Lecture 3

02/04/20

$$\hat{\theta}_{MLE} = \text{argmax}\ \{\ell(\theta; x)\}$$
$$\theta \in \Theta$$

$\hat{\theta}_{MLE} \downarrow$ Point est. "Estimate" $\bar{x}$

In an advanced class, you'll prove:

$$\hat{\theta}_{MLE} \overset{d}{\approx} N(\theta, \underbrace{SE[\hat{\theta}_{MLE}]}_{\text{a function of } \theta}) \approx N(\underbrace{\hat{\theta}_{MLE}}_{\text{Point estimator}}, \underbrace{\hat{SE}[\hat{\theta}_{MLE}]}_{\text{Estimator}})$$

$\hat{\theta}_{MLE} \downarrow$ r.v. "Estimator" $\bar{X}$

Impossible to know

$\boxed{SE[\hat{\theta}_{MLE}]}$ estimator $\Big|_{\theta = \hat{\theta}_{MLE}}$ estimate

MLE's allow for 3 goals of inference:

1) Point Estimator $\hat{\theta}_{MLE}$

2) Confidence Sets: $CI = [\hat{\theta}_{MLE} \pm z_{\frac{\alpha}{2}} SE[\hat{\theta}_{MLE}]|_{\theta = \hat{\theta}_{MLE}}]$

3) Hypothesis Test

$H_0: \theta = \theta_0$

$H_a: \theta \neq \theta_0$

$$\text{Retainment Region}_{\alpha} = [\theta_0 \pm z_{\frac{\alpha}{2}} SE[\hat{\theta}_{MLE}]|_{\theta = \theta_0}]$$

"Trouble in Paradise"
what are some problems

Examples

1) $\tilde{F}$ = iid Bernoulli $X_1 \ldots X_n \overset{iid}{\sim} Bern(\theta)$
$X = <0,0,0>$
$\hat{\theta}_{MLE} = \bar{X} = 0$

$CI_{\theta,\alpha} = [\bar{X} \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\bar{X}(1-\bar{X})}{3}}] = \{0\}$

$RR_\alpha = \{\theta_0\}$ All theta are rejected

2) What if you know $\theta \in [0.1, 0.2]$
Is there any way to make use of this formula?
NO.

3) Let's interpret the confidence interval:

$CI_{\theta, 95\%} = [0.37, 0.43]$

What is the interpretation? (Our assumption was θ is a fixed value (Parameter))

$P(0.392 \in [0.37, 0.43]) = 1$

$P(0.36 \in [0.37, 0.43]) = 0$

Valid Interpretation

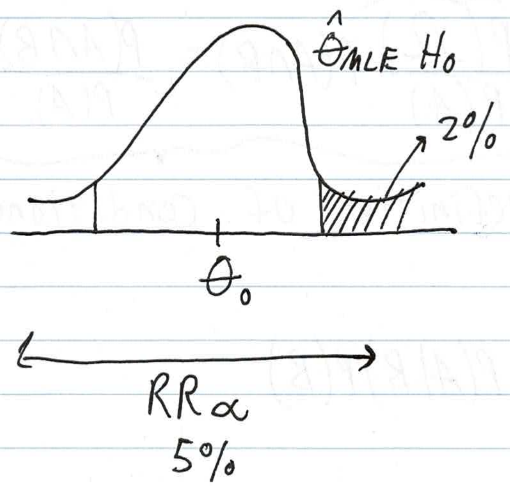
I) If I repeat the experiment many times $\approx 95\%$ of the CI's will include θ

II) Before you do the experiment:
$P(\theta \in CI_{\theta, 1-\alpha}) = 1-\alpha$

4) In a hypothesis test, you either reject H_0 or retain H_0

$$\hat{\theta}_{MLE} \in RR_\alpha \Rightarrow \text{Retain } H_0$$

$$\hat{\theta}_{MLE} \notin RR_\alpha \Rightarrow \text{Reject } H_0$$



"P-Value" is defined as:

$$P_{val} = P(\text{Seeing } \hat{\theta}_{MLE} \text{ "or more estimate"} \mid H_0)$$

$$\neq P(H_0 \mid X)$$

→ Probability my theory is true

5) $\tilde{F}$ = iid Bernoulli $(\Theta = (0,1))$

$X = \langle 0, 1, 0 \rangle \qquad \hat{\theta}_{MLE} = \bar{X} = \frac{1}{3}$

$$CI_{\theta, 95\%} = \left[\frac{1}{3} \pm 2\sqrt{\frac{\frac{1}{3} \cdot \frac{2}{3}}{3}}\right] = [-0.20, 0.87]$$

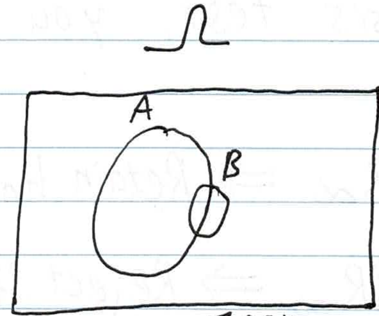
We know that $\theta \nless 0$. This is a bad confidence Interval, due to the parameter space.

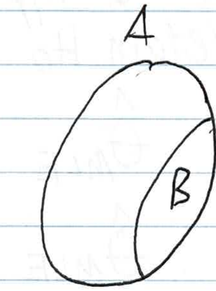
Why did this break? n is small...

$\hat{\theta}_{MLE} \nsim N(\ ,\ ) \Rightarrow$ Game Over

Let: A be the event of Smoking
B is the event lung cancer

Assume: $P(A) = .2$
$P(B) = .06$





$P(A \cap B) = .036$

$P(B|A) = ?$

$$P(B|A) \propto P(A \cap B) = cP(A \cap B) = \underbrace{\frac{\overset{\text{zoom}}{\overbrace{P(\Omega)}}}{P(A)} P(A \cap B) = \frac{P(A \cap B)}{P(A)}}_{\text{Definition of Conditional Probability}}$$

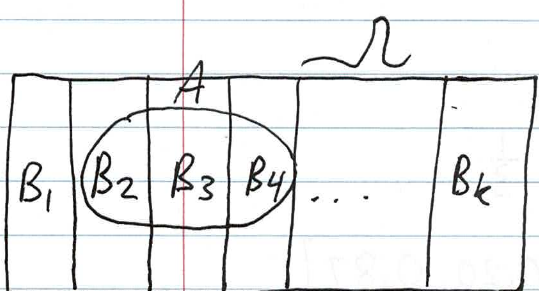
$$\Rightarrow P(A \cap B) = P(B|A)P(A)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)P(B)$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$ Bayes Rule

$$A = (A \cap B) \cup (A \cap B^c)$$

$$P(A) = P(A \cap B) + P(A \cap B^c) \Rightarrow \text{Addition Rule}$$



s.t. $B_1 \cup B_2 \cup \dots \cup B_k = \Omega$ collectively exclusive

but $B_i \cap B_j = \emptyset$ mutually exclusive

I can prove this... $A = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_k)$

$$\Rightarrow P(A) = \sum_{k=1}^{k} P(A|B_k)P(B_k) = \sum_{k=1}^{k} P(A, B_k)$$

see next page

$$\Rightarrow P(B_i|A) = \frac{P(A|B_i)\cdot P(B_i)}{\sum_{k=1}^{k} P(A\cap B)}$$

Bayes Thm.

$$P(B_i|A) = P(A|B_i)\,P(B_i) = \sum_{k=1}^{k} P(A|B_k)\,P(B_k)$$

Imagine two r.v.'s: X, Y

$Supp[X] = \{1, 2, 3, 4\}$
$Supp[Y] = \{1, 2, 3, 4, 5, 6\}$

Y

X	1	2	3	4	5	6
1						
2						
3						
4					$P(Y=5)$	

This table gives the $P(X=x, Y=y)$ i.e. the joint mass function

$$\underbrace{P(Y=5)}_{\text{Marginal Probability}} = P(Y=5, X=1) + P(Y=5, X=2) + \dots + P(Y=5, X=4)$$

$$= \sum_{x \in Supp(X)} P(Y=5, X=x)$$

$$P(X=2|Y=5) = \frac{P(X=2, Y=5)}{P(Y=5)}$$

$$P(X=x|Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$\underbrace{P(X,Y)}_{\substack{\text{CMF} \\ \text{(Conditional Mass Function)}}} = \frac{P(X,Y)^{\to \text{JMF}}}{P(X)_{\searrow \text{PMF}}} = \frac{P(Y|X)\,P(Y)}{P(X)}$$

Can I write the following?

$$P(\theta|x) = \frac{P(x|\theta)P(\theta)}{P(x)}$$

You cannot calculate the probability of X without knowing θ

θ is a constant i.e. a degenerate r.v.

$\theta \sim = \{ \theta$ w.p. 1

$\tilde{F} = \{ p(x;\theta) : \theta \in \Theta \}$

$\theta | X \sim \{ \theta$ w.p. 1

The formula is not useful.

Denominator is a problem

$$p(x) = \sum_{\theta \in \Theta} P(x|\theta_0)P(\theta_0)$$

$$= \int_{\Theta} P(x|\theta_0)P(\theta_0)$$