Lecture 15

F: iid normal (θ, σ^2) with σ^2 known.

$$P(\theta \mid \sigma^2) = N(\mu_0, \tau^2) = N\left(\mu_0, \frac{\sigma^2}{n_0}\right)$$

Imagine Predation $y_1, \ldots, y_{n_0} \overset{iid}{\sim} N(\mu_0, \sigma^2) \Rightarrow \bar{Y} \sim N\left(\mu_0, \frac{\sigma^2}{n_0}\right)$

$$\Rightarrow P(\theta \mid X, \sigma^2) = N(\hat{\theta}_p, \sigma_p^2) \text{ where } \hat{\theta}_p = \frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$= \frac{n\bar{X} + n_0\mu_0}{n + n_0}$$

$$\sigma_p^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{\sigma^2}{n + n_0}$$

$n_* = 1$

$$P(X_* \mid X, \sigma^2) = \int_{\Theta} P(X_* \mid \theta, \sigma^2)\, P(\theta \mid X, \sigma^2)\, d\theta$$

$$= \int_{R} N(\theta, \sigma^2)\, N(\hat{\theta}_p, \sigma_p^2)\, d\theta$$

$$= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_* - \theta)^2}\right) \cdot \left(\frac{1}{\sqrt{2\pi\sigma_p^2}} e^{-\frac{1}{2\sigma_p^2}(\theta - \hat{\theta}_p)^2}\right) d\theta$$

my that's not a func of x_* is kind here.

$$\propto \int_R e^{-\frac{1}{2\sigma^2}(x_*-\theta)^2} e^{-\frac{1}{2\sigma_p^2}(\theta-\hat{\theta}_p)^2} d\theta$$

$$= \int_R e^{-\frac{x_*^2}{2\sigma^2}} e^{\frac{x_*\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} \cdot e^{-\frac{\theta^2}{2\sigma_p^2}} e^{\frac{\theta\hat{\theta}_p}{\sigma_p^2}} \cdot e^{-\frac{\hat{\theta}_p^2}{2\sigma_p^2}} d\theta$$

$$\propto e^{-\frac{x_*^2}{2\sigma^2}} \int_R e^{\frac{x_*\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} e^{-\frac{\theta^2}{2\sigma_p^2}} e^{\frac{\theta\hat{\theta}_p}{\sigma_p^2}} d\theta$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} \int_R e^{\overbrace{\left(\frac{x_*}{\sigma^2}+\frac{\hat{\theta}_p}{\sigma_p^2}\right)}^{a}\theta - \overbrace{\left(\frac{1}{2\sigma^2}+\frac{1}{2\sigma_p^2}\right)}^{b}\theta^2} d\theta.$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} \sqrt{\frac{\pi}{b}} \, e^{\frac{a^2}{4b}} \overbrace{\int \sqrt{\frac{b}{\pi}} e^{-\frac{a^2}{4b}} e^{a\theta-b\theta^2} d\theta}^{1}.$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} \sqrt{\frac{\pi}{b}} e^{\frac{a^2}{4b}} \propto e^{-\frac{x_*^2}{2\sigma^2}} e^{\left(\frac{x_*}{\sigma^2}+\frac{\hat{\theta}_p}{\sigma_p^2}\right)^2 / 4b}$$

Next page.

$$\boxed{\theta \sim N\left(\frac{a}{2b}, \frac{1}{2b}\right) = \frac{1}{\sqrt{2\pi\frac{1}{(2b)}}} e^{-\frac{1}{2\left(\frac{1}{2b}\right)}\left(\theta-\frac{a}{2b}\right)^2}}$$

$$= \sqrt{\frac{b}{\pi}} e^{-b\left(\theta^2 - \frac{a\theta}{b} + \frac{a^2}{4b^2}\right)}$$

$$= \sqrt{\frac{b}{\pi}} e^{b\theta^2} e^{a\theta} e^{-\frac{a^2}{4b}}$$

$$= \underbrace{\sqrt{\frac{b}{\pi}} e^{-\frac{a^2}{4b}}}_{c} \underbrace{e^{a\theta-b\theta^2}}_{K(\theta)}.$$

$$\propto e^{a\theta-b\theta^2}.$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} e^{\frac{x_*^2}{4\sigma^{2^2} b}} e^{\frac{x_* \hat{\theta}_p}{2\sigma_p^2 \sigma^2 b}} e^{\frac{\hat{\theta}_p^2}{4\sigma_p^{2^2} b}}$$

$$\propto e^{\underbrace{\frac{\hat{\theta}_p}{2\sigma^2\sigma_p^2 b}}_{A} x_* - \underbrace{\left(\frac{1}{2\sigma^2} - \frac{1}{4\sigma^{2^2} b}\right)}_{B} x_*^2}$$

$$= e^{A x_* - B x_*^2} \propto N\left(\frac{A}{2B}, \frac{1}{2B}\right)$$

finally $\rightarrow$ $\boxed{p(x_* \mid x, \sigma^2) = N(\hat{\theta}_p, \sigma_p^2 + \sigma^2)}$.

$$= N\left(\hat{\theta}_p, \left(1 + \frac{1}{n+n_0}\right)\sigma^2\right) \longrightarrow \text{Next page.}$$

$$* \; 2b\sigma^2 = 2\left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma_p^2}\right)\sigma^2$$

$$= 1 + \frac{\sigma^2}{\sigma_p^2}$$

$$* \; \frac{1}{2B} = \frac{1}{2\left(\frac{1}{2\sigma^2} - \frac{1}{4\sigma^{2^2} b}\right)} = \frac{1}{\frac{1}{\sigma^2} - \frac{1}{2b\sigma^2\sigma^2}}$$

$$= \frac{\sigma^2}{\sigma^2 - \frac{1}{2b}} = \frac{2b\sigma^2\sigma^2}{2b\sigma^2 - 1} = \frac{\left(1 + \frac{\sigma^2}{\sigma_p^2}\right)\sigma^2}{\left(1 + \frac{\sigma^2}{\sigma_p^2}\right) - 1}$$

$$= \frac{\left(1 + \frac{\sigma^2}{\sigma_p^2}\right)\sigma^2}{\frac{\sigma^2}{\sigma_p^2}} = \sigma_p^2\left(1 + \frac{\sigma^2}{\sigma_p^2}\right)$$

$$= \sigma_p^2 + \sigma^2$$

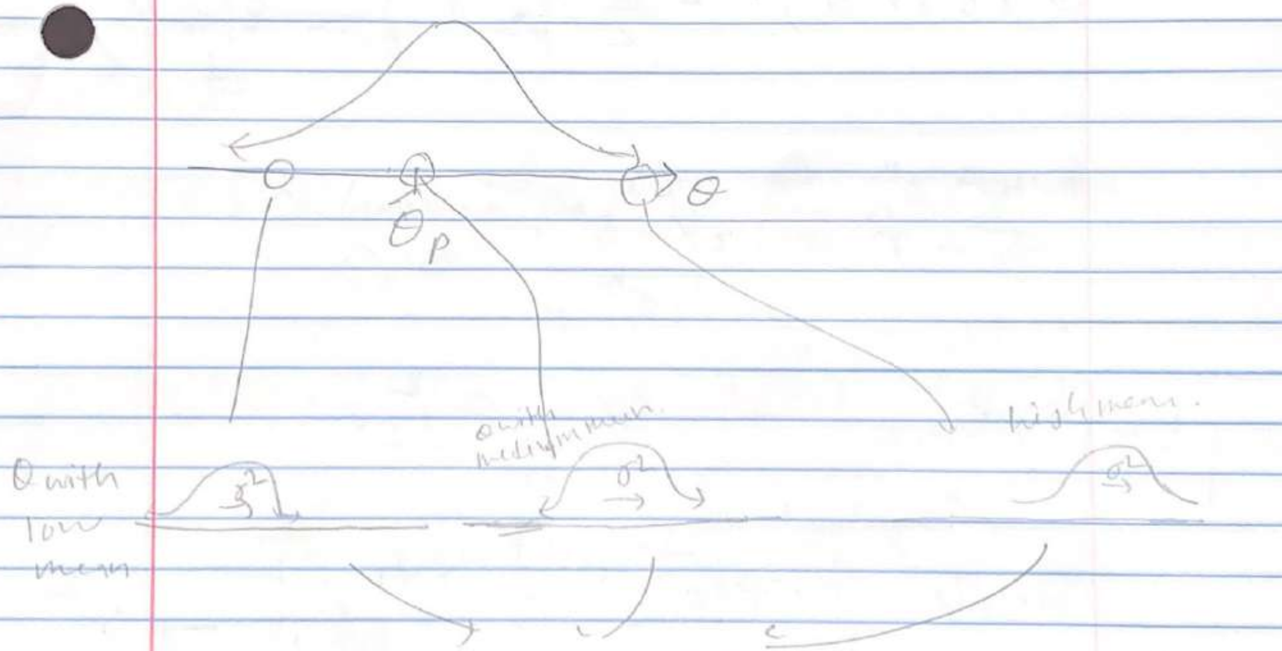
$$\frac{A}{2B} = A\left(\sigma_p^2+\sigma^2\right) = \frac{\hat{\theta}_p}{2b_0\sigma^2\sigma_p^2}\left(\sigma_p^2+\sigma^2\right)$$

$$= \frac{\hat{\theta}_p}{\left(1+\frac{\sigma^2}{\sigma_p^2}\right)\sigma_p^2}\left(\sigma_p^2+\sigma^2\right) = \frac{\hat{\theta}_p}{\cancel{\sigma_p^2+\sigma^2}}\cancel{\left(\sigma_p^2+\sigma^2\right)}$$

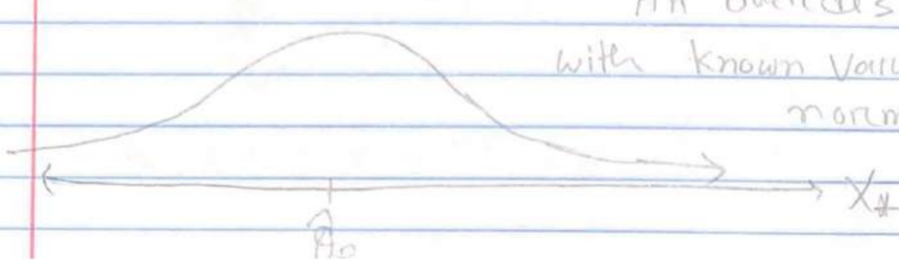
$$= \hat{\theta}_p$$

$$p(x_* \mid x, \sigma^2) = N\left(\hat{\theta}_p, \left(1+\frac{1}{n+n_0}\right)\sigma^2\right) \to N(\theta, \sigma^2)$$

$$\hat{\theta}_p \to \theta, \quad \sigma_p^2+\sigma^2 \to \sigma^2.$$



An overdispersed normal with known variance is normal.



New unit:

F, iid (N, σ^2) with θ known

Log likelihood $L(\sigma^2; x, \theta) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2}$

$$\ell(\sigma^2; x, \theta) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2$$

$$\ell'(\sigma^2; x, \theta) = -\frac{n}{\cancel{2}}\frac{1}{\sigma^2} + \frac{1}{\cancel{2}\sigma^4}\Sigma(x_i-\theta)^2 \overset{set}{=} 0$$

multiply both sides by σ^2

$$\Rightarrow -n + \frac{\Sigma(x_i-\theta)^2}{\sigma^2} = 0$$

$$\Rightarrow \hat{\sigma}^2_{MLE} = \frac{\Sigma(x_i-\theta)^2}{n}$$

↑similar to this

Sample variance: $S^2 = \frac{\Sigma(x_i-\bar{x})^2}{(n-1)} \to \sigma^2$

Laplace Prior: $P(\sigma^2|\theta) \propto 1$, improper

Since $\sigma^2 \in (0, \infty)$

$$P(\sigma^2 | X, \theta) \propto P(X|\theta, \sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2_{MLE}}{2\sigma^2}}$$

$$\propto \underbrace{(\sigma^2)^{-n/2}}_{\alpha} e^{-\frac{n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}$$

$$= (\sigma^2)^{-(n/2-1)-1} e^{-\frac{n\hat{\sigma}^2_{MLE}/2 - \beta}{\sigma^2}}$$

$$= \text{InvGamma}\left(\frac{n-2}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$$

$$U \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u}$$

$$V = \frac{1}{U} = t(U) \sim f_U(t^{-1}(V)) \left| \frac{d}{dV}[t^{-1}(V)] \right|$$

$$\boxed{t^{-1}(V) = \frac{1}{V}} \quad = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{V}\right)^{\alpha-1} e^{-\beta/V} \left| -\frac{1}{V^2} \right|$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} V^{-\alpha+1} V^{-2} e^{-\beta/V}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} V^{-\alpha-1} e^{-\beta/V} = \text{InvGamma}(\alpha, \beta)$$

$$\propto k(\alpha) = V^{-\alpha-1} e^{-\beta/V}$$

inverse gamma r.v.

$$P(\sigma^2|X,\theta) \propto P(X|\theta,\sigma^2)\, P(\sigma^2|\theta) \propto (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}\, K(\sigma^2|\theta)$$

$K(\sigma^2|\theta) = ?$ to get conjugacy?

$$= (\sigma^2)^a e^{-\frac{b}{\sigma^2}} \quad \propto \text{Inverse Gamma in the conjugate prior.}$$

$$= (\sigma^2)^{-n/2+a} e^{-\frac{n\hat{\sigma}^2/2+b}{\sigma^2}}$$

Let $P(\sigma^2|\theta) = \text{InvGamma}(\alpha,\beta)$

$$\Rightarrow P(\sigma^2|X,\theta) \propto \left((\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2_{MLE}}{\sigma^2}}\right)\cdot \left((\sigma^2)^{-\alpha-1} e^{-\beta/\sigma^2}\right)$$

$$\Rightarrow (\sigma^2)^{-(n/2+\alpha)-1} e^{-\frac{(n\hat{\sigma}^2_{MLE}/2+\beta)}{\sigma^2}}$$

$$\propto \text{InvGamma}\left(\frac{n}{2}+\alpha, \frac{n\hat{\sigma}^2_{MLE}}{2}+\beta\right)$$

Let $\alpha = \frac{n_0}{2}$, $\beta = \frac{n_0\sigma_0^2}{2}$

$$\Rightarrow P(\sigma^2|\theta) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$$

$$\Rightarrow p(\sigma^2 \mid \theta, X) = \text{InvGamma}\left(\frac{n+n_0}{2}, \frac{n\hat{\sigma}^2_{MLE} + n_0\sigma_0^2}{2}\right)$$

known, belief

Pseudodata: $Y_1, \ldots, Y_{n_0} \sim N(\theta, \sigma_0^2)$

$$\Rightarrow n_0\sigma_0^2 = \Sigma(y_i - \theta)^2$$

$$\Rightarrow \sigma_0^2 = \frac{\Sigma(y_i - \theta)^2}{n_0}$$

n_0 small means uninformative

Heldin: $n_0 = 0$, $\sigma_0^2 = ?$

$$\Rightarrow p(\sigma^2 \mid \theta) = \text{InvGamma}(0, 0)$$