$Y \sim InvGamma(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-\alpha-1} e^{-\frac{\beta}{y}} \propto y^{-\alpha-1} e^{-\frac{\beta}{y}}$

$E[Y] = \frac{\beta}{\alpha-1}$, $Med[Y] = qinvgamma(0.5, \alpha, \beta)$, $Mode[Y] = \frac{\beta}{\alpha+1}$ always

if $\alpha > 1$.

$\mathcal{F}$: iid $N(\theta, \sigma^2)$ with θ known

$P(\sigma^2 | \theta) = InvGamma\left(\frac{n_0}{2}, \frac{n_0 \sigma_0^2}{2}\right)$

$Y_1, \ldots, Y_n \overset{iid}{\sim} N(\theta, \sigma^2)$

$n_0 \sigma_0^2 = \sum (Y_i - \theta)^2$

$E[\sigma^2 | \theta] = \frac{n_0 \sigma_0^2 / 2}{n_0/2 - 1} = \frac{n_0 \sigma_0^2}{n_0 - 2} \approx \sigma_0^2$ if n_0 large

$Mode[\sigma^2 | \theta] = \frac{n_0 \sigma_0^2}{n_0 + 2} \approx \sigma_0^2$

$P(\sigma^2 | X, \theta) = InvGamma\left(\frac{n_0+n}{2}, \frac{n_0 \sigma_0^2 + n \hat{\sigma}^2_{MLE}}{2}\right)$

$\hat{\sigma}^2_{MMSE} = \frac{(n_0 \sigma_0^2 + n \hat{\sigma}^2)/2}{(n_0+n)/2 - 1} = \frac{n_0 \sigma_0^2 + n \hat{\sigma}^2}{n_0 + n - 2}$

$\hat{\sigma}^2_{MMAE} = qinvgamma\left(0.5, \frac{n_0+n}{2}, \frac{n_0 \sigma_0^2 + n \hat{\sigma}^2}{2}\right)$

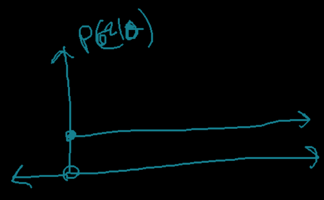
$\hat{\sigma}^2_{MAP} = \frac{n_0 \sigma_0^2 + n \hat{\sigma}^2}{n_0 + n + 2}$

Uninformative Priors

① Laplace / Indifference

$P(\sigma^2 | \theta, X) \propto P(X | \theta, \sigma^2) \propto (\sigma^2)^{-\left(\frac{n}{2} - 1\right) - 1} e^{-\frac{n\hat{\sigma}^2/2}{\sigma^2}} \propto InvGamma\left(\frac{n}{2} - 1, \frac{n \hat{\sigma}^2}{2}\right)$

$\frac{n-2}{2}$ ''

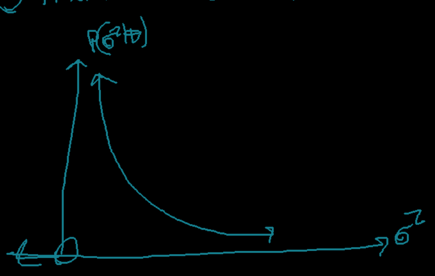


$\Rightarrow n_0 = -2, \sigma_0^2 = 0$

$\Rightarrow P(\sigma^2 | \theta) = InvGamma(-1, 0) \propto 1$

σ^2_{MMSE} is only defined for $n \geq 5$.

② Haldane $n_0 = 0, \sigma_0^2 = ?$ $\Rightarrow InvGamma(0, 0) \propto (\sigma^2)^{-1}$



③ Jeffreys

$l(\sigma^2; \theta, X) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum (X_i - \theta)^2$

$l'(\sigma^2; \theta, X) = -\frac{n}{2} \frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum (X_i - \theta)^2$

$-l''(\sigma^2; \theta, X) = -\frac{n}{2} \frac{1}{(\sigma^2)^2} + \frac{1}{(\sigma^2)^3} \sum (X_i - \theta)^2$

$E_X (X_i - \theta)^2$
$= E_X [X_i^2 - 2\theta X_i + \theta^2]$
$= E_X[X_i^2] - 2\theta E[X_i] + \theta^2$
$= (\sigma^2 + \cancel{\theta^2}) - \cancel{2\theta^2} + \cancel{\theta^2} = \sigma^2$

$I(\sigma^2; \theta) = E_X[-l''(\sigma^2; \theta, X)] = E_X[\checkmark] = -\frac{n}{2} \frac{1}{(\sigma^2)^2} + \frac{1}{(\sigma^2)^3} \sum E_X[(X_i - \theta)^2]$

$= -\frac{n}{2} \frac{1}{(\sigma^2)^2} + \frac{1}{(\sigma^2)^3} n \sigma^2 = n\left(-\frac{1}{2(\sigma^2)^2} + \frac{1}{(\sigma^2)^2}\right) = n \frac{1}{2(\sigma^2)^2} = \frac{n}{2} (\sigma^2)^{-2}$

$P_J(\sigma^2 | \theta) \propto \sqrt{I(\sigma^2; \theta)} = \sqrt{\frac{n}{2} (\sigma^2)^{-2}} \propto (\sigma^2)^{-1} \propto InvGamma(0, 0)$ (default)

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let $\alpha' = \frac{n + n_0}{2}$, $\beta' = \frac{n\hat{\sigma}^2 + n_0 \sigma_0^2}{2}$

Posterior Predictive Distr.

$P(X_* | X, \theta) = \int_0^\infty P(X_* | \theta, \sigma^2) P(\sigma^2 | X, \theta) \, d\sigma^2 = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_* - \theta)^2} \frac{\beta'^{\alpha'}}{\Gamma(\alpha')} (\sigma^2)^{-\alpha' - 1} e^{-\frac{\beta'}{\sigma^2}} \, d\sigma^2$

$\propto \underbrace{\int_0^\infty (\sigma^2)^{-\frac{1}{2}} e^{-\frac{(X_* - \theta)^2/2}{\sigma^2}} (\sigma^2)^{-\alpha'-1} e^{-\frac{\beta'}{\sigma^2}} \, d\sigma^2}_{1} = \int_0^\infty \underbrace{(\sigma^2)^{-\overbrace{(\alpha' + \frac{1}{2})}^{A} - 1} e^{-\frac{\overbrace{(X_* - \theta)^2/2 + \beta'}^{B}}{\sigma^2}}}_{\text{Kernel of InvGamma}(A, B)} \, d\sigma^2$

$= \frac{\Gamma(A)}{B^A} \underbrace{\int_0^\infty \frac{B^A}{\Gamma(A)} (\sigma^2)^{-A-1} e^{-\frac{B}{\sigma^2}} \, d\sigma^2}_{1} = \Gamma(A) B^{-A} = \Gamma\left(\frac{n + n_0 + 1}{2}\right) \left(\frac{n\hat{\sigma}^2 + n_0 \sigma_0^2 + (X_* - \theta)^2}{2}\right)^{-\frac{n + n_0 + 1}{2}}$

$\propto \left(\frac{\overbrace{n\hat{\sigma}^2 + n_0 \sigma_0^2}^{a} + (X_* - \theta)^2}{2}\right)^{-\overbrace{\frac{n + n_0 + 1}{2}}^{\nu}} = \left(\frac{a}{2}\right)^{-\frac{\nu+1}{2}} \left(\frac{2}{a} \frac{a + (X_* - \theta)^2}{2}\right)^{-\frac{\nu+1}{2}} \propto \left(\frac{2}{a} \frac{a + (X_* - \theta)^2}{2}\right)^{-\frac{\nu+1}{2}}$

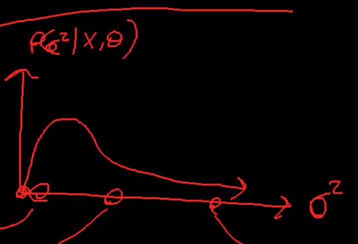
$= \left(1 + \frac{(X_* - \theta)^2}{a}\right)^{-\frac{\nu+1}{2}} = \left(1 + \frac{1}{\nu} \frac{(X_* - \theta)^2}{a/\nu}\right)^{-\frac{\nu+1}{2}} \propto T_\nu\left(\theta, \frac{a}{\nu}\right)$

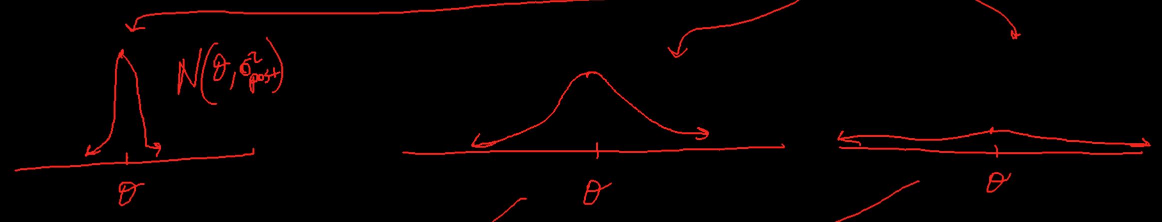
In Stat 101, $\mathcal{F}: X_1, \ldots, X_n; \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$H_0: \theta = 0 \Rightarrow \frac{\bar{X} - \theta}{\frac{s}{\sqrt{n}}} = \frac{\bar{X}\sqrt{n}}{s} \sim T_{n-1}(0, 1)$ Standard T

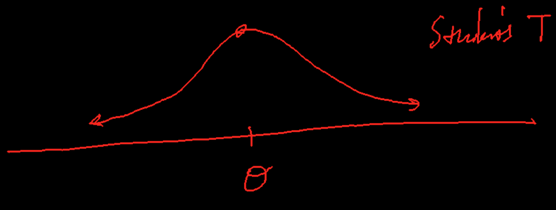
Student's T distribution with ν degrees of freedom and location parameter θ (mean) and scale a/ν.

$= T_{n+n_0}\left(\theta, \frac{n\hat{\sigma}^2 + n_0 \sigma_0^2}{n + n_0}\right)$





Student's T which is Normal with thicker tails



$\lim_{n \to \infty} P(X_* | X, \theta) = N(\theta, \sigma^2)$

$= T_\infty\left(\theta, \lim_{n \to \infty} \hat{\sigma}^2_{MLE}\right) = N(\theta, \sigma^2)$

$\hat{\sigma}^2_{MMSE} = \frac{n\hat{\sigma}^2 + n_0 \sigma_0^2}{n + n_0 - 2} = \underbrace{\frac{n}{n + n_0 - 2}}_{1-\rho} \hat{\sigma}^2 + \frac{n_0 \sigma_0^2}{n + n_0 - 2} \cdot \frac{n_0 - 2}{n_0 - 2}$

$= (1 - \rho)\hat{\sigma}^2 + \underbrace{\frac{n_0 - 2}{n + n_0 - 2}}_{\rho} \underbrace{\frac{n_0 \sigma_0^2}{n_0 - 2}}_{E[\sigma^2 | \theta]}$

Midterm 2 ↑

Final ↓

$\mathcal{F}$: $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ but neither θ or σ^2 known

$P(\theta, \sigma^2) = P(\theta | \sigma^2) P(\sigma^2)$
$= N(0, \sigma^2) \, InvGamma(1, 1)$

$P(\theta, \sigma^2 | X) \propto P(X | \theta, \sigma^2) P(\theta, \sigma^2)$

$= (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum (X_i - \theta)^2} P(\theta, \sigma^2)$

if Laplace

$\propto (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum (X_i - \theta)^2} \propto (\sigma^2)^{-n/2} e^{-\frac{\sum (X_i - \theta)^2 / 2}{\sigma^2}}$

$\overset{?}{\propto}$ InvGamma NO!

this is two-dimensional