$$= n\left(\frac{\frac{1-\theta}{\theta}}{(1-\theta)^2} + \frac{1}{\theta^2}\right) = n\left(\frac{1}{\theta(1-\theta)} + \frac{1}{\theta^2}\right)$$

$$= n\left(\frac{1}{\theta^2(1-\theta)}\right) \propto Beta(0, \tfrac{1}{2})$$

$$p(\theta) \propto \sqrt{I(\theta)} = \sqrt{n \frac{1}{\theta^2(1-\theta)}} \propto \theta^{-1}(1-\theta)^{-1/2}$$

Jeffrey's $\theta \sim Beta(0, \tfrac{1}{2}) \Rightarrow \alpha = 0$, $\beta = \tfrac{1}{2}$

(improper)

Similar to Wilson estimate

3/2/17

$X_1, \dots, X_n | \theta \overset{i.i.d.}{\sim} Geom(\theta) = (1-\theta)^x \theta$,

$\theta \sim Beta(\alpha, \beta)$

$\Rightarrow \theta | X_1, \dots, X_n \sim Beta(\underbrace{n+\alpha}_{\alpha'}, \underbrace{\Sigma x_i + \beta}_{\beta'})$

α: # of pseudotrials

β: # of pseudofailures

w/ θ α, β have Haldane (prior of complete ignorance)

$$\hat{\theta}_{MMSE} = \frac{n+\alpha}{n+\alpha+\Sigma x_i + \beta}$$ } Is there a shrinkage interpretation?

Haldane $\theta \sim Beta(0,0) \Rightarrow \hat{\theta}_{MMSE} = \frac{n}{n+\Sigma x_i} = \frac{1}{1+\frac{\Sigma x_i}{n}} = \frac{1}{1+\bar{x}}$

$= \hat{\theta}_{MLE}$

Laplace $\theta \sim Beta(1,1) \Rightarrow \hat{\theta}_{MMSE} = \frac{n+1}{n+1+\Sigma x_i + 1} = \frac{1}{1 + \frac{\Sigma x_i + 1}{n+1}}$

Jeffrey's $\theta \sim Beta(0, \tfrac{1}{2}) \Rightarrow \hat{\theta}_{MMSE} = \frac{n}{n+\Sigma x_i + \frac{1}{2}} = \frac{1}{1 + \frac{\Sigma x_i + \frac{1}{2}}{n}}$

Weighted arithmetic average

$$\frac{1}{\hat{\theta}_{MMSE}} = \frac{n+\alpha+\sum x_i+\beta}{n+\alpha} = \frac{\alpha+\beta}{n+\alpha}\cdot\frac{\alpha}{\alpha} + \frac{n+\sum x_i}{n+\alpha}\cdot\frac{n}{n}$$

$$= \underbrace{\frac{\alpha+\beta}{\alpha}}_{\frac{1}{E(\theta)}} \cdot \underbrace{\frac{\alpha}{n+\alpha}}_{\rho} + \underbrace{\frac{n+\sum x_i}{n}}_{\frac{1}{\hat{\theta}_{MLE}}} \cdot \underbrace{\frac{n}{n+\alpha}}_{(1-\rho)}$$ } weighted harmonic average

$\bar{x} = \frac{1}{n}\sum x_i$

$\frac{1}{\bar{x}_H} = \frac{1}{n}\sum\frac{1}{x_i}$ } Harmonic sample/average

No credible regions + hypothesis test b/c same thing

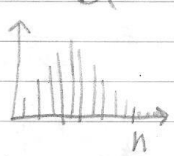
$$P(X^*|X) = \int_{\Theta} p(x^*|\theta)\underbrace{p(\theta|X)}_{dP(\theta|x)}\,d\theta$$

$\int f(x)\,dg$ Riemann Stieltjes integral

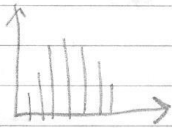
$$= \int_0^1 \left((1-\theta)^{x^*}\theta\right)\frac{1}{B(n+\alpha, \sum x_i+\beta)}\,\theta^{n+\alpha-1}(1-\theta)^{\sum x_i+\beta-1}\,d\theta$$

$$= \frac{1}{B(n+\alpha, \sum x_i+\beta)}\int_0^1 \theta^{\overbrace{n+\alpha+1}-1}(1-\theta)^{\overbrace{x^*+\sum x_i+\beta}-1}\,d\theta$$

$$= \frac{B(n+\alpha+1, x^*+\sum x_i+\beta)}{B(n+\alpha, \sum x_i+\beta)} := \text{BetaGeometric}(n+\alpha, \sum x_i+\beta)$$



Over-dispersed binomial (Beta-Binomial)



(Beta-Geom)

looks different

Er-lang, χ^2, Gamma

$X_1,\dots,X_n|\theta \overset{iid}{\sim} NegBin(r,\theta) = \binom{x+r-1}{x}(1-\theta)^x\theta^r$

$\theta \sim Beta(\alpha,\beta)$

Everything is same except extra r term

$X \sim Bin(n,\theta) := \binom{n}{x}\theta^x(1-\theta)^{n-x}$

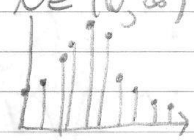
If n is large, θ is small but they're "pushed" to be $\lambda = n\theta$

$$\lim_{n\to\infty} \frac{n!}{x!(n-x)!}\underbrace{\left(\frac{\lambda}{n}\right)^x}_{=\frac{\lambda^x}{n^x}}\left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x}{x!}\lim \underbrace{\frac{\overbrace{n\cdot n\cdots\cdots n}^{n}}{\underbrace{(n-x)(n-x-1)\cdots 1}_{n-x}\cdot \underbrace{n\cdots n}_{n}}}_{1}\ \underbrace{\lim\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}}\ \underbrace{\lim\left(1-\frac{\lambda}{n}\right)^x}_{1}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} = \underbrace{Poisson(\lambda)}_{X}$$

$Supp(X) = \{0,1,\dots\} = \mathbb{N}_0$

$\lambda \in (0,\infty)$



$E(X) = \lambda$

$Var(X) = \lambda$

$X|\theta \sim Poisson(\theta) = \frac{e^{-\theta}\theta^x}{x!}$

Conjugacy

$\theta \sim$

$\theta|x \sim$

$P(\theta|x) \propto P(x|\theta)P(\theta) = \frac{e^{-\theta}\theta^x}{x!}P(\theta) \propto e^{-\theta}\theta^x P(\theta)$

$P(\theta) \propto e^{-b\theta}\theta^a$

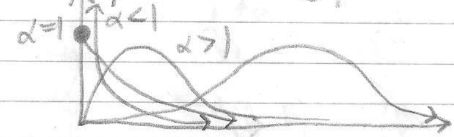
$\Rightarrow P(\theta) = \frac{b^{a+1}}{\Gamma(a+1)}\overbrace{e^{-b\theta}\theta^a}^{kernel}$

$\theta \sim Gamma(\alpha,\beta) := \frac{\beta^\alpha}{\Gamma(\alpha)}e^{-\beta\theta}\theta^{\alpha-1}$

$Supp(\theta) = (0,\infty)$ $\quad\quad \lambda \in (0,\infty)$

↳ Support of prior has to equal sample space of likelihood

Param. Space $\alpha > 0, \beta > 0$



$$E(\theta) = \frac{\alpha}{\beta} \qquad Var(\theta) = \frac{\alpha}{\beta^2}$$

$$Mode(\theta) = \frac{\alpha - 1}{\beta} \text{ if } \alpha \geq 1$$

TEST $Med[\theta] = qgamma(0.5, \alpha, \beta)$

$$P(\theta|X) \propto P(X|\theta)\, P(\theta) \propto e^{-\theta}\theta^{x}$$

$$X|\theta \sim Poisson(\theta) = \frac{e^{-\theta}\theta^{x}}{x!}$$

$$\theta \sim Gamma(\alpha, \beta)$$

$$\theta | X \sim Gamma(x + \alpha, \beta + 1)$$

$$P(\theta|x) \propto P(X|\theta) P(\theta) \propto \left(\frac{e^{-\theta}\theta^{x}}{x!}\right)\left(\frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\beta\theta}\theta^{\alpha-1}\right)$$

$$\propto e^{-\theta}\theta^{x} e^{-\beta\theta}\theta^{\alpha-1} = e^{-(\beta+1)\theta}\theta^{x+\alpha-1}$$

$$\underbrace{X_1, \dots, X_n}_{x} | \theta \overset{i.i.d.}{\sim} Poisson(\theta)$$

$$\theta | X_1, \dots, X_n \sim Gamma(x + \alpha, \beta + 1)$$

$$P(\theta|x) \propto P(X|\theta) P(\theta) \propto \left(\prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!}\right)\left(\frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\beta\theta}\theta^{\alpha-1}\right)$$

$$= \frac{e^{-n\theta}\theta^{\Sigma x_i}}{\Pi x_i!} \cdot \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\beta\theta}\theta^{\alpha-1} \propto e^{-(\beta+n)\theta}\theta^{\Sigma x_i + \alpha - 1}$$

$$\propto Gamma(\Sigma x_i + \alpha, \beta + n)$$

$$\theta | X_1, \dots, X_n \sim Gamma(\underbrace{\Sigma x_i + \alpha}, \underbrace{\beta + n}_{\beta'})$$

α = # of pseudosuccesses

β = # of pseudotrials

Prior of indifference - Final

$\Rightarrow \hat{\theta}_{MMSE} = \dfrac{\sum x_i + \alpha}{\beta + n}$, $\hat{\theta}_{MAE} = q\text{gamma}(0.5, \sum x_i + \alpha, \beta + n)$

$\hat{\theta}_{MAP} = \dfrac{\sum x_i + \alpha - 1}{\beta + n}$ if $\sum x_i + \alpha \geq 1$

Laplace $\theta \sim U$.... Can we do this?

Can't put uniform on infinite support

$\int_0^\infty c \, d\theta \neq 1$

$P(\theta) \propto 1$ Clearly improper

$P(\theta|X) \propto P(X|\theta)P(\theta) \propto e^{-n\theta}\theta^{\sum x_i} P(\theta) \propto e^{-n\theta}\theta^{\sum x_i}$

$\propto \text{Gamma}(\sum x_i, n) \Rightarrow \theta \sim \text{Gamma}(0, 0)$

"Really gets silly" "Haldane" = "Laplace"

This is improper when $\sum x_i = 0$

$\mathcal{L}(\theta; x) = \prod_{i=1} \dfrac{e^{-\theta}\theta^{x_i}}{x_i!} = \dfrac{e^{-n\theta}\theta^{\sum x}}{\prod x_i!}$

$\ell(\theta; x) = -n\theta + \sum x_i \ln(\theta) - \ln(\prod_{i=1} x_i!)$

$\ell'(\theta; x) = -n + \dfrac{\sum x_i}{\theta} \overset{set}{=} 0 \Rightarrow \dfrac{\sum x_i}{\theta} = n$

$\Rightarrow \hat{\theta}_{MLE} = \dfrac{\sum x_i}{n} = \bar{X}$

$\ell''(\theta; x) = -\dfrac{\sum x_i}{\theta^2}$, $I(\theta) = E[-\ell''(\theta; x)] = E\left(\dfrac{\sum x_i}{\theta^2}\right)$

Jeffrey's Prior

$\propto \sqrt{I(\theta)} = \sqrt{\dfrac{n}{\theta}}$

$= \dfrac{E(\sum X_i)}{\theta^2} = \dfrac{\sum E(X_i)}{\theta^2}$

$\propto \sqrt{\dfrac{1}{\theta}} = \theta^{-1/2}$

$= \dfrac{\sum \theta}{\theta^2} = \dfrac{n\theta}{\theta^2} = \dfrac{n}{\theta}$

$\propto \text{Gamma}(\frac{1}{2}, 0)$ (Improper) Can we do this?

$I(\theta) = \text{Var}_x[S(\theta; x)^2]$

$\hat{\theta}_{MMSE} = \dfrac{\sum x_i}{\beta + n} \cdot \dfrac{n}{n} + \dfrac{\alpha}{\beta + n} \cdot \dfrac{\beta}{\beta} = \underbrace{\dfrac{\sum x_i}{n}}_{\hat{\theta}_{MLE}} \cdot \underbrace{\dfrac{n}{\beta + n}}_{1-p} + \underbrace{\dfrac{\alpha}{\beta}}_{E(\theta)} \cdot \underbrace{\dfrac{\beta}{\beta + n}}_{p}$

$$P(X^*|X) = \int_{\Theta} P(X^*|\theta)\, P(\theta|X)\, d\theta$$

$n^* = 1$

$$= \int_0^{\infty} \left(\frac{e^{-\theta}\, \theta^{x^*}}{x^*!} \right) \left(\frac{\beta'^{\alpha'}}{\Gamma(\alpha')} e^{-\beta'\theta}\, \theta^{\alpha' - 1} \right) d\theta$$