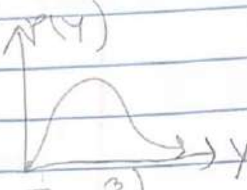
Lecture 16

$Y \sim \text{InvGamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} Y^{-\alpha-1} e^{-\beta/Y}$

$\propto y^{-\alpha-1} e^{-\beta/y}$



mean = $E[Y] = \frac{\beta}{\alpha-1}$ (always $\alpha > 1$), $\text{Med}[Y] = \text{qinvgamma}(0.5, \alpha, \beta)$

$\text{Mode}[Y] = \frac{\beta}{\alpha+1}$

$Y \sim$ iid $N(\theta, \sigma^2)$ with θ known

$P(\sigma^2 \mid \theta) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0 \sigma_0^2}{2}\right)$

why this is dependent on θ?

$Y_1, \ldots, Y_n \overset{iid}{\sim} N(\theta, \sigma^2)$

$n_0 \sigma_0^2 = \Sigma (y_i - \theta)^2$

Prior expectation:

$$E[\sigma^2 \mid \theta] = \frac{n_0 \sigma_0^2}{2} \Big/ \frac{n_0}{2} - 1$$

$$= \frac{n_0 \sigma_0^2}{n_0 - 2} \approx \sigma_0^2 \quad [\text{if } n_0 \text{ is large}]$$

Prior mode:

$$\text{Mode}[\sigma^2 \mid \theta] = \frac{n_0 \sigma_0^2}{n_0 + 2} = \sigma_0^2 \quad [\text{if } n_0 \text{ large}]$$

Posterior

$$P(\sigma^2 | X, \theta) = \text{InvGamma}\left(\frac{n_0+n}{2}, \frac{n_0\sigma_0^2 + n\hat{\sigma}^2_{MLE}}{2}\right)$$

$$\hat{\sigma}^2_{MMSE} = \frac{n_0\sigma_0^2 + n\hat{\sigma}^2)/2}{(n_0+n)/2 - 1}$$

$$= \frac{n_0\sigma_0^2 + n\hat{\sigma}^2}{n_0+n-2}$$

$$\hat{\theta}_{MMAE} = \text{qinvgamma}\left(0.5, \frac{n_0+2}{2}, \frac{n_0\sigma_0^2 + n\hat{\sigma}^2}{2}\right)$$

$$\hat{\sigma}^2_{MAP} = \frac{n_0\sigma_0^2 + n\hat{\sigma}^2}{n_0+n+2}$$

Uninformative priors,

(1) Laplace / Indifference:

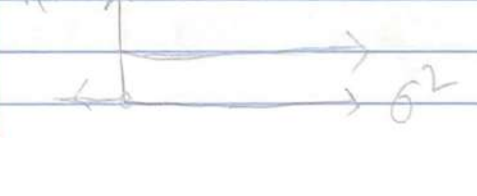
$$P(\sigma^2 | \theta, X) \propto P(X|\theta, \sigma^2) \propto (\sigma^2)^{-(n/2 - 1) - 1} e^{-\frac{n\hat{\sigma}^2/2}{\sigma^2}}$$

$$\propto \text{InvGamma}\left(\frac{n}{2} - 1, \frac{n\hat{\sigma}^2}{2}\right)$$

$$= \text{InvGamma}\left(\frac{n-2}{2}, \frac{n\hat{\sigma}^2}{2}\right)$$

$$\Rightarrow n_0 = -2, \sigma_0^2 = 0$$

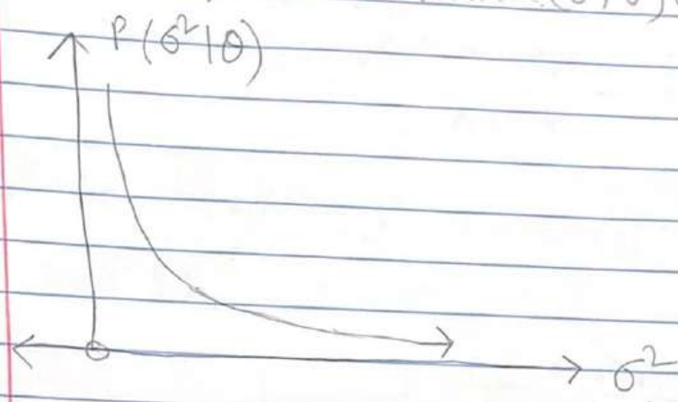
$$\Rightarrow P(\sigma^2 | \theta) = \text{InvGamma}(-1, 0) \propto 1.$$



$\hat{\sigma}^2_{MMSE}$ is only defined for $n \geq 5$.

② <u>Haldin</u> - $n_0 = 0$, $\sigma_0^2 = ?$

$$\Rightarrow \text{InvGamma}(0,0) \propto (\sigma^2)^{-1}$$



③ <u>Jeffery's.</u>

$$l(\sigma^2; \theta, x) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2$$

$$l'(\sigma^2; \theta, x) = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2}\Sigma(x_i-\theta)^2$$

$$l''(\sigma^2; \theta, x) = \frac{n}{2}\cdot\frac{1}{(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}\Sigma(x_i-\theta)^2$$

$$-l''(\sigma^2; \theta, x) = -\frac{n}{2}\frac{1}{(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}\Sigma(x_i-\theta)^2$$

$$I(\sigma^2; \theta) = E\left[-l''(\sigma^2; \theta, x)\right]$$

$$= -\frac{n}{2}\frac{1}{(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}\Sigma E_x\left[(x_i-\theta)^2\right]$$

$$= -\frac{n}{2}\frac{1}{(\sigma^2)^2} + \frac{1}{(\sigma^2)^3} n\sigma^2$$

$$= n\left(-\frac{1}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^2}\right)$$

$$= n\frac{1}{2(\sigma^2)^2} = \frac{n}{2}(\sigma^2)^{-2}$$

$$E_x(x_i-\theta)^2$$
$$= E_x\left(x_i^2 - 2x_i\theta + \theta^2\right)$$
$$= E_x[x_i^2] - 2\theta E[x_i] + \theta^2$$
$$= (\sigma^2 + \cancel{\theta^2}) - \cancel{2\theta^2} + \cancel{\theta^2}$$
$$= \sigma^2$$

$$P_J(\sigma^2|\theta) \propto \sqrt{I(\sigma^2;\theta)} = \sqrt{\frac{n}{2}(\sigma^2)^{-2}} \propto (\sigma^2)^{-1}$$

$$\propto \text{InvGamma}(0,0)$$

(default)

Posterior Predictive distribution:-

Let $\alpha' = \frac{n+n_0}{2}$, $\beta' = \frac{n\hat{\sigma}^2 + n_0\sigma_0^2}{2}$

$$P(X_*|X,\theta) = \int_0^{\infty} P(X_*|\theta,\sigma^2)\, P(\sigma^2|X,\theta)\, d\sigma^2$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_*-\theta)^2} \frac{\beta'^{\alpha'}}{\Gamma(\alpha')} (\sigma^2)^{-\alpha'-1} e^{-\beta'/\sigma^2}\, d\sigma^2$$

$$\propto \int_0^{\infty} (\sigma^2)^{-\frac{1}{2}} e^{-\frac{(x_*-\theta)^2/2}{\sigma^2}} (\sigma^2)^{-\alpha'-1} e^{-\beta'/\sigma^2}\, d\sigma^2$$

$$= \underbrace{\int_0^{\infty} (\sigma^2)^{-\overbrace{(\alpha'+1/2)}^{A}-1} e^{-\frac{\overbrace{(x_*-\theta)^2/2+\beta'}^{B}}{\sigma^2}}\, d\sigma^2}_{\text{Kernel of InvGamma}(A,B)}$$

$$= \frac{\Gamma A}{B^A} \underbrace{\int \frac{B^A}{\Gamma(A)} (\sigma^2)^{-A-1} e^{-B/\sigma^2}\, d\sigma^2}_{1}$$

$$= \Gamma(A)\, B^{-A}$$

$$= \Gamma\left(\frac{n+n_0+1}{2}\right)\left(\frac{n\hat{\sigma}^2 + n_0\sigma_0^2 + (x_* - \theta)^2}{2}\right)^{-\frac{n+n_0+1}{2}}$$

$$\propto \left(\frac{\overbrace{n\hat{\sigma}^2 + n_0\sigma_0^2}^{a} + (x_* - \theta)^2}{2}\right)^{-\frac{\overbrace{n+n_0}^{\mu}+1}{2}}$$

$$= \left(\frac{a}{2}\right)^{-\frac{\mu+1}{2}}\left(\frac{2}{a} \cdot \frac{a + (x_* - \theta)^2}{2}\right)^{-\frac{\mu+1}{2}}$$

$$\propto \left(\frac{2}{a} \cdot \frac{a + (x_* - \theta)^2}{2}\right)^{-\frac{\mu+1}{2}}$$

$$= \left(1 + \frac{(x_* - \theta)^2}{a}\right)^{-\frac{\mu+1}{2}}$$

$$= \left(1 + \frac{1}{\mu}\frac{(x_* - \theta)^2}{a/\mu}\right)^{-\frac{\mu+1}{2}}$$

$$\propto T_\mu\left(\theta, \frac{a}{\mu}\right)$$ Next Page

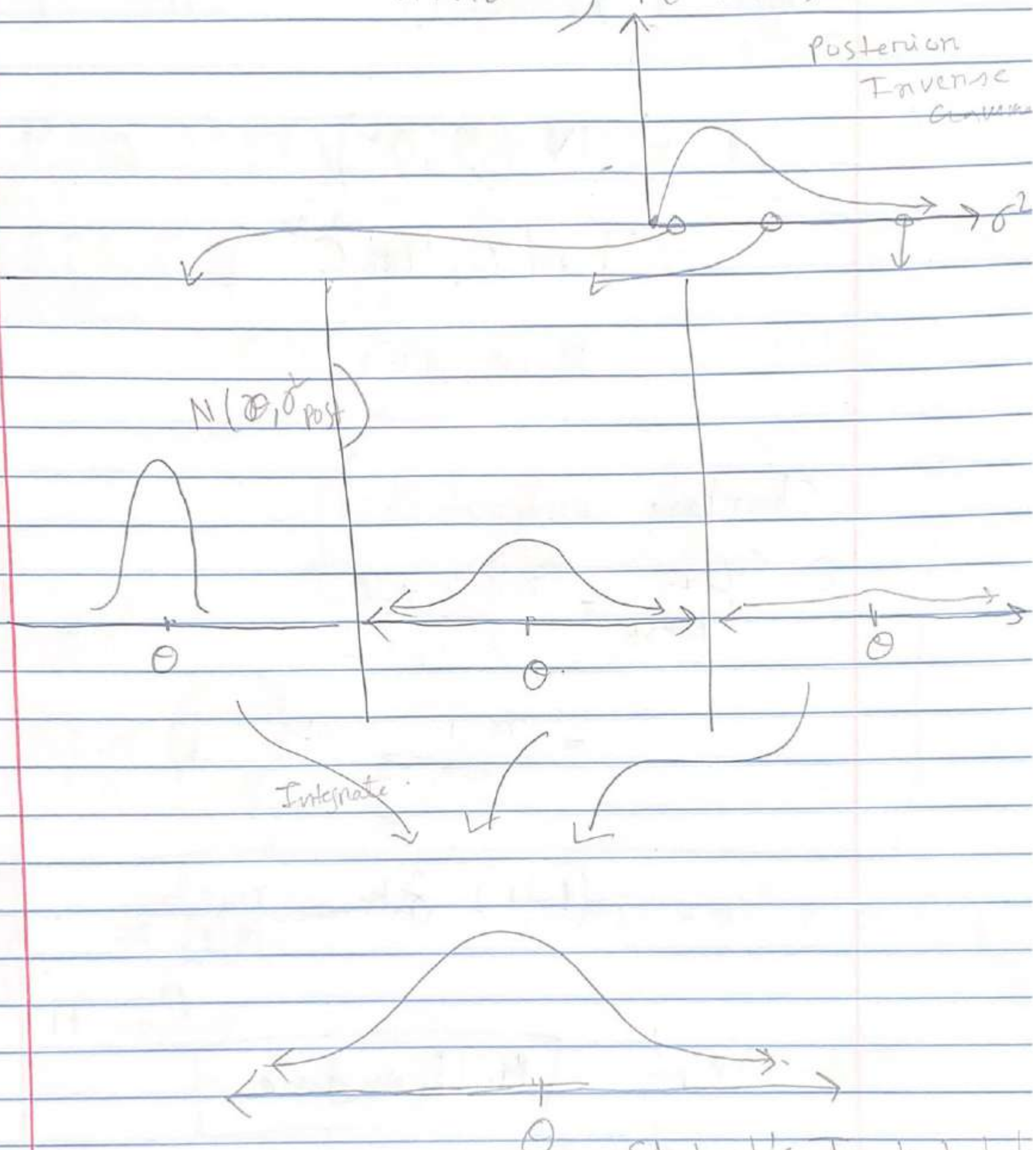
[Student's T distribution
μ degrees of freedom
and location parameter θ (mean)
and scale $\frac{a}{\mu}$]

In stat 101,

$\bar{x}: x_1, \ldots, x_n; \theta, \sigma^2 \stackrel{iid}{\sim} N(\theta, \sigma^2)$

$$H_0: \theta = 0 \Rightarrow \frac{\bar{x} - \theta}{\frac{s}{\sqrt{n}}} = \frac{\bar{x}\sqrt{n}}{s} \sim T_{n-1}(0, 1)$$

$$= T_{n+n_0}\left(\theta, \frac{n\hat{\sigma}^2 + n_0\sigma_0^2}{n+n_0}\right)$$



Student's T distribution which is Normal with thicker tails.

Let's see How it operates under certain limit

$$\lim_{n\to\infty} p(X_* | X, \theta)$$

$$= N(\theta, \sigma^2)$$

$$= T_\infty(\theta, \lim_{n\to\infty} \hat{\sigma}^2)$$

$$= N(\theta, \sigma^2)$$

Shrinkage estimator:-

$$\hat{\sigma}^2_{MMSE} = \frac{n\hat{\sigma}^2 + n_0\sigma_0^2}{n+n_0-2}$$

$$= \underbrace{\frac{n}{n+n_0-2}}_{1-p} \hat{\sigma}^2 + \frac{n_0\sigma_0^2}{n+n_0-2} \cdot \frac{n_0-2}{n_0-2}$$

$$= (1-p)\,\hat{\sigma}^2 + \underbrace{\frac{n_0-2}{n_0+n-2}}_{p} \underbrace{\frac{n_0\sigma_0^2}{m_0-2}}_{E[\sigma^2|\theta]}$$

(Prior expectation)

$$\boxed{\text{Midterm } \#2 \text{ done}}$$

I donot know what todo

Final:

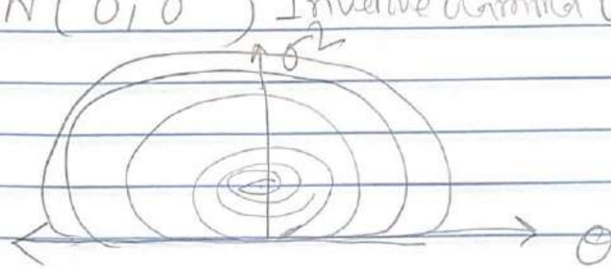
T~: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ but neither θ or σ^2 known.

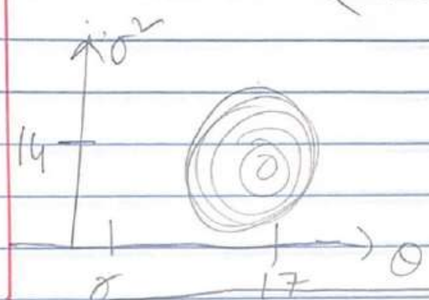
$$P(\theta, \sigma^2 | X) \propto P(X | \theta, \sigma^2)\, P(\theta, \sigma^2)$$

Arna. This is myname

$$P(\theta, \sigma^2) = P(\theta | \sigma^2)\, P(\sigma^2)$$

$$\propto N(0, \sigma^2)\ \text{Inverse Gamma}(1,1)$$





$$= (2\pi\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2}\, P(\theta, \sigma^2)$$

if Laplace

$$\propto (2\pi\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2}$$

$$\propto (\sigma^2)^{-n/2}\, e^{-\frac{\Sigma(x_i-\theta)^2/2}{\sigma^2}}$$

$\overset{?}{\propto}$ Invgamma No!

this is two-dimension