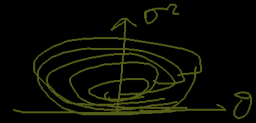
X: iid $N(\theta,\sigma^2)$ but both θ,σ^2 unknown



$$P(\theta,\sigma^2|X) \propto P(X|\theta,\sigma^2)\,P(\theta,\sigma^2)$$

$$= (2\pi\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}\, P(\theta,\sigma^2)$$

$$\propto \underbrace{(\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}}_{\text{normal-inverse gamma kernel}}\, k(\theta,\sigma^2)$$

$$\sum(x_i-\theta)^2 = \sum\left((x_i-\bar{x})+(\bar{x}-\theta)\right)^2 = \sum(x_i-\bar{x})^2 + 2\sum(x_i-\bar{x})(\bar{x}-\theta) + \sum(\bar{x}-\theta)^2$$

$$= (n-1)s_x^2 + 2\sum(x_i\bar{x} - \bar{x}^2 - x_i\theta + \bar{x}\theta) + n(\bar{x}-\theta)^2$$

$$= (n-1)s^2 + n(\bar{x}-\theta)^2 + 2(\bar{x}\cancel{n\bar{x}} - \cancel{n\bar{x}^2} - \cancel{\theta n\bar{x}} + \cancel{n\bar{x}\theta}) = (n-1)s^2 + n(\bar{x}-\theta)^2$$

$$= (\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{x}-\theta)^2\right)}\, k(\theta,\sigma^2)$$

$$P(\theta,\sigma^2|X) = \underbrace{(\sigma^2)^{-\underbrace{(\frac{n}{2}-1)}_{\alpha}-1}\, e^{-\frac{\overbrace{(n-1)s^2/2}^{\beta}}{\sigma^2}}\left(e^{-\frac{\overbrace{n}^{\lambda}}{2\sigma^2}(\theta-\underbrace{\bar{x}}_{\mu})^2}\right)}\, k(\theta,\sigma^2)$$

$$= P(\theta|X,\sigma^2)\,P(\sigma^2|X) \propto \text{NormInvGamma}(\alpha,\beta,\lambda,\mu)\; k(\theta,\sigma^2)$$

Conjugate prior $\propto k(\theta,\sigma^2) = (\sigma^2)^{-\alpha_0-1}\, e^{-\frac{\beta_0}{\sigma^2}}\, e^{-\frac{\lambda_0}{\sigma^2}(\theta-\mu_0)^2} \propto \text{NormInvGamma}(\alpha_0,\beta_0,\lambda_0,\mu_0)$

$P(\theta,\sigma^2) = k(\theta|\sigma^2)\,k(\sigma^2) \propto (\text{Normal})(\text{InvGamma})$

we won't study this, the general prior

$$P_J(\theta,\sigma^2) = \underbrace{P_J(\theta|\sigma^2)}_{\propto 1}\,\underbrace{P_J(\sigma^2)}_{\propto \frac{1}{\sigma^2}} \propto (1)\left(\frac{1}{\sigma^2}\right) = \frac{1}{\sigma^2}$$

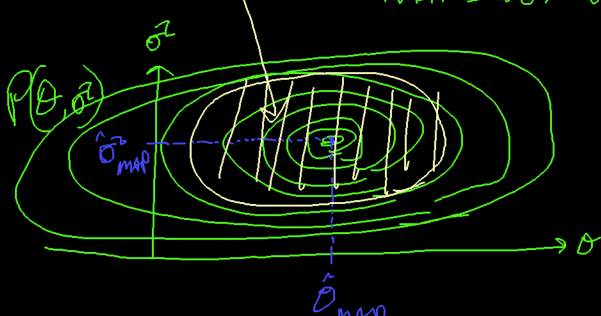
only principled uninformative prior we will use.

$$P(\theta,\sigma^2|X) \propto P(X|\theta,\sigma^2)\,P_J(\theta,\sigma^2) \propto \left((\sigma^2)^{-n/2}\, e^{-\frac{(n-1)s^2/2}{\sigma^2}}\, e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}\right)\left(\frac{1}{\sigma^2}\right)$$

$$= (\sigma^2)^{-n/2-1}\, e^{-\frac{(n-1)s^2/2}{\sigma^2}}\, e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}$$

HDR 95%

$$\propto \text{NormInvGamma}\left(\frac{n}{2}, \frac{(n-1)s^2}{2}, \frac{n}{2}, \bar{x}\right)$$



$\langle \hat{\theta}_{MAP}, \hat{\sigma}^2_{MAP} \rangle$

we will not study $\langle \hat{\theta}_{MMAE}, \hat{\sigma}^2_{MMAE} \rangle$ or $\langle \hat{\theta}_{MMSE}, \hat{\sigma}^2_{MMSE} \rangle$

we also won't study confidence sets or hypothesis tests