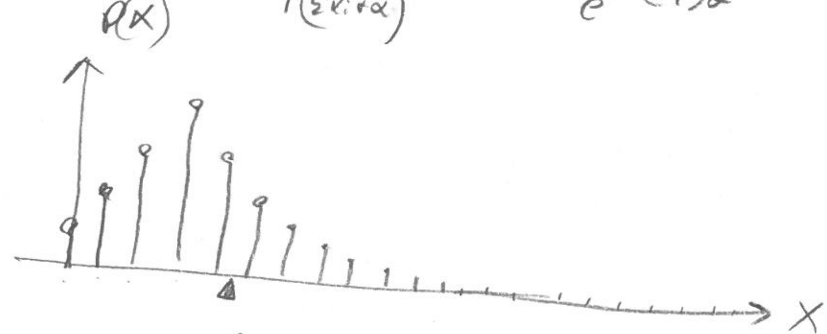
Math 341 Lec 13 3/21/19

Ex: $\overbrace{X_1, \ldots, X_n}^{X} | \theta \overset{iid}{\sim} \text{Poisson}(\theta) := \overbrace{\frac{e^{-\theta}\theta^x}{x!}}^{PMF}$

$$P(\theta) = \text{Gamma}(\alpha, \beta) := \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha - 1} e^{-\beta\theta}}_{PDF}$$

$$\Rightarrow P(\theta | X) = \text{Gamma}\left(\sum x_i + \alpha, n + \beta\right) = \frac{(n+\beta)^{\sum x_i + \alpha}}{\Gamma(\sum x_i + \alpha)} \theta^{\sum x_i + \alpha - 1} e^{-(n+\beta)\theta}$$

$\Rightarrow$ Gamma distr. is conj. prior for Poisson likelihood



$\theta = 4.4$ for example

Pt. Est's

$$\hat{\theta}_{MMSE} = \frac{\sum x_i + \alpha}{n+\beta}$$

$$\hat{\theta}_{MAP} = \frac{\sum x_i + \alpha - 1}{n+\beta} \quad \text{if } \sum x_i + \alpha > 1$$

$$\hat{\theta}_{MMAE} = qgamma(0.5, \sum x_i + \alpha, n+\beta)$$

$$P(\theta|x) = Gamma(\sum x_i + \alpha,\ n+\beta)$$

pseudosuccesses (x_0) # pseudotrials (n_0)

MLE?

$$\ell(\theta;x) = -n\theta + \sum x_i \ln(\theta) - \ln\left(\prod x_i!\right)$$

$$\ell'(\theta;x) = -n + \frac{\sum x_i}{\theta} \overset{set}{=} 0$$

$$\sum x_i = n\theta \Rightarrow \hat{\theta}_{MLE} = \bar{X}$$

Cred. Region? SAME

$$CR_{\theta,1-\alpha_0} = \left[qgamma\left(\frac{\alpha_0}{2}, \sum x_i+\alpha, n+\beta\right),\ qgamma\left(1-\frac{\alpha_0}{2}, \sum x_i+\alpha, n+\beta\right)\right]$$

Hyp. Tests? SAME!

Shrinkage? $\mathcal{L}(\theta;x) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i} = \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!}$

$$\hat{\theta}_{MMSE} = \frac{\sum x_i}{n+\beta}\cdot\frac{n}{n} + \frac{\alpha}{n+\beta}\cdot\frac{\beta}{\beta} = \underbrace{\frac{n}{n+\beta}}_{1-\rho}\hat{\theta}_{MLE} + \underbrace{\frac{\beta}{n+\beta}}_{\rho} E[\theta]$$

Laplace Prior ... not possible as a proper PDF. $P(\theta) \propto 1$

$$\Rightarrow P(\theta|x) \propto e^{-n\theta}\theta^{(\sum x_i+1)-1}(1) \propto Gamma(\sum x_i+1, n)$$

$$\Rightarrow P(\theta) = Gamma(1,0) \text{ improper}$$

Improper priors are a limit of proper priors

What about $P(\theta) = Gamma(1,\varepsilon) = \frac{\overset{\approx 0}{\varepsilon^2}}{\underbrace{\Gamma(2)}_{=1\Gamma(1)=1}}\overbrace{\theta^{1-1}}^{1}\overbrace{e^{-\varepsilon\theta}}^{\approx 1} \approx \varepsilon^2$ everywhere

$\Rightarrow P(\theta) = Gamma(1,0)$ is the closest thing to prior of indifference. It is improper!

$\Rightarrow P(\theta|x) = Gamma(\sum x_i + 1, n)$ always proper!!! since $n \geq 1$.

Laplace prior: I've seen $X_0 = 1$ (i.e. an obs. with one success) with the weight of $n_0 = 0$. Strange!

Haldane prior (total ignorance) $P(\theta) = \text{Gamma}(0,0)$ improper of course!

From pseudocounts... $x_0 = 0,\ n_0 = 0$ $\Rightarrow$

$$\Rightarrow P(\theta|x) = \text{Gamma}(\textstyle\sum x_i, n) \quad \text{proper only if } \sum x_i > 0.$$

$$\Rightarrow \hat{\theta}_{MMSE} = \frac{\sum x}{n} = \hat{\theta}_{MLE}$$

like before

Jeffreys Prior?

$$-\ell''(\theta; x) = +\frac{\sum x_i}{\theta^2}$$

$$I(\theta) = E[-\ell''(\theta;x)] = E\left[\frac{\sum x_i}{\theta^2}\right] = \frac{1}{\theta^2} E\left[\textstyle\sum x_i\right] = \frac{n}{\theta^2} E[X_i]$$

$$E[X_i] = \sum_{x=0}^{\infty} x \frac{e^{-\theta}\theta^x}{x!} = e^{-\theta}\sum_{x=1}^{\infty} x\frac{\theta^x}{x!} = e^{-\theta}\sum_{x=1}^{\infty}\frac{\theta^x}{(x-1)!} = e^{-\theta}\sum_{y=0}^{\infty}\frac{\theta^{y+1}}{y!}$$

$$= \theta e^{-\theta} \underbrace{\boxed{\sum_{y=0}^{\infty}\frac{\theta^y}{y!}}}_{\text{Taylor series for } e^{\theta}} = \theta \cancel{e^{-\theta}} \cancel{e^{\theta}} = \boxed{\theta}$$

$$\Rightarrow I(\theta) = \frac{n}{\theta^2}(\theta) = \frac{n}{\theta} \Rightarrow P_J(\theta) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\theta}} \propto \sqrt{\theta^{-1}} = \theta^{-1/2} = \theta^{\frac{1}{2}-1}$$

$$\propto \text{Gamma}\left(\tfrac{1}{2}, 0\right)$$

Improper!

$$\Rightarrow P(\theta|x) = \text{Gamma}\left(\textstyle\sum x_i + \frac{1}{2},\ n\right)$$

always proper!!

(4)

Posterior predictive distribution for $n^* = 1$ i.e. one future observation. X^* s.t.

$$P(X^*|X) = \int_\Theta P(X^*|\theta)\, P(\theta|x)\, d\theta = \int_0^\infty \left(\frac{e^{-\theta}\theta^{x^*}}{x^*!}\right)\left(\frac{(n+\beta)^{\Sigma x_i+\alpha}}{\Gamma(\Sigma x_i+\alpha)}\theta^{\Sigma x_i+\alpha-1} e^{-(n+\beta)\theta}\right) d\theta$$

$$= \frac{(n+\beta)^{\Sigma x_i+\alpha}}{x^*!\,\Gamma(\Sigma x_i+\alpha)} \int_0^\infty \theta^{(x^*+\Sigma x_i+\alpha)-1} e^{-(n+\beta+1)\theta}\, d\theta$$

let $t = (n+\beta+1)\theta \Rightarrow \theta = \frac{t}{n+\beta+1} \Rightarrow d\theta = \frac{dt}{n+\beta+1}$

$$= \frac{(n+\beta)^{\Sigma x_i+\alpha}}{x^*!\,\Gamma(\Sigma x_i+\alpha)} \int_0^\infty \frac{t^{(x^*+\Sigma x_i+\alpha)-1}}{(n+\beta+1)^{x^*+\Sigma x_i+\alpha-1}} e^{-t} \frac{dt}{n+\beta+1}$$

cancel

$$= \frac{(n+\beta)^{\Sigma x_i+\alpha}}{x^*!\,\Gamma(\Sigma x_i+\alpha)\,(n+\beta+1)^{x^*+\Sigma x_i+\alpha}}\, \Gamma(x^*+\Sigma x_i+\alpha)$$

$$= \left(\frac{n+\beta}{n+\beta+1}\right)^{\Sigma x_i+\alpha} \left(\frac{1}{n+\beta+1}\right)^{x^*} \frac{\Gamma(x^*+\Sigma x_i+\alpha)}{x^*!\,\Gamma(\Sigma x_i+\alpha)} = \frac{\Gamma(x^*+r)}{x^*!\,\Gamma(r)} p^r (1-p)^{x^*}$$

let $p := \frac{n+\beta}{n+\beta+1} \in (0,1)$ and $1-p := \frac{1}{n+\beta+1} = 1 - \frac{n+\beta}{n+\beta+1}$

$$= \text{Extended NegBin}\Big(\underbrace{\Sigma x_i+\alpha}_{r},\ \underbrace{\tfrac{n+\beta}{n+\beta+1}}_{p}\Big)$$

~~Extended~~

let $r = \Sigma x_i + \alpha$

NegBin is the sum of r geometric r.v.'s i.e. wait for r successes for iid Bern(p) experiments