Lecture - 13 03/24/2020

$F \overset{iid}{\sim} \text{poisson}(\theta),\ \theta \sim \text{Gamma}(\alpha, \beta)$

(cont.) $\Rightarrow \theta | x \sim \text{Gamma}(\Sigma x_i + \alpha, n+\beta)$

$$n_* = 1\ ;\ p(x_*|x) = \int_{\Theta} p(x_*|\theta)\, p(\theta|x)\, d\theta$$

$$= \int_0^{\infty} \left(\frac{e^{-\theta}\theta^{x_*}}{x!}\right)\left(\frac{(n+\beta)^{\Sigma x_i+\alpha}}{\Gamma(\Sigma x_i+\alpha)}\theta^{\Sigma x_i+\alpha-1} e^{-(n+\beta)\theta}\right) d\theta$$

$$= \frac{(n+\beta)^{\Sigma x_i+\alpha}}{x_*!\,\Gamma(\Sigma x_i+\alpha)} \int_0^{\infty} e^{-\overset{t}{(n+\beta+1)\theta}}\, \theta^{\Sigma x_i + x_* + \alpha - 1}\, d\theta.$$

let $t = (n+\beta+1)\theta$

$\Rightarrow \frac{dt}{d\theta} = n+\beta+1\ ;\ \theta = \frac{t}{n+\beta+1}$, $d\theta = \frac{dt}{n+\beta+1}$

$$= \frac{(n+\beta)^{\Sigma x_i+\alpha}}{x_*!\,\Gamma(\Sigma x_i+\alpha)} \int_0^{\infty} \left(\frac{t}{(n+\beta+1)}\right)^{\Sigma x_i + x_* + \alpha - 1} e^{-t}\, \frac{dt}{n+\beta+1}$$

$$= \frac{(n+\beta)^{\Sigma x_i+\alpha}}{x_*!\,\Gamma(\Sigma x_i+\alpha)} \cdot \frac{1}{(n+\beta+1)^{\Sigma x_i + x_* + \alpha - 1}} \cdot \frac{1}{(n+\beta+1)} \cdot \int_0^{\infty} \underbrace{t^{\Sigma x_i + x_* + \alpha - 1} e^{-t}\, dt}_{\Gamma(\Sigma x_i + x_* + \alpha)}$$

$$= \frac{(n+\beta)^{\Sigma x_i+\alpha}}{x_*!\,\Gamma(\Sigma x_i+\alpha)} \cdot \frac{\Gamma(\Sigma x_i + x_* + \alpha)}{\underbrace{(n+\beta+1)^{\Sigma x_i + x_* + \alpha}}_{(n+\beta+1)^{\Sigma x_i+\alpha}\,(n+\beta+1)^{x_*}}} = \left(\frac{n+\beta}{n+\beta+1}\right)^{\Sigma x_i+\alpha} \left(\frac{1}{n+\beta+1}\right)^{x_*} \frac{\Gamma(\Sigma x_i + x_* + \alpha)}{x_*!\,\Gamma(\Sigma x_i+\alpha)}$$

let $p := \frac{n+\beta}{n+\beta+1} \in (0,1) \Rightarrow 1-p = \frac{1}{n+\beta+1} \in (0,1)$

let $r := \Sigma x_i + \alpha$

$$= \frac{\Gamma(x_*+r)}{x_*!\,\Gamma(r)} p^r (1-p)^{x_*} \doteq \text{ExtNegBin}(r, p)$$

extended negative binomial model.

If $r = \sum x_i + \alpha \in \mathbb{N}_0 \implies \alpha \in \mathbb{N}$

$\Downarrow$

$= \underbrace{\binom{x_* + r - 1}{x_*}} p^r (1-p)^{x_*} = \text{NegBin}(r, p)$

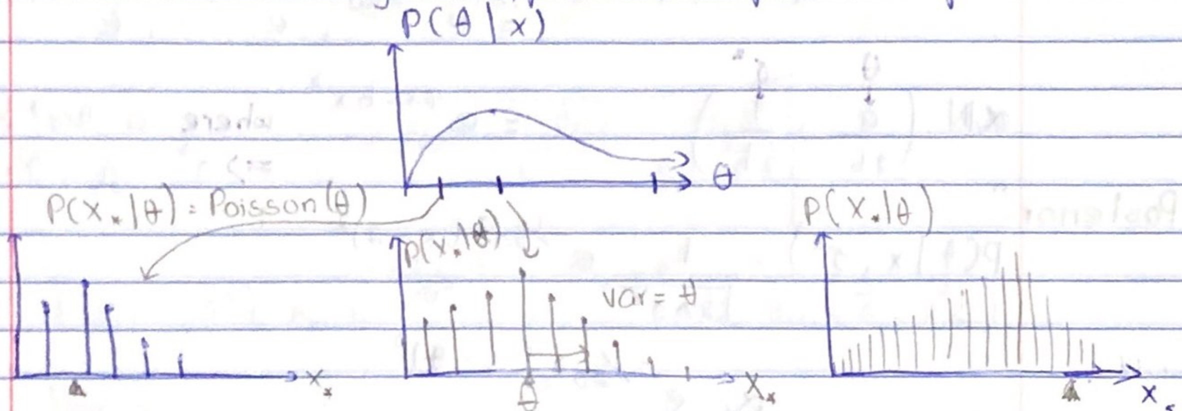
Recall $\Gamma(x) = (x-1)!$ if $x \in \mathbb{N}$

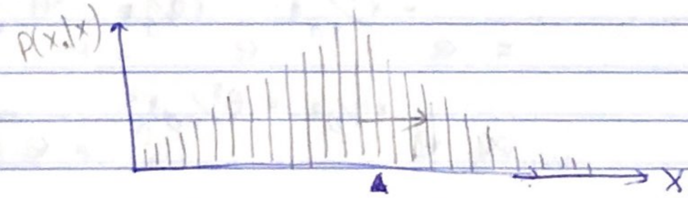
$x_1, \dots, x_r ; p \overset{iid}{\sim} \text{Geom}(p) := (1-p)^x p$ $\{ E[x_i] = \frac{p}{1-p}$

$\sum_{i=1}^{r} x_i \sim \text{NegBin}(r, p) \implies E[\sum x_i] = r \frac{p}{1-p}$ It's mistake; will fix next class.

(Negative Binomial)

$x_* | x \sim \text{ExtNegBin}(r, p)$ Overdisperssed poisson





$E[x_* | x] = r \frac{p}{1-p} = \mu$

$\text{Var}[x_* | x] = \frac{pr}{(1-p)^2} = \frac{1}{1-p} \mu$

$\frac{1}{1-p} \in (1, \alpha)$

Normal model $x \sim N(\theta, \sigma^2) = N(\theta, \theta_2)$

$\dim[\vec{\theta}] = 2$

Pretend we know σ^2; we want inferance for the mean θ.

$x \sim$ kernal practice

$$N(\theta, \sigma^2) \Rightarrow p(x|\theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} \propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$$

$$= e^{-x^2/2\sigma^2 + x\theta/\sigma^2 - \theta^2/2\sigma^2}$$

$$= e^{-x^2/2\sigma^2} \cdot e^{x\theta/\sigma^2} \cdot e^{-\theta^2/2\sigma^2}$$

$$\propto N\left(\frac{\overset{\theta}{\downarrow}}{a \atop 2b}, \frac{\overset{\sigma^2}{\downarrow}}{1 \atop 2b}\right) \qquad = e^{ax - bx^2}$$

where $a = \theta/\sigma^2$ & $b = 1/2\sigma > 0$
$\Rightarrow \sigma^2 = 1/2b$, $\theta = a/2b$

"Posterior"

$$p(\theta|x, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-1/2\sigma^2 (x-\theta)^2}$$

$$\propto e^{-1/2\sigma^2 (x-\theta)^2}$$

$$= e^{-x^2/2\sigma^2} e^{x\theta/\sigma^2} e^{-\theta^2/2\sigma^2}$$

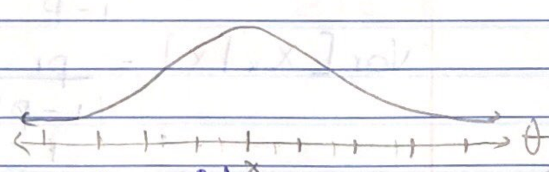
$$\propto e^{x\theta/\sigma^2 - \theta^2/2\sigma^2} = e^{a\theta - b\theta^2}$$

where $a = x/\sigma^2$, $b = \frac{1}{2}\sigma^2$.

Normal distribution is self conjugate

$$\propto N(a/2b, 1/2b) = N(x, \sigma^2)$$

$$\Rightarrow a/2b = \frac{x/\sigma^2}{2/2\sigma^2} = x$$



$$\Rightarrow p(\theta|x, \sigma^2) = \theta \sim N(x, \sigma^2)$$

$F: X_1, \ldots, X_n ; \theta, \sigma^2 \overset{iid}{\sim} \mathbb{N}(\theta, \sigma^2)$

$$\boxed{P(x|\theta,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2}$$

$= \sum x_i^2 - 2\theta\sum x_i + \sum\theta^2$

$= \sum x_i^2 - 2\theta\sum x_i + n\theta^2$

$= \sum x_i^2 - 2\theta n\bar{x} + n\theta^2$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}(\sum x_i^2 - 2\theta n\bar{x} + n\theta^2)}$$

$$\boxed{= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum x_i^2}{2\sigma^2}} \cdot e^{\frac{\theta n\bar{x}}{\sigma^2}} \cdot e^{\frac{-n\theta^2}{2\sigma^2}}}$$

$$P(\theta|x,\sigma^2) \propto e^{\overset{a}{n\bar{x}/\sigma^2}\,\theta} \cdot e^{\overset{b}{-n/2\sigma^2}\,\theta^2} \propto \mathbb{N}(\bar{x}, \sigma^2/n)$$

$$Var = \frac{1}{2b} = \frac{1}{2(n/2\sigma^2)} = \frac{\sigma^2}{n}$$

$$mean = \frac{a}{2b} = \frac{\frac{n\bar{x}}{\sigma^2}}{2\left(\frac{n}{2\sigma^2}\right)} = \bar{x}$$

Laplace uninformative prior

$P(\theta) = U(\Theta)$ if $\Theta = (0,1)$

$P(\theta) \propto 1 \implies P(\theta) = 1$

$\Theta = (0,100)$

Poisson - Gamma

Amazing trick

$P(\theta) = 1/100 \propto 1$

$$P(\theta|x) = \frac{P(x|\theta)\,P(\theta)}{P(x)} \propto P(x|\theta)\,\overset{\propto 1}{P(\theta)} \propto P(x|\theta)$$

$$= \frac{e^{-n\theta}\,\theta^{\Sigma x_i}}{\prod x_i!} \propto e^{-n\theta}\,\theta^{\Sigma x_i + 1 - 1}$$

$\alpha = 1$

$\propto \text{Gamma}(\Sigma x_i + 1, n)$

$\Rightarrow \theta \sim \text{Gamma}(1, 0)$

$\beta = 0$

Generally $P(\theta) = \text{Gamma}(\alpha, \beta)$

$\Longrightarrow P(\theta \mid x) = \text{Gamma}(\Sigma x_i + \alpha,\ n + \beta)$

F: $x_1, \ldots, x_n ; \theta, \sigma^2 \overset{\text{iid}}{\sim} N(\theta, \sigma^2)$, σ^2 known

prior on θ is Laplace $\Rightarrow p(\theta) \propto 1$, improper!

$p(\theta \mid x, \sigma^2) \propto P(x \mid \theta, \sigma^2) \propto N(\bar{x}, \sigma^2/n)$