Lecture 8 - Inference Goal #2

• Frequentist - Confidence Intervals

$$CI_{\theta, 1-\alpha_0} = \left[\hat{\theta}_{MLE} \pm Z_{\frac{\alpha}{2}} \cdot SE(\hat{\theta}_{MLE}) \right]$$

θ: for θ value
$1-\alpha_0$: with this confidence level

* for iid Bernoulli: $= \left[\hat{p} \pm Z_{\frac{\alpha_0}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right]$

ex: $n=2$

$x=1$

$\hat{\theta}_{MLE} = 0.5$

$$CI_{0,95\%} = \left[\frac{1}{2} \pm \underbrace{Z_{25\%}}_{=2} \sqrt{\frac{(0.5)(1-0.5)}{2}} \right]$$

95%: generally considered credible

$= [-0.21, 1.21]$

* which is invalid since $\theta \in \Theta = (0,1)$
* Breaksdown at Z; p is not normal-like

•• Bayesian - Credible Regions

$$CR_{\theta, 1-\alpha_0} := \left[\text{Quantile}\left[\theta|X, \frac{\alpha_0}{2}\right], \text{Quantile}\left[\theta|X, 1-\frac{\alpha_0}{2}\right] \right]$$

* for Beta of binomial $= \left[\text{qbeta}\left(\frac{\alpha_0}{2}, x+\alpha, n-x+\beta\right), \text{qbeta}\left(1-\frac{\alpha_0}{2}, x+\alpha, n-x+\beta\right) \right]$

Note:

qBeta(q, α, β) solves for x in the integral $\int_0^x \frac{1}{B(\alpha,\beta)} t^{\alpha-1}(1-t)^{\beta-1} dt = q$

similarly

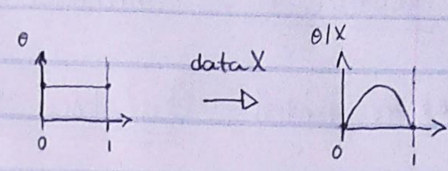
qnormal(q, μ, σ^2) solves for x in $\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(t-\mu)^2} dt = q$ ← quantile

* Why credible regions work:

prior: $\theta \sim Beta(\alpha, \beta)$

likelihood: $X|\theta \sim Binomial(n, \theta)$ (n fixed)

posterior: $\theta|X \sim Beta(\alpha + x, \beta + n - x)$



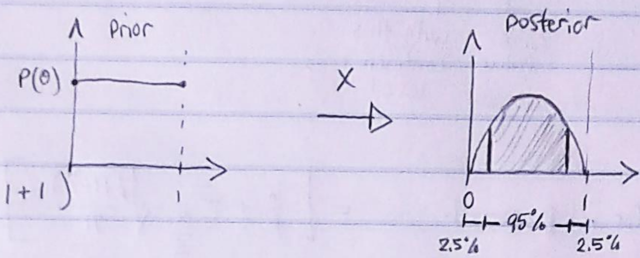
ex: $n = 2$

$x = 1$

$\theta \sim U(0,1) = Beta(1,1)$

so: $\theta|X \sim Beta(1+1, 2-1+1)$

$\sim Beta(2,2)$



- [2 sided credible region]

So $CR_{\theta, 95\%} = [\,qbeta(0.025, 2, 2),\ qbeta(0.975, 2, 2)\,]$

$= [0.094, 0.906] \subset [0, 1]$

** Better at increasing certainty

- [1 sided credible regions]

- Left sided /

$CR_{L,\theta,1-\alpha_0} := [\,-\infty,\ Quantile(\theta|X, 1-\alpha_0)\,]$ — up to $1-\alpha_0$

** Beta binomial $= [\,0,\ qbeta(1-\alpha_0,\ x+\alpha,\ n-x+\beta)\,]$

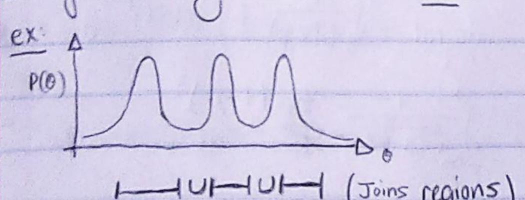
- Right sided /

$CR_{R,\theta,1-\alpha_0} := [\,Quantile(\theta|X, \alpha_0),\ \infty\,]$

** Beta binomial $= [\,qbeta(\alpha_0,\ x+\alpha,\ n-x+\beta),\ 1\,]$

- [High Density Regions] - HDR

ex:



disadvantages

- computationally expensive
- non-continuous region

Note: $P(\theta \in CR_{\theta, 1-\alpha_0}) = 1 - \alpha_0$ and $P(\theta \in HDR_{\theta, 1-\alpha_0}) = 1 - \alpha_0$

End of Midterm 1

Goal 3 – Hypothesis / Theory testing

/ Midterm 2 Begins Here

Generally we want to show/prove H_α (alternative hypothesis) is true

- 1st/ Assume the opposite : $\underline{\underline{H_0}}$ (the null hypothesis)
- 2nd/ deduce evidence against H_0 via data
  an "overwhelming" amount (accounted for by $\underline{\underline{\alpha_0}}$)
- 3rd/ if data is overwhelming against : Reject H_0, accept H_α
  if data is <u>not</u> against H_0 : we don't reject H_0

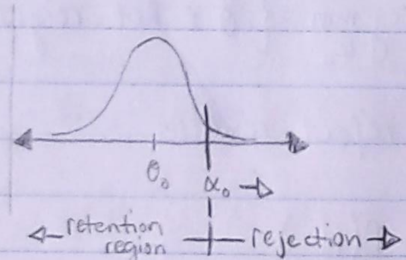
frequentist tests

– one sided / Right sided test :

$H_0 : \theta \leq (0.5 = \theta_0)$

$H_a : \theta > (0.5 = \theta_0)$

at level α_0



needs lots of data for Normal

$$\text{Ret.Reg.} = \left[0,\ \theta_0 + Z_{\alpha_0}\, SE(\hat{\theta}_{MLE})\right]$$

$\hat{\theta} \in$ Ret.Reg. $\Rightarrow$ Keep H_0

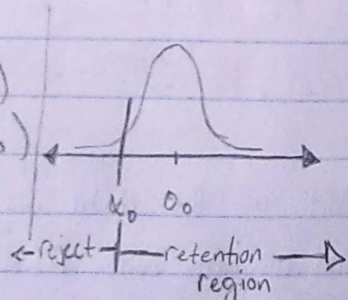
$\hat{\theta} \notin$ Ret.Reg. $\Rightarrow$ Reject H_0

– one sided / left sided test

$H_0 : \theta \geq (0.5 = \theta_0)$

$H_a : \theta < (0.5 = \theta_0)$

at level α_0



$$\text{Ret.Region} = \left[\theta_0 - Z_{\alpha_0}\, SE(\hat{\theta}_{MLE}),\ 1\right]$$

$\hat{\theta} \in$ Ret.Reg $\Rightarrow$ Keep H_0

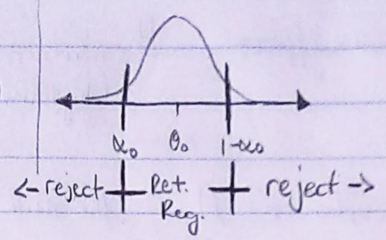
$\hat{\theta} \notin$ Ret.Reg $\Rightarrow$ Reject H_0

– two sided

$H_0 \quad \theta = (\theta_0 = 0.5)$

$H_a \quad \theta \neq (\theta_0 = 0.5)$

at level α_0



$\text{Ret.Reg} = \left[\theta_0 \pm Z_{\frac{\alpha_0}{2}} SE(\hat{\theta}_{MLE})\right]$

$\theta \in \text{Ret.Reg} \Rightarrow \text{keep } H_0$

$\theta \notin \text{Ret.Reg} \Rightarrow \text{rejet } H_0$

* The problem with all models is that they can't distinguish the degree of uncertainty:

** frequentist do this with p-values

$$P_{val} := P\left(\begin{array}{c}\text{seeing data } X \\ \text{"more extreme"}\end{array} \middle| H_0 \text{ is true}\right)$$

$$= P\left(\hat{p} > \hat{\theta} \,\middle|\, \hat{p} \sim N\left(\theta_0, \sqrt{\frac{\theta_0(1-\theta_0)}{n}}\right)\right)$$

$$= \arg\min_{\alpha_0} \{\hat{\theta} \in \text{Ret. region}\}$$

if $P_{val} < \alpha_0$, reject results.

** Problem is

$$\left.\begin{array}{l} P_{val} \neq P(H_0) \\ \quad\;\; \neq P(H_0 \mid X) \\ \quad\;\; \neq P(H_a) \\ \quad\;\; \neq P(H_a \mid X) \end{array}\right\} \text{* want all of this}$$

• Bayesian Hyp testing

$H_0: \theta \leq (0.5 = \theta_0)$

$H_a: \theta > (0.5 = \theta_0)$

we can find all ☆

easier, more reliable than p values

with all the data we know $P(\theta \mid X)$

so we can calc. $P(H_0 \mid X) = P(\theta \leq \theta_0 \mid X)$

$$= \int_0^{\theta_0} \frac{1}{B(x+\alpha, n-x+\beta)} \theta^{\alpha+x-1} (1-\theta)^{n-x+\beta-1} \, d\theta$$

$$= \text{qbeta}(\theta_0, x+\alpha, n-x+\beta)$$