Lecture 14

Correction from Lecture 13: 03/26/20

$\tilde{F}$: iid Poisson(θ) $X_1, \dots, X_n$; $\theta \sim$ Poisson(θ)

$P(X_* | X) = \text{Ext. Neg. Bin.}(r, p)$ where $r = \sum X_i + \alpha$

$$p = \frac{n+\beta}{n+\beta+1}$$

$$= \text{Neg. Bin}(r, p) = \binom{X_* + r - 1}{r - 1} (1-p)^{X_*} p^r$$

if $\alpha \in \mathbb{N}$

$$X_1, \dots, X_r \overset{iid}{\sim} \text{Geom}(p) \Rightarrow \sum_{i=1}^{r} X_i \sim \text{Neg.Bin}(r, p)$$

*corrected version $E[X_i] = \frac{1-p}{p} \neq \frac{p}{1-p} \Rightarrow E[\sum X_i] = r\frac{1-p}{p}$

$$E[X_* | X] = r\frac{1-p}{p} = (\sum X_i + \alpha) \frac{\frac{1}{n+\beta+1}}{\frac{n+\beta}{n+\beta+1}}$$

$$= \frac{\sum X_i + \alpha}{n+\beta} = E[\theta | X]$$

$\tilde{F}$: iid $N(\theta, \sigma^2)$ with σ^2 known

$$P(\theta | X, \sigma^2) \propto e^{a\theta - b\sigma^2} \propto N\left(\frac{a}{2b}, \frac{1}{2b}\right)$$

$$a = \frac{n\bar{X}}{\sigma^2}, \quad b = \frac{n}{2\sigma^2} \quad = N\left(\bar{X}, \frac{\sigma^2}{n}\right)$$

Under Laplace Prior i.e, $P(\theta) \propto 1$ 3

$$P(\theta | X, \sigma^2) \propto P(X | \theta, \sigma^2) \propto N\left(\bar{X}, \frac{\sigma^2}{n}\right)$$

$$P(\theta | \sigma^2) \propto k(\theta | \sigma^2)$$

$$P(\theta | X, \sigma^2) = \frac{P(X | \theta, \sigma^2) P(\theta | \sigma^2)}{P(X | \sigma^2)} \propto \frac{P(X | \theta, \sigma^2) P(\theta | \sigma^2)}{P(X | \sigma^2)}$$

$$\propto k(X | \theta, \sigma^2) k(\theta | \sigma^2)$$

If $k(\theta | \sigma^2) = e^{\alpha\theta - \beta\theta^2} \propto N\left(\frac{\alpha}{2\beta}, \frac{1}{2\beta}\right)$

$$= e^{a\theta - b\theta^2} k(\theta | \sigma^2) = e^{a\theta - b\theta^2} e^{\alpha\theta - \beta\theta^2}$$

$$= e^{(a+\alpha)\theta - (1+\beta)\theta^2} \propto N\left(\frac{a+\alpha}{2(b+\beta)}, \frac{1}{2(b+\beta)}\right) \Rightarrow P(\theta | X, \sigma^2)$$

$$\frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{1}{\sigma^2} + \frac{1}{\tau^2}} \qquad \frac{1}{2\left(\frac{n}{2\sigma^2} + \frac{1}{2\tau^2}\right)} = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$a = \frac{n\bar{x}}{\sigma^2}, \quad b = \frac{1}{2\sigma^2}, \quad \alpha = \frac{\mu_0}{\tau^2}, \quad \beta = \frac{1}{2\tau^2}$$

$$N\left(\frac{\alpha}{2\beta}, \frac{1}{2\beta}\right) = N(\mu_0, \tau^2)$$

$$\mu_0 = \frac{\alpha}{2\beta} \Rightarrow \alpha = \mu_0(2\beta) = \frac{\mu_0}{\tau^2}$$

$$\tau^2 = \frac{1}{2\beta} \Rightarrow \beta = \frac{1}{2\tau^2}$$

$\tilde{F}$: iid $N(\theta, \sigma^2)$ σ^2 known, $P(\theta) = N(\mu_0, \tau^2)$

$$\Rightarrow P(\theta|x) = N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right)$$

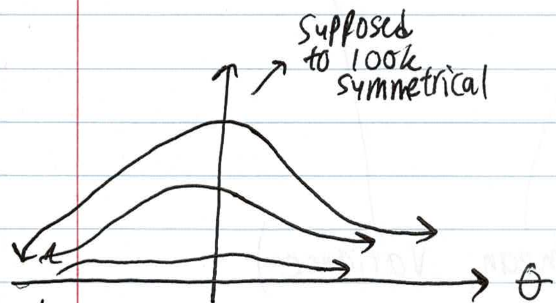
Under Laplace prior

$$= N\left(\bar{X}, \frac{\sigma^2}{n}\right) \Rightarrow \mu_0, \tau^2$$

Laplace

$$P(\theta) \propto 1 \propto N(0, \infty)$$

$$P(\theta|\sigma^2) = N(\mu_0, \tau^2)$$



raise variance to ∞ to make flat.

If $\tau^2 \to \infty \Rightarrow N(\mu_0, \tau^2) \propto 1$

$$\hat{\theta}_{MMSE} = E[\theta|x] = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$\hat{\theta}_{MAE} = Med[\theta|x] = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$\hat{\theta}_{MAP} = mode[\theta|x] = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

Credible Region

$$CR_{\theta,1-\alpha} = \left[qnorm\left(\frac{\alpha_0}{2}, \frac{\frac{n\bar{x}}{\sigma^2} - \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2}, \frac{1}{\tau^2}}\right)\right.$$

$$\left., qnorm\left(1-\frac{\alpha}{2}, \frac{\frac{n\bar{x}}{\sigma^2} - \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2}, \frac{1}{\tau^2}}\right)\right]$$

Hypothesis Tests

$$H_0: \theta \le \theta_0 \quad , \quad H_A: \theta > \theta_0$$

$$Pval = P(H_0|X) = pnorm(\theta, \text{mean}, \text{variance})$$

$$= \int_{-\infty}^{\theta_0} PDF \, d\theta$$

Jeffrey's Prior

$$P_J(\theta|\sigma^2) \propto \sqrt{I(\theta;\sigma^2)}$$

$$\mathcal{L}(\theta; X, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{\sum x_i^2}{2\sigma^2} + \frac{n\bar{X}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}}$$

$$\ell(\theta; X, \sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{\sum x_i}{2\sigma^2} + \frac{n\bar{X}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}$$

$$\ell'(\theta; X, \sigma^2) = \frac{n\bar{X}}{\sigma^2} - \frac{n\theta}{\sigma^2} \implies -\ell''(\theta; X, \sigma^2) = \frac{n}{\sigma^2}$$

$$I(\theta;\sigma^2) = E[-\ell''(\theta; X, \sigma^2)] = E_X\left[\frac{n}{\sigma^2}\right] = \frac{n}{\sigma^2}$$

$$p_J(\theta|\sigma^2) \propto \sqrt{I(\theta;\sigma^2)} = \sqrt{\frac{n}{\sigma^2}} \propto 1 \propto N(0,\infty)$$

(Jeffrey's Prior is the same as Laplace)

Haldane: pure ignorance Recall: $Beta(0,0) \rightarrow$ comes from pseudo-counts

$$\hat{\theta} = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{n_0\mu_0}{\sigma_0^2}}{\frac{n}{\sigma^2} + \frac{n_0}{\sigma_0^2}} = \frac{\frac{n\bar{x} + n_0\mu_0}{\sigma^2}}{\frac{n+n_0}{\sigma^2}} =$$

Let $n_0 = \#$ of pseudo observations

Let $\tau^2 = \frac{\sigma_0^2}{n_0} = \frac{\sigma^2}{n_0}$ since σ^2 is known

Avg. of pseudo-data

$$= \frac{n\bar{x} + n_0\mu_0}{n+n_0} = \frac{\Sigma x_i + \Sigma y_i}{n+n_0}$$

$P(\theta|\sigma^2) = N(\mu_0, \tau^2) = N\left(\mu_0, \frac{\sigma^2}{n_0}\right)$ weight of prior

Haldane: $n_0 = 0 \Rightarrow \tau^2 = \infty,\ \mu_0 = 0$

$P(\theta|\sigma^2) = N(0,\infty)$ same as laplace

Let $y_1, \ldots, y_2, \ldots, y_{n_0}$ be pseudo-data

$$\mu_0 = \bar{y} = \frac{1}{n_0}\Sigma y_i$$

$n_* = 1$

$$P(X_*|X) = \int_{\Theta} P(X_*|\theta,\sigma^2)\,P(\theta|X,\sigma^2)\,d\theta$$

$$= \int_{\mathbb{R}} \underbrace{\text{PDF}}_{\text{Normal}(\theta,\sigma^2)} \quad \underbrace{\text{PDF}}_{\text{Normal}(\theta_p,\sigma_p^2)}$$

$$\theta_p = \frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \quad \sigma_p^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$= \sigma_p^2\left(\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\tau^2}\right)$$

Shrinkage

$$\hat{\theta}_{MMSE} = \frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{\frac{n\bar{X}}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} + \frac{\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$= \frac{\frac{n}{\sigma^2}\hat{\theta}_{MLE}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} + \frac{\frac{1}{\tau^2}E[\theta]}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$\hat{\theta}_{MLE} = \bar{X}$

See previous HW/notes

$$= \underbrace{\frac{1}{1 + \frac{\sigma^2}{n\tau^2}}}_{1-\rho}\hat{\theta}_{MLE} + \underbrace{\frac{1}{1 + \frac{\tau^2}{n\sigma^2}}}_{\rho}E[\theta]$$

If $n \to \infty \Rightarrow \rho \to 0$