Lec 19 Math 341 4/18/19 $\mathcal{F}: X_1, \dots, X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

We can see this another way...

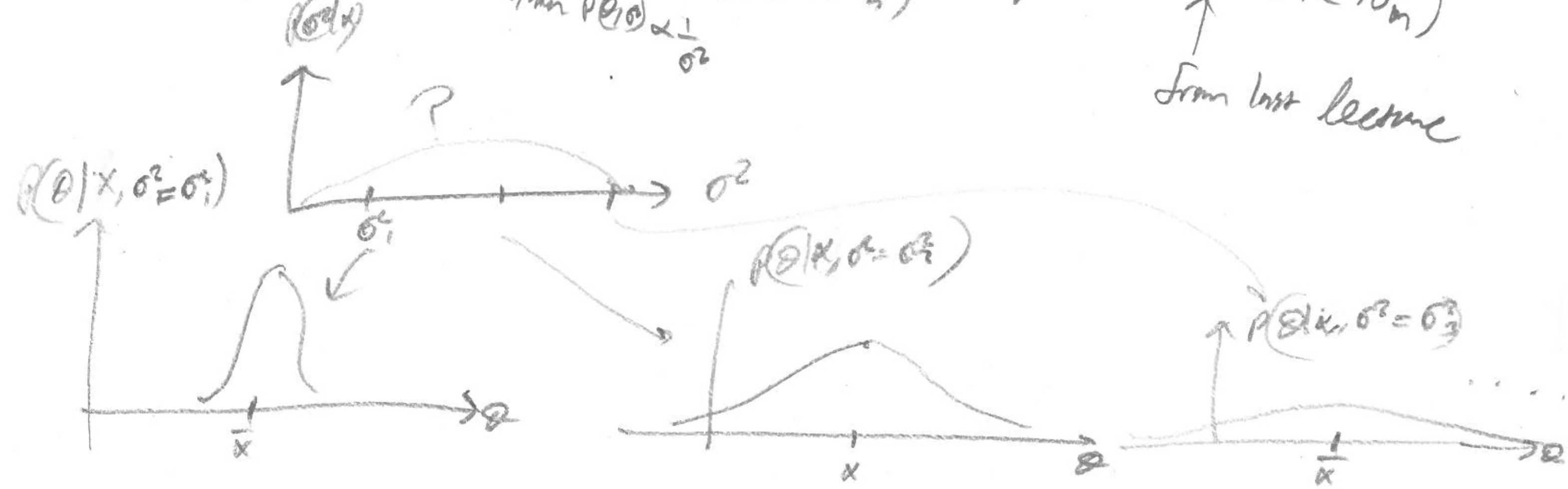
$$P(\theta|X) = \int_0^\infty P(\theta, \sigma^2|X)\, d\sigma^2 = \int_0^\infty \underbrace{P(\theta | X, \sigma^2)}_{N(\bar{x}, \frac{\sigma^2}{n})} \underbrace{P(\sigma^2|X)}_{?}\, d\sigma^2$$

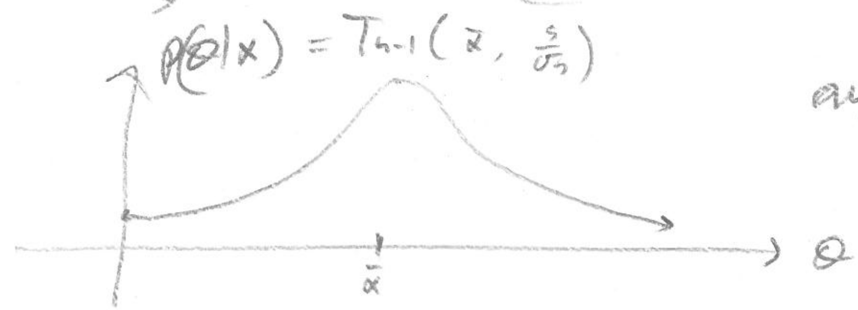
given $P(\theta, \sigma^2) \propto \frac{1}{\sigma^2}$

what do you think this will be?

$= T_{n-1}\left(\bar{x}, \frac{s}{\sqrt{n}}\right)$

from last lecture





averaging Normals with same center but different variances..

Assuming $P(\theta, \sigma^2) \propto \frac{1}{\sigma^2}$ again...

$$P(\sigma^2|X) = \int_{\mathbb{R}} P(\theta, \sigma^2|X)\, d\theta \propto \int_{\mathbb{R}} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}\, d\theta$$

$$= (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \int_{\mathbb{R}} \underbrace{e^{-\frac{1}{2\frac{\sigma^2}{n}}(\theta-\bar{x})^2}}_{\propto N(\bar{x}, \frac{\sigma^2}{n})}\, d\theta$$

$$= (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \sqrt{2\pi\frac{\sigma^2}{n}} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} e^{-\frac{1}{2\sigma^2/n}(\theta-\bar{x})^2}\, d\theta}_{=1}$$

$$\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} (\sigma^2)^{\frac{1}{2}} = (\sigma^2)^{-\frac{n-1}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}$$

$$\propto \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

Easy

$$\boxed{P(\theta,\sigma^2|X) = \underbrace{P(\theta|X,\sigma^2)}_{N(\bar{x}, \frac{\sigma^2}{n})} \underbrace{P(\sigma^2|X)}_{\text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)} = \text{NormInvGamma}\left(\bar{x}, n, \frac{n}{2}+1, \frac{(n-1)s^2}{2}\right)}$$

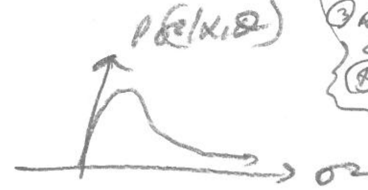
Sample
① Draw $\sigma^2 \sim P(\sigma^2|X)$
② Draw $\theta \sim P(\theta|X,\sigma^2)$
③ Return (θ,σ^2)
④ Repeat

If you just care about inferring variance with the mean as a nuisance parameter it is still inv. gamma

If $P(\sigma^2) \propto \frac{1}{\sigma^2} = \text{InvGamma}(0,0)$

$$P(\sigma^2|X,\theta) = \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$$

$$P(\sigma^2|X) = \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$



the smaller α is, the more dispersed the dist is

$$P(x^*|X) = \int_0^\infty \int_{\mathbb{R}} P(x^*|\theta,\sigma^2)\, P(\theta,\sigma^2|X)\, d\theta\, d\sigma^2$$

Assuming $P(\theta,\sigma^2) \propto \frac{1}{\sigma^2}$...

$$\propto \int_0^\infty \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x^*-\theta)^2}\right) (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}\, d\theta\, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-\frac{1}{2}} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \left(\int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}\left((x^*-\theta)^2 + n(\theta-\bar{x})^2\right)} d\theta\right) d\sigma^2$$

$$e^{-\frac{1}{2\sigma^2}\left(x^{*2} - 2x^*\theta + \theta^2 + n\theta^2 - 2n\theta\bar{x} + n\bar{x}^2\right)}$$

$$= \int_0^\infty (\sigma^2)^{-\frac{n+1}{2}-1} e^{-\frac{(n-1)s^2/2 + x^{*2}/2 + n\bar{x}^2/2}{\sigma^2}} \left(\int_{\mathbb{R}} e^{\frac{x^*+n\bar{x}}{\sigma^2}\theta - \frac{n+1}{2\sigma^2}\theta^2} d\theta\right) d\sigma^2$$

We know $\int_{\mathbb{R}} e^{a\theta - b\theta^2}\, d\theta = \sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}}$

$$\Rightarrow = \int_0^\infty (\sigma^2)^{-\frac{n+1}{2}-1} e^{-\frac{(n-1)s^2/2 + x^{*2}/2 + n\bar{x}^2/2}{\sigma^2}} \sqrt{\frac{\pi}{(n+1)/\cancel{2}\sigma^2}}\; e^{\frac{(x^* + n\bar{x})^2/(\sigma^2)^2}{2 \cdot \frac{(n+1)}{2\sigma^2}}}\, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{\overbrace{-\frac{n}{2}-\frac{1}{2}-1-\frac{1}{2}}^{\alpha:\ -\frac{n+2}{2}-1}} e^{-\frac{\overbrace{(n-1)s^2/2 + x^{*2}/2 + n\bar{x}^2/2 - (x^* + n\bar{x})^2/2(n+1)}^{\beta}}{\sigma^2}}\, d\sigma^2$$

$= \Gamma(\alpha)\beta^{-\alpha}$

$$= \Gamma\left(-\frac{n+2}{2}\right) \beta^{-\frac{n+2}{2}} \propto \beta^{-\frac{(n-1)+1}{2}} = \left(\frac{(n-1)s^2}{2} + \frac{x^{*2}}{2} + \frac{n\bar{x}^2}{2} + \frac{-x^{*2} - 2nx^*\bar{x} - n^2\bar{x}^2}{2n+2}\right)^{-\frac{(n-1)+1}{2}}$$

$$= (ax^{*2} + bx^* + c)^{-\frac{(n-1)+1}{2}} \propto \left(x^{*2} + \frac{b}{a}x^* + \frac{c}{a}\right)^{-\frac{(n-1)+1}{2}} = \left(\left(x^* + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)^{-\frac{(n-1)+1}{2}}$$

$$\propto \left(1 + \frac{\left(x^* + \frac{b}{2a}\right)^2}{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^{-\frac{(n-1)+1}{2}} = \left(1 + \frac{1}{n-1}\frac{\left(x^* + \frac{b}{2a}\right)^2}{\underbrace{\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)/n-1}}\right)^{-\frac{(n-1)+1}{2}} \propto T_{n-1}\left(\frac{-b}{2a}, \sqrt{\ }\right)$$

$$a = \frac{1}{2} - \frac{1}{2n+2} = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) = \frac{1}{2}\frac{n}{n+1}$$

$$b = -\frac{2n\bar{x}}{2n+2} = -\frac{n\bar{x}}{n+1}$$

$$c = \frac{(n-1)s^2}{2} + \frac{n\bar{x}^2}{2} - \frac{n^2\bar{x}^2}{2n+2} = \frac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)$$

$$-\frac{b}{2a} = \frac{\frac{n\bar{x}}{n+1}}{\cancel{2}\left(\frac{1}{\cancel{2}}\frac{n}{n+1}\right)} = \bar{x} \qquad \frac{b^2}{4a^2} = \bar{x}^2$$

$$\frac{c}{a} = \frac{\cancel{\frac{1}{2}}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)}{\cancel{\frac{1}{2}}\ \frac{n}{n+1}} = \frac{(n+1)(n-1)}{n}s^2 + (n+1)\bar{x}^2 - n\bar{x}^2 = \frac{(n+1)(n-1)}{n}s^2 + \bar{x}^2$$

$$\frac{c}{a} - \frac{b^2}{4a^2} = \frac{(n+1)(n-1)}{n}s^2 \Rightarrow \frac{\frac{c}{a} - \frac{b^2}{4a^2}}{n-1} = \frac{n+1}{n}s^2$$

$$= T_{n-1}\left(\bar{x}, s\sqrt{\frac{n+1}{n}}\right)$$

$$\approx N(\bar{x}, s^2)$$

n large...

makes sense!!