Lecture - 08:

$H_0: \theta = \theta_0$

$H_a: \theta \neq \theta_0 = 0.5$ two sided test

if $P_{val} := P(H_0|X) < \alpha \rightarrow$ Reject H_0 / Accept H_a.

$= P(\theta = \theta_0 | X) = 0$ = Problem

$P(\theta) = U(0,1)$.

Two ideas:

① You declare δ e.g $\delta = 0.01$, a "margin of equavalence" then you modify the hypothesis;

$H_0: \theta \in [\theta_0 \pm \delta] \overset{e.g}{=} [.49, .51]$

$H_a: \theta \notin [\theta_0 \pm \delta]$.

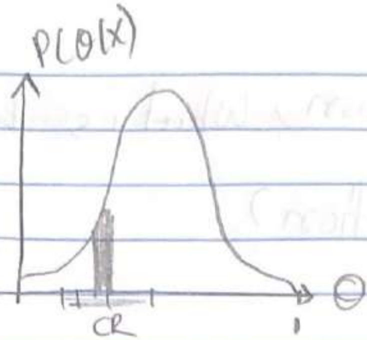
$P_{val} = P(H_0|X) = P(\theta \in [\theta_0 \pm \delta] | X)$.

$n = 100$
$X = 61$

$= qbeta(0.51, 62, 40) - qbeta(.49, 62, 40)$

$=$ (small) $0.609 - 0.607$

$= 0.002 < \alpha = 5\% \Rightarrow$ Reject H_0.



② if $\theta \in CR_{\theta, 1-\alpha} \Rightarrow$ Retain H_0 else Reject

Downside: no Pvalue!

Modeling → Explanation (inference)

Modeling → Prediction ⇒ seen $x_1, \dots, x_n$ and you want to know how future x_* (future data) will be distributed.

$P(X_* | X)$ ~~distributed~~

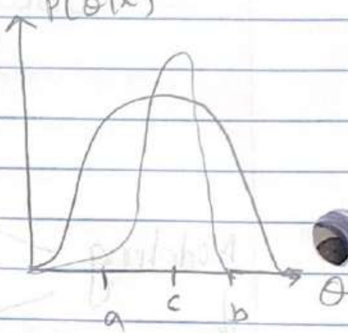
↳ Posterior predictive distribution

if Θ is known, What is the posterior predictive distribution?

$P(x_* | \theta)$: the best you can do

but it is not possible. Since θ is unknown.

$$P(X^* | X) = P(X^* | \Theta = a) P(\theta = a) + P(X^* | \theta = b) P(\theta = b)$$



Continuous posterior weighted ave:-

$$P(X^* | X) = \int_{\Theta} P(x^*, \theta | X) \, d\theta$$

if Θ is known, x doesn't add any information

$$= \int_{\Theta} P(X^* | \theta, x) P(\theta | X) \, d\theta$$

$$= \int_{\Theta} \underbrace{P(X^* | \theta)}_{\text{Likelihood}} \underbrace{P(\theta | X)}_{\text{Posterior}} \, d\theta$$

discrete posterior weighted ave:

$$P(x^* | X) = \sum_{\theta \in \Theta} P(x^* | \theta) P(\theta | X)$$

Example:

$$\Theta_0 = \{0.5, 0.75\}, \quad X = \langle 0, 1, 1 \rangle$$

$x^* \sim ?$ one future observation

$$P(x^* | X) = \text{Bern}(?)$$

$$\text{Sup}[x^* | X] = [0, 1]$$

Consider rich's theory

$$x^* | X \sim \text{Bern}(\hat{\theta}_{MLE} = 2/3)$$

Problems:

1. $\hat{\theta}_{MLE}$ may not be $\in \Theta_0$
2. $\hat{\theta}_{MLE}$ could be 0 or 1.
3. if multiple future observation

$X_* \sim \text{Bin}(n_*, \hat{\theta}_{MLE})$ (BAD Idea.)

Assume prior of indifference posterior

$P(\theta = .75 | X) = 0.53$, $P(\theta = 0.5 | X) = .47$

$$P(X^* | X) = P(X^* | \theta = .75) P(\theta = 0.75 | X) + P(X^* | \theta = 0.5) P(\theta = 0.5 | X)$$

$$= P(X^* | \theta = 0.75)(\times 0.53) + P(X^* | \theta = 0.5)(0.47)$$

$$= (.75)^{X^*} (1 - .75)^{X^*} (.53) + (0.5)^{X^*} (0.5)^{1-X_*} (0.47)$$

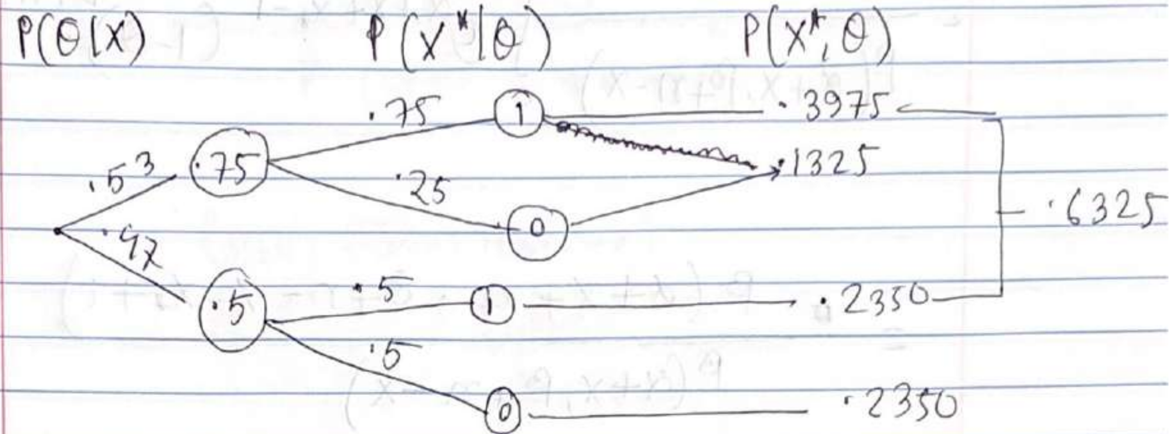
$P(X^* | X) = \text{Bern}(?) = \text{Bern}(.6325)$

trick, Compute $P(X^* = 1 | X) = (.75)^1 (.25)^0 (.53)$

$= (0.5)^0 (0.5)^1 (0.47)$

$= 0.6325.$

Posterior after see the data



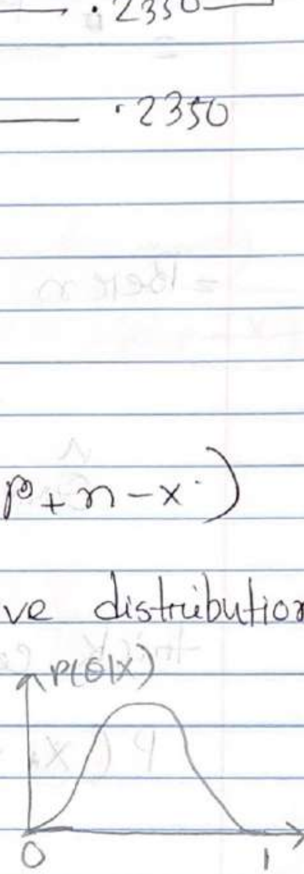
F: Binomial Fixed n,

$P(\Theta) = Beta(\alpha, \beta)$

$\Rightarrow P(\Theta|X) = Beta(\alpha + x, \beta + n - x)$

What is the posterior predictive distribution

for $n^* = 1$ (sure na)



$$P(X_* | X) = \int_{\Theta} P(X_* | \Theta)\, P(\Theta | X)\, d\Theta$$

$$= \int_0^1 \left(\Theta^{X_*} (1-\Theta)^{1-X_*}\right) \cdot \left(\frac{1}{B(\alpha+x, \beta+n-x)} \Theta^{\alpha+x-1} (1-\Theta)^{\beta+n-x-1}\right) d\Theta$$

$$= \frac{1}{\beta(\alpha+x, \beta+n-x)} \int_0^1 \theta^{\alpha+x+x_*-1} (1-\theta)^{\beta+n-x-x_*+1-1} d\theta$$

$$= \frac{\beta(\alpha+x+x_*, \beta+n-x-x_*+1)}{\beta(\alpha+x, \beta+n-x)}$$

$$= \text{Bern}(\quad) = \text{Bern}\left(\frac{\alpha+x}{\alpha+\beta+n}\right)$$

$$\hat{\theta}_{MMSE} = E[\theta | x]$$

trick compute

$$P(x_* = 1 | x) = \frac{\beta(\alpha+x+1, \beta+n-x-1+1)}{\beta(\alpha+x, \beta+n-x)}$$

$$= \frac{\beta(\alpha+x-1, \beta+n-x)}{\beta(\alpha+x, \beta+n-x)}$$

$$= \frac{\dfrac{\Gamma(\alpha+x+1)\,\Gamma(\beta+n-x)}{\Gamma(\alpha+\beta+n+1)}}{B(\alpha+x,\ \beta+n-x)}.$$

$$= \frac{\dfrac{(\alpha+x)\,\Gamma(\alpha+x)\,\Gamma(\beta+n-x)}{(\alpha+\beta+n)\,\Gamma(\alpha+\beta+n)}}{B(\alpha+x,\ \beta+n-x)}$$

$$= \frac{\alpha+x}{\alpha+\beta+n} \cdot \frac{\cancel{B(\alpha+x,\ \beta+n-x)}}{\cancel{B(\alpha+x,\ \beta+n-x)}}$$

$$= \frac{(\alpha+x)}{\alpha+\beta+n}.$$

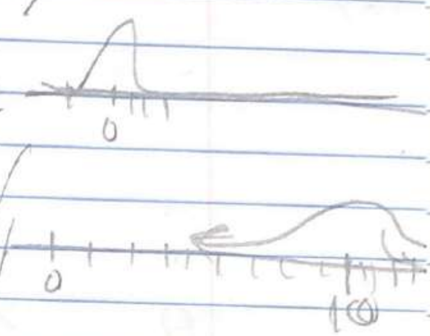
Midterm 2 :o

Component distri

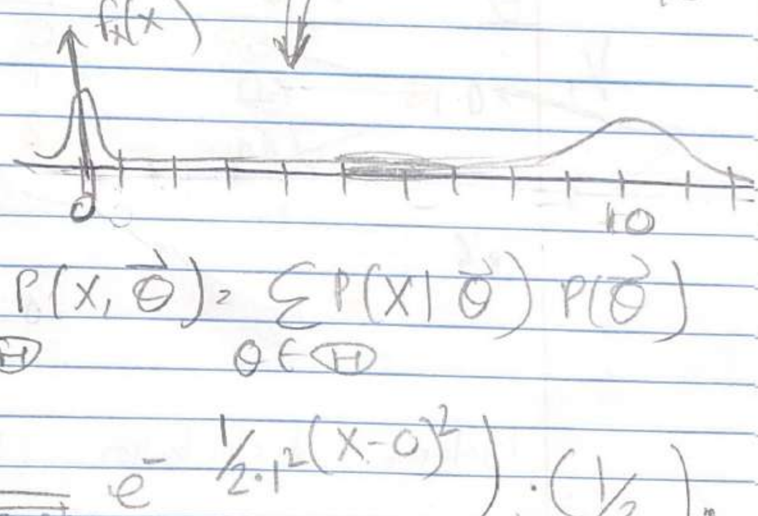
mixing proportion

Mixture distribution :-)

$$X \sim \begin{cases} N(0,1^2) & \text{wp } 1/2 \\ N(10,2^2) & \text{wp } 1/2 \end{cases}$$



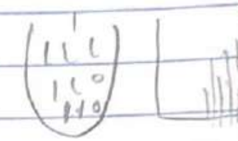
$f_X(x) = ?$



$$P(X) = \sum_{\theta \in \Theta} P(X, \vec{\theta}) = \sum_{\theta \in \Theta} P(X \mid \vec{\theta})\, P(\vec{\theta})$$

$$= \left(\frac{1}{\sqrt{2\pi(1^2)}}\, e^{-\frac{1}{2\cdot 1^2}(x-0)^2}\right)\cdot\left(\frac{1}{2}\right) + \left(\frac{1}{\sqrt{2\pi(2^2)}}\, e^{-\frac{1}{2\cdot 2^2}(x-10)^2}\right)\cdot\frac{1}{2}$$

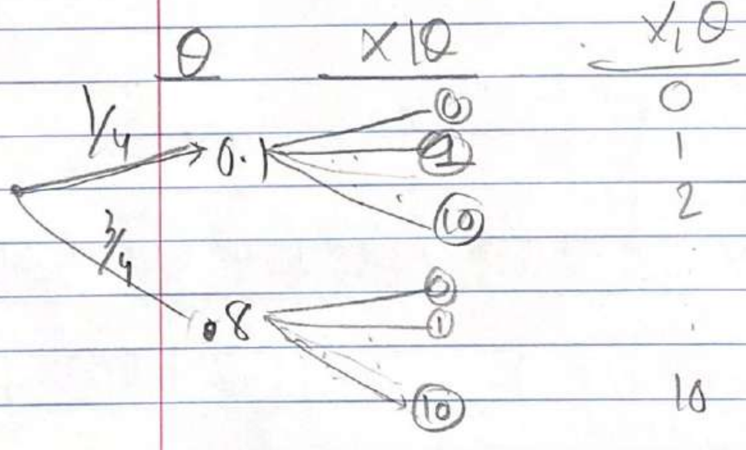
$$X \sim \begin{cases} Bin(10, 0.1) & w.p\ 1/4 \\ Bin(10, 0.8) & w.p\ 3/4 \end{cases}$$



$$P_X(x) = \left(\binom{10}{x}(.1)^x(.9)^{10-x}\right)\left(\tfrac{1}{4}\right)$$

(3) $$+\left(\binom{10}{x}(0.8)^x(.2)^{10-x}\right)\cdot\left(\tfrac{3}{4}\right)$$

$\neq$ Binomial

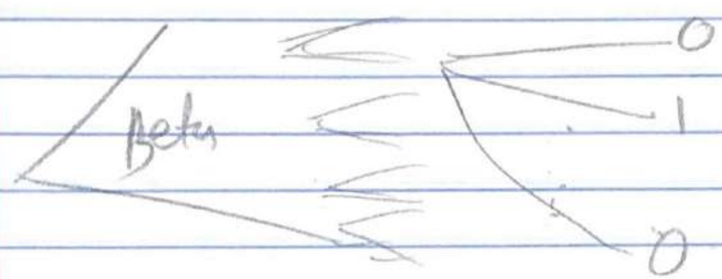


Mixture distribution has a ~~d~~iscrete # of Components.

Compound distri do not.

$$X \sim \begin{cases} Bin(n, \theta) \\ Bin(n, \theta) \end{cases}$$ θ's come from a Beta.

$\underline{\theta}$ $\underline{X|\theta}$



$$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)}$$

$$= \frac{P(X|\theta)P(\theta)}{\int_{\Theta} P(X|\theta)P(\theta)\,d\theta}$$

Compound distri