Lecture - 17. 04/21/2020

$F \overset{iid}{\sim} N(\theta, \sigma^2)$ but both θ, σ^2 unknown

$$P(\theta,\sigma^2|x) \propto P(x|\theta,\sigma^2)\, P(\theta,\sigma^2)$$

$$= (2\pi\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}\, P(\theta,\sigma^2)$$

$$\propto \underbrace{(\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2}}_{\text{normal-inverse gamma kernal}}\, K(\theta,\sigma^2)$$

$$\sum(x_i-\theta)^2 = \sum((x_i-\bar{x})+(\bar{x}-\theta))^2$$

$$= \sum(x_i-\bar{x})^2 + 2\sum(x_i-\bar{x})(\bar{x}-\theta) + \sum(\bar{x}-\theta)^2 \quad \text{constant}$$

$$= (n-1)S_x^2 + 2\sum(x_i\bar{x} - \bar{x}^2 - x_i\theta + \bar{x}\theta) + n(\bar{x}-\theta)^2$$

$$= (n-1)S^2 + n(\bar{x}-\theta)^2 + 2(\bar{x}n\bar{x} - n\bar{x}^2 - \theta n\bar{x} + n\bar{x}\theta)$$

$$= (n-1)S^2 + n(\bar{x}-\theta)^2$$

$$= (\sigma^2)^{-n/2}\, e^{-\frac{1}{2\sigma^2}((n-1)S^2 + n(\bar{x}-\theta)^2)}\, K(\theta,\sigma^2)$$

$$= \underbrace{\left((\sigma^2)^{-(n/2-1)-1}\, e^{-\frac{(n-1)S^2/2}{\sigma^2}}\right)\left(e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}\right)}_{\propto \text{ NormInvGamma}(\alpha,\beta,\lambda,\mu)} K(\theta,\sigma^2)$$

$$\Rightarrow \underbrace{P(\theta|x,\sigma^2)\, P(\sigma^2|x)}_{= P(\theta,\sigma^2|x)} \propto \text{NormInvGamma}(\alpha,\beta,\lambda,\mu)\, K(\theta,\sigma^2)$$

Conjugate prior $\propto K(\theta,\sigma^2) = (\sigma^2)^{-\alpha_0-1}\, e^{-\beta_0/\sigma^2}\, e^{-\frac{\lambda_0}{2\sigma^2}(\theta-\mu_0)^2}$ — we won't study this the general prior

$P(\theta,\sigma^2) = K(\theta,\sigma^2)$

$$\propto \text{NormInvGamma}(\alpha_0,\beta_0,\lambda_0,\mu_0)$$

$$= K(\theta|\sigma^2)\, K(\sigma^2) \propto (\text{Normal})(\text{InvGamma}).$$

$$P_J(\theta, \sigma^2) = \underset{\propto 1}{P_J(\theta|\sigma^2)}\ \underset{\propto 1/\sigma^2}{P_0(\sigma^2)} \propto (1)(1/\sigma^2) = 1/\sigma^2$$

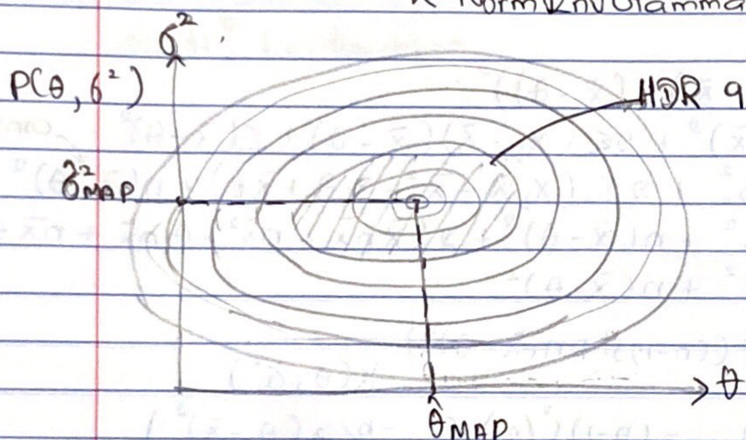
Only principled uninformative prior we will use.

[Posterior] $P(\theta, \sigma^2|x) \propto P(x|\theta, \sigma^2)\, P_J(\theta, \sigma^2)$

$$\propto \left((\sigma^2)^{-n/2} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}\right)(1/\sigma^2)$$

$$= (\sigma^2)^{-n/2-1} e^{\frac{-(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}$$

$$\propto \text{NormInvGamma}\left(n/2, \ (n-1)s^2/2, \ n/2, \ \bar{x}\right)$$



$\langle \hat{\theta}_{MAP}, \hat{\sigma}^2_{MAP} \rangle$

We will not study

$\langle \hat{\theta}_{MMSE}, \hat{\sigma}^2_{MMAE} \rangle$

or

$\langle \hat{\theta}^2_{MMSE}, \hat{\sigma}^2_{MMSE} \rangle$

We won't study confidense sets or hypothesis sets.