Lecture - 11 03/10/2020

$n=10$

$$x \sim Bin(n,\theta) := \binom{n}{x}\theta^x(1-\theta)^{n-x}.$$

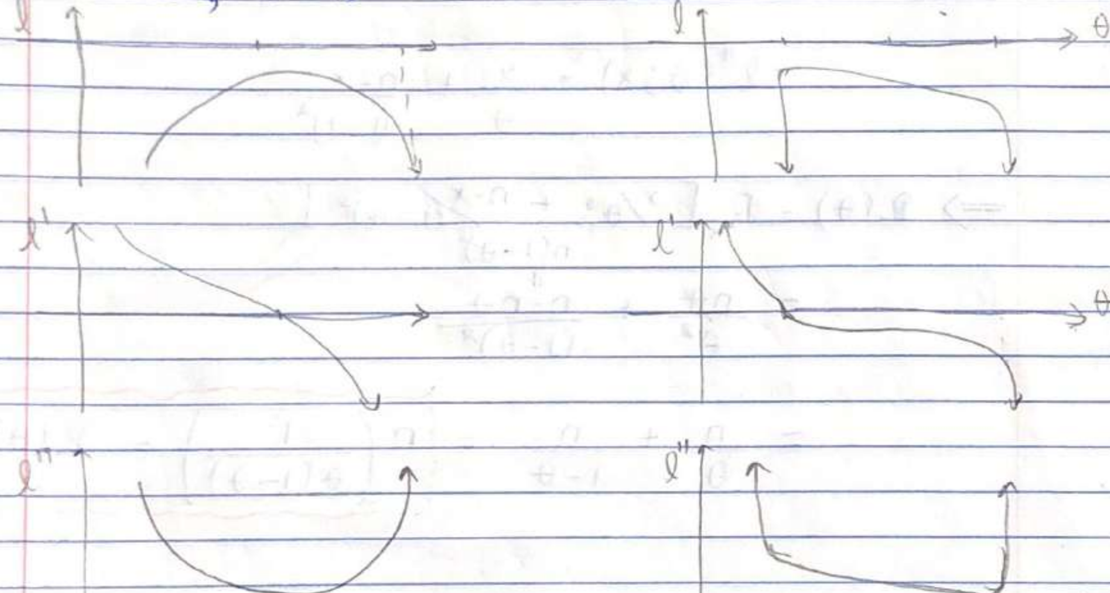
$$\ell(\theta;x) = \ln\left(\binom{n}{x}\right) + x\ln(\theta) + (n-x)\ln(1-\theta)$$

$$S(\theta;x) = \ell'(\theta;x) = x/\theta - n-x/1-\theta$$

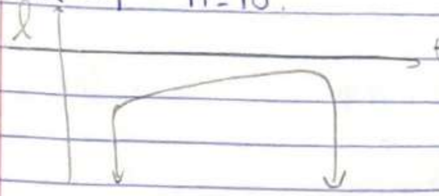
$$S'(\theta;x) = \ell''(\theta;x) = \frac{x}{\theta^2} + \frac{n-x}{(1-\theta)^2};$$

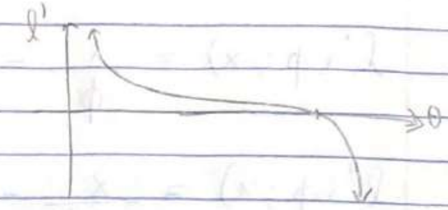
$$\mathcal{I}(\theta) = E_x[\ell''(\theta;x)] = n\left(\frac{1}{\theta(1-\theta)}\right) = n\,\mathcal{I}(\theta)$$

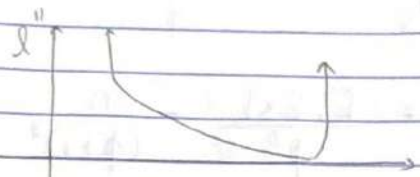
$X=5,\ n=10$ $x=1 \quad n=n$



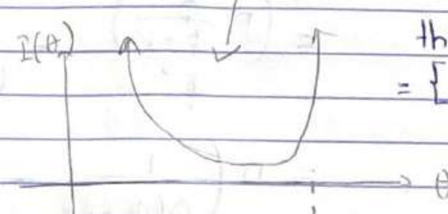
$x = 9 \quad n = 10.$







prob. weighted over the avg.



the Supp [x] $= \{0, 1, \ldots, 10\}$

How much "information is" in X?

$$\mathcal{I}_1(\theta = 1/2) = 4$$
$$\mathcal{I}_2(\theta = 1/100) \approx 101$$

let $p(\theta) \propto \sqrt{\mathcal{I}(\theta)}$

If F: Bin. fixed n

$$P_J(\theta) \propto \sqrt{n\left(\frac{1}{\theta(1-\theta)}\right)} \propto \sqrt{\frac{1}{\theta(1-\theta)}} = \theta^{-1/2}(1-\theta)^{-1/2}$$
$$= \theta^{1/2-1}(1-\theta)^{1/2-1}$$

Jeffrey's prior

The Jeffrey's prior $\propto$ Beta $(1/2, 1/2)$

$$\phi = \frac{\theta}{1-\theta}$$

$\Updownarrow$ one to one function

$$\theta = \frac{\phi}{\phi+1}$$

$$\mathcal{L}(\phi; x) = \binom{n}{x}\left(\frac{\phi}{\phi+1}\right)^{x}\left(1 - \frac{\phi}{\phi+1}\right)^{n-x}$$
$$= \binom{n}{x}\left(\frac{\phi^{x}}{(\phi+1)^{n}}\right)$$

$$\ell(\phi; x) = \ln\binom{n}{x} + x\ln(\phi) - n\ln(\phi+1)$$

$$\ell'(\phi;x) = \frac{x}{\phi} - \frac{n}{\phi+1}$$

$$\ell''(\phi;x) = \frac{x}{\phi^2} - \frac{n}{(\phi+1)^2}$$

$$I(\phi) = E_x\left[-\ell''(\phi;x)\right] = \frac{E_x[x]}{\phi^2} - \frac{n}{(\phi+1)^2}$$

$$= \frac{\left(n\frac{\phi}{\phi+1}\right)}{\phi^2} - \frac{n}{(\phi+1)^2}$$

$$= n\left(\frac{1}{\phi(\phi+1)} - \frac{1}{(\phi+1)^2}\right)$$

$$= n\left(\frac{1}{\phi(\phi+1)^2}\right)$$

Exam question $P_J(\theta) \propto \sqrt{I(\theta)} = \sqrt{n\frac{1}{\phi(\phi+1)^2}} \propto \phi^{-1/2}(\phi+1)^{-1}$

$$\propto \frac{1}{\pi}\phi^{-1/2}(\phi+1)^{-1}$$

trivia $\stackrel{d}{=}$ Beta prime $(1/2, 1/2)$

$$= F_{1,1}$$

$$\begin{tikzcd}
P(X \mid \theta) \arrow[r, "\text{Jeffry's protocol}"] \arrow[d, leftrightarrow, "\substack{\phi = f(\theta) \\ \text{(transfermation of } \theta) \\ \text{(parameterizing)}}"'] & P_J(\theta) \arrow[d, leftrightarrow, "\substack{\text{respect} \\ \phi = f(\theta)}"] \\
P(X \mid \phi) \arrow[r, "\text{Jeffry's protocol}"] & P_J(\phi)
\end{tikzcd}$$

$$\frac{d}{d\phi}\left[\frac{\phi}{\phi+1}\right] = \frac{\phi+1-\phi}{(\phi+1)^2} = (\phi+1)^{-2}$$

Change of variable formula

$$p(\phi) = P_\theta(t^{-1}(\phi)) \left| d/d\theta \left[t^{-1}(\phi) \right] \right|$$

$$= \frac{1}{\beta(1/2, 1/2)} \left(\frac{\phi}{\phi+1} \right)^{-1/2} \left(\frac{1}{\phi+1} \right)^{-1/2} (\phi+1)^{-2}$$

$$= \frac{1}{\pi} \phi^{-1/2} (\phi+1)^{-1} = \text{Beta prime}(1/2, 1/2)$$

Assume

$$P_J(\theta) \propto \sqrt{I(\theta)}$$

prove $P_J(\phi) \propto \sqrt{I(\phi)}$ using $p(\phi) = P_\theta(t^{-1}(\phi)) \left| \frac{d}{d\phi}[t^{-1}(\phi)] \right|$

$$p(\phi) = P_\theta(\theta) \left| d\theta/d\phi \right|$$

$$\propto \sqrt{I(\theta)} \left| d\theta/d\phi \right|$$

$$= \sqrt{I(\theta) \frac{d\theta}{d\phi} \cdot \frac{d\theta}{d\phi}}$$

$$= \sqrt{E_x[\underbrace{l'(\theta:x)^2}_{dl/d\theta \cdot dl/d\theta}] \, d\theta/d\phi \cdot d\theta/d\phi}$$

$$= \sqrt{E_x[dl/d\theta \cdot dl/d\theta \cdot d\theta/d\phi \cdot d\theta/d\phi]}$$

$$= \sqrt{E_x[dl/d\phi \cdot dl/d\phi]}$$

$$= \sqrt{E_x[l'(\phi;x)]}$$

$$= \sqrt{I(\phi)} \propto P_J(\phi)$$

Laplace's $p(\theta) = \text{Beta}(1,1)$
Haldane $p(\theta) = \text{Beta}(0,0)$
Jeffrey's $p_J(\theta) = \text{Beta}(\frac{1}{2},\frac{1}{2})$

principal uninformative priors.

$P(\theta) = \text{Beta}(\alpha, \beta)$
pick α, β
based on previous data.

Empirical Bayes informative priors.

$$X \sim \text{Bin}(n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}.$$

$n \longrightarrow \infty$, $\theta \longrightarrow 0$ but $\lambda = n\theta \Longrightarrow \theta = \lambda/n$

$$= \lim_{n \to \infty} \binom{n}{x} (\lambda/n)^x (1 - \lambda/n)^{n-x}$$

$$= \lim_{n \to \infty} \frac{n!}{x!(n-x)!} \frac{\lambda^x}{n^x} (1 - \lambda/n)^n (1 - \lambda/n)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n \to \infty} \frac{\overbrace{n(n-1)(n-2)\ldots(1)}^{n \text{ terms}}}{\underbrace{(n-x)!(n-x-1)!\ldots(1)}_{n-x \text{ terms}} \underbrace{(n)(n)\ldots(n)}_{\cancel{x} \text{ terms}}} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \overbrace{\lim \frac{n}{n} \lim \frac{n-1}{n} \ldots \lim \frac{n-x+1}{n}}^{x \text{ terms}} \lim \frac{n-x}{n-x} \lim \frac{n-x-1}{n-x-1} \ldots \lim \frac{2}{2} \lim \frac{1}{n}$$

$$\underbrace{\lim (1-\lambda/n)^n}_{e^{-\lambda}} \lim (1-\lambda/n)^x \quad (\lim 1-\lambda/n)^{-x}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} = x \sim \text{poisson}(\lambda)$$

We will work with θ in place of $\lambda = n\theta$ (differed

parameter space

$$\text{Supp}[X] = \{0, 1, \ldots\} = \mathbb{N}_0$$
$$\lambda \in (0, \infty)$$

$E[X] = \theta$
$Var[X] = \theta$

F ; poisson , n = 1 ; we want to find the conjugate prior.

$$P(\theta|x) = \frac{P(x|\theta)\,P(\theta)}{P(x)}$$

$$K(\theta|x) = P(x|\theta)\,P(\theta) \propto K(x|\theta)\,K(\theta)$$

$$K(\theta|x) \propto \underbrace{\left(\frac{\theta^x e^{-\theta}}{x!}\right)}_{\text{PMF to poisson}} P(\theta) \propto \theta^x e^{-\theta}\, K(\theta) \doteq$$

Pattern match to conjugative what the conjugate prior looks like

$$= \theta^x e^{-\theta} \underbrace{(\theta^a e^{-b\theta})}_{K(\theta)} = \theta^{x+a} e^{-\theta(1+b)}$$

to get $p(\theta)$, I need the normalization constant C

$$C = \frac{1}{\int_0^\infty \theta^a e^{-b\theta}\, d\theta}$$

$$= \frac{1}{\int_0^\infty \left(t/b\right)^a e^{-b\cdot t/b}\, \frac{1}{b}\, dt}$$

$$= \frac{1}{\frac{1}{b^{a+1}}\int_0^\infty t^{(a+1)-1} e^{-t}\, dt}$$

$$= \frac{b^{a+1}}{\Gamma(a+1)} \Longrightarrow P(\theta) = \frac{b^{a+1}}{\Gamma(a+1)}\,\theta^a e^{-b\theta}$$

$$\Gamma(y) = \int_0^\infty t^{y-1} e^{-t}\, dt$$

let $t = b\theta$

$\Rightarrow dt/d\theta = b \Rightarrow \theta = t/b$

$\Rightarrow d\theta = \frac{1}{b}\, dt$

Conjugate prior for poisson model $= \text{Gamma}\,(a+1, b)$

$$P(\theta) = \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\,\theta^{\alpha-1} e^{-\beta\theta}$$

$Y \sim \text{Gamma}(\alpha, \beta)$

$\text{Supp}[Y] = (0, \infty)$

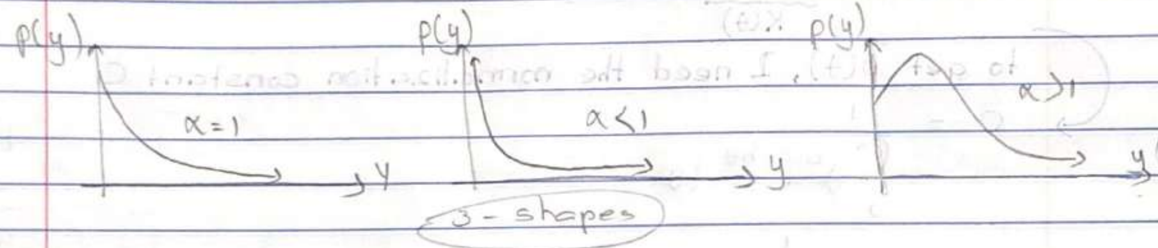
$E[Y] = \dfrac{\alpha}{\beta}$

parameter space $= \alpha, \beta > 0$

$\text{Var}[Y] = \dfrac{\alpha}{\beta^2}$, $\text{mode}[Y] = \dfrac{\alpha - 1}{\beta}$ of $\alpha \geq 1$

$\text{Med}[Y] = \text{qgamma}(0.5, \alpha, \beta)$

no closed form



Poisson - Gamma conjugate model.

F: poisson

$n = 1$ we want to find the conjugate prior

$$P(\theta|x) = \frac{P(x|\theta)\,P(\theta)}{P(X)} \overset{?}{=} \text{Gamma}(x+\alpha, 1+\beta)$$

If $p(\theta) = \text{Gamma}(\alpha, \beta)$