Math 341 / 650 Spring 2017
Midterm Examination One Solutions

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Full Name ____

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

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signature date

Instructions

This exam is seventy five minutes and closed-book. You are allowed **one** page (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If the question reads "compute," this means the solution will be a number otherwise you can leave the answer in *any* widely accepted mathematical notation which could be resolved to an exact or approximate number with the use of a computer. I advise you to skip problems marked "[Extra Credit]" until you have finished the other questions on the exam, then loop back and plug in all the holes. I also advise you to use pencil. The exam is 100 points total plus extra credit. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. Good luck!

Distribution of r.v.	Quantile Function	PMF / PDF function	CDF function	Sampling Function
beta	qbeta(p, α, β)	d-(x, α, β)	p-(x, α, β)	r-(α, β)
betabinomial	qbetabinom(p, n, α, β)	d-(x, n, α, β)	p-(x, n, α, β)	r-(n, α, β)
betanegativebinomial	qbeta_nbinom(p, r, α, β)	d-(x, r, α, β)	p-(x, r, α, β)	r-(r, α, β)
binomial	qbinom(p, n, θ)	d-(x, n, θ)	p-(x, n, θ)	r-(n, θ)
exponential	qexp(p, θ)	d-(x, θ)	p-(x, θ)	r-(θ)
gamma	qgamma(p, α, β)	d-(x, α, β)	p-(x, α, β)	r-(α, β)
geometric	qgeom(p, θ)	d-(x, θ)	p-(x, θ)	r-(θ)
inversegamma	qinvgamma(p, α, β)	d-(x, α, β)	p-(x, α, β)	r-(α, β)
negative-binomial	qnbinom(p, r, θ)	d-(x, r, θ)	p-(x, r, θ)	r-(r, θ)
normal (univariate)	qnorm(p, θ, σ)	d-(x, θ, σ)	p-(x, θ, σ)	r-(θ, σ)
normal (multivariate)		dmvnorm($\boldsymbol{x}$, $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$)		r-($\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$)
poisson	qpois(p, θ)	d-(x, θ)	p-(x, θ)	r-(θ)
T (standard)	qt(p, ν)	d-(x, ν)	p-(x, ν)	r-(ν)
T (nonstandard)	qt.scaled(p, ν, μ, σ)	d-(x, ν, μ, σ)	p-(x, ν, μ, σ)	r-(ν, μ, σ)
uniform	qunif(p, a, b)	d-(x, a, b)	p-(x, a, b)	r-(a, b)

Table 1: Functions from R (in alphabetical order) that can be used on this exam. The hyphen in colums 3, 4 and 5 is shorthand notation for the full text of the r.v. which can be found in column 2.

Problem 1 This question continues our discussion about extrasensory perception. A famous series of experiments called the Ganzfeld experiments work as follows. A human subject called the "receiver" sits in a room in darkness with their eyes covered and with headphones to filter out any noise. An example subject is pictured below:

Another human subject called the transmitter sits in another room. In the beginning of the experiment, the transmitter is shown an image. The transmitter than tries to "telephathically transmit" the image to the receiver. At the conclusion of the experiment, the receiver is shown four images — three decoys and the true image the transmitter was given — and is asked to choose one. If the receiver chooses the true image, this is termed a "hit".

(a) [4 pt / 4 pts] Under the null hypothesis of "no ESP", the receiver chooses one image randomly from four images where one image make the "hit". Create a r.v. X for the "hit" under the null. Indicate the type of r.v. and the value of θ.

$X \sim Bern\left(\theta = \frac{1}{4}\right)$

(b) [5 pt / 9 pts] A psychic claims that his ability to identify the target image is 50%. Create a prior on both the null θ and the psychic's claimed θ. Use the principle of indifference.

$$\theta \sim \begin{cases} \frac{1}{4} & wp\ \frac{1}{2} \\ \frac{1}{2} & wp\ \frac{1}{2} \end{cases}$$

(c) [7 pt / 16 pts] We run three independent experimental trials of which the psychic gets two of the three correct. Calculate $\mathbb{P}(X)$.

$n=3,\ x=2$

$$P(X) = \sum_{\theta \in \Theta} P(X|\theta)P(\theta) = P(X|\theta=\tfrac{1}{4})P(\theta=\tfrac{1}{4}) + P(X|\theta=\tfrac{1}{2})P(\theta=\tfrac{1}{2})$$

$$P(X=2) = \binom{3}{2}\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^1\left(\tfrac{1}{2}\right) + \binom{3}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)$$

$$= 3\,\tfrac{1}{16}\,\tfrac{3}{4}\,\tfrac{1}{2} + 3\,\tfrac{1}{4}\,\tfrac{1}{2}\,\tfrac{1}{2}$$

$$= \frac{9}{128} + \frac{3}{16} = .2578$$

(d) [6 pt / 22 pts] Find the probability that the psychic's assessment of his abilities is correct given this data.

$$P(\theta=0.5|X=2) = \frac{P(X=2|\theta=0.5)P(\theta=0.5)}{P(X=2)} = \frac{\frac{3}{16}}{\frac{9}{128}+\frac{3}{16}} = .7273$$

(e) [7 pt / 29 pts] Calculate the probability the next experiment for this psychic (i.e. the fourth, unobserved experiment) will be a "hit".

$$P(X^*|X) = \sum_{\theta \in \Theta} P(X^*|\theta)P(\theta|X)$$

$$P(X^*=1|X=2) = P(X^*=1|\theta=\tfrac{1}{4})P(\theta=\tfrac{1}{4}|X=2) + P(X^*=1|\theta=\tfrac{1}{2})P(\theta=\tfrac{1}{2}|X=2)$$

$$= \tfrac{1}{4}(1-.7273) + \tfrac{1}{2}(.7273)$$

$$= .068182 + .363636 = .4318$$

(f) [6 pt / 35 pts] We now put a uniform prior on θ across all values in the support. Find the probability that this psychic has "better than normal" abilities given the data. You are free to leave your answer in notation from Table 1.

$\theta \sim U(0,1) = \text{beta}(1,1)$

$X|\theta \sim \text{Bin}(n,\theta)$

$\Rightarrow \theta|X \sim \text{beta}(\alpha+x, \beta+n-x) = \text{beta}(1+2, 1+(3-2)) = \text{beta}(3, 2)$

$P(\theta \geq .25 | x=2) = \boxed{1 - \text{pbeta}(.25, 3, 2)}$ " $\int_{.25}^{1} \text{beta}(3,2)\, d\theta$ "

(g) [5 pt / 40 pts] Instead of a uniform prior, you want to factor in what you've seen previously. You figure your prior experience is worth 18 trials and you haven't seen any evidence of ESP in those trials. Using the conjugate prior we discussed in class, what is the prior for θ now?

$\theta \sim \text{beta}(\alpha, \beta)$

α = "# prior successes"

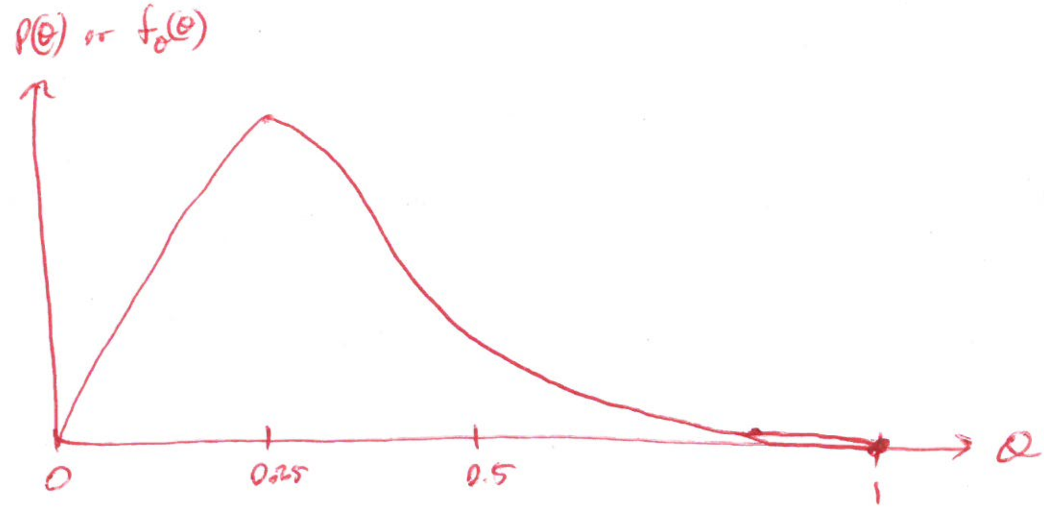
β = "# prior failures"

$\alpha = 18 \cdot \frac{1}{4} = 4.5$

$\beta = 18 \cdot \frac{3}{4} = 13.5$

$\Rightarrow \theta \sim \text{beta}(4.5, 13.5)$

(h) [7 pt / 47 pts] Illustrate this prior the best you can. No need for perfect scale. Make sure to label axes appropriately.



(i) [4 pt / 51 pts] Compute the prior mean.

$E(\theta) = \frac{\alpha}{\alpha+\beta} = \frac{4.5}{18} = \frac{1}{4}$ (by construction)

(j) [5 pt / 56 pts] Given this prior and the data, calculate the shrinkage proportion towards $\mathbb{E}[\theta]$ if you were to use $\hat{\theta}_{\text{MMSE}}$ for your point estimation strategy.

$\rho = \frac{\alpha+\beta}{n+\alpha+\beta} = \frac{18}{3+18} = \frac{18}{21} = .857143$

(k) [5 pt / 61 pts] If there was a lot of data, what would be the influence of the prior on the point estimate of θ? Try to use only one word.

"insignificant", "negligible", "small", etc

(l) [6 pt / 67 pts] Given this prior and the data, fill in the following boxes with only the following symbols: $<$, $\leq$, $>$, $\geq$, $=$ to indicate the numerical relationships.

$$\hat{\theta}_{\text{MAP}} \boxed{<} \hat{\theta}_{\text{MMSE}}, \quad \hat{\theta}_{\text{MAE}} \boxed{<} \hat{\theta}_{\text{MMSE}}, \quad \hat{\theta}_{\text{MAP}} \boxed{<} \hat{\theta}_{\text{MAE}}$$

(m) [10 pt / 77 pts] In 2010, Lance Storm, Patrizio Tressoldi, and Lorenzo Di Risio analyzed 29 ganzfeld studies from 1997 to 2008 amassing data on 1,498 trials with different receiver and transmitter subjects. Of the 1,498 trials, 483 were hits. Test the existence of ESP in the Ganzfeld experiments using the Frequentist (i.e. the *non-Bayesian*) two-sided test. You must (1) write your hypotheses clearly, (2) choose your own significance level and (3) state clearly the conclusion of the test.

H_0: no ESP i.e. $\theta = \theta_0 = \frac{1}{4}$

H_a: ESP i.e. $\theta \neq \theta_0 = \frac{1}{4}$

$\alpha = 5\%$

Ret Region $= \left[\theta_0 \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right] = \left[\frac{1}{4} \pm 2\sqrt{\frac{\frac{1}{4}\left(\frac{3}{4}\right)}{1498}}\right] = [.2276, .2724]$

$\hat{\theta}_{MLE} = \frac{483}{1498} = .32243 \notin$ Ret Region $\Rightarrow$ Reject H_0 $\Rightarrow$ ESP exists

(n) [4 pt / 81 pts] Using the Storm et al. (2010) Ganzfeld data and a uniform prior, create a 95% credible region for θ. You are free to leave your answer in notation from Table 1.

$\theta \sim U(0,1) \Rightarrow \theta|X \sim \text{Beta}(1+x, 1+n-x) = \text{Beta}(1+483, 1+1498-483) = \text{Beta}(484, 1016)$

$CR_{\theta, 95\%} = [\text{qbeta}(.025, 484, 1016), \text{qbeta}(.975, 484, 1016)]$

(o) [4 pt / 85 pts] Describe in one sentence how you would use the answer from the previous question to test the existence of ESP in the Storm et al. (2010) Ganzfeld data using one of the Bayesian two-sided hypothesis test strategies discussed in class.

If $\hat{\theta}_{MLE} = .32243 \in CR_{\theta,1-\alpha}$ then retain H_0; if not, reject H_0

(p) [10 pt / 95 pts] Assess the evidence for the ESP phenomenon in the Ganzfeld experiments using Bayes Factors. You will not be able to solve for B numerically but you need to get as far as you can. 5 points extra credit if you can solve numerically and interpret your result. You can use $\ln(n!) \approx n \ln(n) - n + 1$. $H_0: \theta = \frac{1}{4}$, $H_a: \theta \sim U(0,1)$

$$B := \frac{\int_{\theta \in \Theta_a} P_a(X|\theta)\, P_a(\theta)\, d\theta}{\int_{\theta \in \Theta_0} P_0(X|\theta)\, P_0(\theta)\, d\theta} = \frac{\int_0^1 \binom{1498}{483} \theta^{483} (1-\theta)^{1015}\, d\theta}{\binom{1498}{483} \left(\frac{1}{4}\right)^{483} \left(\frac{3}{4}\right)^{1015}} = \frac{B(484, 1016)}{\frac{3^{1015}}{4^{1498}}} = \frac{\Gamma(484)\,\Gamma(1016)\, 4^{1498}}{\Gamma(1500)\, 3^{1015}} = \frac{483!\, 1015!\, 4^{1498}}{1499!\, 3^{1015}}$$

for extra credit

$$\begin{aligned}
\ln(B) &= \ln(483!) + \ln(1015!) + 1498 \ln(4) - \ln(1499!) - 1015 \ln(3) \\
&\approx 483 \ln(483) - 483 + 1 + 1015 \ln(1015) - 1015 + 1 + 1498 \ln(4) - 1499 \ln(1499) + 1499 - 1 - 1015 \ln(3) \\
&= 2 + 483 \ln(483) + 1015 \ln(1015) + 1498 \ln(4) - 1499 \ln(1499) - 1015 \ln(3) \\
&\approx 13.4914
\end{aligned}$$

$\Rightarrow B \approx e^{13.4914} = 723156 \Rightarrow$ there is absolutely decisive evidence in favor of ESP vs. null

Problem 2 This is a theoretical question.

(a) [5 pt / 100 pts] Consider the case of assessing the evidence of θ_a versus θ_0. Show that B is the odds ratio of the posterior odds to the prior odds.

$$= \frac{\int_{\{\theta_a\}} P_a(X|\theta_a)\, P_a(\theta_a)\, d\theta}{\int_{\{\theta_0\}} P_0(X|\theta_0)\, P_0(\theta_0)\, d\theta} = \frac{P(X|\theta_a)}{P(X|\theta_0)} = B$$

Recall

$$P(\theta_a|X) = \frac{P(X|\theta_a)\, P(\theta_a)}{P(X)}, \quad P(\theta_0|X) = \frac{P(X|\theta_0)\, P(\theta_0)}{P(X)} \Rightarrow \frac{P(\theta_a|X)}{P(\theta_0|X)} = \underbrace{\frac{P(X|\theta_a)}{P(X|\theta_0)}}_{B} \frac{P(\theta_a)}{P(\theta_0)} \Rightarrow B = \frac{\frac{P(\theta_a|X)}{P(\theta_0|X)}}{\frac{P(\theta_a)}{P(\theta_0)}}$$

posterior odds

prior odds