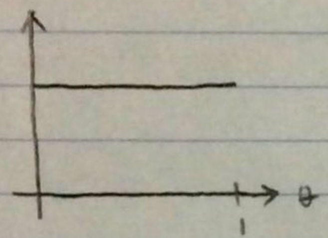
2/16/17

$\tilde{F}$ = Bernoulli

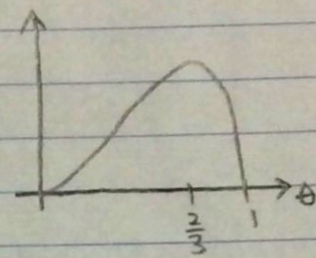
$X = \langle 0,1,1 \rangle$

$\theta \sim U(0,1)$ $\Rightarrow$ support for prior to have <u>all</u> possibilities

$$P(\theta|X) = 12\theta^2(1-\theta)$$



$\overset{x}{\Rightarrow}$



prior distribution

posterior distribution

$\hat{\theta}_{MLE} = \hat{\theta}_{MAP} = \frac{2}{3}$

F = Bernoulli

$X_1, \dots, X_n$

$\theta \sim U(0,1)$

$$P(\theta|\overset{\text{all data}}{X}) = \frac{P(X|\theta)P(\theta)}{P(X)} = \frac{P(X|\theta)\overset{1}{\overset{\shortparallel}{P(\theta)}}}{\int_0^1 P(X|\theta)\underset{1}{\underset{\shortparallel}{P(\theta)}}\,d\theta}$$

$$P(X|\theta) = \prod_{i=1}^{n} P(x_i|\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

Let $x = \sum x_i \Rightarrow = \theta^x(1-\theta)^{n-x}$

$$= \frac{P(X|\theta)P(\theta)}{\int_0^1 P(X|\theta)P(\theta)\,d\theta} = \frac{\theta^x(1-\theta)^{n-x}}{\underbrace{\int_0^1 \theta^x(1-\theta)^{n-x}\,d\theta}_{\text{famous integral}}}$$

$$= \frac{\theta^x(1-\theta)^{n-x}}{\underbrace{B(x+1,\, n-x+1)}_{\text{famous}}}$$

$$\boxed{\underset{\substack{\uparrow \\ \text{capital} \\ \text{beta}}}{B}(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt}$$

"Beta function"

A new brand name random variable is:

$$X \sim \text{Beta}(\alpha,\beta) := \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$\text{supp}[X] = (0,1)$

if $f(x)$ is a PDF, $\int_{supp(x)} f(x)\,dx = 1$

$$\int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx = \cancel{\frac{1}{B(\alpha,\beta)}} \overbrace{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx}^{\cancel{B(\alpha,\beta)}} = 1 \checkmark$$

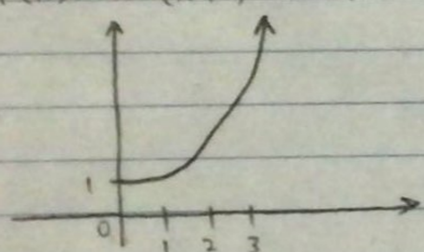
<u>Parametric Space</u>: $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$

when finite? $\alpha > 0,\ \beta > 0$

<u>Gamma Function</u>:

$$\Gamma(x) := \int_0^\infty e^{-t}\, t^{x-1}\,dt$$

$f(x) = \Gamma(x+1)$ ⇒ fill in holes of factorial function



<u>Properties of Gamma Function</u>:

① $B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

② $\Gamma(x) = (x-1)!\quad x \in \mathbb{N}$

$\Gamma(x) = (x-1)\Gamma(x-1)$ valid $\forall x$

$\Gamma(x+1) = x\,\Gamma(x)$

$$E[X] = \int_{supp[x]} x\, f(x)\,dx = \int_0^1 x\, \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha}(1-\alpha)^{\beta-1}\,dx$$

$$= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} = \frac{\frac{\Gamma(\alpha+1)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+1)}}{\frac{\Gamma(\alpha)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta)}} = \frac{\frac{\alpha\Gamma(\alpha)}{(\alpha+\beta)\Gamma(\alpha+\beta)}}{\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}}$$

$$= \boxed{\frac{\alpha}{\alpha+\beta}}$$

2/16/17 - 2

$$\text{Mode}[X] := \underset{x \in \text{Supp}[X]}{\text{argmax}} \{f(x)\} = \text{argmax} \left\{ \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} \right\}$$

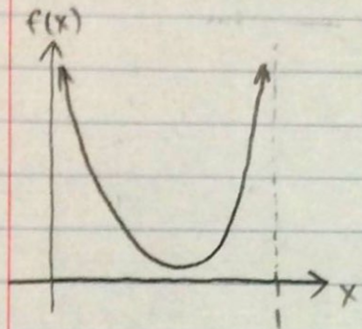
$$= \text{argmax} \left\{ x^{\alpha-1} (1-x)^{\beta-1} \right\}$$

$$\log \rightarrow \quad = \text{argmax} \{ (\alpha-1)\ln(x) + (\beta-1)\ln(1-x) \}$$

$$\Rightarrow \frac{d}{dx}\left[(\alpha-1)\ln(x) + (\beta-1)\ln(x) \right] \overset{\text{set}}{=} 0$$

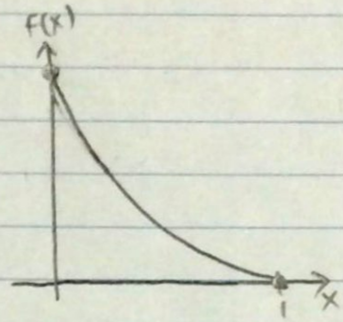
$$\Rightarrow \frac{\alpha-1}{x} - \frac{\beta-1}{1-x} = 0$$

$$\Rightarrow x = \frac{\alpha-1}{\alpha+\beta+2} \quad \text{only for } \alpha>1,\ \beta>1$$

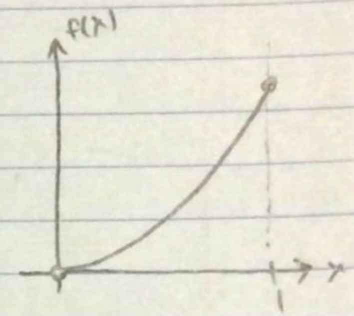


$\alpha = \beta = \frac{1}{2}$

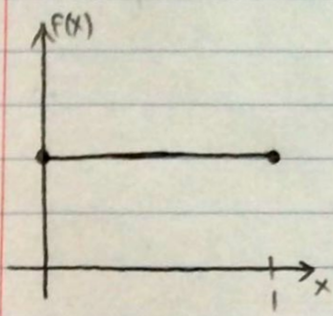
AKA "arcsine" r.v.



$\alpha = 1,\ \beta > 1$

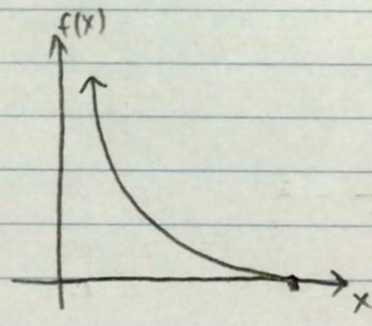


$\alpha > 1,\ \beta = 1$

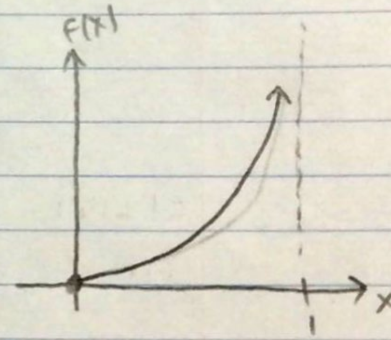


$\alpha = \beta = 1$

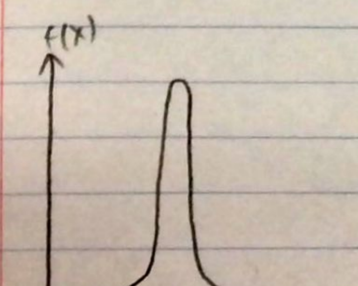
AKA "standard uniform"



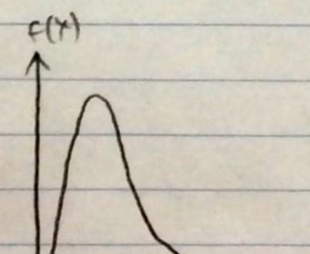
$\alpha < 1,\ \beta > 1$



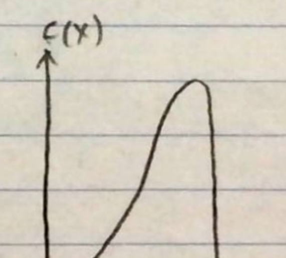
$\alpha > 1,\ \beta < 1$



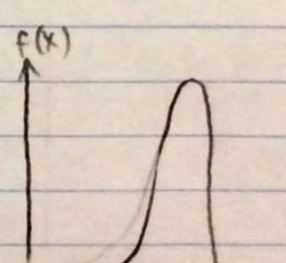
$\alpha = \beta = 100$



$\alpha = 1.01,\ \beta \gg 1$



$\alpha \gg 1,\ \beta = 1.01$



$\alpha = 100,\ \beta = 10$

$\hat{F}$ = Binomial

n known

$\theta \sim U(0,1) = Beta(1,1)$

Binomial PDF = $Binom(n,\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$

$$P(\theta|X) = \frac{P(X|\theta)\overbrace{P(\theta)}^{=1}}{\underbrace{P(X)}_{\int_{\Theta=(0,1)} P(X|\theta)P(\theta)d\theta}} = \frac{\binom{n}{x}\theta^x(1-\theta)^{n-x}}{\int_0^1 \binom{n}{x}\theta^x(1-\theta)^{n-x}d\theta} = Beta(x+1, n-x+1)$$

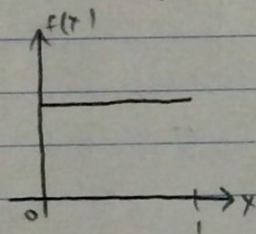
$$P(\theta) \xrightarrow{x} P(\theta|X)$$

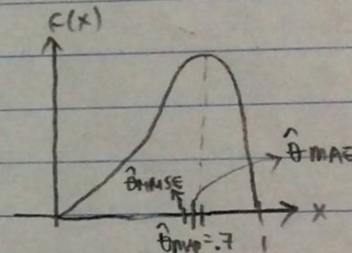
$$\underset{\alpha\ \ \beta}{Beta(1,1)} \xrightarrow{x} Beta(\underset{\alpha'}{x+1}, \underset{\beta'}{n-x+1})$$

n = 10

x = 7

$\theta|x \sim Beta(8,4)$





$$\boxed{\hat{\theta}_{MAP}} = argmax\{P(\theta|x)\}$$

$$\hat{\theta}_{MLE} = \hat{\theta}_{MAP} = mode[\theta|x] = \frac{\alpha'-1}{\alpha'+\beta'-2} = \frac{7}{10} = 0.7$$

$$\boxed{\hat{\theta}_{MMSE}} := E[\theta|x] = \frac{\alpha'}{\alpha'+\beta'} = \frac{2}{3} = 0.67$$

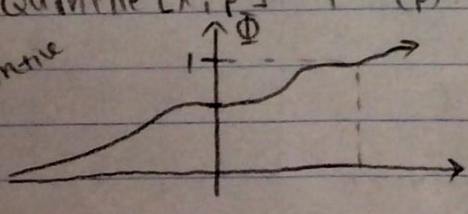
posterior mean/expectation

$$\boxed{\hat{\theta}_{MAE}} := Med[\theta|x] = \overbrace{qbeta(0.5, \underset{8}{\alpha}, \underset{4}{\beta})}^{R} \approx 0.676$$

posterior median *

if X is a continuous r.v.

Quantile$[X, p] = F^{-1}(p)$ Med$[X]$ = Quantile$[X, 0.5] = F^{-1}(\frac{1}{2})$



2/16/17 - ③

$\theta \sim Beta(\alpha, \beta)$ with appropriately chosen α, β.

$$P(\theta|x) = \frac{P(x|\theta)P(\theta)}{\underbrace{P(x)}_{\int_{\Theta} P(x|\theta)P(\theta)d\theta}} = \frac{\cancel{\binom{n}{x}} \theta^{x}(1-\theta)^{n-x} \cancel{\frac{1}{B(\alpha,\beta)}} \theta^{\alpha-1}(1-\theta)^{\beta-1}}{\int_0^1 \cancel{\binom{n}{x}} \theta^{x}(1-\theta)^{n-x} \cancel{\frac{1}{B(\alpha,\beta)}} \theta^{\alpha-1}(1-\theta)^{\beta-1} d\theta}$$

$$= \frac{\theta^{x-\alpha-1}(1-\theta)^{n-x+\beta-1}}{\int_0^1 \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1} d\theta}$$

$$= \frac{1}{B(x+\alpha, n-x+\beta)} \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}$$

$$= Beta(x+\alpha, n-x+\beta)$$

$Beta \overset{x}{\Rightarrow} Beta$ "conjugacy"

prior & posterior are same family

"the beta is conjugate prior for the binomial model"