Math 341 Lec 8 2/26/19 CK

$H_0: \theta = \theta_0, \; H_1: \theta \neq \theta_0$

Two ideas

(a) $\alpha = 5\%$, declare $\delta = 0.01$, the margin of equivalence, the margin that you assume is a fair coin. Makes sense since there is a little extra weight on heads or tails.

$H_0: \theta \in [\theta_0 \pm \delta] \Rightarrow \theta \in [0.49, 0.51]$ New def of fair

$H_1: \theta \notin [\theta_0 \pm \delta] \Rightarrow \theta \in [0, 0.49) \cup (0.51, 1]$

$pval = P(H_0 | x) = F_{\theta | x}(.51) - F_{\theta | x}(.49) = pbeta(.51, 49, 50) - pbeta(.49, 49, 50)$

$= 0.06 \Rightarrow$ Fail to reject $H_0 \Rightarrow$ not enough evidence coin is unfair

Downside: need δ! Subjective!

(b) If theta_0 $\in CR_{\theta, 1-\alpha} \Rightarrow$ Retain H_0 else Reject H_0

This is highly contested. Downside: No p-val!

$CR_{\theta, 1-\alpha} = [qbeta(.025, 49, 50), qbeta(.975, 49, 50)] = [.337, .528]$

theta_0 = 0.5 $\in CR_{\theta, 1-\alpha} \Rightarrow FTR \; H_0$

There is another way of comparing models called Bayes Factors. We will do after midterm. Now I want to move towards prediction, another goal of statistics besides for inference.

See $X_1, \ldots, X_n$ but there will be a new X^* (X_{n+1}) realization

You don't know what it is ... but you want to know its distr.

Let X = Binomial. Let x^* have length $n^* = 1$. Prediction for 1 future obs.

$P(X^*|X)$. Frequentist ... what can you do?

$X^* \sim Bern(\hat{\theta}_{MLE})$. Good idea! But can fail badly if $\hat{\theta}_{MLE} = 0$ or 1

Another problem: what if $n^* > 1$

$X^* \sim Bin(n^*, \hat{\theta}_{MLE})$ correct? No... uncertainty in $\hat{\theta}_{MLE}$ not taken into account.

we will do this right after we do the $n^* = 1$ case.

Bayesian approach: posterior predictive distr.

$$P(X^*|X) = \int_{\Theta} P(X^*, \theta | X)\, d\theta = \int_{\Theta} \underbrace{P(x^*|\theta, x)}_{= P(x^*|\theta)} P(\theta|x)\, d\theta = \int_{\Theta} P(x^*|\theta) P(\theta|x)\, d\theta$$

likelihood of x^*

if $P(\theta|x)$ discrete

$$= \sum_{x \in \Theta} P(x^*|\theta) P(\theta|x)$$

Posterior: Everything known about θ after data is seen

This integral is an average of $P(X^*|\theta)$ weighted by the different probs of $\theta|X$.

$\Theta_0 = \{0.5, 0.75\}$

$X = \langle 0, 1, 1 \rangle$ $\quad P(\theta = 0.75 | X) = 0.53, \quad P(\theta = 0.5|X) = 0.47$

$X^* \sim ?$

$$P(x^*|x) = P(X^*|\theta = 0.75) P(\theta = 0.75|x) + P(X^*|\theta = 0.5) P(\theta = 0.5|x)$$
$$= (0.75)^{x^*}(0.25)^{1-x^*}(0.53) + (0.5)^{x^*}(1-.5)^{1-x^*}(0.47)$$

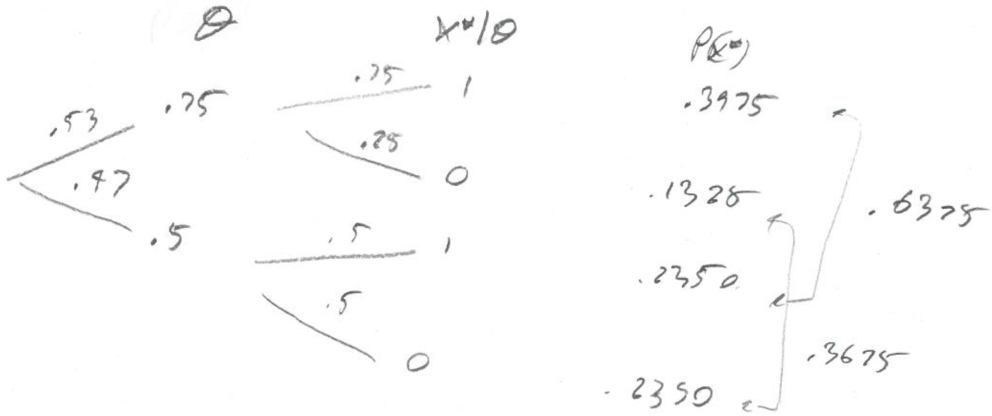
this is the answer... we can simplify it by plugging in 1, 0

we know $supp[x^*|x] = \{0,1\}$

$\Rightarrow$ If $x^* = 1$

$$P(x^*=1|x) = 0.75 \cdot 0.53 + 0.5 \cdot 0.47 = .6325$$

$$\Rightarrow P(x^*|x) = Bern(.6325)$$



Generally... $P(\theta) = Beta(\alpha, \beta)$, f = Binomial fixed n, length of x^* is $n^* = 1$ $\Rightarrow P(\theta|x) = Beta(x+\alpha, n-x+\beta)$

predictive for one obs.

$$P(x^*|x) = \int_\Theta P(x^*|\theta)\, P(\theta|x)\, d\theta = \int_0^1 \theta^{x^*}(1-\theta)^{1-x^*} \frac{1}{B(x+\alpha, n-x+\beta)} \theta^{x+\alpha-1} (1-\theta)^{n-x+\beta-1}\, d\theta$$

$$= \frac{1}{B(x+\alpha, n-x+\beta)} \int_0^1 \theta^{(x^*+x+\alpha)-1} (1-\theta)^{(n-x+\beta-x^*+1)-1}\, d\theta = \frac{B(x^*+x+\alpha,\ n-x+\beta-x^*+1)}{B(x+\alpha,\ n-x+\beta)}$$

this is the answer... but we can simplify...

we know $Supp(x^*|x) = \{0,1\}$

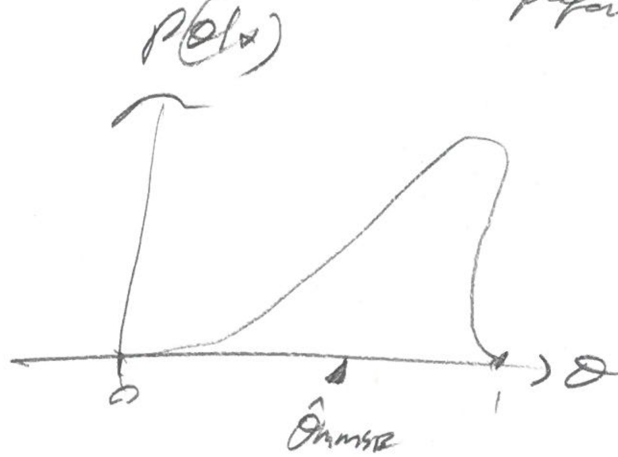
If $x^* = 1$ ...

$$P(X^*=1|x) = \frac{B(x+\alpha+1,\, n-x+\beta)}{B(x+\alpha,\, n-x+\beta)} = \frac{\dfrac{\Gamma(x+\alpha+1)\,\Gamma(n-x+\beta)}{\Gamma(n+\alpha+\beta+1)}}{B(x+\alpha,\, n-x+\beta)}$$

$$= \frac{\dfrac{(x+\alpha)}{(n+\alpha+\beta)}\overbrace{\dfrac{\Gamma(x+\alpha)\,\Gamma(n-x+\beta)}{\Gamma(n+\alpha+\beta)}}^{\cancel{B(x+\alpha,\, n-x+\beta)}}}{\cancel{B(x+\alpha,\, n-x+\beta)}} = \frac{x+\alpha}{n+\alpha+\beta} = \hat{\theta}_{MMSE}$$

... posterior version why this estimate is preferred

$$\Rightarrow P(X^*|x) = \text{Bern}\left(\frac{x+\alpha}{n+\alpha+\beta}\right)$$



Best guess of θ for next obs.

↑

Midterm I

Midterm II

↓

What if $n^* > 1$ See $n=7$ data pts and want data for $n^* = 5$ new observations

Would it be $X^* \sim \text{Bin}\left(n^*, \frac{x+\alpha}{n+\alpha+\beta}\right)$? No...