Math 341 Lec 13 3/26/18

$X \sim \text{Poisson}(\theta) := \dfrac{e^{-\theta}\theta^x}{x!}$ $\quad$ $\text{Supp}(X) = \{0, 1, \ldots\} = \mathbb{N}_0, \ \theta \in (0, \infty)$

Consider one Poisson $\quad$ $E(X) = \theta, \ \text{Var}(X) = \theta$

$$P(\theta \mid X) = \frac{P(X \mid \theta)\, P(\theta)}{P(X)} \propto P(X \mid \theta)\, P(\theta) = \frac{e^{-\theta}\theta^x}{x!} P(\theta) \propto e^{-\theta}\theta^x P(\theta)$$

$$\propto \underbrace{e^{-\theta}\theta^x}_{k(\theta \mid x)} k(\theta)$$

if $P(\theta)$ is conjugate then it will be of the same form as kernel of likelihood

$$k(\theta) = e^{-b\theta}\theta^a$$

$$P(\theta) = \frac{1}{c} k(\theta), \quad c := \int_\Theta k(\theta)\, d\theta = \int_0^\infty e^{-b\theta}\theta^a = \int_0^\infty \left(\frac{u}{b}\right)^a e^{-u} \frac{du}{b} = \frac{1}{b^{a+1}} \int_0^\infty u^{\overset{a+1-1}{a}} e^{-u} du = \frac{1}{b^{a+1}} \Gamma(a+1)$$

let $u = b\theta \Rightarrow \frac{du}{d\theta} = b \Rightarrow d\theta = \frac{du}{b}$, $\quad \Rightarrow \theta = \frac{u}{b}$

$\theta = 0 \Rightarrow u = 0$

$\theta = \infty \Rightarrow u = \infty$

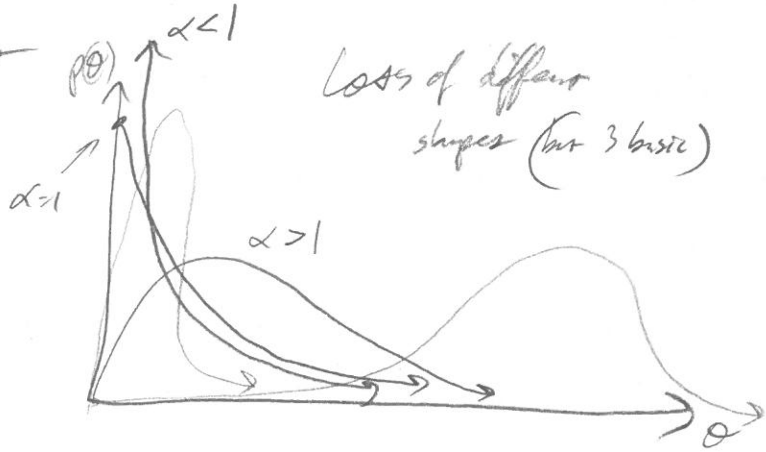
$$\Rightarrow P(\theta) = \frac{b^{a+1}}{\Gamma(a+1)} \theta^a e^{-b\theta} = \text{Gamma distr.}$$

usually parametrized as $\theta \sim \text{Gamma}(\alpha, \beta) := \dfrac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$

Now let's talk about properties of the gamma distr. and come back to the Bayesian model @ hand

$Supp(\theta) = (0, \infty)$ exactly where we were

param space $\alpha, \beta > 0$ just as before



Lots of different shapes (but 3 basic)

$$E(\theta) = \frac{\alpha}{\beta}$$

$$Var(\theta) = \frac{\alpha}{\beta^2}$$

$$Mode(\theta) = \frac{\alpha-1}{\beta} \text{ if } \alpha \geq 1$$

Calculus exercises

$Med(\theta) = qgamma(0.5, \alpha, \beta)$ (in R) (i.e. no closed form)

$$p(\theta|x) \propto p(x|\theta)\,p(\theta) = \frac{e^{-\theta}\theta^x}{x!} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\theta} \theta^{\alpha-1} \propto e^{-\theta}\theta^x e^{-\beta\theta}\theta^{\alpha-1} = e^{-(\beta+1)\theta}\theta^{x+\alpha-1}$$

$X|\theta \sim Poisson(\theta)$

$\theta \sim Gamma(\alpha, \beta)$

$\propto Gamma(x+\alpha, \beta+1)$

$\theta|X \sim Gamma(x+\alpha, \beta+1)$

"gamma is conjugate prior for Poisson likelihood"

Now lets say you have n iid Poissons

$X_1, \ldots, X_n | \theta \overset{iid}{\sim} Poisson(\theta)$

$\theta \sim Gamma(\alpha, \beta)$

$\theta | X_1, \ldots, X_n \sim$?

$$p(\theta|x) \propto p(x|\theta)\,p(\theta) = \left(\prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!}\right)\left(\frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\theta}\theta^{\alpha-1}\right) = \frac{e^{-\sum_{i=1}^{n}\theta_i}\,\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\theta}\theta^{\alpha-1}$$

$$\propto e^{-n\theta}\theta^{\sum x_i} e^{-\beta\theta}\theta^{\alpha-1} \propto Gamma\left(\sum x_i + \alpha, n+\beta\right)$$

Interpretation of pseudocounts

α = sum total of successes seen previously

β = # of pseudo trials performed

$E(\theta) = \frac{\alpha}{\beta}$ = avg # of successes / trial

$\hat{\theta}_{MMSE} = \frac{\Sigma x_i + \alpha}{n+\beta}$, $\hat{\theta}_{MAE} = qgamma(0.5, \Sigma x_i + \alpha, n+\beta)$, $\hat{\theta}_{MAP} = \frac{\Sigma x_i + \alpha - 1}{n+\beta} \approx \hat{\theta}_{MMSE}$ 3

Can we say $\theta \sim U$? No..

But... what if $P(\theta) \propto 1$ Improper since $\int_0^\infty c\, d\theta = \infty$

What happens?

$P(\theta|x) \propto P(x|\theta)\, P(\theta) \propto e^{-n\theta}\theta^{\Sigma x_i} \propto Gamma(\Sigma x_i + 1, n)$ - always proper!! NICE!

$\theta \sim Gamma(1, 0) \propto 1$ Principle of Indifference = Laplace prior (improper)

What is MLE? Back to math stats...

$$\mathcal{L}(\theta; x) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\theta^{\Sigma x_i}}{\Pi x_i!}$$

$$\ell(\theta; x) = -n\theta + \Sigma x_i \ln(\theta) - \ln(\Pi x_i!)$$

$$\ell'(\theta; x) = -n + \frac{\Sigma x_i}{\theta} \overset{set}{=} 0 \Rightarrow \frac{\Sigma x_i}{\theta} = n \Rightarrow \boxed{\hat{\theta}_{ml} = \bar{x}}$$

Also...

$\theta \sim Gamma(0, 0)$ Haldane (improper)

$\Rightarrow \theta|x \sim Gamma(\Sigma x_i, n)$

improper if $x_1 = x_2 = \dots = x_n = 0$ zero successes!

Jeffrey's Prior?

$$\ell''(\theta; x) = -\frac{\Sigma x_i}{\theta^2}$$

$$I(\theta) = E[-\ell''(\theta; x)] = E\left(\frac{\Sigma x_i}{\theta^2}\right) = \frac{E(\Sigma x_i)}{\theta^2} = \frac{\Sigma E(x_i)}{\theta^2} = \frac{\Sigma \theta}{\theta^2} = \frac{n\theta}{\theta^2} = \frac{n}{\theta}$$

$P(\theta) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\theta}} \propto \sqrt{\frac{1}{\theta}} = \theta^{-\frac{1}{2}} \propto Gamma\left(\frac{1}{2}, 0\right)$ improper $\Rightarrow \theta|x \sim Gamma\left(\Sigma x_i + \frac{1}{2}, 0\right)$ always proper!

Same idea as before.. see 0 trials ... but know there's 0.5 successes somewhere!

So two choices for uniformative prior: Haldane, Jeffrey's. Jeffrey's leads to proper post. always.