Lecture 14 03/26/2020

Correction from Lecture 13:

$F \overset{iid}{:}$ poisson(θ) $X_1, \ldots, X_n$; $\theta \sim$ poisson(θ),
$\theta \sim$ Gamma(α, β)

$P(X_* | X) = \text{ExtNegBin}(r, p)$ where $r = \sum x_i + \alpha$, $p = \dfrac{n+\beta}{n+\beta+1}$

if $\alpha \in \mathbb{N}_0$ $\rightarrow$ $= \text{NegBin}(r,p) = \dbinom{X_* + r - 1}{r-1} (1-p)^{X_*} p^r$

$X_1, \ldots, X_r \overset{iid}{\sim} \text{Geom}(p) \Longrightarrow \sum_{i=1}^{r} X_i \sim \text{NegBin}(r,p)$

* corrected version: $E[X_i] = \dfrac{1-p}{p} \neq \dfrac{p}{1-p} \Longrightarrow E[\sum X_i] = r \dfrac{1-p}{p}$

$E[X_* | X] = r \dfrac{1-p}{p} = (\sum x_i + \alpha) \dfrac{\frac{1}{n+\beta+1}}{\frac{n+\beta}{n+\beta+1}} = \dfrac{\sum x_i + \alpha}{n+\beta} = E[\theta | X]$

$F \overset{iid}{\sim} N(\theta, \sigma^2)$ with σ^2 known

$P(\theta | x, \sigma^2) \propto e^{a\theta - b\theta^2} \propto N(a/2b, 1/2b)$ $\quad a = n\bar{x}/\sigma^2$, $b = n/2\sigma^2$

$= N(\bar{x}, \sigma^2/n)$

Under Laplace prior i.e. $P(\theta) \propto 1$

$P(\theta | x, \sigma^2) \propto P(x | \theta, \sigma^2) \propto N(\bar{x}, \sigma^2/n)$

$$P(\theta|\sigma^2) \propto k(\theta|\sigma^2)$$

Conjugate prior

$$P(\theta|x,\sigma^2) = \frac{P(x|\theta,\sigma^2)\,P(\theta|\sigma^2)}{P(x|\sigma^2)} \propto P(x|\theta,\sigma^2)\,P(\theta|\sigma^2)$$

$$\propto k(x|\theta,\sigma^2)\,k(\theta|\sigma^2)$$

$$\text{if } k(\theta|\sigma^2) = e^{\alpha\theta-\beta\theta^2} \propto \mathbb{N}(\alpha/2\beta, 1/2\beta)$$

$$= e^{a\theta-b\theta^2} \qquad k(\theta|\sigma^2) \Rightarrow e^{a\theta-b\theta^2}e^{\alpha\theta-\beta\theta^2} = e^{(a+\alpha)\theta-(b+\beta)\theta^2}$$

$$\Rightarrow P(\theta|x,\sigma^2)$$

$$\propto \mathbb{N}\left(\frac{a+\alpha}{2(b+\beta)}, \frac{1}{2(b+\beta)}\right)$$

$$a = \frac{n\bar{x}}{\sigma^2},\quad b = \frac{1}{2\sigma^2},\quad \alpha = \frac{\mu_0}{t^2},\quad \beta = \frac{1}{2t^2}$$

$$\frac{a+\alpha}{2(b+\beta)} = \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{t^2}}{\frac{n}{\sigma^2}+\frac{1}{t^2}}, \qquad \frac{1}{2(b+\beta)} = \frac{1}{2\left(\frac{n}{2\sigma^2}+\frac{1}{2t^2}\right)} = \frac{1}{\frac{n}{\sigma^2}+\frac{1}{t^2}}$$

*correction

$$\frac{\alpha}{2\beta} = \mu_0;\quad \mu_0 = \frac{\alpha}{2\cdot\frac{1}{2t^2}} = \alpha t^2 \Rightarrow \alpha = \frac{\mu_0}{t^2}$$

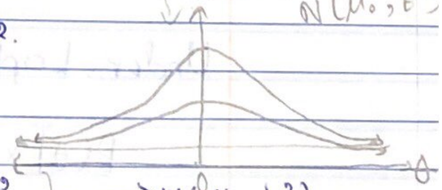
$$\mathbb{N}(\alpha/2\beta, 1/2\beta) = \mathbb{N}(\mu_0, t^2)$$

$F \overset{iid}{:} \mathbb{N}(\theta,\sigma^2)$; σ^2 known, $P(\theta) = \mathbb{N}(\mu_0, t^2)$

$$\Rightarrow P(\theta|x) = \mathbb{N}\left(\frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{t^2}}{\frac{n}{\sigma^2}+\frac{1}{t^2}}, \frac{1}{\frac{n}{\sigma^2}+\frac{1}{t^2}}\right)$$

supposed to look symmetrical

$P(\theta|\sigma^2) = \mathbb{N}(\mu_0, t^2)$



Under Laplace prior $\cong \mathbb{N}(\bar{x}, \sigma^2/n) \Rightarrow \mu_0, t^2$.

$$P(\theta) \propto 1 \propto N(0,\infty) \text{ Laplace}$$

$$\hat{\theta}_{MMSE} = E[\theta|x] = \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{t^2}}{\frac{n}{\sigma^2}+\frac{1}{t^2}}$$

if $t^2 \to \infty \Rightarrow N(\mu_0, t^2) \propto 1$

raise variance to ∞ to make flat

$$\hat{\theta}_{MAE} = \text{med}[\theta | x] = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{t^2}}{\frac{n}{\sigma^2} + \frac{1}{t^2}}$$

$$\hat{\theta}_{MAP} = \text{Mode}[\theta | x] = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{t^2}}{\frac{n}{\sigma^2} + \frac{1}{t^2}}$$

$$CR_{\theta, 1-\alpha} = \left[\text{qnorm}\left(\alpha/2, \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{t^2}}{\frac{n}{\sigma^2} + \frac{1}{t^2}}, \frac{1}{n/\sigma^2 + 1/t^2} \right), \text{qnorm}\left(1 - \frac{\alpha}{2}, \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{t^2}}{\frac{n}{\sigma^2} + \frac{1}{t^2}}, \frac{1}{n/\sigma^2 + 1/t^2} \right) \right]$$

Hypothesis Tests

$H_0 : \theta \leq \theta_0 , \quad H_a : \theta > \theta_0$

$$\text{pval} = P(H_0 | x) = \text{pnorm}(\theta_0, \text{mean}, \text{variance}) = \int_{-\infty}^{\theta_0} \text{PDF} \, d\theta$$

Jeffrey's Prior

$$P_J(\theta | \sigma^2) \propto \sqrt{I(\theta ; \sigma^2)} = \sqrt{n/\sigma^2} \propto 1 \propto N(0, \infty) \Rightarrow$$ Jeffrey's prior in the same as Laplace's prior

$$\mathcal{L}(\theta ; x, \sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum x_i^2}{2\sigma^2} + \frac{n\bar{x}}{\sigma^2}\theta - \frac{n\theta^2}{2\sigma^2}}$$

$$l(\theta ; x, \sigma^2) = -n/2 \ln(2\pi\sigma^2) - \frac{\sum x_i^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}$$

$$l'(\theta ; x, \sigma^2) = \frac{n\bar{x}}{\sigma^2} - \frac{n\theta}{\sigma^2} \Rightarrow -l''(\theta ; x, \sigma^2) = \frac{n}{\sigma^2}$$

$$I(\theta ; \sigma^2) = E_x[l''(\theta ; x, \sigma^2)] = E_x[n/\sigma^2] = n/\sigma^2$$

let n_0 = # of pseudo observations

Haldane: pure ignorance

since σ^2 known

$$\text{let } t^2 = \frac{\sigma_0^2}{n_0} = \frac{\sigma^2}{n_0}$$

$$P(\theta|\sigma^2) = N(\mu_0, t^2) = N(\mu_0, \sigma^2/n_0)$$

$$\hat{\theta} = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{t^2}}{n/\sigma^2 + 1/t^2} = \frac{n\bar{x}/\sigma^2 + n_0\mu_0/\sigma_0^2}{n/\sigma^2 + n_0/\sigma_0^2}$$

average of pseudo-data

$$= \frac{\frac{n\bar{x} + n_0\mu_0}{\sigma^2}}{\frac{n+n_0}{\sigma^2}} = \frac{n\bar{x} + n_0\mu_0}{n+n_0} = \frac{\Sigma x_i + \Sigma y_i}{n+n_0}$$

Haldane: $n_0 = 0 \implies t^2 = \infty$, $\mu_0 = 0$

$P(\theta|\sigma^2) = N(0, \infty)$ same as Laplace

let $y_1, y_2, \ldots, y_{n_0}$ be pseudo-data

$\mu_0 = \bar{y} = 1/n_0 \, \Sigma y_i$

$n_* = 1$

$$P(x_*|x, \sigma^2) = \int_\Theta P(x_*|\theta, \sigma^2)\, P(\theta|x, \sigma^2)\, d\theta$$

$$= \int_{\mathbb{R}} \underset{N(\theta,\sigma^2)}{\text{PDF}}\ \underset{N(\theta_p,\sigma_p^2)}{\text{PDF}}\ d\theta$$

$$\theta_p = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{t^2}}{\frac{n}{\sigma^2} + \frac{1}{t^2}}, \quad \sigma_p^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{t^2}}$$

$$= \sigma_p^2\left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{t^2}\right)$$

$\hat{\theta}_{MLE} = \bar{x}$

$$\hat{\theta}_{NMAE} = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{t^2}}{\frac{n}{\sigma^2} + \frac{1}{t^2}} = \frac{\frac{n\bar{x}}{\sigma^2}}{n/\sigma^2 + 1/t^2} + \frac{\frac{\mu_0}{t^2}}{n/\sigma^2 + 1/t^2} = \frac{\frac{n}{\sigma^2}\hat{\theta}_{MLE}}{\frac{n}{\sigma^2} + \frac{1}{t^2}} + \frac{\frac{1}{t^2}E[\theta]}{\frac{n}{\sigma^2} + \frac{1}{t^2}}$$

$$= \underbrace{\frac{1}{1 + \frac{\sigma^2}{nt^2}}}_{(1-\rho)}\hat{\theta}_{MLE} + \underbrace{\frac{1}{1 + \frac{nt^2}{\sigma^2}}}_{\rho} E[\theta]$$

If $n \to \infty \implies \rho \to 0$