Lecture II

Uninfomative Priors (objective, weak)

indifference / $\theta \sim U(0,1) = Beta(1,1) \xrightarrow{X} \theta | X \sim Beta(1-x, 1+n-x)$

$\hat{\theta}_{MMSE} = \frac{x+1}{n+2}$ (Wilson Estimate)

zero Knowledge / [Haldane, 1932] $\theta \sim Beta(0,0)$ $\xrightarrow{X}$ $\theta | X \sim Beta(x, n-x)$

*Not a proper prior

Not a legal distribution

$\alpha, \beta = 0 \notin$ Parameter space

$\hat{\theta}_{MMSE} = \frac{X}{n} = \hat{\theta}_{MLE}$

*Posterior is Proper if $\begin{matrix} x > 0 \\ n-x > 0 \end{matrix}$

Informative Priors :

ex/ $\theta \sim Beta(\alpha, \beta)$ where α, β are "large"

ex: Batting Averages: $BA = \frac{\#\text{ hits}}{\#\text{ at bat}} = \frac{X}{n}$ | $X_1, \ldots, X_n \overset{iid}{\sim} Bern(\theta)$

BA is a $\hat{\theta}_{MLE} = \frac{X}{n}$; weak when n is small

ex: if $n = 2$; $x = 0, 1, 2$ all give unrealistic values

** use $\hat{\theta}_{MMSE} = \frac{X+\alpha}{n+\alpha+\beta}$, Shrinkage towards $\frac{\alpha}{\alpha+\beta}$ with weight $\left(\frac{\alpha+\beta}{\alpha+\beta+n}\right)$

- What prior? $\theta \sim U(0,1)$? No b/c then it shrink to 0.5 (not realistic)
- Use "Empirical Bayes" θ is based on previous observations.

steps

① def. sample of previous players

② def. N_0, min. sample for each

③ $\hat{\theta}_{MLE}$ for all previous players

④ Plot distribution

"fit" data, find α, β

ex/ "pseudotrials"

$N_0 = 500$ $\alpha = 78.7$ $\beta = 224.8$

$\hat{\theta}_{MMSE} = \frac{X + 78.2}{n + 303.5}$

$\hookrightarrow \frac{\alpha+\beta}{\alpha+\beta+n} \cdot \frac{\alpha}{\alpha+\beta}$

$e = \frac{\alpha+\beta}{\alpha+\beta+n} = \frac{303.5}{303.5+2} = 99.5\%$

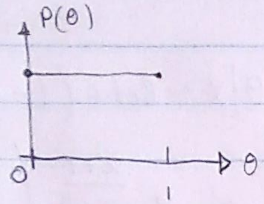
$\frac{\alpha}{\alpha+\beta} = \frac{78.7}{303.5} = 0.258$

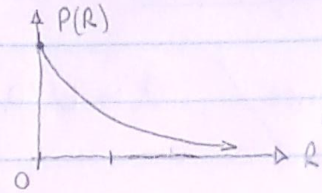
(Shrinkage Estimate) = e % towards $\frac{\alpha}{\alpha+\beta}$ $E(\theta)$

(for new data - Not shown)

- uninformatives viewed differently

$\theta \sim U(0,1)$ $\xrightarrow{\text{transformation}}$ Odds For : $R := \frac{\theta}{1-\theta} \in (0, \infty)$





$\lim_{\theta \to 1} \frac{\theta}{1-\theta} = \infty$

$\theta \in (0,1)$

* this signifying smaller odds are more likely

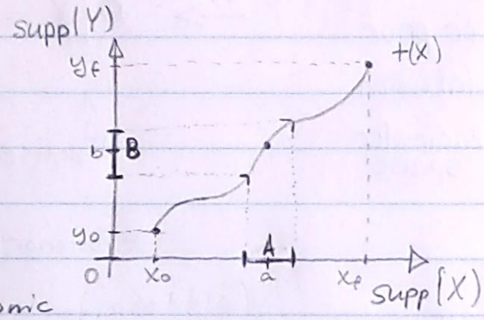
* prior indifference for R is not possible

ex 2 R.V.s X, Y

w/ PDF f_x, f_y

- f_x known
- $Y = t(X)$

  t is monotonic (invertible)



$P(X \in A) \approx f_x(a)\, A$

$P(X \in B) \approx f_y(b)\, B$

...

if $A > 0$, $B > 0$ but are really small

$A = |dx|$; $B = |dy|$

...

$f_y(b) \approx f_x(a) \left| \frac{dx}{dy} \right|$

- $f_y(y) = f_x(t^{-1}(y)) \left| \frac{d}{dy}[t^{-1}(y)] \right|$

* transformation of Variable formula

ex ~~**~~ for odds for: $\left[R = t(\theta) = \frac{\theta}{1-\theta}\right] \xRightarrow{\text{invert function}} \left[\theta = t^{-1}(R) = \frac{R}{R+1}\right]$

$$f_R(r) = f_\theta(t^{-1}(r)) \left| \frac{d}{dr}[t^{-1}(r)] \right|$$

$$= f_\theta\left(\frac{R}{R+1}\right) \left| \frac{d}{dr}\left(\frac{R}{R+1}\right) \right|$$

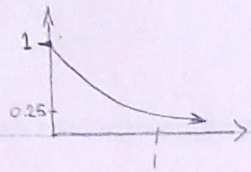
$$= f_\theta(\theta) \cdot \left| \frac{d}{dr}\left(r(r+1)^{-1}\right) \right|$$

$$= \underbrace{1}_{\text{since } \theta \sim (0,1)} \cdot \frac{1}{(r+1)^2}$$

$f_R(r) = 1/(r+1)^2$

* Checkable : $\int_0^\infty \frac{1}{(r+1)^2}\, dr = \frac{r}{(r+1)} \Big|_0^\infty = 1 - 0 = 1$

- Odds $f_R(r) = \frac{1}{(r+1)^2}$:



- Protocol to Choose Uniform prior thats the same under transformation?

let ϕ be the transformation $= f(\theta)$
$1:1$, monotonic

$P(X|\theta) \longrightarrow P(\theta)$
"Protocol"
$P(X|\phi) \longrightarrow P(\phi)$
Transform

again: $p(\phi) = p\left(f^{-1}(\phi)\right) \left| \frac{d}{d\phi}\left[f^{-1}(\phi)\right] \right|$

** Jeffery's Prior : to obtain priors from each likelihood
[Requires ① Kernals ② fisher information]

Kernals $P(\theta|X) = \frac{P(X|\theta)\,P(\theta)}{P(X)} \propto P(X|\theta)\,P(\theta)$

$$f(x;\theta) \propto \underbrace{\{g(x;\theta)\}}_{\text{The Kernal}}$$

- if $\exists\, c \in \mathbb{R}$ $f(x;\theta) \propto c\, g(x;\theta)$

* finding c : $\int_{supp(X)} f(X)\,dx = 1$

so $\int_{supp(X)} c \cdot g(X)\,dx = \int f(x)\,dx \Rightarrow \frac{1}{c} = \overbrace{\int g(x;\theta)\,dx}^{\text{finite and positive}}$

ex: Binomial (n,θ)
$P(x;\theta) = \frac{n!}{x!(n-x)!}\theta^x (1-\theta)^{\cancel{n-x}} \quad (1-\theta)^n (1-\theta)^{-x}$ at a fixed n

- $P(X;\theta) \propto \underbrace{\frac{1}{x!(n-x)!}\left(\frac{\theta}{1-\theta}\right)^x}_{\text{Kernal of Binomial}}$