Lecture 18

Note $X_1 \dots X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ θ, σ^2 unknown

$P(\theta, \sigma^2) \propto 1/\sigma^2$ (Jeffrey's prior) - Prior

$P(\theta, \sigma^2 | X) = \text{Norm Inv Gamma}(\mu, \lambda, \alpha, \beta)$ - Posterior

$P(\theta | X, \sigma^2) = N\left(\bar{x}, \frac{\sigma^2}{n}\right)$

$P(\sigma^2 | X, \theta) = \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2}{2}\right)$

$$\left.\begin{aligned} P(\sigma^2 | x) &= \text{Inv Gamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right) \\ P(\theta | x) &= T_{n-1}\left(\bar{x}, s/\sqrt{n}\right) \end{aligned}\right\} \text{Marginal Posteriors}$$

$CR_{95\%, \theta} = \left[qt\left(0.025, n-1, \bar{x}, \frac{s}{\sqrt{n}}\right), qt\left(0.975, n-1, \bar{x}, \frac{s}{\sqrt{n}}\right)\right]$

- Posterior Predictive :

$x^* | \theta, \sigma^2 \sim N(\theta, \sigma^2)$

$x^* | x \sim ?$

$$P(x^* | x) = \int_{\sigma^2=0}^{\infty} \int_{\theta=-\infty}^{\infty} P(x^* | \theta, \sigma^2) \underbrace{P(\theta, \sigma^2 | x)}_{\text{Bayes Rule}} \, d\theta \, d\sigma^2$$

easier one

$$= \int_{\sigma^2}\int_{\theta} \Big\{ \underbrace{P(x^* | \theta, \sigma^2)}_{\text{Norm}} \underbrace{P(\theta | x \sigma^2)}_{\text{Norm}} \underbrace{P(\sigma^2 | x)}_{\text{Inv Gamma}} \Big\} \text{ sampling from here}$$

$$= \int_{\sigma^2}\int_{\theta} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(x^*-\theta)^2} \right) \left(\frac{1}{\sqrt{2\pi \frac{\sigma^2}{2}}} e^{\frac{-1}{2\frac{\sigma^2}{n}}(\theta - \bar{x})^2} \right) \left(\frac{\left(\frac{(n-1)s^2}{2}\right)^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} (\sigma^2)^{-\left(\frac{n-1}{2}\right)-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \right) d\theta \, d\sigma^2$$

$$\propto \iint \sigma^{2^{\frac{-1}{2}}} \sigma^{2^{\frac{-1}{2}}} \sigma^{2^{-\left(\frac{n-1}{2}+\frac{2}{2}\right)}} e^{\frac{-1}{2\sigma^2}(x^*-\theta)^2} e^{\frac{-n}{2\sigma^2}(\theta-\bar{x})^2} e^{\frac{-1}{2\sigma^2}(n-1)s^2} \, d\theta \, d\sigma^2$$

$\vdots$

$$= \int \sigma^{2^{-\left(\frac{n+1}{2}\right)-1}} e^{-\frac{n}{2\sigma^2}} \int \underbrace{e^{\left(\frac{x^*+n\bar{x}}{\sigma^2}\right)\theta} e^{-\frac{n+1}{2\sigma^2}\theta^2}}_{\text{Kernel of normal}} \, d\theta \, d\sigma^2 \quad \propto$$

$$\boxed{P(x^*|X) \propto T_{n-1}\left(\bar{x}, \sqrt{s^2 \frac{n+1}{n}}\right)}$$

Note

As $n \to \infty$

$P(x^*|X) \Rightarrow N(\theta, s^2)$

Sampling $X^*|X$:

1) draw σ_0^2 from $\text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$ $P(\sigma^2|X)$
2) draw θ_0 from $N(\bar{x}, \sigma_0^2/n)$ $P(\theta|\sigma^2 X)$
3) draw X^* from $N(\theta_0, \sigma_0^2)$
4) Return X^*

Note: $P(\theta, \sigma^2|X) = \underset{\text{NormInvGamma}}{P(X|\theta,\sigma^2)} \cdot P(\theta,\sigma^2)$ ← Jeffrey's prior must ALSO be NormInvGamma

- $P(\theta,\sigma^2) = \underset{N\left(\mu_0,\ \tau^2 = \left(\frac{\sigma}{\sqrt{n_0}}\right)^2\right)}{P(\theta|\sigma^2)}\ \underset{\text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)}{P(\sigma^2)}$

- Consider θ and σ^2 are disjoint such that:

$$P(\sigma^2) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$$

but

$$P(\theta|\sigma^2) = P(\theta) = N\left(\mu_0, \tau^2 \neq \frac{\sigma^2}{n_0}\right)$$ * Not dependent on σ^2

$P(\theta,\sigma^2) = P(\theta)\cdot P(\sigma^2) \neq \text{NormInvGamma}$ so,

$P(\theta,\sigma^2|X) \neq \text{NormInvGamma}$

$$P(\theta,\sigma^2|X) \propto P(X|\theta,\sigma^2)\, P(\theta,\sigma^2)$$

$$\propto \underset{\substack{X_1\ldots X_n \sim N(\theta,\sigma^2) \\ \text{NormInvGamma} \\ (\theta \text{ variable}, \ \sigma^2 \text{ variable})}}{P(X|\theta,\sigma^2)}\ \underset{\substack{N(\mu_0,\tau^2) \\ (\theta \text{ variable}, \ \sigma^2 \text{ fixed})}}{P(\theta)}\ \underset{\substack{\text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right) \\ \propto P(\sigma^2|\theta) \\ \theta \text{ is fixed}, \ \sigma^2 \text{ variable}}}{P(\sigma^2)}$$

[AND $\propto P(\theta|\sigma^2 X)\, P(\sigma^2|X)$]

Note: θ, σ^2 are variables

$$= \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(x_i-\theta)^2} \right) \left(\frac{1}{\sqrt{2\pi\tau^2}} e^{\frac{-1}{2\tau^2}(\theta-\mu_0)^2} \right) \left(\frac{\left(\frac{n_0\sigma_0^2}{2}\right)^{\frac{n_0}{2}}}{\Gamma\left(\frac{n_0}{2}\right)} \sigma^{2^{-\left(\frac{n_0}{2}\right)-1}} e^{-\left(\frac{n_0\sigma_0^2}{2}\right)\cdot\frac{1}{\sigma^2}} \right)$$

$P(X|\theta,\sigma^2)$ $\quad P(\theta)$ $\quad P(\sigma^2)$

$$\propto \left(\sigma^{2^{\frac{-n}{2}}} e^{\frac{-1}{2\sigma^2}\overbrace{\Sigma(x_i-\theta)^2}^{SSE}} \right) \left(e^{\frac{-1}{2\tau^2}(\theta-\mu_0)^2} \right) \left(\sigma^{2^{-\left(\frac{n_0}{2}\right)-1}} e^{-\left(\frac{n_0\sigma_0^2}{2\sigma^2}\right)} \right)$$

$$\propto \sigma^{2^{-\frac{n}{2}-\left(\frac{n_0}{2}+1\right)}} e^{\frac{-1}{2\sigma^2}\left((n-1)s^2+\underbrace{n(\bar{x}-\theta)^2}_{SSE} + n_0\sigma_0^2\right)} e^{\frac{-1}{2\tau^2}(\theta-\mu_0)^2}$$

$$\propto \sigma^{2^{-\frac{n}{2}-\left(\frac{n_0}{2}+1\right)}} e^{\frac{-1}{2\sigma^2}\left((n-1)s^2+n_0\sigma_0^2\right)} \underbrace{e^{\frac{-1}{2\sigma^2}\left(n(\bar{x}-\theta)^2\right)}}_{\text{(foil out)}} \underbrace{e^{\frac{-1}{2\tau^2}(\theta-\mu_0)^2}}_{\text{(foil out)}}$$

$$\propto \sigma^{2^{\frac{-n}{2}-\left(\frac{n_0}{2}+1\right)}} e^{\frac{-1}{2\sigma^2}\left((n-1)s^2+n_0\sigma_0^2\right)} e^{\frac{-n\bar{x}^2}{2\sigma^2}+\frac{2n\bar{x}\theta}{2\sigma^2}-\frac{n\theta^2}{2\sigma^2}-\frac{\theta^2}{2\tau^2}+\frac{2\theta\mu_0}{2\tau^2}-\frac{\mu_0^2}{2\tau^2}}$$

$$\propto \sigma^{2^{\frac{-n}{2}-\left(\frac{n_0}{2}+1\right)}} e^{\frac{-1}{2\sigma^2}\left((n-1)s^2+n_0\sigma_0^2-n\bar{x}^2\right)} \underbrace{e^{-\frac{\mu_0^2}{2\tau^2}}}_{\text{(constant)}} \underbrace{\left[e^{-\left(\frac{n}{2\sigma^2}+\frac{1}{2\tau^2}\right)\theta^2} e^{\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}\right)\theta} \right]}_{\propto N(\theta_p,\, \sigma^2_p)}$$

$$\propto \sigma^{2^{\frac{-n}{2}-\left(\frac{n_0}{2}+1\right)}} e^{\frac{-1}{2\sigma^2}\left((n-1)s^2+n_0\sigma_0^2-n\bar{x}^2\right)} \left[\frac{1}{\sqrt{2\pi\sigma_p^2}} e^{\frac{-1}{2\sigma_p^2}(\theta-\theta_p)^2} \right]$$

Note:

$$P(\theta,\sigma^2|X) \propto \underbrace{\sigma^{2^{\frac{-n}{2}-\left(\frac{n_0}{2}+1\right)}} e^{\frac{-1}{2\sigma^2}\left((n-1)s^2+n_0\sigma_0^2-n\bar{x}^2\right)} \left[(\sigma_p^2)^{\frac{-1}{2}} e^{\frac{-\theta_p^2}{2\sigma^2_p}} \right]}_{\text{Kernel of some r.v.}}$$

- $\theta_p = \dfrac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}$
- $\sigma_p^2 = \dfrac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}$

recall: $P(\theta,\sigma^2|X) \propto P(\theta|X,\sigma^2)\,P(\sigma^2|X)$

$\propto P(\sigma^2|X,\theta)\,P(\theta|X)$

* One part of kernel is Normal, only $P(\theta|X,\sigma^2)$ fits, but $P(\sigma^2|X)$ is now an unknown kernel

** disjoint θ, σ^2 creates a <u>Semi-conjugate Model</u>

• Sampling from $P(\theta, \sigma^2 | X)$ when θ, σ^2 are disjoint

1) choose σ_0^2 from $K(\sigma^2 | X)$ ** unknown portion
2) Choose θ_0 from $N(\theta_p, \sigma_p^2)$
3) return $< \theta_0, \sigma_0^2 >$

** Sampling from $K(\sigma^2 | X)$; note $P(\sigma^2 | X) = c K(\sigma^2 | X)$

Grid Sampling

1) pick Grid: σ^2_{min}, σ^2_{max}, g, $\Delta\sigma^2$

$$G = < \sigma^2_{min}, \sigma^2_{min} + \Delta\sigma^2, \dots, \sigma^2_{min} + (g-1)\Delta\sigma^2, \sigma^2_{max} >$$

2) Compute $K(\sigma_g^2 | X)$ $\forall \sigma_g^2 \in G$

3) Approximate c ; $c \approx \dfrac{1}{\sum_{\sigma_g^2 \in G} K(\sigma_g^2 | X)}$

4) CDF: $F(\sigma_0^2 | x) \approx \sum c K(\sigma^2 | x)$ $\{\sigma^2 \in G : \sigma^2 < \sigma_0^2\}$

5) draw y from $U(0,1)$

compute $\sigma_0^2 = \min\limits_{\sigma^2 \in G} F(\sigma^2) \geq y$