Lecture 13

X_1 /

$X \sim \text{Poisson}(\theta) = \dfrac{\theta^x e^{-\theta}}{x!}$ recall $\theta \to 0$, $n \to \infty$

$(P(X|\theta))$ (for a single X result)

- We want to find form of $P(\theta|X)$

- $P(\theta|X) = \dfrac{P(X|\theta)\,P(\theta)}{P(X)} \propto P(X|\theta)\,P(\theta) = \dfrac{e^{-\theta}\theta^{x}}{X!}P(\theta)$

$$\propto \underbrace{e^{-\theta}\theta^{x}}_{K(X|\theta)}\ \underbrace{P(\theta)}_{K(\theta)}$$

we want $K(\theta)$ to match $K(X|\theta)$ form

* conjugate prior engineering *

so $\propto \left(e^{-\theta}\theta^{x}\right)\left(e^{-b\theta}\theta^{a}\right)$ such that $\left[= \underbrace{e^{-\theta(b+1)}\theta^{a+x}}_{K(\theta|X)}\right]$

Note: $P(\theta) = \frac{1}{c}K(\theta) \Rightarrow \int_{\Theta} P(\theta) = \frac{1}{c}\int_{\Theta} K(\theta) \Rightarrow 1 = \frac{1}{c}\int K(\theta)\,d\theta$

- find constant for kernel

- $c = \int_{\Theta} K(\theta)\,d\theta = \int_{\theta=0}^{\infty} e^{-b\theta}\theta^{a}\,d\theta \overset{\text{u-sub.}}{\Rightarrow} \int_{u=0}^{\infty} e^{-u}\dfrac{u^a}{b^a}\cdot\dfrac{1}{b}\,du$

let $u = b\theta$, $\theta = \frac{u}{b}$, $d\theta = \frac{1}{b}du$

limits

$\theta = 0$; $u = b\cdot 0$ $u = 0$

$\theta = \infty$; $u = b\cdot\infty$ $u = \infty$

$$= \frac{1}{b^{(a+1)}}\int_0^{\infty} e^{-u}u^{a}\,du = \frac{1}{b^{(a+1)}}\underbrace{\int_0^{\infty} u^{(\alpha+1)-1}e^{-u}}_{\Gamma(a+1)}$$ gamma function

$c = \dfrac{\Gamma(a+1)}{b^{(a+1)}}$ and $\dfrac{1}{c} = \dfrac{b^{(a+1)}}{\Gamma(a+1)}$

since $\alpha - 1$ is defined in gamma we can drop "+1" in $P(\theta)$ such that

- $\theta \sim \text{Gamma}(\alpha, \beta) := \dfrac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}$

- $\theta|X \propto \underbrace{\left(\theta^{x}e^{-\theta}\right)}_{K(X|\theta)}\underbrace{\left(\theta^{\alpha-1}e^{-\beta\theta}\right)}_{P(\theta)} \propto \text{Gamma}(\underbrace{x+\alpha}_{\alpha'}, \underbrace{1+\beta}_{\beta'})$

$\theta \sim \text{Gamma}(\alpha, \beta)$

OR $\theta | X \sim \text{Gamma}(\alpha', \beta')$

For: $Y \sim \text{Gamma}(\alpha', \beta') = \frac{\beta^{\alpha'}}{\Gamma(\alpha')} Y^{\alpha'-1} e^{-\beta' Y}$

$supp[Y] = (0, \infty)$
$\alpha > 0$
$\beta > 0$

- Point Estimation

find by:

* $\hat{\theta}_{MMSE} = E[Y] = E[\theta | X] = \frac{\alpha'}{\beta'}$ {• $\int y f(y)$

* $\hat{\theta}_{MAP} = \text{Mode}[Y] = \text{Mode}[\theta | X] = \frac{\alpha' - 1}{\beta'}$ $(\alpha \geq 1)$ {• argmax $\{f(y)\}$ • OR $\frac{d}{dy}[\ell(y)] = 0$

* $\hat{\theta}_{MAE} = \text{Med}[Y] = \text{Med}[\theta | X] =$ qgamma$(0.5, \alpha', \beta')$ {• Run in 'R'

Note: $Var[Y] = \frac{\alpha'}{\beta'^2}$ {• using $E[Y^2] - E[Y]^2$

$X_1, \dots X_n$

- $X_1, \dots, X_n \sim \text{Poisson}(\theta) = \frac{e^{-\theta} \theta^X}{X!}$

$$P(\theta | X) = \frac{P(X|\theta) P(\theta)}{P(X)} \propto P(X|\theta) P(\theta) = \left(\prod_{i=1}^{n} P(X_i | \theta) \right) P(\theta)$$

$$= \left(\prod_{i=1}^{n} \frac{\theta^{X_i} e^{-\theta}}{X_i!} \right) P(\theta) \propto \left(\frac{\theta^{\Sigma X_i} e^{-n\theta}}{\prod_{i=1}^{n} X_i!} \right) \cdot K(\theta)$$

$$\propto \underbrace{\left(\theta^{\Sigma X_i} e^{-n\theta} \right)}^{K(X|\theta)} \underbrace{\left(\theta^{\alpha-1} e^{-\beta\theta} \right)}^{k(\theta)} = \theta^{\Sigma X_i + \alpha - 1} e^{-(n+\beta)\theta}$$

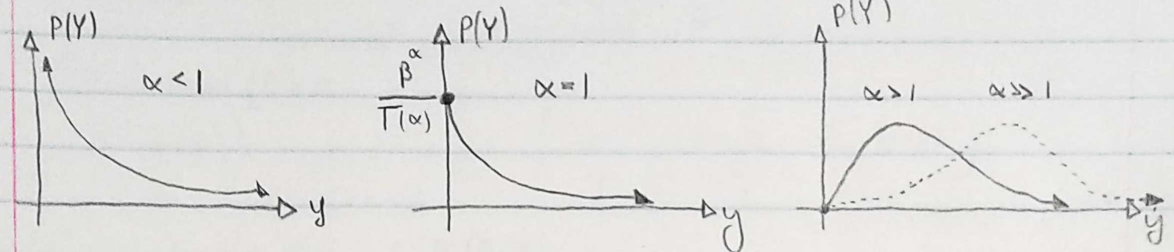
- $\boxed{\theta | X_1 \dots X_n \sim \text{Gamma}(\underbrace{\Sigma X_i + \alpha}_{\alpha'}, \underbrace{n + \beta}_{\beta'})}$

* $\hat{\theta}_{MMSE}$ (Avg) $= \frac{\Sigma X_i + \alpha}{n + \beta} = \frac{\alpha'}{\beta'}$

* $\hat{\theta}_{MAP}$ (Mode) $= \frac{\Sigma X_i + \alpha - 1}{n + \beta} = \frac{\alpha' - 1}{\beta'}$ $(\alpha' \geq 1)$

* $\hat{\theta}_{MAE}$ (Median) $=$ qgamma$(0.5, \underbrace{\Sigma X_i + \alpha}_{\alpha'}, \underbrace{n + \beta}_{\beta'})$

$$Y \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} Y^{\alpha-1} e^{-\beta Y}$$



- We want $P(\theta) \propto 1$; that is $\theta \sim U(0, \infty)$ which is invalid.
  Assume " " is true, then

$$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)} \propto P(X|\theta)P(\theta) \propto P(X|\theta)\cdot 1 \propto P(X|\theta) \propto \theta^{\Sigma X_i} e^{-n\theta}$$

$\theta | X \sim \text{Gamma}(\Sigma X_i + 1, n)$ * Posterior is proper

since $\Sigma X_i + \alpha$, $n + \beta$

$\theta \sim \text{Gamma}(1, 0)$ * since $\beta > 0$; Improper Prior

*** laplace / Indifference prior single event

*** haldane Prior : No events, Zero-Knowledge

$\theta \sim \text{Gamma}(0, 0)$

- MLE / Fisher info for Poisson

$$f(\theta; X) = \frac{e^{-n\theta}\theta^{\Sigma X_i}}{\prod_{i=1} X_i!}$$

$$l(\theta, X) = -n\theta + \Sigma X_i \ln(\theta) - \ln\left(\prod_{i=1}^{n} X_i!\right)$$

$$\left(\frac{d}{d\theta}\right) = l' = -n + \frac{\Sigma X_i}{\theta} - 0$$

- MLE $l' = -n + \frac{\Sigma X_i}{\theta} = 0$

$$n = \frac{\Sigma X_i}{\theta}$$

$$\theta = \frac{\Sigma X_i}{n} = E[X] = \bar{x}$$

$$\left(\frac{d}{d\theta}\right)^2 = l'' = -\frac{\Sigma X_i}{\theta^2}$$

*** Jeffrey's Prior

- Fisher info.

$$I(\theta) = E[-l''(\theta; X)] = E\left[\frac{\Sigma X_i}{\theta^2}\right] = \frac{1}{\theta} E[\Sigma X_i]$$

$$I(\theta) = \frac{1}{\theta^2} \cdot n\theta$$

$$P_J(\theta) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\theta}} \propto \sqrt{\frac{1}{\theta}} = \theta^{-1/2}$$

$$\propto \theta^{-1/2} e^0$$

$$\propto \text{Gamma}\left(\tfrac{1}{2}, 0\right)$$

* improper