$\mathcal{F}$: iid Poisson(θ) $\quad X_1,\dots,X_n;\theta \sim \text{Poisson}(\theta)$, $\theta \sim \text{Gamma}(\alpha,\beta)$

$p(X_*|X) = \text{ExtNegBin}(r,p)$ where $r = \underbrace{\sum x_i}+\alpha$, $p = \frac{n+\beta}{n+\beta+1}$

$= \text{NegBin}(r,p) = \binom{x_*+r-1}{r-1}(1-p)^{x_*}p^r$

if $\alpha \in \mathbb{N}_0$
$$\left(\begin{array}{l} X_1,\dots,X_r \overset{iid}{\sim} \text{Geom}(p) \Rightarrow \sum_{i=1}^{r} X_i \sim \text{NegBin}(r,p) \\ E[X_i] = \frac{1-p}{p} \neq \frac{p}{1-p} \Rightarrow E[\sum X_i] = r\frac{1-p}{p} \end{array} \right)$$

$$E[X_*|X] = r\frac{1-p}{p} = (\sum x_i+\alpha)\frac{\frac{1}{\cancel{n+\beta+1}}}{\frac{n+\beta}{\cancel{n+\beta+1}}} = \frac{\sum x_i+\alpha}{n+\beta} = E[\theta|X]$$

$\mathcal{F}$: iid $N(\theta,\sigma^2)$ with σ^2 known

$$p(\theta|X,\sigma^2) \propto e^{a\theta - b\theta^2} \propto N\left(\frac{a}{2b},\frac{1}{2b}\right) \leftarrow a = \frac{n\bar{x}}{\sigma^2},\ b = \frac{n}{2\sigma^2}$$
$$= N\left(\bar{x},\frac{\sigma^2}{n}\right)$$

Under Laplace prior i.e. $p(\theta) \propto 1$

$$p(\theta|X,\sigma^2) \propto p(X|\theta,\sigma^2) \propto N\left(\bar{x},\frac{\sigma^2}{n}\right)$$

$p(\theta|\sigma^2) \propto k(\theta|\sigma^2)$ $\qquad$ conjugate prior

$$p(\theta|X,\sigma^2) = \frac{p(X|\theta,\sigma^2)\,p(\theta|\sigma^2)}{p(X|\sigma^2)} \propto p(X|\theta,\sigma^2)\,p(\theta|\sigma^2) \propto k(X|\theta,\sigma^2)\,k(\theta|\sigma^2)$$

if $k(\theta|\sigma^2) = e^{\alpha\theta-\beta\theta^2} \propto N\left(\frac{\alpha}{2\beta},\frac{1}{2\beta}\right)$

$$= e^{a\theta-b\theta^2}\,k(\theta|\sigma^2) = e^{a\theta-b\theta^2}\,e^{\alpha\theta-\beta\theta^2} = e^{(a+\alpha)\theta-(b+\beta)\theta^2}$$
$$\propto N\left(\frac{a+\alpha}{2(b+\beta)},\frac{1}{2(b+\beta)}\right)$$

$$\frac{a+\alpha}{2(b+\beta)} = \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}, \qquad \frac{1}{2(b+\beta)} = \frac{1}{2\left(\frac{n}{2\sigma^2}+\frac{1}{2\tau^2}\right)} = \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}$$

$a = \frac{n\bar{x}}{\sigma^2}$, $b = \frac{1}{2\sigma^2}$, $\alpha = \frac{\mu_0}{\tau^2}$, $\beta = \frac{1}{2\tau^2}$

$$N\left(\frac{\alpha}{2\beta},\frac{1}{2\beta}\right) = N(\mu_0,\tau^2)$$

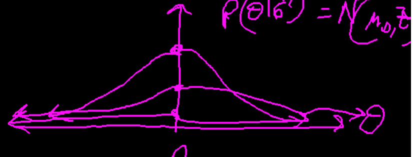
$\frac{\alpha}{2\beta} = \mu_0 \quad \mu_0 = \Rightarrow \frac{\alpha}{\cancel{2}\frac{1}{\cancel{2}\tau^2}} = \alpha\tau^2 \Rightarrow \alpha = \frac{\mu_0}{\tau^2}$

$\mathcal{F}$: iid $N(\theta,\sigma^2)$ σ^2 known, $p(\theta) = N(\mu_0,\tau^2)$ $\qquad$ under Laplace

$$\Rightarrow p(\theta|X) = N\left(\frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}},\ \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\right) = N\left(\bar{x},\frac{\sigma^2}{n}\right) \Rightarrow \mu_0,\tau^2$$

$p(\theta) \propto 1 \propto N(0,\infty)$ Laplace

$p(\theta|\sigma^2) = N(\mu_0,\tau^2)$



if $\tau^2 \to \infty \Rightarrow N(\mu_0,\tau^2) \propto 1$

$$\hat{\theta}_{MMSE} = E[\theta|X] = \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}$$

$$\hat{\theta}_{MAE} = \text{Med}[\theta|X] = \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}},$$

$$\hat{\theta}_{MAP} = \text{Mode}[\theta|X] = \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}$$

$$CR_{\theta,1-\alpha} = \left[\text{qnorm}\left(\frac{\alpha}{2},\ \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}},\ \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\right),\ \text{qnorm}\left(1-\frac{\alpha}{2},\ \cdot,\ \cdot\right)\right]$$

$H_0: \theta \le \theta_0$, $H_a: \theta > \theta_0$

$$\text{pval} = P(H_0|X) = \text{pnorm}(\theta_0,\cdot,\cdot) = \int_{-\infty}^{\theta_0} \text{PPF}\, d\theta$$

$p_J(\theta|\sigma^2) \propto \sqrt{I(\theta;\sigma^2)} = \sqrt{n/\sigma^2} \propto 1 \propto N(0,\infty) \Rightarrow$ Jeffrey's prior is the same as Laplace's prior

$$\mathcal{L}(\theta;X,\sigma^2) = (2\pi\sigma^2)^{-n/2}\,e^{-\frac{\sum x_i^2}{2\sigma^2}+\frac{n\bar{x}\theta}{\sigma^2}-\frac{n\theta^2}{2\sigma^2}}$$

$$\ell(\theta;X,\sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{\sum x_i^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}$$

$$\ell'(\theta;X,\sigma^2) = \frac{n\bar{x}}{\sigma^2} - \frac{n\theta}{\sigma^2} \Rightarrow -\ell''(\theta;X,\sigma^2) = \frac{n}{\sigma^2}$$

$$I(\theta;\sigma^2) = E_X[-\ell''(\theta;X,\sigma^2)] = E_X\left[\frac{n}{\sigma^2}\right] = \frac{n}{\sigma^2}$$

Haldane: pure ignorance

$p(\theta|\sigma^2) = N(\mu_0,\tau^2) = N\left(\mu_0,\frac{\sigma^2}{n_0}\right)$

let n_0 = # of pseudo~~ob~~observations

since σ^2 known

let $\tau^2 = \frac{\sigma_0^2}{n_0} = \frac{\sigma^2}{n_0}$

average of pseudodata

$$\hat{\theta} = \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} = \frac{\frac{n\bar{x}}{\sigma^2}+\frac{n_0\mu_0}{\sigma_0^2}}{\frac{n}{\sigma^2}+\frac{n_0}{\sigma_0^2}} = \frac{\frac{n\bar{x}+n_0\mu_0}{\sigma^2}}{\frac{n+n_0}{\sigma^2}} = \frac{n\bar{x}+n_0\mu_0}{n+n_0} = \frac{\sum x_i+\sum y_i}{n+n_0}$$

Haldane: $n_0 = 0 \Rightarrow \tau^2 = \infty$, $\mu_0 = 0$

$p(\theta|\sigma^2) = N(0,\infty)$ same as Laplace

let $y_1,y_2,\dots,y_{n_0}$ be the pseudodata

$\mu_0 = \bar{y} = \frac{1}{n_0}\sum y_i$

$n_* = 1$

$N(\theta,\sigma^2)$ $\quad N(\theta_p,\sigma_p^2)$

$$p(X_*|X,\sigma^2) = \int_{\Theta} p(X_*|\theta,\sigma^2)\,p(\theta|X,\sigma^2)\,d\theta = \int_{\mathbb{R}} \text{PDF}\ \text{PDF}\ d\theta$$

$$\theta_p = \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}},\quad \sigma_p^2 = \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}$$

$$= \sigma_p^2\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}\right)$$

$$\hat{\theta}_{MMSE} = \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} = \frac{\frac{n\bar{x}}{\sigma^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} + \frac{\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} = \frac{\frac{n}{\sigma^2}\hat{\theta}_{MLE}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} + \frac{\frac{1}{\tau^2}E[\theta]}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}$$

$$= \underbrace{\frac{1}{1+\frac{\sigma^2}{n\tau^2}}}_{1-\rho}\hat{\theta}_{MLE} + \underbrace{\frac{1}{1+\frac{n\tau^2}{\sigma^2}}}_{\rho}E[\theta]$$