$\mathcal{F}$: $\text{poisson}(\theta) \overset{i.i.d}{}$, $\theta \sim \text{Gamma}(\alpha, \beta)$ continuous $\theta | x \sim \text{Gamma}(\alpha + \sum x_i, n+\beta)$

$n^* = 1$ $\quad P(X^* | X) = \int_{\Theta} P(X^* | \theta) P(\theta | x) \, d\theta$

$$= \int_0^\infty \left(\frac{e^{-\theta} \theta^{x^*}}{x^*!} \right) \left(\frac{(n+\beta)^{\sum x_i + \alpha}}{\Gamma(\alpha + \sum x_i)} \theta^{\alpha - 1 + \sum x_i} e^{-(n+\beta)\theta} \right) d\theta$$

$$= \frac{(n+\beta)^{\alpha + \sum x_i}}{x^*! \, \Gamma(\alpha + \sum x_i)} \int_0^\infty \theta^{\alpha - 1 + x^* + \sum x_i} e^{-(n+\beta+1)\theta} d\theta \sim \frac{\Gamma(\alpha + x^* + \sum x_i)}{(n+\beta+1)^{\alpha + x^* + \sum x_i}}$$

let $t = (n+\beta+1)\theta \Rightarrow \frac{dt}{d\theta} = n+\beta+1$

$\theta = \frac{t}{n+\beta+1}$

$$= \frac{(n+\beta)^{\alpha + \sum x_i}}{x^*! \, \Gamma(\alpha + \sum x_i)} \cdot \int_0^\infty \left(\frac{t}{n+\beta+1} \right)^{\alpha + x^* - 1 + \sum x_i} e^{-t} \frac{dt}{n+\beta+1} = \frac{(n+\beta)^{\alpha + \sum x_i}}{x^*! \, \Gamma(\alpha + \sum x_i)} \cdot \frac{1}{(n+\beta+1)^{\alpha + x^* + \sum x_i}} \cdot \int_0^\infty t^{\alpha + x^* - 1 + \sum x_i} e^{-t} dt$$

$$= \frac{(n+\beta)^{\alpha + \sum x_i}}{x^*! \, \Gamma(\alpha + \sum x_i)} \cdot \frac{\Gamma(\alpha + x^* + \sum x_i)}{(n+\beta+1)^{\alpha + \sum x_i} \cdot (n+\beta+1)^{x^*}} = \left(\frac{n+\beta}{n+\beta+1} \right)^{\alpha + \sum x_i} \left(\frac{1}{n+\beta+1} \right)^{x^*} \frac{\Gamma(\alpha + x^* + \sum x_i)}{x^*! \, \Gamma(\alpha + \sum x_i)}$$

let $p := \frac{n+\beta}{n+\beta+1} \in (0,1)$

$\Rightarrow 1-p = \frac{1}{n+\beta+1} \in (0,1)$

let $r := \alpha + \sum x_i$

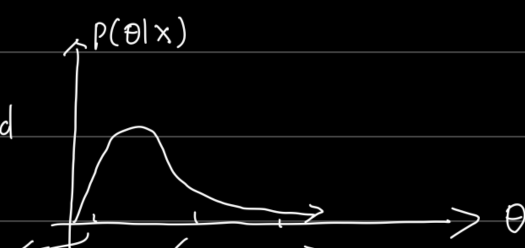
$$\Rightarrow \frac{\Gamma(x^* + r)}{x^*! \, \Gamma(r)} \cdot p^r (1-p)^{x^*} = \underset{\text{extend Negative Binomial}}{\text{Ext NegBin}(r, p)}$$

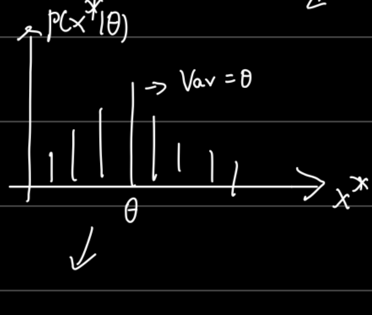
If $r = \alpha + \sum x_i \in \mathbb{N}_0 \Rightarrow \alpha \in \mathbb{N}_0$ then $\frac{\Gamma(x^* + r)}{x^*! \, \Gamma(r)} = \frac{(x^* + r - 1)!}{x^*! \, (r-1)!} = \binom{x^* + r - 1}{x^*}$, then it becomes $\text{NegBin}(r, p)$

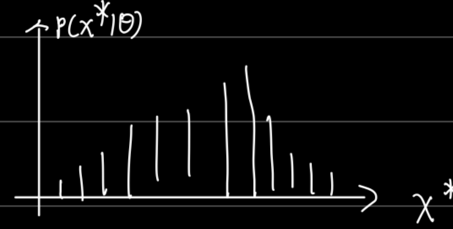
what is NegBin

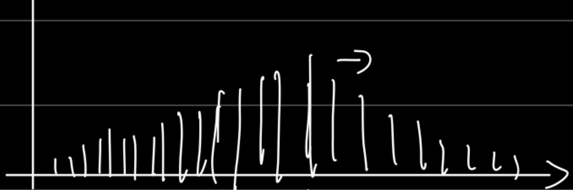
$X_1, \dots, X_r ; p \overset{iid}{\sim} \text{Geom}(p) := (1-p)^x p \quad E(X_i) = \frac{p}{1-p} \quad \sum_{i=1}^{r} X_i \sim \text{NegBin}(r, p) \Rightarrow E[\sum x_i] = \frac{rp}{1-p}$

$X^* | X \sim \text{Ext NegBin}(r, p)$ overdispersed (过度分布)









$E[x^* | x] = \frac{rp}{1-p} = \mu \quad \text{Var}[x^* | x] = \frac{pr}{(1-p)^2} = \frac{1}{1-p} \mu$

$\frac{1}{1-p} \in (1, +\infty)$ so lowest Var is μ, mean have absolutely certainty of my θ

negtive Binomial is the generalization of poisson.

Examine normal model

$X \sim N(\theta, \sigma^2) = N(\theta_1, \theta_2)$ dimension $=2$ / $\dim[\vec{\theta}]=2$.

a could negtive, b positive

$\sigma^2 = \frac{1}{2b}$ $\theta = \frac{a}{2b}$

$\Uparrow$

where $a = \frac{\theta}{\sigma^2}$ $b = \frac{1}{2\sigma^2}$

Pretended to know σ^2 We want inference for the norm θ

kernel factor:

$$p(x \mid \theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-N)^2}{2\sigma^2}} \propto e^{-\frac{(x-N)^2}{2\sigma^2}} = e^{-\frac{x^2}{2\sigma^2}} \cdot e^{\frac{x\theta}{\sigma^2}} \cdot e^{-\frac{\theta^2}{2\sigma^2}} \propto e^{-\frac{x^2-2x\theta}{2\sigma^2}} = e^{ax-bx^2}$$

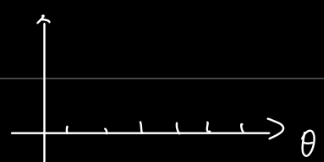
why need to do? $\Rightarrow$ If I see a kernel e^{ax-bx^2}, you will realize that it's a norm / $\propto N(\frac{a}{2b}, \frac{1}{2b})$

like a posterior, but do kernel practice now

$N(\theta,\sigma^2)_2$

$$P(\theta \mid x, \sigma^2) \overset{\text{same distribution}}{=} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} \propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2} = e^{\frac{-x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}} e^{\frac{-\theta^2}{2\sigma^2}} \propto e^{\frac{x\theta}{\sigma^2} - \frac{\theta^2}{2\sigma^2}}$$

$$= e^{a\theta - b\theta^2} \propto N(\frac{a}{2b}, \frac{1}{2b}) = N(x, \sigma^2)$$ where $a = \frac{x}{\sigma^2}$ $b = \frac{1}{2\sigma^2}$ (because you are finding $p(\theta \mid x, \sigma^2)$ so it similar to $p(x\mid\theta,\sigma^2)$ $N(\theta,\sigma^2)_2$ $N(x,\sigma^2)$



normal is self conjugate.

test gamma (video recording 58-67 min)

F : $X_1, \dots, X_n ; \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$ $\bar{x} = \frac{\sum x_i}{n}$

$$P(x \mid \theta, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2} = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2}$$

$$\sum_{i=1}^{n}(x_i-\theta)^2 = \sum x_i^2 - 2n\theta\bar{x} + n\theta^2$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum x_i^2}{2\sigma^2}} e^{\frac{\theta n \bar{x}}{\sigma^2}} e^{-\frac{n\theta^2}{2\sigma^2}}$$

flip it

$$P(\theta \mid x, \sigma^2) \propto e^{\overbrace{-\frac{n}{2\sigma^2}}^{-b}\theta^2} e^{\overbrace{\frac{n\bar{x}}{\sigma^2}}^{a}\cdot\theta} \quad N(\bar{x}, \frac{\sigma^2}{n}) \quad Var = \frac{1}{2b} = \frac{1}{2(\frac{n}{2\sigma^2})} = \frac{\sigma^2}{n} \quad mean = \frac{a}{2b} = \bar{x}$$

review of Laplace prior $P(\theta) = U(\theta)$ $P(\theta) \propto 1$ if $\Theta = (0,1) \Rightarrow P(\theta) = 1$ $\Theta = (0, 100) \Rightarrow P(\theta) = \frac{1}{100}$

poisson Gamma

$$P(\theta \mid x) = \frac{P(x\mid\theta)P(\theta)}{P(x)} \propto P(x\mid\theta)P(\theta) \propto P(x\mid\theta) = \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod x_i!} \propto e^{-n\theta}\theta^{\sum x_i} \propto Gamma(\sum x_i + 1, n) \Rightarrow Gamma(1, 0)$$

$\propto 1$

so, generally $p(\theta) = Gamma(\alpha, \beta) \Rightarrow P(\theta \mid x) = Gamma(\alpha + \sum x_i, n + \beta)$

It prove laplace is Gamma (1, 0)

(recording 82, 81)

go back to $\hat{F}$: $X_1, \dots, X_n; \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 known. Prior on θ is Laplace $\Rightarrow p(\theta) \propto 1$ improper!

$$p(\theta \mid x, \sigma^2) \propto p(x \mid \theta, \sigma^2) \propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$