Lecture 14:

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$X_1, \dots, X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

Assume σ^2 known $\theta | \sigma^2 \sim N(\mu_0, \tau^2)$ or $\theta | \sigma^2 \sim N(\mu_0, \frac{\sigma^2}{n_0})$

$\Rightarrow \theta | X_1, \dots, X_n, \sigma^2 \sim N(\theta_p, \sigma^2_p)$

$X_1, \dots, X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

Assume θ known $\sigma^2 \sim \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0 \sigma_0^2}{2}\right)$

$\Rightarrow \sigma^2 | X_1, \dots, X_n, \theta \sim \text{InvGamma}\left(\frac{n+n_0}{2}, \frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{2}\right)$

$Y \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta y} y^{\alpha - 1}$

$Y^{-1} \sim \text{InvGamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\frac{\beta}{y}} y^{-\alpha-1}$

$$P(\sigma^2 | X_1, \dots, X_n, \theta) \propto P(X_1, \dots, X_n | \theta, \sigma^2) P(\sigma^2 | \theta)$$

$$= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i - \theta)^2}\right) P(\sigma^2 | \theta)$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sqrt{\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\Sigma(x_i - \theta)^2} \quad P(\sigma^2 | \theta)$$

$$\propto \underbrace{(\sigma^2)^{-\frac{n}{2}} e^{-\frac{n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}}_{\substack{\text{Kernel of} \\ \text{InvGamma}\left(\frac{n}{2} - 1, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)}} P(\sigma^2 | \theta) \qquad \hat{\sigma}^2_{MLE} = \frac{1}{n} \underbrace{\sum_{i=1}^{n} (x_i - \theta)^2}_{SSE} = \frac{SSE}{n}$$

$\Rightarrow \sigma^2 \sim \text{InvGamma}(\alpha, \beta)$

$$\propto (\sigma^2)^{-\frac{n}{2}} e^{-\frac{n\hat{\sigma}^2/2}{\sigma^2}} (\sigma^2)^{-\alpha-1} e^{-\frac{\beta}{\sigma^2}}$$

$$= (\sigma^2)^{-\frac{n}{2}-\alpha-1} e^{-\left(\frac{n\hat{\sigma}^2}{2} + \beta\right)/\sigma^2}$$

$$\propto \text{InvGamma}\left(\frac{n}{2} + \alpha, \frac{n\hat{\sigma}^2_{MLE}}{2} + \beta\right)$$

$\hookrightarrow$ messy

$\sigma^2 \sim \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$

$$\Rightarrow \text{InvGamma}\left(\frac{n+n_0}{2}, \frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{2}\right) = \text{InvGamma}\left(\frac{n+n_0}{2}, \frac{SSE + SSE_0}{2}\right)$$

Hyperparameter Interpretations: n_0: # of prior trials seen

$\uparrow$SSE, $\sigma^2\uparrow$; if $\downarrow$SSE then $\sigma^2\downarrow$ $\qquad n_0\sigma_0^2$: prior SSE

Point Estimation for σ^2

$$\hat{\sigma}^2_{mmse} = E[\sigma^2 | X, \theta] = \frac{\beta}{\alpha - 1} = \frac{\frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{2}}{\frac{n+n_0}{2} - 1} = \frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{n + n_0 - 2}$$

$$\hat{\sigma}^2_{MAP} = \frac{\beta}{\alpha+1} = \frac{n\hat{\sigma}_{MLE} + n_0\sigma_0^2}{n + n_0 + 2}$$

$$\hat{\sigma}^2_{MAE} = \text{qinvgamma}(0.5, \alpha, \beta) = \left(0.5, \frac{n+n_0}{2}, \frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{2}\right)$$

Uninformative Prior:

let $n_0 = 0$ $\Rightarrow \sigma^2 \sim \text{InvGamma}(0,0) \Rightarrow$ improper } Haldane

$\Rightarrow \sigma^2 | X, \theta \sim \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right) \Rightarrow$ always proper

If $\sigma^2 \sim \text{InvGamma}(2, 0) \Rightarrow \sigma^2 | X, \theta \sim \text{InvGamma}\left(\frac{n+2}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$

$\Rightarrow \hat{\sigma}^2_{mmse} = \frac{1}{n}\sum(X_i - \theta)^2 = \hat{\sigma}^2_{MLE}$

Jeffrey's Prior

$$P(\sigma^2) \propto \sqrt{I(\sigma^2)}$$

$$l'(\sigma^2; x, \theta) = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} SSE = -\frac{n}{2}(\sigma^2)^{-1} + \frac{SSE}{2}(\sigma^2)^{-2}$$

$$l''(\sigma^2; x, \theta) = \frac{n}{2}(\sigma^2)^{-2} - SSE(\sigma^2)^{-3}$$

$$I(\sigma^2) = E[-l''(\sigma^2; x, \theta)] = E\left[-\frac{n}{2}(\sigma^2)^{-2} + SSE(\sigma^2)^{-3}\right]$$

$$\left.\begin{aligned} E[SSE] &= E\left[\sum_{i=1}^n (X_i - \theta)^2\right] \\ &= \sum_{i=1}^n E[(X_i - \theta)^2] \\ &= n E[(X_1 - \theta)^2] \\ &= n \text{Var}[X_1] \\ &= n\sigma^2 \end{aligned}\right\} \rightarrow$$

$$= -\frac{n}{2}(\sigma^2)^{-2} + (\sigma^2)^{-3} E[SSE]$$

$$= -\frac{n}{2}(\sigma^2)^{-2} + (\sigma^2)^{-3} \cdot n\sigma^2$$

$$= (\sigma^2)^{-2}\left(n - \frac{n}{2}\right) = \frac{n}{2}(\sigma^2)^{-2}$$

$$\Rightarrow P(\sigma^2) \propto \sqrt{I(\sigma^2)} = \sqrt{\frac{n}{2}(\sigma^2)^{-2}} \propto \sqrt{(\sigma^2)^{-2}} = \frac{1}{\sigma^2}$$

$\propto \text{InvGamma}(0,0)$

improper $\Rightarrow \int_0^\infty \frac{1}{\sigma^2} \, d\sigma^2 = \infty$

diverges

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$X_1, \dots, X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

θ and σ^2 unknown

$$P(\theta,\sigma^2|x_1,\dots,x_n) \propto P(x_1,\dots,x_n|\theta,\sigma^2)P(\theta,\sigma^2)$$

$$\propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2} P(\theta,\sigma^2)$$

$$\propto \underbrace{(\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2}}_{\text{not kernel of inverse Gamma b/c } \theta \text{ is not fixed.}} P(\theta,\sigma^2)$$

$$SSE = \sum_{i=1}^{n}(x_i-\theta)^2 = \sum_{i=1}^{n}((x_i-\bar{x})+(\bar{x}-\theta))^2$$

$$= \sum_{i=1}^{n}\left((x_i-\bar{x})^2 + 2(x_i-\bar{x})(\bar{x}-\theta) + (\bar{x}-\theta)^2\right)$$

$$= \sum_{i=1}^{n}\left((x_i-\bar{x})^2 + 2(x_i\bar{x} - x_i\theta - \bar{x}^2 + \bar{x}\theta) + (\bar{x}-\theta)^2\right)$$

$$= \sum(x_i-\bar{x})^2 + 2\left(\sum x_i\bar{x} - \sum x_i\theta - \sum\bar{x}^2 + \sum\bar{x}\theta\right) + \sum(\bar{x}-\theta)^2$$

$$= (n-1)s^2 + 2(\cancel{n\bar{x}^2} - \cancel{\theta n\bar{x}} - \cancel{n\bar{x}^2} + \cancel{n\bar{x}\theta}) + n(\bar{x}-\theta)^2$$

Define: $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$

↳ sample variance

$s^2(n-1) = \sum(x_i-\bar{x})^2$

$$= (\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}((n-1)s^2 + n(\bar{x}-\theta)^2)} P(\theta,\sigma^2)$$

$$= \underbrace{(\sigma^2)^{-\frac{n}{2}} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}_{\text{kernel of NormInvGamma}} P(\theta,\sigma^2)$$

$$\text{NormInvGamma}\left(\mu=\bar{x},\ \lambda=n,\ \alpha=\frac{n}{2}+1,\ \beta=\frac{(n-1)s^2}{2}\right)$$

Thus $P(\theta,\sigma^2) = \text{NormInvGamma}$ <conjugate model>

- NormInvGamma is the conjugate prior for the normal likelihood where both θ, σ^2 are unknown.

Jeffreys's Prior: $P(\theta,\sigma^2) \propto \frac{1}{\sigma^2} = \underset{\propto 1}{P(\theta)}\ \underset{\propto \frac{1}{\sigma^2}}{P(\sigma^2)}$

$$\Rightarrow P(\theta,\sigma^2|x_1,\dots,x_n) = \text{NormInvGamma}\left(\bar{x}, n, \frac{n}{2}, \frac{(n-1)s^2}{2}\right)$$

How to simulate from NormInvGamma? Assuming Jeffrey's prior

$$P(\theta,\sigma^2|X) = \underbrace{P(\theta|X,\sigma^2)}_{\text{norm}}\ \underbrace{P(\sigma^2|X)}_{\text{InvGamma}}$$

$$P(\sigma^2|X) = \frac{P(\theta,\sigma^2|X)}{P(\theta|X,\sigma^2)}$$

$$\propto \frac{(\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}{\frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}\cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}$$

$$\propto \frac{(\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}}{(\sigma^2)^{-\frac{1}{2}}} = (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}} e^{-\frac{(n-1)s^2/2}{\sigma^2}}$$

$$= (\sigma^2)^{-(\frac{n}{2}-\frac{1}{2})-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}$$

$$\propto \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

$$\left.\begin{aligned} P(\sigma^2|X) &= \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right) \\ P(\sigma^2|X,\theta) &= \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)\end{aligned}\right\}$$

difference?
margined out θ to get σ^2;