03/05/2020 (Class #10)

Informative Priors.

Let θ be the career prob. of guessing a hit. for a batter in baseball.

$\hat{\theta}_{MLE} = \frac{x}{n}$ ← # hit, ← # at bats.

Born θ in history is 0.366, assuming is 0.260

$n = 3$; $x = 2$

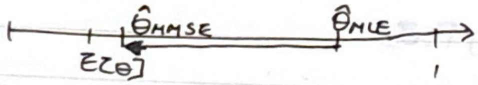
$\hat{\theta}_{MLE} = 0.667 = \hat{\theta}_{MMSE}$ if $\theta \sim Beta(0, 0)$.

If $\theta \sim Beta(1, 1)$

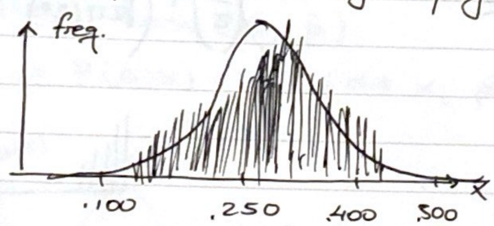
$\hat{\theta}_{MMSE} = \frac{1+2}{3+\cancel{2}+1} = 0.600$

Design a prior s.t. pick α, β

$E[\theta] = 0.260$



Look at previous data e.g. all players ≥ 500 at bats and we examine $\bar{X}$'s



Now try to fit a Beta dist. to the data. Via maximum likelihood,

$\hat{\alpha}_{MLE} = 78.7$

$\hat{\beta}_{MLE} = 224.8$

$\Rightarrow E[X] = .260$

$\Rightarrow n_0 = 303.5$

This prosess is called 'Empirical Bayes'

$\Rightarrow \rho = \frac{303.5}{303.5 + 3} = 99\%$

$\hat{\theta}_{MMSE} = (1\%)(.667) + (99\%)(.260) = .263$

$\quad\quad (1-\rho)\,\hat{\theta}_{MLE} \quad\quad \rho\, E[\theta]$

$\theta \sim Beta(\frac{1}{2}, \frac{1}{2})$ is called the Jeffrey's Prior

$Odds(A) = \frac{P(A)}{P(A^c)} = \frac{P(A)}{1 - P(A)} \in [0, \infty)$

$$\text{OddsAgainst}(A) = \text{Add}(A)^{-1} = \frac{1 - P(A)}{P(A)}. \in [0, \infty]$$

Example

$$x \sim \begin{cases} 5 & \text{wp } \frac{1}{6} \\ -1 & \text{wp } \frac{5}{6} \end{cases}$$

$$E[x] = (5)\left(\frac{1}{6}\right) + (-1)\left(\frac{5}{6}\right) = 0$$

$$\text{Odds}(A, B) = \frac{P(A)}{P(B)}$$

$$P(\theta = \theta_a | x) = \frac{P(x | \theta = \theta_a) P(\theta | \theta_a)}{P(x)}$$

$$P(\theta = \theta_b | x) = \frac{P(x | \theta = \theta_b) P(\theta | \theta_b)}{P(x)}$$

$$\text{Odds}(\theta_a, \theta_b)$$

$$\underbrace{\text{Odds}(\theta_a, \theta_b | x)}_{\text{Posterior Odds}} = \frac{P(\theta = \theta_a | x)}{P(\theta = \theta_b | x)} = \overbrace{\frac{P(x / \theta = \theta_a)}{P(x / \theta = \theta_b)}}^{\text{likelihood ratio}} \overbrace{\frac{P(\theta = \theta_a)}{P(\theta = \theta_b)}}^{\text{prior odds}}$$

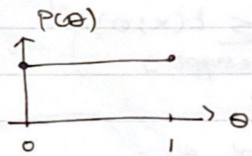
$\Rightarrow$ e.g. $1:1 \xrightarrow{x} 5:1$

$$\text{Odds}(\theta_a, \theta_b) \xrightarrow{x} \text{Odds}(\theta_a, \theta_b | x)$$

f: Binomial, fixed n

Let $\phi(\theta)$ be odds θ

$$\phi(\theta) = \frac{\theta}{1 - \theta} \qquad P(\theta) = U(0, 1)$$



What is prior of indifference of ϕ?

$P(\phi) \overset{?}{=} U(0, \infty) = 0 \neq$ not a valid PDF. $\Rightarrow \int_0^\infty \text{odd} \neq 1.$

If $P(\theta) = U(0, 1)$

where is $P(\phi) = ?$ $\Rightarrow$ use transformation.

For a continuous r.v. X,

If $Y = t(X)$ where t is invertible and $f_X(x)$ known

$\rightarrow X = t^{-1}(Y)$

$$\Rightarrow f_Y(y) = f_X(t^{-1}(y)) \left| \frac{d}{dy}[t^{-1}(y)] \right|$$

"change of variable formula"

$$\phi = \phi(\theta) = \frac{\theta}{1 - \theta} = t(\theta)$$

$\phi(1 - \phi) = \theta$

$\phi - \theta\phi = \theta$

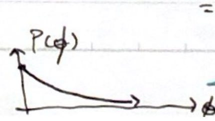
$\phi = \theta + \theta\phi$

$\phi = \theta(1 + \phi)$

$\Rightarrow \theta = \frac{\phi}{1 + \phi} = t^{-1}(\phi)$

$$\frac{d}{d\phi}[t^{-1}(\phi)] = \frac{\overbrace{(1 + \phi)(1) - \phi(1)}^{1}}{(1 + \phi)^2} = \frac{1}{(1 + \phi)^2}$$

$$\Rightarrow f_\phi(\phi) = \overbrace{f_\theta\left(\frac{\phi}{1 + \phi}\right)}^{1 \text{ due to indiff.}} \left| \frac{1}{(1 + \phi)^2} \right| = \frac{1}{(1 + \phi)^2}$$



after change of variable.

$$\begin{tikzcd}[column sep=huge] P(x|\theta) \arrow[r, "\text{Jeffrey's Protocal}"] \arrow[d, "t"] & P(\theta) \arrow[d, leftrightarrow, "\text{trasformation of var. formula works.}"] \\ P(x|\phi) \arrow[r, "\text{Jeffrey's Protocal}"] & P(\phi) \end{tikzcd}$$

$\phi = \phi(\theta) = \frac{\theta}{1-\theta} = t(\theta) \Rightarrow \theta = \frac{\phi}{1+\phi} = t^{-1}(\phi)$

$P(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$

$\Updownarrow$

$P(x|\phi) = \binom{n}{x}\left(\frac{\phi}{1+\phi}\right)^x\left(1-\frac{\phi}{1+\phi}\right)^{n-x}$

$= \binom{n}{x}\left(\frac{\phi^x}{(1+\phi)^n}\right)$

Let X be continuous with density $f(x;\theta)$

$f(x;\theta) \propto \underbrace{k(x;\theta)}_{\text{is unique}} \longrightarrow$ by def. $\exists\ c>0$ not a function of x

$f(x;\theta) = \underleftarrow{c}\, k(x;\theta)$ normalization constant.

$$1 = \int_{supp[x]} f(x,\theta)\,dx = \int_{supp[x]} c\,k(x;\theta)\,dx$$

$$\Rightarrow \left(\int_{supp[x]} k(x;\theta)\,dx\right)^{-1} = c$$

$p(x;\theta) \propto k(x;\theta)$ which means $\exists\ c>0$

$p(x;\theta) = c\,k(x;\theta)$

$$c = \left(\sum_{x\in supp[x]} k(x;\theta)\right)^{-1}$$

$$Y \sim Beta(\alpha,\beta) = \overbrace{\underbrace{\frac{1}{B(\alpha,\beta)}}_{c}\underbrace{y^{\alpha-1}(1-y)^{\beta-1}}_{k(y;\alpha,\beta)}}^{f_Y(y)}$$

$$\propto y^{\alpha-1}(1-y)^{\beta-1}$$

$\vdash$: Binomial, $P(\theta) = Beta(\alpha,\beta) \Rightarrow P(\theta|x) = Beta(x+\alpha,\ n-x+\beta)$

$$P(\theta|x) = \frac{P(x|\theta)P(\theta)}{P(x)} \propto P(x|\theta)P(\theta) = \left(\binom{n}{x}\theta^x(1-\theta)^{n-x}\right)\left(\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\right)$$

$$\propto \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}$$

$$\propto Beta(x+\alpha,\ n-x+\beta)$$

$$Y \sim N(\theta, \quad = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{1}{-2\sigma^2}(y-\theta)^2}$$

$$\propto e^{-\frac{1}{2\sigma^2}(y-\theta)^2} = e^{-\frac{1}{2\sigma^2}(y-2\theta y+\sigma^2)}$$

$$= e^{-\frac{y^2}{2\sigma^2}+\frac{\theta y}{\sigma^2}} e^{-\frac{\sigma^2}{2\sigma^2}}$$

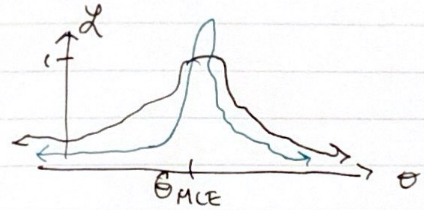
$$\propto e^{\frac{\theta y}{\sigma^2}-\frac{y^2}{2\sigma^2}} = k(y;\theta,\sigma^2)$$

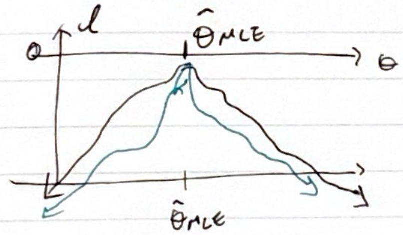
$$c = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{\theta^2}{2\sigma^2}}$$

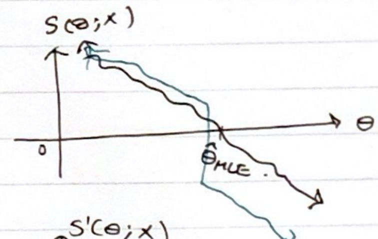
Remember $\overset{\text{likelihood}}{\mathcal{L}(\theta;x)} = P(x;\theta)$, $\ell(\theta;x) := \ln(\mathcal{L})$

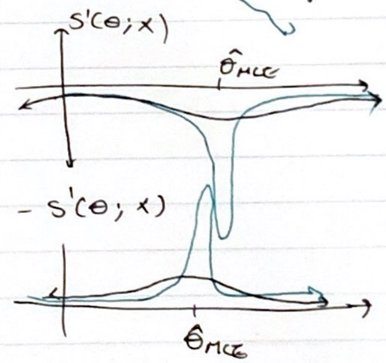
Score function $s(\theta;x) := \ell'(\theta;x)$

Fisher's Information $\rightarrow I(\theta) := Var_x[s(\theta;x)] = \ldots = -E_x[\underbrace{\ell''(\theta;x)}_{s'(\theta;x)}]$









$X \sim Bin(n, \theta)$

$P(x;\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$

$\ell(\theta;x) = \ln\left(\binom{n}{x}\right) + x\ln(\theta) + (n-x)\ln(1-\theta)$

$s(\theta;x) = \ell'(\theta,x) = \frac{x}{\theta} - \frac{n-x}{1-\theta}$

$\ell''(\theta;x) = -\frac{x}{\theta^2} - \frac{n-x}{(1-\theta)^2}$

$-\ell''(\theta;x) = \frac{x}{\theta^2} + \frac{n-x}{(1-\theta)^2}$

$I(\theta) = E_x\left[\frac{x}{\theta^2} + \frac{n-x}{(1-\theta)^2}\right]$

$= \frac{n\cancel{\theta}}{\theta^{\cancel{2}}} + \frac{\overbrace{n-n\theta}^{n(\cancel{1-\theta})}}{(1-\theta)^{\cancel{2}}}$

$= \frac{n}{\theta} + \frac{n}{1-\theta}$

$= n\left(\frac{1}{\theta(1-\theta)}\right) = I(\theta)$