Lecture 15:

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$X_1 \ldots X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$P(\theta, \sigma^2) \propto \frac{1}{\sigma^2}$ $\Rightarrow$ jeffrey's prior

↳ both θ, σ^2 unknown.

If σ^2 known,

$$P(\theta | X, \sigma^2) = N(\bar{x}, (\tfrac{\sigma}{\sqrt{n}})^2)$$

If θ known,

$$P(\sigma^2 | X, \theta) = \text{InvGam}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right) = \text{InvGam}\left(\frac{n}{2}, \frac{\sum_{i=1}^{n}(x_i-\theta)^2}{2}\right)$$

If both unknown,

$$P(\theta, \sigma^2 | X) \propto P(X | \theta, \sigma^2) P(\theta, \sigma^2)$$

$$= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2}\right)\left(\frac{1}{\sigma^2}\right)$$

$$\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}.$$

$$\propto \text{NormInvGam}\left(\mu = \bar{x}, \lambda = n, \alpha = \frac{n}{2}, \beta = \frac{(n-1)s^2}{2}\right)$$

Sampling: r —( )

↳ pull or draw a realization, make data

$X \sim \text{Bern}(0.5)$? flip a coin

$X \sim \text{Bin}(10, 0.5)$? Flip 10 coins

$X \sim \text{Bin}(10, 0.2385976)$? } harder to get realizations

$X \sim N(11.2, 3.7^2)$?

LaPlace:

$F(X) := P(X \le x)$ $\Rightarrow$ CDF

↳ For a continuous r.v. X, what is the distribution of $Y := F_X(X)$?

$= t(X)$

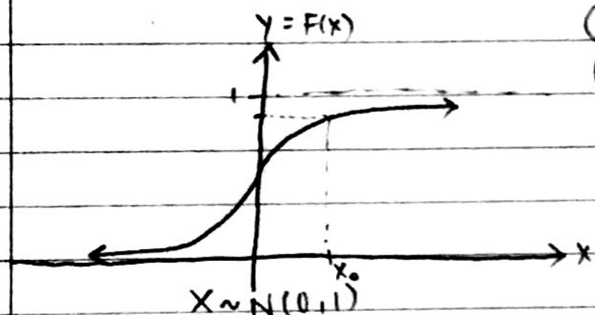
$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$
$$= f_X(x)\frac{1}{\left|\frac{dy}{dx}\right|}$$
$$= f_X(x)\frac{1}{\left|\frac{d}{dx}[F(x)]\right|} = \cancel{f_X(x)}\frac{1}{\cancel{|f_X(x)|}} = 1$$

$Supp[Y] = [0,1]$

$f_Y(y) = 1$

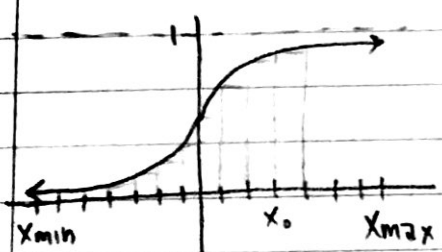
$\left.\begin{array}{l}\Rightarrow Y \sim U(0,1) \\ \Rightarrow X = F_X^{-1}(Y)\end{array}\right\}$ To sample X,

1. sample y_0 from $U(0,1)$
2. compute $x_0 = F_X^{-1}(y_0)$
3. return x_0



If F^{-1} is not available,

pick Xmin, Xmax, ΔX using these create a "grid"



$G := \langle X_{min}, X_{min}+\Delta X, X_{min}+2\Delta x, X_{min}+3\Delta X, \ldots, X_{max}\rangle$

compare $F(x)\ \forall x \in G$

Approximate $x_0 \approx \min_{y_0 \in G} F(x_0) \geq y$

What if X is discrete? because support is not infinite.

$G := Supp[X]$

We know how to sample from $f(x)$ but how to sample from $f(x, y)$?

		Y: 2	4	6	$f(x)$ ($f(x,y)$)
X	1	0.2	0.05	0.1	0.35
	3	0.1	.05	0.1	0.25
	5	0.2	0.1	0.1	0.4
	$f(y)$	0.5	0.2	0.3	

$\Rightarrow \frac{\{0.2, 0.05, 0.1\}}{0.35} \Rightarrow f(y|X=1)$

Recall Bayes Rule

$$f(x,y) = f(y|x)f(x)$$

sampling:

① Draw x_0 from $f(x)$

② Draw y_0 from $f(y|x=x_0)$

③ return $\langle x_0, y_0 \rangle$

$$P(\theta, \sigma^2 | X) = P(\theta | X, \sigma^2)\, P(\sigma^2 | X)$$

we know $P(\theta | X, \sigma^2)$ so,

$$P(\sigma^2|X) = \frac{P(\theta,\sigma^2|X)}{P(\theta|X,\sigma^2)} \propto \frac{(\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}{\underbrace{\frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}}}_{(\sigma^2)^{-\frac{1}{2}}} \cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}$$

$$\propto (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}} e^{-\frac{(n-1)s^2/2}{\sigma^2}}$$

$$-\frac{n}{2}-\frac{1}{2} = -\frac{n}{2}-\frac{1}{2}+1-1 = -\left(\frac{n}{2}+\frac{1}{2}-1\right)-1 = -\left(\frac{n}{2}-\frac{1}{2}\right)-1 = -\left(\frac{n-1}{2}\right)-1$$

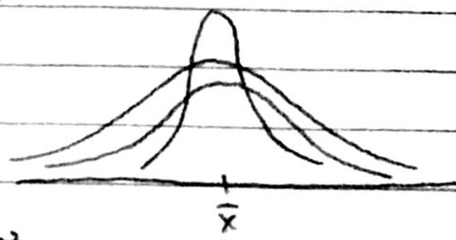
$$\propto \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

Sampling from NormInvGam

① Sample σ_0^2 from $\text{InvGam}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

② Sample θ_0 from $N\left(\bar{x}, \left(\frac{\sigma_0}{\sqrt{n}}\right)^2\right)$ => posterior

③ Return $\langle \theta_0, \sigma_0^2 \rangle$



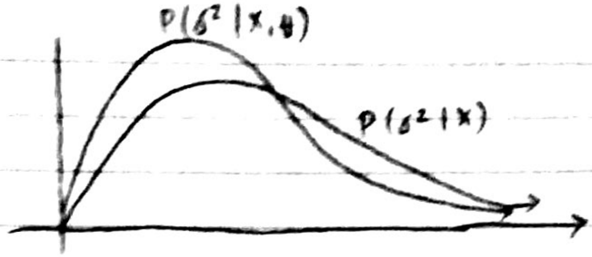
If $P(\theta,\sigma^2) \propto \frac{1}{\sigma^2}$

What is $P(\sigma^2|X)$? $\Rightarrow \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$ } not equal

Is it $P(\sigma^2|X,\theta)$? $\Rightarrow \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$

Recall $P(\sigma^2|X) = \int_{\mathbb{R}} P(\theta,\sigma^2|X)\,d\theta$ => averaging over all uncertainties of θ

"marginal distribution"



more variance in $P(\sigma^2 | X)$
because of uncertainty

What is $P(\theta | X)$?

$$P(\theta | X) = \int_0^\infty P(\theta, \sigma^2 | X)\, d\sigma^2$$

σ^2 is the "nuisance parameter"
↳ because you don't care about it

$$X_1, \dots, X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$$

$$Z = \frac{\bar{X} - \theta}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

$$V = \frac{\bar{X} - \theta}{\frac{s}{\sqrt{n}}} \sim T_{n-1} \Rightarrow \text{Student's T distribution}$$

$$V \sim T_n := \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{v^2}{n}\right)^{-\frac{n+1}{2}}$$

↳ Students' T distribut or "standard T distribution"

$$W = \sigma V + \mu = t(v)$$
$$v = t^{-1}(\mu) = \frac{W - \mu}{\sigma}$$

$$f_W(w) = f_V(t^{-1}(w)) \left| \frac{d}{dw}[t^{-1}(w)] \right|$$
$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n \sigma^2}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{1}{n}\left(\frac{\mu - n}{\sigma}\right)^2\right)^{-\frac{n+1}{2}} = T_n(\mu, \sigma)$$

↳ non-standard distribution.

$$P(\theta | X) = \frac{P(\theta, \sigma^2 | X)}{P(\sigma^2 | \theta, X)} = \frac{\left(\prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i - \theta)^2}\right)\left(\frac{1}{\sigma^2}\right)}{\frac{\left(\frac{n\hat{\sigma}^2}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{n\hat{\sigma}^2/2}{\sigma^2}}}$$

$$= \frac{\cancel{e^{-\frac{1}{2\sigma^2}\sum(X_i - \theta)^2}}}{\left(\frac{n\hat{\sigma}^2}{2}\right)^{n/2} \cancel{e^{-\frac{1}{2\sigma^2}\sum(X_i - \theta)^2}}} = \left(\frac{n\hat{\sigma}^2}{2}\right)^{-n/2}$$

$$= \left(\frac{(n-1)s^2}{2} + \frac{n(\bar{X} - \theta)^2}{2}\right)^{-n/2}$$

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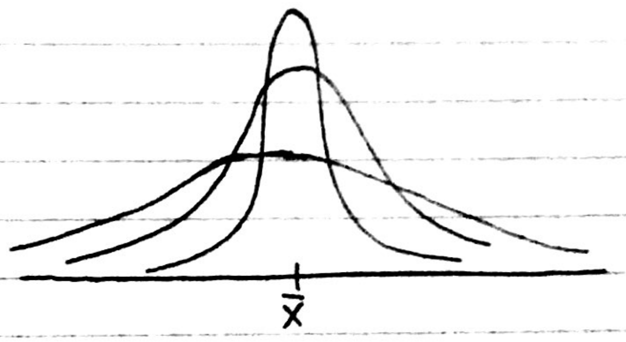
$$\propto \left(\frac{1}{\frac{(n-1)s^2}{2}} \right)^{-n/2} \left(\frac{(n-1)s^2}{2} + \frac{n(\bar{x}-\theta)^2}{2} \right)^{-n/2} = \left(1 + \frac{\frac{n(\bar{x}-\theta)^2}{\cancel{2}}}{\frac{(n-1)s^2}{\cancel{2}}} \right)$$



$$= \left(1 + \frac{1}{n-1} \left(\frac{\bar{x}-\theta}{\frac{s}{\sqrt{n}}} \right)^2 \right)^{-n/2}$$

$$\propto T_{n-1}\left(\bar{x}, \frac{s}{\sqrt{n}} \right)$$

$$= \int_0^\infty p(\theta, \sigma^2 \mid x)\, d\sigma^2$$