Lecture 13

03/24/20

$$\Rightarrow \theta | x \sim Gamma(\Sigma x_i + \alpha, n+\beta)$$

$\tilde{F}$: Poisson (θ) iid, $\theta \sim Gamma(\alpha, \beta)$

$n_* = 1 \quad P(X_*|X) = \int_{\Theta} P(X_*|\theta) P(\theta|X)\, d\theta$

$$= \int_0^\infty \left(\frac{e^{-\theta}\theta^{X_*}}{X_*!}\right)\left(\frac{(n+\beta)^{\Sigma x_i+\alpha}}{\Gamma(\Sigma x_i+\alpha)}\theta^{\Sigma x_i+\alpha-1} e^{-(n+\beta)\theta}\right) d\theta$$

$$= \frac{(n+\beta)^{\Sigma x_i+\alpha}}{X_*!\,\Gamma(\Sigma x_i+\alpha)} \int_0^\infty \theta^{\Sigma x_i + X_* + \alpha - 1} e^{-(n+\beta+1)\theta}\, d\theta$$

let $t = (n+\beta+1)\theta$

$\frac{dt}{d\theta} = n+\beta+1, \quad \theta = \frac{t}{n+\beta+1}$

$$= \frac{(n+\beta)^{\Sigma x_i+\alpha}}{X_*!\,\Gamma(\Sigma x_i+\alpha)} \int_0^\infty \left(\frac{t}{n+\beta+1}\right)^{\Sigma x_i + X_* + \alpha - 1} e^{-t}\, \frac{dt}{n+\beta+1}$$

$$= \frac{(n+\beta)^{\Sigma x_i+\alpha}}{X_*!\,\Gamma(\Sigma x_i+\alpha)} \frac{1}{(n+\beta+1)^{\Sigma x_i + X_* + \alpha - 1}} \cdot \frac{1}{n+\beta+1}$$

$$\underbrace{\int_0^\infty t^{\Sigma x_i + X_* + \alpha - 1} e^{-t}\, d\theta}_{\Gamma(\Sigma x_i + X_* + \alpha)}$$

$$= \frac{(n+\beta)^{\Sigma x_i+\alpha}}{X_*!\,\Gamma(\Sigma x_i+\alpha)} \cdot \frac{\Gamma(\Sigma x_i + X_* + \alpha)}{\underbrace{(n+\beta+1)^{\Sigma x_i + X_* + \alpha}}_{(n+\beta+1)^{\Sigma x_i+\alpha}(n+\beta+1)^{X_*}}}$$

$$\left(\frac{n+\beta}{n+\beta+1}\right)^{\sum x_i + \alpha} \left(\frac{1}{n+\beta+1}\right)^{X_*} \frac{\Gamma(\sum x_i + X_* + \alpha)}{X_*!\,\Gamma(\sum x_i + \alpha)}$$

Let $p = \frac{n+\beta}{n+\beta+1} \in (0,1)$, $1-p = \frac{1}{n+\beta+1} \in (0,1)$

Let $r = \sum x_i + \alpha$

$$= \frac{\Gamma(X_* + r)}{X_* \Gamma(r)} p^r (1-p)^{X_*} = \text{Ext. Neg. Binomial}(r, p)$$

Extended Negative Binomial Model

If $r = \sum x_i + \alpha \in \mathbb{N}_0 \Rightarrow \alpha \in \mathbb{N}$

$$= \binom{X_* + r - 1}{X_*} p^r (1-p)^{\alpha_*} = \text{NegBinomial}(r, p)$$

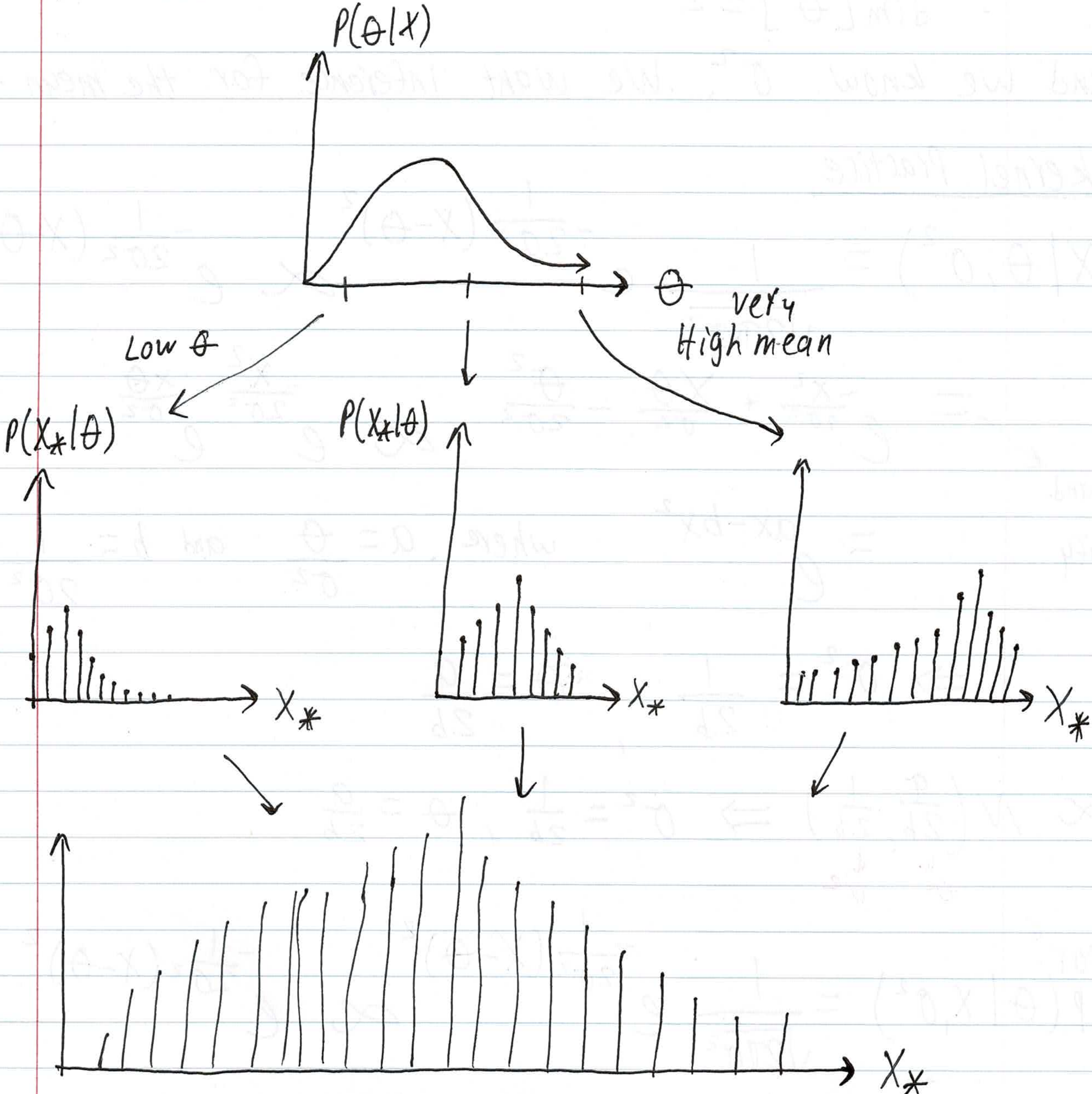
Recall: $\Gamma(X) = (X-1)!$ if $X \in \mathbb{N}$

$X_1, \dots, X_n ; p \overset{iid}{\sim} \text{Geometric}(p) = (1-p)^x p$, $E[X_i] = \frac{p}{1-p}$

$$\sum_{i=1}^{r} X_i \sim \text{NegBin}(r, p) \Rightarrow E[\sum x_i] = r \frac{p}{1-p}$$

$X_* | X \sim \text{Ext. Neg. Bin}(r, p)$ Overdispersed Poisson

Overdispersed Poisson



$$X \sim \text{Neg. Bin.}(r, p) \Rightarrow E[X_*|X] = r\frac{p}{1-p} = \mu$$

$$Var[X_*|X] = \frac{pr}{(1-p)^2} = \frac{1}{1-p}\mu$$

$$\frac{1}{1-p} \in (1, \infty)$$

Normal Model $X \sim N(\theta, \sigma^2) = N(\theta_1, \theta_2)$

$\dim[\vec{\theta}\,] = 2$

Pretend we know σ^2. We want inference for the mean θ

<u>kernel Practice</u>

$$p(X \mid \theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X-\theta)^2} \propto e^{-\frac{1}{2\sigma^2}(X-\theta)^2}$$

Expand + simplify

$$= e^{\frac{-X^2}{2\sigma^2} + \frac{X\theta}{\sigma^2} - \frac{\theta^2}{2\sigma^2}} \propto e^{\frac{-X^2}{2\sigma^2}} e^{\frac{X\theta}{\sigma^2}}$$

$$= e^{ax - bx^2} \quad \text{where } a = \frac{\theta}{\sigma^2} \text{ and } b = \frac{1}{2\sigma^2} > 0$$

$$\Rightarrow \sigma^2 = \frac{1}{2b}, \quad \theta = \frac{a}{2b}$$

$$\propto N\left(\underset{\theta}{\frac{a}{2b}}, \underset{\sigma^2}{\frac{1}{2b}}\right) \Rightarrow \sigma^2 = \frac{1}{2b}, \quad \theta = \frac{a}{2b}$$

"Posterior"

$$p(\theta \mid X, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X-\theta)^2} \propto e^{-\frac{1}{2\sigma^2}(X-\theta)^2}$$

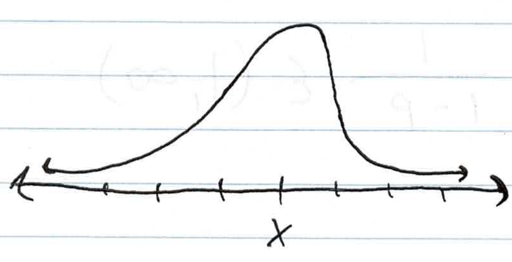
$$= e^{\frac{-X^2}{2\sigma^2} + \frac{X\theta}{\sigma^2} - \frac{\theta^2}{2\sigma^2}} \propto e^{\frac{X\theta}{\sigma^2} - \frac{\theta^2}{2\sigma^2}} = e^{a\theta - b\theta^2}$$

$$\propto N\left(\frac{a}{2b}, \frac{1}{2b}\right) = N(X, \sigma^2)$$

$$\frac{a}{2b} = \frac{\frac{\theta}{\sigma^2}}{2\left(\frac{1}{2\sigma^2}\right)}$$

$$\frac{a}{2b} = X$$

$$\frac{1}{2b} = \frac{1}{2\left(\frac{1}{2\sigma^2}\right)} = \sigma^2$$



$\tilde{F}: X_1, \ldots, X_n ; \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$$P(X|\theta,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i-\theta)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\underbrace{\sum_{i=1}^{n}(X_i-\theta)^2}}$$

$\sum(X_i^2 - 2\theta X_i + \theta^2)$

$= \sum X_i^2 - \sum 2\theta X_i + \sum\theta^2$

$= \sum X_i^2 - 2\theta\sum X_i + n\theta^2$

$= \sum X_i^2 - 2\theta n\bar{X} + n\theta^2$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}(\sum X_i^2 - 2\theta n\bar{X} + n\theta^2)}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{\frac{-\sum X_i^2}{2\sigma^2}} e^{\frac{\theta n\bar{X}}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}}$$

$$P(\theta|X,\sigma^2) \propto e^{\frac{n\bar{X}}{\sigma^2}\theta - \frac{n}{2\sigma^2}\theta^2} \propto N(?,?)$$

$$= N\left(\frac{a}{2b}, \frac{1}{2b}\right)$$

$$Var[\theta] = \frac{1}{2b^2} = \frac{1}{2\left(\frac{n}{2\sigma^2}\right)} = \frac{1}{\frac{n}{\sigma^2}} = \frac{\sigma^2}{n}$$

$$E[\theta] = \frac{a}{2b} = \frac{\frac{n\bar{X}}{\sigma^2}}{2\left(\frac{n}{2\sigma^2}\right)} = \bar{X}$$

$$P(\theta|X,\sigma^2) = N\left(\bar{X}, \frac{\sigma^2}{n}\right)$$

Laplace Uninformative Prior

$$P(\theta) = U(\Theta) \Rightarrow P(\theta) = 1 \quad \text{if } \Theta = (0,1)$$

$$\Theta = (0, 100), \quad P(\theta) = \frac{1}{100} \propto 1$$

Poisson - Gamma

$P(\theta) \propto 1$

$$P(\theta|x) = \frac{P(x|\theta)P(\theta)}{P(x)} \propto P(x|\theta)\underbrace{P(\theta)}_{\propto 1} \propto P(x|\theta)$$

$$= \frac{e^{-n\theta}\theta^{\Sigma x_i}}{\Pi x_i!} \propto e^{-n\theta}\theta^{\Sigma x_i} \propto \text{Gamma}(\Sigma x_i + 1, n)$$

Generally $P(\theta) = \text{Gamma}(\alpha, \beta)$

$$\Rightarrow P(\theta|x) = \text{Gamma}(\Sigma x_i + \alpha, n + \beta)$$

proves Laplace prior

$$\Rightarrow \theta \sim \text{Gamma}(1, 0)$$

$\tilde{F}: X_1, \ldots, X_n ; \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$ σ^2 known

Prior on θ is Laplace $\Rightarrow P(\theta) \propto 1$

improper

$$P(\theta|x, \sigma^2) \propto P(x|\theta, \sigma^2) \propto N\left(\bar{X}, \frac{\sigma^2}{n}\right)$$