$\Rightarrow X_1, \ldots, X_n \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$ where σ^2 is considered known (unrealistic, but go with it)

$\theta \mid \sigma^2 \sim N(\mu_0, \tau^2)$

$\Rightarrow \theta \mid X_1, \ldots, X_n, \sigma^2 \sim N\left(\dfrac{\frac{\bar{x} n}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \dfrac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \right)$

$$\hat{\theta}_{MMSE} = \frac{\frac{\bar{x} n}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$\hat{\theta}_{MAE} =$ "

$\hat{\theta}_{MAP} =$ "

why? Normal is symmetric & unimodal $\Rightarrow$ mean = median = mode

Shrinkage...

$$\hat{\theta}_{MMSE} = \frac{\frac{\bar{x} n}{\sigma^2}}{\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)} \cdot \frac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}} + \frac{\frac{\mu_0}{\tau^2}}{\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)} \frac{\tau^2}{\tau^2}$$

$$= \frac{1}{1 + \frac{\sigma^2}{n\tau^2}} \bar{X} + \frac{1}{\frac{n\tau^2}{\sigma^2} + 1} \mu_0$$

$$= \underbrace{\frac{\sigma^2}{n\tau^2 + \sigma^2}}_{\rho} \underbrace{\mu_0}_{E[\theta]} + \underbrace{\frac{n\tau^2}{n\tau^2 + \sigma^2}}_{(1-\rho)} \underbrace{\bar{X}}_{\hat{\theta}_{MLE}}$$

$= \rho \quad E[\theta] + (1-\rho) \quad \hat{\theta}_{MLE}$

weighted arithmetic avg

shrinkage

Hyperparameter interpretation

μ_0 : prior mean

τ^2 : prior variance

do they represent pseudodata?

Imagine pseudodata $Y_1, \ldots, Y_{n_0}$ you "saw" before

let $\mu_0 = \bar{Y} = \frac{1}{n_0}\sum_{k=1}^{n_0} Y_k$

Remember σ^2 is known...

let $\tau^2 = \frac{\sigma^2}{n_0} \Rightarrow n_0 = \frac{\sigma^2}{\tau^2}$. So μ_0, τ^2 can be thought of as $\bar{Y}$ and n_0 pseudodata

using this reparam....

$$\hat{\theta}_{MMSE} = \frac{\frac{\bar{X} n}{\sigma^2} + \frac{\bar{Y} n_0}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{n_0}{\sigma^2}} \cdot \frac{\sigma^2}{\sigma^2} = \frac{\bar{X} n + \bar{Y} n_0}{n + n_0} = \frac{\sum x_i + \sum y_i}{n + n_0}$$

$\Rightarrow$ the avg of all obs's ... prior and at hand

$X_1, \ldots, X_n \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$\Rightarrow \quad \theta \mid \sigma^2 \sim N(\mu_0, \frac{\sigma^2}{n_0})$

$\theta \mid X_1, \ldots, X_n, \sigma^2 \sim N\left(\frac{\bar{x} n + \mu_0}{n + n_0}, \left(\frac{\sigma}{\sqrt{n+n_0}}\right)^2\right)$

What prior should be chosen? Jeffreys / Laplace?

lets play the trick again...

$P(\theta \mid \sigma^2) \propto 1$

Improper!!!

$$P(\theta \mid X, \sigma^2) \propto P(X \mid \theta, \sigma^2)\, P(\theta \mid \sigma^2) \propto P(X \mid \theta, \sigma^2)$$

$$\propto \underbrace{e^{\frac{\bar{X} n}{\sigma^2}\theta}}_{\frac{c}{v^2}} \underbrace{e^{-\frac{n}{2\sigma^2}\theta^2}}_{\frac{1}{2v^2}} \propto N\left(\bar{X}, \frac{\sigma^2}{n}\right)$$

always proper!!!

$$\frac{1}{2v^2} = \frac{n}{2\sigma^2} \Rightarrow v^2 = \frac{\sigma^2}{n}$$

$$\frac{c}{v^2} = \frac{\bar{X} n}{\sigma^2} \Rightarrow c = \frac{\bar{X} n}{\sigma^2} v^2 = \frac{\bar{X} \cancel{n}}{\cancel{\sigma^2}} \cdot \frac{\cancel{\sigma^2}}{\cancel{n}}$$

Under Laplace $\hat{\theta}_{MMSE} = \hat{\theta}_{MAE} = \hat{\theta}_{MAP} = \hat{\theta}_{MLE} = \bar{X}$

the ultimate confluence!

Jeffreys prior...

$$\ell'(\theta; X, \sigma^2) = \frac{\bar{X} n}{\sigma^2} - \frac{n\theta}{\sigma^2}$$

$$\ell''(\theta; X, \sigma^2) = -\frac{n}{\sigma^2}$$

$$I(\theta) = E\left[-\ell''(\theta; X, \sigma^2)\right] = E\left[\frac{n}{\sigma^2}\right] = \frac{n}{\sigma^2}$$

Jeffreys...

$$P(\theta \mid \sigma^2) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\sigma^2}} \propto 1 = \text{Laplace prior!}$$

Improper priors can be thought of as the limit of proper priors

$X|\theta \sim Bin(n,\theta)$

$\theta \sim Beta(\alpha,\beta)$

$\theta|X \sim Beta(x+\alpha, n-x+\beta)$

$$\lim_{\substack{\alpha\to 0\\ \beta\to 0}} p(\theta|x) = Beta(x, n-x)$$

$X_1,\dots,x_n|\theta,\sigma^2 \overset{iid}{\sim} N(\theta,\sigma^2)$

$\theta|\sigma^2 \sim N(\mu_0,\tau^2)$

$$\theta|x_1,\dots,x_n,\sigma^2 \sim N\Bigg(\underbrace{\frac{\frac{\bar{x}n}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}}_{\hat{\theta}_{mmse}}, \underbrace{\frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}}_{\sigma^2_p}\Bigg) = N(\hat{\theta}_{mmse},\sigma^2_p)$$

What's this lim?

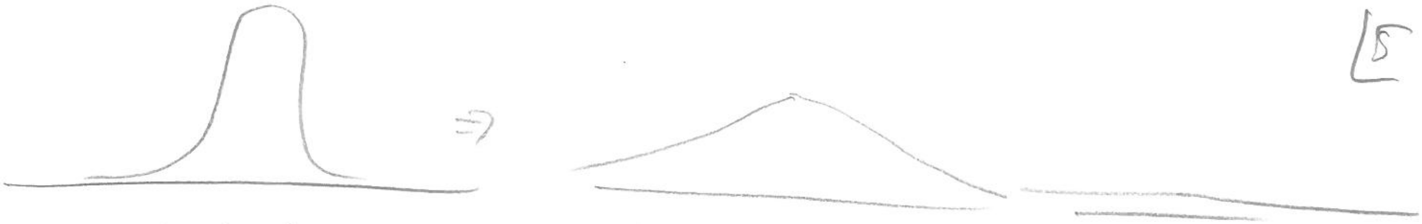
$$\lim_{\tau^2\to\infty} p(\theta|x_1,\dots,x_n,\sigma^2) = N\left(\bar{x},\frac{\sigma^2}{n}\right)$$

let's see

$$\lim_{\tau^2\to\infty} \frac{\frac{\bar{x}n}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\cdot\frac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}} = \lim \frac{\bar{x}+\frac{\mu_0\sigma^2}{\tau^2 n}}{1+\frac{\sigma^2}{n\tau^2}} = \bar{x} \quad ✓$$

$$\lim_{\tau^2\to\infty} \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} = \frac{1}{\frac{n}{\sigma^2}} = \frac{\sigma^2}{n} \quad ✓$$

$$\lim_{\tau^2\to\infty} \frac{1}{\sqrt{2\pi\tau^2}} e^{-\frac{1}{2\tau^2}(\theta-\mu_0)^2} \underbrace{= 0 \propto 1}_{\text{weird!!!}} \Rightarrow p(\theta|\sigma^2)\propto 1$$



$$P(X^* \mid X, \sigma^2) = \int_{\Theta} \underbrace{P(X^* \mid \theta, \sigma^2)}_{N(\theta, \sigma^2)} \underbrace{P(\theta \mid X, \sigma^2)}_{N(\theta_p, \sigma^2_p)} d\theta$$

for $n^* = 1$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X^* - \theta)^2} \frac{1}{\sqrt{2\pi\sigma^2_p}} e^{-\frac{1}{2\sigma^2_p}(\theta - \theta_p)^2} d\theta$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2_p}} \underbrace{\int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}(X^* - \theta)^2 - \frac{1}{2\sigma^2_p}(\theta - \theta_p)^2} d\theta}$$

Difficult!
Easier way?

$X_1, X_2 \overset{iid}{\sim} U(1,2,3,4,5,6)$

$S = X_1 + X_2 \sim ?$

Not covered..

$P(S=1) = 0$

$P(S=2) = P(X_1 = 1) \cdot P(X_2 = 1) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

$P(S=3) = P(X_1 = 1) P(X_2 = 2) + P(X_1 = 2) P(X_1 = 1)$

$$= \sum_{x \in \mathrm{Supp}(X_1)} P(X_1 = x) P(X_2 = 3 - x)$$

$$\Rightarrow P(S=s) = \sum_{x \in Supp(X_1)} P(X_1 = x)\, P(X_2 = s-x)$$

$$= \sum_{x \in Supp(X_2)} P(X_2 = x)\, P(X_1 = s-x)$$

for iid, doesn't matter the order

For cont. r.v.'s,

$$S = X_1 + X_2 \sim \underbrace{\int_{Supp[S]} f_{X_1}(x)\, f_{X_2}(s-x)\, dx}_{=: f_{X_1} * f_{X_2}}$$

convolution operator

Recall fact from Math 241

$X_1 \sim N(\mu_1, \sigma_1^2)$

$X_2 \sim N(\mu_2, \sigma_2^2)$

$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ we proved this using MGF's

thus

$$f_{X_1} * f_{X_2} = \int_{\mathbb{R}} f_{X_1}(x)\, f_{X_2}(s-x)\, dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}(s-x-\mu_2)^2}\, dx$$

Note: HARD PROOF $\longrightarrow$

$$\overset{\text{must}}{=} \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-\frac{1}{2(\sigma_1^2+\sigma_2^2)}(x-\mu_1-\mu_2)^2}$$

let's return to our problem...

$$P(X^* \mid X, \sigma^2) = \int_{\mathbb{R}} \underbrace{\frac{1}{\sqrt{2\pi\sigma_p^2}} e^{-\frac{1}{2\sigma_p^2}(\theta - \theta_p)^2}} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X^* - \theta - 0)}} \, d\theta$$

$S = X^*$

$\sigma_1^2 = \sigma_p^2$ $\quad$ $\sigma_2^2 = \sigma^2$

$\mu_1 = \theta_p$ $\quad$ $\mu_2 = 0$

$$= N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

$$= N(\theta_p, \sigma_p^2 + \sigma^2)$$

$\Rightarrow$ A mixture of normal where mean is draw normal is normal

Sum up.....

$\Rightarrow X_1, \dots, X_n \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$\theta \mid \sigma^2 \sim N(\mu_0, \tau^2)$

$\theta \mid X_1, \dots, X_n, \sigma^2 \sim N(\theta_p, \sigma_p^2)$ $\qquad$ $\theta_p = \dfrac{\frac{\bar{X}n}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$, $\sigma_p^2 = \dfrac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$

$X^* \mid X_1, \dots, X_n, \sigma^2 \sim N(\theta_p, \sigma_p^2 + \sigma^2)$

if $\tau^2 \to 0$

$\Rightarrow \theta \mid X_1, \dots X_n, \sigma^2 \sim N\left(\bar{X}, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$

$X^* \mid X_1, \dots, X_n, \sigma^2 \sim N\left(\bar{X}, \left(\sqrt{\frac{\sigma^2}{n} + \sigma^2}\right)^2\right)$

$$\left(\sigma\sqrt{\tfrac{1}{n} + 1}\right)^2$$

$$\left(\sigma\sqrt{\tfrac{n+1}{n}}\right)^2 \qquad \sqrt{\frac{n+1}{n}} \to 1$$

Mid 2 ↑↑

FINAL ↓↓

$X_1, \ldots, X_n \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

if θ known, σ^2 unknown, what is MLE of σ^2?

$$\mathcal{L}(\sigma^2; X, \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i-\theta)^2}$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n}_{(2\pi\sigma^2)^{-\frac{n}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\theta)^2}$$

$$\ell(\sigma^2; X, \theta) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum(X_i-\theta)^2$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2)$$

$$\ell'(\sigma^2; X, \theta) = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum(X_i-\theta)^2 = 0$$

$$\Rightarrow -n + \frac{1}{\sigma^2}\sum(X_i-\theta)^2 = 0 \Rightarrow \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i-\theta)^2$$

$$\Rightarrow n\hat{\sigma}^2_{MLE} = \sum(X_i-\theta)^2$$ AKA sum of sqd error (SSE)

$$\theta \sim \text{Gamma}(\alpha, \beta) := \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\theta}\theta^{\alpha-1}$$

$Y = \frac{1}{\theta} \sim$? Use c.o.v. $y = t(\theta) = \frac{1}{\theta} \Rightarrow \theta = t^{-1}(y) = \frac{1}{y} = y^{-1}$

InvGamma(α, β) !!

$$\left|-y^{-2}\right| = y^{-2}$$

$$f_Y(y) = f_\theta(t^{-1}(y))\left|\frac{d}{dy}\left[t^{-1}(y)\right]\right| = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\frac{\beta}{y}} \underbrace{\left(\frac{1}{y}\right)^{\alpha-1}}_{y^{1-\alpha}} \left|\frac{d}{dy}\left[y^{-1}\right]\right| = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\frac{\beta}{y}} y^{-\alpha-1}$$

$E[Y] = \frac{\beta}{\alpha-1}$ if $\alpha > 1$, Mode$(Y) = \frac{\beta}{\alpha+1}$, Median$(Y)$ = qinvgamma(0.5, α, β)

Supp$(Y) = (0, \infty)$

Param Space $\alpha, \beta > 0$