Lec 17 Math 341 4/23/10

We were talking about sampling realizations from a discrete distr.

$X \sim \text{Binom}(6, 0.4)$

$\text{Supp}(X) = \{0, 1, \ldots, 6\}$

Calculate:

$F(0) = p(0) = .047$

$F(1) = p(1) + F(0) = .233$

$F(2) = p(2) + F(1) = .544$

$F(3) = .821$

$F(4) = .959$

$F(5) = .996$

$F(6) = 1.00$

draw u from $U \sim U(0,1)$.

if $u \in [0, .047) \Rightarrow x = 0$

$u \in [.047, .233) \Rightarrow x = 1$

$\vdots$

$u \in [.996, 1.00] \Rightarrow x = 6$

What about $X \sim$ cont. distr?

Note $F(x)$ cont.

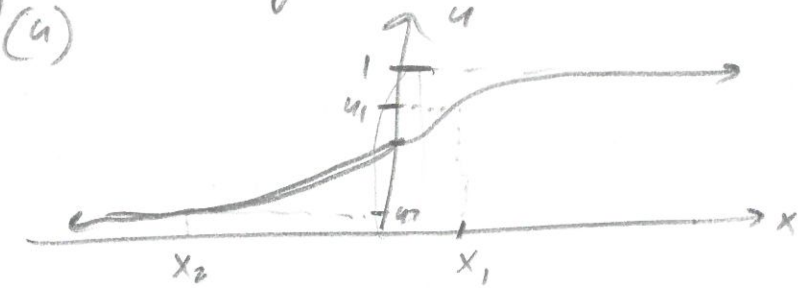
How is $F(X)$ distr? $\quad Y = t(X) = F(X)$

$$\overset{\nearrow \text{supp} [0,1]}{F_Y(y)} = P(Y \le y) = P(F_X(X) \le y) = P(X \le F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

What r.v. has CDF y and support $[0,1]$? $Y \sim U(0,1) \Rightarrow F_X(X) \sim U(0,1)$

e.g $X \sim N(0,1)$

$\Rightarrow \quad X = F^{-1}(u)$



Draw u from $U(0,1)$ and let $x = F^{-1}(u)$.

This is useful if F^{-1} is known. What if it's not?

$\Rightarrow$ Approx. the conv. r.v. with a discrete r.v.. First pick $x_{min}, x_{max}, \Delta x$

Let $G = \langle x_{min}, x_{min} + \Delta x, x_{min} + 2\Delta x, \ldots, x_{max} \rangle$ with g total elements

$\uparrow$ the grid

Calc $F(x_{min}) = \int_{-\infty}^{x_{min}} f_X(x)\,dx \approx$ Riemann sum

$F(x_{min} + \Delta x) = \int_{x_{min}}^{x_{min}+\Delta x} f_X(x)\,dx + F(x_{min}) \approx$ Riemann sum

$\vdots$

$F(x_{max}) = \int_{x_{min}+(g-1)\Delta x}^{x_{max}} f_X(x)\,dx + F(x_{min} + (g-1)\Delta x) \approx$ Riemann sum

then use previous algorithm for discrete r.v.'s.

How to sample from $P(x,y)$ when X, Y is discrete?

Imagine...

$P(x, y)$

$P(Y|X=1)$

		Y: 2	4	6	
X	1	0.2	0.05	0.1	0.35
	3	0.1	0.05	0.1	0.25
	5	0.2	0.1	0.1	0.4
					1

$\}$ $P(X)$

Recall: $P(x,y) = P(y|x)\,P(x)$ Bayes Rule

To sample...

① Sample x_0 from $P(x)$ using discrete alg.

② sample y_0 from $P(y \mid x = x_0)$

③ Return $\langle x_0, y_0\rangle$ as the 2-d realization

Can we do this with the Norm Inv Gamma? If $P(\theta,\sigma^2) \propto \frac{1}{\sigma^2}$ σ affecting prior, nothing else considered

$$P(\theta,\sigma^2|x) \propto (\sigma^2)^{-n/2-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{1}{2\sigma^2/n}(\bar{x}-\theta)^2} \propto \text{NormInvGamma}\left(\bar{x}, n, \frac{n}{2}, \frac{(n-1)s^2}{2}\right)$$

$$P(\theta,\sigma^2|x) = P(\theta|x,\sigma^2)\,P(\sigma^2|x) \quad \text{Bayes Rule}$$

$$\Rightarrow P(\theta|x,\sigma^2) = \frac{P(\theta,\sigma^2|x)}{P(\sigma^2|x)} \propto P(\theta,\sigma^2|x) \propto e^{-\frac{1}{2\sigma^2/n}(\bar{x}-\theta)^2} \propto N\left(\bar{x}, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$$

$$\Rightarrow P(\sigma^2|x) = \frac{P(\theta,\sigma^2|x)}{P(\theta|x,\sigma^2)} \propto \frac{(\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}{\frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} \cancel{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\bar{x}-\theta)^2}}}$$

$$-\frac{n}{2}+\frac{1}{2}-1 = -\frac{n-1}{2}-1$$

$$-\frac{n}{2}-\frac{1}{2}+1-1$$

$$\propto \frac{(\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}}{(\sigma^2)^{-\frac{1}{2}}} = (\sigma^2)^{-\frac{n}{2}-\frac{1}{2}} e^{-\frac{(n-1)s^2/2}{\sigma^2}}$$

$$\propto \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

To sample from $N(\theta, \sigma^2 | X)$

① Sample σ_0^2 from $\text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

② Sample θ_0 from $P(\theta | X, \sigma^2 = \sigma_0^2) = N\left(\bar{x}, \left(\frac{\sigma_0}{\sqrt{n}}\right)^2\right)$

③ Return $\langle \theta_0, \sigma_0^2 \rangle$

$\Rightarrow$ No need to ever work with NormInvGamma directly!

Also note, we solved for $P(\sigma^2 | X)$. What is that?

$$P(\sigma^2 | X) = \int_{\mathbb{R}} P(\sigma^2, \theta | X) \, d\theta$$

It's the posterior of σ^2 with the uncertainty inherent in ignorance of θ "averaged" out or marginalized out

Note: $Y \sim \text{InvGamma}(\alpha, \beta)$
$\Rightarrow \text{Var}(Y) = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$

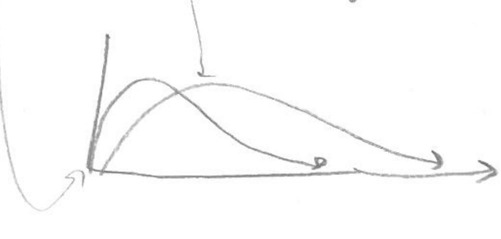
$$P(\sigma^2 | X, \theta) = \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2}{2}\right)$$

$$P(\sigma^2 | X) = \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

need θ to compute...

If $s^2 = \hat{\sigma}^2$

$$\text{Var}[\sigma^2 | X, \theta] = \frac{\left(\frac{n\hat{\sigma}^2}{2}\right)^2}{\left(\frac{n}{2}-1\right)^2\left(\frac{n}{2}-2\right)} = \ldots =$$



more uncertainty!

do for HW...

Also...
What is
$P(\theta|x)$?

It's the posterior of θ with the ~~uncertainty~~ in σ^2 averaged/marginalized out..

This is usually of great interest. σ^2 is a nuisance parameter.

$$P(\theta|x) = \int_0^\infty P(\theta,\sigma^2|x)\, d\sigma^2 \quad \text{or} \quad P(\theta|x) = \frac{P(\theta,\sigma^2|x)}{P(\sigma^2|\theta,x)}$$

Before we get there... let's do some math 241...

If $X_1, \ldots X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$$\Rightarrow \frac{\bar{X} - \theta}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

$$\Rightarrow \frac{\bar{X} - \theta}{\frac{s}{\sqrt{n}}} \sim ?$$

Makes "sense" to use if σ unknown. "Student" did this in the early 1900's.

Define $V \sim T_n := \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{v^2}{n}\right)^{-\frac{n+1}{2}}$ the Student's T distr.

AKA
Student
T distr

It can be shown that

$$\frac{\bar{X} - \theta}{\frac{s}{\sqrt{n}}} \sim T_{n-1}$$

let $W = \sigma V + \mu = t(v)$

$$v = t^{-1}(w) = \frac{w - \mu}{\sigma}$$

$$f_W(w) = f_V\left(t^{-1}(w)\right)\left|\frac{d}{dw}\left[t^{-1}(w)\right]\right|$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{\left(\frac{w-\mu}{\sigma}\right)^2}{n}\right)^{-\frac{n+1}{2}} \frac{1}{\sigma}$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n \sigma^2}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{1}{n}\left(\frac{w-\mu}{\sigma}\right)^2\right)^{-\frac{n+1}{2}} := T_n(\mu, \sigma)$$

the noncentered and scaled
T distr.

AKA
non-standard
T distr

$$= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2}\right)\frac{1}{\sigma^2}$$

$$p(\theta|x) = \frac{p(\theta,\sigma^2|x)}{p(\sigma^2|\theta,x)} \propto \frac{(\sigma^2)^{-\frac{n}{2}}\, e^{-\frac{1}{2\sigma^2}\overbrace{\sum(x_i-\theta)^2}^{n\hat{\sigma}^2}}\,(\sigma^2)^{-1}}{\frac{\left(\frac{n\hat{\sigma}^2}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)}(\sigma^2)^{-\frac{n}{2}-1}\, e^{-\frac{n\hat{\sigma}^2/2}{\sigma^2}}}$$

$$\propto \frac{\cancel{(\sigma^2)^{-\frac{n}{2}-1}}\ \cancel{e^{-\frac{n\hat{\sigma}^2}{2\sigma^2}}}}{\left(\frac{n\hat{\sigma}^2}{2}\right)^{n/2}\cancel{(\sigma^2)^{-\frac{n}{2}-1}}\ \cancel{e^{-\frac{n\hat{\sigma}^2/2}{\sigma^2}}}} = \left(\frac{n\hat{\sigma}^2}{2}\right)^{-n/2}$$

Recall $n\hat{\sigma}^2 = \ldots = (n-1)s^2 + n(\bar{x}-\theta)^2$

$$= \left(\frac{(n-1)s^2}{2} + \frac{n(\bar{x}-\theta)^2}{2}\right)^{-n/2}$$

const ... only a $f(x)$!!!

$$\propto \left(\frac{1}{\frac{(n-1)s^2}{2}}\right)^{-n/2}\left(\frac{(n-1)s^2}{2} + \frac{n(\bar{x}-\theta)^2}{2}\right)^{-n/2}$$

$$= \left(1 + \frac{\frac{n(\bar{x}-\theta)^2}{2}}{\frac{(n-1)s^2}{2}}\right)^{-n/2}$$

$$= \left(1 + \frac{1}{n-1}\left(\frac{\bar{x}-\theta}{\frac{s}{\sqrt{n}}}\right)^2\right)^{-n/2} \propto T_{n-1}\left(\bar{x}, \frac{s}{\sqrt{n}}\right)$$