4/24/17

$X_1, \dots, X_n \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$p(\theta, \sigma^2) \propto \frac{1}{\sigma^2}$

Both θ, σ^2 unknown.

If σ^2 known $(\theta = ?)$, $p(\theta \mid x, \sigma^2) = N\left(\bar{x}, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

If θ known $(\sigma^2 = ?)$, $p(\sigma^2 \mid x, \theta) = \text{Invgamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$

If both unknown, $p(\theta, \sigma^2 \mid X) \propto p(X \mid \theta, \sigma^2)\, p(\theta, \sigma^2)$

$$= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i - \theta)^2}\right)\left(\frac{1}{\sigma^2}\right)$$

$$\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}$$

$$\propto \text{Norm Inv Gamma}\left(\mu = \bar{x}, \lambda = n, \alpha = \frac{n}{2}, \beta = \frac{(n-1)s^2}{2}\right)$$

Sampling

("r" function on formula sheet) w "rnorm..."

$X \sim \text{Bern}(0.5)$? Sampling means pull or draw a realization. Make data

Flip a coin.

$X \sim \text{Binomial}(10, 0.5)$? → Flip 10 coins.

$X \sim \text{Binomial}(10, 0.238597642)$? gets harder... how do you find a weighted coin?

$X \sim \text{Normal}(11.2, 3.7^2)$? same thing...

Recall: $F(x) := P(X \leq x)$ CDF.

For a cont. r.v. X, what is the distribution of $Y := F_x(X)$? not on exam just proof for next thing

$t''(X)$

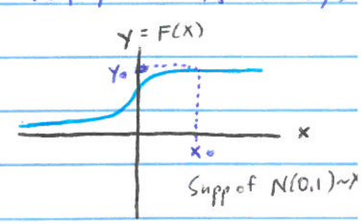
$$f_y(y) = f_x(x)\left|\frac{dx}{dy}\right| = f_x(x)\frac{1}{\left|\frac{dy}{dx}\right|} = f_x(x)\frac{1}{\frac{d}{dx}[F(x)]}$$

$$= f_x(x)\frac{1}{|f_x(x)|}$$ ← always (+) so ≠ need abs. value.

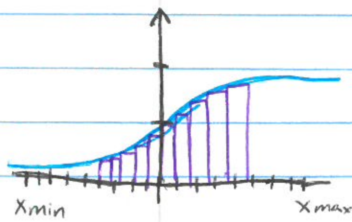
$$= 1$$

So, $\text{Supp}\{Y\} = [0,1]$, $f_y(y) = 1$, $\Rightarrow Y \sim \text{Uniform}(0,1) \Rightarrow X = F^{-1}(y)$

To sample x,
1. Sample y_0 from Uniform(0,1)
2. Compute $x_0 = F_x^{-1}(y_0)$
3. Return x_0.



If F^{-1} is not available, pick X_{min}, X_{max}, ΔX. Using these, create a "grid". $G := \langle X_{min}, X_{min} + \Delta X, X_{min} + 2\Delta X, X_{min} + 3\Delta X, \ldots, X_{max} \rangle$



Compute $F(X)$ $\forall$ $x \in G$

Approximate $x_0 \approx \min_{y_0 \in G} F(x_0) \geq y$

- can lower Δx, stretch X_{min}, X_{max} but comp. has to do more work.

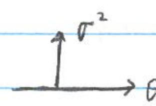
What if X is discrete?

- Can set $G := \text{Supp}\{X\}$. and do the same thing.

for 1-Dimensional

But NormInvGamma is 2-Dimensional

$\theta, \sigma^2 | X \sim \text{NormInvGamma}(\cdots)$



We know how to sample from $f(x)$, but how do we sample from $f(x,y)$?

Y

X \ Y	2	4	6	Σ
1	0.2	0.05	0.1	0.35
3	0.1	0.05	0.1	0.25
5	0.2	0.1	0.1	0.4
Σ	0.5	0.2	0.3	

$f(x,y)$ joint probability

$f(x)$

$f(y)$

if $\frac{\{0.2 + 0.05 + 0.1\}}{0.35} = f(y | x=1)$

Recall Bayes Rule: $f(x,y) = f(y|x) f(x)$

Sampling from 2-Dimension:
1. ~~Sample~~ Draw x_0 from $f(x)$.
2. Draw y_0 from $f(y | x = x_0)$
3. Return $\langle x_0, y_0 \rangle$

$$p(\theta, \sigma^2 \mid X) = p(\theta \mid X, \sigma^2)\, p(\sigma^2 \mid X)$$

$$p(\theta \mid X, \sigma^2) = \text{Normal}\left(\bar{x}, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$$

solve for $p(\sigma^2 \mid X)$

$$p(\sigma^2 \mid X) = \frac{p(\theta, \sigma^2 \mid X)}{p(\theta \mid X, \sigma^2)} \propto \frac{(\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}{\underbrace{\frac{1}{\sqrt{2\pi \frac{\sigma^2}{n}}}}_{(\sigma^2)^{-1/2}} \cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}$$

$$\propto (\sigma^2)^{\underbrace{-\frac{n}{2}-\frac{1}{2}}_{\text{need } -\alpha-1}} e^{-\frac{(n-1)s^2/2}{\sigma^2}}$$

$$\propto$$

$$\left(-\frac{n}{2} - \frac{1}{2} + 1\right) - 1$$

$$-\left(\frac{n}{2} + \frac{1}{2} - 1\right) - 1$$

$$-\left(\frac{n}{2} - \frac{1}{2}\right) - 1$$

$$-\left(\frac{n-1}{2}\right) - 1$$

$$\propto (\sigma^2)^{-\left(\frac{n-1}{2}\right)-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}$$ kernel of Inv Gam ☑

$$\propto \text{Invgamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

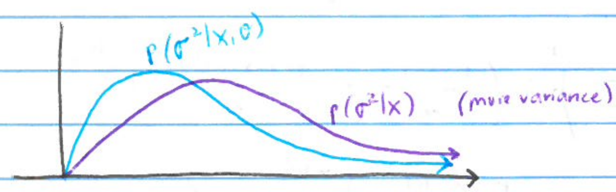
Sampling from Norm Inv Gamma:
1. Sample σ_0^2 from Invgamma$\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$
2. Sample θ_0 from $N\left(\bar{x}, \left(\frac{\sigma_0}{\sqrt{n}}\right)^2\right)$
3. Return (θ_0, σ_0^2)

What is $p(\sigma^2 \mid X)$? Is it $p(\sigma^2 \mid X, \theta)$?

↳ No. $p(\sigma^2 \mid X) = \text{Inv Gamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$ ($\leftarrow$ ≠ care about θ)

$p(\sigma^2 \mid X, \theta) = \text{Inv Gamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$ ← check w/ notes. ≠ ($\leftarrow$ under knowledge of θ)

Recall, $p(\sigma^2 \mid X) = \int_{\mathbb{R}} p(\theta, \sigma^2 \mid x)\, d\theta$ "marginal distribution".



What is $p(\theta \mid X)$?

↳ $p(\theta \mid X) = \int_0^\infty p(\theta, \sigma^2 \mid X)\, d\sigma^2$ HW...

supp $\{\sigma^2\}$

σ^2 is the "nuissance parameter": Don't care about the σ^2 (variance). but you have to use it to get what you want.

$X_1, \ldots, X_n$

$X_1, \ldots, X_n \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$\frac{\bar{X} - \theta}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$ (Math 241)

$\frac{\bar{X} - \theta}{\frac{S}{\sqrt{n}}} \sim T_{n-1}$ (Math 242) "Student's T distribution" (Beer Story)

$$V \sim T_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{v^2}{n}\right)^{-\frac{n+1}{2}}$$

Student's T distribution or "Standard T distr."

$W = \sigma V + \mu = t(v)$

$V = t^{-1}(W) = \frac{W - \mu}{\sigma}$

$f_W(w) = f_V(t^{-1}(w)) \left| \frac{d}{dw}[t^{-1}(w)] \right|$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n \sigma^2}\,\Gamma\left(\frac{n}{2}\right)} \cdot \left(1 + \frac{1}{n}\left(\frac{w-\mu}{\sigma}\right)^2\right)^{-\frac{n+1}{2}}$$

$= T_n(\mu, \sigma)$ Non-Standard T dist.

$$P(\theta \mid X) = \frac{P(\theta, \sigma^2 \mid X)}{P(\sigma^2 \mid \theta, X)} = \frac{\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i - \theta)^2}\right)\left(\frac{1}{\sigma^2}\right)}{\frac{\left(\frac{n\hat{\sigma}^2}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} (\sigma^2)^{-\frac{n}{2}-1} e^{\frac{-n\hat{\sigma}^2/2}{\sigma^2}}}$$

$\sim InvGamma\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$

$\sim InvGamma\left(\frac{n}{2}, \frac{\sum_{i=1}^{n}(X_i - \theta)^2}{2}\right)$

Also $Q \sim InvGamma(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma\alpha} e^{-\frac{\beta}{q}} q^{-\alpha\ldots}$

$$\propto \frac{e^{-\frac{1}{2\sigma^2}\sum(X_i - \theta)^2}}{\left(\frac{n\hat{\sigma}^2}{2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum(X_i - \theta)^2}}$$

$$= \left(\frac{n\hat{\sigma}^2}{2}\right)^{n/2}$$

Recall: $n\hat{\sigma}^2_{MLE} = \cdots = (n-1)s^2 + n(\bar{X} - \theta)^2$

$$= \left(\frac{(n-1)s^2}{2} + \frac{n(\bar{X}-\theta)^2}{2}\right)^{-n/2}$$

$$\propto \overset{\text{multiply by}}{\left(\frac{1}{\frac{(n-1)s^2}{2}}\right)^{-n/2}} \left(\frac{(n-1)s^2}{2} + \frac{n(\bar{X}-\theta)^2}{2}\right)^{-n/2}$$

$$\propto \left[1 + \frac{\frac{n(\bar{X}-\theta)^2}{2}}{\frac{(n-1)s^2}{2}}\right)^{-n/2} = \left(1 + \frac{1}{n-1}\left(\frac{\bar{X}-\theta}{\frac{s}{\sqrt{n}}}\right)^2\right)^{-n/2}$$

$$\propto T_{n-1}\left(\bar{X}, \frac{s}{\sqrt{n}}\right) = \int_0^\infty P(\theta, \sigma^2 \mid X)\, d\theta^2$$

θ is free variable

Summary: $P(\theta, \sigma^2 \mid X)$ ✓ $P(\sigma^2 \mid X, \theta)$ ✓

$P(\theta \mid X, \sigma^2)$ ✓ $P(\theta \mid X)$ ✓

$P(\sigma^2 \mid X)$ ✓

qt conf. int.

pt p-value.

rt draw out.