$$P(X^*|X) = \int_{\Theta} P(X^*|\theta)\, P(\theta|X)\, d\theta$$

$n^* = 1$

$$= \int_0^\infty \left(\frac{e^{-\theta}\theta^{x^*}}{x^*!}\right)\left(\frac{\beta'^{\alpha'}}{\Gamma(\alpha')} e^{-\beta'\theta}\theta^{\alpha'-1}\right) d\theta$$

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$X_1, \ldots, X_n | \theta \overset{iid}{\sim} \text{Poisson}(\theta)$

$\theta \sim \text{Gamma}(\alpha, \beta)$

$\theta | X_1, \ldots, X_n \sim \text{Gamma}(\underbrace{\sum x_i + \alpha}_{\alpha'}, \underbrace{n+\beta}_{\beta'})$

$X_1, \ldots, X_n$

$\uparrow$

$$P(X^*|X) = \int_{\Theta} P(X^*|\theta)\, P(\theta|X)\, d\theta$$

for $n^* = 1$

$$= \int_0^\infty \left(\frac{e^{-\theta}\theta^{x^*}}{x^*!}\right)\left(\frac{\beta'^{\alpha'}}{\Gamma(\alpha')} e^{-\beta'\theta}\theta^{\alpha-1}\right) d\theta$$

$$\frac{\beta'^{\alpha'}}{\Gamma(\alpha')x^*!}\int_0^\infty \underbrace{e^{-(\beta'+1)\theta}\,\theta^{x^*+\alpha'-1}}_{\substack{\text{kernel of} \\ \text{Gamma}(x^*+\alpha', \beta'+1)}} d\theta$$

$$= \frac{\beta'^{\alpha'}}{\Gamma(\alpha')x^*!}\underbrace{\frac{\Gamma(x^*+\alpha')}{(\beta'+1)^{x^*+\alpha'}}}_{(\beta'+1)^{x^*}(\beta'+1)^{\alpha'}}\underbrace{\int_0^\infty \frac{(\beta'+1)^{x^*+\alpha'}}{\Gamma(x^*+\alpha')} e^{-(\beta'+1)\theta}\,\theta^{x^*+\alpha'-1}\, d\theta}_{=1}$$

$$= \left(\frac{\beta'}{\beta'+1}\right)^{\alpha'}\left(\frac{1}{\beta'+1}\right)^{x^*}\frac{\Gamma(x^*+\alpha')}{x^*!\,\Gamma(\alpha')} = \frac{\Gamma(x^*+\alpha')}{x^*!\,\Gamma(\alpha')}(1-p)^{x^*}p^{\alpha}$$

Note $\frac{\beta'}{\beta'+1} \in (0,1)$ Note $1 - \frac{\beta'}{\beta'+1} = \frac{1}{\beta'+1} \in (0,1)$

Let $p := \frac{\beta'}{\beta'-1} \Rightarrow 1-p = \frac{1}{\beta'+1}$

If $\alpha' \in \mathbb{N}$

$\Rightarrow \Gamma(x^*+\alpha') = (x^*+\alpha'-1)!$

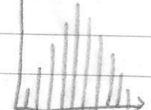
$\Gamma(\alpha') = (\alpha'-1)!$

$$\Rightarrow = \frac{(x^*+\alpha'-1)!}{\underbrace{x^*!\,(\alpha'-1)!}} (1-p)^{x^*} p^{\alpha'} = \binom{x^*+\alpha'-1}{x^*}(1-p)^{x^*}p^{\alpha'}$$

$$= NegBin(\alpha', p) = NegBin\left(\sum x_i + \alpha, \frac{n+\beta}{n+\beta+1}\right) = P(X^*|X)$$

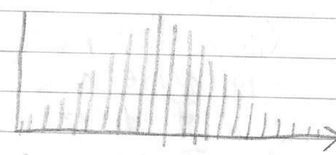
$X^*|\theta \sim Poisson(\theta)$

$X^*|X \sim NegBin(\ldots)$



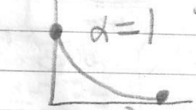
$Exp = \theta$

$Var = \theta$



Consider

$$X|\theta \sim Gamma(1,\theta) = \underbrace{\frac{\theta^1}{\Gamma(1)}}_{1} e^{-\theta x}\underbrace{\theta^{1-1}}_{1} = \theta e^{-\theta x} = Exp(\theta)$$



$$P(\theta|X) \propto P(X|\theta)P(\theta) = \underbrace{\theta e^{-\theta x}}_{\text{Gamma kernel}} \underbrace{P(\theta)}_{\text{Gamma kernel}} = \theta e^{-\theta x}\frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\theta}\theta^{\alpha-1}$$

$\propto e^{-(\beta+x)\theta}\theta^{\alpha+1-1}$

$\propto Gamma(\alpha+1, \beta+X)$

Let $\theta \sim Gamma(\alpha,\beta)$

$\Rightarrow X|\theta \sim Exp(\theta)$ $\quad$ $\theta|X \sim Gamma(\alpha+1, \beta+x)$

$\theta \sim Gamma(\alpha,\beta)$ $\quad$ Gamma is conjugate for the exponential likelihood

MMR model

Consider $X|\theta \sim Gamma(r,\theta) = \frac{\theta^r}{\Gamma(r)} e^{-\theta x} x^{r-1}$

$= \frac{\theta^r}{(r-1)!} e^{-\theta x} x^{r-1} = Erlang(r,\theta)$ if $r \in \mathbb{N}$

$p(\theta|X) \propto p(X|\theta) p(\theta)$

$= \left(\frac{\theta^r}{(r-1)!} e^{-\theta x} x^{r-1}\right) p(\theta) \Rightarrow$ Gamma is conjugate for the gamma likelihood w/ fixed α

$\propto \underbrace{\theta^r e^{-\theta x} p(\theta)}_{gamma}$

$p(\theta|x,r) \propto p(x|\theta,r) p(\theta,r)$

because r is considered known

$X|\theta,n \sim Bin(n,\theta)$

$\theta|n \sim Beta(\alpha,\beta)$

$\theta|X,n \sim Beta(x+\alpha, n-x+\beta)$

Consider

$X|\theta,\sigma^2 \sim N(\theta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$

$E(X) = \theta$ $\quad$ $Supp(X) = \mathbb{R}$

$Var(X) = \sigma^2$ $\quad$ $\theta \in \mathbb{R}$

$\sigma^2 \in (0,\infty)$

$\propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$

$= e^{\frac{-x^2}{2\sigma^2} + \frac{\theta x}{\sigma^2} - \frac{\theta^2}{2\sigma^2}}$

$= e^{\frac{-x^2}{2\sigma^2}} e^{\frac{\theta x}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} \propto \underbrace{e^{\frac{-x^2}{2\sigma^2}}}_{e^{-ax^2}} \underbrace{e^{\frac{\theta x}{\sigma^2}}}_{e^{bx}}$

$(x-\theta)^2 = x^2 - 2\theta x + \theta^2$

given $X_1, \ldots, X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta,\sigma^2)$

What is $\hat{\theta}_{MLE}$?

Assume σ^2 known

$\mathcal{L}(\theta; x, \sigma^2) = \prod_{i=1}^{n} p(x_i|\theta,\sigma^2)$

$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(x-\theta)^2} = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{\frac{-1}{2\sigma^2}\sum(x_i-\theta)^2}$

$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{\frac{-1}{2\sigma^2}} \underbrace{\sum_{i=1}^{n} x_i^2 - 2\theta x_i + \theta^2}_{(\sum x_i^2 - 2\theta\sum x_i + \sum\theta^2)}$

$\sum x_i^2 - 2\theta\bar{x}n + n\theta^2$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\left(\sum x_i^2 - 2\theta \bar{x} n + n\theta^2\right)}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum x_i^2}{2\sigma^2}} e^{\frac{\theta \bar{x} n}{2}} e^{\frac{-n\theta^2}{2\sigma^2}}$$

$$\ell(\theta; x, \sigma^2) = n \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{\sum x_i^2}{2\sigma^2} + \frac{\theta \bar{x} n}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}$$

$$\ell'(\theta; x, \sigma^2) = \frac{\bar{x} n}{\sigma^2} - \frac{n\theta}{\sigma^2} \overset{\text{set}}{=} 0$$

$$\Rightarrow \frac{\sigma^2}{n}\left(\dots\right) = \frac{\sigma^2}{n} 0 \Rightarrow \bar{x} - \theta = 0 \Rightarrow \boxed{\hat{\theta}_{MLE} = \bar{x}}$$

$$P(\theta | X, \sigma^2) = \frac{P(X|\theta, \sigma^2)\, P(\theta|\sigma^2)}{P(X|\sigma^2)}$$

$$\propto P(X|\theta, \sigma^2) P(\theta|\sigma^2) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum x_i^2}{2\sigma^2}} e^{\frac{\theta \bar{x} n}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}} P(\theta|\alpha^2)$$

$$\propto e^{\frac{\theta \bar{x} n}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}} P(\theta|\sigma^2) = \underbrace{e^{\frac{-n}{2\sigma^2}\theta^2} e^{\frac{\bar{x} n}{\sigma^2}\theta}}_{\text{kernel for normal}} P(\theta|\alpha^2)$$

$$P(\theta|\sigma^2) = N(\mu_0, \tau^2)$$

$$= \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{1}{2\tau^2}(\theta - \mu_0)^2} \propto e^{-\frac{1}{2\tau^2}(\theta^2 - 2\theta\mu_0 + \mu_0^2)}$$

$$\propto e^{-\frac{1}{2\tau^2}\theta^2} e^{\frac{\mu_0}{\tau^2}\theta}$$

$$= \left(e^{\frac{-n}{2\sigma^2}\theta^2} e^{\frac{\bar{x} n}{\sigma^2}\theta}\right)\left(e^{\frac{-1}{2\tau^2}\theta^2} e^{\frac{\mu_0}{\tau^2}\theta}\right)$$

$$= e^{-\left(\frac{n}{2\sigma^2} + \frac{1}{2\tau^2}\right)\theta^2} e^{\left(\frac{\bar{x} n}{\sigma^2} + \frac{\mu_0}{\tau^2}\right)\theta} \propto N($$

Consider $X | c, v^2 \sim N(c, v^2) = \frac{1}{\sqrt{2\pi v^2}} e^{\frac{-1}{2v^2}(\theta - c)^2}$

$$\propto e^{\frac{-1}{2v^2}\theta^2} e^{\frac{c}{v^2}\theta} e^{\frac{-c^2}{2v^2}}$$

$$\cancel{-}\frac{1}{\cancel{2}v^2} = \cancel{-}\left(\frac{n}{\cancel{2}\sigma^2} + \frac{1}{\cancel{2}\tau^2}\right) \Rightarrow \frac{1}{v^2} = \frac{n}{\sigma^2} + \frac{1}{\tau^2} \Rightarrow$$

Symmetric, unimodal
Pseudo-interpretations

$$v^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$\frac{c}{v^2} = \frac{\bar{X} n}{\sigma^2} + \frac{\mu_0}{\tau^2}$$

$$c = \left(\frac{\bar{X} n}{\sigma^2} + \frac{\mu_0}{\tau^2}\right) v^2 = \frac{\frac{\bar{X} n}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$X_1, \dots, X_n \mid \theta, \sigma^2 \overset{\text{i.i.d.}}{\sim} N(\theta, \sigma^2)$$

$$\theta \mid \sigma^2 \sim N(\mu_0, \tau^2)$$

TEST $\theta \mid X_1, \dots, X_n, \sigma^2 \sim N\left(\frac{\frac{\bar{X} n}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right)$

Normal is conjugate for the normal likelihood when σ^2 known.

Posterior expectation $\hat{\theta}_{MMSE}$ (Mean) $= \frac{\frac{\bar{X} n}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$

(Median) $\hat{\theta}_{MAE} =$ " " , $\hat{\theta}_{MAP}$ (Mode) $=$ " "

θ-normal; hypothesis testing; credible region

$\hat{\theta}_{MMSE}$ as a shrinkage estimator

$$\hat{\theta}_{MMSE} = \frac{\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \cdot \frac{\tau^2}{\tau^2} + \frac{\frac{\bar{X} n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \cdot \frac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}}$$

$$= \frac{1}{\frac{n\tau^2}{\sigma^2} + 1} \cdot \frac{\sigma^2}{\sigma^2} \mu_0 + \frac{1}{1 + \frac{\sigma^2}{n\tau^2}} \cdot \frac{n\tau^2}{n\tau^2} \bar{X}$$

$$= \underbrace{\frac{\sigma^2}{n\tau^2 + \sigma^2}}_{\rho} \underbrace{\mu_0}_{E(\theta)} + \underbrace{\frac{n\tau^2}{n\tau^2 + \sigma^2}}_{(1-\rho)} \underbrace{\bar{X}}_{MLE}$$

Weighted arithmetic average
shrinkage $\lim_{n \to \infty} \rho = 0$