4/21 (thu) [MATH341] Lecture 17

F: iid $N(\theta, \sigma^2)$ but both θ, σ^2 unknown.

$$P(\theta,\sigma^2|X) \propto P(X|\theta,\sigma^2)\,P(\theta,\sigma^2)$$
$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2} \cdot P(\theta,\sigma^2)$$
$$= \underbrace{(\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2}}_{\text{normal-inverse gamma kernel}} \; k(\theta,\sigma^2)$$

- $$\Sigma(x_i-\theta)^2 = \Sigma((x_i-\bar{x})+(\bar{x}-\theta))^2$$
$$= \Sigma(x_i-\bar{x})^2 + 2\Sigma(x_i-\bar{x})(\bar{x}-\theta) + \Sigma(\bar{x}-\theta)^2 \quad \text{constant}$$
$$= (n-1)s_x^2 + 2\Sigma(x_i\bar{x} - \bar{x}^2 - x_i\theta + \bar{x}\theta) + n(\bar{x}-\theta)^2$$
$$= (n-1)s^2 + n(\bar{x}-\theta)^2 + 2(\cancel{\bar{x}\,n\bar{x}} - \cancel{n\bar{x}^2} - \theta n\bar{x} + n\bar{x}\theta)$$
$$= (n-1)s^2 + n(\bar{x}-\theta)^2$$

$$= (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}((n-1)s^2 + n(\bar{x}-\theta)^2)} k(\theta,\sigma^2)$$
$$= \underbrace{(\sigma^2)^{-\underbrace{(\frac{n}{2}-1)}_{\alpha}-1} e^{-\frac{\overbrace{(n-1)s^2/2}^{\beta}}{\sigma^2}} e^{-\frac{\overbrace{n}^{\lambda}}{2\sigma^2}(\theta-\overbrace{\bar{x}}^{\mu})^2}}_{\propto \text{NormInvGamma}(\alpha,\beta,\lambda,\mu)} \; k(\theta,\sigma^2).$$

$$\Rightarrow \underbrace{P(\theta|X,\sigma^2)\,P(\sigma^2|X)}_{=\,p(\theta,\sigma^2|X)} \propto \text{NormInvGamma}(\alpha,\beta,\lambda,\mu)\; k(\theta,\sigma^2)$$

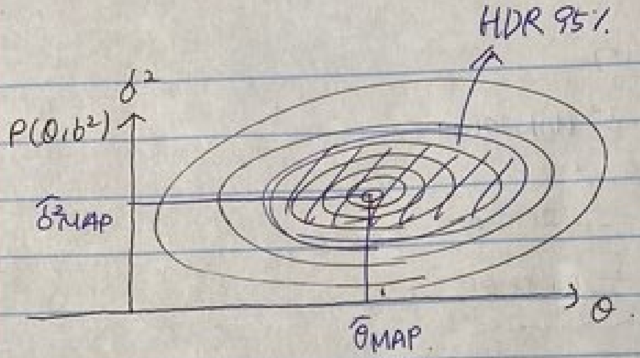
Conjugate prior $\propto k(\theta,\sigma^2) = (\sigma^2)^{-\alpha_0-1} e^{-\frac{\beta_0}{\sigma^2}} e^{-\frac{\lambda_0}{2\sigma^2}(\theta-\mu_0)^2}$ ← we won't study this, the general prior.

$= P(\theta,\sigma^2)$ $\quad \propto \text{NormInvGamma}(\alpha_0, \beta_0, \lambda_0, \mu_0)$

$$= k(\theta|\sigma^2)\,k(\sigma^2) \propto (\text{Normal})(\text{InvGamma})$$

$$P_J(\theta,\sigma^2) = \underbrace{P_J(\theta|\sigma^2)}_{\propto 1}\underbrace{P_J(\sigma^2)}_{\propto \frac{1}{\sigma^2}} \propto (1)\left(\frac{1}{\sigma^2}\right) = \frac{1}{\sigma^2}$$: only principled uninformative prior we will use.

[posterior] $P(\theta,\sigma^2|X) \propto P(X|\theta,\sigma^2)\,P_J(\theta,\sigma^2)$
$$\propto \left((\sigma^2)^{-n/2} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}\right)\left(\frac{1}{\sigma^2}\right)$$
$$= (\sigma^2)^{-n/2-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}$$
$$\propto \text{NormInvGamma}\left(\frac{n}{2}, \frac{(n-1)s^2}{2}, \frac{n}{2}, \bar{x}\right)$$



two dimentional.

we will not study

- $\hat{\theta}_{MAP}$ $\quad$ $\hat{\theta}_{MMSE}$ $\quad$ $\hat{\theta}_{MMAE}$ ...
- $\hat{\sigma}^2_{MAP}$ $\quad$ $\hat{\sigma}^2_{MMSE}$ $\quad$ $\hat{\sigma}^2_{MMSE}$

· we won't study confidence sets or hypothesis sets.