$2\theta - 3\theta^2 = 0$

$2 - 3\theta = 0$

$\Rightarrow \hat{\theta}_{MAP} = .67 = \hat{\theta}_{MLE}$

$P(\theta \in [0.6\ 0.7] \mid X)$

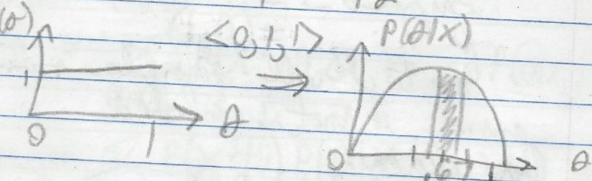
frequentists can't do this

$= \int_{.6}^{.7} P(\theta|x)\,d\theta = \int_{.6}^{.7} 12(\theta^2 - \theta^3) = 12\left[\frac{\theta^3}{3} - \frac{\theta^4}{4}\right]_{.6}^{.7}$

$= 12(.0543 - .0396) = \boxed{.1765}$

$$\frac{\theta^2 - \theta^3}{\int_0^1 P(X|\theta)\underbrace{P(\theta)}_{=1}d\theta} = \frac{\theta^2 - \theta^3}{\int_0^1 (\theta^2 - \theta^3)d\theta} = 12(\theta^2 - \theta^3)$$

$$\left.\frac{\theta^3}{3} - \frac{\theta^4}{4}\right|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$



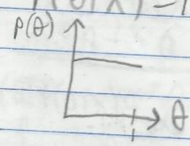
2/16/17

F = Bernoulli $\quad X = \langle 0, 1, 1 \rangle$

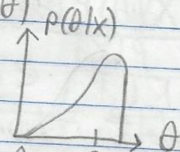
$\theta \sim U(0,1)$

$P(\theta|X) = 12\theta^2(1-\theta)$

M A T
a p h
x e e
i x o
m r
a y
l



$\xRightarrow{X}$



$\hat{\theta}_{MLE} = \hat{\theta}_{MAP}$

F = Bernoulli

$X_1, \dots, X_n$

$\theta \sim U(0,1)$

$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)} = \frac{P(X|\theta)\overset{=1}{P(\theta)}}{\int_{\Theta_0} P(X|\theta)P(\theta)d\theta}$

all data

$= \theta^X(1-\theta)^{n-X}$

$P(X|\theta) = \prod_{i=1}^{n} P(X_i|\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$

$= \theta^{\sum x_i}(1-\theta)^{n - \sum x_i}$

$X_i = \sum x_i$

$$P(\theta|x) = \frac{\theta^x(1-\theta)^{n-x}}{\underbrace{\int_0^1 \theta^x(1-\theta)^{n-x}\,d\theta}_{\text{famous integral}}}$$

$\int f(x)dx = F(x)$

capital β $\rightarrow$ $B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$

"Beta function"

$$= \frac{\theta^x(1-\theta)^{n-x}}{\underbrace{B(x+1, n-x+1)}}$$

A new famous brand-name r.v. is

$$X \sim Beta(\alpha,\beta) := \frac{1}{\beta(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

Supp$(X) = (0,1)$

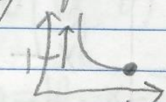
If $f(x)$ is a PDF, $\int_{Supp(X)} f(x)dx = 1$

$$\int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{1}{B(\alpha,\beta)}\overbrace{\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx}^{\cancel{B(\alpha,\beta)}}$$

$= 1$ ✓

Param. Space

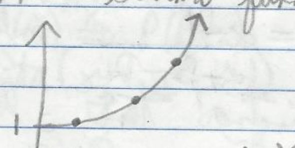
$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$ Why finite?



$\alpha > 0, \beta > 0$

$\Gamma(x) := \int_0^\infty e^{-t}t^{x-1}dt$ Gamma function

$f(x) = x!$



$f(x) = \Gamma(x+1) = x!$

Properties of gamma

① $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

② $\Gamma(x) = (x-1)!$ $\quad x \in \mathbb{N}$

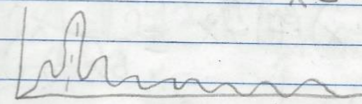
$\Gamma(x) = x-1\,\Gamma(x-1)!$ valid $\forall x$

$$E[X] := \int_{Supp(x)} x f(X)\,dx = \int_0^1 x \overbrace{\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}}$$

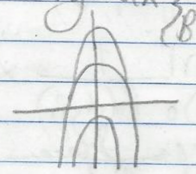
$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha} (1-\alpha)^{\beta-1}\,dx = \frac{B(\alpha+1,\beta)}{\Gamma(\alpha,\beta)}$$

$$= \frac{\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}} = \frac{\frac{\alpha\Gamma(\alpha)}{(\alpha+\beta)\Gamma(\alpha+\beta)}}{\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}} = \frac{\alpha}{\alpha+\beta}$$

$$\text{Mode}[X] := \underset{x \in Supp(X)}{\text{argmax}} \{f(x)\} = \text{argmax}\left\{\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}\right\}$$



A



$= \text{argmax}\{x^{\alpha-1}(1-x)^{\beta-1}\}$

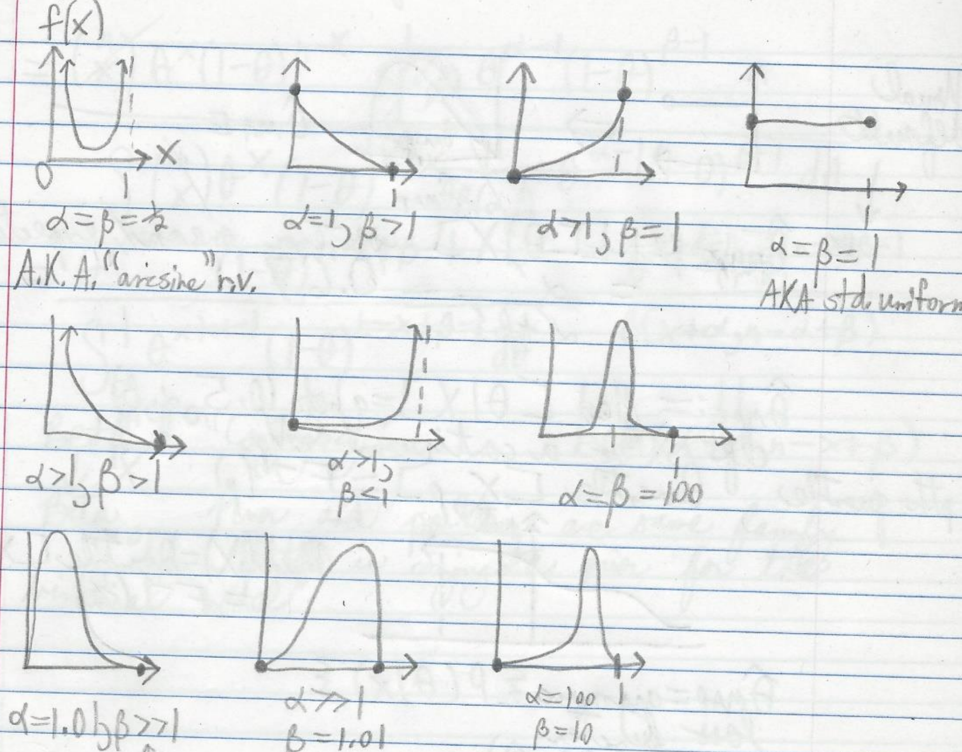
$= \text{argmax}\{(\alpha-1)\ln(x) + (\beta-1)\ln(1-x)\}$

$\frac{d}{dx}\left[\quad\right] \overset{set}{=} 0$

$$\frac{\alpha-1}{x} - \frac{\beta-1}{1-x} = 0$$

$\Rightarrow x = \frac{\alpha-1}{\alpha+\beta-2}$ only for $\alpha>1, \beta>1$

Second derivative test, local min.



much greater

F = Binomial

n known

$$\theta \sim U(0,1) = \text{Beta}(1,1)$$

$$\text{Binom}(n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{\underset{\sim}{P(X)}} = \frac{\binom{n}{x}\theta^x(1-\theta)^{n-x}}{\int_0^1 \binom{n}{x}\theta^x(1-\theta)^{n-x}\,d\theta}$$

$$P(X) = \int_{\theta=(0,1)} P(X|\theta)P(\theta)\,d\theta$$

$$= \text{Beta}(x+1, n-x+1)$$

$n=10$, $x=7$, $\bar{x}=.7$

$\theta|x \sim \text{Bern}(8,4)$

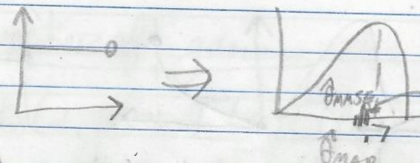
$$P(\theta) \xRightarrow{X} P(\theta|x)$$

$$\text{Beta}(\underset{\alpha}{1}, \underset{\beta}{1}) \Rightarrow \text{Beta}(\underbrace{x+1}_{\alpha'}, \underbrace{n-x+1}_{\beta'})$$

$$\hat{\theta}_{MLE} = \hat{\theta}_{MAP} = \text{Mode}[\theta|X] = \frac{\alpha'-1}{\alpha'+\beta'-2} = \frac{7}{10} = .7$$

Incomplete Beta Function
rfiddle.com

Usual default ↓

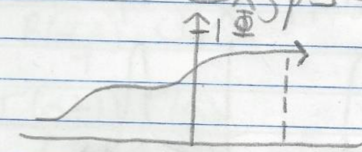


* $\hat{\theta}_{MMSE} := E[\theta | X]$ posterior mean/expectation
$= \frac{\alpha'}{\alpha' + \beta'} = 0.67$

$\hat{\theta}_{MAE} := Med[\theta | X] = qbeta(0.5, \alpha, \beta)$ ($\alpha = 8$, $\beta = 4$)

If X is a continuous R.V.,

pth quantile: $Quantile[X, p] = F^{-1}(p)$



$Med(X) = Quantile[X, 0.5]$
$= F^{-1}(\frac{1}{2})$

$\hat{\theta}_{MAP} = argmax\{P(\theta | x)\}$

loss function

$\ell(\theta, \hat{\theta})$

real → θ, $\hat{\theta}$ ← estimate

$R(\theta, \hat{\theta}) = E[\ell(\theta, \hat{\theta})]$

$\ell(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2$ squared loss

$\ell(\theta, \hat{\theta}) = |\theta - \hat{\theta}|$ abs. loss

→ $\hat{\theta}_{MMSE}$ gives you min. risk

→ $\hat{\theta}_{MAE}$

MAE = min. abs. error

MMSE = min. mean squared error

$\theta \sim U(0, 1) = Beta(1, 1)$
$\Downarrow X$
$\theta | x \sim Beta(x+1, n-x+1)$

$\theta \sim Beta(\alpha, \beta)$ w/ appropriately chosen α, β

$$P(\theta | X) = \frac{P(X|\theta) P(\theta)}{P(X)}$$

$P(X) = \int_{\Theta} P(X|\theta) P(\theta) \, d\theta$

$$= \frac{\cancel{\binom{n}{x}}\,\theta^{x}(1-\theta)^{n-x}\,\frac{1}{\cancel{B(\alpha,\beta)}}\,\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\int \cancel{\binom{n}{x}}\,\theta^{x}(1-\theta)^{n-x}\,\frac{1}{\cancel{B(\alpha,\beta)}}\,\theta^{\alpha-1}(1-\theta)^{\beta-1}\,d\theta}$$

$$= \frac{\theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}}{\int_0^1 \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}\,d\theta} = \frac{\theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}}{B(x+\alpha, n-\alpha+\beta)}$$

$$\| $$

$$\text{Beta}(\alpha+\alpha, n-x+\beta)$$

$$\text{Beta} \xrightarrow{X} \text{Beta}$$

"Conjugacy"

Prior and posterior are same family

"The Beta is conjugate prior for the binomial model"