Lecture 15

Note for fixed σ^2, θ as inferential target

$$X_1 \ldots X_n \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2/n) \quad \text{where } \theta = \bar{x}n$$

mean, variance

$$\theta \mid \sigma^2 \sim N(\mu_0, \tau^2)$$

$$\theta \mid X_1 \ldots X_n, \sigma^2 \sim N(\theta_p, \sigma_p^2)$$

where: $\theta_p = \left(\dfrac{\frac{\bar{x}n}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \right) \qquad \sigma_p^2 = \left(\dfrac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \right)$

- Point Estimation

$\hat{\theta}_{MMSE} = \theta_p$

$\hat{\theta}_{MAE} = \theta_p$

$\hat{\theta}_{MAP} = \theta_p$

$$= \frac{\frac{\bar{x}n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \overset{(\times 1)}{\left(\frac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}} \right)} + \frac{\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \overset{(\times 1)}{\left(\frac{\tau^2}{\tau^2} \right)}$$

$$= \frac{\bar{x}}{1 + \frac{\sigma^2}{n\tau^2}} \overset{(\times 1)}{\left(\frac{n\tau^2}{n\tau^2} \right)} + \frac{\mu_0}{\frac{n\tau^2}{\sigma^2} + 1} \overset{(\times 1)}{\left(\frac{\sigma^2}{\sigma^2} \right)}$$

$$= \underbrace{\left(\frac{\tau^2}{n\tau^2 + \sigma^2} \right)}\ \underbrace{\bar{x}} + \underbrace{\left(\frac{\sigma^2}{n\tau^2 + \sigma^2} \right)}\ \underbrace{\mu_0}$$

Shrinkage Estimator $= (1 - \rho)\ \hat{\theta}_{MLE} + (\rho)\ E(\theta)$

$$X|\theta \sim N(\theta, \sigma^2) \quad \theta = \bar{X}n$$

Reparameterization for Prior $\sim N(\mu_0, \tau^2)$

let: $\mu_0 = \bar{y} = \dfrac{\Sigma y_i}{n_0}$, $\tau^2 := \dfrac{\sigma^2}{n_0}$ again, where σ^2 is known

so $$\hat{\theta}_{MMSE} = \theta_p = \frac{\frac{\bar{X}n}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \underset{(\text{reparameter})}{=} \frac{\frac{\bar{X}n}{\sigma^2} + \frac{\bar{y}n_0}{\sigma^2}}{\underbrace{\frac{n}{\sigma^2} + \frac{n_0}{\sigma^2}}_{\frac{1}{var}}} \times \underbrace{\left(\frac{\sigma^2}{\sigma^2}\right)}_{(\times 1)}$$

$$= \hat{\theta}_p = \frac{\Sigma X_i + \Sigma Y_i}{n + n_0}$$ } sum of all data past (Y) + present (X)
} all trials past + present

so $$\theta | \sigma^2 \sim N\left(\mu_0 = \bar{y}, \tau^2 = \frac{\sigma^2}{n}\right)$$

$$\theta | X_1 \dots X_n, \sigma^2 \sim N\left(\tilde{\theta}_p = \frac{\Sigma X_i + \Sigma Y_i}{n + n_0}, \sigma_p^2 = \left(\sqrt{\frac{\sigma^2}{n+n_0}}\right)^2\right)$$

<u>Uninformative Priors</u>

- <u>laplace</u> prior / $$P(\theta | X_i, \sigma^2) = \frac{P(X_i|\theta,\sigma^2)P(\theta|\sigma^2)}{P(X|\sigma^2)} \propto P(X_i|\theta,\sigma^2)P(\theta|\sigma^2)$$

  <u>let</u> $P(\theta|\sigma^2) \propto 1$

  • improper $\sim N(\mu_0, \tau^2 \to \infty) = 0$ should

  $$\propto P(X_i|\theta,\sigma^2)\cdot 1 \propto \underbrace{e^{\left(\frac{\bar{x}n}{\sigma^2}\right)\theta} e^{-\left(\frac{n}{2\sigma^2}\right)\theta^2}}_{(\text{Kernal of Normal})} \propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

- so: laplace $(\theta|\sigma^2) \sim \lim\limits_{\tau^2 \to \infty} N(\mu_0, \tau^2) \propto 1$ • improper
  $n_0 > 0$

- $(\theta | X_i, \sigma^2) \sim N\left(\bar{x}, \frac{\sigma^2}{n}\right)$ • always proper

- Jeffrey's prior / $\mathcal{L} = k(X|\theta) = e^{\left(\frac{\bar{x}n}{\sigma^2}\right)\theta} e^{-\left(\frac{n}{2\sigma^2}\right)\theta^2}$

$\ln \mathcal{L}$ $\quad l = \left(\frac{\bar{x}n}{\sigma^2}\right)\theta - \left(\frac{n}{2\sigma^2}\right)\theta^2$

$(\theta; x, \sigma^2)$ $\quad l' = \frac{\bar{x}n}{\sigma^2} - \frac{2n\theta}{2\sigma^2}$

$(\theta; x, \sigma^2)$ $\quad l'' = -\frac{n}{\sigma^2}$

Fisher Info

$$I(\theta) = E[-l''(\theta; x)]$$
$$= E[n/\sigma^2] = \frac{n}{\sigma^2}$$

Jeffrey Protcol

$$P_J(\theta|\sigma^2) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\sigma^2}} \propto 1$$

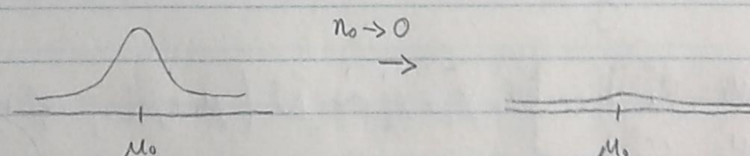
since σ^2 is constant
n is fixed

- so Jeffrey's $(\theta|\sigma^2) \sim \lim_{\tau^2 \to \infty} N(\mu_0, \tau^2) \propto 1$ $\quad n_0 > 1$ $\quad$ improper

- $(\theta | x_i, \sigma^2) \sim N(\bar{x}, \sigma^2/n)$ $\quad$ proper

- Haldane prior / assume no prior trials

- $\theta/\sigma^2 \sim \lim_{n_0 \to 0} N\left(\mu_0, \frac{\sigma^2}{n_0}\right) \sim N(\mu_0, \infty) \propto 1$ $\quad$ Improper



- $(\theta / x_i, \sigma^2) \sim N\left(\bar{x}, \frac{\sigma^2}{n}\right)$ $\quad$ proper

* Note: improper priors are similar to limits of proper priors

* Note: Laplace $\propto$ Jeffrey $\propto$ Haldane $\propto 1$

- <u>Posterior Predictive distribution</u>

given σ^2, $X_1 \dots X_n$, X^* (new data) with $n^* = 1$

$$P(X^* \mid X_1 \dots X_n, \sigma^2) = \int_\Theta P(X^* \mid \theta, \sigma^2)\, P(\theta \mid X_1 \dots X_n, \sigma^2)\, d\theta$$

$$= \int_{\mathbb{R}} N(\theta, \sigma^2) \cdot N(\theta_p, \sigma_p^2)\, d\theta$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(X^* - \theta)^2} \cdot \frac{1}{\sqrt{2\pi\sigma_p^2}} e^{\frac{-1}{2\sigma_p^2}(\theta - \theta_p)^2} d\theta \quad \Big\}$$ possible integral

By convolution

$$S = X_1 + X_2 \sim \int_{\mathrm{supp}[S]} f_{X_1}(x)\, f_{X_2}(S - x)\, dx$$

$$X_1 + X_2 = f_{X_1} * f_{X_2} = \text{"} \qquad \text{"}$$

By moment Generating functions

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^2) \\ X_2 \sim N(\mu_2, \sigma_2^2) \end{array}\right\} X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

so $$X^* \mid X_1 \dots X_n, \sigma^2 \sim N(\overset{\mu_1}{\theta_p} + \overset{\mu_2}{0}, \overset{\sigma_1^2}{\sigma_p^2} + \overset{\sigma_2^2}{\sigma^2})$$

$$\sim N(\underset{\text{mean}}{\bar{x}}, \underset{\text{var}}{\frac{\sigma^2}{n} + \sigma^2})$$

as $\lim_{n \to \infty} N(\bar{x}, \frac{\sigma^2}{n} + \sigma^2) = N(\bar{x}, \sigma^2)$

// End of Midterm 2

B. $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

Now: θ is Known

σ^2 is inferred target

- MLE of σ^2?

$$f(\sigma^2; X_i, \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i - \theta^2)} = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{\frac{-1}{2\sigma^2}\sum(X_i-\theta)^2}$$

$$\ln(f) = l = n \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}\left[\sum(X_i-\theta)^2\right]$$

$$\frac{-n}{2}\left[\ln(2\pi) + \ln(\sigma^2)\right]$$

SSE : Sum of squared error

$$\frac{dl}{d\sigma^2} = l' = 0 - \frac{n}{2}\cdot\frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2}\cdot SSE = 0 \quad \left[\begin{matrix}\text{to find} \\ \theta_{MLE}\end{matrix}\right]$$

$$\left(\frac{n}{2\sigma^2} = \frac{1}{2(\sigma^2)^2}\cdot SSE\right)_{\times 2\sigma^2} \Rightarrow \left(n = \frac{SSE}{\sigma^2}\right) \Rightarrow \boxed{\left(\hat{\sigma}^2_{MLE} = \frac{\sum(X_i-\theta)^2}{n}\right)}$$

if $\theta \sim Gamma(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\theta} \theta^{\alpha-1}$, if $y = \frac{1}{\theta}$

then $Y \sim$??

$$y = t(\theta) = \frac{1}{\theta}$$

$$\theta = t^{-1}(y) = \frac{1}{y}$$

$$\frac{d}{dy}\left[t^{-1}(y)\right] = \frac{-1}{y^2}$$

$$f_y(y) = f_\theta\left(t^{-1}(y)\right)\left|\frac{d}{dy}\left[t^{-1}(y)\right]\right|$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\left(\frac{1}{y}\right)} \underbrace{\left(\frac{1}{y}\right)^{\alpha-1}}_{y^{1-\alpha}} \underbrace{\left|-\frac{1}{y^2}\right|}_{y^{-2}}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{\frac{-\beta}{y}} y^{-(\alpha+1)}$$

$\propto$ Inverse Gamma (α, β)