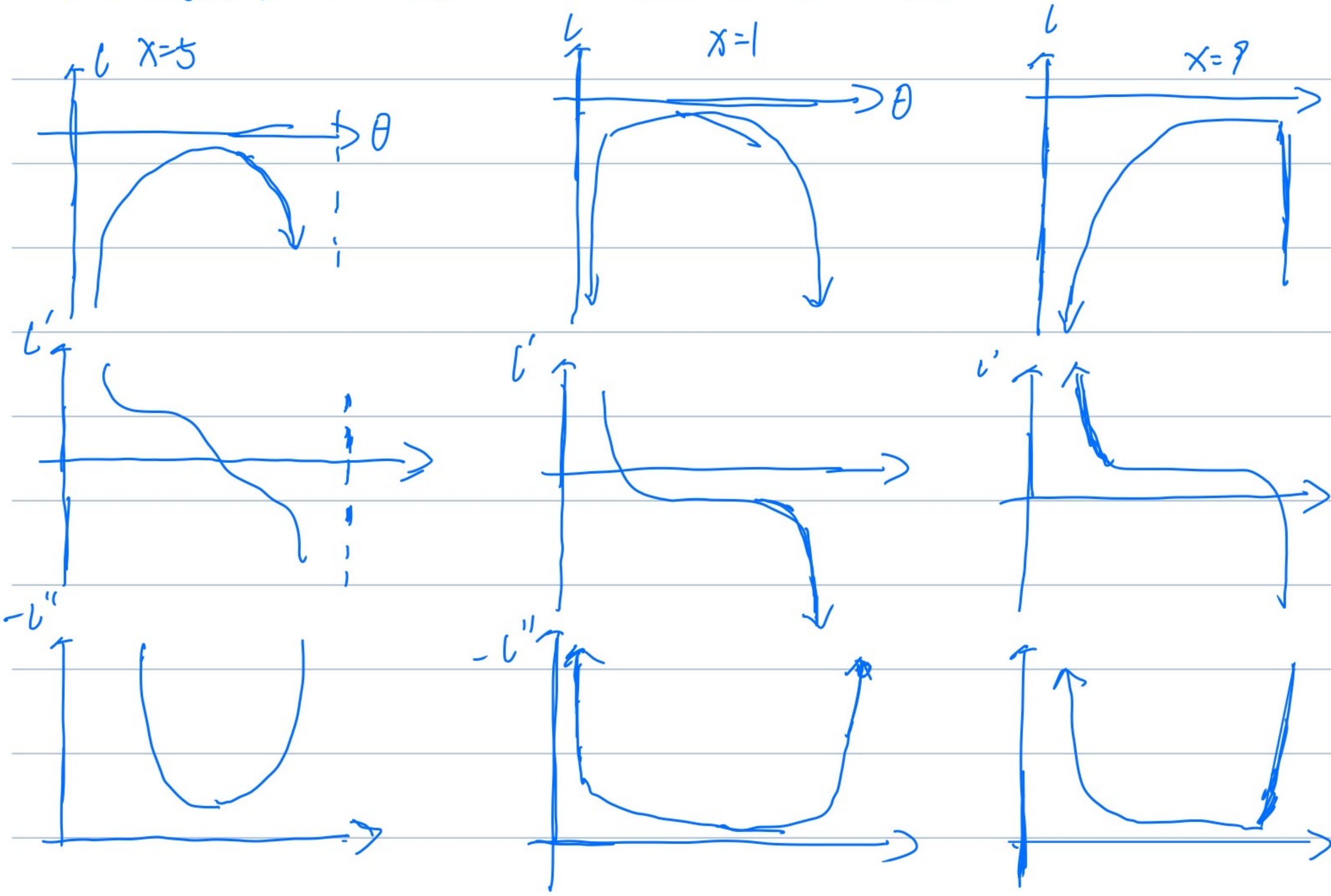
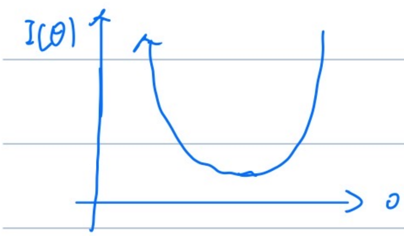
$n=10 \quad X \sim Bin(n;\theta) := \binom{n}{x}\theta^{x}(1-\theta)^{n-x} \qquad l(\theta;x) = \ln\binom{n}{x} + x\ln\theta + (n-x)\ln(1-\theta)$

$s(\theta;x) = l'(\theta;x) = \frac{x}{\theta} - \frac{n-x}{1-\theta} \qquad -l''(\theta;x) = \frac{x}{\theta^2} + \frac{n-x}{(1-\theta)^2}$



prob weight average out.
$supp\{x\} = \{0, 1, \dots 10\}$

How much "informative" is in X? $I_1(\theta=\frac{1}{2}) = 4 \quad I_1(\theta=\frac{1}{100}) \approx 101$



Let $P(\theta) \propto \sqrt{I(\theta)}$ If F: Bin fixed n.

$P(\theta) \propto \sqrt{n \cdot \frac{1}{\theta(1-\theta)}} \propto \sqrt{\frac{1}{\theta(1-\theta)}} = \theta^{\frac{1}{2}-1}(1-\theta)^{\frac{1}{2}-1} \propto Beta(\frac{1}{2},\frac{1}{2})$ the Jefferson prior

$\phi = \tau(\theta) = \frac{\theta}{1-\theta}$

$\phi = \frac{\theta}{1-\theta} \rightarrow \theta = \frac{\phi}{1+\phi} \qquad L(\phi;x) = \binom{n}{x}\left(\frac{\phi}{\phi+1}\right)^{x}\left(\frac{1}{\phi+1}\right)^{n-x} = \binom{n}{x}\frac{\phi^{x}}{(\phi+1)^{n}}$

$l(\phi;x) = \ln\binom{n}{x} + x\ln(\phi) - n\ln(\phi+1) \qquad l'(\phi;x) = \frac{x}{\phi} - \frac{n}{\phi+1} \qquad -l''(\phi;x) = \frac{x}{\phi^2} - \frac{n}{(\phi+1)^2}$

$I(\phi) = E_x[-l''(\phi;x)] = \frac{n}{\phi(\phi+1)} - \frac{n}{(\phi+1)^2} = \frac{n}{\phi(\phi+1)^2}$

$P_J(\theta) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\phi(\phi+1)^2}} \propto \phi^{-\frac{1}{2}}(\phi+1)^{-1} \propto \frac{1}{\pi}\phi^{-\frac{1}{2}}(\phi+1)^{-1} \overset{\text{trivia}}{=} \text{Betaprime}(\frac{1}{2},\frac{1}{2}) = F_{1,1}$

$$\begin{array}{ccc} P(x|\theta) & \xrightarrow{\text{Jefferson's prior}} & P_J(\theta) \\ \updownarrow \phi = t(\theta) & & \updownarrow \phi = t(\theta) \\ P(x|\phi) & \xrightarrow{\text{Jefferson's prior}} & P_J(\phi) \end{array}$$

change of variable $\quad \theta = k(\phi) = \frac{\phi}{1+\phi}$

$P(\phi) = P_\theta(t^{-1}(\phi)) \left|\frac{d}{d\phi}[t^{-1}(\phi)]\right|$

$= \frac{1}{B(\frac{1}{2},\frac{1}{2})}\left(\frac{\phi}{1+\phi}\right)^{-\frac{1}{2}}\left(\frac{1}{1+\phi}\right)^{-\frac{1}{2}}(\phi+1)^{-2} = \frac{1}{\pi}\phi^{-\frac{1}{2}}(\phi+1)^{-1} = \text{Betaprime}(\frac{1}{2},\frac{1}{2})$

you can't use prior Beta (1, 1) ; is unique

Assume $P(\theta) \propto \sqrt{I(\theta)}$ prove:

$P(\phi) \propto \sqrt{I(\phi)}$ using $P(\phi) = P_\theta(t^{-1}(\phi)) \left|\frac{d}{d\phi}(t^{-1}(\phi))\right|$

$\sqrt{E_X(l'(\phi;x)^2)} = \sqrt{I(\phi)} \propto P_J(\phi)$

$\phi = \frac{\theta}{1-\theta}$

$P(\phi) = P_\theta(\theta)\left|\frac{d\theta}{d\phi}\right| \propto \sqrt{I(\theta)}\left|\frac{d\theta}{d\phi}\right|$

$= \sqrt{I(\theta)\frac{d\theta\, d\theta}{d\phi \cdot d\phi}}$

$= \sqrt{E_X[l'(\theta;x)^2]\frac{d\theta \cdot d\theta}{d\phi \cdot d\phi}}$

$\longleftarrow \quad = \sqrt{E_X\left[\frac{dl}{d\theta}\cdot\frac{dl}{d\theta}\right]\frac{d\theta \cdot d\theta}{d\phi}}$

$P(\theta) = \text{Beta}(1,1)$ Laplace'

$P(\theta) = \text{Beta}(0,0)$

$P_J(\theta) = \text{Beta}(\frac{1}{2},\frac{1}{2})$ Jefferson

} principle of uninformative priors

$P(\theta) = \text{Beta}(\alpha,\beta) \longrightarrow$ empirical Bayes informative prior, α, β based on provided data.

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new distribution: poisson

$X \sim Bin(n;\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} \quad n\to\infty \quad \theta \to 0 \quad$ But $\lambda = n\theta \quad \theta = \frac{\lambda}{n}$

$= \lim_{n\to\infty}\binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(\frac{n-\lambda}{n}\right)^{n-x} = \lim_{n\to\infty}\frac{n!}{x!(n-x)!}\frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$

$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\frac{n!}{(n-x)!}\frac{1}{n^x}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x} = \frac{\lambda^x}{x!}\lim_{n\to\infty}\frac{n}{n}\cdot\frac{(n-1)}{n}\cdots\frac{(n-x)}{(n-x)}\cdot\frac{1}{1}\cdot\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{=e^{-\lambda}}\left(1-\frac{\lambda}{n}\right)^{-x}$

$= \frac{e^{-\lambda}\lambda^x}{x!}$ poisson(λ)

$\infty \cdot 0 \quad$ L'H rule

$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{\lim_{n\to\infty} n\ln(1-\frac{\lambda}{n})} = e^{\lim_{n\to\infty}\ln(1-\frac{\lambda}{n}) + \frac{n^2}{n-\lambda}(\frac{\lambda}{n^2})} = e^0 ?$ X

F: poisson $n=1$ we want to find the conjugate prior

$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)}$ $\quad k(\theta|x) \propto p(x|\theta)p(\theta) \propto k(x|\theta)k(\theta)$

$k(\theta|x) \propto \left(\frac{\theta^x e^{-\theta}}{x!}\right)p(\theta) \propto \theta^x e^{-\theta} k(\theta) = \theta^x e^{-\theta}(\underbrace{\theta^a e^{-b\theta}}_{k(\theta)}) = \theta^{x+a} e^{-b(\theta+1)}$

posterior match to conjecture to / what conjugate prior looks like

let $t = b\theta$

$C = \frac{1}{\int_0^\infty \theta^a e^{-b\theta} d\theta} = \frac{1}{\int_0^\infty \left(\frac{t}{b}\right)^a e^{-t} \frac{1}{b} dt} = \frac{1}{\frac{1}{b^{a+1}} \int_0^\infty t^{a+1-1} e^{-t} dt} = \frac{b^{a+1}}{\Gamma(a+1)} \rightarrow p(\theta) = \frac{b^{a+1}}{\Gamma(a+1)}\theta^a e^{-b\theta}$

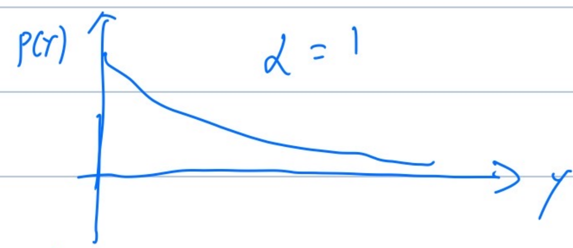
Gamma$(a+1, b)$

conjugate prior in poisson model

$p(\theta) = \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}$ $\quad E(Y) = \frac{\alpha}{\beta}$ $\quad \text{Supp}[Y] = (0, \infty)$

$Y \sim \text{Gamma}(\alpha, \beta)$ $\quad Var(Y) = \frac{\alpha}{\beta^2}$

$\text{Mode}[Y] = \frac{\alpha-1}{\beta}$ if $\alpha \geq 1$ $\quad \text{Med}[Y] = \text{qGamma}(0.5, \alpha, \beta)$ no closed form.



3 shape: