't care about it. Want θ, but need to handle σ^2.

$X_1, \ldots, X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$$Z = \frac{\bar{X} - \theta}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \quad \text{(Math 241)}$$

$$V = \frac{\bar{X} - \theta}{s/\sqrt{n}} \sim T_{n-1} \quad \text{(Math 242) Cochran's thm}$$

Student's T dist.

$$V \sim T_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{v^2}{n}\right)^{-\frac{n+1}{2}}$$

Student's T distr. or "Standard T distr."

$$W = \sigma V + \mu = t(v)$$

$$v = t^{-1}(w) = \frac{w - \mu}{\sigma}$$

$$f_w(w) = f_v(t^{-1}(w)) \left| \frac{d}{dw}\left[t^{-1}(w)\right]\right|$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n \sigma^2}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{1}{n}\left(\frac{w-\mu}{\sigma}\right)^2\right)^{-\frac{n+1}{2}} = T_n(\mu, \sigma)$$

non-standard T distr.

What is $P(\theta | X)$?

$$P(\theta | X) = \frac{P(\theta, \sigma^2 | X)}{P(\sigma^2 | \theta, X)}$$

$$\sigma^2 | X, \theta \sim \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right) = \text{InvG}\left(\frac{n}{2}, \frac{\sum_{i=1}^{n}(X_i - \theta)^2}{2}\right)$$

$$Q \sim \text{InvG}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\frac{\beta}{q}} q^{-\alpha - 1}$$

$$= \frac{\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i - \theta)^2}\right)\left(\frac{1}{\sigma^2}\right)}{\frac{\left(\frac{n\hat{\sigma}^2}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \left(\sigma^2\right)^{-\frac{n}{2}-1} e^{-\frac{\frac{n\hat{\sigma}^2}{2}}{\sigma^2}}}$$

Semi-conjugacy

No ridge regression

$$\propto \frac{e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}}{\left(\frac{n\hat{\sigma}^2}{2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}} = \left(\frac{n\hat{\sigma}^2}{2}\right)^{-n/2}$$

$$= \left(\frac{(n-1)s^2}{2} + \frac{n(\bar{X}-\theta)^2}{2}\right)^{-n/2}$$

$$\propto \left(\frac{1}{\frac{(n-1)s^2}{2}}\right)^{-n/2} \left(\frac{(n-1)s^2}{2} + \frac{n(\bar{X}-\theta)^2}{2}\right)^{-n/2}$$

$$= \left(1 + \frac{n(\bar{X}-\theta)^2}{2} \Big/ \frac{(n-1)s^2}{2}\right)$$

$$\frac{n(\bar{X}-\theta)^2}{2} \Big/ \frac{(n-1)s^2}{2} = \frac{(\bar{X}-\theta)^2}{s^2/n}$$

$$\left(1 + \frac{1}{n-1}\left(\frac{\bar{X}-\theta}{s/\sqrt{n}}\right)^2\right)^{-n/2} \propto T_{n-1}\left(\bar{X}, \frac{s}{\sqrt{n}}\right)$$

$$= \int_0^\infty p(\theta, \sigma^2|X)\, d\sigma^2$$

$P(\theta, \sigma^2|X)$, $p(\theta|X, \sigma^2)$, $p(\sigma^2|X, \theta)$, $P(\theta|X)$, $P(\sigma^2|X)$

4/27/17

Grid sampling

$X_1, \dots, X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$P(\theta, \sigma^2) \propto \frac{1}{\sigma^2}$

$= P(\theta) P(\sigma^2)$

$\propto 1 \qquad \propto \frac{1}{\sigma^2}$ } Uninformative / Jeffreys

$\Rightarrow P(\theta, \sigma^2|X) = N\text{InvG}$

$P(\theta | X, \sigma^2) = N\left(\bar{X}, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

$P(\sigma^2 | X, \theta) = \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2}{2}\right)$

$P(\theta|X) = T_{n-1}\left(\bar{X}, \frac{s}{\sqrt{n}}\right)$

$P(\sigma^2|X) = \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

CR? Hyp. Test?