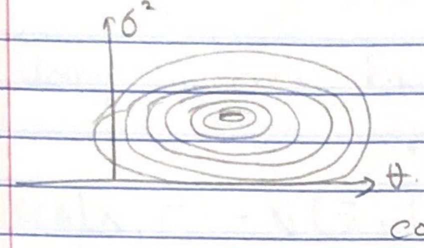
Lecture 18 04/28/2020

F: iid Normal and θ, σ^2 unknown

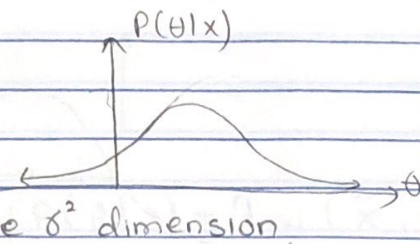
$P_J(\theta, \sigma^2) \propto 1/\sigma^2$

$\propto \text{NormInvGamma}\left(\bar{x}, n, n/2, \frac{(n-1)S}{2}\right)$

$$\Rightarrow P(\theta, \sigma^2 | x) \propto P(x|\theta, \sigma^2) P_J(\theta, \sigma^2) \propto (\sigma^2)^{-n/2-1} e^{-\frac{(n-1)S^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}$$



$\Rightarrow$



collapsing the σ^2 dimension

"marging out σ^2"

$$P(\theta|x) = \int_{\sigma^2} P(\theta, \sigma^2|x) d\sigma^2 \propto \int_0^{\infty} (\sigma^2)^{-n/2-1} e^{-\frac{(n-1)S^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2} d\sigma^2$$

$$= \int_0^{\infty} (\sigma^2)^{\overbrace{-n/2}^{\alpha}-1} e^{\frac{-\overbrace{(n-1)S^2/2 + n(\theta-\bar{x})^2/2}^{\beta}}{\sigma^2}} d\sigma^2$$

$$= \frac{\Gamma(\alpha)}{\beta^{\alpha}} \int_0^{\infty} \overbrace{\frac{\beta^{\alpha}}{\Gamma(\alpha)} (\sigma^2)^{-\alpha-1} e^{-\beta/\sigma^2}}^{1} d\sigma^2$$

$$= \Gamma(\alpha)\beta^{-\alpha} = \Gamma\left(\frac{n}{2}\right)\left(\frac{(n-1)S^2 + n(\theta-\bar{x})^2}{2}\right)^{-n/2}$$

$$\propto \left(\frac{2}{(n-1)S^2}\right)^{-n/2} \left(\frac{(n-1)S^2 + n(\theta-\bar{x})^2}{2}\right)^{-n/2}$$

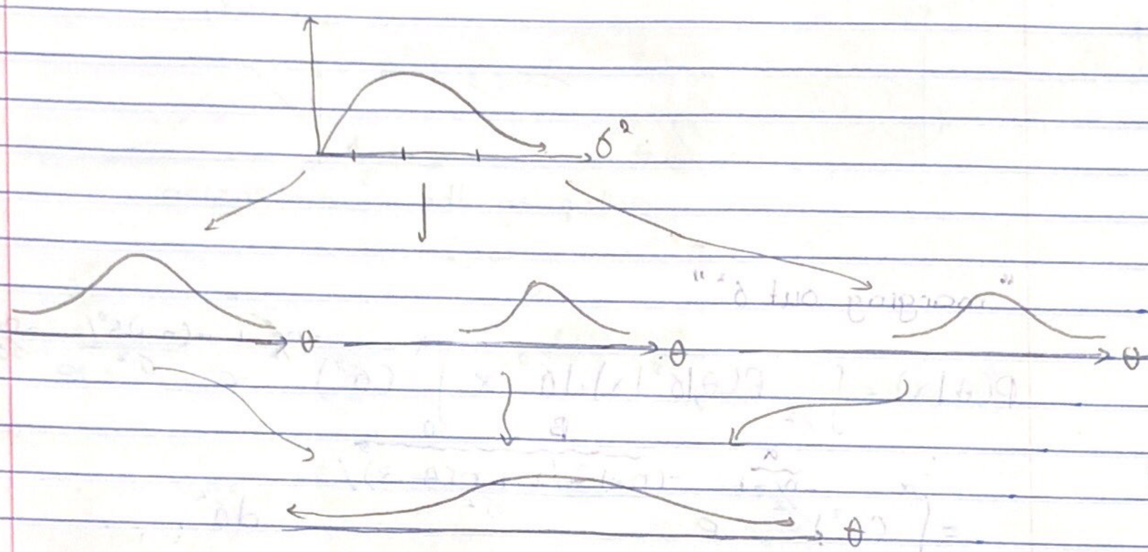
$$= \left(1 + \frac{n(\theta-\bar{x})^2}{(n-1)S^2}\right)^{-n/2}$$

$$= \left(1 + \frac{1}{n-1}\frac{(\theta-\bar{x})^2}{(S/\sqrt{n})^2}\right)^{-\left(\frac{(n-1)+1}{2}\right)} \propto T_{n-1}(\bar{x}, S/\sqrt{n})$$

$$\overset{n \text{ large}}{\approx} N(\bar{x}, S^2/n)$$

$$P(\theta|x) = \int_{\sigma^2} P(\theta, \sigma^2|x)\, d\sigma^2$$

$$= \int_{\sigma^2} \underbrace{P(\theta|x,\sigma^2)}_{\text{normal}}\, \underbrace{P(\sigma^2|x)}_{\text{inv-gamma}}\, d\sigma^2$$ compound distribution



$$P(\sigma^2|x) = \int_\theta P(\theta,\sigma^2|x)\, d\theta$$

$$\propto \int_{\mathbb{R}} (\sigma^2)^{-n/2-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-n/2\sigma^2(\theta-\bar{x})^2}\, d\theta.$$

$$= (\sigma^2)^{-n/2-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \int_{\mathbb{R}} e^{-n/2\sigma^2(\theta-\bar{x})^2}\, d\theta.$$

$$= (\sigma^2)^{-n/2-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \sqrt{2\pi\sigma^2/n} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2/n}} e^{-\frac{1}{2\sigma^2/n}(\theta-\bar{x})^2}\, d\theta}_{1}$$

$$\propto (\sigma^2)^{-n/2-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} (\sigma^2)^{1/2}$$

$$= (\sigma^2)^{-\left(\frac{n-1}{2}\right)-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}$$

$$\propto \text{Inv Gamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

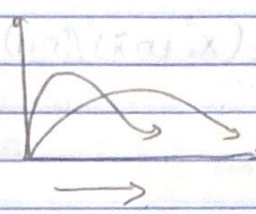
$$P(\sigma^2|x) = \int_\theta \overbrace{P(\sigma^2|x,\theta)}^{\text{Inv Gamma}} \overbrace{P(\theta|x)}^{T} d\theta = \text{Inv Gamma}$$

Using Jeffrey's Prior

$P(\theta|x,\sigma^2) = N(\bar{x}, (\sigma/\sqrt{n})^2)$ & $P(\theta|x) = T_{n-1}(\bar{x}, s/\sqrt{n})$

$P(\sigma^2|x,\theta) = \text{Inv Gamma}(n/2, n\hat{\sigma}^2_{MLE}/2)$ &

$P(\sigma^2|x) = \text{Inv Gamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

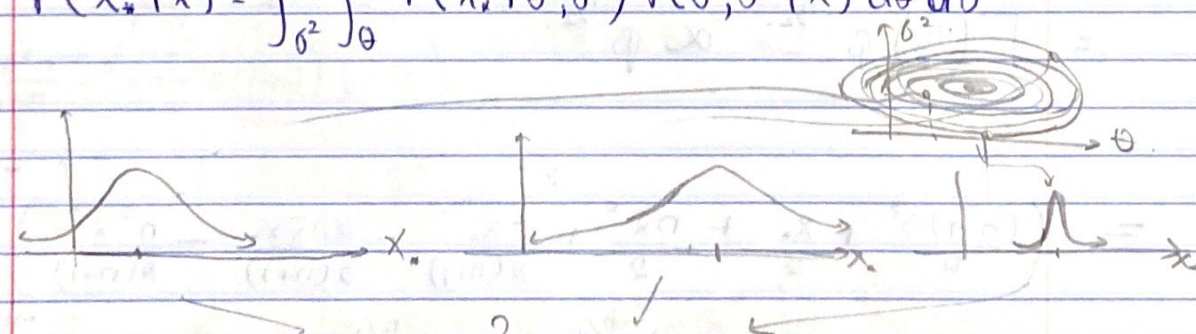


$n\hat{\sigma}^2_{MLE} = \sum(x_i - \theta)^2$

$(n-1)s^2 = \sum(x_i - \bar{x})^2$ similar

Posterior predictive distribution.

$$P(X_*|x) = \int_{\sigma^2}\int_\theta \overbrace{P(X_*|\theta,\sigma^2)}^{\text{normal}} \overbrace{P(\theta,\sigma^2|x)}^{\text{norm-inv-gamma}} d\theta\, d\sigma^2$$



$$\propto \int_0^\infty \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_*-\theta)^2}\right)\left((\sigma^2)^{-n/2-1} e^{\frac{-(n-1)s^2/2}{\sigma^2}} e^{\frac{-n(\theta-\bar{x})^2}{2\sigma^2}}\right) d\theta\, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-1/2} (\sigma^2)^{-n/2-1} e^{\frac{-(n-1)s^2}{2\sigma^2}} \int_{\mathbb{R}} e^{-1/2\sigma^2\left((x_*-\theta)^2+n(\theta-\bar{x})^2\right)} d\theta\, d\sigma^2$$

$$x_*^2 - 2x_*\theta + \underbrace{\theta^2 + n\theta^2}_{(n+1)\theta^2} - 2n\theta\bar{x} + n\bar{x}^2 \qquad e^{-x_*^2/2\sigma^2}\, e^{x_*\theta/\sigma^2}\, e^{\frac{-(n+1)\theta^2}{2\sigma^2}}\, e^{\frac{n\theta\bar{x}}{\sigma^2}}\, e^{\frac{-n\bar{x}^2}{2\sigma^2}}$$

$$= \int_0^\infty (\sigma^2)^{-\left(\frac{n+1}{2}\right)-1} e^{\frac{-\left((n-1)s^2+x_*^2+n\bar{x}^2\right)/2}{\sigma^2}} \int_{\mathbb{R}} e^{\overbrace{\frac{x_*+n\bar{x}}{\sigma^2}}^{a}\theta - \overbrace{\frac{n+1}{2\sigma^2}}^{b}\theta^2} d\theta \cdot \sqrt{\frac{2\sigma^2}{n+1}} \propto (\sigma^2)^{-1/2}$$

$$-\frac{n+1}{2} - 1 + \frac{1}{2} = -\frac{n}{2} - \frac{1}{2} - 1 + \frac{1}{2}$$

$$\int_{\mathbb{R}} e^{a\theta - b\theta^2} d\theta = \sqrt{\pi/b}\, e^{a^2/4b} \propto b^{-1/2} e^{a^2/4b}$$

$$\propto \int_0^\infty (\sigma^2)^{-\left(\frac{n+1}{2}\right)-1} e^{-\frac{\left((n-1)s^2+x_*^2+n\bar{x}^2\right)/2}{\sigma^2}} (\sigma^2)^{1/2} e^{\left(\frac{(x_*+n\bar{x})^2}{(\sigma^2)^2\, 4\, \frac{n+1}{2\sigma^2}}\right)} d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-(n/2)-1} e^{-\frac{\left((n-1)s^2+x_*^2+n\bar{x}^2-(x_*+n\bar{x})^2/(n+1)\right)/2}{\sigma^2}} d\sigma^2$$

$$= \frac{\Gamma(\alpha)}{\beta^\alpha} \int_0^\infty \overbrace{\frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-\alpha-1} e^{-\beta/\sigma^2}}\, d\sigma^2 = \Gamma(\alpha)\beta^{-\alpha}$$

$$= \Gamma(n/2)\beta^{-n/2} \propto \beta^{-n/2}$$

$a = \frac{1}{2} - \frac{1}{2n+2} = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) = \frac{1}{2}\frac{n}{n+1}$

$b = -\frac{n\bar{x}}{n+1}$

$c = \frac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)$

$$= \left(\frac{(n-1)s^2}{2} + \frac{x_*^2}{2} + \frac{n\bar{x}^2}{2} + \frac{-x_*^2}{2(n+1)} - \frac{2n\bar{x}x_*}{2(n+1)} - \frac{n^2\bar{x}^2}{2(n+1)}\right)^{-n/2}$$

$$= (ax_*^2 + bx_* + c)^{-n/2} \propto (1/a)^{-n/2} (ax_*^2 + bx_* + c)^{-n/2}$$

$$= \left(x_*^2 + \frac{b}{a}x_* + \frac{c}{a}\right)^{-n/2}$$

$$= \left(\left(x_* + \frac{b}{2a} \right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \right)^{-n/2} \ldots$$

$$\propto \left(\frac{1}{c/a - b^2/4a^2} \right)^{-n/2} \left(\left(x_* + b/2a \right)^2 + \left(c/a - b^2/4a^2 \right) \right)^{-n/2}$$

$$= \left(1 + \frac{\left(x_* + b/2a \right)^2}{\left(c/a - b^2/4a^2 \right)} \right)^{-n/2} = \left(1 + \frac{1}{n-1} \frac{\left(x_* + b/2a \right)^2}{\left(\frac{c/a + b^2/4a^2}{n-1} \right)} \right)^{-\frac{(n-1)+1}{2}}$$

$P(x_* | x, \sigma^2) = N(\hat{\theta}_p, \sigma^2 + \sigma^2/_{n+n_0})$

$$\propto T_{n-1}\left(\frac{-b}{2a}, \sqrt{\frac{c/a - b^2/4a^2}{n-1}} \right) = T_{n-1}\left(\bar{x}, \sqrt{\tfrac{n+1}{n}}\, s \right) \overset{n \text{ large}}{\approx} N(\bar{x}, s^2)$$

$\| \quad \sqrt{s^2 + s^2/n}$

Exactly what you expect

$$\frac{-b}{2a} = \frac{\frac{n\bar{x}}{n+1}}{2\left(\frac{1}{2} \frac{n}{n+1} \right)} = \bar{x}$$

$$\frac{c}{a} = \frac{\frac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1} \right)}{\frac{1}{2}\, n/n+1} \cdot \frac{\frac{n+1}{n}}{\frac{n+1}{n}}$$

$$= \frac{(n-1)(n+1)}{n} s^2 + \underbrace{(n+1)\bar{x}^2}_{n\bar{x}^2 + \bar{x}^2} - n\bar{x}^2 = \frac{(n-1)(n+1)}{n} s^2 + \bar{x}^2$$

$$\frac{b^2}{4a^2} = \left(\frac{-b}{2a} \right)^2 = \bar{x}^2$$

$$\frac{c}{a} - \frac{b^2}{4a^2} = \frac{(n-1)(n+1)}{n} s^2 + \bar{x}^2 - \bar{x}^2 \qquad \sqrt{\frac{c/a - b^2/4a^2}{n-1}} = \sqrt{n+1/n}\; s$$