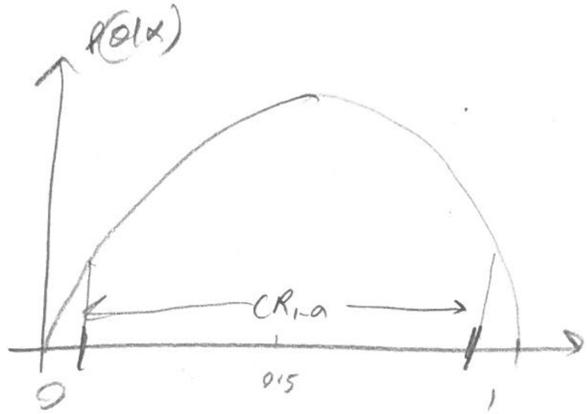
Math 3?? Lec 7 2/21/17

We still have only been focusing on point estimation. The second goal of inference is "confidence sets" (regions of reasonable values for θ)

consider

$$P(\theta) = U(0,1), \quad X=1, \ n=2 \Rightarrow P(\theta|X) = \text{Beta}(2,2)$$



What if I want a set R s.t. $P(\theta \in R \mid X) = 1-\alpha$

that represents the "middle of the dist"

What is a procedure that would do this?

the "Bayesian credible region"

Define $CR_{\theta,1-\alpha} = \left[\text{Quantile}\left[\theta|X, \frac{\alpha}{2}\right], \text{Quantile}\left[\theta|X, 1-\frac{\alpha}{2}\right]\right]$

e.g. $CR_{\theta,95\%} = \left[\text{Quantile}\left[\theta|X, 2.5\%\right], \text{Quantile}\left[\theta|X, 97.5\%\right]\right]$

In our setup,

$$CR_{\theta,95\%} = \left[\text{qbeta}(2.5\%, 2, 2), \text{qbeta}(97.5\%, 2, 2)\right]$$

$$= [0.094, 0.906] \quad \text{Makes sense?}$$

Interpretation: Assuming the data and prior, the prob of θ being between 0.094 and 0.906 is 95%.

Contrast to frequentist $CI_{\theta, 95\%}$

$$= \left[\hat{\theta}_{mle} \pm z_{\frac{\alpha}{2}} SE[\hat{\theta}_{mle}]\Big|_{\theta=\hat{\theta}_{mle}}\right] = \left[0.5 \pm 2\sqrt{\frac{0.5(1-0.5)}{2}}\right] = [-0.21, 1.21]$$

Absurd... And interpretation? NONE!

You can also do one-sided CR's

$$CR_{L,\theta,1-\alpha} = \left[\text{smallest value in support or } -\infty,\ \text{Quantile}[\theta|X, 1-\alpha]\right]$$

In our example..

$$CR_{L,\theta,1-\alpha} = [0,\ \text{qbeta}(0.95, 2, 2)] = [0, 0.867]$$

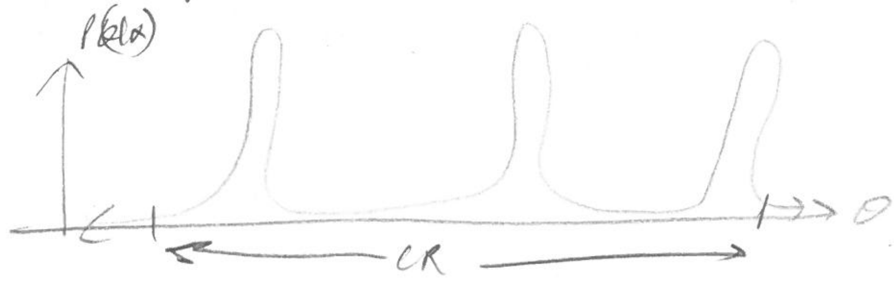
There's a 95% chance θ is less than .867 given the data & prior

$$CR_{R,\theta,1-\alpha} = \left[\text{Quantile}[\theta|X, \alpha],\ \text{largest value or } \infty\right]$$

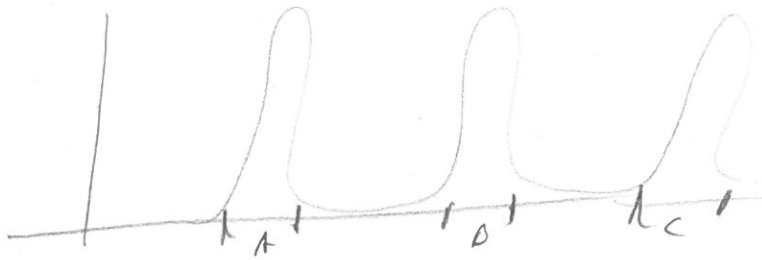
$$= [\text{qbeta}(0.05, 2, 2),\ 1] = [.135, 1]$$

95% chance θ is greater than .135 given the data & prior.

Consider a posterior that looks like this.



Is there another idea?



high density region

$$HDR_{0,1-\alpha} = A \cup B \cup C$$ so that $\mu(A)+\mu(B)+\mu(C)$ minimizes the total length $\mu(R)$

length of interval

s.t. $P(\theta \in HDR \mid x) = 1-\alpha$

Disadvantages of HDR

① Very computationally intense ② Strange to have non-contiguous

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3rd goal of inference: theory testing

Let's review logical structure first

You would like to convince others of your own theory (H_1) the alt. but people universe believe a business-as-usual theory (H_0, the null)
Two ways to do so.

I Assume your theory is true and wait for people to disprove you.
II Assume your theory is wrong and demonstrate contradictory evidence to H_0 until people are forced to see that you're right. Which is the preferred way? II is intellectually honest. It's forcing you to come up with evidence beyond a reasonable doubt. I forces people to start off with believing you. If it's a crazy theory.. there's not too many of those people.

I) H_0: UFO's exist and aliens have visited Earth

H_a: " " do not " " " " " " not " " " " " "

II) H_0: UFO's do not exist _ _ _

H_a: _ _ _ _ _ _ _ _

There's a level of ~~evidence~~ data / α which can convince people to ditch H_0 and accept H_a. If data doesn't meet this standard, no reason to ditch H_0, so retain it.

In statistics, we test theories about θ. For example, we would like to demonstrate beyond a reasonable doubt that

$H_a: \theta \neq \theta_0 \Rightarrow H_0: \theta = \theta_0$ "two-sided test"

$H_a: \theta < \theta_0 \Rightarrow H_0: \theta \geq \theta_0$ "left-sided test"

$H_a: \theta > \theta_0 \Rightarrow H_0: \theta \leq \theta_0$ "right-sided test"

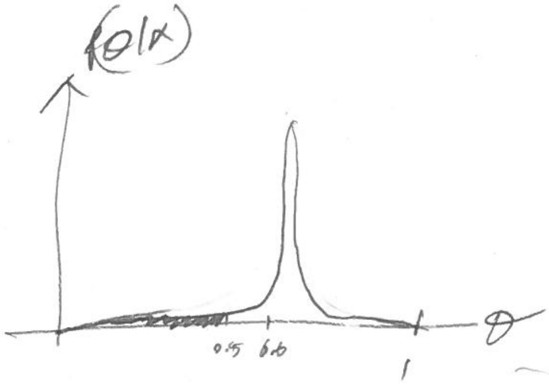
these are the only 3 examples we will do. Of course there's others

In the Bayesian worldview, we can ask directly

Is $P(H_0 | X) < \alpha$ If so $\Rightarrow$ reject H_0 since it is an improbable / unreasonable theory

For example let's say flip a coin 100 times, 61 heads. Is the coin unfairly weighted towards heads? $\alpha = 5\%$ (scientific standard)

$P(\theta) = U(0,1) = \text{Beta}(1,1) \Rightarrow P(\theta | X) = \text{Beta}(62, 40)$



has to be done with computer

here's the function for CDF

$$\text{pval} = P(H_0|x) = P(\theta \le 0.5|x) = \int_0^{0.5} \frac{1}{B(62,40)}\theta^{61}(1-\theta)^{39}\,d\theta = \text{pbeta}(0.5, 62, 39)$$

$$= F_{\theta|x}(0.5)$$

the Bayesian "p-val"

$= 0.018 \Rightarrow$ Reject H_0 $\Rightarrow$ coin certainly weighted

Uber driver does 200 rides and gets 37 non-5 star ratings. If $\theta < 25\%$ $\Rightarrow$ fired but we have to prove beyond a reasonable doubt.

$H_0: \theta \ge 25\%$, $H_a: \theta < 25\%$.

$f(\theta) = U(0,1)$, $\alpha = 5\%$ $\Rightarrow$ $f(\theta|x) = \text{Beta}(38, 164)$

$$P(H_0|x) = P(\theta > .25|x) = \int_{.25}^{1} \frac{1}{B(38,164)}\theta^{37}(1-\theta)^{163}\,d\theta = 1 - \text{pbeta}(.25, 38, 164) = .017$$

$$= 1 - F_{\theta|x}(.25)$$

$\Rightarrow$ Reject H_0
$\Rightarrow$ Fire driver

What if $H_0: \theta = \theta_0$ e.g. $n = 100$, $x = 43$ wish to test if coin is fair

$H_0: \theta = 0.5$
$H_a: \theta \ne 0.5$

$f(\theta) = U(0,1) \Rightarrow f(\theta|x) = \text{Beta}(44, 58)$

$\text{pval} = P(H_0|x) = P(\theta = 0.5|x) = 0$ Why? $\theta|x$ is continuous. Prob of single value is zero!