Lec 12

- Fisher's info / $\mathcal{L}(\theta;X) = P(X;\theta)$

$$l(\theta;X) = \ln\left(\mathcal{L}(\theta;X)\right)$$

$$S(\theta;X) := \frac{d}{d\theta}\left[l(\theta;X)\right]$$

score function

$$I(\theta) := Var_X\left[S(\theta;X)\right]$$

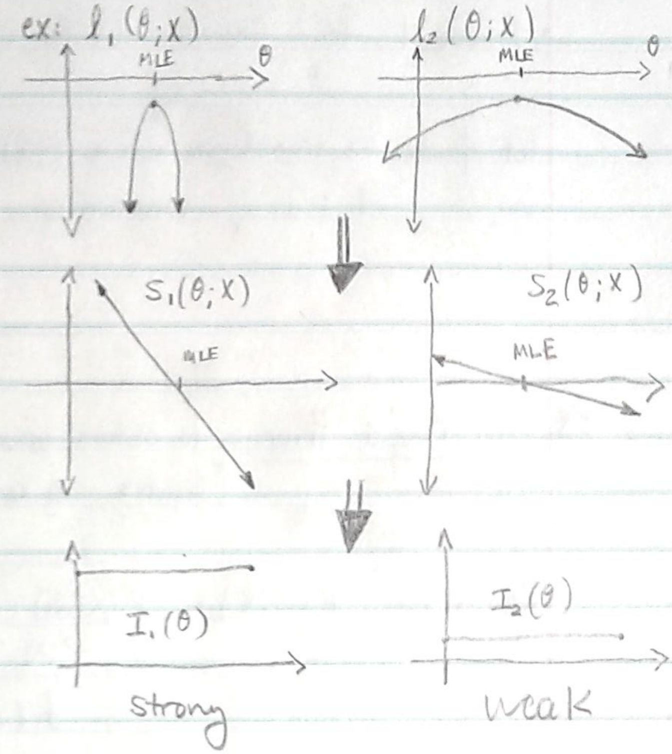
fisher Info

$$= \dots$$

$$= E_X\left[S(\theta,X)^2\right]$$

$$= \dots$$

$$= E_X\left[-l''(\theta;X)\right]$$



ex: $X \sim Binomial(n,\theta)$ (fisher info)

$$\mathcal{L}(\theta,X) = P(X;\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

$$l(\theta;X) = \ln\left(\binom{n}{x}\right) + x\ln(\theta) + (n-x)\ln(1-\theta)$$

$$S(\theta;X) = l'(\theta;X) = \frac{x}{\theta} - \frac{n-x}{1-\theta}$$

Note:
$E(X)$ of Binomial $= n\theta$

$$l''(\theta;X) = \frac{-x}{\theta^2} - \frac{(n-x)}{(1-\theta)^2}$$

$$I(\theta) = E_X\left[-l''(\theta;X)\right] = E_X\left[\frac{+x}{\theta^2} + \frac{(n-x)}{(1-\theta)^2}\right] = \frac{E_X[X]}{\theta^2} + \frac{n-E_X[X]}{(1-\theta)^2}$$

$$= \frac{n\theta}{\theta^2} + \frac{n-n\theta}{(1-\theta)^2} = n\left(\frac{1}{\theta} + \frac{1}{(1-\theta)}\right) = n\left(\frac{1-\theta}{\theta(1-\theta)} + \frac{\theta}{\theta(1-\theta)}\right)$$

$$\boxed{I(\theta) = n\left(\frac{1}{\theta(1-\theta)}\right)}$$

ex: if $n=1$ $\quad \theta_1 = 0.5 \quad \theta_2 = 0.01$

$I(\theta_1) = 4 \quad I(\theta_2) = 101.01$

Note: Not a function of X anymore

Relatively: (weak) (strong)

Jeffrey's Prior

Idea: $\boxed{P(\theta) \propto \sqrt{I(\theta)}}$ (Kernel of Prior)

- Applied to the binomial where:

- $X \sim \text{Binomial}(n, \theta)$
  $= \binom{n}{x}(\theta)^x(1-\theta)^{n-x}$
- $\theta \sim$ Jeffrey's Prior
  $\theta = f^{-1}(R) = \frac{R}{R+1}$
- $\frac{\theta}{1-\theta} = R$ (odds) $\sim$ Jeffrey's Prior
  $R = f(\theta)$

So we want to find Jeffrey's prior for θ given $\phi = f(\theta) = \frac{\theta}{1-\theta} = R$

Since $P(\theta) \propto \sqrt{I(\theta)}$ and $\theta = \frac{R}{R+1}$

- $X \sim \text{Binomial}(n, \theta = \frac{R}{R+1}) = \binom{n}{x}\left(\frac{R}{R+1}\right)^x\left(1 - \frac{R}{R+1}\right)^{n-x} = \binom{n}{x}\left(\frac{R}{R+1}\right)^x\left(\frac{1}{R+1}\right)^{n-x} = \binom{n}{x}\frac{R^x}{(R+1)^n}$

$$f(R; X) = \binom{n}{x}\frac{R^x}{(R+1)^n}$$

$$\ell(R; x) = \ln\left(\binom{n}{x}\right) + x\ln(R) - n\ln(R+1)$$

$\frac{d}{dR}$
$$\ell' = 0 + \frac{x}{R} - \frac{n}{R+1}$$

$\frac{d}{dR}$
- $$\ell'' = -\frac{x}{R^2} + \frac{n}{(R+1)^2} \qquad -\ell'' = \frac{x}{R^2} - \frac{n}{(R+1)^2}$$

$\boxed{\text{* find fisher info for } \theta = \frac{R}{R+1}}$

- $$I(R) = E_x\left[-\ell''(R; x)\right] = E_x\left[\frac{x}{R^2} - \frac{n}{(R+1)^2}\right] = \frac{E_x(x)}{R^2} - \frac{n}{(R+1)^2}$$

$$= \frac{n\left(\frac{R}{R+1}\right)}{R^2} - \frac{n}{(R+1)^2} = \frac{n}{R(R+1)} - \frac{n}{(R+1)^2}$$

$$= \frac{n(R+1)}{R(R+1)^2} - \frac{nR}{R(R+1)^2}$$

Note: again
$E(X)$, $X \sim \text{Bin}(n, \theta)$
$= n\theta$ or now
$= n\left(\frac{R}{R+1}\right)$

$$\boxed{I(R) = \frac{n}{R(R+1)^2}}$$

Fisher info for $R = \frac{\theta}{1-\theta}$

Note: $I(\theta) = n \cdot \left(\frac{1}{\theta(1-\theta)}\right)$ $\quad I(R) = n \cdot \left(\frac{1}{R(R+1)^2}\right)$

so $P(X \mid \theta)$ — Protocol $\rightarrow$ $P_J(\theta)$

$P(X \mid \phi)$ $\quad P_J \propto \sqrt{I(\theta)}$ $\quad P_J(\phi)$ } Transformation of variables

- $P_J(\theta) \propto \sqrt{I(\theta)} = \sqrt{n \frac{1}{\theta(1-\theta)}} \propto \frac{1}{\sqrt{\theta(1-\theta)}} = \theta^{-1/2}(1-\theta)^{-1/2}$ } Kernal $K(\theta)$ of a Jeffrey PRIOR

**Note: $\theta^{-\frac{1}{2}}(1-\theta)^{-\frac{1}{2}} = \theta^{\frac{1}{2}-1}(1-\theta)^{\frac{1}{2}-1} \propto \frac{\theta^{\frac{1}{2}-1}(1-\theta)^{\frac{1}{2}-1}}{B(\frac{1}{2},\frac{1}{2})}$ } constant

* so $\boxed{P_J(\theta) = \text{Beta}\left(\frac{1}{2},\frac{1}{2}\right) \propto \sqrt{I(\theta)} \propto K(\theta)}$

- $P_J(R) \propto \sqrt{I(R)} = \sqrt{n \cdot \frac{1}{R(R+1)^2}} \propto \frac{1}{\sqrt{R}} \cdot \frac{1}{\sqrt{(R+1)^2}} = R^{-1/2}\left((R+1)^2\right)^{-1/2}$ } $K(R)$

**Note: $P_J(R) \leftrightarrow P_J(\theta)$ Requires a transformation of variables

$f_y(y) = f_x(f^{-1}(y)) \left| \frac{d}{dy}[f^{-1}(y)] \right|$

- $P_R^J(R) = \underbrace{P_\theta^J(f^{-1}(R))} \left| \frac{d}{dr} f^{-1}(R) \right|$ $\quad$ } $f^{-1}(R) = \theta = \frac{R}{R+1}$

$$= \left[\frac{1}{B(\frac{1}{2},\frac{1}{2})} \cdot \left(\frac{R}{R+1}\right)^{-1/2}\left(1 - \frac{R}{R+1}\right)^{-1/2}\right]\left[\left|\frac{d}{dR}\left[\frac{R}{R+1}\right]\right|\right]$$

$$= \left[\frac{1}{\pi} \cdot \frac{R+1}{\sqrt{R}}\right]\left[\frac{1}{(R+1)^2}\right]$$

$$\boxed{P_J(R) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{R}} \cdot \frac{1}{R+1} \propto \sqrt{I(R)} \propto K(R)}$$

Note

$B\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 \theta^{-1/2}(1-\theta)^{-1/2}\,d\theta$

$= \int_0^1 K(\theta)$

$= \int_0^\infty K(R)$

$B\left(\frac{1}{2},\frac{1}{2}\right) = \pi$

- Proof of Jeffrey's Protocol

given $P_J(\theta) \propto \sqrt{I(\theta)}$; $P_J(\phi) \propto \sqrt{I(\phi)}$; $\phi = f(\theta)$; $\theta = f^{-1}(\phi)$

want to show $P^J_\phi(\phi) = P^J_\theta\left(f^{-1}(\phi)\right)\left|\frac{d}{d\phi}\left[f^{-1}(\phi)\right]\right| \propto \sqrt{I(\phi)}$

$$P^J_\phi(\phi) = P^J_\theta(\theta)\left|\frac{d\theta}{d\phi}\right|$$

$$\propto \sqrt{I(\theta)\left|\frac{d\theta}{d\phi}\right|^2} = \sqrt{E_X\left[S(\theta;X)^2\right]\frac{d\theta}{d\phi}\frac{d\theta}{d\phi}}$$

$$= \sqrt{E_X\left[\left(\frac{d\ell}{d\theta}\right)^2\right]\frac{d\theta}{d\phi}\frac{d\theta}{d\phi}} = \sqrt{E_X\left[\frac{d\ell}{d\theta}\cdot\frac{d\ell}{d\theta}\cdot\frac{d\theta}{d\phi}\frac{d\theta}{d\phi}\right]}$$

$$= \sqrt{E_X\left[\left(\frac{d\ell}{d\phi}\right)^2\right]} = \sqrt{E_X\left[S(\phi,X)^2\right]}$$

so $P^J_\phi(\phi) \propto \sqrt{I(\phi)}$ *after transformation*

*****

Now: Given $X \sim \text{Binomial}(n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$

$$\text{supp}(X) = \{0, 1, \dots, n\}$$

$$\theta \in (0,1)$$

let $n \to \infty$ and $\lambda = n\theta$, so $\theta = \frac{\lambda}{n}$

$\theta \to 0$ (new variable)

then, $$\lim_{n \to \infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} = \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!\, n^x} \cdot \underbrace{\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n}_{= 1^\infty} \cdot \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left[\frac{n \cdot (n-1) \cdots (n-x)(n-x-1) \cdots (2)(1)}{\underbrace{n \cdot n \cdots}_{n^x} \quad (n-x)(n-x-1) \cdots (2)(1)} \right] \cdot e^{-\lambda} \cdot 1$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left[\frac{n\,(n-1) \dots (n-x+1)}{n \quad n \quad \dots \quad n} \right] \cdot e^{-\lambda} = \boxed{\frac{\lambda^x}{x!} e^{-\lambda} = \text{Poisson}(\lambda)}$$

** we will work with θ in place of $\lambda = n\theta$ (different θ)

$$\left[X \sim \text{Poisson}(\theta) := \frac{\theta^x e^{-\theta}}{x!} \right]$$

$n \to \infty$
$x =$
$\theta \to 0$

$$\text{Supp}(X) = \{0, 1, \dots, \dots\} = \mathbb{N}_0$$

$$\theta \in \Theta = (0, \infty)$$

$E[X] = \theta$
$Var[X] = \theta$