Let $\mathcal{F}$ be Binomial, with $\theta \sim U(0,1)$, $n = 2$ and $x = 1$. Then $\theta|X \sim \text{Beta}(2,2)$. At an alpha level of 5%, the 2 sided is $CR_{\theta,1-\alpha} = [\text{Quantile}[\theta|X, \frac{\alpha}{2}], \text{Quantile}(\theta|X, 1-\frac{\alpha}{2})] = [\text{qbeta}(0.025, 2, 2), \text{qbeta}(0.975, 2, 2)] = [0.094, 0.906]$. However since $n = 2$, asymptotic normaling breaks down and we can't do this.

One Sided Credible Region:

$$CR_{L,\theta,1-\alpha} = [0, \text{Quantile}[\theta|X, 1-\alpha]]$$

$$CR_{R,\theta,1-\alpha} = [\text{Quantile}[\theta|X, 1], 1]$$

The left credible region is for the lower 95% while the right credible region is for the higher 95%.

In the above example,

$$\begin{aligned} CR_{L,\theta,1-\alpha} &= [0, \text{qbeta}(0.95, 2, 2)] \\ &= [0, 0.865] \end{aligned}$$

and

$$\begin{aligned} CR_{R,\theta,1=\alpha} &= [\text{qbeta}(0.05, 2, 2), 1] \\ &= [0.135, 1] \end{aligned}$$

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Hypothesis Test (Theory Testing): "theory" - research hypothesis or alternative hypothesis - $H_A$
Null hypothesis - assuming the theory is opposite - $H_0$
We reject the null hypothesis (accept theory) if "overwhelming" evidence. "Overwhelming" is the "level" of $\alpha$ that is chosen. If data is sufficient at $\alpha$, reject $H_0$ and accept $H_A$. If it is not sufficient, retain $H_0$ (fail to reject).

One Sided Hypothesis Test: $H_0 : \theta \leq \theta_0 = 0.5$, $H_A : \theta > \theta_0 = 0.5$ where $\hat{P} = N(\theta_0, (\sqrt{\frac{\theta(1-\theta)}{n}})^2)$.
If $\theta \in$ retainment region, retain $H_0$ (fail to reject). If $\theta \notin$ retainment region, reject $H_0$.
P-value = P(seeing the data or more extreme $|H_0$ true) = $\underset{\alpha}{\text{argmax}}\{\hat{\theta} \in \text{Retainment region}\}$
If the p-value $< \alpha$, reject $H_0$. If the p-value $> \alpha$, retain $H_0$.

Two Sided Hypothesis Test: $H_0 : \theta = \theta_0 = 0.5$, $H_A : \theta \neq \theta_0 = 0.5$. This is the same as asking if $\{\theta > 0.5 \bigcup \theta < 0.5\}$.
Note:

- p-value $\neq \mathbb{P}(H_0)$
- p-value $\neq \mathbb{P}(H_A)$
- p-value $\neq \mathbb{P}(H_0 \mid X)$
- p-value $\neq \mathbb{P}(H_A \mid X)$

Let's say $H_0 : \theta \leq \theta_0 = 0.5$, $H_A : \theta > \theta_0 = 0.5$ and $\alpha = 5\%$, $n = 2$, $x = 1$ and $\theta \sim U(0, 1)$.
Bayesian P-value:

$$\begin{aligned}\text{p-value} &= \mathbb{P}\left(H_0 \mid X\right) = \mathbb{P}\left(\theta \leq \theta_0 \mid X\right) \\ &= \int_0^1 \frac{1}{\mathrm{B}(\alpha + x, \beta + n - x)} \theta^{\alpha+x-1}(1-\theta)^{n-x+\beta-1}\, d\theta = \text{pbeta}(\theta_0, x + \alpha, n - x + \beta)\end{aligned}$$

For this example, p-value $= \mathbb{P}\left(\theta < 0.5 \mid X\right) = \int_0^{0.5} \text{Beta}(2, 2)\, d\theta = \text{pbeta}(0.5, 2, 2) = 0.5$
Since this is $\not< \alpha = 5\%$, retain $H_0$. Note that here, we said $U(0, 1) = \text{Beta}(2, 2)$.

$$\mathbb{P}\left(H_0 \mid X\right) = \frac{\mathbb{P}\left(X \mid H_0\right)\mathbb{P}\left(H_0\right)}{\mathbb{P}\left(X\right)} = \frac{\mathbb{P}\left(X \mid H_0\right)\mathbb{P}\left(H_0\right)}{\mathbb{P}\left(X \mid H_0\right)\mathbb{P}\left(H_0\right) + \mathbb{P}\left(X \mid H_A\right)\mathbb{P}\left(H_A\right)}$$

This puts more weight on $H_A$ than desired.
Point Null: $H_0 : \theta = \theta_0 = 0.5$, $H_A : \theta = \theta \neq 0.5$. Then

$$\text{p-value} = \mathbb{P}\left(H_0 \mid X\right) = \mathbb{P}\left(\theta = 0.5 \mid X\right) = \int_{0.5}^{0.4} \text{Beta}(2, 2)\, d\theta = 0$$

This integral will always be zero..

Solution: (1) $H_0 : \theta \in (\theta_0 \pm \delta)$, $H_A : \theta \notin (\theta_0 \pm \delta)$. The parenthesis is the region of equivalence. (2) $H_0 : \theta = \theta_0 = 0.5$, $H_A : \theta = \theta_0 \neq 0.5$, if $\theta_0 \in CR_{\theta,1-\alpha}$, retain $H_0$

Let's say $\alpha = 5\%$, $n = 100$ and $x = 61$.
In the frequentist approach: Retainment Region =

$$[\theta_0 \pm z_{\alpha/1}\sqrt{\frac{\theta_0(1-\theta_0)}{n}}] = [0.5 \pm 2\sqrt{\frac{0.5^2}{100}}] = [0.4, 0.6]$$

Since $\hat{\theta} = \frac{61}{100} = 0.61$, $0.61 \in$ retainment region, thus reject $H_0$.
P-value $= \mathbb{P}\left(|z| > \frac{0.61-0.5}{0.05}\right) = 2\mathbb{P}\left(z > 2.2\right) = 2(1 - \text{pnorm}(2.2)) = 0.278$. This is less than $\alpha = 5\%$ thus reject $H_0$.

In the Bayesian approach, $\theta \sim U(0, 1)$ and $\delta = 0.01$. Then $H_0 : \theta \in (0.49, 0.51)$ and $H_A : \theta \notin (0.49, 0.51)$. Since $\theta|X \sim \text{Beta}(62, 40)$,

$$\begin{aligned}\text{p-value} &= \mathbb{P}\left(H_0 \mid X\right) \\ &= \mathbb{P}\left(\theta \in (0.49, 0.51) \mid X\right) \\ &= \int_{0.49}^{0.51} \text{Beta}(62, 40)\, d\theta \\ &= \text{qbeta}(.51, 62, 40) - \text{qbeta}(0.49, 62, 40) = 0.0147\end{aligned}$$

This value is $< \alpha - .05$. Thus retain $H_0$.

$$CR_{\theta,1-\alpha} = [\text{qbeta}(0.025, 62, 40), \text{qbeta}(0.975, 62, 40)] = (0.511, 0.700)$$

Thus $\theta_0 = 0.5 \notin CR$, therefore reject $H_0$.

Let's say $H_0 : \theta = \theta_0 = 0.5$ and $H_A : \theta \neq \theta_0 = 0.5$ with $\theta \sim U(0,1)$.

Bayesian Factor: tells the relativity of $P_{H_A}(X)$ to $P_{H_0}(X)$

$$\begin{aligned} B &= \frac{P_{H_A}(X)}{P_{H_0}(X)} \\ &= \frac{\int_{\Theta \in H_A} \mathbb{P}\left(X \mid \theta\right) P_{H_A}(\theta)\, d\theta}{\int_{\Theta \in H_0} \mathbb{P}\left(X \mid \theta\right) P_{H_0}(\theta)\, d\theta} \\ &= \frac{\int_0^1 \binom{n}{x} \theta^x (1-\theta)^{n-x}\, d\theta}{\int_{0.5} \binom{n}{x} \theta^x (1-\theta)^{n-x}\, d\theta} \\ &= \frac{\int_0^1 \theta^{0.61} (1-\theta)^{0.39}\, d\theta}{0.5^{0.61}(1-0.5)^{0.39}} \\ &= \frac{\mathrm{B}(62,40)}{0.5^{100}} = 1.39 \end{aligned}$$

This tells us that $P_{H_A}$ is not too far from $P_{H_0}$.