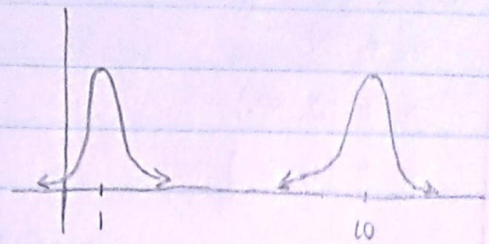
Lecture 10

•• Mixture distributions:

ex1 $$X \sim \begin{cases} N(1, 1^2) & \text{w.p } \tfrac{1}{2} \\ N(10, 1^2) & \text{w.p } \tfrac{1}{2} \end{cases}$$



PDF of X?
$$F_x(x) = \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} \cdot e^{-\left(\frac{x-1}{2}\right)^2}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} \cdot e^{-\left(\frac{x-10}{2}\right)^2}\right)$$
$\theta = 1$ $\qquad$ $\theta = 10$

$$P(X) = \sum_{\theta \in \Theta} P(X|\theta)\, P(\theta) = \sum_{\theta \in \Theta} N(\theta, 1^2) \quad \theta \sim \begin{cases} 1 & \text{w.p } \tfrac{1}{2} \\ 10 & \text{w.p } \tfrac{1}{2} \end{cases}$$

ex2 $X_1 \sim Bin(10, 0.1)$
$X_2 \sim Bin(10, 0.9)$
$$\Rightarrow \quad X \sim \begin{cases} Bin(10, 0.1) & \text{w.p } \tfrac{1}{2} \\ Bin(10, 0.9) & \text{w.p } \tfrac{1}{2} \end{cases}$$

$$P(X) = \frac{1}{2}\left(\binom{10}{x}(0.1)^x (0.9)^{10-x}\right) + \frac{1}{2}\left(\binom{10}{x}(0.9)^x (0.1)^{10-x}\right)$$

- if $\theta \sim (0,1)$ of $Bin(n, \theta)$

$$* \quad P(X) = \int_{\theta \in \Theta} P(X|\theta) P(\theta)\, d\theta = \int_0^1 \binom{n}{x} \theta^x (1-\theta)^{n-x}\, d\theta = \binom{n}{x} B(x+1, n-x+1)$$

- if $\theta \sim Beta(\alpha, \beta)$

$$* \quad P(X) = \int_{\theta \in \Theta} \left(\binom{n}{x} \theta^x (1-\theta)^{n-x}\right)\left(\frac{1}{B(\alpha,\beta)} \cdot \theta^{\alpha-1}(1-\theta)^{\beta-1}\right) d\theta = \left[\binom{n}{x} \frac{B(x+\alpha, n-x+\beta)}{B(\alpha, \beta)}\right] **$$

Beta Binomial (n, α, β)

- Beta Binomial $(n, \alpha, \beta) \sim X$ (Overdispersed Binomial)

$$\text{supp}[X] = \{0, 1, 2, \ldots, n\}$$

$n \in \mathbb{N}$

$\alpha \in (0, \infty)$

$\beta \in (0, \infty)$

$$E[X] = \sum x f(x) = \sum_{1}^{n} x \left[\binom{n}{x} \frac{B(x+\alpha, n-x+\beta)}{B(\alpha, \beta)} \right] = \boxed{n \frac{\alpha}{\alpha+\beta}}$$

Beta Binomial

$$\text{Var}[X] = E[X^2] - E[X]^2 = \ldots = \boxed{\frac{n\alpha\beta\,(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

Binomial

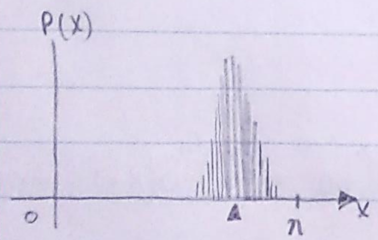
BetaBinomial if $\theta = \frac{\alpha}{\alpha+\beta}$

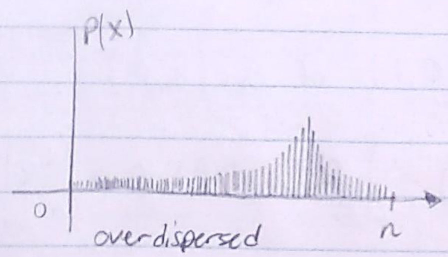
$E[X] = n\theta$

$E[X] = n\frac{\alpha}{\alpha+\beta} \Rightarrow n\theta$

$\text{Var}[X] = n\theta(1-\theta)$

$\text{Var}[X] = \frac{n\alpha\beta\,(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)} \Rightarrow n\theta(1-\theta) \cdot \frac{\alpha+\beta+n}{\alpha+\beta+1}$





Note

Beta Bin$(n, 1, 1)$

resembles principle of indifference

ex. $P(\text{male Births}) = 0.511$; look at women with at least 12 children (6115 total)

# of boys	0	1	2	3	4	5	6	7	8	9	10	11	12
X	3	24	104	286	670	1033	1343	1112	829	478	181	45	7
expected by Binomial $n(P(X))$	1	12	72	259	628	1085	1367	1266	854	* too small 410	152	26	2
Betabin(12, 34, 32) (* Better fit disperses data)	2	23	105	311	656	1036	1258	1182	854	462	178	47	5

* indicates each woman has a different propensity of having boys

Posterior predictive distribution $n, x, \theta \sim Beta(\alpha,\beta)$

$$X^* \mid X \sim Bern\left(\frac{x+\alpha}{n+\alpha+\beta}\right)$$ when $* = n^* = 1$

# of future observations.

Let $n^* \in \mathbb{N}$

$$P(X^*|X) = \int_\Theta \underbrace{P(X^*|\theta)}_{[Bin(n^*,\theta)]} \underbrace{P(\theta|X)}_{[Beta(x+\alpha, n-x+\beta)]} d\theta$$

$$= \int_0^1 \binom{n^*}{x^*}\left(\theta^{x^*}(1-\theta)^{n^*-x^*}\right) \cdot \left(\frac{1}{B(x+\alpha, n-x+\beta)} \cdot \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}\right) d\theta$$

$$= \binom{n^*}{x^*}\left(\frac{1}{B(x+\alpha, n-x+\beta)}\right) \int_0^1 \theta^{x^*+\alpha+x-1}(1-\theta)^{n^*-x^*+n-x+\beta-1} d\theta$$

$$= \binom{n^*}{x^*} \frac{B(x^*+x+\alpha,\ n^*-x^*+n-x+\beta)}{B(\alpha+x,\ \beta+n-x)}$$

$$** \quad = Beta\,Bin(n^*, \alpha+x, \beta+n-x)$$

compared to

$$P(X|\{\}) = Beta\,Bin(n, \alpha, \beta)$$