# Lec 20 Math 341 4/30/14

$\mathcal{F}$: $X_1, \ldots X_n \mid \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$$P(X^* \mid X) = \iint P(X^* \mid \theta, \sigma^2)\, P(\theta, \sigma^2 \mid X)\, d\theta\, d\sigma^2 = T_{n-1}\left(\bar{x}, s\sqrt{\tfrac{n+1}{n}}\right) \approx N(\bar{x}, s^2) \to N(\theta, \sigma^2)$$

Without doing the integral, could $P(X^* \mid X)$ be sampled from?

$$= \iint P(X^* \mid \theta, \sigma^2)\, P(\theta \mid X, \sigma^2)\, P(\sigma^2 \mid X)\, d\theta\, d\sigma^2 \quad \text{if } P(\theta, \sigma^2) \propto \frac{1}{\sigma^2} \ldots$$

Step I: sample $\sigma^2_{samp}$ from $\text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

II: sample $\theta_{samp}$ from $N\left(\bar{x}, \frac{\sigma^2}{n}\right)$

III: sample $X^*_{samp}$ from $N(\theta, \sigma^2)$

IV: repeat steps I–III many times

V: ship all the $X^*_{samp}$

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Back to inference..

$$P(\theta, \sigma^2 \mid X) \propto P(X \mid \theta, \sigma^2)\, P(\theta, \sigma^2) \overset{\checkmark}{=} \text{NormInvGamma}$$

If $P(\theta, \sigma^2) \propto \frac{1}{\sigma^2}$ only prior considered

What is the general conjugate prior? Note.

$$P(\theta, \sigma^2 \mid X) = \underbrace{(\sigma^2)^{-\frac{n}{2}}\, e^{-\frac{(n-1)s^2/2}{\sigma^2}}}_{\substack{K(\sigma^2 \mid X) \\ \propto \text{InvGamma}}}\ \underbrace{e^{-\frac{1}{2\sigma^2/n}(\theta - \bar{x})^2}}_{\substack{K(\theta \mid X, \sigma^2) \\ \propto \text{Norm}}}\ P(\theta, \sigma^2)$$

$$\Rightarrow\ = \underbrace{K(\sigma^2 \mid X)\, P(\sigma^2 \mid X)}_{\propto \text{InvGamma}}\ \overset{\propto \text{InvG}}{}\ K(\theta \mid X, \sigma^2)\, P(\theta \mid \sigma^2) \overset{\propto \text{Norm}}{} \propto \text{Norm}$$

$$P(\theta, \sigma^2) = P(\theta|\sigma^2)\,P(\sigma^2)$$

$$\Rightarrow P(\theta|\sigma^2) = N\left(\mu_0, \frac{\sigma^2}{n_0}\right),\ P(\sigma^2) = \text{InGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$$

$$\Rightarrow P(\theta, \sigma^2|X) = \text{Norm In Gamma (complicated)}$$ ← we won't bother!!

The $P(\theta, \sigma^2) \propto \frac{1}{\sigma^2} \Rightarrow P(\theta|\sigma^2) = N(\theta, \infty),\ P(\sigma^2) = \text{InGamma}(0,0)$

the only one that works!

But...

$\tau^2 \neq \frac{\sigma^2}{n_0}$ since $\sigma^2$ unknown to prior

What if $P(\theta, \sigma^2) = P(\theta)P(\sigma^2)$ where $P(\theta) = \underbrace{N(\mu_0, \tau^2)}_{\text{conj prior for model with } \sigma^2 \text{ known}},\ P(\sigma^2) = \underbrace{\text{InGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)}_{\text{conj prior for model with } \theta \text{ known}}$

What happens?

$$P(\theta, \sigma^2|X) \propto P(X|\theta, \sigma^2)\,P(\theta)\,P(\sigma^2) \propto k(X|\theta, \sigma^2)\,k(\theta)\,k(\sigma^2)$$

$$= \left((\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{x}-\theta)^2\right)}\right)\left(e^{-\frac{1}{2\tau^2}(\theta-\mu_0)^2}\right)\left((\sigma^2)^{-\frac{n_0}{2}-1} e^{-\frac{n_0\sigma_0^2/2}{\sigma^2}}\right)$$

Note $-\frac{n(\bar{x}-\theta)^2}{2\sigma^2} = -\frac{n\bar{x}^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}$, $-\frac{1}{2\tau^2}(\theta-\mu_0)^2 = -\frac{\theta^2}{2\tau^2} + \frac{\theta\mu_0}{\tau^2} - \frac{\mu_0^2}{2\tau^2}$

$$= (\sigma^2)^{-\frac{n}{2}-\frac{n_0}{2}-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n_0\sigma_0^2 + n\bar{x}^2\right)} \underbrace{e^{\overbrace{\left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}\right)}^{a}\theta - \overbrace{\left(\frac{n}{2\sigma^2} + \frac{1}{2\tau^2}\right)}^{b}\theta^2}}_{\propto N\left(\frac{a}{2b}, \frac{1}{2b}\right)}$$

From midterm 2...

$$N\left(\frac{a}{2b}, \frac{1}{2b}\right) = \sqrt{\frac{b}{\pi}}\, e^{-\frac{a^2}{4b}}\, e^{a\theta - b\theta^2}$$

$$\Rightarrow = \underbrace{(\sigma^2)^{-\left(\frac{n+n_0}{2}\right)-1} e^{-\frac{1}{2\sigma^2}\left(\overbrace{(n-1)s^2 + n_0\sigma_0^2 + n\bar{x}^2}^{d}\right)} \sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}}}_{= k(\sigma^2 | x)} \underbrace{\left(\sqrt{\frac{b}{\pi}}\, e^{-\frac{a^2}{4b}}\, e^{a\theta - b\theta^2}\right)}_{= P(\theta | \sigma^2 | x)}$$

try to find $P(\sigma^2|x)$ now...

$$k(\sigma^2|x) \propto (\sigma^2)^{-\left(\frac{n+n_0}{2}\right)-1} e^{-\frac{d}{2\sigma^2}} \underbrace{\left(\frac{n}{2\sigma^2} + \frac{1}{2\tau^2}\right)^{-\frac{1}{2}}} e^{\left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}\right)^2 / 4\left(\frac{n}{2\sigma^2} + \frac{1}{2\tau^2}\right)}$$

$\not\propto$ InvGamma Problem!!

Actually it's not anything!!

First time Bayes has failed us!!!

What do we do now?

If we want to sample from $P(\theta, \sigma^2 | x) = P(\theta | \sigma^2, x)\, P(\sigma^2 | x)$

we need a way to sample from $P(\sigma^2|x)$ only given $k(\sigma^2|x)$.

Luckily we can use computers!

Recall: $P(\sigma^2|x) = c\,k(\sigma^2|x)$, we can try to approximate the norm const $c$ through a procedure called "grid sampling"!

## Grid Sampling Algorithm

(I) Create a grid by setting $\sigma^2_{min}$, $\sigma^2_{max}$, $\Delta$ and computing...

$$G := \{\sigma^2_{min}, \sigma^2_{min} + \Delta, \sigma^2_{min} + 2\Delta, \ldots, \sigma^2_{max}\}$$

e.g. $\sigma^2_{min} = 0$, $\sigma^2_{max} = 1{,}000$, $\Delta = 0.1 \Rightarrow G = \{0, 0.1, 0.2, \ldots, 999.8, 999.9, 1000\}$

(II) Compute $c \approx \dfrac{1}{\sum_{\sigma^2 \in G} k(\sigma^2 | X)}$

Note: $P(\sigma^2 \in [\sigma^2_0 \pm \frac{\Delta}{2}] | X) \approx c\, k(\sigma^2_0 | X)$
kind of like a PMF!

(III) Compute the sampling CDF $F(\sigma^2_0 | X) := P(\sigma^2 \leq \sigma^2_0 | X) \approx \sum_{\{\sigma^2 \in G \,:\, \sigma^2 \leq \sigma^2_0\}} c\, k(\sigma^2 | X)$

(IV) Draw $u$ from $U(0,1)$ and locate $\sigma^2_{samp} = \min_{\sigma^2 \in G} \{F(\sigma^2 | X) \geq u\}$

(V) Repeat step IV as many times as you wish.

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Sampling from posterior (we did this last class)

$$P(\theta, \sigma^2 | X) \propto P(\theta | X, \sigma^2)\, P(\sigma^2 | X)$$

(I) Sample $\sigma^2_{samp}$ from grid sampler!

(II) Sample $\theta_{samp}$ from $N(\bar{X}, \sigma^2/n)$ where $\sigma^2 \overset{set}{=} \sigma^2_{samp}$

(III) Ship $\langle \theta_{samp}, \sigma^2_{samp} \rangle$

(IV) Repeat I–III as many times as desired