Lec 15 4/25/17 Math 341

$$X_1, \ldots, X_n \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$$

$P(\theta, \sigma^2) \propto \frac{1}{\sigma^2}$

Both θ, σ^2 unknown

If σ^2 known

$\theta \mid X_1, \ldots, X_n, \sigma^2 \sim N\left(\bar{x}, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

If θ known

$\sigma^2 \mid X_1, \ldots, X_n, \theta \sim \text{InvG}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$

$$p(\theta, \sigma^2 \mid X_1, \ldots, X_n) \propto \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i - \theta)^2}\right)\left(\frac{1}{\sigma^2}\right)$$

$\vdots$

$$\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}$$

$$\propto \text{NormInvGamma}\left(\mu = \bar{x}, \lambda = n, \alpha = \frac{n}{2}, \beta = \frac{(n-1)s^2}{2}\right)$$

How to sample from this? This is $f(x, y)$

Let's talk about sampling...

How to sample $X \sim \text{Bern}(0.5)$? Coin

How to sample $X \sim \text{Bin}(10, 0.5)$? 10 flips

$X \sim \text{Bin}\left(10, \frac{1}{6}\right)$? 10 die rolls

$X \sim \text{Bin}(10, 0.23896)$? ... ?

$X \sim N(10, 0.3^2)$? ... ?

Recall $F(x) = P(X \leq x)$ the "CDF"

What is the distr of $F(X)$ for any cont. r.v. X?

let $Y = F(X) = t(x)$. Recall...

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = f_X(x)\frac{1}{\left|\frac{dy}{dx}\right|} = f_X(x)\left|\frac{1}{\frac{d}{dx}[F(x)]}\right| = \cancel{f_X(x)}\frac{1}{\cancel{f_X(x)}} = 1$$

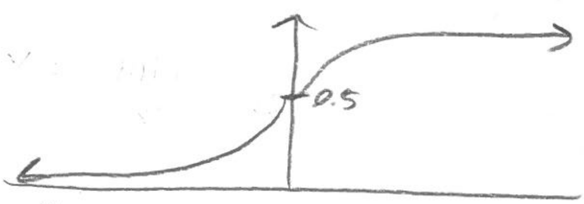
Since $Y \in [0,1]$ and $f_Y(y) = 1 \Rightarrow Y \sim U(0,1)$

$\Rightarrow X = F^{-1}(Y)$

Sampling:

1. draw y_0 from $U(0,1)$ (done with computers)
2. Compute $X_0 = F^{-1}(y_0)$

What if F^{-1} is not available in closed form? e.g. $X \sim N(0,1)$



Do it numerically. Here's a basic algorithm:

Pick $X_{min} = -10$, $X_{max} = 10$

Pick step $\Delta X = 0.1$. Create grid $G := \langle X_{min}, X_{min} + \Delta x, X_{min} + 2\Delta x, \ldots, X_{max} \rangle$

Compute $F(x)\ \forall x^* \in G$: $F(X_{min}), F(X_{min} + \Delta x), F(X_{min} + 2\Delta x), \ldots, F(X_{max})$

Approximate X_0 as the $\min\limits_{x^* \in G} F(x^*) \geq y$

What if X is discrete? Let $G = Supp[X]$. X is not approximate

How to sample from $f(x,y)$

Imagine...

$f(x,y)$

$f(y \mid x=1)$

X \ Y	2	4	6	$f(x)$
1	0.2	0.05	0.1	0.35
3	0.1	0.05	0.1	0.25
5	0.2	0.1	0.1	0.4
				1

Recall: $f(x,y) = f(y \mid x)\, f(x)$ Bayes Rule

To sample...
1. Sample x_0 from $f(x)$
2. sample y_0 from $f(y \mid x = x_0)$?
3. Return $\langle x_0, y_0 \rangle$

Can we do this with the Norm Inv Gamma?

$$P(\theta, \sigma^2 \mid x) = P(\theta \mid x, \sigma^2)\, P(\sigma^2 \mid x) \quad \text{Bayes Rule}$$

$$P(\theta \mid x, \sigma^2) = N\left(\bar{x}, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$$

$$\Rightarrow P(\sigma^2 \mid x) = \frac{P(\theta, \sigma^2 \mid x)}{P(\theta \mid x, \sigma^2)} \propto \frac{(\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}{\frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} \cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}$$

$$-\frac{n}{2}+\frac{1}{2}-1 = -\frac{n-1}{2}-1$$

$$\parallel$$

$$-\frac{n}{2}-\frac{1}{2}+1-1$$

$$\parallel$$

$$\propto \frac{(\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}}{(\sigma^2)^{-\frac{1}{2}}} = (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}} e^{-\frac{(n-1)s^2/2}{\sigma^2}}$$

$$\propto \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

To sample from $N(\theta, \sigma^2 | X)$

① Sample σ_0^2 from $\text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

② Sample θ_0 from $P(\theta | X, \sigma^2 = \sigma_0^2) = N\left(\bar{x}, \left(\frac{\sigma_0}{\sqrt{n}}\right)^2\right)$

③ Return $\langle \theta_0, \sigma_0^2 \rangle$

$\Rightarrow$ No need to ever work with NormInvGamma directly!

Also note, we solved for $P(\sigma^2 | X)$. What is that?

$$P(\sigma^2 | x) = \int_{\mathbb{R}} P(\sigma^2, \theta | x)\, d\theta$$

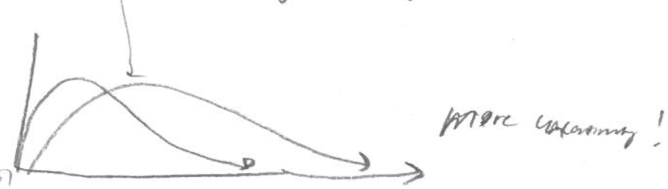
It's the posterior of σ^2 with the uncertainty inherent in ignorance of θ "averaged" out or marginalized out

$P(\sigma^2 | X, \theta) = \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2}{2}\right)$

$P(\sigma^2 | X) = \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

need θ to compute...

If $s^2 = \hat{\sigma}^2$



Also...
What is
$P(\theta|x)$?

It's the posterior of θ with the ~~uncertainty~~ in σ^2 marginalized out..

This is usually of great interest. σ^2 is a nuisance parameter.

$$P(\theta|x) = \int_0^\infty P(\theta, \sigma^2|x)\, d\sigma^2 \quad \text{or} \quad P(\theta|x) = \frac{P(\theta, \sigma^2|x)}{P(\sigma^2|\theta, x)}$$

Before we get there... let's do some Math 241...

If $X_1, \dots X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$$\Rightarrow \frac{\bar{X} - \theta}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

$$\Rightarrow \frac{\bar{X} - \theta}{\frac{s}{\sqrt{n}}} \sim ?$$

Makes "sense" to use if σ unknown. "Student" did this in the early 1900's.

Define $V \sim T_n := \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{v^2}{n}\right)^{-\frac{n+1}{2}}$ the Student's T distr.

AKA
Standard
T distr

It can be shown that

$$\frac{\bar{X}-\theta}{\frac{s}{\sqrt{n}}} \sim T_{n-1}$$

let $W = \sigma V + \mu = t(v)$

$$v = t^{-1}(w) = \frac{w-\mu}{\sigma}$$

$$f_W(w) = f_V\left(t^{-1}(w)\right)\left|\frac{d}{dw}\left[t^{-1}(w)\right]\right|$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{\left(\frac{w-\mu}{\sigma}\right)^2}{n}\right)^{-\frac{n+1}{2}}\frac{1}{\sigma}$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n \sigma^2}\,\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{1}{n}\left(\frac{w-\mu}{\sigma}\right)^2\right)^{-\frac{n+1}{2}} := T_n(\mu, \sigma)$$

the noncentered and scaled T distr.

AKA
non-standard
T distr

$$p(\theta|x) = \frac{p(\theta,\sigma^2|x)}{p(\sigma^2|\theta,x)} = \frac{\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2}\right)\left(\frac{1}{\sigma^2}\right)}{\frac{\left(\frac{n\hat{\sigma}^2}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{n\hat{\sigma}^2/2}{\sigma^2}}}$$

The sigsq terms are constants anyway and do not need to be written

$$\propto \frac{\cancel{(\sigma^2)^{-\frac{n}{2}-1}} \cancel{e^{-\frac{n\hat{\sigma}^2}{2\sigma^2}}}}{\left(\frac{n\hat{\sigma}^2}{2}\right)^{n/2} \cancel{(\sigma^2)^{-\frac{n}{2}-1}} \cancel{e^{-\frac{n\hat{\sigma}^2/2}{\sigma^2}}}} = \left(\frac{n\hat{\sigma}^2}{2}\right)^{-n/2}$$

Recall $n\hat{\sigma}^2 = \dots = (n-1)s^2 + n(\bar{x}-\theta)^2$

$$= \left(\frac{(n-1)s^2}{2} + \frac{n(\bar{x}-\theta)^2}{2}\right)^{-n/2}$$

const ... only a $f(x)$!!!

$$\propto \left(\frac{1}{\frac{(n-1)s^2}{2}}\right)^{-n/2} \left(\frac{(n-1)s^2}{2} + \frac{n(\bar{x}-\theta)^2}{2}\right)^{-n/2}$$

$$= \left(1 + \frac{\frac{n(\bar{x}-\theta)^2}{2}}{\frac{(n-1)s^2}{2}}\right)^{-n/2}$$

$$= \left(1 + \frac{1}{n-1}\left(\frac{\bar{x}-\theta}{\frac{s}{\sqrt{n}}}\right)^2\right)^{-n/2} \propto T_{n-1}\left(\bar{x}, \frac{s}{\sqrt{n}}\right)$$