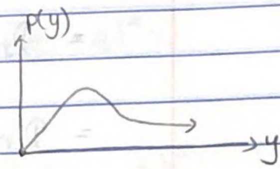
Lecture - 16 04/14/2020

$$Y \sim \text{InvGamma}(\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} y^{-\alpha-1} e^{-\beta/y}$$
$$\propto y^{-\alpha-1} e^{-\beta/y}$$



$E[Y] = \frac{\beta}{\alpha-1}$, $\text{med}[Y] = \text{qinvgamma}(0.5, \alpha, \beta)$, $\text{mode}[Y] = \frac{\beta}{\alpha+1}$ always

if $\alpha > 1$.

F: $\overset{iid}{\sim} N(\theta, \sigma^2)$ with θ known; $P(\sigma^2 \mid \theta) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$

$Y_1, \dots, y_n \overset{iid}{\sim} N(\theta, \sigma^2)$; $n_0\sigma_0^2 = \sum (y_i - \theta)^2$

$$E[\sigma^2 \mid \theta] = \frac{n_0\sigma_0^2/2}{n_0/2 - 1} = \frac{n_0\sigma_0^2}{n_0 - 2} \approx \sigma_0^2 \text{ if } n_0 \text{ large}$$

$$\text{Mode}[\sigma^2 \mid \theta] = \frac{n_0\sigma_0^2}{n_0 + 2} \approx \sigma_0^2$$

$$P(\sigma^2 \mid x, \theta) = \text{InvGamma}\left(\frac{n_0+n}{2}, \frac{n_0\sigma_0^2 + n\hat{\sigma}^2_{MLE}}{2}\right)$$

$$\hat{\sigma}^2_{MMSE} = \frac{(n_0\sigma_0^2 + n^2\hat{\sigma}^2)/2}{(n_0+n)/2 - 1} = \frac{n_0\sigma_0^2 + n\hat{\sigma}^2}{n_0 + n - 2}$$

$$\hat{\sigma}^2_{MMAE} = \text{qinvgamma}\left(0.5, (n_0+n)/2, (n_0\sigma_0^2 + n\hat{\sigma}^2)/2\right)$$

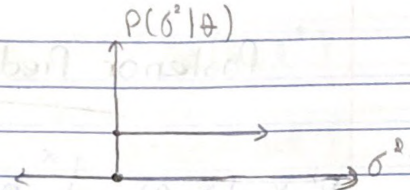
$$\hat{\sigma}^2_{MAP} = \frac{n_0\sigma_0^2 + n\hat{\sigma}^2}{n_0 + n + 2}$$

Uninformative Priors

① Laplace / Indifferance

$$P(\sigma^2 \mid \theta, x) \propto P(X \mid \theta, \sigma^2) \propto (\sigma^2)^{-(n/2 - 1) - 1} e^{\frac{-n\hat{\sigma}^2/2}{\sigma^2}}$$

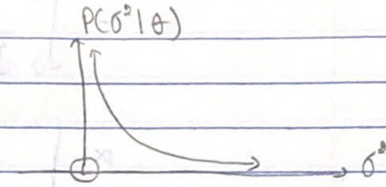
$$\propto \text{InvGamma}\Big(\overset{\frac{n-2}{2}}{\overset{\parallel}{\tfrac{n}{2}-1}}, \tfrac{n\hat{\sigma}^2}{2}\Big)$$



$\Longrightarrow n_0 = -2 \; ; \; \sigma_0^2 = 0$

$\Longrightarrow P(\sigma^2|\theta) = \text{InvGamma}(-1, 0) \propto 1$

σ^2_{MMSE} is only defined for $n \geq 5$.



② Haldane $n_0 = 0, \sigma_0^2 = ?$

$\Longrightarrow \text{InvGamma}(0,0) \propto (\sigma^2)^{-1}$

③ Jeffery's prior

$$l(\sigma^2;\theta,x) = -\tfrac{n}{2}\ln(2\pi) - \tfrac{n}{2}\ln(\sigma^2) - \tfrac{1}{2\sigma^2}\sum(x_i-\theta)^2$$

$$l'(\sigma^2;\theta,x) = -\tfrac{n}{2}\tfrac{1}{\sigma^2} + \tfrac{1}{2(\sigma^2)^2}\sum(x_i-\theta)^2$$

$$l''(\sigma^2;\theta,x) = \tfrac{n}{2}\tfrac{1}{(\sigma^2)^2} - \tfrac{1}{(\sigma^2)^3}\sum(x_i-\theta)^2$$

$$-l''(\sigma^2;\theta,x) = \underbrace{-\tfrac{n}{2}\tfrac{1}{(\sigma^2)^2} + \tfrac{1}{(\sigma^2)^3}\sum(x_i-\theta)^2}$$

$$I(\sigma^2;\theta) = E[-l''(\sigma^2;\theta,x)] = E_x[\quad] = -\tfrac{n}{2}\tfrac{1}{(\sigma^2)^2} + \tfrac{1}{(\sigma^2)^3}\sum E_x[(x_i-\theta)^2]$$

$$E_x(x_i-\theta)^2$$
$$= E_x[x_i^2 - 2\theta x_i + \theta^2]$$
$$= E_x[x_i^2] - 2\theta E[x_i] + \theta^2$$
$$= (\sigma^2+\theta^2) - 2\theta^2 + \theta^2 = \sigma^2$$

$$= -\tfrac{n}{2}\tfrac{1}{(\sigma^2)^2} + \tfrac{1}{(\sigma^2)^3}n\sigma^2$$
$$= n\left(-\frac{1}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^2}\right)$$
$$= n\frac{1}{2(\sigma^2)^2} = \frac{n}{2}(\sigma^2)^{-2}$$

$$P_J(\sigma^2|\theta) \propto \sqrt{I(\sigma^2;\theta)} = \sqrt{\tfrac{n}{2}(\sigma^2)^{-2}} \propto (\sigma^2)^{-1}$$
$$\propto \text{InvGamma}(0,0) \text{ (default)}$$

let $\alpha' = \frac{n+n_0}{2}$

$\beta' = \frac{n\hat{\sigma}^2 + n_0\sigma_0^2}{2}$

Posterior Predictive distribution

$$P(X_*|X,\theta) = \int_0^\infty P(X_*|\theta,\sigma^2)\, P(\sigma^2|X,\theta)\, d\sigma^2$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_*-\theta)^2} \cdot \frac{\beta'^{\alpha'}}{\Gamma(\alpha')} (\sigma^2)^{-\alpha'-1} e^{-\beta'/\sigma^2}\, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-1/2} \underbrace{e^{-\frac{(X_*-\theta)^2/2}{\sigma^2}}}_{A} \underbrace{(\sigma^2)^{-\alpha'-1} e^{-\beta'/\sigma^2}}_{B}\, d\sigma^2$$

$$= \int_0^\infty \underbrace{(\sigma^2)^{-(\alpha'+1/2)-1} e^{-\frac{(X_*-\theta)^2/2+\beta'}{\sigma^2}}}_{\text{Kernal of InvGamma}(A,B)}\, d\sigma^2$$

$$= \frac{\Gamma(A)}{\beta^A} \int_0^\infty \frac{B^A}{\Gamma(A)} (\sigma^2)^{-A-1} e^{-B/\sigma^2}\, d\sigma^2$$

$$= \Gamma(A)\, \beta^{-A}$$

$$= \Gamma\left(\frac{n+n_0+1}{2}\right) \left(\frac{\underbrace{n\hat{\sigma}^2 + n_0\sigma_0^2}_{a} + (X_*-\theta)^2}{2}\right)^{-\frac{n+n_0+1}{2}}$$

$$\propto \left(\frac{n\hat{\sigma}^2 + n_0\sigma_0^2 + (X_*-\theta)^2}{2}\right)^{-\frac{n+n_0+1}{2}}$$

$$= \left(\frac{a}{2}\right)^{-\frac{\nu+1}{2}} \left(\frac{2}{a}\, \frac{a+(X_*-\theta)^2}{2}\right)^{-\frac{\nu+1}{2}}$$

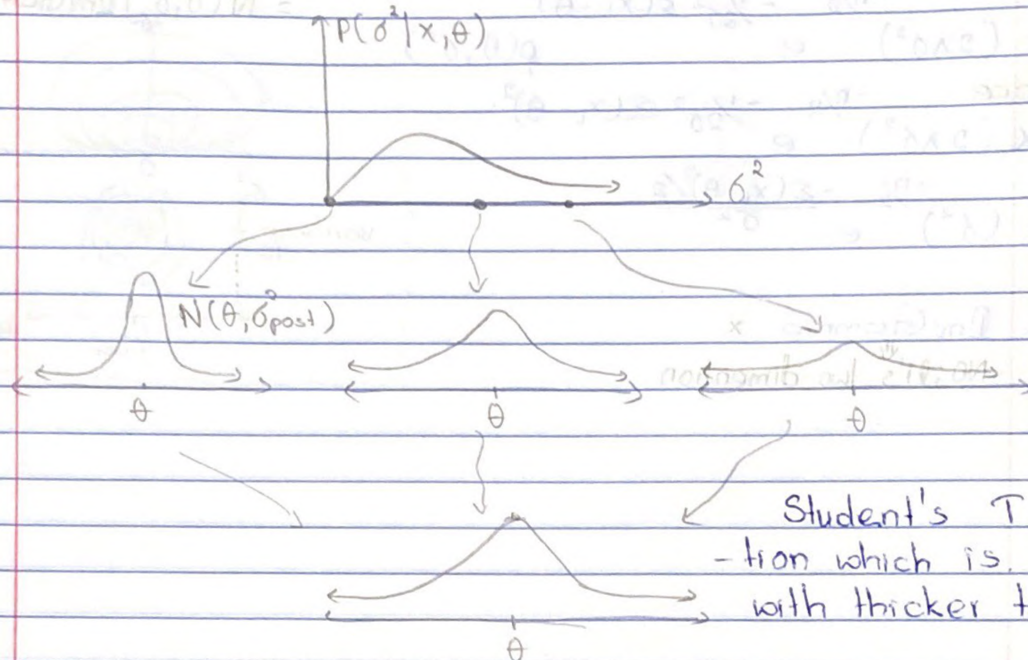
$$\propto \left(\frac{2}{a}\, \frac{a+(X_*-\theta)^2}{2}\right)^{-\frac{\nu+1}{2}}$$

$$= T_{n+n_0}\left(\theta, \frac{n\hat{\sigma}^2 + n_0\sigma_0^2}{n+n_0}\right) = \left(1 + \frac{(X_*-\theta)^2}{a}\right)^{-\frac{\nu+1}{2}} = \left(1 + \frac{1}{\nu}\, \frac{(X_*-\theta)^2}{a/\nu}\right)^{-\frac{\nu+1}{2}}$$

$\propto T_\nu(\theta, a/\nu)$ Student's T distribution with ν degrees of freedom and location parameter θ (mean) and scale a/ν.

Instant 101, F: $x_1, \ldots, x_n ; \theta, \sigma^2 \overset{iid}{\sim} \mathbb{N}(\theta, \sigma^2)$

$H_0: \theta = 0 \implies \dfrac{\bar{x} - \theta}{s/\sqrt{n}} = \dfrac{\bar{x}\sqrt{n}}{s} \sim T_{n-1}(0,1)$ Standard T



Student's T distribution which is Normal with thicker tails.

$$\lim_{n \to \infty} P(x_* | x, \theta) = \mathbb{N}(\theta, \sigma^2)$$

$$= T_\infty(\theta, \lim_{n \to \infty} \hat{\sigma}^2_{MLE}) = \mathbb{N}(\theta, \sigma^2)$$

Shrinkage

$$\hat{\sigma}^2_{MMSE} = \frac{n\hat{\sigma}^2 + n_0\sigma_0^2}{n + n_0 - 2} = \underbrace{\frac{n}{n + n_0 - 2}}_{1-\rho} \hat{\sigma}^2 + \frac{n_0\sigma_0^2}{n + n_0 - 2} \cdot \frac{n_0 - 2}{n_0 - 2}$$

$$= (1-\rho)\hat{\sigma}^2 + \underbrace{\frac{n_0 - 2}{n + n_0 - 2}}_{\rho} \cdot \underbrace{\frac{n_0\sigma_0^2}{n_0 - 2}}_{E[\sigma^2|\theta]}$$

$$= (1-\rho)\hat{\sigma}^2 + \rho \cdot E[\sigma^2|\theta]$$

END OF MIDTERM 02

T: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ but neither θ or σ^2 known

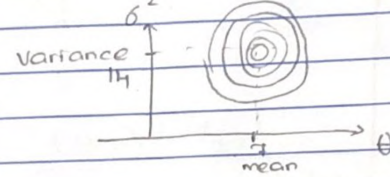
$P(\theta,\sigma^2|x) \propto P(X|\theta,\sigma^2)\,P(\theta,\sigma^2)$; $P(\theta,\sigma^2) = P(\theta|\sigma^2)\,P(\sigma^2)$

$= N(0,\sigma^2)\,\text{InvGamma}(1,1)$

$= (2\pi\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}\, p(\theta,\sigma^2)$

if Laplace $\propto (2\pi\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}$

$\propto (\sigma^2)^{-n/2}\, e^{-\frac{\sum(x_i-\theta)^2/2}{\sigma^2}}$



? $\propto$ InvGamma ×

NO; It's two-dimention.