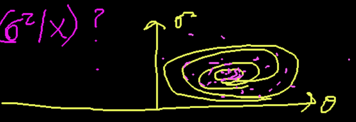
sample from $P(\sigma^2|X)$?



F: $\overset{iid}{\sim} N(\theta, \sigma^2)$ where both θ, σ^2 are unknown

if $P_J(\theta,\sigma^2) \propto \frac{1}{\sigma^2}$

$$P(\theta,\sigma^2|X) = \underbrace{P(\theta|X,\sigma^2)}\ \underbrace{P(\sigma^2|X)}_{\text{realizations}} = \left(N\left(\bar{x}, \frac{\sigma^2}{n}\right)\right)\left(\text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)\right)$$

How would I sample $\langle \theta, \sigma^2 \rangle$ from $P(\theta,\sigma^2|X)$?

samples are realization

Step I: Draw a σ^2_{smp} realization from $P(\sigma^2|X)$ using rinvgamma$\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

Step II: Draw a θ_{smp} realization from $P(\theta|X, \sigma^2 = \sigma^2_{smp})$ using rnorm$\left(\bar{x}, \sqrt{\frac{\sigma^2_{smp}}{n}}\right)$

return $\langle \theta_{smp}, \sigma^2_{smp} \rangle$. To sample n realizations, repeat n times.

How to sample from $P(X_*|X) = T_{n-1}\left(\bar{x}, \sqrt{\frac{n+1}{n}}\, s\right)$? rt.scaled$\left(n-1, \bar{x}, \sqrt{\frac{n+1}{n}}\, s\right)$

$$P(X_*|X) = \int_\theta\int_{\sigma^2} \underbrace{P(X_*,\theta,\sigma^2|X)}\, d\sigma^2 d\theta = \int_\theta\int_{\sigma^2} \underbrace{P(X_*|\theta,\sigma^2)\,P(\theta,\sigma^2|X)}\, d\sigma^2 d\theta = \int_\theta\int_{\sigma^2} \underbrace{P(X_*|\theta,\sigma^2)\,P(\theta|X,\sigma^2)\,P(\sigma^2|X)}\, d\sigma^2 d\theta$$

How to sample from $P(X_*, \theta, \sigma^2 | X)$?

(I) Sample σ^2_{smp} from $P(\sigma^2|X)$ via rinvgamma$\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

(II) Sample θ_{smp} from $P(\theta|X, \sigma^2 = \sigma^2_{smp})$ via rnorm$\left(\bar{x}, \sqrt{\frac{\sigma^2_{smp}}{n}}\right)$

(III) Sample X_{*smp} from $P(X_*|\theta = \theta_{smp}, \sigma^2 = \sigma^2_{smp})$ via rnorm$(\theta_{smp}, \sigma^2_{smp})$

return $\langle X_{*smp}, \theta_{smp}, \sigma^2_{smp} \rangle$

To sample from $P(X_*|X)$ you sample from $P(X_*, \theta, \sigma^2|X)$ and ignore $\theta_{smp}, \sigma^2_{smp}$ to leave you with X_{*smp}.

To sample n realizations, repeat n times.

$$P(\theta,\sigma^2|X) \propto \underbrace{P(\theta|X,\sigma^2)}\ \underbrace{P(\sigma^2|X)} \overset{\text{If } P(\theta,\sigma^2) = \text{NormInvGamma}}{=} \text{NormInvGamma due to conjugacy}$$

If $P(\theta,\sigma^2) \neq$ NormInvGamma $\Rightarrow$ non conjugate

dependence

NormInvGamma

"$P(\theta,\sigma^2) = P(\theta|\sigma^2)P(\sigma^2)$ where $P(\theta|\sigma^2) = N\left(\mu_0, \frac{\sigma^2}{n_0}\right)$, $P(\sigma^2) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$

then model is conjugate.

What if i.e. θ, σ^2 independent

$$P(\theta,\sigma^2) = P(\theta)P(\sigma^2) \quad \text{where} \quad P(\theta) = N(\mu_0, \tau^2),\ P(\sigma^2) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right) \quad \text{s.t. } \tau^2 \neq \frac{\sigma^2}{n_0}$$

$$\Rightarrow P(\theta,\sigma^2|X) \propto P(X|\theta,\sigma^2)P(\theta,\sigma^2) = P(X|\theta,\sigma^2)P(\theta)P(\sigma^2) \propto k(X|\theta,\sigma^2)\,k(\theta)\,k(\sigma^2)$$

$$= \left((\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{x}-\theta)^2\right)}\right)\left(e^{-\frac{\theta^2}{2\tau^2}} e^{\frac{\theta\mu_0}{\tau^2}}\right)\left((\sigma^2)^{-\frac{n_0}{2}-1} e^{-\frac{n_0\sigma_0^2}{2\sigma^2}}\right)$$

$$= (\sigma^2)^{-\frac{n+n_0}{2}-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n\bar{x}^2 + n_0\sigma_0^2\right)}\ e^{\overbrace{\left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}\right)}^{a}\theta - \overbrace{\left(\frac{n}{2\sigma^2} + \frac{1}{2\tau^2}\right)}^{b}\theta^2}$$

$e^{a\theta - b\theta^2} \propto N\left(\frac{a}{2b}, \frac{1}{2b}\right)$

$n(\bar{x}-\theta)^2 = n\bar{x}^2 - 2n\bar{x}\theta + n\theta^2$

$$= (\sigma^2)^{-\frac{n+n_0}{2}-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n\bar{x}^2 + n_0\sigma_0^2\right)}\left(\sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}}\, N\left(\frac{a}{2b}, \frac{1}{2b}\right)\right)$$

$$\propto \underbrace{(\sigma^2)^{-\frac{n+n_0}{2}-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n\bar{x}^2 + n_0\sigma_0^2\right)}\left(\frac{n}{2\sigma^2} + \frac{1}{2\tau^2}\right)^{-\frac{1}{2}} e^{\frac{\left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}\right)^2}{2\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)}}}_{}\ \underbrace{\left(N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right)\right)}_{P(\theta|\sigma^2, X)}$$

$k(\sigma^2|X)$, the kernel of some unknown distr. and we don't know how to draw realizations from it.