Math 341 / 650 Spring 2019
Midterm Examination One

Solutions

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Full Name ______________________________

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

______________________________ signature ______________ date

Instructions

This exam is seventy five minutes and closed-book. You are allowed **one** page (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If the question reads "compute," this means the solution will be a number otherwise you can leave the answer in *any* widely accepted mathematical notation which could be resolved to an exact or approximate number with the use of a computer. I advise you to skip problems marked "[Extra Credit]" until you have finished the other questions on the exam, then loop back and plug in all the holes. I also advise you to use pencil. The exam is 100 points total plus extra credit. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. Good luck!

Distribution of r.v.	Quantile Function	PMF / PDF function	CDF function	Sampling Function
beta	qbeta$(p,\ \alpha,\ \beta)$	d-$(x,\ \alpha,\ \beta)$	p-$(x,\ \alpha,\ \beta)$	r-$(\alpha,\ \beta)$
betabinomial	qbetabinom$(p,\ n,\ \alpha,\ \beta)$	d-$(x,\ n,\ \alpha,\ \beta)$	p-$(x,\ n,\ \alpha,\ \beta)$	r-$(n,\ \alpha,\ \beta)$
betanegativebinomial	qbeta_nbinom$(p,\ r,\ \alpha,\ \beta)$	d-$(x,\ r,\ \alpha,\ \beta)$	p-$(x,\ r,\ \alpha,\ \beta)$	r-$(r,\ \alpha,\ \beta)$
binomial	qbinom$(p,\ n,\ \theta)$	d-$(x,\ n,\ \theta)$	p-$(x,\ n,\ \theta)$	r-$(n,\ \theta)$
exponential	qexp$(p,\ \theta)$	d-$(x,\ \theta)$	p-$(x,\ \theta)$	r-(θ)
gamma	qgamma$(p,\ \alpha,\ \beta)$	d-$(x,\ \alpha,\ \beta)$	p-$(x,\ \alpha,\ \beta)$	r-$(\alpha,\ \beta)$
geometric	qgeom$(p,\ \theta)$	d-$(x,\ \theta)$	p-$(x,\ \theta)$	r-(θ)
inversegamma	qinvgamma$(p,\ \alpha,\ \beta)$	d-$(x,\ \alpha,\ \beta)$	p-$(x,\ \alpha,\ \beta)$	r-$(\alpha,\ \beta)$
negative-binomial	qnbinom$(p,\ r,\ \theta)$	d-$(x,\ r,\ \theta)$	p-$(x,\ r,\ \theta)$	r-$(r,\ \theta)$
normal (univariate)	qnorm$(p,\ \theta,\ \sigma)$	d-$(x,\ \theta,\ \sigma)$	p-$(x,\ \theta,\ \sigma)$	r-$(\theta,\ \sigma)$
poisson	qpois$(p,\ \theta)$	d-$(x,\ \theta)$	p-$(x,\ \theta)$	r-(θ)
T (standard)	qt$(p,\ \nu)$	d-$(x,\ \nu)$	p-$(x,\ \nu)$	r-(ν)
uniform	qunif$(p,\ a,\ b)$	d-$(x,\ a,\ b)$	p-$(x,\ a,\ b)$	r-$(a,\ b)$

Table 1: Functions from R (in alphabetical order) that can be used on this exam with their arguments. The hyphen in colums 3, 4 and 5 is shorthand notation for the full text of the r.v. which can be found in column 2.

Problem 1 Let θ represent the true probability a specific couple (i.e. a mother and father) will have a male baby. Assume the model $\mathcal{F} = \text{Binomial}\,(n,\ \theta)$ where n is the number of children this mother-father couple have.

(a) [6 pt / 6 pts] Describe what the assumption $\mathcal{F}$ means in this real-world context for the couple John and Susan. Indicate what is known/unknown.

(1) The gender of each of J&S's children is independent. (2) The probability of each child being male is the same (θ) and (3) unknown.

(b) [5 pt / 11 pts] John and Susan do not have any kids yet. Using the principle of indifference, what is a distribution that describes the uncertainty in their θ?

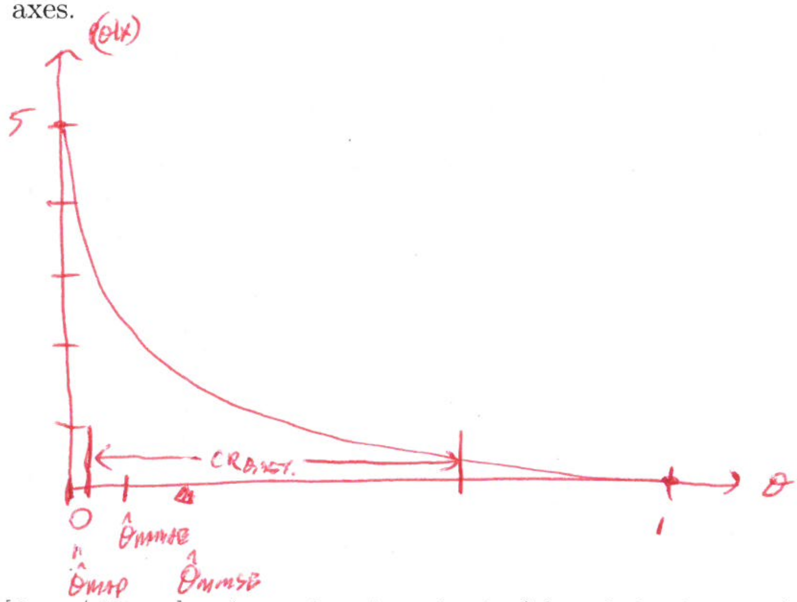
$$P(\theta) = U(0,1) = \text{Beta}(1,1)$$

(c) [6 pt / 17 pts] John and Susan have four kids which are all girls. Assuming the prior in (b), provide the explicit PMF or PDF of the distribution that describes the uncertainty in their θ after you see the gender of their first four kids. Simplify.

$$P(\theta|x) = \text{Beta}(1,5) = \frac{1}{B(1,5)}\,\theta^{1-1}(1-\theta)^{5-1} = 5(1-\theta)^4$$

$$\frac{1}{B(1,5)} = \frac{\Gamma(6)}{\Gamma(1)\Gamma(5)} = \frac{5\,\Gamma(5)}{\Gamma(5)}$$

(d) [7 pt / 24 pts] Plot this PMF or PDF (as best as you can) in the space below. It does not have to be drawn to scale but label the axes and important points on the axes.



(e) [6 pt / 30 pts] Assuming the prior in (b) and the data in (c), find all three Bayesian point estimates for John and Susan's θ and notate them appropriately. Mark them on the plot in (d).

$\hat{\theta}_{MMSE} = \frac{1}{1+5} = \frac{1}{6}$

$\hat{\theta}_{MAP} = 0$ (see plot)

$\hat{\theta}_{MMAE} = \text{qbeta}(0.5, 1, 5)$

(f) [4 pt / 34 pts] Assuming the prior in (b) and the data in (c), what is the distribution of X^*, the r.v. modeling the gender of John and Susan's fifth child?

$P(X^* | x) = \text{Bern}\left(\frac{1}{6}\right)$

(g) [4 pt / 38 pts] Assuming the prior in (b) and the data in (c), what is your best guess (the one that minimizes mean squared error) of the gender of John and Susan's fifth child?

0 (female)

(h) [10 pt / 48 pts] Assuming the prior in (b) and the data in (c), find a credible region for θ. Notate your answer appropriately. Then estimate where this region would be on your plot in (d) *as best as you can.* Assume 95% region

$$CR_{\theta, 95\%} = \left[\text{qbeta}(0.025, 1, 5), \text{qbeta}(0.975, 1, 5) \right]$$

(i) [10 pt / 58 pts] Assuming the prior in (b) and the data in (c), test the theory that their preference for girl or boy births is uneven *as best as you can.* State the hypotheses clearly using mathematical notation. Find the Bayesian p_{val} if possible.

$H_0: \theta = 0.5$
$H_a: \theta \neq 0.5$

If $0.5 \in CR_{\theta, 95\%}$ $\Rightarrow$ Retain H_0, otherwise reject H_0. No pval available.

↑ from (h)

(j) [5 pt / 63 pts] Although the prior in (b) is reasonable and the data in (c) is very possible, scientists believe the point estimates you found in (e) are too low. They can say this because they examined tons of data from couples before John and Susan. In this case, we can incorporate that prior observation into a new prior: $\mathbb{P}(\theta) =$

Beta (34, 32). Assuming this prior and the data in (c), find all three Bayesian point estimates for John and Susan's θ and notate them appropriately. Round to two decimal places.

$$\hat{\theta}_{MMSE} = \frac{34}{4+32+34} = \frac{34}{\cancel{36}\,70} = 0.49$$

$$\hat{\theta}_{MAP} = \frac{34-1}{70-2} = 0.49$$

$$\hat{\theta}_{MMAE} = \text{qbeta}(0.5, 34, 36)$$

(k) [5 pt / 68 pts] One of the three Bayesian point estimates in (j) is a "shrinkage estimator". Which one? Calculate the degree of shrinkage. Round to two decimal places.

$\hat{\theta}_{MMSE}$ $$\rho = \frac{\alpha+\beta}{n+\alpha+\beta} = \frac{66}{70} = 0.94$$

(l) [7 pt / 75 pts] Is the prior in (j) "informative" or "uninformative" in context of the data in (c)? Provide mathematical evidence for your answer.

Informative. The prior is equivalent to seeing $n_0 = 70$ children and J&S only have 4 children. $\rho = 0.95$ means that the prior expectation is weighted 95% in the MMSE pt. estimate with the data only weighted at 5%.

Problem 2 Consider $X_1, \ldots, X_n \stackrel{iid}{\sim}$ Beta $(\theta, 1)$. $= \frac{1}{B(\theta,1)} x^{\theta-1}(1-x)^{1-1}$

(a) [6 pt / 81 pts] Find $\mathcal{L}(\theta; X_1, \ldots, X_n)$. Simplify so that your answer does not include the $B(\cdot, \cdot)$ function or the $\Gamma(\cdot)$ function.

$$\mathcal{L}(\theta|x) = \prod_{i=1}^{n} p(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{B(\theta,1)} x_i^{\theta-1}(1-x_i)^{1-1} = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$$

$$\frac{1}{B(\theta,1)} = \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} = \frac{\theta\,\cancel{\Gamma(\theta)}}{\cancel{\Gamma(\theta)}}$$

(b) [4 pt / 85 pts] Find $\ell(\theta; X_1, \ldots, X_n)$. Simplify as much as possible.

$$\ell(\theta|x) = \ln(\mathcal{L}(\theta|x)) = n\ln(\theta) + (\theta-1)\ln\left(\prod_{i=1}^{n} x_i\right) = n\ln(\theta) + \theta\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\ln(x_i)$$

(c) [3 pt / 88 pts] Find $\hat{\theta}_{\text{MLE}}$. Set $\ell'(\theta; x) = 0$ and solve for θ.

$$\ell'(\theta; x) : \frac{n}{\theta} + \sum_{i=1}^{n} \ln(x_i) \overset{\text{set}}{=} 0 \Rightarrow n = -\theta \sum_{i=1}^{n} \ln(x_i) \Rightarrow \hat{\theta}_{\text{MLE}} = -\frac{n}{\sum_{i=1}^{n} \ln(x_i)}$$

Problem 3 Below are some theoretical problems that are independent from each other.

(a) [6 pt / 94 pts] In the frequentist perspective where no randomness is allowed in θ, prove that $\mathbb{P}(X) = \mathbb{P}(X \mid \theta = \theta^*)$ where θ^* is the true value of θ.

$$\Theta = \{\theta^*\}$$

$$P(X) = \sum_{\theta \in \Theta} P(X|\theta) P(\theta) = P(X | \theta = \theta^*) \underbrace{P(\theta = \theta^*)}_{1} = P(X | \theta = \theta^*)$$

(b) [6 pt / 100 pts] If $\mathcal{F} = \underbrace{\overset{iid}{\sim} \text{Bernoulli}(\theta)}_{\text{not necessary}}$, prove that $\mathbb{P}(X_2 \mid \theta, X_1) = \mathbb{P}(X_2 \mid \theta)$.

by $\mathcal{F}$ = iid [something]

$$P(X_2 | \theta, X_1) = \frac{P(\theta, X_1, X_2)}{P(\theta, X_1)} = \frac{P(X_1, X_2 | \theta) \cancel{P(\theta)}}{P(X_1 | \theta) \cancel{P(\theta)}} = \frac{\cancel{P(X_1 | \theta)} P(X_2 | \theta)}{\cancel{P(X_1 | \theta)}} = P(X_2 | \theta)$$

(c) [3 pt / 103 pts] [Extra credit] Demonstrate for $H_0 : \theta \leq \theta_0$ or $H_0 : \theta \geq \theta_0$ that $\mathbb{P}(X \mid H_0) = \frac{1}{\mathbb{P}(H_0)} \int_{H_0} \mathbb{P}(X \mid \theta) \mathbb{P}(\theta) \, d\theta$.