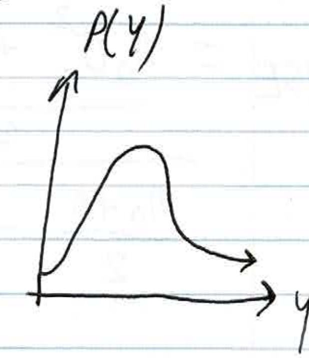
Lecture 16

04/14/20

$Y \sim \text{InvGamma}(\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} y^{-\alpha-1} e^{\frac{-\beta}{y}}$

$\propto y^{-\alpha-1} e^{\frac{-\beta}{y}}$



$E[Y] = \frac{\beta}{\alpha - 1}$ for $\alpha > 1$

$\text{med}[Y] = \text{qinvgamma}(0.5, \alpha, \beta)$

$\text{mode}[Y] = \frac{\beta}{\alpha + 1}$

$\tilde{F}$: iid $N(\theta, \sigma^2)$ with θ known

$p(\sigma^2 | \theta) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0 \sigma_0^2}{2}\right)$ → σ_0^2 is mean prior belief

$Y_1, \ldots, Y_n \overset{iid}{\sim} N(\theta, \sigma^2)$

$n_0 \sigma_0^2 = \Sigma (Y_i - \theta)^2$

$E[\sigma^2 | \theta] = \frac{\frac{n_0 \sigma_0^2}{2}}{\frac{n_0}{2} - 1} = \frac{n_0 \sigma_0^2}{n_0 - 2} \approx \sigma_0^2$

If no is large

$\text{mode}[\sigma^2 | \theta] = \frac{n_0 \sigma_0^2}{n_0 + 2} \approx \sigma_0^2$

$$P(\sigma^2 | X, \theta) = \text{InvGamma}\left(\frac{n_0 + n}{2}, \frac{n_0 \sigma_0^2 + n \hat{\sigma}^2_{MLE}}{2}\right)$$

$$\hat{\sigma}^2_{MMSE} = \frac{\frac{n_0 \sigma_0^2 + n \hat{\sigma}^2}{2}}{\frac{n_0 + n}{2} - 1} = \frac{n_0 \sigma_0^2 + n \hat{\sigma}^2}{n_0 + n - 2}$$

$$\hat{\sigma}^2_{MMAE} = \text{qinvgamma}\left(0.5, \frac{n_0 + n}{2}, \frac{n_0 \sigma_P^2 + n \hat{\sigma}^2}{2}\right)$$

$$\hat{\sigma}^2_{MAP} = \frac{n_0 \sigma_0^2 + n \hat{\sigma}^2}{n_0 + n + 2}$$

Credible Regions / Hyp. Test — Same as before

<u>Uninformative Priors</u>

1) Laplace / Indifference

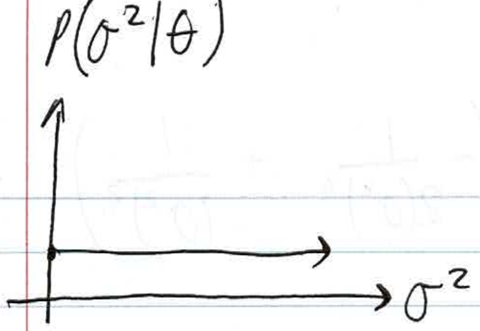
$$P(\sigma^2 | \theta, x) \propto P(x | \theta, \sigma^2) \propto (\sigma^2)^{-(\frac{n}{2} - 1) - 1} e^{\frac{-n\hat{\sigma}^2/2}{\sigma^2}}$$

$$\propto \text{InvGamma}\Big(\underbrace{\frac{n}{2} - 1}_{\frac{n-2}{2}}, \frac{n\hat{\sigma}^2}{2}\Big) \qquad n_0 = -2, \quad \sigma_0^2 = 0$$

$$P(\sigma^2 | \theta) = \text{InvGamma}(-1, 0) \propto 1$$

n_0 = pseudo observations (# of pseudo observations)

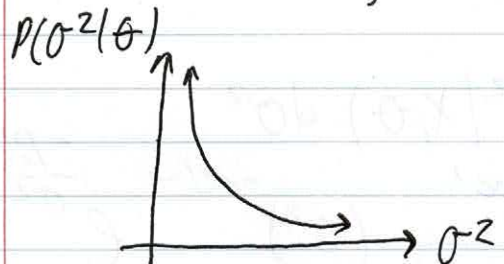
σ_0^2 = variance of the pseudo observations



σ^2_{MMSE} is only defined for $n \geq 5$

2) Haldane $\quad n_0 = 0 \quad , \sigma_0^2 = \,? \quad \Rightarrow \text{InvGamma}(0,0)$

$\text{InvGamma}(0,0) \propto (\sigma^2)^{-1}$



3) Jeffrey's Prior

$$\ell(\sigma^2; \theta, x) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}(\sigma^2) - \frac{1}{2\sigma^2}\Sigma(x-\theta)^2$$

$$\cancel{\ell'(\sigma^2, x, \theta} \quad \ell'(\sigma^2; \theta, x) = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2}\Sigma(x_i-\theta)^2$$

$$-\ell''(\sigma^2; \theta, x) = -\frac{n}{2}\frac{1}{(\sigma^2)^2} + \frac{1}{(\sigma^2)^3}\Sigma(x_i-\theta)^2$$

$$I(\sigma^2; \theta, x) = E_x\left[\frac{-n}{2(\sigma^2)^2} + \frac{\Sigma(x_i-\theta)^2}{(\sigma^2)^3}\right]$$

$$= \frac{-n}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^3}\Sigma E_x[(x_i-\theta)^2]$$

$$E(x_i-\theta)^2 = E_x[x_i^2] - 2\theta E[x_i] + \theta^2$$
$$= (\sigma^2+\theta^2) - 2\theta^2 + \theta^2$$
$$= \sigma^2$$

$$= -\frac{n}{2}\frac{1}{(\sigma^2)^2} + \frac{1}{(\sigma^2)^3} n\sigma^2 = n\left(-\frac{1}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^2}\right)$$

$$= n\frac{1}{2(\sigma^2)^2} = \frac{n}{2}(\sigma^2)^{-2}$$

$$P_J(\sigma^2|\theta) \propto \sqrt{I(\sigma^2;\theta)} = \sqrt{\frac{n}{2}(\sigma^2)^{-2}} \propto (\sigma^2)^{-1}$$

$\propto$ InvGamma $(0,0)$ Default

Prior Predictive Distribution

$$P(X_*|X,\theta) = \int_0^\infty P(X_*|\theta,\sigma^2)\, P(\sigma^2|X,\theta)\, d\sigma^2$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_*-\theta)^2} \frac{\beta'^{\alpha'}}{\Gamma(\alpha')}(\sigma^2)^{-\alpha'-1} e^{\frac{-\beta'}{\sigma^2}} d\sigma^2$$

$\boxed{\text{Let } \alpha' = \frac{n+n_0}{2} \quad \beta' = \frac{n\hat{\sigma}^2 + n_0\sigma_0^2}{2}}$

$$\propto \int_0^\infty (\sigma^2)^{-\frac{1}{2}} e^{\frac{-(X_*-\theta)^2/2}{\sigma^2}} (\sigma^2)^{-\alpha'-1} e^{\frac{-\beta'}{\sigma^2}} d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\overbrace{(\alpha+\frac{1}{2})}^{A}-1} e^{-\frac{\overbrace{(X_*-\theta)^2/2+\beta'}^{B \text{ (only the numerator)}}}{\sigma^2}} d\sigma^2$$

kernel of InvGamma(A,B)

$$= \frac{\Gamma(A)}{B^A} \underbrace{\int_0^\infty \frac{B^A}{\Gamma(A)} (\sigma^2)^{-A-1} e^{-\frac{B}{\sigma^2}} d\sigma^2}_{1}$$

$$= \Gamma(A) B^{-A} = \Gamma\left(\frac{n+n_0+1}{2}\right)\left(\frac{n\hat{\sigma}^2 + n_0\sigma_0^2 + (X_* - \theta)^2}{2}\right)^{-\frac{n+n_0+1}{2}}$$

$$\propto \left(\frac{n\hat{\sigma}^2 + n_0\sigma_0^2 + (X_* - \theta)^2}{2}\right)^{-\frac{n+n_0+1}{2}}$$

lowercase "nu"

$$= \left(\frac{a}{2}\right)^{-\frac{\nu+1}{2}}\left(\frac{2}{a}\,\frac{a+(X_* - \theta)^2}{2}\right)^{-\frac{\nu+1}{2}}$$

$$\propto \left(\frac{2}{a}\,\frac{a+(X_* - \theta^2)}{2}\right)^{-\frac{\nu+1}{2}} = \left(1+\frac{(X_* - \theta)^2}{a}\right)^{-\frac{\nu+1}{2}}$$

$$= \left(1+\frac{1}{\nu}\,\frac{(X_* - \theta)^2}{\frac{a}{2}}\right)^{-\frac{\nu+1}{2}} \propto T_\nu\left(\theta, \frac{a}{\nu}\right)$$

Student's T distribution with ν degrees of freedom and location parameter θ (mean) and scale $\frac{a}{\nu}$

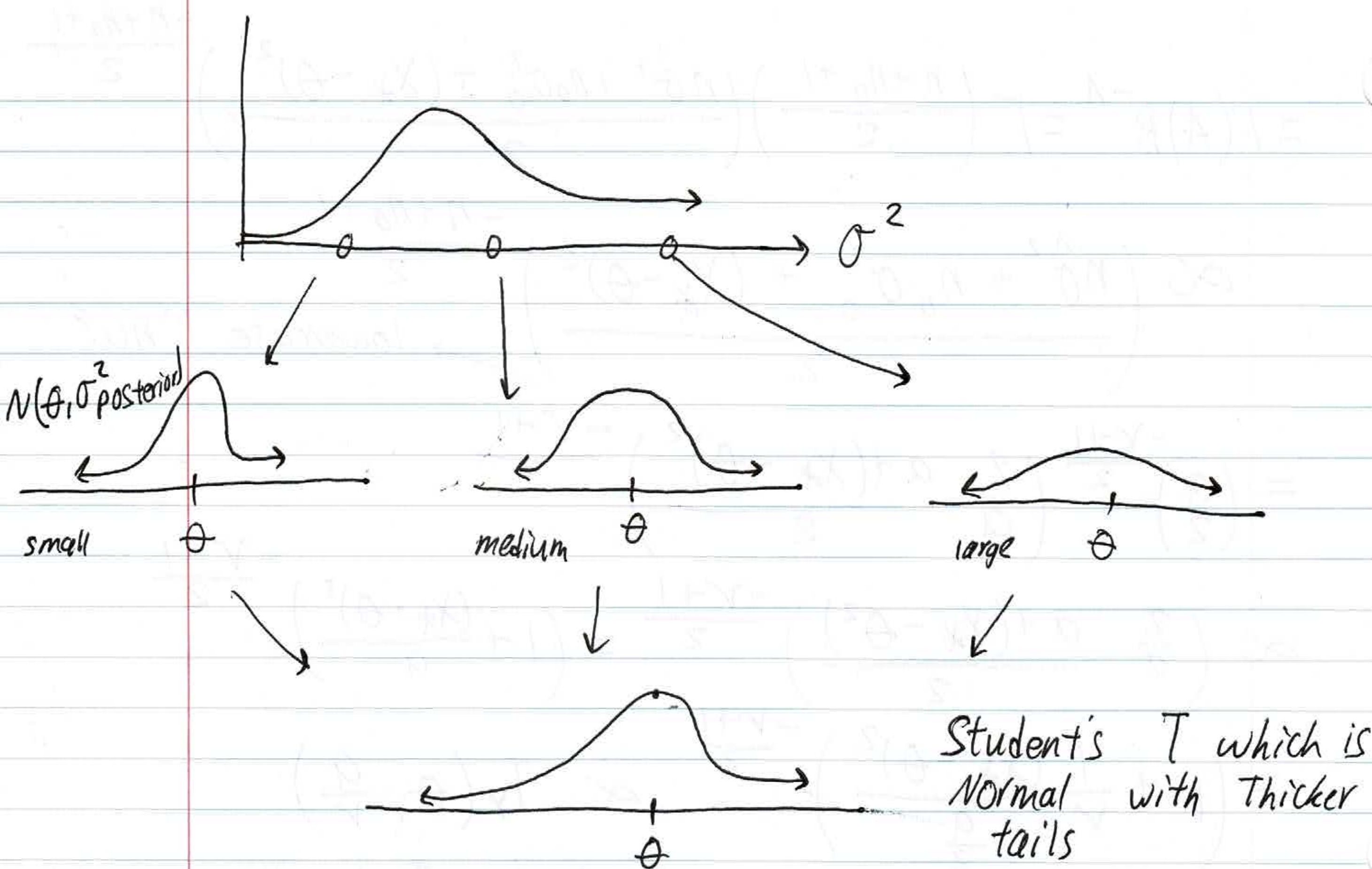
In Math 242... $\tilde{F}: X_1, \ldots, X_n; \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$$H_0: \theta = 0 \Rightarrow \frac{\bar{X} - \theta}{\frac{S}{\sqrt{n}}} = \frac{\bar{X}\sqrt{n}}{S} \sim T_{n-1}(0,1)$$

(Standard T)

$$= T_{n+n_0}\left(\theta, \frac{n\hat{\sigma}^2 + n_0\sigma_0^2}{n+n_0}\right)$$

$p(\sigma^2 | X, \theta)$



$$\lim_{n \to \infty} p(X_* | X, \theta) = N(\theta, \sigma^2)$$

$$= T_\infty\left(\theta, \lim_{n \to \infty} \hat{\sigma}^2_{MLE}\right) = N(\theta, \sigma^2)$$

$$\hat{\sigma}^2_{MMSE} = \frac{n\hat{\sigma}^2 + n_0\sigma_0^2}{n + n_0 - 2} = \underbrace{\frac{n\hat{\sigma}^2}{n + n_0 - 2}}_{1-\rho} + \frac{n_0\sigma_0^2}{n + n_0 - 2}$$

$$= \underbrace{\frac{n}{n + n_0 - 2}}_{1-\rho}\hat{\sigma}^2 + \frac{n_0\sigma_0^2}{n + n_0 - 2} \cdot \frac{n_0 - 2}{n_0 - 2}$$

$$= (1-\rho)\hat{\sigma}^2 + \underbrace{\frac{n_0 - 2}{n + n_0 - 2}}_{\rho} \underbrace{\frac{n_0\sigma_0^2}{n_0 - 2}}_{E[\sigma^2 | \theta]}$$

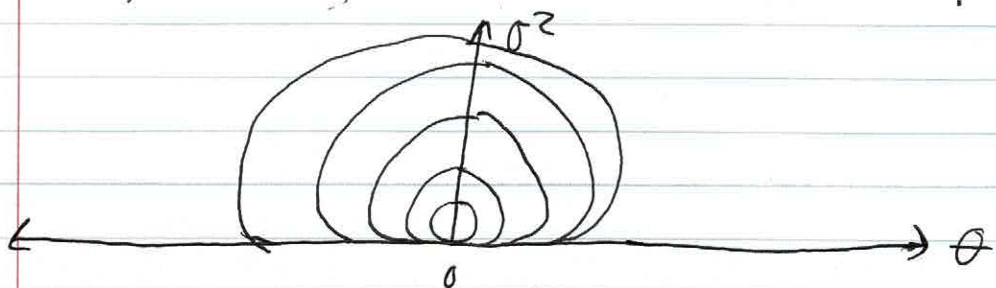
End of Midterm 2 Material

Start of Final Material

$\tilde{F}: X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ but neither θ or σ^2 known

$$P(\theta, \sigma^2 | x) \propto P(x | \theta, \sigma^2) \, P(\theta, \sigma^2)$$

$$P(\theta, \sigma^2) = P(\theta | \sigma^2) \, P(\sigma^2)$$
$$= N(0, \sigma^2) \, \text{InvGamma}(1,1)$$



$$= (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\Sigma(X_i - \theta)^2} \qquad P(\theta, \sigma^2)$$

If Laplace

$$\propto (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\Sigma(X_i - \theta)^2}$$
$$\propto (\sigma^2)^{-\frac{n}{2}} e^{\frac{-\Sigma(X_i - \theta)^2/2}{\sigma^2}}$$

Is this $\propto$ InvGamma? NO, there are two dimensions here, while for the distribution, there is only 1-dimension.

- This is Normal InvGamma dist.
- Will have 4 parameters