Bayes Factor:

$$B := \frac{P_{H_A}(X)}{P_{H_0}(X)} = \frac{\int_{\Theta_{H_A}} P_{H_A}(X \mid \theta) P_{H_A}(\theta)\, d\theta}{\int_{\Theta_{H_0}} P_{H_0}(X \mid \theta) P_{H_0}(\theta)\, d\theta}$$

Note: If $B > 1$, H_A is supported. The bigger B is, the better H_A is.

Let $H_0 : \theta = 0.5$ and $H_A : \theta \neq 0.5$. Assume $\mathcal{F}$ is Binomial. For H_0: $\theta \sim \text{Deg}(0.5)$ and for H_A: $\theta \sim U(0,1)$. $n = 100$ and $x = 61$. In the frequentist approach, H_0 is rejected because $p = 0.61$ which is too far from 0.5.

$$B = \frac{\int_0^1 \binom{n}{x} \theta^x (1-\theta)^{n-x} \cdot (1)\, d\theta}{\int_{\{0.5\}} \binom{n}{x} 0.5^x (1-0.5)^{n-x} \cdot (1)\, d\theta} = \frac{B(x+1, n-x+1)}{0.5^n} = \frac{B(62, 98)}{0.5^{100}} = 1.39$$

Difference Conclusions:

- If $B < 1$, then no evidence
- If $B \in [1 : 1.3 : 1]$, then barely worth mentioning
- If $B \in [3 : 1, 10 : 1]$, then substantial
- If $B \in [10 : 1, 30 : 1]$, then strong
- If $B \in [30 : 1, 100 : 1]$, then very strong
- If $B > 100\%$, then decisive

Suppose $H_0 : \theta = 0.5$ and $H_A : \theta \neq 0.5$. Let $n = 104490000$, $x = 52263920$ and $\hat{\theta} = 0.50001768$. In the frequentist approach, the p-value is 0.0003, which is less than 0.05 and thus H_0 is rejected. In the Bayesian approach, assuming $\theta \sim \text{Beta}(1,1)$,

$$B = \frac{B(52263921, 104490000 - 52263920 + 1)}{0.50001768^{104490000}} = \frac{1}{12}$$

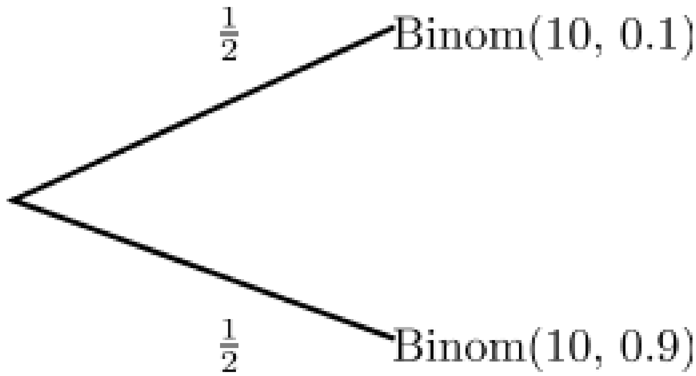
According to this, since $B < 1$, there is no evidence. This gives conflicting results. This happened because as n becomes large, H_0 cannot be true and thus is rejected.

End of Midterm 1 Material

Mixture Distribution: Let $X \sim \begin{cases} N(0,1)^2 & 0.5 \\ N(10, 1^2) & 0.5 \end{cases}$.

$$\begin{aligned} P(X) &= \sum_{\theta \in \Theta} \mathbb{P}(X \mid \theta)\, \mathbb{P}(\theta) \\ &= \mathbb{P}(X \mid \theta = 0)\, \mathbb{P}(\theta = 0) + \mathbb{P}(X \mid \theta = 10)\, \mathbb{P}(\theta = 10) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-10)^2} \cdot \frac{1}{2} \end{aligned}$$

Suppose the following:



Then

$$\begin{aligned}
\mathbb{P}\left(X\right) &= \sum_{\theta\in\Theta} \mathbb{P}\left(X \mid \theta\right)\mathbb{P}\left(\theta\right) \\
&= \mathbb{P}\left(X \mid \theta = 0.1\right)\mathbb{P}\left(\theta = 0.1\right) + \mathbb{P}\left(X \mid \theta = 0.9\right)\mathbb{P}\left(\theta = 0.9\right) \\
&= \binom{10}{x} 0.1^x (1-0.1)^{10-x} \cdot \frac{1}{2} + \binom{10}{x} 0.9^x (1-0.9)^{10-x} \cdot \frac{1}{2}
\end{aligned}$$

What we did here is that we went from $\theta \sim \text{Beta}(\alpha, \beta)$ to $X \mid \theta \sim \text{Binom}(n, \theta)$. Since θ is continuous:

$$\begin{aligned}
\mathbb{P}\left(X\right) &= \int_\Theta \mathbb{P}\left(X \mid \theta\right)\mathbb{P}\left(\theta\right)\, d\theta \\
&= \int_0^1 \left(\binom{n}{x}\theta^x(1-\theta)^{n-x}\right)\cdot\left(\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\right) d\theta \\
&= \binom{n}{x}\frac{1}{B(\alpha,\beta)}\int_0^1 \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}\, d\theta \\
&= \binom{n}{x}\frac{B(x+\alpha, n-x+\beta)}{B(\alpha,\beta)} \\
&= \text{BetaBinom}(n, \alpha, \beta)
\end{aligned}$$

This is the Beta-Binomial model. Let X is a random variable of this model; then X $\sim$BetaBinom(n, α, β). Supp$[X] = \{0.1.\ldots, n\}$ and the parameter spaces are: $n \in \mathbb{N}$, $\alpha > 0$ and $\beta > 0$.

$$\begin{aligned}
\text{E}[X] &= n\frac{\alpha}{\alpha+\beta} \\
\text{Var}[X] &= \frac{n\alpha\beta}{(\alpha+\beta)^2}\underbrace{\frac{\alpha+\beta+n}{\alpha+\beta+1}}_{\in[1,n]}
\end{aligned}$$

Thus the variance is an inflated binomial variance. Let $\theta = \frac{\alpha}{\alpha+\beta}$, then $\mathrm{E}[X] = n\theta$. Let $B = \frac{\alpha}{\theta} - \alpha$. Then

$$
\begin{aligned}
\lim_{\alpha\to\infty} \mathrm{E}[X] &= n\theta \\
\lim_{\alpha\to\infty} \mathrm{Var}[X] &= \lim_{\alpha\to\infty} n \, \frac{\overbrace{\alpha}^{\theta}}{\alpha+\beta} \, \frac{\overbrace{\beta}^{1-\theta}}{\alpha-\beta} \\
&= \frac{\alpha+\beta+n}{\alpha+\beta+1} \\
&= \underbrace{n\theta(1-\theta)}_{\text{variance of binom}} \lim_{\alpha\to\infty} \frac{\alpha + \frac{\alpha}{\theta} - \alpha + n}{\alpha + \frac{\alpha}{\theta} - \alpha + 1} \\
&= n\theta(1-\theta) \lim_{\alpha\to\infty} \frac{\alpha + n\theta}{\alpha+\theta} = n\theta(1-\theta)\cdot 1 \\
&= n\theta(1-\theta)
\end{aligned}
$$

From this, as α gets higher, θ gets tighter and becomes degenerate and more like a binomial model.

Suppose $X \mid \theta \sim \mathrm{Binom}(n,\theta)$, $\theta \sim \mathrm{Beta}(\alpha,\beta)$ and $\theta \mid X \sim \mathrm{Beta}(\alpha+x, \beta+n-x)$. Suppose $X^* \mid X \sim \mathrm{Bern}(\frac{x+\alpha}{n+\alpha+\beta})$ where $n^* = 1$. Then:

$$
\begin{aligned}
\mathbb{P}(X^* \mid X) &= \int_\Theta \underbrace{\mathbb{P}(X^* \mid \theta)}_{\text{binom}} \underbrace{\mathbb{P}(\theta \mid X)}_{\text{beta}} \, d\theta \\
&= \int_0^1 \binom{n^*}{x^*} \theta^{x^*} (1-\theta)^{n^*-x^*} \cdot \frac{1}{B(\alpha+x, \beta+n-x)} \theta^{x+\alpha-1} (1-\theta)^{n-x+\beta-1} \, d\theta \\
&= \mathrm{BetaBinom}(n^*, \alpha+x, \beta+n-x)
\end{aligned}
$$