Lec 9 Math 391 3/7/18

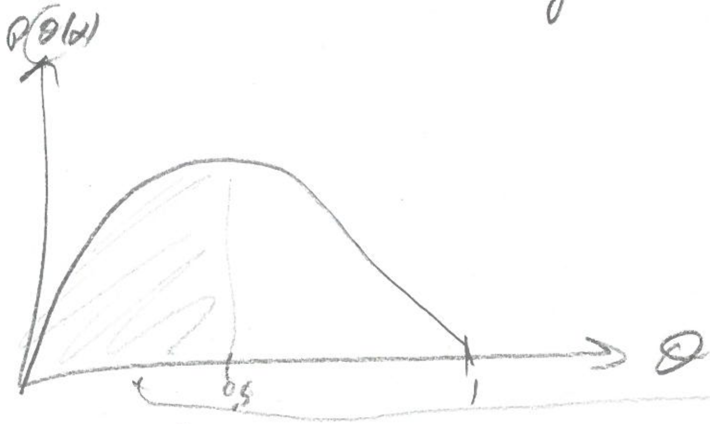
Bayesian Hypothesis Testing!

Consider the right-sided test...

$H_0: \theta \le \theta_0 \stackrel{e.g.}{=} 0.5$

$H_a: \theta > \theta_0 \stackrel{e.g.}{=} 0.5$ , $\alpha = 5\%$

In Bayes... we know $P(\theta|x)$ e.g.




thus we know $P(H_0|x) = P(\theta \le \theta_0 | x) \stackrel{e.g.}{=} P(\theta \le 0.5 | x)$

pval :=

what you want!!

$= \int_0^{0.5=\theta_0} \frac{1}{B(\alpha', \beta')} \theta^{\alpha'-1} (1-\theta)^{\beta'-1} \, d\theta$

the Bayesian "p-value"

$= pbeta(\theta_0, \alpha', \beta')$

What prob. is it?? P(Null Hyp. being true)

Also know $P(H_a|\theta) = 1 - pval$!

if pval $< \alpha \Rightarrow H_0$ not likely enough! Ditch it! Reject $H_0$!

if $p_{val} < \alpha_0 \Rightarrow$ Reject $H_0$

$p_{val} \geq \alpha_0 \Rightarrow$ FTR $H_0$

$$P(H_0 \mid X) = \frac{P(X \mid H_0)\, P(H_0)}{P(X)}$$ what's this? prior prob of $H_0$

$$P(\theta \leq \theta_0 \mid x) = \frac{P(x \mid \theta > \theta_0)\, P(\theta \leq \theta_0)}{P(x \mid \theta > \theta_0)\, P(\theta \leq \theta_0) + P(x \mid \theta \leq \theta_0)\, P(\theta > \theta_0)}$$

if $\theta \sim U(0,1)$ ...

$P(H_0) = P(\theta \leq \theta_0) = \theta_0$ Prior matters, ...

Role of $\hat{\theta} = \bar{x}$? No... no longer as test statistic

Note: this scheme equivalent to $\theta_0 \in CR_{R,\theta,1-\alpha} \Rightarrow$ Retain $H_0$

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$H_0: \theta \geq \theta_0 = 0.5$

$H_1: \theta < \theta_0 = 0.5$

$$p_{val} = P(H_0 \mid X) = P(\theta \geq \theta_0 \mid X) = \int_{\theta_0}^{1} \frac{1}{B(\alpha+x, \beta+n-x)}\, \theta^{\alpha+x-1} (1-\theta)^{\beta+n-x-1}\, d\theta$$

$$= 1 - \text{pbeta}(\theta_0, \alpha+x, \beta+n-x)$$

if $p_{val} < \alpha \Rightarrow$ Reject $H_0$

$p_{val} \geq \alpha \Rightarrow$ Retain $H_0$

Note: equivalent to $\theta_0 \in CR_{L,\theta,1-\alpha} \Rightarrow$ Retain $H_0$

$H_0: \theta = \theta_0$ "point null" or "precise null"

$H_a: \theta \neq \theta_0$ lower dim. than $H_a$

$p_{val} = P(H_0 \mid X) = P(\theta = \theta_0 \mid X) = 0$ (always) why? Peruker Math 241 ... ?

why? In reality all ~~point~~ sniffle nulls are absurd ... not plausible!

prob coin flip $= \frac{1}{2}$? No.. 0.500001 real or Oc scale

Two ideas ① but 0.500001 = 0.5 for all practical purposes

$\Rightarrow H_0: \theta \in [\theta_0 \pm \delta]$ margin of equivalence

$H_a: \theta \notin [\theta_0 \pm \delta]$ this is usually what you mean!

What is margin of equivalence for coin ~~flip~~ heads $\delta = 0.01$?

② $H_0: \theta = \theta_0$ if plausible

$H_a: \theta \neq \theta_0$

$\theta_0 \in CR_{\theta, 1-\alpha} \Rightarrow$ Retain

$\theta_0 \notin CR_{\theta, 1-\alpha} \Rightarrow$ Reject

p-value calculation here not clear ...

This seems to be contested ...

which is why we use Bayes Factors (soon)

Examples

$H_0: \theta \le .1$

$H_a: \theta > .1$

$\alpha = 5\%$, $n = 150$, $X = 23$

$$\text{Rej Region} = \left[0, .1 + 1.67\overbrace{\sqrt{\frac{.1 \cdot .9}{150}}}^{=.024}\right] = [0, .140]$$

$\hat\theta = .153 \notin$ Rej Region $\Rightarrow$ Reject $H_0$

$pval := P(\hat p > .153 \mid \theta = .1)$

$P\left(Z > \frac{.153 - .1}{.024}\right) = 2.16$

$= 1 - pnorm(2.16)$

$= .01539 < \alpha \Rightarrow$ Reject

Bayesian ... $\theta \sim U(0,1) = Beta(1,1)$

$\theta \mid X \sim Beta(\alpha + x, \beta + n - x) = Beta(24, 128)$

$$pval := P(\theta \le .1 \mid X) = \int_0^{.1} Beta(24,128)\, d\theta = pbeta(.1, 24, 128) = .01544$$

Similar!

n is high
Freq $\approx$ Bayes

$H_0: \theta = .5$

$H_a: \theta \ne .5$

$\alpha = 5\%$

$n = 100$

$X = 61$

$$\text{Rej Reg} = \left[.5 \pm 2\sqrt{\frac{.5 \cdot .5}{100}}\right] = [.4, .6]$$

$\hat\theta = .61 \notin$ Rej Region $\Rightarrow$ Reject

$$pval := P\left(Z > \frac{.61 - .5}{.05}\right) = 2P(Z > 2.2) = 2(1 - pnorm(2.2)) = .0278$$

Bayesian $\theta \sim U(0,1)$

$\theta \mid X \sim Beta(62, 40)$

I $H_0: \theta \in [.49, .51]$ i.e. $\delta = .01$ equivalence margin

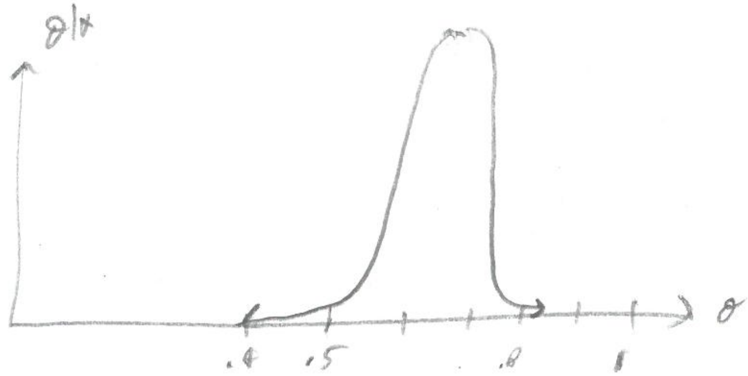
$H_a: \theta \notin [.49, .51]$

$$p_{val} = P(\theta \in [.49, .51] \mid x) = \int_{.49}^{.51} Beta(62,40)\, d\theta = pbeta(.51, 62, 40) - pbeta(.49, 62, 40) = .0147$$

II $CR_{\theta,1-\alpha} = [\,qbeta(.025, 62, 40),\ qbeta(.975, 62, 40)\,]$

$= [.511, .700]$

$\theta_0 = .5 \notin CR_{\theta,1-\alpha} \Rightarrow$ Reject $H_0$




Another way to "test". Bayesians like this better

$H_0: \theta = \theta_0 = 0.5$

$H_a: \theta \neq \theta_0 = 0.5$

what is this really?

maybe

$H_a: \theta \sim U(0,1)$

$\theta$ could be "anything"

$n = 100$

$x = 61$

$\Rightarrow \hat{\theta} = .61$

Define $\mathcal{B} := \dfrac{P_{H_a}(X)}{P_{H_0}(X)}$

Bayes Factor

denominator in Bayes Rule!!
prob of data!

if $\mathcal{B}$ big $\Rightarrow$ $H_a$ is a better model for the data, $X$.

$$= \frac{\displaystyle\int_{\Theta_{H_a}} P(X|\theta)\, P_{H_a}(\theta)\, d\theta}{\displaystyle\int_{\Theta_{H_0}} P(X|\theta)\, P_{H_0}(\theta)\, d\theta}$$

oops...

$$= \frac{\displaystyle\int_{\{.5\}} \cancel{\binom{100}{61}}\, .5^{61}(1-.5)^{100-61}\,(1)\, d\theta}{\displaystyle\int_0^1 \cancel{\binom{100}{61}}\, \theta^{61}(1-\theta)^{100-61}\,(1)\, d\theta}$$

for a diff prior

$$= \frac{B(62, 40)}{.5^{100}} = 1.39$$

$H_a$ better model... but is it decisive?

Jeffreys 1961 scale of Bayes Factors categories for validity of $H_a$

$B < 1 \Rightarrow$ no evidence whatsoever

$B \in [1:1, 3:1] \Rightarrow$ barely worth mentioning

$B \in [3:1, 10:1] \Rightarrow$ Substantial

$B \in [10:1, 30:1] \Rightarrow$ strong

$B \in [30:1, 100:1] \Rightarrow$ very strong

$B \geq 100:1 \Rightarrow$ absolutely decisive

our results here .. barely worth mentioning

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Famous data

testing psychokinesis (ESP)

$H_0: \theta = 0.5$

$H_a: \theta \neq 0.5$

$\alpha = 5\%$

$n = 104{,}490{,}000$

$\hat{\theta} = .50001768$

$X = 52{,}263{,}470$

successes

$p_{val} = .0003 < 5\% \Rightarrow$ this guy has the psychokinesis ability!!

But ... Bayesian ...

$\theta | X \sim \text{Beta}(52263471, 52226531)$

$H_0: \theta = 0.5$

$H_a: \theta \sim U(0,1)$