Lecture 19

04/30 목 [MATH341]

$a = \frac{1}{2} - \frac{1}{2n+2} = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) = \frac{1}{2}\frac{n}{n+1}$

$b = -\frac{n\bar{x}}{n+1}$

$c = \frac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)$

$P(x_*|X) \propto T_{n-1}\left(-\frac{b}{2a}, \sqrt{\frac{\frac{c}{a} - \frac{b^2}{4a^2}}{n-1}}\right) = T_{n-1}\left(\bar{x}, \sqrt{\frac{n+1}{n}}\, s\right) \overset{n \text{ large}}{\approx} N(\bar{x}, s^2)$

exactly what you expect.

$\cdot\ -\frac{b}{2a} = \frac{\frac{n\bar{x}}{n+1}}{\frac{n}{n+1}} = \bar{x}$

$\cdot\ \frac{c}{a} = \frac{\cancel{\frac{1}{2}}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)}{\cancel{\frac{1}{2}}\frac{n}{n+1}} \cdot \frac{\frac{n+1}{n}}{\frac{n+1}{n}} = \frac{\frac{(n-1)(n+1)}{n}s^2 + (n+1)\bar{x}^2 - \cancel{n\bar{x}^2}}{}$

$\cancel{n\bar{x}^2} + \bar{x}^2$

$= \frac{(n-1)(n+1)}{n}s^2 + \bar{x}^2$

$\cdot\ \frac{b^2}{4a^2} = \left(-\frac{b}{2a}\right)^2 = \bar{x}^2$

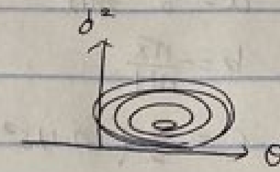
$\cdot\ \frac{c}{a} - \frac{b^2}{4a^2} = \frac{(n-1)(n+1)}{n}s^2 + \cancel{\bar{x}^2} - \cancel{\bar{x}^2}$

$\cdot\ \sqrt{\frac{\frac{c}{a} - \frac{b^2}{4a^2}}{n-1}} = \sqrt{\frac{\frac{\cancel{(n-1)}(n+1)}{n}s^2}{\cancel{n-1}}} = \sqrt{\frac{(n+1)s^2}{n}} = \sqrt{\frac{n+1}{n}}\cdot s$

cf. $P(x_*|X, \sigma^2) = N\left(\hat{\theta}_p,\ \sigma^2 + \frac{\sigma^2}{n+n_0}\right)$

$\cdot\ P(x_*|X) = T_{n-1}\left(\bar{x}, \sqrt{s^2 + \frac{s^2}{n}}\right)$

F: $\overset{iid}{\sim} N(\theta, \sigma^2)$ where both θ, σ^2 are unknown.



$P(\theta, \sigma^2|X) = P(\theta|X, \sigma^2)\, P(\sigma^2|X)$

$= \left(N(\bar{x}, \frac{\sigma^2}{n})\right)\left(\text{InvGamma}(\frac{n-1}{2}, \frac{(n-1)s^2}{2})\right)$

If $P(\theta, \sigma^2) \propto \frac{1}{\sigma^2}$,

$P(\sigma^2|X) = \text{InvGamma}(\frac{n-1}{2}, \frac{(n-1)s^2}{2})$

How would I sample $\langle \theta, \sigma^2 \rangle$ from $P(\theta, \sigma^2|X)$?

realization.

samples are realization

Step I: Draw a σ^2_{samp} realization from $P(\sigma^2|X)$ using rinvgamma($\frac{n-1}{2}$, $\frac{(n-1)s^2}{2}$)

Step II: Draw a θ_{samp} realization from $P(\theta|X, \sigma^2 = \sigma^2_{samp})$ using rnorm($\bar{x}$, $\sqrt{\frac{\sigma^2_{samp}}{n}}$)

return $\langle \theta_{samp}, \sigma^2_{samp} \rangle$ To sample n realizations, repeat n times.

How to sample from $P(X_*|X) = T_{n-1}(\bar{x}, \sqrt{\frac{n+1}{n}}\, s)$? rt.scaled($n-1$, $\bar{x}$, $\sqrt{\frac{n+1}{n}}\, s$)

$P(x_*|X) = \int_\theta \int_{\sigma^2} P(X_*|\theta, \sigma^2) \cdot P(\theta, \sigma^2|X)\, d\sigma^2\, d\theta$

$= \iint_{\theta\,\sigma^2} P(X_*, \theta, \sigma^2|X)\, d\sigma^2\, d\theta$ $= \int_\theta \int_{\sigma^2} P(X_*|\theta, \sigma^2)\, P(\theta|X, \sigma^2)\, P(\sigma^2|X)\, d\sigma^2\, d\theta$.

How to sample from $P(X_*, \theta, \sigma^2|X)$?

① sample σ^2_{samp} from $P(\sigma^2|X)$ via rinvgamma($\frac{n-1}{2}$, $\frac{(n-1)s^2}{2}$)

② sample θ_{samp} from $P(\theta|X, \sigma^2_{samp})$ via rnorm($\bar{x}$, $\frac{\sqrt{\sigma^2_{samp}}}{n}$)

③ sample X_{*samp} from $P(X_*|\theta = \theta_{samp}, \sigma^2 = \sigma^2_{samp})$ via rnorm(θ_{samp}, σ^2_{samp}).

return $\langle X_{*samp}, \theta_{samp}, \sigma^2_{samp} \rangle$

To sample from $P(X_*|X)$ you sample from $P(X_*, \theta, \sigma^2|X)$ and ignore θ_{samp}, σ^2_{samp} to leave you with X_*.

To sample n realizations, repeat n times.

$$P(\theta,\sigma^2|x) \propto P(\theta|x,\sigma^2)\,P(\sigma^2|x)$$

$$= \text{NormInvGamma} \quad \text{due to conjugacy.}$$

$\uparrow$ if $P(\theta,\sigma^2) = \text{NormInvGamma}$

If $P(\theta,\sigma^2) \neq \text{NormInvGamma} \Rightarrow$ non conjugate.

dependence.

$$\overset{\text{NormInvGamma}}{\overset{\parallel}{P(\theta,\sigma^2)}} = P(\theta|\sigma^2)P(\sigma^2) \quad \text{where } P(\theta|\sigma^2) = N\left(\mu_0, \frac{\sigma^2}{n_0}\right),\ P(\sigma^2) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$$

then model is conjugate.

What if (i.e. θ, σ^2 independent)

$$P(\theta,\sigma^2) = P(\theta)P(\sigma^2) \quad \text{where } P(\theta) = N(\mu_0, \tau^2),\ P(\sigma^2) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$$

such that $\tau^2 \neq \frac{\sigma^2}{n_0}$

$$\Rightarrow P(\theta,\sigma|x) \propto P(x|\theta,\sigma^2)P(\theta,\sigma^2)$$

$$= P(x|\theta,\sigma^2)P(\theta)\cdot P(\sigma^2)$$

$$\propto k(x|\theta,\sigma^2)\,k(\theta)\,k(\sigma^2)$$

$n\bar{x}^2 - 2n\bar{x}\theta + n\theta^2$

$$= \left((\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}((n-1)s^2 + n(\bar{x}-\theta)^2)}\right)\left(e^{-\frac{1}{2\tau^2}(\theta-\mu_0)^2}\right)\left((\sigma^2)^{-\frac{n_0}{2}-1} e^{-\frac{n_0\sigma_0^2}{2\sigma^2}}\right)$$

$$\left(e^{-\frac{\theta^2}{2\tau^2}} e^{\frac{\theta\mu_0}{\tau^2}}\right)$$

$$= (\sigma^2)^{-\frac{n+n_0}{2}-1} e^{-\frac{1}{2\sigma^2}((n-1)s^2 + n\bar{x}^2 + n_0\sigma_0^2)} \underbrace{e^{\underbrace{\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}\right)}_{=a}\theta - \underbrace{\left(\frac{n}{2\sigma^2}+\frac{1}{2\tau^2}\right)}_{=b}\theta^2}}_{= e^{a\theta - b\theta^2}}$$

$$\propto N\left(\frac{a}{2b}, \frac{1}{2b}\right)$$

$$= (\sigma^2)^{-\frac{n+n_0}{2}-1} e^{-\frac{1}{2\sigma^2}((n-1)s^2 + n\bar{x}^2 + n_0\sigma_0^2)} \left(\sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}}\, N\left(\frac{a}{2b}, \frac{1}{2b}\right)\right)$$

$$\parallel$$

$$e^{a\theta - b\theta^2}.$$

$$\propto (\sigma^2)^{-\frac{n+n_0}{2}-1} \underbrace{e^{-\frac{1}{2\sigma^2}\left((n-1)s^2+n\bar{x}^2+n_0\sigma_0^2\right)} \left(\frac{n}{2\sigma^2}+\frac{1}{2\tau^2}\right)^{-\frac{1}{2}} e^{\frac{\left(\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}\right)^2}{2\left(\frac{n}{\sigma^2}+\frac{1}{\tau^2}\right)}}}_{k(\sigma^2|x),\ (\neq \text{InvGamma } x)} \underbrace{\left(N\left(\frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\right)\right)}_{p(\theta|\sigma^2,x)}$$

the kernel of some unknown. distr,

and we don't know how to draw realization from it.