Lecture 12

07/19 목 [Math341]

$X \sim \text{Poisson}(\theta) = \dfrac{e^{-\theta}\theta^{x}}{x!} = p(x;\theta) \propto e^{-\theta}\theta^{x}$

· discrete

$Y \sim \text{Gamma}(\alpha, \beta) = \dfrac{\beta^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \propto y^{\alpha-1} e^{-\beta y}$

· continuous

F : iid Poisson $\quad X_1 \dots X_n ; \theta \overset{iid}{\sim} \text{Poisson}(\theta)$

$P(X|\theta) = \prod_{i=1}^{n} \dfrac{e^{-\theta}\theta^{x_i}}{x_i!} = \dfrac{e^{-n\theta}\theta^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!}$, $\quad \theta \sim \text{Gamma}(\alpha, \beta)$

put a prior on θ. Last time, we learned that the gamma is conjugate for the Poisson likelihood model.

posterior $P(\theta|X) = \dfrac{P(X|\theta) \cdot P(\theta)}{P(X)} \propto P(X|\theta) \cdot P(\theta) = \left(\dfrac{e^{-n\theta}\theta^{\Sigma x_i}}{\cancel{\prod_{i=1}^{n} x_i!}} \right) \cdot \left(\dfrac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \right)$

$\propto e^{-n\theta}\theta^{\Sigma x_i}\theta^{\alpha-1}e^{-\beta\theta}$

$= \theta^{\Sigma x_i + \alpha - 1} e^{-(n+\beta)\theta} \qquad \alpha' = \Sigma x_i + \alpha, \ \beta' = n + \beta$

$\propto \text{Gamma}(\Sigma x_i + \alpha, n + \beta)$

$P(\theta) \xrightarrow{X} P(\theta|X)$

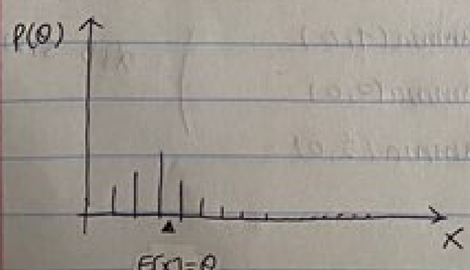
$\text{Gamma}(\alpha, \beta) \xrightarrow{X} \text{Gamma}(\Sigma x_i + \alpha, n+\beta)$

$X \sim \text{Poisson}(\theta)$

· $\text{Supp}[X] = \mathbb{N}_0$ $\quad$ " np (n: # of trials, p: prob of successes)

· Parameter Space : $\theta \in (0, \infty)$

· $E[X] = \theta$



$E[\theta] = \dfrac{\alpha}{\beta}$, $\quad \text{Mode}[\theta] = \dfrac{\alpha-1}{\beta}$ if $\alpha \geq 1$

$\hat{\theta}_{MMSE} = E[\theta|X] = \dfrac{\Sigma x_i + \alpha}{n+\beta}$

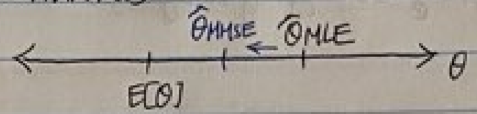
$\hat{\theta}_{MMAE} = \text{Med}[\theta|X] = \text{qgamma}(0.5, \Sigma x_i + \alpha, n+\beta)$

$\hat{\theta}_{MAP} = \text{Mode}[\theta|X] = \dfrac{\Sigma x_i + \alpha - 1}{n+\beta}$ if $\Sigma x_i + \alpha \geq 1$.

$$\theta|x \sim \text{Gamma}(\underbrace{\textstyle\sum x_i+\alpha}_{\#\text{ of total successes}}, \underbrace{n+\beta}_{\#\text{ of trials}})$$

\# of pseudo successes (α); \# of pseudo trials (n_0)

Shrinkage



$$\hat\theta_{MMSE} = \frac{\sum x_i+\alpha}{n+\beta}$$

$$= \frac{\sum x_i}{n+\beta}\cdot\frac{n}{n} + \frac{\alpha}{n+\beta}\cdot\frac{\beta}{\beta}$$

$$= \underbrace{\frac{n}{n+\beta}}_{1-\rho}\cdot\underbrace{\bar{x}}_{\hat\theta_{MLE}} + \underbrace{\frac{\beta}{n+\beta}}_{\rho}\cdot E[\theta]$$

$$\lim_{n\to\infty}\rho = 0$$

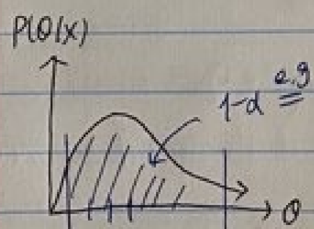
$$\mathcal{L}(\theta;x) = \frac{e^{-n\theta}\cdot\theta^{\sum x_i}}{\prod x_i!}, \quad \ell(\theta;x) = -n\theta + \sum x_i \ln(\theta) - \ln(\prod x_i!)$$

$$\ell'(\theta;x) = -n + \frac{\sum x_i}{\theta} \overset{set}{=} 0$$

$$\Rightarrow \frac{\sum x_i}{\theta} = n.$$

$$\Rightarrow \sum x_i = n\theta$$

$$\therefore \hat\theta_{MLE} = \frac{\sum x_i}{n} = \bar{x}$$



$CR_{\theta,1-\alpha_0}: = [\text{qgamma}(\frac{\alpha_0}{2}, \sum x_i+\alpha, n+\beta),$
$\text{qgamma}(1-\frac{\alpha_0}{2}, \sum x_i+\alpha, n+\beta)]$

<u>Uninformative Priors</u> (principle)

① Laplace Indifference prior — Gamma(1,0)
② H____ Ignorance prior — Gamma(0,0)
③ Jeffrey's — Gamma($\frac{1}{2}$,0)

α, β small.

$\theta \sim \text{Gamma}(\alpha,\beta)$. $Supp[\theta] = (0,\infty)$

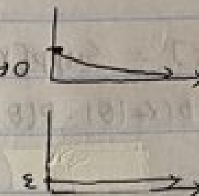
①' $\theta \overset{?}{\sim} U(0,\infty)$ NO

$P(\theta) = \frac{1}{\infty} = 0$. not a valid pdf!

$P(\theta) = Gamma(1, \varepsilon) = \frac{\varepsilon}{\Gamma(1)} \theta^{1-1} e^{-\varepsilon\theta}$ $\quad P(0) = \varepsilon \cdot e^{-\varepsilon(0)} = \varepsilon$.

$= \varepsilon e^{-\varepsilon\theta}$ $\quad \varepsilon \to 0$

If ε small, $\longrightarrow \approx 0$



Laplace $\theta \sim Gamma(1, 0)$ proper? No $\quad B \neq 0$.

$\Rightarrow P(\theta|X) = Gamma(\sum x_i + 1, n)$ proper? Yes Always. (n은 항상 1이상)

shrinkage is undefined.

H — $B = 0, \alpha = 0$. $\theta \sim Gamma(0, 0)$

$\Rightarrow P(\theta|X) = Gamma(\sum x_i + \alpha, n + B)$

$= Gamma(\sum x_i, n)$ Proper? Only if $\exists x_i > 0$.

$\hat{\theta}_{MMSE} = E[\theta|X] = \frac{\sum x_i}{n} = \bar{x} = \hat{\theta}_{MLE}$

Jeffrey's Prior $p_J(\theta) \propto \sqrt{I(\theta)} = \sqrt{n \cdot \frac{1}{\theta}} \propto \theta^{-\frac{1}{2}} = \theta^{\frac{1}{2}-1} \propto Gamma(\frac{1}{2}, 0)$

proper? No

b/c $B = 0$.

$l'(\theta; x) = -n + \frac{\sum x_i}{\theta}$

$-l''(\theta; x) = \frac{\sum x_i}{\theta^2}$

$I(\theta) = E_x[-l''(\theta; x)] = E_x\left[\frac{\sum x_i}{\theta^2}\right] = \frac{1}{\theta^2} \sum_{i=1}^{n} E X_i$

$= \frac{1}{\theta^2} n\theta$

$= n \cdot \frac{1}{\theta}$

$\Rightarrow P(\theta|X) = Gamma(\sum x_i + \frac{1}{2}, n)$ proper? Yes. Always.

$X \sim Poisson(\theta)$

X_* is the next observation that you want to predict.

$X_*|X \sim ?$

- $Supp[X_*|X] = Supp[X] = \{0, 1, \dots\}$

$$P(X_*|X) = \int_{\Theta} p(X_*|\theta) \cdot p(\theta|X)\, d\theta$$

$$= \int_0^{\infty} \left(\frac{e^{-\theta} \cdot \theta^{x_*}}{x_*!} \right) \cdot \left(\frac{(n+\beta)^{\Sigma x_i + \alpha}}{\Gamma(\Sigma x_i + \alpha)} \cdot \theta^{\Sigma x_i + \alpha - 1} \cdot e^{-(n+\beta)\theta} \right) d\theta.$$