Math 341 Spring 2020
Midterm Examination Two

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Thursday, April 23, 2020

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

Instructions

This exam is 80 minutes (variable time per question) and closed-book. You are allowed **one** page (front and back) of a "cheat sheet", one table of reference and scrap paper but no graphing calculator. Please read the questions carefully. No food is allowed, only drinks.

Problem 1 [8min] We return the topic of inference for a new player's baseball batting average, the θ in a binomial likelihood for n at bats. Using previous data, we create an informative prior of $\alpha = 42.3$ and $\beta = 127.7$.

- [10 pt / 10 pts] A new player enters baseball and he has 3 hits out of 7 at bats. What is the probability of the next 40 at bats he will have 11 or more hits. Record the letter(s) of all the following that are **true**.

  (a) pbinom(11, 40, 3 / 7)

  (b) 1 - pbinom(10, 40, 3 / 7)

  (c) pbetabinom(11, 40, 3 / 7)

  (d) 1 - pbetabinom(10, 40, 3 / 7)

  (e) pbetabinom(11, 40, 42.3, 127.7)

  (f) 1 - pbetabinom(10, 42.3, 127.7)

  (g) pbetabinom(11, 40, 45.3, 131.7)

  (h) 1 - pbetabinom(10, 45.3, 131.7)

  (i) $\displaystyle\sum_{x_*=11}^{40} \binom{40}{x_*} \frac{B(45.3 + x_*, 171.7 - x_*)}{B(45.3, 131.7)}$

  (j) 1 - $\displaystyle\sum_{x_*=1}^{10} \binom{40}{x_*} \frac{B(45.3 + x_*, 171.7 - x_*)}{B(45.3, 131.7)}$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 2 [8min] Let $X \sim \mathcal{N}(\theta, \sigma^2)$ and let k denote the *fully reduced* kernel of a distribution.

- [12 pt / 22 pts] Record the letter(s) of all the following that are **true**.

(a) $k(X \mid \theta, \sigma^2) = \left(\sigma^2\right)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$

(b) $k(\sigma^2 \mid \theta, X) = \left(\sigma^2\right)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$

(c) $k(\theta \mid \sigma^2, X) = \left(\sigma^2\right)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$

(d) $k(X \mid \theta, \sigma^2) = e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$

(e) $k(\sigma^2 \mid \theta, X) = e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$

(f) $k(\theta \mid \sigma^2, X) = e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$

(g) $k(X \mid \theta, \sigma^2) = e^{-\frac{x^2}{2\sigma^2}} e^{\frac{\theta x}{\sigma^2}}$

(h) $k(\sigma^2 \mid \theta, X) = e^{-\frac{x^2}{2\sigma^2}} e^{\frac{\theta x}{\sigma^2}}$

(i) $k(\theta \mid \sigma^2, X) = e^{-\frac{x^2}{2\sigma^2}} e^{\frac{\theta x}{\sigma^2}}$

(j) $k(X \mid \theta, \sigma^2) = e^{-\frac{\theta^2}{2\sigma^2}} e^{\frac{\theta x}{\sigma^2}}$

(k) $k(\sigma^2 \mid \theta, X) = e^{-\frac{\theta^2}{2\sigma^2}} e^{\frac{\theta x}{\sigma^2}}$

(l) $k(\theta \mid \sigma^2, X) = e^{-\frac{\theta^2}{2\sigma^2}} e^{\frac{\theta x}{\sigma^2}}$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 3 [8min] Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\theta)$ and use the Jeffrey's prior for θ. We wish to test if $\theta > 1$.

- [10 pt / 32 pts] Record the letter(s) of all the following expressions that correctly compute the Bayesian p-value at the $\alpha_0 = 5\%$ level.

  (a) pgamma(0.05, $\sum x_i + 0.5, n$)

  (b) 1 - pgamma(0.05, $\sum x_i + 0.5, n$)

  (c) pgamma(1, $\sum x_i + 0.5, n$)

  (d) 1 - pgamma(0.05, $\sum x_i + 0.5, n$)

  (e) qgamma(0.05, $\sum x_i + 0.5, n$)

  (f) [qgamma(0.025, $\sum x_i + 0.5, n$), qgamma(0.975, $\sum x_i + 0.5, n$)]

  (g) pgamma(0.05, $\sum x_i, n$)

  (h) 1 - pgamma(0.05, $\sum x_i, n$)

  (i) pgamma(1, $\sum x_i, n$)

  (j) 1 - pgamma(1, $\sum x_i, n$)

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 4 [8min] Let $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\theta)$ and use the prior of indifference for θ. We wish to find the probability that $x_* = 5$, i.e. the realization of the next observation is equal to 5.

- [8 pt / 40 pts] Record the letter(s) of all the following expressions that correctly compute the probability of interest.

  (a) pgamma($5, \sum x_i + 1, n$)

  (b) dgamma($5, \sum x_i + 1, n$)

  (c) qnbinom($n/(n+1), \sum x_i + 1, 5$)

  (d) dnbinom($5, \sum x_i + 1, n/(n+1)$)

  (e) ppois(5, $\bar{x}$)

  (f) ppois(5, θ)

  (g) dpois(5, θ)

  (h) dpois(5, $\bar{x}$)

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 5 [8min] Let $\mathcal{F}$ be the $\stackrel{iid}{\sim}$ normal model with σ^2 known. You wish to design a prior that has the strength of 3 observations centered at 1.

- [10 pt / 50 pts] Record the letter(s) of all the following expressions that correctly specify this prior.

  (a) $\mathbb{P}(\theta \mid \sigma^2) = \mathcal{N}(0, 3)$
  (b) $\mathbb{P}(\theta \mid \sigma^2) = \mathcal{N}(0, \infty)$
  (c) $\mathbb{P}(\theta \mid \sigma^2) = \mathcal{N}(1, \sigma^2/3)$
  (d) $\mathbb{P}(\theta \mid \sigma^2) = \mathcal{N}(1, \sigma^2/3^2)$
  (e) $\mathbb{P}(\theta \mid \sigma^2) = \mathcal{N}(1, \sigma^2)$
  (f) $\mathbb{P}(\theta \mid \sigma^2) = \text{InvGamma}(3/2, 3/2)$
  (g) $\mathbb{P}(\theta \mid \sigma^2) = \text{InvGamma}(3/2, 3\sigma^2/2)$
  (h) $\mathbb{P}(\theta \mid \sigma^2) = \text{InvGamma}(3, 3\sigma^2)$
  (i) $\mathbb{P}(\theta \mid \sigma^2) = \text{InvGamma}(3, \infty)$
  (j) $\mathbb{P}(\theta \mid \sigma^2) = \text{InvGamma}(1, \infty)$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 6 [8min] Let $\mathcal{F}$ be the $\overset{iid}{\sim}$ normal model with σ^2 known. Using the Laplace prior, find the probability that θ is greater than $\bar{x}$.

- [8 pt / 58 pts] Record the letter(s) of all the following expressions that correctly compute this probability.

  (a) 0.5

  (b) $\mathbb{P}\left(\theta > \hat{\theta}_{\text{MMSE}}\right)$

  (c) $\mathbb{P}\left(\theta > \hat{\theta}_{\text{MMAE}}\right)$

  (d) $\mathbb{P}\left(\theta > \hat{\theta}_{\text{MAP}}\right)$

  (e) pnorm(0.5, $\bar{x}$, $\sigma/\sqrt{n}$)

  (f) qnorm(0.5, $\bar{x}$, $\sigma/\sqrt{n}$)

  (g) pnorm($\bar{x}$, $\bar{x}$, $\sigma/\sqrt{n}$)

  (h) 1 - pnorm($\bar{x}$, $\bar{x}$, $\sigma/\sqrt{n}$)

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 7 [8min] Let $\mathcal{F}$ be the $\stackrel{iid}{\sim}$ normal model with $n = 20$ observations, a data average of $\bar{x} = 13.9$ and $\sigma^2 = 5^2$. Using the Jeffrey's prior, find the distribution that the next observation x_* is realized from.

- [12 pt / 70 pts] Record the letter(s) of all the following expressions that correctly compute this probability.

  (a) $\mathcal{N}(13.9,\, 5^2)$

  (b) $\mathcal{N}(13.9,\, 5^2/20)$

  (c) $\mathcal{N}(13.9,\, 5^2 + 5^2/20)$

  (d) $\mathcal{N}\left(13.9,\, (5 + 5/\sqrt{20})^2\right)$

  (e) $T_{13.9}(5^2)$

  (f) $T_{13.9}(5^2/20)$

  (g) $T_{13.9}(5^2 + 5^2/20)$

  (h) $T_{13.9}\left((5 + 5/\sqrt{20})^2\right)$

  (i) $T_{20}(13.9, 5^2)$

  (j) $T_{20}(13.9, 5^2/20)$

  (k) $T_{20}(13.9, 5^2 + 5^2/20)$

  (l) $T_{20}\left(13.9, (5 + 5/\sqrt{20})^2\right)$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 8 [8min] Let $\mathcal{F}$ be the $\overset{iid}{\sim}$ normal model with θ known. If you wished to design a prior based on two pseudo-observations, $y_1 = 0$ and $y_1 = 1$, what would the prior form be?

- [10 pt / 80 pts] Record the letter(s) of all the following expressions that correctly specify the prior distribution.

  (a) $\mathbb{P}(\sigma^2 \mid \theta) = \text{InvGamma}(0,\, 0)$
  (b) $\mathbb{P}(\sigma^2 \mid \theta) = \text{InvGamma}(1,\, 0)$
  (c) $\mathbb{P}(\sigma^2 \mid \theta) = \text{InvGamma}(1/2,\, 1/2)$
  (d) $\mathbb{P}(\sigma^2 \mid \theta) = \text{InvGamma}(1/2,\, \theta^2(1-\theta)^2/2)$
  (e) $\mathbb{P}(\sigma^2 \mid \theta) = \text{InvGamma}(1,\, \theta^2(1-\theta)^2/2)$
  (f) $\mathbb{P}(\sigma^2 \mid \theta) = \text{InvGamma}(1,\, \theta^2(1-\theta)^2)$
  (g) $\mathbb{P}(\sigma^2 \mid \theta) = \mathcal{N}(0,\, 0.5)$
  (h) $\mathbb{P}(\sigma^2 \mid \theta) = \mathcal{N}(0,\, \infty)$
  (i) $\mathbb{P}(\sigma^2 \mid \theta) = \mathcal{N}(0.5,\, \sigma^2/2)$
  (j) $\mathbb{P}(\sigma^2 \mid \theta) = \text{Deg}(0)$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 9 [8min] Let $\mathcal{F}$ be the $\stackrel{iid}{\sim}$ normal model with $\theta = 0$ i.e. the mean is known to be zero. Let the prior be Jeffrey's and you then observe $x_1 = 3.45, x_2 = 1.87$ and $x_3 = 5.03$.

- [10 pt / 90 pts] Record the letter(s) of all the following expressions that provide a Bayesian estimate (rounded to the nearest two decimals). This Bayesian estimate must be one we studied in this class.

  (a) 2.50

  (b) 1.58

  (c) qgamma(0.5, 3, 40.70)

  (d) qgamma(0.5, 1.5, 20.35)

  (e) qinvgamma(0.5, 3, 40.70)

  (f) qinvgamma(0.5, 1.5, 20.35)

  (g) 40.70

  (h) 13.57

  (i) 8.14

  (j) 0

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

Problem 10 [8min] Let $\mathcal{F}$ be the $\stackrel{iid}{\sim}$ normal model with $\theta = 0$ i.e. the mean is known to be zero. Let the prior be Jeffrey's and you then observe $x_1 = 3.45, x_2 = 1.87$ and $x_3 = 5.03$. We are interested in the distribution that x_*, the next observation, is drawn from.

- [10 pt / 100 pts] Record the letter(s) of all the following expressions that provide the correct posterior predictive distribution with parameters specified to the nearest two decimal places.

  (a) $T_3\left(3.45, 2.50\right)$

  (b) $T_3\left(3.45, 13.57\right)$

  (c) $T_3\left(0, 2.50\right)$

  (d) $T_3\left(0, 13.57\right)$

  (e) $T_3\left(0, 1.58^2 + 1.58^2/3\right)$

  (f) $\mathcal{N}\left(0,\ 1.58^2\right)$

  (g) $\mathcal{N}\left(3.45,\ 1.58^2\right)$

  (h) $\mathcal{N}\left(0,\ 1.58^2 + 1.58^2/3\right)$

  (i) $\mathcal{N}\left(3.45,\ 1.58^2 + 1.58^2/3\right)$

  (j) $\mathcal{N}\left(0,\ 13.57\right)$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.