Lec 18 Math 341 3/26/13

Pr: $X_1, \dots, X_n | \theta \overset{iid}{\sim}$ Poisson (θ)

$p(\theta) = \text{Gamma}(\alpha, \beta)$. then for the one next observation X^*,

$$p(X^* | X) = \int_{\Theta} p(X^* | \theta)\, p(\theta | X)\, d\theta = \dots = \frac{(n+\beta)^{\sum x_i + \alpha}\, \Gamma(x^* + \sum x_i + \alpha)}{x^*!\, \Gamma(\sum x_i + \alpha)\, (n+\beta+1)^{x^* + \sum x_i + \alpha}}$$

$$= \frac{\Gamma(x^* + \sum x_i + \alpha)}{x^*!\, \Gamma(\sum x_i + \alpha)} \left(\frac{n+\beta}{n+\beta+1}\right)^{\sum x_i + \alpha} \left(\frac{1}{n+\beta+1}\right)^{x^*}$$

Subst's: let $p := \frac{n+\beta}{n+\beta+1} \in (0,1) \Rightarrow 1 - p = \frac{1}{n+\beta+1}$

let $r := \sum x_i + \alpha$

"extended negative binomial"

$$= \frac{\Gamma(x+r)}{x^*!\, \Gamma(r)}\, p^r (1-p)^{x^*} = \text{Ext NegBin}(r, p)$$

Back to 241..

Negative Binomial

$X_1, \dots X_n | \theta \overset{iid}{\sim} \text{Geom}(p) = (1-p)^x p \quad T = \sum X_i \sim \text{NegBin}(r, p)$

Waiting time to first success

Count # of failures

$r \in \mathbb{N}$ # of waiting processes

$E[X_i] = \frac{p}{1-p}$

$p \in (0,1)$ prob of success

Why "extended neg binomial". We allow for fractional success r.

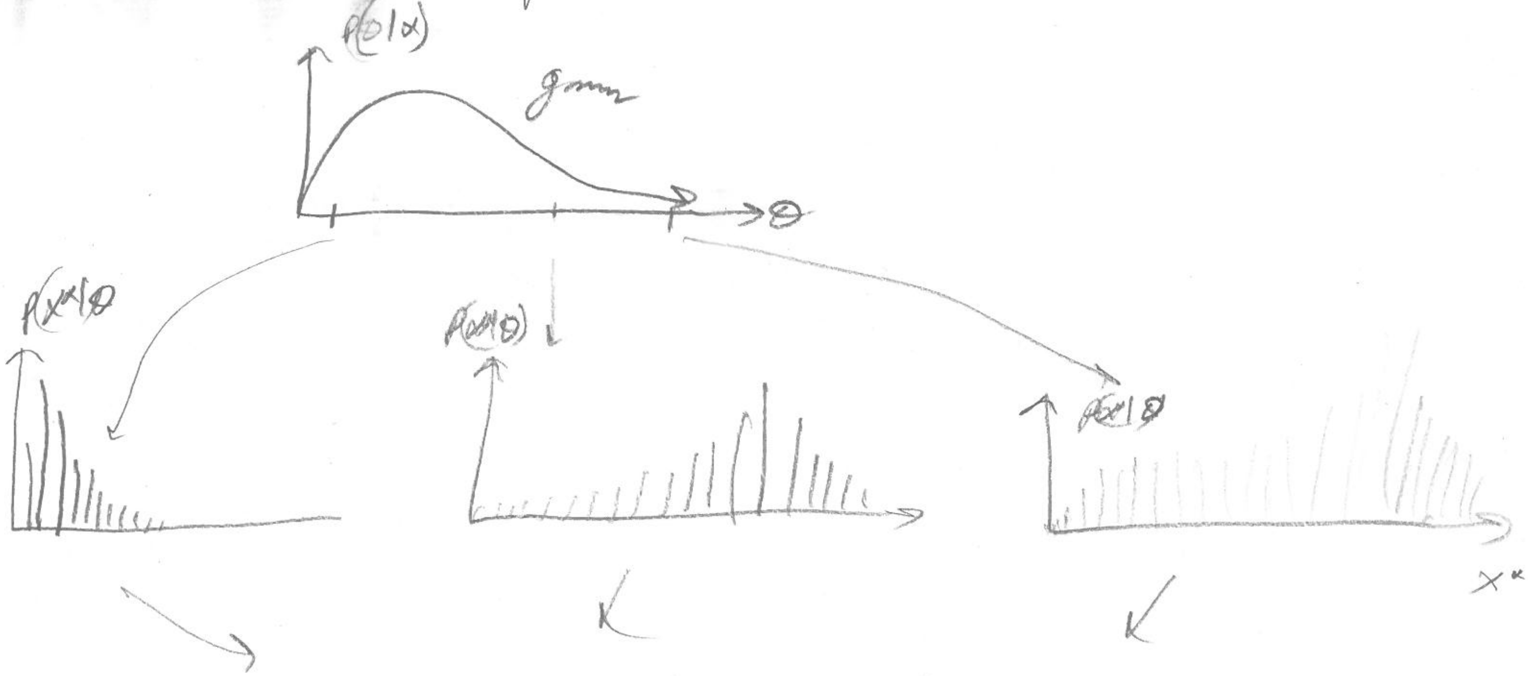
Param space: $r \in (0,\infty)$, $p \in (0,1)$

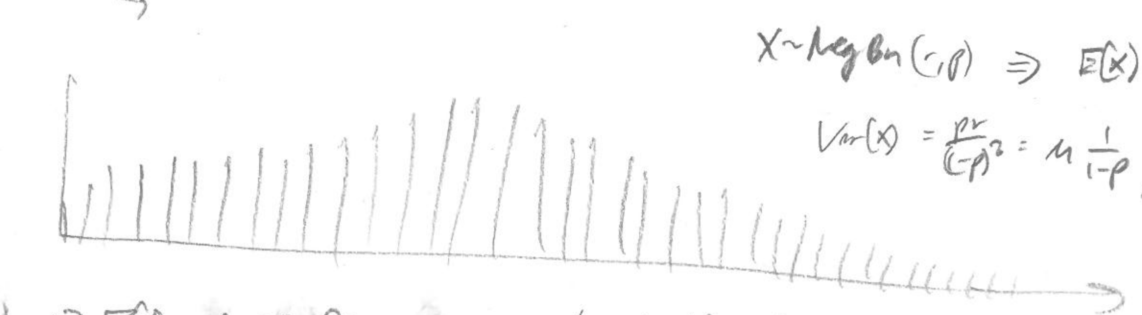
If $r \in \mathbb{N}$,

$$\frac{\Gamma(x^*+r)}{x^*!\,\Gamma(r)} = \frac{(x^*+r-1)!}{x^*!\,(r-1)!} = \binom{x^*+r-1}{x^*}$$

$$\Rightarrow p(X^*|X) = \text{NegBin}(r,p) = \binom{x^*+r-1}{x^*} p^r (1-p)^{x^*} \quad \text{(from Book 291).}$$

NegBin is Poisson dispersed:





$X \sim \text{NegBin}(r,p) \Rightarrow E(X) = r\frac{p}{1-p} = \mu$

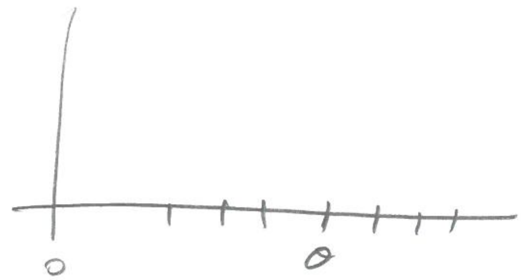
$Var(X) = \frac{pr}{(1-p)^2} = \mu \frac{1}{1-p}$, $\frac{1}{1-p} \in (1,\infty)$

You can disperse as much as you wish!

$X \sim \text{Poisson}(\mu) \Rightarrow E(X) = \mu$, $Var(X) = \mu$ no choice! Like Binomial

~~$X \sim \text{NegBin}(r,p) \Rightarrow E(X) = \frac{pr}{1-p} = \theta$, $SD(X) = \frac{pr}{(1-p)^2} = \theta \frac{1}{1-p}$ an increase by~~

Normal Model! One observation:



$$P(X|\underbrace{\theta,\sigma^2}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X-\theta)^2}$$

$\dim[\vec{\theta}] = 2$

$X \sim \text{Bin}(n,\theta)$

also dim 2. But n treated as fixed.

Here, we will do the same thing.

Find Kernels:

$$P(x|\theta,\sigma^2) \propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2} = e^{-\frac{1}{2\sigma^2}(x^2-2\theta x+\theta^2)} = e^{-\frac{x^2}{2\sigma^2}} e^{\frac{\theta x}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}}$$

$$\propto \underbrace{e^{-\frac{x^2}{2\sigma^2}} e^{\frac{\theta x}{\sigma^2}}}_{k(x|\theta,\sigma^2)} = e^{ax-bx^2}$$

where $a = \frac{\theta}{\sigma^2}$, $b = \frac{1}{2\sigma^2}$

Note: $\text{Var}[X] = \frac{1}{2b} = \frac{1}{2\left(\frac{1}{2\sigma^2}\right)} = \sigma^2$, $E[X] = \frac{a}{2b} = \frac{\frac{\theta}{\sigma^2}}{2\left(\frac{1}{2\sigma^2}\right)} = \theta$

$$P(\theta|x,\sigma^2) \propto \underbrace{e^{\frac{\theta x}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}}}_{k(\theta|x,\sigma^2)} = e^{a\theta-b\theta^2}$$

where $a = \frac{x}{\sigma^2}$, $b = \frac{1}{2\sigma^2}$

$\Rightarrow \text{Var}[\theta] = \sigma^2$, $E[\theta] = x$

$\propto N(x,\sigma^2)$ strange!! Switch x,θ!

$P(\sigma^2|x,\theta)$ ... next week we will do this!

F: $\underbrace{X_1,\ldots,X_n}_{X} | \theta,\sigma^2 \overset{iid}{\sim} N(\theta,\sigma^2)$ Many observations

$\Sigma x_i^2 - 2x_i\theta + \theta^2 = \Sigma x_i^2 - 2n\bar{x}\theta + n\theta^2$

$$P(X|\theta,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2}$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{\Sigma x_i^2}{2\sigma^2}} e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{-\frac{n\theta^2}{2\sigma^2}}$$

$$= \mathcal{L}(\theta; x, \sigma^2)$$

Find kernels:

$$P(\underset{q}{X}|\theta,\sigma^2) \propto e^{-\frac{\Sigma x_i^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2}}$$ this form will be important next week

n dim r.v.!

$$P(\theta|X,\sigma^2) \propto e^{\frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}} = e^{a\theta - b\theta^2}$$

where $a = \frac{n\bar{x}}{\sigma^2}$, $b = \frac{n}{2\sigma^2}$

$$\propto N(?, ?) = N\left(\frac{a}{2b}, \frac{1}{2b}\right)$$

$$Var[\theta] = \frac{1}{2b} = \frac{1}{2\left(\frac{n}{2\sigma^2}\right)} = \frac{\sigma^2}{n}$$

$$E[\theta] = \frac{a}{2b} = \frac{\frac{n\bar{x}}{\sigma^2}}{2\left(\frac{n}{2\sigma^2}\right)} = \bar{x}$$

$$\Rightarrow P(\theta|X,\sigma^2) = N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$ wow!

$P(\sigma^2|X,\sigma^2)$ ... next week we will do this

Back to Bayes... Assume σ^2 fixed, infer θ: $\mathcal{F}: X_1 \ldots X_n | \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

Let's try to find conjugate prior for inferring θ:

σ^2 considered fixed for prior as well

$$P(\theta|X,\sigma^2) \propto P(X|\theta,\sigma^2)\, P(\theta|\sigma^2)$$

$$\propto \left(e^{a\theta} e^{-b\theta^2}\right) ?$$

$$\Rightarrow P(\theta|\sigma^2) = N(?, ?)$$

$$\propto \left(e^{a\theta} e^{-b\theta^2}\right)\left(e^{\alpha\theta} e^{-\beta\theta^2}\right)$$

$$= e^{(a+\alpha)\theta - (b+\beta)\theta^2} \propto N(?, ?)$$

$$\cancel{Var(\theta) = \frac{1}{2(b+\beta)} = \frac{1}{2\left(\frac{n}{2\sigma^2} + \beta\right)}}$$

$\Rightarrow$ the normal distr is conjugate for the iid normal model with known variance ("normal-normal model")

Let $P(\theta|\sigma^2) = N(\mu_0, \tau^2)$ $\Rightarrow$ $\tau^2 = \frac{1}{2\beta} \Rightarrow \beta = \frac{1}{2\tau^2}$

$$\mu_0 = \frac{\alpha}{2\beta} \Rightarrow \alpha = \mu_0(2\beta) = \frac{\mu_0}{\tau^2}$$

Since $P(\theta|X,\sigma^2) = e^{(a+\alpha)\theta - (b+\beta)\theta^2}$