Lecture 16

Recall: $\hat{\sigma}^2_{MLE} = \frac{\overbrace{\Sigma(X_i-\theta)^2}^{SSE}}{n}$ so $SSE = \hat{\sigma}^2_{MLE}\, n$; * Note: θ is fixed, σ^2 variable

Posterior $P(\sigma^2 | X_i, \theta) \propto P(X_i | \sigma^2, \theta) \cdot P(\sigma^2 | \theta)$

$X_i \sim X_1 \ldots X_n$ iid

$$\propto \left[\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(X_i-\theta)^2}\right] \cdot P(\sigma^2|\theta)$$

$$= \left[\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{\frac{-1}{2\sigma^2}\underbrace{\Sigma(X_i-\theta)^2}_{SSE}}\right] \cdot P(\sigma^2|\theta)$$

$$= \left[\left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sqrt{\sigma^2}}\right)^n \cdot e^{-\frac{\hat{\sigma}^2_{MLE}\, n}{2\sigma^2}}\right] \cdot P(\sigma^2|\theta) \propto \underbrace{\left[\left(\frac{1}{\sqrt{\sigma^2}}\right)^n \cdot e^{-\left(\frac{n\hat{\sigma}^2_{MLE}}{2}\right)\frac{1}{\sigma^2}}\right]}_{K(X|\sigma^2,\theta)} \cdot K(\sigma^2|\theta)$$

$$= \left[{\sigma^2}^{\left(-\frac{n}{2}\right)} e^{-\left(\frac{n\hat{\sigma}^2}{2}\right)\frac{1}{\sigma^2}}\right] \cdot \left[{\sigma^2}^{-\alpha-1} e^{-\beta\frac{1}{\sigma^2}}\right] \Big\} \text{ conjugate prior}$$

$\sigma^2/\theta \sim \text{Inversegamma}(\alpha, \beta)$

$$= \left[{\sigma^2}^{-\left(\frac{n}{2}+\alpha\right)-1} e^{-\left(\frac{n\hat{\sigma}^2}{2}+\beta\right)\frac{1}{\sigma^2}}\right] \propto \text{Invgamma}\left(\frac{n}{2}+\alpha, \frac{n\hat{\sigma}^2}{2}+\beta\right)$$

$P(\sigma^2|X_i, \theta) =$

Note: $\sigma^2 | \theta \sim \text{Inv gamma}(\alpha, \beta)$

$\sigma^2 | \theta, X \sim \text{Inv gamma}\left(\frac{n}{2}+\alpha, \frac{n\hat{\sigma}^2}{2}+\beta\right)$

canonical: let $\alpha = \frac{n_0}{2}$, $\beta = \frac{n_0\hat{\sigma}_0^2}{2}$

Prior: $\sigma^2 / \theta \sim \text{Invgamma}\left(\frac{n_0}{2}, \frac{n_0\hat{\sigma}_0^2}{2}\right)$ } pseudotrials / variance

Posterior: $\sigma^2 | \theta, X \sim \text{Invgamma}\left(\frac{n+n_0}{2}, \frac{n\hat{\sigma}^2 + n_0\hat{\sigma}_0^2}{2}\right)$

- $X_1 \dots X_n \overset{iid}{\sim} N(\theta, \sigma^2)$

Point Estimation /

- $\sigma^2 / \theta, X_i \sim \text{Inv gamma}\left(\underbrace{\frac{n+n_0}{2}}_{\alpha'}, \underbrace{\frac{n\hat{\sigma}^2 + n_0\hat{\sigma}_0^2}{2}}_{\beta'}\right)$

Note

$\hat{\sigma}^2 = \frac{SSE}{n}$

if $Y \sim \text{Invgamma}(\alpha, \beta)$

- $E[Y] = \frac{\beta}{\alpha - 1}$ $\quad \sigma^2_{MMSE} = E[\sigma^2 | \theta, X] = \frac{\beta'}{\alpha' - 1} = \frac{(n\hat{\sigma}^2 + n_0\hat{\sigma}_0^2)/2}{(n+n_0)/2 - 1} \left(\frac{2}{2}\right)$

$$\boxed{\sigma^2_{MMSE} = \frac{n\hat{\sigma}^2 + n_0\hat{\sigma}_0^2}{n + n_0 - 2}}$$

- $\text{Mode}[Y] = \frac{\beta}{\alpha + 1}$ $\quad \sigma^2_{MAP} = \text{Mode}[\sigma^2 | \theta, X] = \frac{(n\hat{\sigma}^2 + n_0\hat{\sigma}_0^2)/2}{(n+n_0)/2 + 1} \cdot \left(\frac{2}{2}\right)$

$$\boxed{\sigma^2_{MAP} = \frac{n\hat{\sigma}^2 + n_0\hat{\sigma}_0^2}{n + n_0 + 2}}$$

- $\text{Med}[Y] =$ No closed form $\quad \boxed{\sigma^2_{MAE} = \text{qinvgamma}(0.5, \alpha', \beta')}$

Uninformative Priors /

- **Jeffrey's:**

$\mathcal{L}(\sigma^2; \theta, X_i) = e^{-\frac{SSE}{2}\cdot\frac{1}{\sigma^2}} \sigma^{2^{-\frac{n}{2}}}$

$\ell = \ln \mathcal{L} = -\frac{SSE}{2}\frac{1}{\sigma^2} - \frac{n}{2}\ln(\sigma^2)$

$\frac{d}{d\sigma^2}$ $\quad \ell' = \frac{SSE}{2} \frac{1}{(\sigma^2)^2} - \frac{n}{2} \cdot \frac{1}{\sigma^2}$

$\frac{d}{d\sigma^2}$ $\quad \ell'' = -SSE\frac{1}{(\sigma^2)^3} + \frac{n}{2} \cdot \frac{1}{(\sigma^2)^2}$

$I(\theta) = E_X[-\ell''(\sigma^2; X_i, \theta)]$

$= E_X\left[\frac{SSE}{(\sigma^2)^3} - \frac{n}{2(\sigma^2)^2}\right]$

$= \underbrace{E_X[SSE]}(\sigma^2)^{-3} - \frac{n}{2}(\sigma^2)^{-2}$

$E\left[\sum_1^n (X_i - \theta)^2\right] = \sum_1^n E[(X_i - \theta)^2] = \sum_1^n \sigma^2 = n\sigma^2$

$= n\sigma^2(\sigma^2)^{-3} - \frac{n}{2}(\sigma^2)^{-2}$

$= n(\sigma^2)^{-2} - n/2(\sigma^2)^{-2}$

- $I(\theta) = (\sigma^2)^{-2}\left(\frac{2n}{2} - \frac{n}{2}\right) = \boxed{(\sigma^2)^{-2}\left(\frac{n}{2}\right)}$

- $P_J(\sigma^2) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{2}(\sigma^2)^{-2}} \propto \sqrt{(\sigma^2)^{-2}} = (\sigma^2)^{-1} = (\sigma^2)^{0-1} e^0$

- $P_J(\sigma^2) \propto \text{Inv gamma}(0, 0)$

- Haldane: $\sigma^2 | \theta \sim \text{Invgamma}(0,0)$

$$\sigma^2 | \theta, X_i \sim \text{Invgamma}\left(\frac{n}{2}+0, \frac{n\hat{\sigma}^2}{2}+0\right)$$ always proper $n \geq 1$; $\hat{\sigma}^2 > 0$

- Indifference: ?

C. $X_1 \ldots X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ let: θ, σ^2 are both inferred targets

Posterior/ $P(\theta, \sigma^2 | X_i) \propto P(X | \theta, \sigma^2) \cdot P(\theta, \sigma^2)$

$$= \left[\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(X_i-\theta)^2}\right] \cdot P(\theta, \sigma^2)$$

$$\propto \left[{\sigma^2}^{\frac{-n}{2}} e^{\frac{-1}{2\sigma^2}\Sigma(X_i-\theta)^2}\right] \cdot P(\theta, \sigma^2)$$

Not invgamma b/c θ is unknown; Kernel of Normal-Inverse gamma

$$\Sigma(X_i-\theta)^2 = \Sigma(X_i - \bar{x} - \theta + \bar{x})^2 = \Sigma\left((X_i - \bar{x}) + (\bar{x} - \theta)\right)^2$$

$$= \Sigma\left[(X_i-\bar{x})^2 + 2(X_i-\bar{x})(\bar{x}-\theta) + (\bar{x}-\theta)^2\right]$$

$$= \Sigma(X_i-\bar{x})^2 + 2\Sigma(-X_i\theta + X_i\bar{x} + \bar{x}\theta - \bar{x}^2) + \Sigma(\bar{x}-\theta)^2$$

$$+(2)\ -\bar{x}n\theta + n\bar{x}^2 + n\bar{x}\theta - n\bar{x}^2$$

$$= \Sigma(X_i-\bar{x})^2 - 0 + \Sigma(\bar{x}-\theta)^2$$

$$\Sigma(X_i-\theta)^2 = s^2(n-1) + n(\bar{x}-\theta)^2$$

Note

$\bar{x} = \frac{\Sigma X_i}{n}$

- $\Sigma X_i = \bar{x}n$

$s^2 = \frac{\Sigma(X_i-\bar{x})^2}{n-1}$

- $s^2(n-1) = \Sigma(X_i-\bar{x})^2$

s^2: sample variance

$$P(\theta, \sigma^2 | X_i) \propto {\sigma^2}^{\frac{-n}{2}} e^{\frac{-1}{2\sigma^2}\Sigma(X_i-\theta)^2} \cdot \left[P(\theta, \sigma^2)\right]$$

$$\propto \underset{\alpha}{{\sigma^2}^{-\left(\frac{n}{2}\right)}}\ \underset{\beta}{e^{-\left(\frac{s^2(n-1)}{2}\right)\frac{1}{\sigma^2}}}\ \underset{\lambda,\ \mu}{e^{-\frac{n}{2\sigma^2}(\bar{x}-\theta)}} \cdot \left[P(\theta, \sigma^2)\right]$$

- $P(\theta, \sigma^2 | X) \propto \text{NormInvgamma}\left(\mu = \bar{x}, \lambda = n, \alpha = \frac{n}{2}, \beta = \frac{(n-1)s^2}{2}\right)$

** $P(\theta, \sigma^2)$ must be a norminvgamma also