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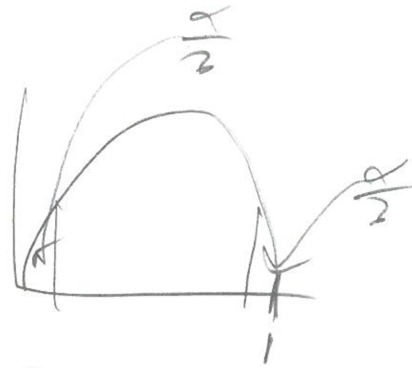
Lec 7 Math 341 2/23/17

$\mathcal{F}$ = Binomial

$\theta \sim U(0,1) = Beta(1,1)$

$n = 2, x = 1$

$\Rightarrow \theta | X \sim Beta(x+\alpha, n-x+\beta) = Beta(2,2)$



$CR_{\theta, 1-\alpha} := \left[Quantile\left[\theta|X, \frac{\alpha}{2}\right], Quantile\left[\theta|X, 1-\frac{\alpha}{2}\right] \right]$

Two-sided CR. The middle $1-\alpha$

exactly what you mean

$CR_{\theta, 95\%} = \left[\underbrace{.094}_{qbeta(.025,2,2)}, \underbrace{.906}_{qbeta(.975,2,2)} \right]$

$P(\theta \in CR) = 1-\alpha$

$P(\theta \in [.094, .906]) = 95\%$!

One-sided CR $\rightarrow$ left-sided CR lowest 95%

$\rightarrow$ right side CR highest 95%

$CR_{L,\theta,1-\alpha} := \left[-\infty, Quantile\left[\theta|X, 1-\alpha\right]\right] = [0, \text{''}]$

in our case, $CR_{L,\theta,95\%} = [0, qbeta(.95, 2, 2)] = [0, .865]$

$CR_{R,\theta,1-\alpha} := \left[Quantile\left[\theta|X, \alpha\right], \infty\right] = [\text{''}, 1]$

in our case $CR_{R,\theta,95\%} = [qbeta(.05, 2, 2), 1] = [.135, 1]$

Review Hypothesis Testing (3rd goal of inference)

↳ Theory Testing

theory: H_a ... which I want to prove

But I want to be sure I'm right.

So I assume the opposite (H_0) and only accept my theory if there is overwhelming evidence.

Evidence is determined by α.

H_0: theory not true

H_a: theory true

data → Sufficient at α? $\xrightarrow{\text{yes}}$ Accept H_a (Reject H_0)

Sufficient at α? $\xrightarrow{\text{no}}$ Fail to reject H_0

$H_0: \theta \le 0.5 = \theta_0$

$H_a: \theta > 0.5 = \theta_0$

$$\hat{p} \sim N\left(\theta_0, \left(\sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right)^2\right) = N\left(0.5, \left(\sqrt{\frac{0.5(1-0.5)}{100}}\right)^2\right)$$

if $n=100$

$$\text{Ret Region} = \left[0,\ \theta + z_\alpha \sqrt{\frac{\theta(1-\theta)}{n}}\right] = [0 + 0.5 + 1.64 \cdot .05] = [0, 0.58]$$

if $\hat{\theta} \in$ Ret Region $\Rightarrow$ F.T.R. H_0

if $\hat{\theta} \notin$ '' $\Rightarrow$ Reject H_0

$$\text{pval} := P\left(\text{seeing data as or more extreme} \mid H_0 \text{ true}\right) = P\left(\hat{p} > \hat{\theta} \mid \hat{p} \sim N\left(\theta_0, \left(\sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right)^2\right)\right) = \underset{\alpha}{\text{argmin}} \left\{\hat{\theta} \in \text{Ret Region}\right\}$$

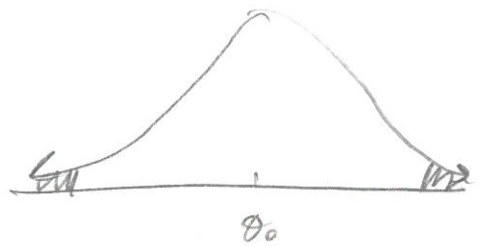
$H_0: \theta \geq 0.5 = \theta_0$

$H_1: \theta < 0.5 = \theta_0$

Rej Region = $\left[\theta_0 - z_\alpha \sqrt{\frac{\theta_0(1-\theta_0)}{n}},\ 1\right]$

$H_0: \theta = 0.5 = \theta_0$

$H_1: \theta \neq 0.5 = \theta_0$



$\theta > 0.5$ or $\theta < 0.5$

two ways to reject

Rej. Region = $\left[\theta_0 \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right]$

pval = $\operatorname{argmin}_{\alpha} \{\hat{\theta} \in \text{Rej. Region}\}$

pval = P(seeing data or more extreme | H_0 true)

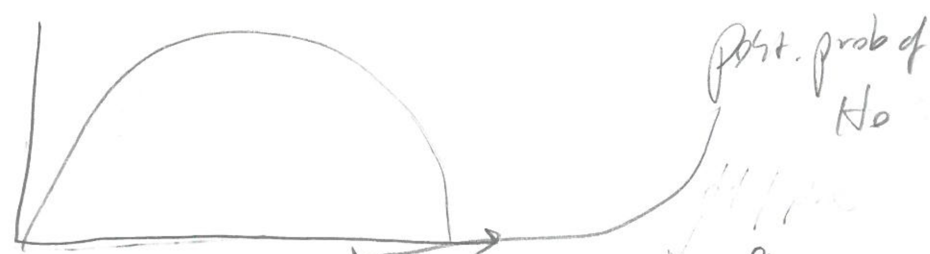
$\neq P(H_0)$
$\neq P(H_0 | X)$
$\neq P(H_1)$
$\neq P(H_1 | X)$

these are things you want!

Bayesian Hyp. Testing

$H_0: \theta \leq 0.5 = \theta_0$

$H_1: \theta > 0.5 = \theta_0$



with data

we know $P(\theta | x)$

we can calc pval := $P(H_0 | X) = P(\theta \leq \theta_0 | X) = \int_0^{\theta_0} \frac{1}{B(\alpha+x, \beta+n-x)} \theta^{\alpha+x-1} (1-\theta)^{\beta+n-x-1} \, d\theta$

$= \text{pbeta}(\theta_0, x+\alpha, n-x+\beta)$

if low.. reject

if $p_{val} < \alpha \Rightarrow$ Reject H_0

$p_{val} \geq \alpha \Rightarrow$ FTR H_0

$$P(H_0 \mid X) = \frac{P(X \mid H_0)\ P(H_0)}{P(X)}$$ ← what's this? prior prob of H_0

$$P(\theta \leq \theta_0 \mid X) = \frac{P(X \mid \theta > \theta_0)\ P(\theta \leq \theta_0)}{P(X \mid \theta > \theta_0)\ P(\theta \leq \theta_0) + P(X \mid \theta \leq \theta_0)\ P(\theta > \theta_0)}$$

if $\theta \sim U(0,1)$ ...

$P(H_0) = P(\theta \leq \theta_0) = \theta_0$ Prior matters, ...

Role of $\hat{\theta} = \bar{X}$? No... no longer a test statistic

Note: this scheme equivalent to $\theta_0 \in CR_{R,\theta,1-\alpha} \Rightarrow$ Retain H_0

$H_0: \theta \geq \theta_0 = 0.5$

$H_A: \theta < \theta_0 = 0.5$

$$p_{val} = P(H_0 \mid X) = P(\theta \geq \theta_0 \mid X) = \int_{\theta_0}^{1} \frac{1}{B(\alpha+x,\ \beta+n-x)}\, \theta^{\alpha+x-1} (1-\theta)^{\beta+n-x-1}\, d\theta$$

$$= 1 - \text{pbeta}(\theta_0,\ \alpha+x,\ \beta+n-x)$$

if $p_{val} < \alpha \Rightarrow$ Reject H_0

$p_{val} \geq \alpha \Rightarrow$ Retain H_0

Note: equivalent to $\theta_0 \in CR_{L,\theta,1-\alpha} \Rightarrow$ Retain H_0

$H_0: \theta = \theta_0$ "point null" or "precise null"

$H_a: \theta \neq \theta_0$ lower dim. than H_a

$\text{pval} = P(H_0 \mid X) = P(\theta = \theta_0 \mid X) = 0$ (always) why? Further Math 241 etc?

why? In reality all point nulls are absurd... smaller

not plausible!

probcoin flip $= \frac{1}{2}$? No.. 0.500001 real on
0c scale

Two ideas ① but 0.500001 = 0.5 for all practical purposes

$\Rightarrow H_0: \theta \in [\theta_0 \pm \delta]$ margin of equivalence

$H_a: \theta \notin [\theta_0 \pm \delta]$ this is usually what you mean!

what is margin of equivalence for coin Heads $\delta = 0.01$?

② $H_0: \theta = \theta_0$
$H_a: \theta \neq \theta_0$ if plausible

$\theta_0 \in CR_{\theta, 1-\alpha} \Rightarrow$ Retain
$\theta_0 \notin CR_{\theta, 1-\alpha} \Rightarrow$ Reject

p-value calculation here not clear...

This seems to be contested...

which is why we use Bayes Factors (soon)

Examples

$H_0: \theta \le .1$

$H_a: \theta > .1$

$\alpha = 5\%$, $n = 150$, $X = 23$

$$\text{Rej Region} = \Big[0, .1 + 1.67\overbrace{\sqrt{\tfrac{.1 \cdot .9}{150}}}^{=.024}\Big] = [0, .140]$$

$\hat\theta = .153 \notin$ Rej Region $\Rightarrow$ Reject H_0

pval $:= P(\hat p > .153 \mid \theta = .1)$

$P\left(Z > \frac{.153 - .1}{.024}\right) = 2.16$

$= 1 - \text{pnorm}(2.16)$

$= .01539 < \alpha \Rightarrow$ Reject

Bayesian ... $\theta \sim U(0,1) = \text{Beta}(1,1)$

$\theta \mid x \sim \text{Beta}(\alpha + x, \beta + n - x) = \text{Beta}(24, 128)$

$$\text{pval} := P(\theta \le .1 \mid x) = \int_0^{.1} \text{Beta}(24,128)\, d\theta = \text{pbeta}(.1, 24, 128) = .01544$$

Similar!

$H_0: \theta = .5$

$H_a: \theta \ne .5$

$\alpha = 5\%$

$n = 100$

$X = 61$

$$\text{Rej Reg} = \Big[.5 \pm 2\sqrt{\tfrac{.5 \cdot .5}{100}}\Big] = [.4, .6]$$

$\hat\theta = .61 \notin$ Rej Region $\Rightarrow$ Reject

$$\text{pval} := P\left(|Z| > \frac{.61 - .5}{.05}\right) = 2P(Z > 2.2) = 2(1 - \text{pnorm}(2.2)) = .0278$$

Bayesian $\theta \sim U(0,1)$

$\theta \mid X \sim \text{Beta}(62, 40)$

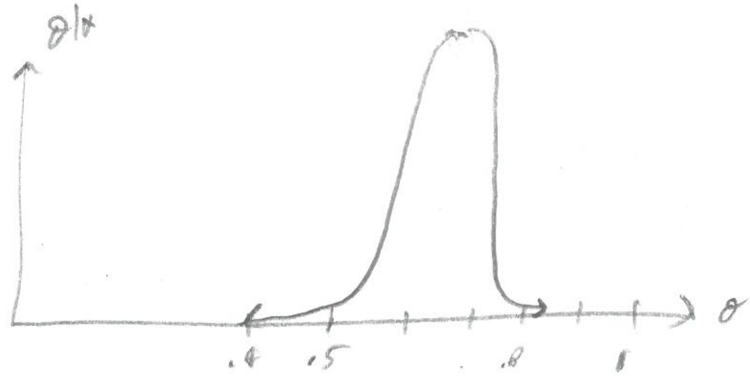
I $H_0: \theta \in [.49, .51]$ i.e. $\delta = .01$ equivalence margin

$H_a: \theta \notin [.49, .51]$

$$p_{val} = P(\theta \in [.49, .51] \mid x) = \int_{.49}^{.51} \text{Beta}(62, 40)\, d\theta = \text{pbeta}(.51, 62, 40) - \text{pbeta}(.49, 62, 40) = .0147$$

(II) $CR_{\theta, 1-\alpha} = [\text{qbeta}(.025, 62, 40), \text{qbeta}(.975, 62, 40)]$
$= [.511, .700]$

$\theta_0 = .5 \notin CR_{\theta, 1-\alpha} \Rightarrow$ Reject H_0



Another way to "test". Bayesian like this better

$H_0: \theta = \theta_0 = 0.5$

$H_a: \theta \neq \theta_0 = 0.5$

what is this really?

maybe

$H_a: \theta \sim U(0,1)$

θ could be "anything"

$n = 100$
$x = 61$
$\Rightarrow \hat{\theta} = .61$

Define $\mathcal{B} := \dfrac{P_{H_a}(X)}{P_{H_0}(X)}$

if $\mathcal{B}$ big $\Rightarrow$ H_a is a better model for the data, X.

Bayes Factor

denominator in Bayes Rule!! prob of data!

$$= \frac{\displaystyle\int_{\Theta_{H_a}} P(X|\theta)\, P_{H_a}(\theta)\, d\theta}{\displaystyle\int_{\Theta_{H_0}} P(X|\theta)\, P_{H_0}(\theta)\, d\theta}$$

oops...

$$= \frac{\displaystyle\int_{\{.5\}} \cancel{\binom{100}{61}}\, .5^{61}(1-.5)^{100-61}\,(1)\, d\theta}{\displaystyle\int_0^1 \cancel{\binom{100}{61}}\, \theta^{61}(1-\theta)^{100-61}\,(1)\, d\theta}$$

for a difference

$$= \frac{B(62, 40)}{.5^{100}} = 1.39$$

H_a better model... but is it decisive?

Jeffreys 1961 scale of Bayes Factors interpretations for validity of H_a