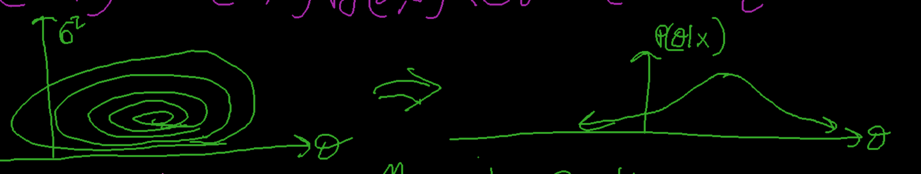
F: iid Normal and θ, σ^2 unknown

$$s^2 := \frac{1}{n-1}\sum(x_i-\bar{x})^2$$

$$P_J(\theta,\sigma^2) \propto \frac{1}{\sigma^2}$$

$$\propto \text{NormInvGamma}\left(\bar{x}, n, \frac{1}{2}, \frac{(n-1)s^2}{2}\right)$$

$$\Rightarrow P(\theta,\sigma^2|x) \propto P(x|\theta,\sigma^2)P_J(\theta,\sigma^2) \propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}$$



"marginizing out σ^2"   collapsing the σ^2 dimension

$$P(\theta|x) = \int_{\sigma^2} P(\theta,\sigma^2|x)\,d\sigma^2 \propto \int_0^\infty (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}\, d\sigma^2$$

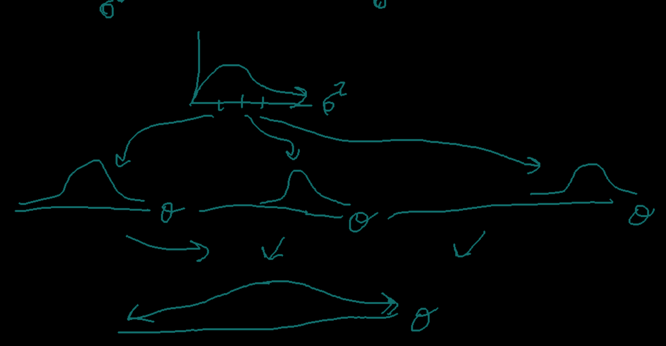
$$= \int_0^\infty (\sigma^2)^{-\overbrace{\frac{n}{2}}^{\alpha}-1} e^{-\frac{\overbrace{(n-1)s^2/2 + n(\theta-\bar{x})^2/2}^{\beta}}{\sigma^2}}\, d\sigma^2 = \frac{\Gamma(\alpha)}{\beta^\alpha}\overbrace{\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-\alpha-1} e^{-\frac{\beta}{\sigma^2}}\, d\sigma^2}^{1}$$

$$= \Gamma(\alpha)\beta^{-\alpha} = \Gamma\left(\frac{n}{2}\right)\left(\frac{(n-1)s^2 + n(\theta-\bar{x})^2}{2}\right)^{-n/2} \propto \left(\frac{2}{(n-1)s^2}\right)^{-n/2}\left(\frac{(n-1)s^2 + n(\theta-\bar{x})^2}{2}\right)^{-n/2}$$

$$= \left(1 + \frac{n(\theta-\bar{x})^2}{(n-1)s^2}\right)^{-n/2} = \left(1 + \frac{1}{n-1}\frac{(\theta-\bar{x})^2}{\left(\frac{s}{\sqrt{n}}\right)^2}\right)^{-\frac{(n-1)+1}{2}} \propto T_{n-1}\left(\bar{x}, \frac{s}{\sqrt{n}}\right)$$

$$\overset{n \text{ large}}{\approx} N\left(\bar{x}, \frac{s^2}{n}\right)$$

$$P(\theta|x) = \int_{\sigma^2} P(\theta,\sigma^2|x)\,d\sigma^2 = \int_{\sigma^2} \overbrace{P(\theta|x,\sigma^2)}^{\text{normal}}\overbrace{P(\sigma^2|x)}^{\text{Inv-gamma}}\, d\sigma^2$$, compound distr.



$$P(\sigma^2|x) = \int_\theta P(\theta,\sigma^2|x)\,d\theta \propto \int_{\mathbb{R}} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}\, d\theta$$

$$= (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \int_{\mathbb{R}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}\, d\theta = (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \sqrt{2\pi\frac{\sigma^2}{n}} \overbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\theta-\bar{x})^2}\, d\theta}^{1}$$

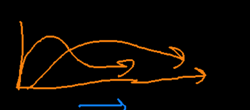
$$\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} (\sigma^2)^{\frac{1}{2}} = (\sigma^2)^{-\left(\frac{n-1}{2}\right)-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \propto \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

$$P(\sigma^2|x) = \int_\theta \overbrace{P(\sigma^2|x,\theta)}^{\text{InvGamma}}\overbrace{P(\theta|x)}^{T}\, d\theta = \text{InvGamma}$$

under Jeffreys priors...

$$P(\theta|x,\sigma^2) = N\left(\bar{x}, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) \quad \& \quad P(\theta|x) = T_{n-1}\left(\bar{x}, \frac{s}{\sqrt{n}}\right)$$

$$P(\sigma^2|x,\theta) = \text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right) \quad \& \quad P(\sigma^2|x) = \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$



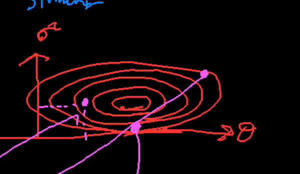
$$n\hat{\sigma}^2_{MLE} = \sum(x_i-\theta)^2$$

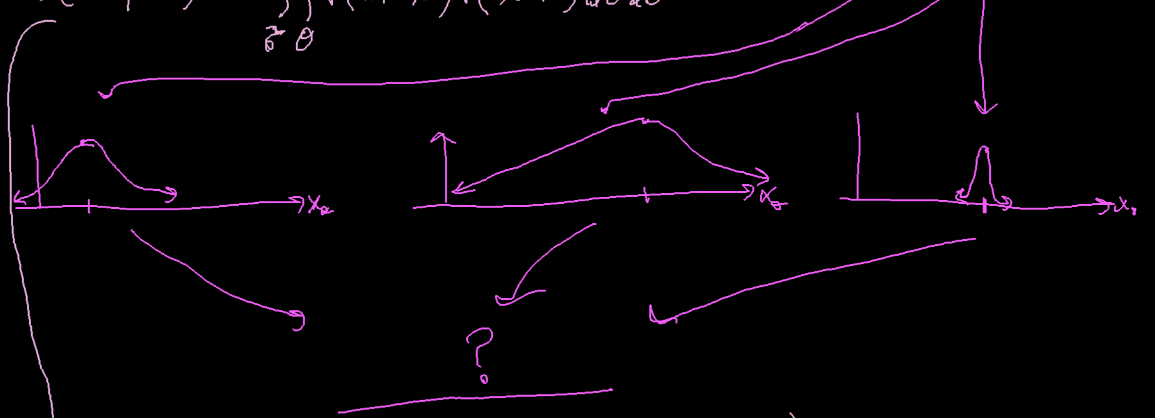
$$(n-1)s^2 = \sum(x_i-\bar{x})^2$$

similar

posterior predictive distr

$$P(x_*|x) = \iint_{\sigma^2\ \theta} \overbrace{P(x_*|\theta,\sigma^2)}^{\text{normal}}\overbrace{P(\theta,\sigma^2|x)}^{\text{norm-inv-gamma}}\, d\theta\, d\sigma^2$$





$$\propto \int_0^\infty \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_*-\theta)^2}\right)\left((\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n(\theta-\bar{x})^2}{2\sigma^2}}\right) d\theta\, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-\frac{1}{2}}(\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2}{2\sigma^2}} \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}\left((x_*-\theta)^2 + n(\theta-\bar{x})^2\right)}\, d\theta\, d\sigma^2$$

$$x_*^2 - 2x_*\theta + \underbrace{\theta^2 + n\theta^2}_{(n+1)\theta^2} - 2n\theta\bar{x} + n\bar{x}^2$$

$$e^{-\frac{x_*^2}{2\sigma^2}} e^{\frac{x_*\theta}{\sigma^2}} e^{-\frac{(n+1)\theta^2}{2\sigma^2}} e^{\frac{n\theta\bar{x}}{\sigma^2}} e^{-\frac{n\bar{x}^2}{2\sigma^2}}$$

$$= \int_0^\infty (\sigma^2)^{-\left(\frac{n+1}{2}\right)-1} e^{-\frac{((n-1)s^2 + x_*^2 + n\bar{x}^2)/2}{\sigma^2}} \int_{\mathbb{R}} e^{\overbrace{\frac{x_*+n\bar{x}}{\sigma^2}}^{a}\theta - \overbrace{\frac{n+1}{2\sigma^2}}^{b}\theta^2}\, d\theta \qquad \sqrt{\frac{2\sigma^2}{n+1}} \propto (\sigma^2)^{\frac{1}{2}}$$

$$-\frac{n+1}{2} - 1 + \frac{1}{2} = -\frac{n}{2} - \frac{1}{2} - 1 + \frac{1}{2}$$

$$\int_{\mathbb{R}} e^{a\theta - b\theta^2}\, d\theta = \sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}} \propto b^{-\frac{1}{2}} e^{a^2/4b}$$

$$\propto \int_0^\infty (\sigma^2)^{-\left(\frac{n+1}{2}\right)-1} e^{-\frac{((n-1)s^2 + x_*^2 + n\bar{x}^2)/2}{\sigma^2}} (\sigma^2)^{\frac{1}{2}} e^{\frac{(x_*+n\bar{x})^2}{(\sigma^2)^2}\frac{\sigma^2}{2(n+1)}}\, d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\left(\overbrace{\frac{n}{2}}^{\alpha}\right)-1} e^{-\frac{\overbrace{\left((n-1)s^2 + x_*^2 + n\bar{x}^2 - (x_*^2+n\bar{x})^2/(n+1)\right)/2}^{\beta}}{\sigma^2}}\, d\sigma^2$$

$$= \frac{\Gamma(\alpha)}{\beta^\alpha}\overbrace{\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)}(\sigma^2)^{-\alpha-1} e^{-\beta/\sigma^2}\, d\sigma^2}^{1} = \Gamma(\alpha)\beta^{-\alpha} = \Gamma\left(\frac{n}{2}\right)\beta^{-\frac{n}{2}} \propto \beta^{-\frac{n}{2}}$$

$$a = \frac{1}{2} - \frac{1}{2n+2} = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) = \frac{1}{2}\frac{n}{n+1}$$

$$b = -\frac{n\bar{x}}{n+1}$$

$$c = \frac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)$$

$$= \left(\frac{(n-1)s^2}{2} + \frac{x_*^2}{2} + \frac{n\bar{x}^2}{2} + \frac{-x_*^2 - 2n\bar{x}x_* - n^2\bar{x}^2}{2(n+1)}\right)^{-n/2}$$

$$= (ax_*^2 + bx_* + c)^{-n/2} \propto \left(\frac{1}{a}\right)^{-n/2}(ax_*^2 + bx_* + c)^{-n/2} = \left(x_*^2 + \frac{b}{a}x_* + \frac{c}{a}\right)^{-n/2}$$

$$= \left(\left(x_* + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)^{-n/2} \propto \left(\frac{1}{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^{-n/2}\left(\left(x_* + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right)^{-n/2}$$

$$= \left(1 + \frac{\left(x_* + \frac{b}{2a}\right)^2}{\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)}\right)^{-n/2} = \left(1 + \frac{1}{n-1}\frac{\left(x_* + \frac{b}{2a}\right)^2}{\left(\frac{\frac{c}{a} - \frac{b^2}{4a^2}}{n-1}\right)}\right)^{-\frac{(n-1)+1}{2}}$$

$$P(x_*|x,\sigma^2) = N\left(\hat{\theta}_p, \sigma^2 + \frac{\sigma^2}{n+n_0}\right)$$

$$\propto T_{n-1}\left(-\frac{b}{2a}, \sqrt{\frac{\frac{c}{a} - \frac{b^2}{4a^2}}{n-1}}\right) = T_{n-1}\left(\bar{x}, \sqrt{\frac{n+1}{n}}\, s\right) \overset{n \text{ large}}{\approx} N(\bar{x}, s^2)$$

$$\sqrt{\frac{n+1}{n}}\, s = \sqrt{s^2 + \frac{s^2}{n}}$$

exactly what you expect

$$-\frac{b}{2a} = \frac{\frac{n\bar{x}}{n+1}}{2\left(\frac{1}{2}\frac{n}{n+1}\right)} = \bar{x}$$

$$\frac{c}{a} = \frac{\frac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)}{\frac{1}{2}\frac{n}{n+1}} \cdot \frac{\frac{n+1}{n}}{\frac{n+1}{n}} = \frac{(n-1)(n+1)}{n}s^2 + \underbrace{(n+1)\bar{x}^2}_{n\bar{x}^2 + \bar{x}^2} - n\bar{x}^2 = \frac{(n-1)(n+1)}{n}s^2 + \bar{x}^2$$

$$\frac{b^2}{4a^2} = \left(-\frac{b}{2a}\right)^2 = \bar{x}^2 \qquad \frac{c}{a} - \frac{b^2}{4a^2} = \frac{(n-1)(n+1)}{n}s^2 + \bar{x}^2 - \bar{x}^2 \qquad \sqrt{\frac{\frac{c}{a} - \frac{b^2}{4a^2}}{n-1}} = \sqrt{\frac{n+1}{n}}\, s$$