Lecture 19 04/30/2020

$$a = \tfrac{1}{2} - \tfrac{1}{2n+2} = \tfrac{1}{2}\left(1 - \tfrac{1}{n+1}\right) = \tfrac{1}{2}\,\tfrac{n}{n+1}$$

$$b = \tfrac{n\bar{x}}{n+1} \quad ; \quad c = \tfrac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)$$

$$P(x_* \mid x) \propto T_{n-1}\left(\tfrac{-b}{2a}, \sqrt{\frac{c/a - b^2/4a^2}{n-1}}\right)$$

$$= T_{n-1}\left(\bar{x}, \sqrt{\tfrac{n+1}{n}}\, s\right) \overset{n \text{ large}}{\approx} N(\bar{x}, s^2)$$ exactly what you expect

$$-\tfrac{b}{2a} = \frac{\frac{n\bar{x}}{n+1}}{\frac{n}{n+1}} = \bar{x}$$

$$\frac{c}{a} = \frac{\tfrac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)}{\tfrac{1}{2}\,\tfrac{n}{n+1}} \cdot \frac{n+1/n}{n+1/n}$$

$$= \frac{(n+1)(n-1)}{n} s^2 + \underbrace{(n+1)\bar{x}^2}_{n\bar{x}^2 + \bar{x}^2} - n\bar{x}^2$$

$$= \frac{(n-1)(n+1)}{n} s^2 + \bar{x}^2$$

$$\frac{b^2}{4a^2} = \left(\frac{-b}{2a}\right)^2 = \bar{x}^2$$

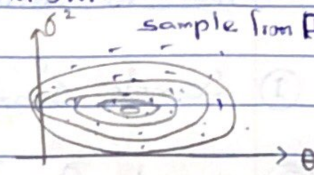
$$\frac{c}{a} - \frac{b^2}{4a^2} = \frac{(n-1)(n+1)}{n} s^2 + \bar{x}^2 - \bar{x}^2$$

$$\sqrt{\frac{c/a - b^2/4a^2}{n-1}} = \sqrt{\frac{\frac{(n-1)(n+1)}{n} s^2}{n-1}} = \sqrt{\frac{(n+1)s^2}{n}} = \sqrt{\frac{n+1}{n}}\, s$$

$$P(x_* \mid x, \sigma^2) = N\left(\theta_p, \sigma^2 + \sigma^2/n+n_0\right)$$

$$P(x_* \mid x) = T_{n-1}\left(\bar{x}, \sqrt{s^2 + s^2/n}\right)$$

$F: \overset{iid}{\sim} N(\theta, \sigma^2)$ where both θ, σ^2 are unknown



$$P(\theta, \sigma^2 \mid x) = \underbrace{P(\theta \mid x, \sigma^2)\, P(\sigma^2 \mid x)}$$

if $P_0(\theta, \sigma^2) \propto 1/\sigma^2$

$$= \left(N(\bar{x}, \sigma^2/n)\right)\left(\text{Inv Gamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)\right)$$

realization

How would I sample $\langle \theta, \sigma^2 \rangle$ from $P(\theta, \sigma^2 \mid x)$?

Step I: Draw a σ^2_{samp} realization from $P(\sigma^2 \mid x)$ using

sample one realization

rinvgamma $\left(n-1/2, (n-1)s^2/2\right)$

Step II: Draw a θ_{samp} realization from $P(\theta \mid x, \sigma^2 = \sigma^2_{samp})$ using rnorm $\left(\bar{x}, \sqrt{\sigma^2_{samp}/n}\right)$

return $\langle \theta_{samp}, \sigma^2_{samp} \rangle$ To sample n realizations, repeat n times.

How to sample from $P(X_* \mid x) = T_{n-1}\left(\bar{x}, \sqrt{\frac{n+1}{n}}\, s\right)$?

rt.scaled $\left(n-1, \bar{x}, \sqrt{\frac{n+1}{n}}\, s\right)$

dependence

$$P(X_* \mid X) = \int_\theta \int_{\sigma^2} P(X_*, \theta, \sigma^2 \mid x)\, d\sigma^2\, d\theta$$

$$= \int_\theta \int_{\sigma^2} P(X_* \mid \theta, \sigma^2)\, P(\theta, \sigma^2 \mid x)\, d\sigma^2\, d\theta$$

$$= \int_\theta \int_{\sigma^2} P(X_* \mid \theta, \sigma^2)\, P(\theta \mid x, \sigma^2)\, P(\sigma^2 \mid x)\, d\sigma^2\, d\theta.$$

How to sample from $P(X_*, \theta, \sigma^2 | x)$?

① Sample σ^2_{samp} from $P(\sigma^2|x)$ via rinvgamma$\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

Ⅱ Sample θ_{samp} from $P(\theta|X, \sigma^2 = \sigma^2_{samp})$ via rnorm$\left(\bar{x}, \frac{\sqrt{\sigma^2_{samp}}}{n}\right)$

Ⅲ Sample X_{*samp} from $P(X_*|\theta = \theta_{samp}, \sigma^2 = \sigma^2_{samp})$ via rnorm $(\theta_{samp}, \sigma^2_{samp})$

return $\langle X_{*samp}, \theta_{samp}, \sigma^2_{samp} \rangle$

To sample from $P(X_*|x)$ you sample from $P(X_*, \theta, \sigma^2|x)$ and ignore θ_{samp}, σ^2_{samp} to leave you with X_{*samp}.

To sample n realizations, repeat n times.

$P(\theta, \sigma^2|x) \propto \underbrace{P(\theta|x, \sigma^2)}\,\underbrace{P(\sigma^2|x)}$ = NormInvGamma due to conjugacy

If $P(\theta, \sigma^2) \neq$ NormInvGamma $\Rightarrow$ non conjugate.

dependence

NormInvGamma $= P(\theta, \sigma^2) = P(\theta|\sigma^2)\,P(\sigma^2)$ where $P(\theta|\sigma^2) = N(\mu_0, \sigma^2/n_0)$, $P(\sigma^2) =$ InvGamma$(n_0/2, n_0\sigma_0^2/2)$ then model is conjugate.

What if.

$P(\theta, \sigma^2) = P(\theta)\,P(\sigma^2)$ where $P(\theta) = N(\mu_0, \tau^2)$, $P(\sigma^2) =$ InvGamma$(n_0/2, n_0\sigma_0^2/2)$ s.t. $\tau^2 \neq \sigma^2/n_0$

$$\Rightarrow p(\theta, \sigma^2 | x) \propto p(x|\theta,\sigma^2)\, p(\theta,\sigma^2)$$

$$= p(x|\theta,\sigma^2)\, p(\theta)\, p(\sigma^2) \propto k(x|\theta,\sigma^2)\, k(\theta)\, k(\sigma^2)$$

$$= \left((\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + \overbrace{n(\bar{x}-\theta)^2}^{n\bar{x}^2 - 2n\bar{x}\theta + n\theta^2}\right)}\right)\left(e^{-\frac{\theta^2}{2\tau^2}} e^{\frac{\theta \mu_0}{\tau^2}}\right)\left((\sigma^2)^{-\frac{n_0}{2}-1} e^{-\frac{n_0\sigma_0^2}{2\sigma^2}}\right)$$

$$= (\sigma^2)^{-\frac{n+n_0}{2}-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n\bar{x}^2 + n_0\sigma_0^2\right)} \underbrace{e^{\overbrace{\left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}\right)}^{a}\theta - \overbrace{\left(\frac{n}{2\sigma^2} + \frac{1}{2\tau^2}\right)}^{b}\theta^2}}_{e^{a\theta - b\theta^2}} = e^{a\theta - b\theta^2} \propto N\left(\frac{a}{2b}, \frac{1}{2b}\right)$$

$$= (\sigma^2)^{-\frac{n+n_0}{2}-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n\bar{x}^2 + n_0\sigma_0^2\right)} \left(\sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}} N\left(\frac{a}{2b}, \frac{1}{2b}\right)\right)$$

$$\underbrace{\propto (\sigma^2)^{-\frac{n+n_0}{2}-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n\bar{x}^2 + n_0\sigma_0^2\right)} \left(\frac{n}{2\sigma^2} + \frac{1}{2\tau^2}\right)^{-1/2} e^{\frac{\left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}\right)^2}{2\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)}}}_{}$$

$K(\sigma^2|X)$, the kernal of some unknown distribution and we don't know how to draw realization from it.

$$\underbrace{\left(N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right)\right)}_{p(\theta|\sigma^2, x)}$$