$H_0: \theta = \theta_0$ $H_a: \theta \neq \theta_0 = 0.5$ two sided toss a coin

If Prob: $= p(H_0 | X) < \alpha \Rightarrow$ Reject H_0 / Accept H_a

$p(\theta) = U(0,1)$ $= P(\theta = \theta_0 | X) = 0 \Rightarrow$ Problem.

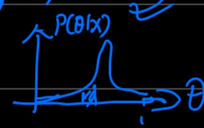
Two Ideas:

① Delta δ e.g $\delta = 0.01$, a margin of "equivalent". Then you change hypothesis.

$H_a: \theta \notin [\theta_0 \pm \delta]$ $H_0: \theta \in [\theta_0 \pm \delta]$ e.g $[0.49, 0.51]$

calculate p-value P-value $= P(H_0 | X) = P(\theta \in [\theta_0 \pm \delta] | x) =$ e.g $n=100$, $x=61$,

qbeta(0.51, 62, 40) − qbeta(0.49, 62, 40)



$= 0.609 - 0.607 = 0.002$ / $0.2\% < \alpha = 5\%$. reject H_0.

② If $\theta_0 \in CR_{\theta, 1-\alpha} \Rightarrow$ Reject H_0 or Retain. (not powerful)

Downside: no pure

Modelling: ① explanation (inference)

↓

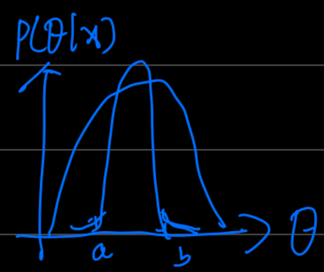
try to approximate reality. ② Prediction

Seen $X_1, \dots, X_n$ and you want to know how X_α (fullture data) will be distributed

$P(X_\alpha | x)$ = the posterior prediction distribution.

If θ was known, what is the posterior predicted distribution.

$P(X^* | \theta)$ The best you can do. But it isn't possible since θ is unknown.



if we know a $P(X=x) = P(x^* | \theta = a) P(\theta = a) + P(x^* | \theta = b) P(\theta = b)$

$p(x) = \int p(x, y)\, dy$

$$P(x^* | x) = \int_\Theta P(x^*, \theta | x)\, d\theta$$

$P(X^* | x) =$

$$= \int P(x^* | \theta, x) P(\theta | x)\, d\theta = \int_\Theta P(x^* | \theta) P(\theta | x)\, d\theta$$

if θ is known, X doesn't provide any information

likelihood $\mathcal{F}$ posterior

discrete $\sum_{\theta \in \Theta} P(X^*|\theta)\, P(\theta|X)$

example: $\Theta_0 = \{0.25, 0.75\}$, $X = <0,1,1>$

$x^* = ?$ One future observation

$P(x^*|x) = \text{Bern}(?)$

$\text{supp}[x^*|x] = \{0,1\}$

Bern

$X^*|x \sim \text{Bern}(\hat{\theta}_{MLE} = \frac{2}{3})$

Problems:

① $\hat{\theta}_{MLE}$ may not be $\in \Theta_0$

② $\hat{\theta}_{MLE}$ could be 0 or 1

③ if multiple future observation

$X_d \sim \text{Bin}(n, \hat{\theta}_{MLE})$ Bad idea (discuss after mid-term)

Assume prior is indifference posterior

$P(\theta = 0.75|x) = 0.53$ $P(\theta = 0.5|x) = 0.47$ $P(x^*|x) = P(x^*|\theta = 0.75)P(\theta = 0.75|x) + P(x^*|\theta = 0.5)P(\theta = 0.5|x)$

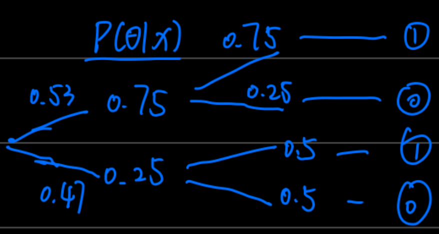
$= P(x^*|\theta = 0.25)\cdot 0.53 + P(x^*|\theta = 0.75)\cdot 0.47$

$= (0.75)^{x^*}(0.25)^{1-x^*}\cdot 0.53 + (0.5)^{x^*}(0.5)^{1-x^*}\cdot 0.47$

P

Compute $P(X^* = 1|x) = 0.75 \cdot 0.25^0 \cdot 0.53 + 0.5^0 \cdot 0.5^1 \cdot 0.47 = 0.6325$

So $P(x^*|x) = 0.6325$



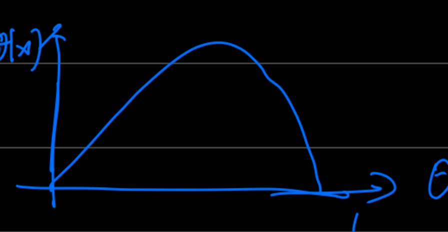
$P(x^*, \theta)$

0.3575
0.1325
0.2350
$+ = 0.6325$
0.2350

general case:

$\mathcal{F}$ is binomial find n $P(\theta) = \text{Beta}(\alpha, \beta) \Rightarrow P(\theta|x) = \text{Beta}(\alpha + x, \beta + n - x)$

what is the posterior prediction for $n_p = 1$



$P(X^*|x) = \int_{\Theta} P(X^*|\theta)\, P(\theta|x)\, d\theta$

$$= \int_0^1 \theta^{x_*}(1-\theta)^{1-x_*}\cdot\left(\frac{1}{B(\alpha+x,\ \beta+n-x)}\theta^{\alpha+x-1}(1-\theta)^{\beta+n-x-1}\right)d\theta$$

$$= \frac{1}{B(\alpha+x,\ \beta+n-x)}\int_0^1 \theta^{x_*+\alpha+x-1}(1-\theta)^{\beta+n-x-x_*}\,d\theta = \frac{B(x_*+\alpha+x,\ \beta+n-x-x_*+1)}{B(\alpha+x,\ \beta+n-x)} = \text{Bern}\left(\frac{\alpha+x}{\alpha+\beta+n}\right)$$

trick compute $P(X_*=1|x) = \dfrac{B(\alpha+x+1,\ \beta+n-x)}{B(\alpha+x,\ \beta+n-x)} = \dfrac{\alpha+x}{\alpha+\beta+n}$ $\qquad \hat{\theta}_{MMSE} = E[\theta|x]$

$B(c_1,c_2) = \dfrac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(c_1+c_2)}$

end of midterm 1.

Mixture Distribution: component dist mixing proportions

$$X \sim \begin{cases} N(0,1^2) & w.p.\ \frac{1}{2} \\ N(10,2^2) & w.p.\ \frac{1}{2} \end{cases} \quad \text{what is } f_X(x)$$

$$p(x) = \sum_{\theta\in\Theta} p(x,\vec{\theta}) = \sum_{\theta\in\Theta} p(x|\vec{\theta})p(\vec{\theta}) = \frac{1}{\sqrt{2\pi\cdot 1^2}}e^{-\frac{1}{2\cdot 1^2}(x-0)^2}\cdot\frac{1}{2} + \frac{1}{\sqrt{2\pi\cdot 2^2}}e^{-\frac{1}{2\cdot 2^2}(x-10)^2}\cdot\frac{1}{2}$$

$$X \sim \begin{cases} \text{Bino}(10, 0.1) & wp\ \frac{1}{4} \\ \text{Bino}(10, 0.8) & wp\ \frac{3}{4} \end{cases} \qquad P_X(x) = \underbrace{\binom{10}{x}0.1^x 0.9^{10-x}}_{\theta\ \ x|\theta}\cdot\underbrace{\tfrac{1}{4}}_{x} + \binom{10}{x}0.8^x 0.2^{10-x}\cdot\frac{3}{4}$$

not a Bin

tree diagram $\frac{1}{4}$ — 0.1 ⋮ 10; $\frac{3}{4}$ — 0.8 ⋮ 10

mixture doesn't has a discrete # of components $\qquad$ compound distr.

$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p_X} = \frac{p(x|\theta)p(\theta)}{\underbrace{\int_\Theta p(x|\theta)p(\theta)d\theta}_{\text{compound.}}}$$