Lecture 12

03/19/20

kernel

$X \sim \text{Poisson}(\theta) = \dfrac{e^{-\theta}\theta^{x}}{x!} = p(x;\theta) \propto e^{-\theta}\theta^{x}$
(Discrete)

$Y \sim \text{Gamma}(\alpha,\beta) = \dfrac{\beta^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}$
(continuous)

$\propto \underbrace{y^{\alpha-1}e^{-\beta y}}_{\text{kernel}}$

$\tilde{F}$: iid Poisson $X_1,\ldots,X_n$; $\theta \overset{iid}{\sim} \text{Poisson}(\theta)$

$$p(X|\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\theta^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!}$$

Put a prior on θ. Last time, we learned the gamma is conjugate for the Poisson likelihood model

$$p(\theta|X) = \frac{p(X|\theta)p(\theta)}{p(X)} \propto p(X|\theta)p(\theta)$$

$$= \left(\frac{e^{-n\theta}\theta^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!}\right)\left(\frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\right)$$

$$\propto e^{-n\theta}\theta^{\Sigma x_i}\theta^{\alpha-1}e^{-\beta\theta} = \theta^{\Sigma x_i+\alpha-1}e^{-(n+\beta)\theta}$$

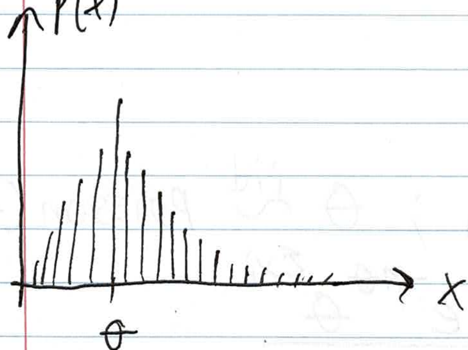
$$\propto \text{Gamma}(\Sigma x_i+\alpha,\ n+\beta)$$

$$P(\theta) \xrightarrow{X} P(\theta|X)$$

$$Gamma(\alpha,\beta) \xrightarrow{X} Gamma(\sum X_i+\alpha,\, n+\beta)$$

$X \sim Poisson(\theta)$ $\quad Supp[X]=\mathbb{N}_0$, Parameter Space $\theta \in (0,\infty)$

$\theta = \underbrace{n}_{\text{\# of trials}}\underbrace{p}_{\text{probability of successes}}$ $\qquad E[X]=\theta$



$E[\theta]=\dfrac{\alpha}{\beta}$, $\quad mode[\theta]=\dfrac{\alpha-1}{\beta}$ if $\alpha\ge 1$

For Poisson inference

$$\hat{\theta}_{MMSE}=E[\theta|\alpha]=\frac{\sum X_i+\alpha}{n+\beta}$$

$$\hat{\theta}_{MMAE}=Med[\theta|\alpha]=qgamma(0.5, \sum X_i+\alpha,\, n+\beta)$$

$$\hat{\theta}_{MAP}=mode[\theta|X]=\frac{\sum X_i+\alpha-1}{n+\beta} \text{ if } \sum X_i+\alpha\ge 1$$

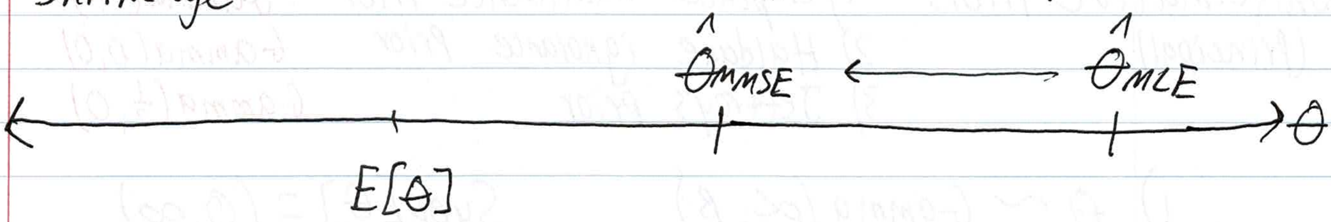
$$\theta|X\sim Gamma(\underbrace{\sum X_i}_{\text{\# of total successes}}+\underbrace{\alpha}_{\text{\# of pseudo-successes}},\ \underbrace{n}_{\text{\# of trials}}+\underbrace{\beta}_{\text{\# of pseudo trials}})$$

~~pseudo success~~

High Beta = Strong Prior

How much you shrink is determined by ρ

Shrinkage



$$\hat{\theta}_{MMSE} = \frac{\sum X_i + \alpha}{n+\beta} = \frac{\sum X_i}{n+\beta}\left(\frac{n}{n}\right) + \frac{\alpha}{n+\beta}\left(\frac{\beta}{\beta}\right)$$

$$= \underbrace{\frac{n}{n+\beta}}_{(1-\rho)} \underbrace{\bar{X}}_{\hat{\theta}_{MLE}} + \underbrace{\frac{\beta}{n+\beta}}_{\rho} E[\theta]$$

$$\lim_{n\to\infty} \rho = 0$$

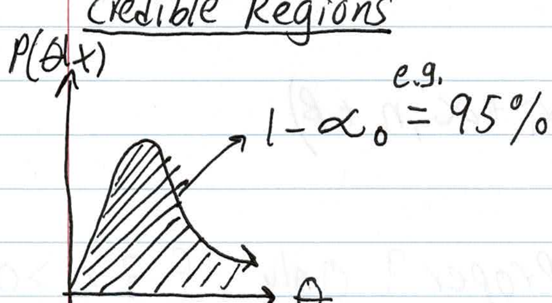
$$\mathcal{L}(\theta; x) = \frac{e^{-n\theta}\,\theta^{\sum X_i}}{\prod X_i!} = -n\theta + (\sum X_i)\ln(\theta) - \underbrace{\ln(\prod X_i!)}$$

When you take this derivative, it becomes a constant

$$\ell'(\theta; X) = -n + \frac{\sum X_i}{\theta} \xRightarrow{\text{set equal to zero}}$$

$$n = \frac{\sum X_i}{\theta} \Longrightarrow \sum X_i = n\theta \Longrightarrow \hat{\theta}_{MLE} = \bar{X}$$

Credible Regions



$$CR_{\theta, 1-\alpha_0} = \left[\text{qgamma}\left(\tfrac{\alpha_0}{2}, \textstyle\sum X_i + \alpha, n+\beta\right), \text{qgamma}\left(1-\tfrac{\alpha_0}{2}, \textstyle\sum X_i + \alpha, n+\beta\right)\right]$$

Bayesian Hypothesis Testing (Same as before)

Uniformative Priors (Principal)

1) Laplace indifference Prior $Gamma(1,0)$
2) Haldane ignorance prior $Gamma(0,0)$
3) Jeffrey's prior $Gamma(\frac{1}{2},0)$

1) $\theta \sim Gamma(\alpha, \beta)$ $\qquad Supp[\theta] = (0,\infty)$

~~$P(\theta)$~~

$\theta \overset{?}{\sim} U(0,\infty)$ NO

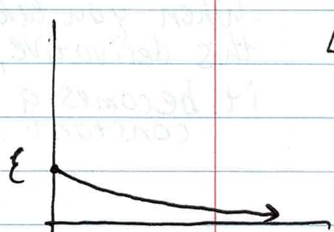
$P(\theta) = \frac{1}{\infty} = 0$

NOT a Valid PDF

$$P(\theta) = Gamma(1, \varepsilon) = \frac{\varepsilon}{\Gamma(1)} \theta^{1-1} e^{-\varepsilon\theta} = \varepsilon e^{-\varepsilon\theta}$$

If ε is small $\longrightarrow \approx 0$

$$P(\theta) = \varepsilon e^{-\varepsilon(0)} = \varepsilon$$



Laplace $\theta \sim Gamma(1,0)$ Proper? NO $\beta \not> 0$

$P(\theta|X) = Gamma(\Sigma x_i + 1, n)$ Proper? Yes (n is greater than 0, must always be ≥ 1)

2) Haldane

States $\beta = 0$, ~~$\alpha =$~~ $\alpha = 0$ (no such thing as a pseudo success)

$$P(\theta|X) = Gamma(\Sigma x_i + \alpha, n + \beta)$$

$$Gamma(0,0) = P(\theta)$$

$\Rightarrow Gamma(\Sigma x_i, n)$ Proper? Only if $\exists\, x_i > 0$

$$\hat{\theta}_{MMSE} = E[\theta|X] = \frac{\Sigma x_i}{n} = \bar{X} = \hat{\theta}_{MLE}$$

3) Jeffrey's Prior $P_J(\theta) \propto \sqrt{I(\theta)}$

$$\ell'(\theta;x) = -n + \frac{\sum x_i}{\theta} \quad , \quad \ell''(\theta;x) = \frac{\sum x_i}{\theta^2}$$

$$I(\theta) = E_X[-\ell''(\theta;x)] = E_X\left[\frac{\sum x_i}{\theta^2}\right] = \frac{1}{\theta^2}\sum_{i=1}^{n} E[X_i]$$

$$= \frac{1}{\theta^2} n\theta = n\frac{1}{\theta}$$

$$\sqrt{I\theta} \leftarrow = \sqrt{n\frac{1}{\theta}} \propto \theta^{-\frac{1}{2}} = \theta^{\frac{1}{2}-1} \propto Gamma(\tfrac{1}{2}, 0)$$

Not proper

$$P(\theta|x) = Gamma\left(\sum x_i + \tfrac{1}{2}, n\right)$$

Always Proper? Yes

Prediction

X_* is the next observation that you want to predict

$X_*|X \sim ?$ $\quad Supp[X_*|x] = Supp[x] = \{0, 1, 2, \dots\}$

$$P(X_*|x) = \int_{\Theta} P(X_*|\theta)\, P(\theta|x)\, d\theta$$

$$= \int_0^\infty \left(\frac{e^{-\theta}\theta^{X_*}}{X_*!}\right)\left(\frac{(n+\beta)^{\sum x_i + \alpha}}{\Gamma(\sum x_i + \alpha)}\theta^{\sum x_i + \alpha - 1} e^{-(n+\beta)\theta}\right) d\theta$$