$X \sim \text{poisson}(\theta) = \frac{e^{-\theta}\theta^{x}}{x!} = P(x;\theta) \propto e^{-\theta}\theta^{x}$ discrete

continuous $Y \sim \text{Gamma}(\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \propto y^{\alpha-1} e^{-\beta y}$

the prior $p(\theta) \sim \text{Gamma}(\alpha, \beta)$

F : iid poisson $X_1, X_2, \dots, X_n ; \theta \overset{iid}{\sim} \text{poisson}(\theta)$

$P(x|\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!}$ this is likelihood.

put a prior on θ. last time, we learn the gamma is conjugate for the poisson likelihood model.

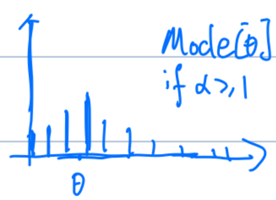
$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)} \propto p(x|\theta)p(\theta) = \left(\frac{e^{-n\theta}\theta^{\sum x_i}}{\prod_{j=1}^{n} x_i!}\right)\left(\frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\right) \propto \underbrace{e^{-n\theta}\theta^{\sum x_i}\theta^{\alpha-1}e^{-\beta\theta}}_{\theta^{\alpha-1+\sum x_i} e^{-(n+\beta)\theta}}$$

$\propto \text{Gamma}(\alpha + \sum x_i, n+\beta)$ posterior

So $p(\theta) \xrightarrow{X} P(\theta|x)$ $\text{Gamma}(\alpha,\beta) \xrightarrow{X} \text{Gamma}(\alpha+\sum x_i, n+\beta)$

$\theta = np$ n: # of success.

$X \sim \text{poisson}(\theta)$ $\text{Supp}[X] = \mathbb{N}_0$ parameter space $\theta \in (0,\infty)$ $E(X) = \theta$



$\text{Mode}[\theta] = \frac{\alpha-1}{\beta}$ if $\alpha \geq 1$ $E(\theta) = \frac{\alpha}{\beta}$ $\hat{\theta}_{MMSE} = E[\theta|x] = \frac{\alpha+\sum x_i}{n+\beta}$ $\hat{\theta}_{MMAE} = \text{Med}[\theta|x] = \text{qgamma}(0.5, \alpha+\sum x_i, n+\beta)$

$\hat{\theta}_{MAP} = \text{Mode}[\theta|x] = \frac{\alpha-1+\sum x_i}{n+\beta}$ if $\alpha + \sum x_i \geq 1$

if $\alpha < 1$

this is Baye's point estimate

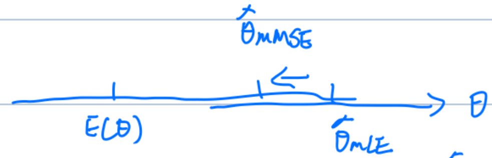
*

$\theta | x \sim \text{Gamma}(\alpha + \sum x_i, n+\beta)$ # of trail β: # of pseudo trials (n_0) $\sum x_i$: number of total success

α: # of pseudo successes

interpre prior

shrinkage:



$$\hat{\theta}_{MMSE} = \frac{\sum x_i + \alpha}{n+\beta} = \frac{\sum x_i}{n+\beta}\cdot\frac{n}{n} + \frac{\alpha}{n+\beta}\cdot\frac{\beta}{\beta}$$

$$= \underbrace{\frac{\sum x_i}{n}}\cdot\underbrace{\frac{n}{\alpha+\beta}}_{1-e} + E(\theta)\cdot\underbrace{\frac{\beta}{n+\beta}}_{e}$$

$\hat{\theta}_{MLE} = \bar{x} = \frac{\sum x_i}{n}$

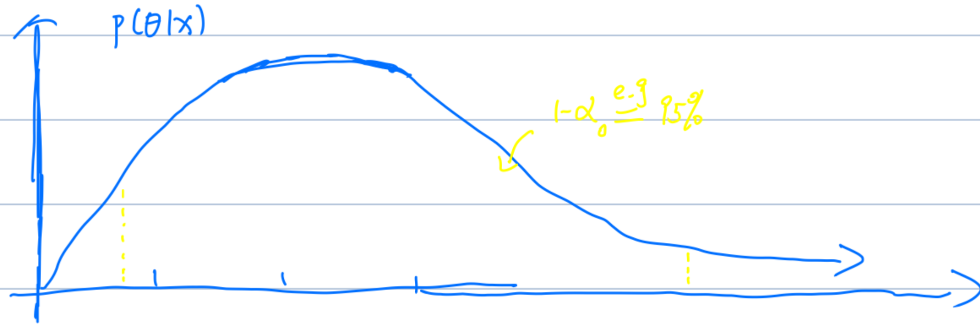
$\lim_{n\to 0} e = 0$

now to prove $\hat{\theta}_{MLE} = \bar{x} = \frac{\sum x_i}{n}$.

$\mathcal{L}(\theta;x) = \frac{e^{-n\theta}\,\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!}$ take ln : $= -n\theta + (\sum x_i)\ln\theta - \ln(\prod_{i=1}^{n} x_i!)$

$l'(\theta;x) = -n + \frac{\sum x_i}{\theta}$ let $l'(\theta;x)=0$ $\hat{\theta}_{MLE} = \frac{\sum x_i}{n} = \bar{x}$

credible region



$CR_{\theta,1-\alpha}: = [qgamma(\frac{\alpha_0}{2}, \alpha+\sum x_i, n+\beta), qgamma(1-\frac{\alpha_0}{2}, \alpha+\sum x_i, n+\beta)]$

hypothesi, leave it in hw.

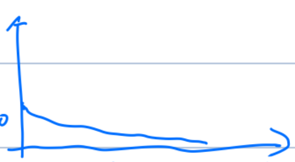
uninformative priors.

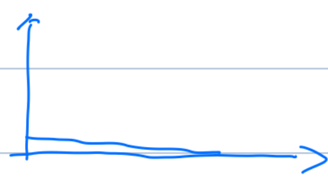
① Laplace indifference priors Gamma(1,0) ② Heldone ignorance prior Gamma(0,0) ③ jeffery Gamma($\frac{1}{2}$,0)

$\theta \sim Gamma(\alpha,\beta)$ Supp $=[0,\infty)$ for ① does $\theta \overset{?}{\sim} U(0,+\infty)$ if we do that $p(\theta) = \frac{1}{\infty} = 0$ not a valid pdf!

let $p(\theta)$ is gamma$(1,\varepsilon) = \frac{\varepsilon}{\Gamma(1)}\theta^0 e^{-\varepsilon\theta} = \varepsilon e^{-\varepsilon\theta}$ if ε is small then $p(\theta) \approx 0$

as $\varepsilon \to 0$





so laplace $\theta \sim Gamma(1,0)$ proper? no, $\beta \neq 0$

under the laplace prior $\Rightarrow p(\theta|x) = Gamma(\sum x_i + 1, n)$ proper? yes always. for n is always ≥ 1

no shrinkage for this case.

② Haldmne? $\beta=0$, $\alpha=0$ $p(\theta|x) = Gamma(\alpha+\sum x_i, n+\beta)$ they would think there is no pseudo success and fail.

so the prior $Gamma(0,0) \propto p(\theta)$

↳ $p(\theta|x) \Rightarrow Gamma(\sum x_i, n)$ proper? only if $\sum x_i > 0$

$\hat{\theta}_{MMSE} = E(\theta|x) = \frac{\Sigma x_i}{n} = \bar{x} = \hat{\theta}_{MLE}$

Jeffery prior: $\quad P_J(\theta) \propto \sqrt{I(\theta)} \quad l'(\theta;x) = -n + \frac{\Sigma x_i}{\theta} \quad -l''(\theta;x) = \frac{\Sigma x_i}{\theta^2}$

$I(\theta) = E_l[-l''(\theta;x)] = E_x\left[\frac{\Sigma x_i}{\theta^2}\right] = \frac{1}{\theta^2}\sum_{i=1}^{n} E(x_i) = \frac{1}{\theta^2}\cdot n\theta = \frac{n}{\theta}$

So $P_J(\theta) \propto \sqrt{I(\theta)} = \sqrt{n/\theta} \propto \theta^{-\frac{1}{2}} = \theta^{\frac{1}{2}-1}\cdot e^{-0\theta} \propto \text{Gamma}(\frac{1}{2}, 0)$

do it as gamma

$P(\theta|x) = \text{Gamma}(\frac{1}{2} + \Sigma x_i, n)$ always proper? yes, cuz $n \geq 1$.

prediction:

x^* is next observation you want to predict $x^*|x \sim ?$ $\text{Supp}[x^*|x] = \text{Supp}[x] = \{0, 1, 2, \dots\}$

$P(x^*|x) = \int_{\Theta} P(x^*|\theta)\,p(\theta|x)\,d\theta = \int_0^{\infty}\left(\frac{e^{-\theta}\theta^{x^*}}{x!}\right)\frac{(n+\beta)^{\alpha+\Sigma x_i}}{\Gamma(\alpha+\Sigma x_i)}$

$= \int_0^{\infty}\left(\frac{e^{-\theta}\theta^{x^*}}{x!}\right)\left(\frac{(n+\beta)^{\alpha+\Sigma x_i}}{\Gamma(\alpha+\Sigma x_i)}\,\theta^{\alpha-1+\Sigma x_i}\,e^{-(n+\beta)\theta}\right)d\theta$