Lec 4.

Bayes's rule for 2 r.v

$$P(\theta|x) = \frac{P(x|\theta)P(\theta)}{P(x)}$$

$$P(Y|x) = \frac{P(x|Y)P(Y)}{P(x)}$$

Assumed: θ was fixed at $\theta \sim Deg(\theta_0)$ JMF, JDF equal the likelihood.

$P(X=x|\theta=\theta)$ $\quad p(\theta) = \begin{cases} 1 & \text{if } \theta=\theta_0 \\ 0 & \text{o.w} \end{cases} \Rightarrow P(\theta|x) = \begin{cases} \frac{P(x|\theta=\theta_0)}{P(x)} & \text{if } \theta=\theta_0 \\ 0 & \text{if } \theta\neq\theta_0 \end{cases}$

$$P(x) = \begin{cases} \sum_{\theta\in\Theta} P(x|\theta)\cdot P(\theta) = P(x|\theta=\theta_0) \\ \int_{\Theta} p(x|\theta)\cdot p(\theta)\,d\theta \quad \text{(marginal)} \end{cases}$$

So this case not useful.

Assume: θ is a non-degenerate r.v

$$\underset{\text{posterior}}{P(\theta|x)} = \frac{P(x|\theta)P(\theta)}{P(x)}$$

$P(\theta)$ → prior (your thought about θ before see any data).

$P(x)$ ↳ prior prediction distribution

posterior (you thoughts about θ after see x)

For example: $\tilde{F}$ = iid Bernoulli $\quad \Theta = \{0.5, 0.75\}$ $\quad X = \langle 0, 1, 1 \rangle$ ($1-\theta$, θ, θ)

$p(\theta=0.75|x) > P(\theta=0.5|x)$

↳ $= \dfrac{P(x|\theta=0.75)P(\theta=0.75)}{P(x|\theta=0.5)p(\theta=0.5)+P(x|\theta=0.75)P(\theta=0.75)}$ $\qquad \dfrac{P(x|\theta=0.5)P(\theta=0.5)}{P(x|\theta=0.5)P(\theta=0.5)+P(x|\theta=0.75)P(\theta=0.75)}$

0.25×0.75^2

we need probability $P(\theta=0.75)$ and $P(\theta=0.5)$ · principle of indifferent

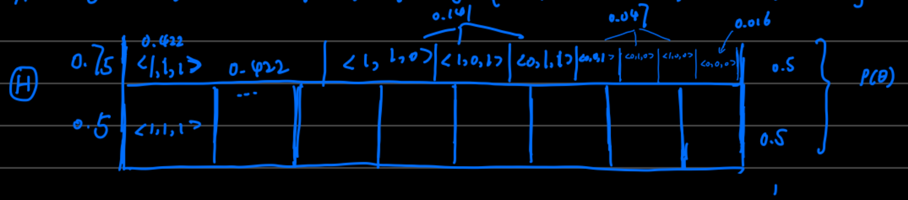
all $\theta\in\Theta$ are equally likely e.g $P(\theta) = \frac{1}{|\Theta|}$

$$P(\theta) = \begin{cases} \frac{1}{2} & \theta=0.5 \\ \frac{1}{2} & \theta=0.75 \\ 0 & \text{o.w} \end{cases}$$

go back. we could get $P(\theta=0.75|x)=0.53$ $P(\theta=0.5|x)=0.47$

3 Bern form a result

$$x \in \mathcal{X} = \{0,1\} \times \{0,1\} \times \{0,1\} = \left\{ \begin{matrix} \langle 1,1,1\rangle, \langle 1,1,0\rangle, \langle 1,0,1\rangle, \langle 0,1,1\rangle, \\ \langle 0,1,0\rangle, \langle 0,0,1\rangle, \langle 1,0,0\rangle, \langle 0,0,0\rangle \end{matrix} \right\}$$



Other case change Ⓗ to $\textcircled{H}_0 = \{0.1, 0.25, 0.5, 0.75, 0.9\}$ prior $p(\theta) = \begin{cases} 0.2 & \text{if } \theta \in \textcircled{H} \\ 0 & \text{o/t} \end{cases}$

what if I want the most likely value of θ given x? $\sum_{x \in \mathcal{X}} P(x) = 1$ $\sum_{\theta \in \textcircled{H}} P(x|\theta) = ?$

$$\hat{\theta} := \underset{\theta \in \textcircled{H}}{\arg\max} \{P(\theta|x)\} = \arg\max \left\{ \frac{P(x|\theta)P(\theta)}{P(x)} \right\}$$ $\sum_{\theta \in \textcircled{H}_0} P(x) = 1$ $\sum_{\theta \in \textcircled{H}_0} P(\theta|x) = 1$

$$\hat{\theta}_{map} = \underset{\theta \in \textcircled{H}}{\arg\max} \{P(x|\theta)P(\theta)\} \;\overset{\text{if } \textcircled{H}_0 = \textcircled{H}}{=\!=\!=}\; \underset{\theta \in \textcircled{H}_0}{\arg\max} \{\underbrace{P(x|\theta)}_{\mathcal{L}(\theta, x)}\} = \hat{\theta}_{MLE}$$

maximum a posterior

$P(x|\theta=0.1) = 0.1^2 \times 0.9 = 0.009$ $P(x|\theta=0.25) = 0.25^2 \times 0.75 = 0.047$

$P(x|\theta=0.5) = 0.125$ $P(x|\theta=0.75) = 0.141$ $P(x|\theta=0.9) = 0.81$ their sum is 0.40371

$\Rightarrow \hat{\theta}_{map} = 0.75$

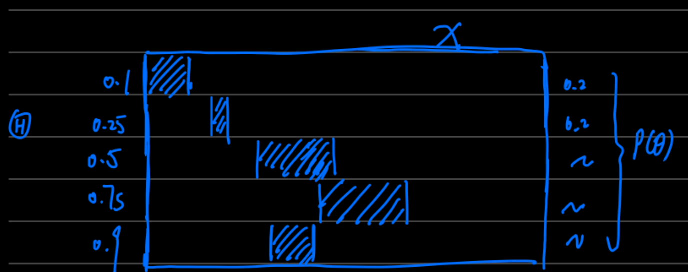
$P(\theta)$ is the fixed value

$$p(\theta|x) = \frac{P(x|\theta)P(\theta)}{\sum_{\theta \in \textcircled{H}_0} P(x|\theta) \cdot P(\theta)} = \frac{P(\theta)\,P(x|\theta)}{P(\theta) \sum_{\theta \in \textcircled{H}_0} P(x|\theta)} = \frac{P(x|\theta)}{\sum_{\theta \in \textcircled{H}_0} P(x|\theta)}$$

take $\theta = 0.75$

$= \frac{0.14}{0.403} = 0.35$ $\underset{=0.2}{P(\theta=0.75)} \xrightarrow{x} \underset{=0.35}{P(\theta=0.75|x)}$

Because just 3 points so the probability not that high.



jump back to $\Theta = \{0.5, 0.75\}$ $\qquad P(\theta | X_1, X_2, X_3)$

after seeing $X_1 = 0$

$$P(\theta = 0.75 | X_1 = 0) = \frac{P(X_1 = 0 | \theta = 0.75) \cdot P(\theta = 0.75)}{P(X_1 = 0 | \theta = 0.75) \cdot P(\theta = 0.75) + P(X_1 = 0 | \theta = 0.5) P(\theta = 0.5)} = \frac{0.25}{0.25 + 0.5} = \frac{1}{3}$$

$$P(\theta = 0.5 | X = 0) = \frac{2}{3}$$

Now, my prior changes $P(\theta) = \begin{cases} \frac{1}{3} & \text{if } \theta = 0.75 \\ \frac{2}{3} & \text{if } \theta = 0.25 \end{cases}$ $\qquad P(\theta | X_2) = \frac{P(X_2 | \theta) P(\theta)}{P(X_2)}$