Lecture15.

04/02 목 [MATH341].

F: iid Normal(θ, σ^2) with σ^2 known.

$$P(\theta|\sigma^2) = N(\mu_0, \tau^2) = N\left(\mu_0, \frac{\sigma^2}{n_0}\right)$$

Imagine pseudodata $Y_1, \dots, Y_n \overset{iid}{\sim} N(\mu_0, \sigma^2)$

$$\Rightarrow \bar{Y} \sim N\left(\mu_0, \frac{\sigma^2}{n_0}\right)$$

$$\Rightarrow P(\theta|X, \sigma^2) = \underset{\text{conjugate}}{N}(\hat{\theta}_p, \sigma_p^2) \text{ where } \hat{\theta}_p = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{n\bar{x} + n_0\mu_0}{n+n_0}.$$

$$\cdot\ \sigma_p^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{\sigma^2}{n+n_0}.$$

$\cdot\ n_* = 1$

$$P(x_*|X, \sigma^2) = \int_\Theta P(x_*|\theta, \sigma^2) P(\theta|X, \sigma^2)\, d\theta$$

$$= \int_{\mathbb{R}} N(\theta, \sigma^2) \cdot N(\hat{\theta}_p, \sigma_p^2)\, d\theta.$$

$$= \int_{\mathbb{R}} \left(\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} e^{-\frac{1}{2\sigma^2}(x_*-\theta)^2}\right)\left(\cancel{\frac{1}{\sqrt{2\pi\sigma_p^2}}} e^{-\frac{1}{2\sigma_p^2}(\theta-\hat{\theta}_p)^2}\right) d\theta.$$

constant

$$\propto \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}(x_*-\theta)^2} e^{-\frac{1}{2\sigma_p^2}(\theta-\hat{\theta}_p)^2}\, d\theta.$$

$$= \int_{\mathbb{R}} \underline{e^{-\frac{x_*^2}{2\sigma^2}}}\, e^{\frac{x_*\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} e^{-\frac{\theta^2}{2\sigma_p^2}} e^{\frac{\theta\hat{\theta}_p}{\sigma_p^2}} \cancel{e^{-\frac{\hat{\theta}_p^2}{2\sigma_p^2}}}\, d\theta.$$

X_* function x

$$\propto e^{-\frac{x_*^2}{2\sigma^2}} \int_{\mathbb{R}} e^{\frac{x_*\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} e^{-\frac{\theta^2}{2\sigma_p^2}} e^{\frac{\theta\hat{\theta}_p}{\sigma_p^2}}\, d\theta.$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} \int_{\mathbb{R}} e^{\underbrace{\left(\frac{x_*}{\sigma^2} + \frac{\hat{\theta}_p}{\sigma_p^2}\right)}_{a}\theta - \underbrace{\left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma_p^2}\right)}_{b}\theta^2}\, d\theta.$$

$$\circ\ \ \theta \sim N\left(\frac{a}{2b}, \frac{1}{2b}\right) = \frac{1}{\sqrt{2\pi\left(\frac{1}{2b}\right)}} e^{-\frac{1}{2\left(\frac{1}{2b}\right)}\left(\theta - \frac{a}{2b}\right)^2}$$

$$= \sqrt{\frac{b}{\pi}}\, e^{-b\left(\theta^2 - \frac{a\theta}{b} + \frac{a^2}{4b^2}\right)}$$

$$= \sqrt{\frac{b}{\pi}}\, e^{-b\theta^2 + a\theta - \frac{a^2}{4b}}$$

$$= \underbrace{\sqrt{\frac{b}{\pi}}\, e^{-\frac{a^2}{4b}}}_{c}\, \underbrace{e^{a\theta - b\theta^2}}_{k(\theta)}$$

$$\propto e^{a\theta - b\theta^2}.$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} \sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}} \underbrace{\int_{\mathbb{R}} \sqrt{\frac{b}{\pi}}\, e^{-\frac{a^2}{4b}} e^{a\theta - b\theta^2}\, d\theta}\,. = e^{-\frac{x_*^2}{2\sigma^2}} \sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}}$$

constant: not a function of x_*

$$\propto e^{-\frac{x_*^2}{2\sigma^2}} e^{\left(\frac{x_*}{\sigma^2}+\frac{\hat\theta_p}{\sigma_p^2}\right)^2/4b}$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} e^{\frac{x_*^2}{4\sigma^4 b}} e^{\frac{x_*\hat\theta_p}{2\sigma_p^2\sigma^2 b}} e^{\cancel{\frac{\hat\theta_p^2}{4\sigma_p^4 b}}}$$

$$\propto e^{\underbrace{\frac{\hat\theta_p}{2\sigma^2\sigma_p^2 b}}_{A} x_* - \underbrace{\left(\frac{1}{2\sigma^2}-\frac{1}{4\sigma^4 b}\right)}_{B} x_*^2}$$

$$= e^{Ax_* - Bx_*^2}$$

$$\propto N\left(\frac{A}{2B}, \frac{1}{2B}\right)$$

$$2b\sigma^2 = 2\left(\frac{1}{2\sigma^2}+\frac{1}{2\sigma_p^2}\right)\sigma^2 = 1+\frac{\sigma^2}{\sigma_p^2}$$

$$\frac{1}{2B} = \frac{1}{2\left(\frac{1}{2\sigma^2}-\frac{1}{4\sigma^{2^2}b}\right)} = \frac{1}{\frac{1}{\sigma^2}-\frac{1}{2b\sigma^{2^2}}}$$

$$= \frac{\sigma^{2^2}}{\sigma^2-\frac{1}{2b}}$$

$$= \frac{2b\sigma^2\sigma^2}{2b\sigma^2-1}$$

$$= \frac{\left(1+\frac{\sigma^2}{\sigma_p^2}\right)\sigma^2}{\left(1+\frac{\sigma^2}{\sigma_p^2}\right)-1} = \frac{\left(1+\frac{\sigma^2}{\sigma_p^2}\right)\cancel{\sigma^2}}{\frac{\cancel{\sigma^2}}{\sigma_p^2}} = \sigma_p^2\left(1+\frac{\sigma^2}{\sigma_p^2}\right)$$

$$= \sigma_p^2+\sigma^2$$

$$\frac{A}{2B} = A(\sigma_p^2+\sigma^2) = \frac{\hat\theta_p}{2b\sigma^2\sigma_p^2}(\sigma_p^2+\sigma^2) = \frac{\hat\theta_p}{\left(1+\frac{\sigma^2}{\sigma_p^2}\right)\sigma_p^2}(\sigma_p^2+\sigma^2)$$

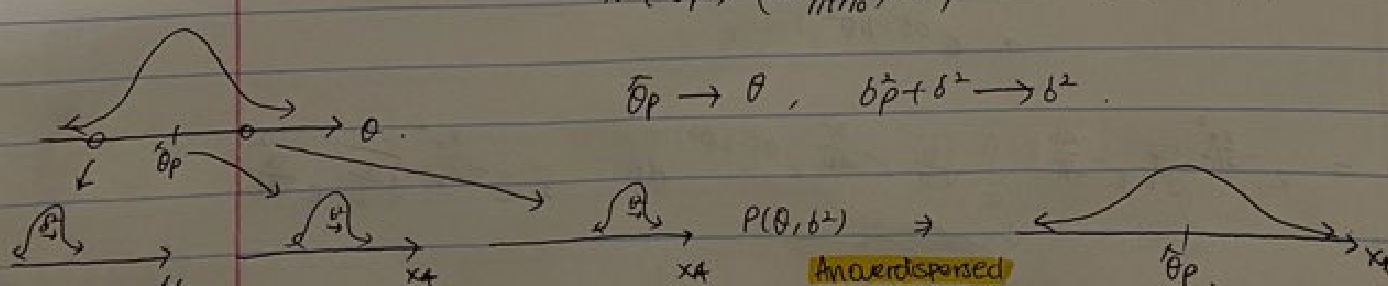
$$= \frac{\hat\theta_p}{\cancel{(\sigma_p^2+\sigma^2)}}\cancel{(\sigma_p^2+\sigma^2)}$$

$$= \hat\theta_p$$

$$\therefore P(x_* \mid x, \sigma^2) = N(\hat\theta_p, \sigma_p^2+\sigma^2)$$

$$= N\left(\hat\theta_p, \left(1+\frac{1}{n+n_0}\right)\sigma^2\right) \rightarrow N(\theta, \sigma^2)$$

$$\hat\theta_p \rightarrow \theta, \quad \sigma_p^2+\sigma^2 \rightarrow \sigma^2$$



An overdispersed normal w/ known variance is normal

F: iid $N(\theta, \sigma^2)$ with θ known.

$\mathcal{L}(\sigma^2; x, \theta) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2}$

$\ell(\sigma^2; x, \theta) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2$

$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2$

$\ell'(\sigma^2; x, \theta) = -\frac{n}{\cancel{2}}\frac{1}{\sigma^2} + \frac{1}{\cancel{2}\sigma^{2^2}}\Sigma(x_i-\theta)^2 \stackrel{set}{=} 0.$

$\Rightarrow -n + \frac{\Sigma(x_i-\theta)^2}{\sigma^2} = 0.$ $\quad\Sigma(x_i-\theta)^2 = n\cdot\hat{\sigma}^2_{MLE}$

$\therefore \hat{\sigma}^2_{MLE} = \frac{\Sigma(x_i-\theta)^2}{n}$ similar to S^2 (sample variance)

$= \frac{\Sigma(x_i-\bar{x})^2}{n-1} \to \sigma^2.$

Laplace Prior: $P(\sigma^2|\theta) \propto 1$, improper since $\sigma^2 \in (0,\infty)$

$P(\sigma^2|x,\theta) \propto P(x|\theta,\sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2_{MLE}}{2\sigma^2}}$

$\propto (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}$

- $U \sim Gamma(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u}$

$V = \frac{1}{U} = t(U) \sim f_U(t^{-1}(v)) \left|\frac{d}{dv}[t^{-1}(v)]\right| = \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{v}\right)^{\alpha-1} e^{-\frac{\beta}{v}}\left|-\frac{1}{v^2}\right|$

$t^{-1}(v) = \frac{1}{v}$

$= \frac{\beta^\alpha}{\Gamma(\alpha)} v^{-\alpha+1} v^{-2} e^{-\frac{\beta}{v}}$

$= \frac{\beta^\alpha}{\Gamma(\alpha)} v^{-\alpha-1} e^{-\beta/v}$

$= InvGamma(\alpha, \beta).$

(inverse gamma r.v.)

$\cancel{\frac{\beta^\alpha}{\Gamma(\alpha)}} v^{-\alpha-1} e^{-\beta/v} \propto k(v)$

$= v^{-\alpha-1} e^{-\beta/v}$

$\Rightarrow (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2_{MLE}/2}{\sigma^2}} = (\sigma^2)^{-(\frac{n}{2}-1)-1} e^{\frac{-n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}$

$= InvGamma\left(\frac{n}{2}-1, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$

$= \frac{n-2}{}$ ...

$$P(\sigma^2|X,\theta) \propto P(X|\theta,\sigma^2)\,P(\sigma^2|\theta,X) \propto (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2_{MLE}/2}{\sigma^2}} k(\sigma^2|\theta).$$

$k(\sigma^2|\theta) = ?$ to get conjugancy? $= (\sigma^2)^a e^{-\frac{b}{\sigma^2}}$

$$= (\sigma^2)^{-n/2+a} e^{-\frac{n\hat{\sigma}^2_{MLE}/2+b}{\sigma^2}}$$

$\propto$ InvGamma is the conjugate prior.

let $P(\sigma^2|\theta) = \text{InvGamma}(\alpha,\beta)$

$$\Rightarrow P(\sigma^2|X,\theta) \propto \left((\sigma^2)^{-n/2} e^{\frac{-n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}\right)\cdot\left((\sigma^2)^{-\alpha-1} e^{-\frac{\beta}{\sigma^2}}\right).$$

$$= (\sigma^2)^{-(\frac{n}{2}+\alpha)-1} e^{\frac{-(n\hat{\sigma}^2_{MLE}/2+\beta)}{\sigma^2}}$$

$$\propto \text{InvGamma}\left(\frac{n}{2}+\alpha,\ \frac{\overbrace{n\hat{\sigma}^2_{MLE}}^{=\Sigma(x_i-\theta)^2}}{2}+\beta\right).$$

let $\alpha = \frac{n_0}{2}$, $\beta = \frac{n_0\sigma_0^2}{2}$ $\Rightarrow P(\sigma^2|\theta) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$

$$\Rightarrow P(\sigma^2|\theta,X) = \text{InvGamma}\left(\frac{n+n_0}{2}, \frac{n\hat{\sigma}^2_{MLE}+n_0\sigma_0^2}{2}\right)$$

Pseudodata : $Y_1, \ldots Y_{n_0} \sim N(\theta, \sigma_0^2)$

known belief.

$$\Rightarrow n_0\sigma_0^2 = \Sigma(Y_i-\theta)^2$$

$$\Rightarrow \sigma_0^2 = \Sigma(Y_i-\theta)^2/(n_0).$$

n_0 strength.

n_0 small $\Rightarrow$ uninformative.

Haldane : n_0 is nothing. $n_0=0$, $\sigma_0^2=?$

$$\Rightarrow P(\sigma^2|\theta) = \text{InvGamma}(0,0)$$