# Lecture 14

- Recall: Priors + Posteriors for Poisson $n \geq 0$ (# of trials), $x_i \geq 0$ (successes)

Indifference / $P(\theta) \propto 1$

$\theta \sim Gamma(1, 0)$ $\quad$ $\theta | X \sim Gamma(\Sigma X_i + 1, n)$

* improper, $\beta \neq 0$ $\quad$ * always proper

Jeffrey's / $P_J(\theta) \propto \sqrt{I(\theta)}$

$\theta \sim Gamma(\frac{1}{2}, 0)$ $\quad$ $\theta / X \sim Gamma(\Sigma X_i + \frac{1}{2}, n)$

* improper, $\beta \neq 0$ $\quad$ * always proper

Haldane / $\theta \sim Gamma(0, 0)$ $\quad$ $\theta / X \sim Gamma(\Sigma X_i, n)$

* improper, $\alpha \neq 0, \beta \neq 0$ $\quad$ * improper when no successes

- $\hat{\theta}_{MMSE}$

$$= E[\theta | X] = \left(\frac{\Sigma X_i + \alpha}{n + \beta}\right) = \left(\frac{\Sigma X_i}{n + \beta}\right)\left(\frac{n}{n}\right) + \left(\frac{\alpha}{n + \beta}\right)\left(\frac{\beta}{\beta}\right) = \left(\frac{\Sigma X_i}{n} \cdot \frac{n}{n+\beta}\right) + \left(\frac{\alpha}{\beta} \cdot \frac{\beta}{n+\beta}\right)$$

let $\left(\frac{\beta}{n+\beta}\right) = \rho$ then $\left(\frac{n}{n+\beta}\right) = 1 - \rho$ AND

- $\Sigma X_i / n = \hat{\theta}_{MLE}$
- $\alpha / \beta = E[\hat{\theta}]$

So

$$= E[\theta | X] = \rho E[\hat{\theta}] + (1 - \rho)\hat{\theta}_{MLE}$$

* Shrinkage Estimator

$$\lim_{n \to \infty} (\rho) = 0$$

so $E[\theta / X]$ shinks to $\hat{\theta}_{MLE}$

- Posterior Prediction Distribution

given $X^*$ and $n^*=1$ new trials, what's the new posterior prediction

$$P(x^*|x) = \int_{\Theta} P(x^*|\theta)\, P(\theta|x)\, d\theta$$

$$= \int_0^\infty \left(\frac{\theta^{x^*} e^{-\theta}}{x^*!}\right)\underbrace{\left(\frac{\beta'^{\alpha'}}{\Gamma(\alpha')}\theta^{\alpha'-1} e^{-\beta'\theta}\right)}^{\text{gamma}(\alpha',\beta')} d\theta$$

$$= \left(\frac{\beta'^{\alpha'}}{\Gamma(\alpha')\cdot x^*!}\right)\int_0^\infty \left(\theta^{x^*+\alpha'-1} e^{-(\beta'+1)\theta}\right) d\theta \quad \Big\} \text{ Kernal of gamma} \qquad \beta'' = \beta'+1,\ \alpha'' = \alpha'+x^*$$

$$= \frac{\beta'^{\alpha'}}{\Gamma(\alpha')\cdot x^*!}\underbrace{\left[\int_0^\infty \underbrace{\left(\frac{(\beta'+1)^{(\alpha'+x^*)}}{\Gamma(\alpha'+x^*)}\right)}_{\text{constant}} \cdot \theta^{(\alpha'+x^*)-1} e^{-(\beta'+1)\theta}\, d\theta\right]}_{\text{pdf}}\underbrace{\left(\frac{\Gamma(\alpha'+x^*)}{(\beta'+1)^{(\alpha'+x^*)}}\right)}_{\text{reciprocal}}$$

$= 1$ of distribution over its support

$$= \frac{\beta'^{\alpha'}}{\Gamma(\alpha')\cdot x^*!}\cdot\frac{\Gamma(\alpha'+x^*)}{(\beta'+1)^{\alpha'}(\beta'+1)^{x^*}}$$

$$= \frac{\Gamma(x^*+\alpha')}{x^*!\,\Gamma(\alpha')}\cdot\left(\frac{\beta'}{\beta'+1}\right)^{\alpha'}\left(\frac{1}{\beta'+1}\right)^{x^*} \quad \Bigg|\ \text{let } \left(\frac{\beta'}{\beta'+1} = \rho\right) \text{ And } \left(\frac{1}{\beta'+1} = 1-\rho\right)$$

$$= \frac{\Gamma(x^*+\alpha')}{x^*!\,\Gamma(\alpha')}\cdot \rho^{\alpha'}(1-\rho)^{x^*} \quad \Big|\ \text{if } \alpha' \in \mathbb{N}$$

$$= \left(\frac{(x^*+\alpha'-1)!}{x^*!\,(\alpha'-1)!}\right)\rho^{\alpha'}(1-\rho)^{x^*} = \left[\binom{x^*+\alpha'-1}{x^*}\rho^{\alpha'}(1-\rho)^{x^*}\right]$$

- $P(x^*|x) = \text{NegBinomial}(\alpha', \rho) = \left[\text{NegBin}\left(\underbrace{\Sigma X_i + \alpha}_{\alpha'},\ \underbrace{\frac{\beta'+n}{\beta'+1+n}}_{p}\right)\right]$

$n^* = 1$

- The Normal distribution $(N)$

$$P(X;\theta;\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(X-\theta)^2}$$

$supp[X] = \mathbb{R}$
$\theta \in \mathbb{R}$
$\sigma^2 \in (0,\infty)$

A. ** let $\sigma^2$ be fixed, find Kernal

$$K(X;\theta;\sigma^2) \propto e^{\frac{-1}{2\sigma^2}(X-\theta)^2} = e^{\frac{-1}{2\sigma^2}(X^2 - 2X\theta + \theta^2)}$$

$$\propto e^{\frac{-X^2}{2\sigma^2} + \frac{X\theta}{\sigma^2}}$$

$$\propto e^{-X^2\left(\frac{1}{2\sigma^2}\right)} e^{X\left(\frac{\theta}{\sigma^2}\right)}$$

$$= e^{\alpha X^2} e^{\beta X}$$

• Note •
$\theta$ is also fixed/constant

$\alpha = {}^{-1}/_{2\sigma^2}$ $\quad \alpha \geq 0$
$\beta = {}^{\theta}/_{\sigma^2}$ $\quad \beta = \mathbb{R}$

- $\hat{\theta}_{MLE}$ for Normal

$$\mathcal{L}(\theta; X_1 \ldots X_n, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i-\theta)^2} = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\theta)^2}$$

$$\ell = \ln(\mathcal{L}) = n \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2} \cancel{\sum_{i=1}^{n}(X_i^2 - 2X_i\theta + \theta^2)}$$
$$\left(\sum X_i^2 - 2\theta\sum X_i + n\theta^2\right)$$

* $d/d\theta$

$$\ell' = 0 - {}^{1}/_{2\sigma^2}\left(0 - 2\sum X_i + 2n\theta\right) = \frac{\sum X_i}{\sigma^2} - \frac{n\theta}{\sigma^2}$$

Find max. $\quad \ell' = 0 = \frac{\sum X_i}{\sigma^2} - \frac{n\theta}{\sigma^2}$

$$\frac{n\theta}{\cancel{\sigma^2}} = \frac{\sum X_i}{\cancel{\sigma^2}}$$

$$\theta = \frac{\sum X_i}{n} = \bar{X} = \hat{\theta}_{MLE}$$

Note: $\sum X_i = n\bar{X}$

- $\underset{\text{fixed}}{P(\theta / X_1 \dots X_n ; \sigma^2)} = \dfrac{P(X \mid \theta, \sigma^2)\, P(\theta \mid \sigma^2)}{P(X \mid \sigma^2)} \propto P(X \mid \theta ; \sigma^2)\, P(\theta \mid \sigma^2)$

$$= \left[ \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^n e^{\frac{-1}{2\sigma^2} \Sigma (X_i - \theta)^2} \right] P(\theta / \sigma^2)$$

$$\propto e^{\frac{-1}{2\sigma^2} \Sigma X_i^2} \, e^{\frac{\theta \Sigma X_i}{\sigma^2}} \, e^{\frac{-n\theta^2}{\sigma^2}} \cdot P(\theta \mid \sigma^2)$$

- **Notes:** $X_i$ is now constant

$$\propto e^{-\left(\frac{n}{\sigma^2}\right)\theta^2} \, e^{\left(\frac{\Sigma X_i}{\sigma^2}\right)\theta} \cdot P(\theta \mid \sigma^2)$$

- $\Sigma X_i = n\bar{X}$

$$\propto \underbrace{e^{-\left(\frac{n}{\sigma^2}\right)\theta^2} \, e^{\left(\frac{n\bar{x}}{\sigma^2}\right)\theta}}_{\text{form of } K(\theta \mid X, \sigma^2)} \cdot P(\theta \mid \sigma^2)$$

- $a = \frac{-n}{2\sigma^2}$
- $b = \frac{n\bar{x}}{\sigma^2}$

- So $P(\theta \mid \sigma^2)$ must resemble $K(\theta \mid X ; \sigma^2)$ for <u>conjugate</u> prior + posterior

$$\propto \underbrace{\left( e^{a\theta^2} e^{b\theta} \right)}_{K(\theta \mid X ; \sigma^2)} \cdot \underbrace{\left( e^{c\theta^2} e^{d\theta} \right)}_{K(\theta \mid \sigma^2)} \propto N(\dots)$$

- let prior be Normal too : so $\theta / \sigma^2 \sim N(\underset{\text{prior mean}}{\mu_0}, \underset{\text{prior variance}}{\tau^2})$

$$P(\theta \mid \sigma^2) = \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{1}{2\tau^2}(\theta - \mu_0)^2} \propto e^{\left(\frac{-1}{\tau^2}\right)\theta^2} \, e^{\left(\frac{\mu_0}{2\tau^2}\right)\theta}$$

since $\mu_0$ fixed
$\tau^2$ fixed

- So

$$P(\theta \mid X_1 \dots X_n ; \sigma^2) \propto \underbrace{\left( e^{a\theta^2} e^{b\theta} \right)}_{\substack{\text{Norm}(\Sigma X_i, \sigma^2) \\ \text{(data)}}} \underbrace{\left( e^{c\theta^2} e^{d\theta} \right)}_{\substack{\text{Norm}(\mu_0, \tau^2) \\ \text{(prior)}}}$$

$a = {-n}/{2\sigma^2}$
$b = n\bar{x}/\sigma^2$
$c = {-1}/{2\tau^2}$
$d = \mu_0/\tau^2$

$$\propto e^{(a+c)\theta^2} \cdot e^{(b+d)\theta}$$

- coverting to form of kernal :

$$P(\theta \mid X ; \sigma^2) \propto N\left( \underbrace{\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}}_{\theta'} , \underbrace{\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}}_{\sigma^{2\prime}} \right)$$

$a + c$ : form of $-\frac{1}{2}\text{var}$
$b + d$ : form of mean/var

** Compare $a+c$ to $\alpha$
$b+d$ to $\beta$

from Kernal of Norm. distribution $K(X ; \theta, \sigma^2)$