Let $X_1, \dots, X_n | \theta \overset{iid}{\sim} \text{Geom}(\theta)$, $\theta \sim \text{Beta}(\alpha, \beta)$. Then $\theta | X_1, \dots, X_n \sim \text{Beta}(n+\alpha, \sum x_i + \beta)$ where α is the number of pseudotrials and β is the number of pseudofailures.

$$\hat{\theta}_{\text{MMSE}} = \frac{n+\alpha}{n+\alpha+\sum x_i + \beta}$$

Haldane Prior: if $\theta \sim \text{Beta}(0,0), \hat{\theta}_{\text{MMSE}} = \frac{n}{n+\sum x_i} = \frac{1}{1+\frac{\sum x_i}{n}} = \frac{1}{1+\bar{x}} = \hat{\theta}_{\text{MLE}}$

Laplace Prior: if $\theta \sim \text{Beta}(1,1), \hat{\theta}_{\text{MMSE}} = \frac{n+1}{n+1+\sum x_i + 1} = \frac{1}{1+\frac{\sum x_i + 1}{n+1}}$

Jeffrey's Prior: if $\theta \sim \text{Beta}(0, \frac{1}{2}), \hat{\theta}_{\text{MMSE}} = \frac{n}{n+\sum x_i + \frac{1}{2}} = \frac{1}{1+\frac{\sum x_i + \frac{1}{2}}{n}}$

Note: Harmonic average: $\frac{1}{\bar{x}} = \frac{1}{n}\sum_i \frac{1}{x}$

In the general case, is there a shrinkage interpretation?

$$\begin{aligned}
\frac{1}{\hat{\theta}_{\text{MMSE}}} &= \frac{n+\alpha+\sum x_i + \beta}{n+\alpha} \\
&= \frac{\alpha+\beta}{n+\alpha} \cdot \frac{\alpha}{\alpha} + \frac{\sum x_i + n}{n+\alpha} \cdot \frac{n}{n} \\
&= \frac{\alpha+\beta}{\alpha} \cdot \frac{\alpha}{n+\alpha} + \frac{n+\sum x_i}{n} \cdot \frac{n}{n+\alpha} \\
&= \frac{1}{\text{E}[\theta]}\rho + \frac{1}{\hat{\theta}_{\text{MLE}}}(1-\rho)
\end{aligned}$$

Note, if n is small, then there is huge shrinkage; if n is large, $\hat{\theta}_{\text{MMSE}} = \hat{\theta}_{\text{MLE}}$.

Under $n^* = 1$,

$$\begin{aligned}
\mathbb{P}(X^* \mid X) &= \int_\Theta \mathbb{P}(X^* \mid \theta)\, \mathbb{P}(\theta \mid X)\, d\theta \\
&= \int_0^1 \Big((1-\theta)^{x^*}\theta\Big)\Big(\frac{1}{B(n+\alpha, \sum x_i + \beta)}\theta^{n+\alpha-1}(1-\theta)^{\sum x_i + \beta - 1}\Big)\, d\theta \\
&= \frac{1}{B(n+\alpha, \sum x_i + \beta)} \int_0^1 \theta^{n+\alpha+1-1}(1-\theta)^{x^*+\sum x_i + \beta - 1}\, d\theta \\
&= \frac{B(n+\alpha+1, x^* + \sum x_i + \beta)}{B(n+\alpha, \sum x_i + \beta)} \\
&= \text{BetaGeom}(n+\alpha, \sum x_i + \beta)
\end{aligned}$$

Let $X_1, \dots, X_n \overset{iid}{\sim} \text{NegBinom}(r, \theta) = \binom{x+r-1}{x}(1-\theta)^x\theta^r$ and $\theta \sim \text{Beta}(\alpha, \beta)$. Then $\theta | X_1, \dots, X_n \sim \text{Beta}(r+\alpha, \sum x_i + \beta)$ and $\mathbb{P}(X^* \mid X) = \text{BetaGeom}(n+\alpha, \sum x_i + \beta)$.

Let $X \sim \text{Binom}(n, \theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$. If n is large and θ is small, let $\lambda = n\theta$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{n!}{x!(n-x)!}(\frac{\lambda}{n})^x(1-\frac{\lambda}{n})^{1-x} &= \frac{\lambda^x}{x!}\lim_{n\to\infty}\frac{n \cdot n-1 \cdot n-2 \cdot \cdots \cdots n-x+1}{n \cdot n \cdot n \cdot \cdots \cdot n}(1-\frac{\lambda}{n})^n(1-\frac{\lambda}{n})^{-x} \\
&= \frac{\lambda^x e^{-\lambda}}{x!}
\end{aligned}$$

Let $X \sim \text{Poisson}(\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$. $\text{Supp}[X] = \{0, 1, \dots\}$, $\lambda \in (0, \infty)$. $\text{E}[X] = \lambda$, $\text{Var}[X] = \lambda$.

Let $X|\theta \sim \text{Poisson}(\theta) = \frac{e^{-\theta}\theta^x}{x!}$.

$$\mathbb{P}(\theta \mid X) \propto \mathbb{P}(X \mid \theta)\,\mathbb{P}(\theta) = \frac{e^{-\theta}\theta^x}{x!}\mathbb{P}(\theta) \propto e^{-\theta}\theta^x\mathbb{P}(\theta)$$

Therefore $\mathbb{P}(\theta) \propto e^{-b\theta}\theta^a$.

$$\mathbb{P}(\theta) = \frac{b^{a+1}}{\Gamma(a+1)}e^{-b\theta}\theta^a$$

Then

$$\theta \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta\theta}\theta^{\alpha-1}$$

$\text{Supp}[\theta] = (0, \infty)$, parameter space: $\alpha > 0, \beta > 0$. $\text{E}[\theta] = \frac{\alpha}{\beta}$, $\text{Var}[\theta] = \frac{\alpha}{\beta^2}$, $\text{Mode}[\theta] = \frac{\alpha-1}{\beta}$ if $\alpha \geq 1$ and $\text{Med}[\theta] = \text{qgamma}(0.5,\, \alpha, \beta)$.

$$\begin{aligned}
\mathbb{P}(\theta \mid X) &\propto \mathbb{P}(X \mid \theta)\,\mathbb{P}(\theta) \\
&= \frac{e^{-\theta}\theta^x}{x!}\frac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta\theta}\theta^{\alpha-1} \\
&\propto e^{-\theta}\theta^x e^{-\beta\theta}\theta^{\alpha-1} \\
&= e^{-(\beta+1)\theta}\theta^{x+\alpha-1} \\
&\propto \text{Gamma}(x+\alpha, \beta+1)
\end{aligned}$$

Therefore when $X|\theta \sim \text{Poisson}(\theta)$ and $\theta \sim \text{Gamma}(\alpha, \beta)$, $\theta|X \sim \text{Gamma}(x+\alpha, \beta+1)$. We say that the gamma is conjugate prior for the Poisson likelihood.

Let $X_1, \dots, X_n|\theta \overset{iid}{\sim} \text{Poisson}(\theta)$ and $\theta \sim \text{Gamma}(\alpha, \beta)$.

$$\begin{aligned}
\mathbb{P}(\theta \mid X) &\propto \mathbb{P}(X \mid \theta)\,\mathbb{P}(\theta) \\
&= \left(\prod_{i=1}^n \frac{e^{-\theta}\theta^{x_i}}{x_i!}\right)\left(\frac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta\theta}\theta^{\alpha-1}\right) \\
&= \frac{e^{-\sum_{i=1}^n \theta_i}\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}\frac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta\theta}\theta^{\alpha-1} \\
&\propto e^{-n\theta}\theta^{\sum x_i}e^{-\beta\theta}\theta^{\alpha-1} \\
&\propto \text{Gamma}(\sum x_i + \alpha, n+\beta)
\end{aligned}$$

Here α is the total number of successes seen previously and β is the number of pseudotrials performed.

$$\hat{\theta}_{\text{MMSE}} = \frac{\sum x_i + \alpha}{n+\beta} \;\; \hat{\theta}_{\text{MAE}} = \text{qgamma}(0.5, \sum x_i + \alpha, n+\beta) \;\; \hat{\theta}_{\text{MAP}} = \frac{\sum x_i + \alpha - 1}{n+\beta} \text{ if } \sum x_i + \alpha \geq 1$$

Can we say that the Laplace prior is $\theta \sim U$? No because the support in infinity and thus not an integratable region. Let's say $\mathbb{P}(\theta) \propto 1$. This is clearly improper and indifferent.

$$\begin{aligned}\mathbb{P}(\theta \mid X) &\propto \mathbb{P}(X \mid \theta)\,\mathbb{P}(\theta) \\ &\propto e^{-n\theta}\theta^{\sum x_i}\mathbb{P}(\theta) \\ &\propto e^{-n\theta}\theta^{\sum x_i} \\ &= \text{Gamma}(\sum x_i, n)\end{aligned}$$

Thus if $\theta \sim \text{Gamma}(0,0)$, then the Haldane prior equals the Laplace prior, both of which are improper.

$$\begin{aligned}\mathcal{L}(\theta;x) &= \prod_{i=1}^{n}\frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod x_i!} \\ l(\theta;x) &= -n\theta + \sum x_i \ln\theta - \ln(\prod x_i!) \\ l'(\theta;x) = -n + \frac{\sum x_i}{\theta} \overset{\text{set}}{=} 0 \to \frac{\sum x_i}{\theta} = n \to \hat{\theta}_{\text{MLE}} = \bar{x} l''(\theta;x) &= -\frac{\sum x_i}{\theta^2} \\ I(\theta) &= \text{E}[-l''(\theta;x)] = \text{E}[\frac{\sum x_i}{\theta^2}] \\ &= \frac{\text{E}[\sum x_i]}{\theta^2} \\ &= \frac{\sum \text{E}[x_i]}{\theta^2} = \frac{\sum \theta}{\theta^2} = \frac{n\theta}{\theta^2} = \frac{n}{\theta} \\ \mathbb{P}(\theta) &\propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\theta}} \propto \sqrt{\frac{1}{\theta}} = \theta^{-\frac{1}{2}} \\ &\propto \text{Gamma}(\frac{1}{2}, 0)\end{aligned}$$

This Jeffrey's prior is improper.

$$\hat{\theta}_{\text{MMSE}} = \frac{\sum x_i + \alpha}{n = \beta} = \frac{\sum x_i}{\beta + n}\cdot\frac{n}{n} + \frac{\alpha}{n+\beta}\cdot\frac{\beta}{\beta} = \frac{n}{n+\beta}\frac{\sum x_i}{n} + \frac{\beta}{n+\beta}\frac{\alpha}{\beta} = \hat{\theta}_{\text{MLE}}(1-\rho) + \rho\text{E}[\theta]$$

For $n^* = 1$,

$$\begin{aligned}\mathbb{P}(X^* \mid X) &= \int_{\alpha}\mathbb{P}(X^* \mid \theta)\,\mathbb{P}(\theta \mid X)\,d\theta \\ &= \int_0^\infty \Big(\frac{e^{-\theta}\theta^{x^*}}{x^*!}\Big)\Big(\frac{\beta'^{\alpha'}}{\Gamma(\alpha')}e^{-\beta'\theta}\theta^{\alpha'-1}\Big)\,d\theta \\ &= \frac{\beta'^{\alpha'}}{\Gamma(\alpha')x^*!}\int_0^\infty e^{-(\beta'+1)\theta}\theta^{x^*+\alpha'-1}\,d\theta\end{aligned}$$

The integrand in the last line above is the kernel of $\text{Gamma}(x^* + \alpha', \beta' + 1)$.