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$X_1, \dots, X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

Assume σ^2 known $\theta | \sigma^2 \sim N(\mu_0, \tau^2)$

$\Rightarrow \theta | X_1, \dots, X_n, \sigma^2 \sim N(\theta_p, \sigma^2_p$

$\theta | \sigma^2 \sim N(\mu_0, \frac{\sigma^2}{n_0})$

$Y_1, \dots, Y_{n_0} | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$\mu_0 = \bar{y}$

Assume θ known $\sigma^2 \sim$

$Y \sim Gamma(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta y} y^{\alpha - 1}$

$Y^{-1} \sim InvGamma(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\frac{\beta}{y}} y^{-\alpha - 1}$

$P(\sigma^2 | X_1, \dots, X_n, \theta) \propto P(X_1, \dots, X_n | \theta, \sigma^2) P(\sigma^2 | \theta)$

$= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i - \theta)^2} \right) P(\sigma^2 | \theta)$

$= \left(\frac{1}{\sqrt{2\pi}} \right)^n \left(\frac{1}{\sqrt{\sigma^2}} \right)^n e^{-\frac{1}{2\sigma^2} \Sigma (x_i - \theta)^2} p(\sigma^3 | \theta)$

$\propto \underbrace{(\sigma^2)^{-\frac{n}{2}} e^{\frac{-n \hat{\sigma}^2_{MLE}/2}{\sigma^2}}} p(\sigma^2 | \theta)$

$\hat{\sigma}^2_{MLE} = \frac{1}{n} \underbrace{\sum_{i=1}^{n} (X_i - \theta)^2}_{SSE} = \frac{SSE}{n}$

kernel of $InvGamma\left(\frac{n}{2} - 1, \frac{n \hat{\sigma}^2_{MLE}}{2} \right)$

✓

$$-\frac{n}{2} = -\frac{n}{2} + 1 - 1 = -\left(\frac{n}{2} - 1\right) - 1$$

$\Rightarrow \sigma^2 \sim InvGamma(\alpha, \beta)$

$$\propto (\sigma^2)^{-\frac{n}{2}} e^{-\frac{\frac{n\hat{\sigma}^2}{2}}{\sigma^2}} (\sigma^2)^{-\alpha-1} e^{-\frac{\beta}{\sigma^2}}$$

$$= (\sigma^2)^{-\frac{n}{2}-\alpha-1} e^{-\left(\frac{n\hat{\sigma}^2}{2}+\beta\right)/\sigma^2}$$

$$\propto InvGamma\left(\frac{n}{2}+\alpha, \frac{n\hat{\sigma}^2_{MLE}}{2}+\beta\right)$$

$$\sigma^2 \sim InvGamma\Big(\underset{\overset{\shortparallel}{\alpha}}{\frac{n_0}{2}}, \underset{\overset{\shortparallel}{\beta}}{\frac{n_0\sigma_0^2}{2}}\Big) \qquad InvGamma\left(\frac{n+n_0}{2}, \frac{SSE+SSE_0}{2}\right)$$

$$\Rightarrow = InvGamma\Bigg(\overbrace{\frac{n+n_0}{2}}^{\alpha'}, \overbrace{\frac{n\hat{\sigma}^2_{MLE}+n_0\sigma_0^2}{2}}^{\beta'}\Bigg)$$

<u>Hyperparameter Interpretations</u>

n – # of trials

n_0 – prior # of trials

n_0: # prior trials seen

$n_0\sigma_0^2$: prior SSE

Imagine "prior data"

$$Y_1, \ldots, Y_{n_0} \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$$

θ is known

$$\hat{\sigma}^2_{MLE,0} = \frac{1}{n_0} \underbrace{\sum_{i=1}^{n_0} (Y_i - \theta)^2}_{SSE_0}$$

<u>Point Estimation for σ^2</u>

$$\hat{\sigma}^2_{MMSE} = E[\sigma^2 \mid X, \theta] \overset{\frac{\alpha}{\beta-1}}{=} \frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{\frac{n+n_0}{2} - 1}$$

$$= \frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{n+n_0-2}, \quad \hat{\sigma}^2_{MAP} = \frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{n+n_0+2}$$

$$\hat{\sigma}^2_{MAE} = \text{qinvgamma}\left(0.5, \frac{n+n_0}{2}, \frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{2}\right)$$

Uninformative prior

Let $n_0 = 0 \Rightarrow \sigma^2 \sim \text{InvGamma}(0,0)$ (improper)

$$\Rightarrow \sigma^2 | X, \theta \sim \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$$

proper when $\hat{\sigma}^2_{MLE} = 0$, measure 0, really

$$\Rightarrow \hat{\sigma}^2_{MMSE} = \frac{1}{n-2}\sum_{i=1}^{n}(X_i-\theta)^2$$

If $\sigma^2 \sim \text{InvGamma}(2, 0)$

$$\Rightarrow \sigma^2 | X, \theta \sim \text{InvGamma}\left(\frac{n+2}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$$

$$\Rightarrow \hat{\sigma}^2_{MMSE} = \frac{1}{n}\sum(X_i-\theta)^2 = \hat{\sigma}^2_{MLE}$$

Jeffrey's Prior

$$P(\theta^2) \propto \sqrt{I(\theta^2)}$$

$$l'(\sigma^2; x, \theta) = \frac{-n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} SSE$$

$$= \frac{-n}{2}(\sigma^2)^{-1} + \frac{SSE}{2}(\sigma^2)^{-2}$$

$$l''(\sigma^2; x, \theta) = \frac{n}{2}(\sigma^2)^{-2} - SSE(\sigma^2)^{-3}$$

$$I(\sigma^2) = E[-l''(\sigma^2; X, \theta)] = E\left[\frac{-n}{2}(\sigma^2)^{-2} + SSE(\sigma^2)^{-3}\right]$$

$$= \frac{-n}{2}(\sigma^2)^{-2} + (\sigma^2)^{-3}E(SSE) =$$

$$E(SSE) = E\left[\sum_{i=1}^{n}(X_i-\theta)^2\right] = \sum_{i=1}^{n} E[(X_i-\theta)^2]$$

Recall $X_i \sim N(\theta, \sigma^2)$

$$= n\,E[(X_1-\theta)^2]$$

$$= n\,Var(X_1) = n\sigma^2$$

$$-\frac{n}{2}(\sigma^2)^{-2} + (\sigma^2)^{-3} E(SSE)$$

$$= -\frac{n}{2}(\sigma^2)^{-2} + (\sigma^2)^{-3} \cdot n\sigma^2$$

$$= (\sigma^2)^{-2}\left(n - \frac{n}{2}\right) = \frac{n}{2}(\sigma^2)^{-2}$$

$$\Rightarrow P(\sigma^2) \propto \sqrt{I(\sigma^2)} = \sqrt{\frac{n}{2}(\sigma^2)^{-2}}$$

$$\propto \sqrt{(\sigma^2)^{-2}} = \frac{1}{\sigma^2} = (\sigma^2)^{-1}$$

$$\propto \text{InvGamma}(0, 0)$$

$$\int_0^\infty \frac{1}{\sigma^2}\, d\sigma^2 = \infty$$

$$\propto \underbrace{(\sigma^2)^{-\frac{n}{2}} e^{-\frac{n\hat{\sigma}^2}{2\sigma^2}}}_{\text{InvGamma}\left(\frac{n}{2}-1,\ \frac{n\hat{\sigma}^2}{2}\right)} \underbrace{P(\sigma^2|\theta)}_{(1)}$$

$$P(\sigma^2|\theta) \propto 1$$

$$\propto \text{InvGamma}(2, 0)$$

Test: Use, not derive, Jeffreys' prior

MIDTERM 2 MATERIAL FINISHED

No posterior predictive inverse gamma
Beta-Bin, post. pred.
Geom. model, Beta-Geom.
Binom., Poisson
Poisson-Gamma, Exponential Gamma, Gamma-Gamma
Normal-normal; known σ^2, inference on θ_i from θ, σ^2
Normal-normal-normal post. pred., credible region, same

Capture-recapture fish

FROM HERE ON IS FINAL MATERIAL

$$X_1, \ldots, X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$$

θ and σ^2 unknown

$$p(\theta, \sigma^2 | X_1, \ldots, X_n) \propto p(X_1, \ldots, X_n | \theta, \sigma^2) P(\theta, \sigma^2)$$

$f(x, y) = \cdots$

$p(\theta | X, \sigma^2)$
$p(\sigma^2 | X, \theta)$

$$\propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i - \theta)^2} P(\theta, \sigma^2)$$

$$\propto \underbrace{(\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \theta)^2}}_{\text{kernel of inverse gamma? NO}} P(\theta, \sigma^2)$$

Normal inverse gamma

$$SSE = \sum_{i=1}^{n} (X_i - \theta)^2 = \sum_{i=1}^{n} ((X_i - \bar{X}) + (\bar{X} - \theta))^2$$

$$= \sum_{i=1}^{n} \left((X_i - \bar{X})^2 + 2\underbrace{(X_i - \bar{X})(\bar{X} - \theta)}_{2(X_i \bar{X} - X_i \theta - \bar{X}^2 + \bar{X}\theta)} + (\bar{X} - \theta)^2 \right)$$

Define $s^2 := \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

sample variance

$$= \sum (X_i - \bar{X})^2 + 2\left(\sum x_i \bar{X} - \sum x_i \theta - \sum \bar{X}^2 + \sum \bar{X}\theta\right) + \sum (\bar{X} - \theta)^2$$

$$= (n-1)s^2 + 2(\cancel{n\bar{X}^2 - \theta n\bar{X} - n\bar{X}^2 + n\bar{X}\theta}) + n(\bar{X} - \theta)^2$$

$$(\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \theta)^2} p(\theta, \sigma^2)$$

$$= (\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{X} - \theta)^2\right)} p(\theta, \sigma^2)$$

$$= \underbrace{(\sigma^2)^{-\frac{n}{2}} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{X} - \theta)^2}}_{\propto \text{ Norm Inv Gamma}\left(\mu = \bar{X},\ \lambda = n,\ \alpha = \frac{n}{2} + 1,\ \beta = \frac{(n-1)s^2}{2}\right)} p(\theta, \sigma^2)$$

$\Rightarrow p(\theta, \sigma^2) = \text{Norm Inv Gamma}$ (conjugate model)

Norm Inv Gamma is the conjugate prior for the normal likelihood where both θ, σ^2 are unknown

Jeffrey's Prior $\quad p(\theta, \sigma^2) \propto \frac{1}{\sigma^2} = \underbrace{p(\theta)}_{\propto 1} \underbrace{p(\sigma^2)}_{\propto \frac{1}{\sigma^2}}$

$\Rightarrow p(\theta, \sigma^2 | X_1, \ldots, X_n)$

$= \text{Norm Inv Gamma}\left(\bar{X}, n, \frac{n}{2}, \frac{(n-1)s^2}{2}\right)$

How to simulate from Norm-Inv Gamma?

$$p(\theta, \sigma^2 | X) = \underbrace{p(\theta | X, \sigma^2)}_{\text{Normal}} \underbrace{p(\sigma^2, X)}_{\text{Inv Gamma}}$$

$$P(\sigma^2|X) = \frac{P(\theta, \sigma^2|X)}{P(\theta|X, \sigma^2)} \propto N\left(\bar{X}, \frac{\sigma^2}{n}\right)$$ assuming Jeffrey's prior

$$\propto \frac{(\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}{\frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} \cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}$$

$$\propto \frac{(\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}}{(\sigma^2)^{-\frac{1}{2}}}$$

$$= (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \qquad \to -\frac{n}{2}-\frac{1}{2}+1-1$$

$$\propto \mathrm{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

$$P(\sigma^2|X) = \mathrm{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

$$P(\sigma^2|X, \theta) = \mathrm{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{m}}{2}\right)$$