Lecture 15

f: iid Normal (θ, σ^2) with σ^2 Known

$P(\theta|\sigma^2) = N(\mu_0, \tau^2) = N\left(\mu_0, \frac{\sigma^2}{n_0}\right)$

Imagine Pseudodata $y_1, \ldots, y_{n_0} \overset{iid}{\sim} N(\mu_0, \sigma^2)$

$$\Rightarrow \bar{y} \sim N\left(\mu_0, \frac{\sigma^2}{n_0}\right)$$

$$\Rightarrow P(\theta|X, \sigma^2) = N(\hat{\theta}_p, \sigma_p^2)$$

Where $\hat{\theta}_p = \dfrac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \dfrac{n\bar{x} + n_0\mu_0}{n + n_0}$

$$\sigma_p^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{\sigma^2}{n + n_0}$$

$n_* = 1$,

$$P(x_*|x, \sigma^2) = \int_{\Theta} P(x_*|\theta, \sigma^2)\, P(\theta|X, \sigma^2)\, d\theta$$

$$= \int_{\mathbb{R}} N(\theta, \sigma^2)\, N(\hat{\theta}_p, \sigma_p^2)\, d\theta$$

$$= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_* - \theta)^2}\right)\left(\frac{1}{\sqrt{2\pi\sigma_p^2}} e^{-\frac{1}{2\sigma_p^2}(\theta - \hat{\theta}_p)^2}\right) d\theta$$

$$\propto \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}(x_* - \theta)^2}\, e^{-\frac{1}{2\sigma_p^2}(\theta - \hat{\theta}_p)^2}\, d\theta$$

$$= \int_{\mathbb{R}} \underbrace{e^{-\frac{x_*^2}{2\sigma^2}}}_{\text{function of } x_*} e^{\frac{x_*\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} e^{-\frac{\theta^2}{2\sigma_p^2}} e^{\frac{\theta\hat{\theta}_p}{\sigma_p^2}} \underline{e^{-\frac{\hat{\theta}_p^2}{2\sigma_p^2}}} \, d\theta$$

Not a function of X_*

$$\propto e^{-\frac{x_*^2}{2\sigma^2}} \int_{\mathbb{R}} e^{\frac{x_*\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} e^{-\frac{\theta^2}{2\sigma_p^2}} e^{\frac{\theta\hat{\theta}_p}{\sigma_p^2}} \, d\theta$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} \int_{\mathbb{R}} e^{\overbrace{\left(\frac{x_*}{\sigma^2} + \frac{\hat{\theta}_p}{\sigma_p^2}\right)}^{a}\theta - \overbrace{\left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma_p^2}\right)}^{b}\theta^2} \, d\theta$$

$$\theta \sim N\left(\frac{a}{2b}, \frac{1}{2b}\right) = \frac{1}{\sqrt{2\pi\left(\frac{1}{2b}\right)}} e^{-\frac{1}{2\left(\frac{1}{2b}\right)}\left(\theta - \frac{a}{2b}\right)^2}$$

$$= \sqrt{\frac{b}{\pi}} e^{-b\left(\theta^2 - \frac{a\theta}{b} + \frac{a^2}{4b^2}\right)}$$

$$= \sqrt{\frac{b}{\pi}} e^{-b\theta^2 + a\theta - \frac{a^2}{4b}}$$

$$= \underbrace{\sqrt{\frac{b}{\pi}} e^{-\frac{a^2}{4b}}}_{C} \underbrace{e^{a\theta - b\theta^2}}_{K(\theta)}$$

$$\propto e^{a\theta - b\theta^2}$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} \sqrt{\frac{\pi}{b}}\, e^{\frac{a^2}{4b}} \overbrace{\int_{\mathbb{R}} \sqrt{\frac{b}{\pi}}\, e^{-\frac{a^2}{4b}} e^{a\theta - b\theta^2}\, d\theta}^{1}$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} \sqrt{\frac{\pi}{b}}\, e^{a^2/4b}$$

$$\propto e^{-\frac{x_*^2}{2\sigma^2}}\, e^{\left(\frac{x_*}{\sigma^2} + \frac{\hat\theta_p}{\sigma_p^2}\right)/4b}$$

$$= e^{-\frac{x_*^2}{2\sigma^2}}\, e^{\frac{x_*^2}{4\sigma^4 b}}\, e^{\frac{x_*\hat\theta_p}{2\sigma_p^2\sigma^2 b}}\, e^{\frac{\hat\theta_p^2}{4\sigma_p^4 b}}$$

$$\propto e^{\underbrace{\frac{\hat\theta_p}{2\sigma^2\sigma_p^2 b}}_{A} x_* - \underbrace{\left(\frac{1}{2\sigma^2} - \frac{1}{4\sigma^4 b}\right)}_{B} x_*^2}$$

$$= e^{Ax_* - Bx_*^2} \propto N\left(\frac{A}{2B}, \frac{1}{2B}\right)$$

$$(*)\quad 2b\sigma^2 = 2\left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma_p^2}\right)\sigma^2 = 1 + \frac{\sigma^2}{\sigma_p^2}$$

$$\frac{1}{2B} = \frac{1}{2\left(\frac{1}{2\sigma^2} - \frac{1}{4\sigma^4 b}\right)} = \frac{1}{\frac{1}{\sigma^2} - \frac{1}{2b\sigma^2\sigma^2}}$$

$$= \frac{\sigma^4}{\sigma^2 - \frac{1}{2b}} = \frac{2b\sigma^2\sigma^2}{2b\sigma^2 - 1}$$

$$= \frac{\left(1 + \frac{\sigma^2}{\sigma_p^2}\right)\sigma^2}{\left(1 + \frac{\sigma^2}{\sigma_p^2}\right) - 1}$$

$$= \frac{\left(1+\frac{\sigma^2}{\sigma_p^2}\right)\sigma^2}{\frac{\sigma^2}{\sigma_p^2}} = \sigma_p^2\left(1+\frac{\sigma^2}{\sigma_p^2}\right) = \sigma_p^2+\sigma^2$$

$$\frac{A}{2B} = A\left(\sigma_p^2+\sigma^2\right) = \frac{\hat{\theta}_p}{2b\sigma^2\sigma_p^2}\left(\sigma_p^2+\sigma^2\right)$$

$$= \frac{\hat{\theta}_p}{\left(1+\frac{\sigma^2}{\sigma_p^2}\right)\sigma_p^2}\left(\sigma_p^2+\sigma^2\right)$$

$$= \frac{\hat{\theta}_p}{\cancel{\left(\sigma_p^2+\sigma^2\right)}}\cancel{\left(\sigma_p^2+\sigma^2\right)}$$

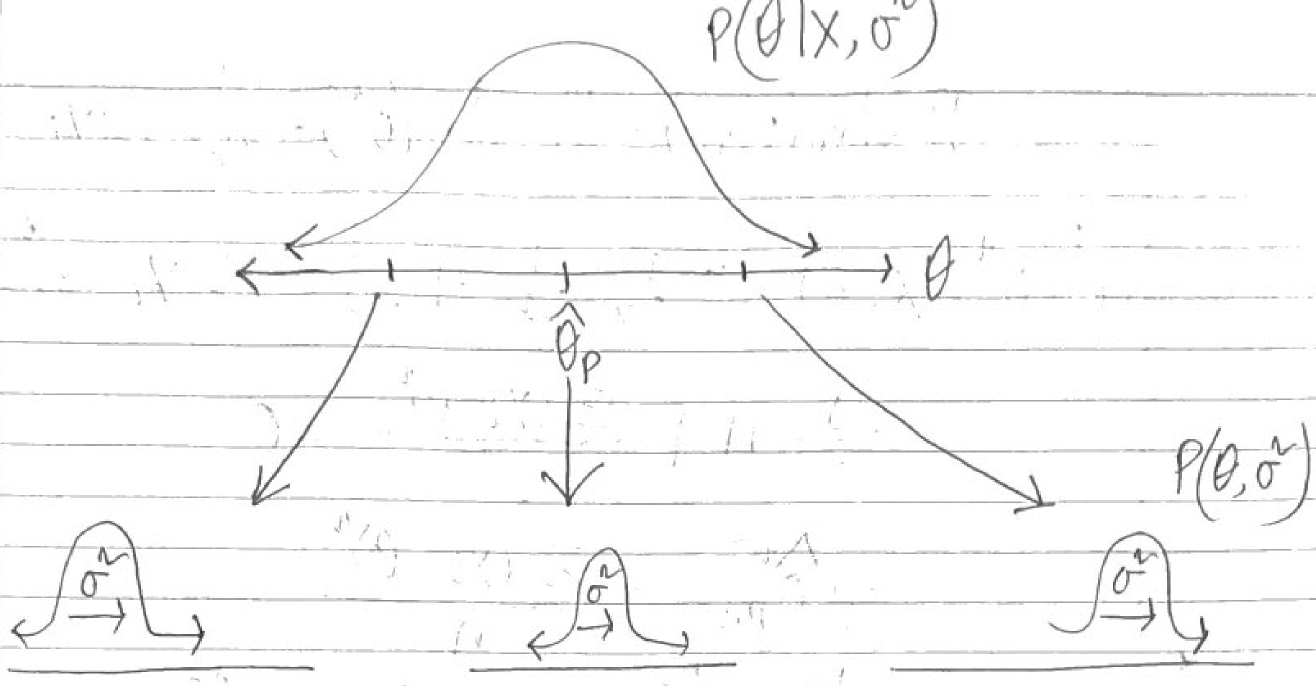
$$= \hat{\theta}_p$$

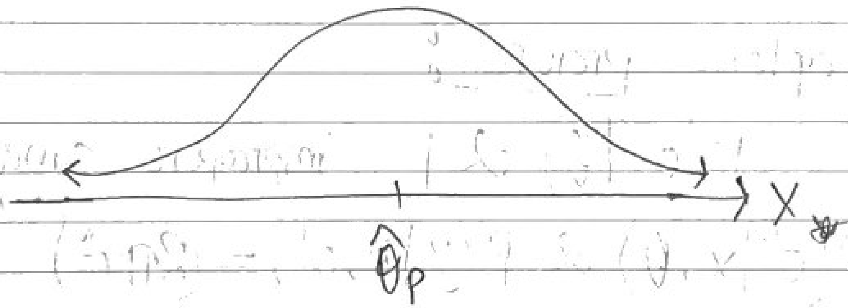
$$P(X_* \mid x, \sigma^2) = N\left(\hat{\theta}_p, \sigma_p^2+\sigma^2\right)$$

$$= N\left(\hat{\theta}_p, \left(1+\frac{1}{n+n_0}\right)\sigma^2\right) \to N(\theta, \sigma^2)$$

$$\hat{\theta}_p \longrightarrow \theta,$$

$$\sigma_p^2+\sigma^2 \longrightarrow \sigma^2$$





An overdisperse normal with known variance is normal.

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f: iid $N(\theta, \sigma^2)$ with θ known

$$\mathcal{L}(\sigma^2; x, \theta) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2}$$

$$l(\sigma^2; x, \theta) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\Sigma(x_i-\theta)^2$$

$$\ell'(\sigma^2; x, \theta) = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^{2\cdot 2}}\Sigma(x_i-\theta)^2 \overset{set}{=} 0$$

$$\Rightarrow -n + \frac{\Sigma(x_i-\theta)^2}{\sigma^2} = 0$$

$$\Rightarrow \hat{\sigma}^2_{MLE} = \frac{\Sigma(x_i-\theta)^2}{n}$$

Similar to, $s^2 = \dfrac{\Sigma(x_i-\bar{x})^2}{n-1} \to \sigma^2$

Laplace prior:

$P(\sigma^2|\theta) \propto 1$ improper since $\sigma^2 \in (0, \infty)$

$$P(\sigma^2|x,\theta) \propto P(x|\theta,\sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2_{MLE}}{2\sigma^2}}$$

$$\propto (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}$$

$$U \sim Gamma(\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u}$$

$$V = \frac{1}{u} = t(u) \sim f_u(t^{-1}(v))\left|\frac{d}{dv}[t^{-1}(v)]\right|$$

$$t^{-1}(v) = \frac{1}{v} \qquad = \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{v}\right)^{\alpha-1} e^{-\frac{\beta}{v}}\left|-\frac{1}{v^2}\right|$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} v^{-\alpha+1} v^{-2} e^{-\frac{\beta}{v}}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} v^{-\alpha-1} e^{-\beta/v}$$

$$= \text{InvGamma}(\alpha, \beta)$$

inverse gamma R.V.

$$\propto K(v) = v^{-\alpha-1} e^{-\frac{\beta}{v}}$$

$$p(\sigma^2 | x, \theta) = (\sigma^2)^{-\overbrace{\left(\frac{n}{2}-1\right)}^{\alpha}-1} e^{\frac{-\overbrace{n \hat{\sigma}^2_{MLE}/2}^{\beta}}{\sigma^2}}$$

$$= \text{InvGamma}\left(\frac{n-2}{2}, \frac{n \hat{\sigma}^2_{MLE}}{2}\right)$$

$$p(\sigma^2 | x, \theta) \propto p(x | \theta, \sigma^2)\, p(\sigma^2 | \theta)$$

$$\propto (\sigma^2)^{-n/2} e^{\frac{-n\hat{\sigma}^2_{MLE}/2}{\sigma^2}} K(\sigma^2 | \theta)$$

$K(\sigma^2 | \theta) = ?$ to get conjugacy ?

$$= (\sigma^2)^a e^{\frac{-b}{\sigma^2}} \checkmark$$

$$= (\sigma^2)^{-n/2+a} e^{-\frac{n\hat{\sigma}^2/2 + b}{\sigma^2}}$$

$\propto$ InvGamma is the conjugate prior.

Let $p(\sigma^2|\theta) = \text{InvGamma}(\alpha, \beta)$

$$\Rightarrow p(\sigma^2|x, \theta) \propto \left((\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}\right)\left((\sigma^2)^{-\alpha-1} e^{-\frac{\beta}{\sigma^2}}\right)$$

$$= (\sigma^2)^{-\left(\frac{n}{2}+\alpha\right)-1} e^{-\frac{(n\hat{\sigma}^2_{MLE}/2)+\beta}{\sigma^2}}$$

$$\propto \text{InvGamma}\left(\frac{n}{2}+\alpha, \frac{n\hat{\sigma}^2_{MLE}}{2}+\beta\right)$$

Let $\alpha = \frac{n_0}{2}$, $\beta = \frac{n_0\sigma_0^2}{2}$

$$\Rightarrow p(\sigma^2|\theta) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$$

$$\Rightarrow p(\sigma^2|\theta, x) = \text{InvGamma}\left(\frac{n+n_0}{2}, \frac{n\hat{\sigma}^2_{MLE}+n_0\sigma_0^2}{2}\right)$$

<u>Pseudodata</u>: $y_1, \ldots, y_{n_0} \sim N(\underbrace{\theta}_{\text{known}}, \underbrace{\sigma_0^2}_{\text{belief}})$

$$\Rightarrow n_0\sigma_0^2 = \Sigma(y_i-\theta)^2$$

$$\Rightarrow \sigma_0^2 = \frac{\Sigma(y_i-\theta)^2}{n_0}$$

n_0 small $\longrightarrow$ uninformative

<u>Haldane</u>: $n_0 = 0$,

$\sigma_0^2 = ?$

$$\Rightarrow p(\sigma^2|\theta) = \text{InvGamma}(0, 0)$$