Lecture 15

04/02/20

$\tilde{F}$: iid Normal(θ, σ^2) with σ^2 known

$P(\theta|\sigma^2) = N(\mu_0, \tau^2) = N(\mu_0, \frac{\sigma^2}{n_0})$

Imagine pseudodata

$Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu_0, \sigma^2) \Rightarrow \bar{Y} \sim N(\mu_0, \frac{\sigma^2}{n_0})$

$P(\theta | X, \sigma^2) = N(\hat{\theta}_p, \sigma_p^2)$ where...

$$\hat{\theta}_p = \frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{n\bar{X} + n_0\mu_0}{n + n_0}$$

$$\sigma_p^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{\sigma^2}{n + n_0}$$

$n_* = 1 \qquad P(X_* | X, \sigma^2) = \int_\Theta P(X_* | \theta, \sigma^2) P(\theta | X, \sigma^2)\, d\theta$

$$= \int_{\mathbb{R}} N(\theta, \sigma^2) N(\hat{\theta}_p, \sigma_p^2)\, d\theta$$

$$= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_* - \theta)^2}\right)\left(\frac{1}{\sqrt{2\pi\sigma_p^2}} e^{-\frac{1}{2\sigma_p^2}(\theta - \hat{\theta}_p)^2}\right) d\theta$$

$$\propto \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}(X_* - \theta)^2} e^{-\frac{1}{2\sigma_p^2}(\theta - \hat{\theta}_p)^2}\, d\theta$$

$$= \int_{\mathbb{R}} e^{\frac{-X_*^2}{2\sigma^2}} e^{\frac{X_*\theta}{\sigma^2}} e^{\frac{-\theta^2}{2\sigma^2}} e^{\frac{-\theta^2}{2\sigma_p^2}} e^{\frac{\theta\hat{\theta}_p}{\sigma_p^2}} e^{\frac{-\hat{\theta}_p^2}{2\sigma_p^2}}\, d\theta$$

Pull out constants

Also not a Function of X_*

$$\propto e^{\frac{-X_*^2}{2\sigma}} \int_{\mathbb{R}} e^{\frac{X_*\theta}{\sigma^2}} e^{\frac{-\theta^2}{2\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2_p}} e^{\frac{\theta\hat{\theta}_p}{\sigma^2_p}} \, d\theta$$

$$= e^{\frac{-X_*^2}{2\sigma}} \int_{\mathbb{R}} e^{\overbrace{\left(\frac{X_*}{\sigma^2} + \frac{\hat{\theta}_p}{\sigma^2_p}\right)}^{a}\theta - \overbrace{\left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma^2_p}\right)}^{b}\theta^2} d\theta$$

$$\theta \sim N\left(\frac{a}{2b}, \frac{1}{2b}\right) = \frac{1}{\sqrt{2\pi\left(\frac{1}{2b}\right)}} e^{-\frac{1}{2\left(\frac{1}{2b}\right)}\left(\theta - \frac{a}{2b}\right)^2}$$

$$= \sqrt{\frac{b}{\pi}}\, e^{-b\left(\theta^2 - \frac{a\theta}{b} + \frac{a^2}{4b^2}\right)} = \sqrt{\frac{b}{\pi}}\, e^{-b\theta^2 + a\theta - \frac{a^2}{4b}}$$

$$= \underbrace{\sqrt{\frac{b}{\pi}}\, e^{\frac{-a^2}{4b}}}_{C} \underbrace{e^{a\theta - b\theta^2}}_{k(\theta)} \propto e^{a\theta - b\theta^2}$$

$$\cancel{= e^{\frac{-X_*^2}{2\sigma^2}} \sqrt{\ }}$$

$$= e^{\frac{-X_*^2}{2\sigma^2}} e^{\left(\frac{X_*}{\sigma^2} + \frac{\hat{\theta}_p}{\sigma^2_p}\right)^2 / 4b}$$

$$= e^{\frac{-X_*^2}{2\sigma^2}} e^{\frac{X_*^2}{4\sigma^2 b}} e^{\frac{X_*\hat{\theta}_p}{2\sigma^2_p\sigma^2 b}} e^{\frac{\hat{\theta}_p^2}{4\sigma^2_p b}}$$

$$\propto e^{\underbrace{\frac{\hat{\theta}_p}{2\sigma^2\sigma^2_p b}}_{A} X_* - \underbrace{\left(\frac{1}{2\sigma^2} - \frac{1}{4\sigma^{22} b}\right)}_{B} X_*^2}$$

$$= e^{AX_* - BX_*^2} \propto N\left(\frac{A}{2B}, \frac{1}{2B}\right)$$

$$2b\sigma^2 = 2\left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma_p^2}\right)\sigma^2 = 1 + \frac{\sigma^2}{\sigma_p^2}$$

$$\Rightarrow \frac{1}{2B} = \frac{1}{2\left(\frac{1}{2\sigma^2} - \frac{1}{4\sigma^2 b}\right)} = \frac{1}{\frac{1}{\sigma^2} - \frac{1}{2b\sigma^2\sigma^2}}$$

$$= \frac{\sigma^{2^2}}{\sigma^2 - \frac{1}{2b}} = \frac{2b\sigma^2\sigma^2}{2b\sigma^2 - 1} = \frac{\left(1 + \frac{\sigma^2}{\sigma_p^2}\right)\sigma^2}{\left(1 + \frac{\sigma^2}{\sigma_p^2}\right) - 1}$$

$$= \frac{\left(1 + \frac{\sigma^2}{\sigma_p^2}\right)\sigma^2}{\frac{\sigma^2}{\sigma_p^2}} = \sigma_p^2\left(1 + \frac{\sigma^2}{\sigma_p^2}\right) = \sigma_p^2 + \sigma^2$$

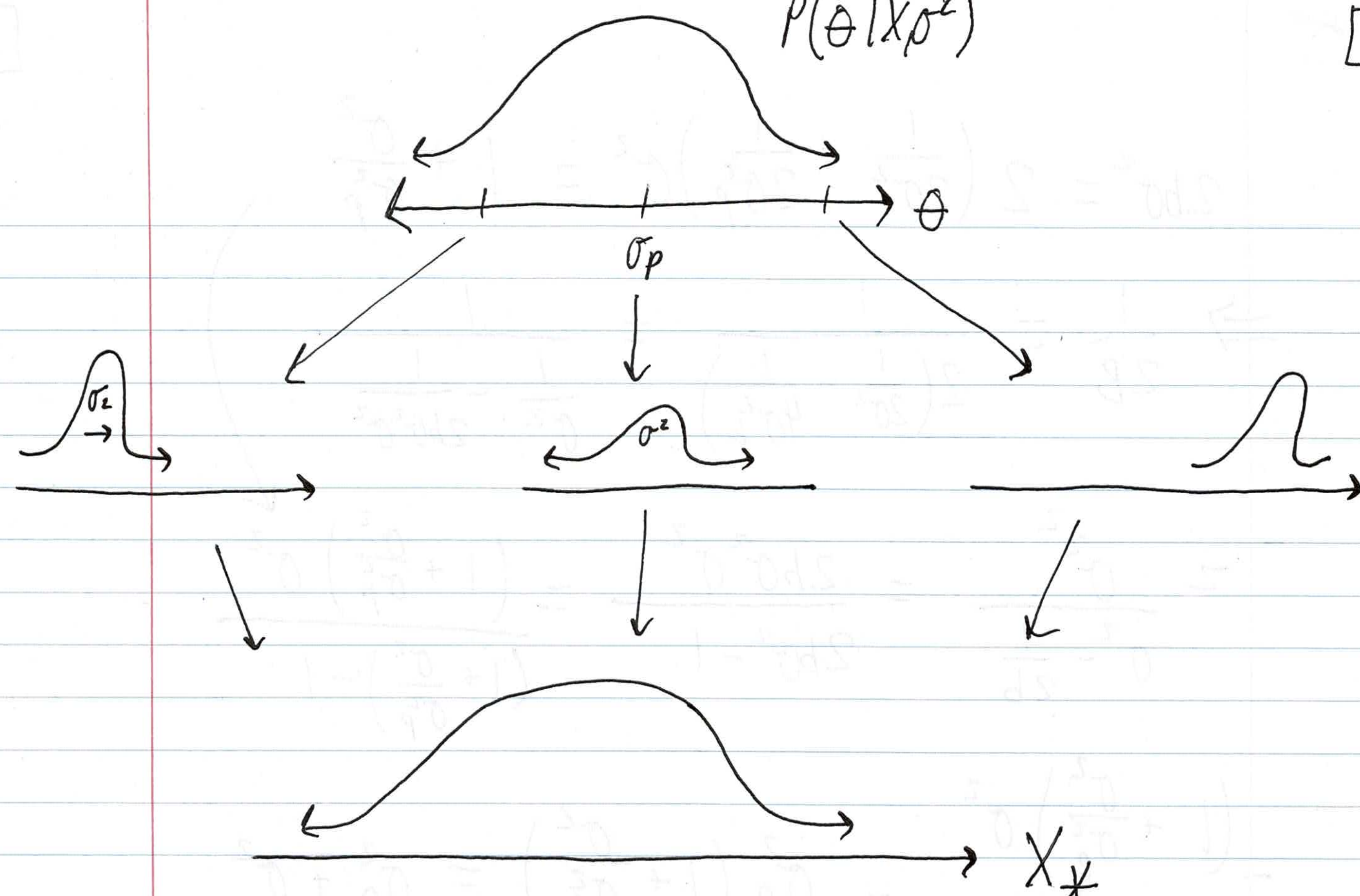
$$\frac{A}{2B} = A(\sigma_p^2 + \sigma^2) = \frac{\hat{\theta}_p}{2\sigma^2\sigma_p^2}(\sigma_p^2 + \sigma^2)$$

$$= \frac{\hat{\theta}_p}{\left(1 + \frac{\sigma^2}{\sigma_p^2}\right)\sigma_p^2}(\sigma_p^2 + \sigma^2) = \frac{\hat{\theta}_p}{\sigma_p^2 + \sigma^2}\,\sigma_p^2 + \sigma^2 = \hat{\theta}_p$$

$$P(X_*|X, \sigma^2) = N(\hat{\theta}_p, \sigma_p^2 + \sigma^2)$$

$$= N\left(\hat{\theta}_p, \left(1 + \frac{1}{n+n_0}\right)\sigma^2\right) \longrightarrow N(\theta, \sigma^2)$$

$$\hat{\theta}_p \longrightarrow \theta, \quad \sigma_p^2 + \sigma^2 \longrightarrow \sigma^2$$



An overdispersed normal with known variance is normal.

F: iid $N(\theta, \sigma^2)$ with θ known

$$\mathcal{L}(\sigma^2; x, \theta) \Rightarrow (2\pi\sigma^2)^{-n/2} e^{\frac{-1}{2\sigma^2}\Sigma(x_i - \theta)^2}$$

$$\ell(\sigma^2; x, \theta) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\Sigma(X_i - \theta)^2$$

$$= -\frac{n}{2}\ln(2\pi) - \frac{n}{2}(\sigma^2) - \frac{1}{2\sigma^2}\Sigma(X_i - \theta)^2$$

$$\ell'(\sigma^2; x, \theta) = \frac{-n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^{2^2}} \overset{\text{set}}{=} 0 \Rightarrow -n + \frac{\Sigma(X_i - \theta)^2}{\sigma^2} = 0$$

$$\hat{\sigma}^2_{MLE} = \frac{\Sigma(X_i - \theta)^2}{n}$$

see previous

$$\Sigma(X_i - \sigma)^2 = n\hat{\sigma}^2_{MLE}$$

Similar to $S^2 = \frac{\Sigma(X_i - \bar{X})^2}{n-1} \longrightarrow \sigma^2$

Laplace Prior: $P(\sigma^2|\theta) \propto 1$ improper since $\sigma^2 \in (0,\infty)$

$$P(\sigma^2|X,\theta) \propto P(X|\theta,\sigma^2) = (2\pi\sigma^2)^{-n/2} e^{\frac{-n\hat{\sigma}^2_{MLE}}{2\sigma^2}}$$

$$\propto (\sigma^2)^{-\frac{n}{2}} e^{\frac{-n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}$$

$$U \sim Gamma(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} U^{\alpha-1} e^{-\beta U}$$

$$V = \frac{1}{U} = t(u) \sim f_u(t^{-1}(v)) \left| \frac{d}{dv}[t^{-1}(v)] \right|$$

$$t^{-1}(v) = \frac{1}{v} \qquad = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{v}\right)^{\alpha-1} e^{-\frac{\beta}{v}} \left| -\frac{1}{v^2} \right|$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} v^{-\alpha+1} v^{-2} e^{-\frac{\beta}{2}} = \frac{\beta^\alpha}{\Gamma(\alpha)} v^{-\alpha-1} e^{-\frac{\beta}{v}}$$

= Inverse Gamma(α, β)

inverse gamma r.v.

$$\propto k(v) = v^{-\alpha-1} e^{-\frac{\beta}{v}}$$

$$P(\sigma^2|X,\theta) = {\sigma^2}^{-\underbrace{(\frac{n}{2}-1)}_{\alpha}-1} e^{\frac{\overbrace{-n\hat{\sigma}^2_{MLE}/2}^{\beta}}{\sigma^2}}$$

$$= InvGamma\left(\frac{n}{2}-1, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$$

$$= InvGamma\left(\frac{n-2}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)$$

$$P(\sigma^2|X,\sigma^2) \propto P(X|\theta,\sigma^2)P(\sigma^2|\theta) \propto (\sigma^2)^{-\frac{n}{2}} e^{\frac{-n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}$$

$k(\sigma^2|\theta)$

$k(\sigma^2|\theta) = ?$ to get conjugacy ?

$$= (\sigma^2)^a e^{\frac{-b}{\sigma^2}} = (\sigma^2)^{-\frac{n}{2}+a} e^{\frac{-\frac{n\sigma^2}{2}+b}{\sigma^2}}$$

$\propto$ InvGamma

is the conjugate prior

Let $P(\sigma^2|\theta) = \text{InvGamma}(\alpha, \beta)$

$$\Rightarrow P(\sigma^2|X,\theta) \propto \left((\sigma^2)^{-n/2} e^{\frac{-n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}\right)\left((\sigma^2)^{-\alpha-1} e^{\frac{-\beta}{\sigma^2}}\right)$$

$$= (\sigma^2)^{-(\frac{n}{2}+\alpha)-1} e^{\frac{-(n\hat{\sigma}^2_{MLE}+\beta}{\sigma^{22}}} \propto \text{InvGamma}\left(\frac{n}{2}+\alpha, \frac{n\hat{\sigma}_{MLE}}{2}+\beta\right)$$

Let $\alpha = \frac{n_0}{2}$, $\beta = \frac{n_0\sigma_0^2}{2}$

$$P(\sigma^2|\theta) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$$

$$\Rightarrow P(\sigma^2|\theta,X) = \text{InvGamma}\left(\frac{n+n_0}{2}, \frac{n\hat{\sigma}^2_{MLE}+n_0\sigma_0^2}{2}\right)$$

Pseudodata: $Y_1, \ldots, Y_{n_0} \sim N(\underset{\text{known}}{\theta}, \underset{\text{belief}}{\sigma_0^2})$

$$\Rightarrow n_0\sigma_0^2 = \Sigma(Y_i-\theta)^2$$

$$\Rightarrow \sigma_0^2 = \frac{\Sigma(Y_i-\theta)^2}{n_0}$$

n_0 small $\longrightarrow$ Uninformative

Haldane: $n_0 = 0$, $\sigma_0^2 = ? \Longrightarrow P(\sigma^2|\theta) = \text{InvGamma}(0,0)$