Lecture 13

03/24 Et [MATH 341]

$\mathcal{F}$: Poisson(θ), $\theta \sim$ Gamma(α, β) (continuous) $\Rightarrow$ $\theta | X \sim$ Gamma$(\Sigma x_i + \alpha, n+\beta)$

$n_* = 1$ $\quad P(X_* | X) = \int_{\Theta} P(X_* | \theta) P(\theta | X) \, d\theta$

$$= \int_0^\infty \left(\frac{e^{-\theta} \theta^{x_*}}{x_*!} \right) \left(\frac{(n+\beta)^{\Sigma x_i + \alpha}}{\Gamma(\Sigma x_i + \alpha)} \theta^{\Sigma x_i + \alpha - 1} e^{-(n+\beta)} \right) d\theta.$$

$$= \frac{(n+\beta)^{\Sigma x_i + \alpha}}{x_*! \, \Gamma(\Sigma x_i + \alpha)} \int_0^\infty \theta^{\Sigma x_i + x_* + \alpha - 1} e^{-(n+\beta+1)\theta} d\theta$$

let $t = (n+\beta+1)\theta \Rightarrow \frac{dt}{d\theta} = n+\beta+1$, $\theta = \frac{t}{n+\beta+1}$, $d\theta = \frac{dt}{n+\beta+1}$

$$= \frac{(n+\beta)^{\Sigma x_i + \alpha}}{x_*! \, \Gamma(\Sigma x_i + \alpha)} \cdot \int_0^\infty \left(\frac{t}{(n+\beta+1)} \right)^{\Sigma x_i + x_* + \alpha - 1} e^{-t} \frac{dt}{n+\beta+1}$$

$$= \frac{(n+\beta)^{\Sigma x_i + \alpha}}{x_*! \, \Gamma(\Sigma x_i + \alpha)} \cdot \frac{1}{(n+\beta+1)^{\Sigma x_i + x_* + \alpha - 1}} \cdot \frac{1}{(n+\beta+1)^1} \left(\int_0^\infty t^{\Sigma x_i + x_* + \alpha - 1} \cdot e^{-t} dt \right)$$

$= \Gamma(\Sigma x_i + x_* + \alpha)$

$$= \frac{(n+\beta)^{\Sigma x_i + \alpha}}{x_*! \, \Gamma(\Sigma x_i + \alpha)} \cdot \frac{\Gamma(\Sigma x_i + x_* + \alpha)}{\underbrace{(n+\beta+1)^{\Sigma x_i + x_* + \alpha}}_{= (n+\beta+1)^{\Sigma x_i + \alpha} (n+\beta+1)^{x_*}}}$$

$$= \left(\frac{n+\beta}{n+\beta+1} \right)^{\Sigma x_i + \alpha} \left(\frac{1}{n+\beta+1} \right)^{x_*} \frac{\Gamma(\Sigma x_i + x_* + \alpha)}{x_*! \, \Gamma(\Sigma x_i + \alpha)}$$

let $p := \frac{n+\beta}{n+\beta+1} \in (0,1)$ $\qquad$ let $r := \Sigma x_i + \alpha$

$\Rightarrow 1-p = \frac{1}{n+\beta+1} \in (0,1)$

$$= \frac{\Gamma(x_* + r)}{x_*! \, \Gamma(r)} p^r (1-p)^{x_*} = \text{ExtNegBin}(r, p)$$

[extended negative binomial model]

- If $r = \Sigma x_i + \alpha \in \mathbb{N}_0 \Rightarrow \alpha \in \mathbb{N}_0$

$$\binom{x_* + r - 1}{x_*} p^r (1-p)^{x_*} = \text{NegBin}(r, p)$$

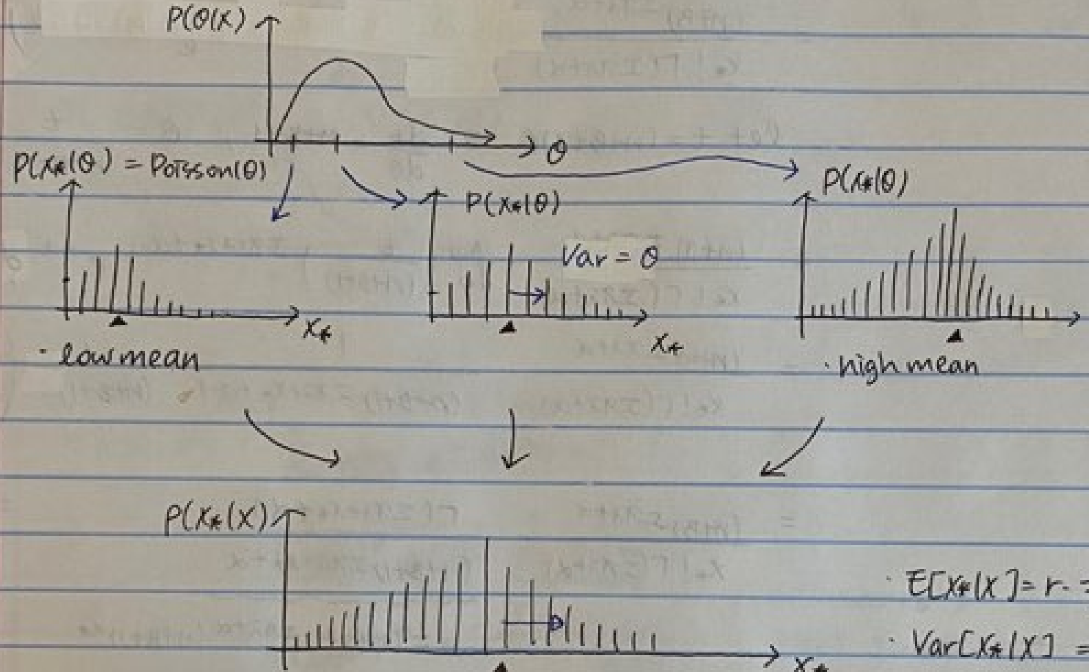
recall $\Gamma(x) = (x-1)!$ if $x \in \mathbb{N}$

$X_1, \ldots X_r ; p \overset{iid}{\sim} Geom(p) := (1-p)^x p \qquad E[X_i] = \frac{p}{1-p}$

$\sum_{i=1}^{r} x_i \sim NegBin(r, p)$

· $E[\sum x_i] = r \cdot \frac{p}{1-p}$

$X_* | X \sim ExtNegBin(r, p)$: Overdispersed Poisson.



· $E[X_*|X] = r \cdot \frac{p}{1-p} = \mu$

· $Var[X_*|X] = \frac{p \cdot r}{(1-p)^2} = \frac{1}{1-p}\mu$

$\frac{1}{1-p} \in (1, \infty)$

Normal Model $X \sim N(\theta, \sigma^2) = N(\theta_1, \theta_2)$

· $\dim[\vec{\theta}] = 2$

∘ Pretend we know σ^2, we want inference for the ____, θ.

kernel practice ..

$$N(\theta, \sigma^2) = P(X|\theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} \propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$$

$$= e^{-\frac{x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}}$$

$$\propto e^{-\frac{x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}}$$

$$= e^{ax - bx^2} \quad \text{where } a = \frac{\theta}{\sigma^2} \text{ and } b = \frac{1}{2\sigma^2} > 0$$

$$\Rightarrow \sigma^2 = \frac{1}{2b}, \ \theta = \frac{a}{2b}$$

$$\propto N\left(\frac{a}{2b}, \frac{1}{2b}\right)$$

$\quad$ "θ" "σ^2"

"Posterior"

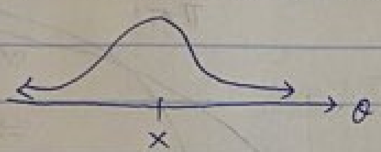
$P(\theta|x,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$

$\propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2} = e^{-\frac{x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}}$

$\propto e^{\frac{x\theta}{\sigma^2}-\frac{\theta^2}{2\sigma^2}} = e^{a\theta - b\theta^2}$, where $a = \frac{x}{\sigma^2}$, $b = \frac{1}{2\sigma^2}$

$\propto N(\frac{a}{2b}, \frac{1}{2b}) = N(x, \sigma^2)$ ↳ $\frac{a}{2b} = \frac{\frac{x}{\sigma^2}}{2\cdot\frac{1}{2\sigma^2}} \cdot \frac{\sigma^2}{\sigma^2} = x$

$\Rightarrow P(\theta|x,\sigma^2) = \overset{\theta_n}{N}(x, \sigma^2)$



F: $x_1, \dots x_n;\ \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$P(x|\theta,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2}$

$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2}$ → $\sum(x_i^2 - 2\theta x_i + \theta^2)$

$= \sum x_i^2 - \sum 2\theta x_i + \sum \theta^2$

$= \sum x_i^2 - 2\theta \sum x_i + n\theta^2$

$= \sum x_i^2 - 2\theta n\bar{x} + n\theta^2$

$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}(\sum x_i^2 - 2\theta n\bar{x} + n\theta^2)}$

$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum x_i^2}{2\sigma^2}} e^{\frac{\theta n\bar{x}}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}}$

$P(\theta|x,\sigma^2) \propto e^{-\overbrace{\frac{n\bar{x}}{\sigma^2}}^{a}\theta - \overbrace{\frac{n}{2\sigma^2}}^{b}\theta^2}$

$\propto N(\bar{x}, \frac{\sigma^2}{n})$

· $Var = \frac{1}{2b} = \frac{1}{2(\frac{n}{2\sigma^2})} = \frac{1}{\frac{n}{\sigma^2}} = \frac{\sigma^2}{n}$

· $mean = \frac{a}{2b} = \frac{\frac{n\bar{x}}{\sigma^2}}{2(\frac{n}{2\sigma^2})} = \bar{x}$

Laplace uninformative prior $P(\theta) = U(\Theta)$ · if $\Theta = (0,1)$

$P(\theta) \propto 1$ $\Rightarrow P(\theta) = 1$

· if $\Theta = (0,100)$

$\Rightarrow P(\theta) = \frac{1}{100} \propto 1$

Poisson-Gamma

amazing trick.

$$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)} \propto P(X|\theta)\underbrace{P(\theta)}_{\propto 1} \propto P(X|\theta)$$

$$= \frac{e^{-n\theta}\theta^{\Sigma x_i}}{\Pi x_i!} \propto e^{-n\theta}\theta^{\Sigma x_i + 1 - 1}$$

$$\propto \text{Gamma}(\Sigma x_i + 1, n)$$

$$\Rightarrow \theta \sim \text{Gamma}(1, 0).$$

Generally $P(\theta) = \text{Gamma}(\alpha, B)$

$$\Rightarrow P(\theta|X) = \text{Gamma}(\Sigma x_i + \alpha, n + B)$$

F: $X_1 \dots X_n ; \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 known.

prior on θ is Laplace $\Rightarrow P(\theta) \propto 1$ improper!

$$P(\theta|X, \sigma^2) \propto P(X|\theta, \sigma^2) \propto N(\bar{X}, \frac{\sigma^2}{n})$$