Lecture 9

- Bayesian Hypothesis testing : Bayes $P_{val} = P(H_0/X)$

- **Right-sided**

$H_0: \theta \leq (\theta_0 = 0.5)$

$H_a: \theta > (\theta_0 = 0.5)$

$$P_{val} = P(H_0|X) = P(\theta \leq \theta_0 / X)$$

$$= \int_0^{\theta_0} \frac{1}{B(x+\alpha, n-x+\beta)} \theta^{\alpha+x-1}(1-\theta)^{n-x+\beta-1} \, d\theta$$

$$= qbeta(\theta_0, x+\alpha, n-x+\beta)$$

- **Left-sided**

$H_0: \theta \geq (\theta_0 = 0.5)$

$H_a: \theta < (\theta = 0.5)$

$$P_{val} = P(H_0|X) = P(\theta \geq \theta_0 | X) = 1 - \mathcal{F}_{\theta|x}(\theta_0)$$

$$1 - P_{val}(\theta_0)$$

Right side

$$= \int_{\theta_0}^{1} \text{posterior Beta} \, d\theta$$

$$= qbeta(1, x+\alpha, n-x+\beta) - qbeta(\theta_0, x+\alpha, n-x+\beta)$$

** Note : By convention

if $P_{val} < \alpha$ (Reject H_0)

$P_{val} \geq \alpha$ (Retain H_0)

- α is generally: 5%

- **2-sided testing**

$H_0 = (\theta_0 = 0.5)$

$H_\alpha \neq (\theta_0 = 0.5)$

$\left\{ \begin{array}{l} \alpha = 0.05 = 5\% \\ \theta \sim \text{some prior} \end{array} \right\}$

$$P_{val} = P(H_0 | X) = P(\theta = \theta_0 | X)$$

$$= \int_{\theta_0}^{\theta_0} Beta(\alpha', \beta')$$

$$= 0 \quad (\theta_0 \text{ is a point value})$$

* So P_{val} always $= 0$ and $p_{val} < \alpha$ always so we would always reject H_0

Methods of 2 sided testing

#1 | Equivalence Regions

Let: $H_0 : \theta \in [\theta \pm \delta]$
$H_a : \theta \notin [\theta \pm \delta]$

$[\theta_0 - \delta, \theta_0 + \delta]$
FAIR

- this gives a range of θ about θ_0 in which H_0 should exist.

However,

- δ is too subjective for some
we set the bounds for H_0 to exist

$$P_{val} = P(H_0|X) = P(\theta \in [\theta_0 \pm \delta] \mid X) = \int_{\theta_0-\delta}^{\theta_0+\delta} Beta(\alpha', \beta') = pbeta(\theta_0 + \delta, \alpha', \beta') - pbeta(\theta_0 + \delta, \alpha', \beta')$$

#2 | Credible Regions

$H_0 = \theta_0$ $\quad \alpha = 5\%$
$H_a \neq \theta_0$ $\quad \theta \sim$ some prior

if $\theta \in CR_{\theta, 1-\alpha} = [qbeta(\alpha, \alpha', \beta'), qbeta(1-\alpha, \alpha', \beta')]$
$\Rightarrow$ Retain H_0
(if not reject H_0)

* cant say $P(H_0|X)$
** No P-value

#3 | Bayes Factor

$$B := \frac{P_{H_a}(X)}{P_{H_0}(X)} = \frac{\int_{\Theta_{H_a}} P_{H_a}(X|\theta) P_{H_a}(\theta)\, d\theta}{\int_{\Theta_{H_0}} P_{H_0}(X|\theta) P_{H_0}(\theta)\, d\theta}$$

(Bayes Rule)

*Note

- Generally we find $P_0(X)$
- if B is large, H_a is more likely
- if B is small, H_0 is more likely

Ratio scale is described By Jeffrey (1961) :

$B < 1$	No evidence for H_a
$B \in [1, 3]$	'Barely worth mentioning'
$B \in [3, 10)$	'substantial'
$B \in (10, 30)$	'strong'
$B \in (30, 100)$	'very strong'
$B \in (100, \infty)$	'absolutely decisive'

Example

$H_0 : \theta = 0.5$ $\quad \alpha_0 = 0.05 = 5\%$

$H_a : \theta \neq 0.5$ $\quad n = 100$

$x = 61$

freq. test

$$\text{Ret. Region} = \left[\theta_0 \pm Z_{\frac{\alpha_0}{2}} SE[\theta_0]\right] = \left[0.5 \pm \underbrace{2}_{Z_{2.5\%}} \left(\underbrace{\sqrt{\tfrac{0.5 \cdot 0.5}{100}}}_{\sqrt{\frac{\theta(1-\theta)}{n}}}\right)\right] = \left[0.5 \pm 0.1\right] = \left[0.4, 0.6\right]$$

- $\hat{\theta} = \frac{61}{100} = 0.61$; $\hat{\theta} \notin [0.4, 0.6]$; reject H_0
- $P_{val} = 0.0278$, $< \alpha_0 = 0.05$; reject H_0

Bayes testing

Let $H_a : \theta \sim U(0,1)$ so θ could be any value

$$\theta \sim Beta(1,1) \overset{x}{\Longrightarrow} \theta | x \sim Beta(62, 40)$$

- Method #1/

$H_0 : \theta = [\theta = 0.5 \pm f]$ $\quad$ let $f = 0.01$

$H_a : \theta \neq [\theta = 0.5 \pm f]$ $\quad$ equivalence Region

p-val = pbeta(0.51, 62, 40) − pbeta(0.49, 62, 40) = 0.0147

$0.0147 < 0.05 \Rightarrow$ reject H_0

- method #2/

$H_0 : \theta = 0.5$ $\quad \alpha_0 = 0.05$

$H_a : \theta \neq 0.5$ $\quad \theta \sim U(0,1)$ $\quad \theta | x \sim Beta(62,40)$

$CR_{\theta, 1-\alpha}$ = [qbeta(0.025, 62, 40) , qbeta(0.975, 62, 40)] = [0.511 , 0.700]

$H_0 : \theta = 0.5 \notin CR_{\theta, 1-\alpha} \Rightarrow$ Reject H_0

$\left[P(\theta);\ \theta \sim U(0,1)\right]$

Method #3 /

$$B \doteq \frac{\int_{\Theta_{H_a}} P_{H_a}(X|\theta)\, P_{H_a}(\theta)}{\int_{\Theta_{H_0}} P_{H_0}(X|\theta)\, P_{H_0}(\theta)} = \frac{\int_0^1 \theta^{61}(1-\theta)^{39}\,(1)\,d\theta}{\int_{0.5}^{0.5} (0.5)^{61}(0.5)^{39}\,(1)\,d\theta}$$

$\left[P(\theta);\ \theta = 0.5\right]$

or

$$\sum_{\theta \in \Theta_{H_0}} P_{H_0}(X|\theta)\, P_{H_0}(\theta) \qquad \text{or} \qquad \sum_{\theta \in \Theta_{H_0}} P(X|\theta)\, P(\theta)$$

so $$B = \frac{B(62,40)}{(0.5)^{100}} = 1.39$$

"Barely worth mentioning"

$H_a > H_0$

* here we see the $H_0 : \theta = 0.5$ is rejected but that our alternative is not much stronger