Math 341 Lec 12 3/19/19 L1

$P(X|\theta) = \text{Binomial}(n, \theta) \Rightarrow \rho_J(\theta) = \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right)$

$\phi = \frac{\theta}{1-\theta}$

$P(X|\phi) = \text{Binomial}\left(1, \frac{\phi}{\phi+1}\right)$

$$P(X|\theta) \xrightarrow{\text{Jeffrey's prior}} \rho_J(\theta) \Rightarrow P(\theta|X)$$

$$\updownarrow \phi = t(\theta) \qquad \updownarrow \text{ respects C.O.V.}$$

$$P(X|\phi) \xrightarrow{\text{Jeffrey's}} \rho_J(\phi) \Rightarrow P(\phi|X)$$

which is $\rho_J(\theta) \propto \sqrt{I(\theta)}$

needs $\ell(\theta; x)$

$$\Rightarrow \rho_J(\phi) = \frac{1}{\pi} \phi^{\overset{\frac{1}{2}-1}{\overset{\shortparallel}{-\frac{1}{2}}}} (1+\phi)^{-1} = F(1,1)$$

$= \text{BetaPrime}\left(\frac{1}{2}, \frac{1}{2}\right)$

F distr. Fisher Snedecor distribution

Now do $P(\phi)$ and $P(\theta)$ respect the change of variables?

$$P(\phi) = P_\theta\left(t^{-1}(\phi)\right) \left|\frac{d}{d\phi}\left[t^{-1}(\phi)\right]\right|$$

$$= \frac{1}{\pi} \left(\frac{\phi}{\phi+1}\right)^{-\frac{1}{2}} \left(\frac{1}{\phi+1}\right)^{-\frac{1}{2}} \left|\frac{1}{(\phi+1)^2}\right|$$

$$= \frac{1}{\pi} \phi^{-\frac{1}{2}} (\phi+1)(\phi+1)^{-2} = \frac{1}{\pi} \phi^{-\frac{1}{2}} (1+\phi)^{-1} \quad \text{YES!}$$

We have a protocol that picks specific priors directly from likelihoods that is the same regardless of parameterization of the likelihood!

How is this possible? $\phi = t(\theta)$

$P(X|\theta) \longrightarrow P(\theta) \propto \sqrt{I(\theta)}$

$P(X|\phi) \longrightarrow P(\phi) \propto \sqrt{I(\phi)}$ with $P(\phi) = P_\theta\left(t^{-1}(\phi)\right) \left|\frac{d}{d\phi}\left[t^{-1}(\phi)\right]\right|$

Assume $P(\theta) \propto \sqrt{I(\theta)}$, prove $P(\phi) \propto \sqrt{I(\phi)}$

$$P(\phi) = P_\theta(\theta) \left|\frac{d\theta}{d\phi}\right| \propto \sqrt{I(\theta)} \left|\frac{d\theta}{d\phi}\right| = \sqrt{I(\theta) \frac{d\theta}{d\phi} \frac{d\theta}{d\phi}} = \sqrt{E\left[\ell'(\theta; X)^2\right] \frac{d\theta}{d\phi} \frac{d\theta}{d\phi}}$$

$$= \sqrt{E\left[\frac{d\ell}{\cancel{d\theta}} \frac{d\ell}{\cancel{d\theta}} \frac{\cancel{d\theta}}{d\phi} \frac{\cancel{d\theta}}{d\phi}\right]} = \sqrt{E\left[\frac{d\ell}{d\phi} \frac{d\ell}{d\phi}\right]} = \sqrt{E\left[\ell'(\phi; X)^2\right]} = \sqrt{I(\phi)} \checkmark$$

Jeffrey's used Fisher's own information against him!

Extra Credit

$$P(\phi|x) \propto P(x|\phi)\,P(\phi) = \left(\binom{n}{x}\left(\frac{\phi}{\phi+1}\right)^{x}\left(\frac{1}{\phi+1}\right)^{n-x}\right)\left(\frac{1}{\pi}\,\phi^{-\frac{1}{2}}(1+\phi)^{-1}\right)$$

$$\propto \phi^{x-\frac{1}{2}}(1+\phi)^{-n-1}$$

$$= \phi^{\overbrace{(x+\frac{1}{2})}^{\alpha}-1}\,(1+\phi)^{-\overbrace{\left(n+1-(x+\frac{1}{2})\right)}^{\beta}-\overbrace{(x+\frac{1}{2})}^{\alpha}}$$

$\propto$ Beta Prime $\left(x+\frac{1}{2},\ n-x+\frac{1}{2}\right)$

$x=2, n=6 \Rightarrow P(\theta|x) = \text{Beta}(2.5, 4.5)$ $\quad \hat{\theta}_{MMSE} = 0.357$ $\quad \hat{\theta}_{MMAE} = 0.3429$

$\Rightarrow P(\phi|x) = \text{Beta Prime}(2.5, 4.5)$ $\quad \hat{\phi}_{MMSE} = 0.714 \Rightarrow \theta =$ $\quad \hat{\phi}_{MMAE} = 0.5217 \Rightarrow \theta = 0.3429$

Because $P(\theta|x)$ and $P(\phi|x)$, the posterior are invariant to t under Jeffreys prior. Quantiles are invariant to monotonic transformations. Expectation is not!

let $q := Q[X,\rho]$, $Y = g(X)$. Is $g(q) = Q[Y,\rho]$? g is strictly increasing.

$$\rho = F_X(q) := P(X \le q) = P(g(X) \le g(q)) = P(Y \le g(q)) = F_Y(g(q))$$ ✓

$P(x|\theta) = \text{Bin}(n, \theta)$

$P(\theta) = \text{Beta}(1,1)$ Laplace / Indifference
$P(\theta) = \text{Beta}(0,0)$ Haldane / total ignorance
$P(\theta) = \text{Beta}(\frac{1}{2},\frac{1}{2})$ Jeffrey's Invariant Prior

} Principled Uninformative Priors

$P(\theta) = \text{Beta}(\alpha,\beta)$ when $\alpha+\beta$ is "large" due to n } Informative prior built using previous data to estimate α, β. You know the shrinkage.

$P(x|\theta) = \text{Bin}(n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$

Model # phone calls in call center

Imagine $n \to \infty$ and $\theta \to 0$ but $n\theta = \lambda$

Imagine $\lambda = 1$ $n = 1000$ & $\theta = \frac{1}{1000}$, $n = 1{,}000{,}000$, $\theta = \frac{1}{1{,}000{,}000}$ etc.

$$\lim_{n\to\infty} \binom{n}{x} \theta^x (1-\theta)^{n-x} = \lim \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x}{x!} \lim \frac{\overbrace{(n-1)(n-2)\cdots(n-x+1)}^{x \text{ terms}}}{(n)(n)\cdots(n)} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim\left(\frac{n-1}{n}\right)^{1} \cdot \lim\left(\frac{n-2}{n}\right)^{1} \cdots \lim\left(\frac{n-x+1}{n}\right)^{1} \underbrace{\lim\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \left(\lim\left(1-\frac{\lambda}{n}\right)\right)^{-x} = \frac{\lambda^x e^{-\lambda}}{x!} = \text{Poisson}(\lambda)$$

$\text{Supp}(X) = \{0, 1, \ldots\} = \mathbb{N}_0$

Param space $\lambda \in (0, \infty)$

Let θ denote λ now

$$P(\theta|x) \propto P(x|\theta) P(\theta) = \frac{\theta^x e^{-\theta}}{x!} P(\theta) \propto \theta^x e^{-\theta} k(\theta)$$

If we use the conjugate prior, then $k(\theta) = \theta^a e^{-b\theta}$

so $P(\theta|x) \propto (\theta^x e^{-\theta})(\theta^a e^{-b\theta}) = \theta^{x+a} e^{-(b+1)\theta}$

Find $P(\theta) \Rightarrow$ Find c!

$$\int_0^\infty \theta^a e^{-b\theta}\, d\theta = \int_0^\infty \left(\frac{t}{b}\right)^a e^{-t} \frac{1}{b}\, dt = \frac{1}{b^{a+1}} \int_0^\infty t^{(a+1)-1} e^{-t}\, dt = \frac{\Gamma(a+1)}{b^{a+1}}$$

let $t = b\theta \Rightarrow \theta = \frac{t}{b} \Rightarrow dt = b\, d\theta \Rightarrow d\theta = \frac{1}{b} dt$

$$\Rightarrow P(\theta) = \frac{1}{c} k(\theta) = \frac{b^{a+1}}{\Gamma(a+1)} \theta^a e^{-b\theta} = \text{Gamma}(a+1, b)$$

Let $P(Y) = \text{Gamma}(\alpha, \beta) = \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} y^{\alpha-1} e^{-\beta\theta}$

three basic shapes:
α is the most important...

$\text{Supp}[Y] = (0, \infty)$

Param space: $\alpha > 0$, $\beta > 0$

$E(Y) = \frac{\alpha}{\beta}$

$\text{Var}(Y) = \frac{\alpha}{\beta^2}$

Calculus Exercises

$\text{Mode}(Y) = \frac{\alpha - 1}{\beta}$ if $\alpha > 0$

$\text{Med}(Y) = \text{qgamma}(0.5, \alpha, \beta)$ numerical integration, no closed form solution

~~$P(x|\theta) = \text{Poisson}(\theta)$~~ $\leftarrow$ One Observation

$P(\theta) = \text{Gamma}(\alpha, \beta)$

Gamma is conj. prior for Poisson lik. model

$$P(\theta|x) \propto P(x|\theta) P(\theta) = \left(\frac{\theta^x e^{-\theta}}{x!}\right)\left(\frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} \theta^{\alpha-1} e^{-\beta\theta}\right) \propto \theta^{x+\alpha-1} e^{-(\beta+1)\theta} \propto \text{Gamma}(x+\alpha, 1+\beta)$$

$X_1, \ldots, X_n | \theta \overset{iid}{\sim} \text{Poisson}(\theta)$

$$P(\theta|x) \propto \left(\prod_{i=1}^n P(x|\theta)\right) P(\theta) = \left(\prod_{i=1}^n \frac{\theta^{x_i} e^{-\theta}}{x!}\right)\left(\frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} \theta^{\alpha-1} e^{-\beta\theta}\right) \propto \left(\theta^{\sum x_i} e^{-n\theta}\right)\left(\theta^{\alpha-1} e^{-\beta\theta}\right) \propto \text{Gamma}\left(\sum x_i + \alpha, n+\beta\right)$$