Math 341 Lec 1 1/31/17

- syllabus

Let X be a r.v. which means it "realizes" to "data" x. The value $x \in Supp(X)$

It can • discrete $|Supp(X)| \le |\mathbb{N}|$ if $|Supp(X)| = 1 \Rightarrow X \sim Deg(c) := \{c$ wp 1

support is ctbl. infinite at most

$\exists\, p(x) := P(X = x)$, a prob mass function (PMF)

$p : Supp(X) \to (0, 1]$

$\exists\, F(x) := P(X \le x)$

• Cont. $|Supp(X)| = |\mathbb{R}|$ uncntbl. infin. each realization contains infinite information

$\exists\, f(x) := F'(x)$ which is the prob density func (PDF)

where $F(x) := P(X \le x)$, the cumul. distr. function (CDF) $\Rightarrow$ all r.v.'s have $F(x)$ but $p(x)$, $f(x) \to$ n.a.

Note: $P(X \in [a, b]) = P(X \le b) - P(X \le a) = F(b) - F(a) = \int_a^b f(x)\,dx$ F.T.C.

if $f(x)$ exists (F needs to be diff.)

r.v.'s are defined by their PMF/PDF/CDF's. Some common ones are below:

Discrete:

$X \sim Bernoulli(p) := p^x (1-p)^{1-x}$, $x \in Supp(X) = \{0, 1\}$

$X \sim Binomial(n, p) := \binom{n}{x} p^x (1-p)^{n-x}$, $x \in Supp(X) = \{0, 1, \ldots, n\}$

Cont.:

$X \sim Exp(\lambda) := \lambda e^{-\lambda x}$, $x \in Supp(X) = (0, \infty)$

$X \sim N(\mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$, $x \in Supp(X) = \mathbb{R}$

normal

$p^x(1-p)^{1-x}$ what is p? If you remember the Bernoulli model,

$P(X=1)=p$, $P(X=0)=1-p$, p is a "tuning knob" AKA a parameter

non-degenerate values of p are $(0,1)$ why not 0 or 1?

~~Bernoulli(p) r.v. is a parametric model i.e. a model with finite standard parameters.~~

Parameter space: all set of possible values of the parameter(s) in the model

From now on, parameters are denoted θ and param. space, Θ.

$X \sim \text{Bern}(\theta) = \theta^x(1-\theta)^{1-x}$

$X \sim N(\theta_1, \theta_2^2) = \frac{1}{\sqrt{2\pi\theta_2^2}} e^{-\frac{1}{2\theta_2^2}(x-\theta_1)^2}$, $\vec{\theta} = \begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix}$

$\dim(\vec{\theta}) = 2 < \infty$

i.e. a parametric model

Parametric Model ($\mathcal{F}$): is a set of r.v. models that are parametrized with finite parameters

$\mathcal{F} := \{ P(X;\theta) : \theta \in \Theta \}$

$P(X;\theta)$ "prob of X assuming a value of θ"

i.e. prob's are different depending on the specific model (specific θ).

For a Bernoulli,

$\mathcal{F} := \{ \theta^x(1-\theta)^{1-x} : \theta \in (0,1) \}$

What is iid?

Assume $x_1, \dots, x_n$ are realizations from an iid model, then:

$$p(x_1, \dots, x_n; \theta) = p(x_1;\theta)\, p(x_2;\theta) \cdots p(x_n;\theta) = \prod_{i=1}^{n} p(x_i;\theta)$$

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plain x means all $x_1 \dots x_n$

In the real world, you see $x \overset{iid}{=} \langle 0,0,1,0,1,0 \rangle$ (the data)
and if you want to model this data, you pick an $\mathcal{F}$,
a **class** of parametric models, but you don't know θ.
Figuring out θ is the goal of "inference" and there are generally 3:

(1) Point estimation. Give best guess of θ.

(2) Confidence set. Give a range of possible θ's.

(3) Theory testing. Evaluate whether or not a theory about θ is true.

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e.g. data above, let $\mathcal{F}$ = Bernoulli model, $\Theta = (0,1)$

$$p(\langle 0,0,1,0,1,0 \rangle; \theta) = \prod_{i=1}^{6} p(x_i;\theta)$$

if $\theta = 0.5$

$$= \prod_{i=1}^{6} 0.5^{x_i} (1-0.5)^{1-x_i} = 0.5^6 = 0.0156$$

if $\theta = 0.25$

$$= \prod_{i=1}^{6} 0.25^{x_i} (1-0.25)^{1-x_i} = 0.25^2\, 0.75^4 = 0.0198$$

$\theta = 0.5$ is more likely than $\theta = 0.25$.

We really want to know how probable the value of θ are.

$$\mathcal{L}(\theta; x) := P(x;\theta) \qquad \text{the "inverse question"}$$

likelihood

The likelihood asks the question what is the likelihood of seeing the parameter. It is equal to the probability of the data under the param value. Higher probs $\Rightarrow$ higher lik of a given θ.

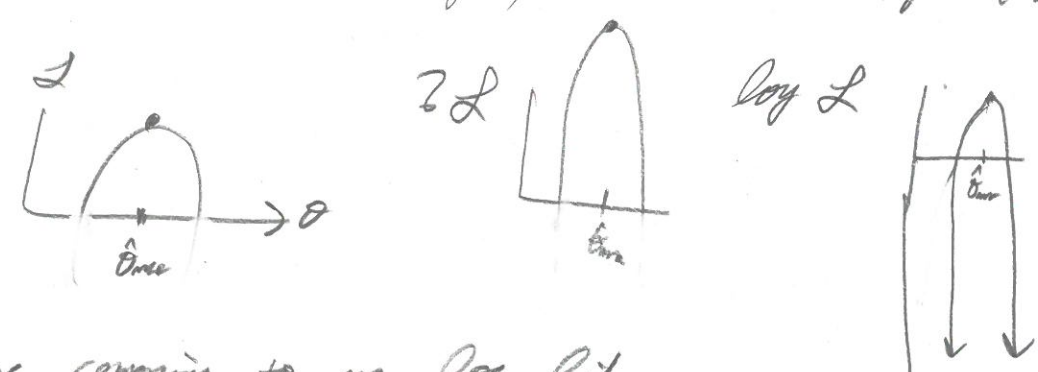
Is $\mathcal{L}$ a PMF/PDF of a r.v.? No... there is no r.v. spanning over θ's.

The most likely value of θ is..

$$\hat{\theta}_{MLE} := \underset{\theta \in \Theta}{\arg\max} \left\{ \mathcal{L}(\theta; x) \right\}$$

maximum likelihood estimator

Note that it remains the same if you take a 1:1 increasing (monotonic) function of $\mathcal{L}$



Usually it's more convenient to use log lik.

$$\ell(\theta; x) := \ln\left(\mathcal{L}(\theta; x)\right)$$

$$\hat{\theta}_{MLE} = \underset{\theta \in \Theta}{\arg\max} \left\{ \ell(\theta; x) \right\}$$