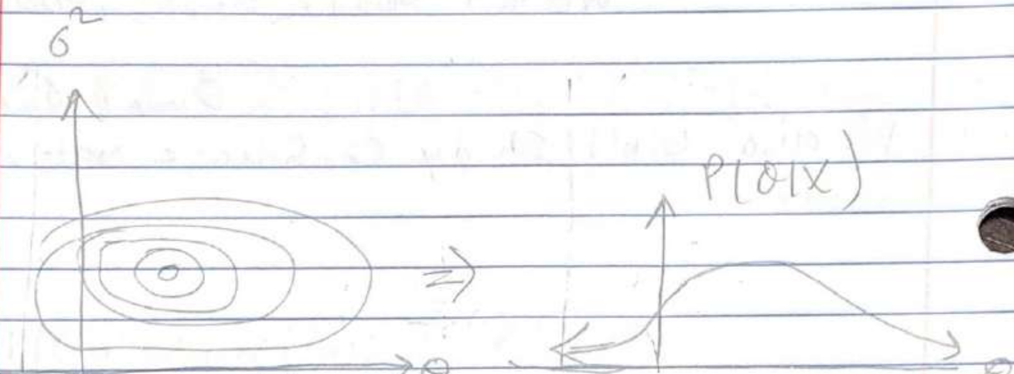
Lecture 18:

F: iid Normal and $\theta, \sigma^2$ unknown    $s^2$: Sample SD $= \frac{1}{n-1}\sum(x_i-\bar{x})^2$

$P_J(\theta,\sigma^2) \propto \frac{1}{\sigma^2}$    $\propto \text{NormInvGamma}\left(\bar{x}, n, \frac{n}{2}, \frac{(n-1)s}{2}\right)$

$\Rightarrow P(\theta, \sigma^2 | x) \propto P(x|\theta,\sigma^2)\, P_J(\theta,\sigma^2)$

$$\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}$$




"marginizing on $\sigma^2$"

collapsing the $\sigma^2$ dimension

$$P(\theta|x) = \int_{\sigma^2} P(\theta,\sigma^2|x)\, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-n/2-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2} d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\overset{\alpha}{\frac{n}{2}}-1} e^{-\frac{\overbrace{(n-1)s^2/2 + n(\theta-\bar{x})^2/2}^{\beta}}{\sigma^2}} d\sigma^2$$

$$= \frac{\Gamma(\alpha)}{\beta^{\alpha}} \underbrace{\int_0^{\infty} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left(\sigma^2\right)^{-\alpha-1} e^{-\frac{\beta}{\sigma^2}} \, d\sigma^2}_{1}$$

$$= \Gamma(\alpha)\, \beta^{-\alpha} = \Gamma\left(\tfrac{n}{2}\right) \left( \frac{(n-1)s^2 + n(\theta-\bar{x})^2}{2} \right)^{-n/2}$$

$$\propto \left(\frac{2}{(n-1)s^2}\right)^{-n/2} \left( \frac{(n-1)s^2 + n(\theta-\bar{x})^2}{2} \right)^{-n/2}$$

$$= \left(1 + \frac{n(\theta-\bar{x})^2}{(n-1)s^2}\right)^{-n/2}$$

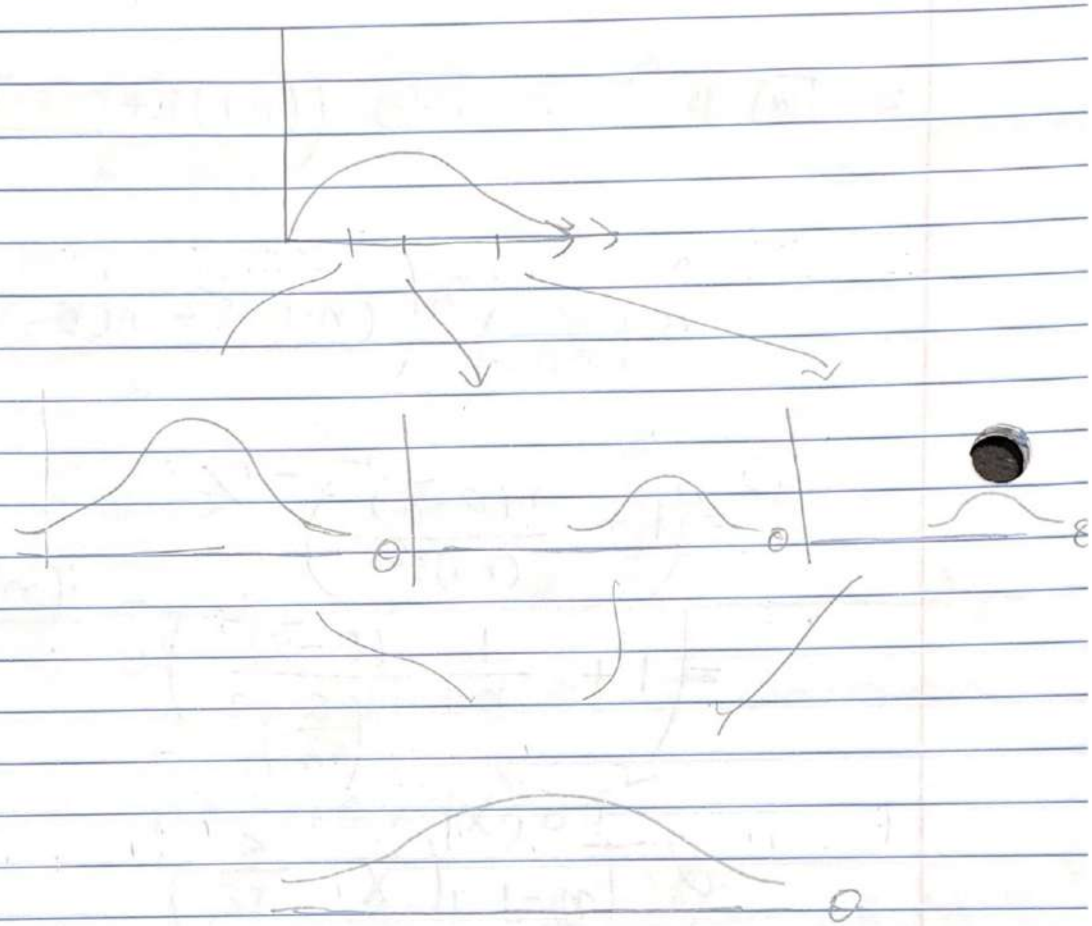
$$= \left(1 + \frac{1}{n-1} \frac{(\theta-\bar{x})^2}{\left(\frac{s}{\sqrt{n}}\right)^2}\right)^{-\frac{((n-1)+1)}{2}}$$

$$\propto T_{n-1}\left(\bar{x}, \frac{s}{\sqrt{n}}\right)$$

$$\overset{n \text{ is large}}{\approx} N\left(\bar{x}, \frac{s^2}{n}\right)$$

$$P(\theta|X) = \int_{\sigma^2} P(\theta,\sigma^2|X)\, d\sigma^2$$

$$= \int_{\sigma^2} \overbrace{P(\theta|X,\sigma^2)}^{\text{Normal}}\ \overbrace{P(\sigma^2|X)}^{\text{inv-Gamma}}\, d\sigma^2,$$ Compound distri




$$P(\sigma^2|X) = \int_{\theta} P(\theta,\sigma^2|X)\, d\theta$$

$$\propto \int_{R} (\sigma^2)^{-n/2-1}\, e^{-\frac{(n-1)s^2/2}{\sigma^2}}\, e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}\, d\theta$$

$$= (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \int_R e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2} d\theta$$

$$= (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \sqrt{2\pi\frac{\sigma^2}{n}} \underbrace{\int_R \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} e^{-\frac{1}{2\frac{\sigma^2}{n}}(\theta-\bar{x})^2} d\theta}_{1}$$

$$\propto (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} (\sigma^2)^{1/2}$$

$$= (\sigma^2)^{-\left(\frac{n-1}{2}\right)-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}}$$

$$\propto \text{Inv Gamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

$$P(\sigma^2|x) = \int_\theta \underbrace{P(\sigma^2|x,\theta)}_{\text{Inv Gamma}} \underbrace{P(\theta|x)}_{T} d\theta$$

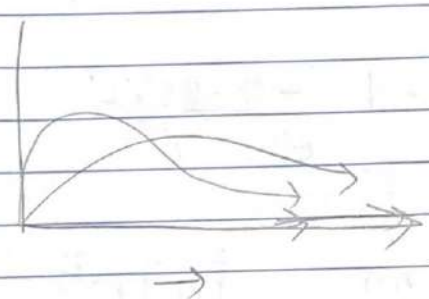
$$= \text{Inv Gamma}$$

Using Jeffery's Prior:

$$P(\theta|X,\sigma^2) = N\left(\bar{X}, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right) \quad \& \quad P(\theta|X) = T_{n-1}\left(\bar{X}, \frac{s}{\sqrt{n}}\right)$$

$$P(\sigma^2|X,\theta) = \text{Inv Gamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$$

$$\& \quad P(\sigma^2|X) = \text{Inv Gamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$



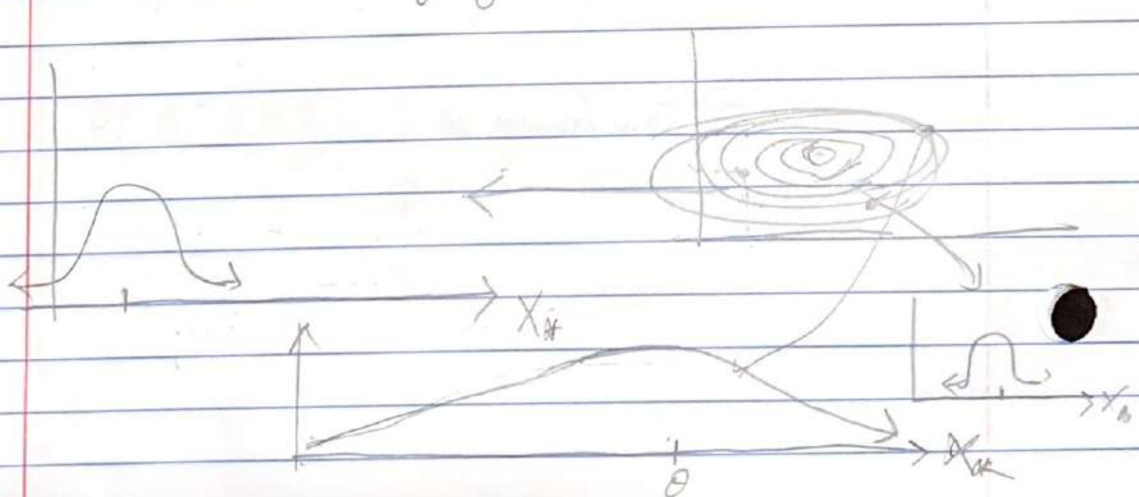
$$n\hat{\sigma}^2_{MLE} = \Sigma(x_i-\theta)^2$$

similar

$$(n-1)s^2 = \Sigma(x_i-\bar{x})^2$$

Posterior Predictive distri.:

$$P(X_*|X) = \int_{\sigma^2}\int_{\theta} \overbrace{P(X_*|\theta,\sigma^2)}^{\text{Normal}}\,\overbrace{P(\theta,\sigma^2|X)}^{\text{Norm-inv-Gamma}}\, d\theta\, d\sigma^2$$



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Last page:

$$P(X_* \mid X) = \int_0^\infty \int_R \left( \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_* - \theta)^2} \right) \cdot \left( (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n(\theta-\bar{x})^2}{2\sigma^2}} \right) d\theta\, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-\frac{1}{2}} (\sigma^2)^{-\frac{n}{2}-1} e^{\frac{-(n-1)s^2}{2\sigma^2}} \int_R e^{-\frac{1}{2\sigma^2}\left((X_*-\theta)^2 + n(\theta-\bar{x})^2\right)} \, d\theta\, d\sigma^2$$

$$(X_*^2 - 2X_*\theta + \underbrace{\theta^2 + n\theta^2}_{(n+1)\theta^2} - 2n\theta\bar{x} + n\bar{x}^2$$

$$e^{-\frac{X_*^2}{2\sigma^2}} e^{\frac{X_*\theta}{\sigma^2}} e^{-\frac{(n+1)\theta^2}{2\sigma^2}} e^{\frac{n\theta\bar{x}}{\sigma^2}} e^{-\frac{n\bar{x}^2}{2\sigma^2}}$$

$$= \int_0^\infty (\sigma^2)^{-\left(\frac{n+1}{2}\right)-1} e^{-\frac{\left((n-1)s^2 + X_*^2 + \bar{x}^2\right)/2}{\sigma^2}} \int_R e^{\overbrace{\frac{X_* + n\bar{x}}{\sigma^2}}^{a}\theta - \overbrace{\frac{n+1}{2\sigma^2}}^{b}\theta^2} d\theta$$

$$\int_R e^{a\theta - b\theta^2} d\theta =$$

From last page.

$$\int_R e^{a\theta - b\theta^2} d\theta = \sqrt{\frac{\pi}{b} e^{\frac{a^2}{4b}}} \propto b^{-1/2} e^{a^2/4b}$$

$$\sqrt{\frac{2\sigma^2}{n+1}} \propto (\sigma^2)^{-1/2}$$

$$\propto \int_0^\infty (\sigma^2)^{-\left(\frac{n+1}{2}\right)-1} \cdot e^{\frac{((n-1)s^2 + X_*^2 + n\bar{x}^2)/2}{\sigma^2}} (\sigma^2)^{-1/2} \cdot e^{\frac{(X_* + n\bar{x})^2}{(\sigma^2) 4\left(\frac{n+1}{2\sigma^2}\right)}} d\sigma^2$$

$\beta$

$$= \int_0^\infty (\sigma^2)^{-\overbrace{\left(\frac{n+2}{2}\right)}^{\alpha}-1} e^{-\frac{\overbrace{((n-1)s^2 + X_*^2 + n\bar{x}^2) - (X_* + n\bar{x})^2/(n+1)}^{}}{\sigma^2}/2} d\sigma^2$$

$$= \frac{\Gamma(\alpha)}{\beta^\alpha} \overbrace{\int_0^\infty (\sigma^2)^{-\alpha-1} e^{-\beta/\sigma^2} d\sigma^2}^{1}$$

$$= \Gamma(\alpha) \beta^{-\alpha}$$

$$= \Gamma\left(\frac{n+2}{2}\right) \beta^{-\frac{n+2}{2}}$$

$c = \frac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)$

$a = \frac{1}{2} - \frac{1}{2n+2}$

$= \frac{1}{2}\left(1 - \frac{1}{n+1}\right)$

$= \frac{1}{2}\left(\frac{n}{n+1}\right)$

$b = \frac{n\bar{x}}{n+1}$

$$= \left(\frac{(n-1)s^2}{2} + \frac{X_*^2}{2} + \frac{n\bar{x}^2}{2} + \frac{-X_*^2 - 2n\bar{x}X_* - n^2\bar{x}^2}{2(n+1)}\right)^{-n/2}$$

$$= (aX_*^2 + bX_* + c)^{-n/2}$$

$$\propto \left(\frac{1}{a}\right)^{-n/3} (aX_*^2 + bX_* + c)^{-n/2}$$

$$= X_*^2 + \left(\frac{b}{a} X_* + \frac{c}{a}\right)^{-n/2}$$

$$= \left(\left(X_* + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)^{-n/2}$$

$$\propto \left(\frac{1}{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^{-n/2} \left(\left(X_* + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right)^{-n/2}$$

$$= \left(1 + \frac{(X_* + b/2a)}{\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)}\right)^{-n/2}$$

$p(x_* | x, \sigma^2) = N\left(\hat{\theta}_{p}, \hat{\sigma} + \frac{\sigma^2}{n + n_0}\right)$

$$= \left(1 + \frac{1}{n-1} \frac{\left(X_* + b/2a\right)^2}{\left(\frac{\frac{c}{a} - \frac{b^2}{4a^2}}{n-1}\right)}\right)^{-\frac{(n-1)+1}{2}}$$

$$\propto T_{n-1}\left(-\frac{b}{2a}, \frac{\frac{c}{a} - \frac{b^2}{4a^2}}{n-1}\right)$$

$$\boxed{-\frac{b}{2a} = \frac{\frac{n\bar{x}}{(n+1)}}{2\left(\frac{1}{2} \cdot n/n+1\right)} = \bar{X}}$$

$$\frac{c}{a} = \frac{\frac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n\bar{x}^2}{n+1}\right)}{\frac{1}{2} \frac{n}{n+1}} \cdot \frac{\frac{n+1}{n}}{n+1}$$

$$= T_{n-1}\left(\bar{X}, \sqrt{\frac{n+1}{n}}\, s\right)$$

$$\frac{\cancel{n}\bar{x} + \bar{x}^2}{(n+1)\bar{X}^2 - \cancel{n}\bar{X}^2} = \frac{(n-1)(n+1)s^2}{} + \quad =$$

$$\frac{c}{a} = \frac{(n-1)(n+1)}{n} s^2 + \bar{x}^2$$

$$\frac{b^2}{4a^2} = (\bar{x})^2$$

$$\frac{c}{a} - \frac{b^2}{4a^2} = \frac{(n-1)(n+1)}{s^2} + \bar{x}^2 - \bar{x}^2$$

$$\sqrt{\frac{\frac{c}{a} - \frac{b^2}{4a^2}}{n-1}} = \sqrt{\frac{n+1}{n}} \cdot s$$

n is large

$$\approx N(\bar{x}, s^2)$$

exactly what you expect.

$$T_{n-1}\left(\bar{x}, \sqrt{\frac{n+1}{n}}\, s\right)$$

$$\downarrow$$

$$\sqrt{s^2 + \frac{s^2}{n}}$$