F: iid Normal (θ, σ^2) with σ^2 known

$P(\theta \mid \sigma^2) = N(M_0, \tau^2) = N\left(M_0, \frac{\sigma^2}{n_0}\right)$

Imagine pseudodata $Y_1, \ldots, Y_{n_0} \overset{iid}{\sim} N(M_0, \sigma^2) \Rightarrow \bar{Y} \sim N\left(M_0, \frac{\sigma^2}{n_0}\right)$

$\Rightarrow P(\theta \mid X, \sigma^2) = N(\hat{\theta}_P, \sigma_P^2)$ where $\hat{\theta}_P = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{M_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{n\bar{x} + n_0 M_0}{n + n_0}$

$\sigma_P^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{\sigma^2}{n + n_0}$

$P(X_* \mid X, \sigma^2) = \int_\Theta \theta(x_* \mid \theta, \sigma^2) P(\theta \mid X, \sigma^2) d\theta = \int_R N(\theta, \sigma^2) N(\hat{\theta}_P, \sigma_P^2) d\theta$

$= \int_R \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_* - \theta)^2}\right)\left(\frac{1}{\sqrt{2\pi\sigma_P^2}} e^{-\frac{1}{2\sigma_P^2}(\theta - \hat{\theta}_P)^2}\right) d\theta$

$\propto \int_R e^{-\frac{1}{2\sigma^2}(x_* - \theta)^2} e^{-\frac{1}{2\sigma_P^2}(\theta - \hat{\theta}_P)^2} d\theta = \int_R e^{-\frac{x_*^2}{2\sigma^2}} e^{\frac{x_* \theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} e^{-\frac{\theta^2}{2\sigma_P^2}} e^{\frac{\hat{\theta}_P \theta}{\sigma_P^2}} e^{-\frac{\hat{\theta}_P^2}{2\sigma_P^2}} d\theta$

$\propto e^{-\frac{x_*^2}{2\sigma^2}} \int_R e^{\frac{x_* \theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} e^{-\frac{\theta^2}{2\sigma_P^2}} e^{\frac{\hat{\theta}_P \theta}{\sigma_P^2}} d\theta = e^{-\frac{x_*^2}{2\sigma^2}} \int_R e^{\left(\frac{x_*}{\sigma^2} + \frac{\hat{\theta}_P}{\sigma_P^2}\right)\theta - \left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma_P^2}\right)\theta^2} d\theta$

$\boxed{\theta \sim N\left(\frac{a}{2b}, \frac{1}{2b}\right) = \frac{1}{\sqrt{2\pi\left(\frac{1}{2b}\right)}} e^{-\frac{1}{2\left(\frac{1}{2b}\right)}\left(\theta - \frac{a}{2b}\right)^2} = \sqrt{\frac{b}{\pi}} e^{-b\left(\theta^2 - \frac{a\theta}{b} + \frac{a^2}{4b^2}\right)}}$

$= \sqrt{\frac{b}{\pi}} e^{-b\theta^2 + a\theta - \frac{a^2}{4b}} = \sqrt{\frac{b}{\pi}} e^{-\frac{a^2}{4b}} \underbrace{e^{a\theta - b\theta^2}}_{k(\theta)} \propto e^{a\theta - b\theta^2}$

$= e^{-\frac{x_*^2}{2\sigma^2}} \sqrt{\frac{\pi}{b}} e^{\frac{a^2}{4b}} \int \sqrt{\frac{b}{\pi}} e^{-\frac{a^2}{4b}} e^{a\theta - b\theta^2} d\theta = e^{-\frac{x_*^2}{2\sigma^2}} \sqrt{\frac{\pi}{b}} e^{\frac{a^2}{4b}}$

$\propto e^{-\frac{x_*^2}{2\sigma^2}} e^{\left(\frac{x_*}{\sigma^2} + \frac{\hat{\theta}_P}{\sigma_P^2}\right)^2 / 4b} = e^{-\frac{x_*^2}{2\sigma^2}} e^{\frac{x_*^2}{4b\sigma^4}} e^{\frac{x_* \hat{\theta}_P}{2\sigma^2 \sigma_P^2 b}} e^{\frac{\hat{\theta}_P^2}{4\sigma_P^4 b}}$

$\propto e^{\frac{\hat{\theta}_P}{2b\sigma^2\sigma_P^2} x_* - \left(\frac{1}{2\sigma^2} - \frac{1}{4b\sigma^4}\right)x_*^2} = e^{Ax_* - Bx_*^2} \propto N\left(\frac{A}{2B}, \frac{1}{2B}\right)$

$2b\sigma^2 = 2\sigma^2 \cdot \frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{\sigma_P^2}\right)\sigma^2 = 1 + \frac{\sigma^2}{\sigma_P^2}$

$\frac{1}{2B} = \frac{1}{2\left(\frac{1}{2\sigma^2} - \frac{1}{4b\sigma^4}\right)} = \frac{1}{\frac{1}{\sigma^2} - \frac{1}{2b\sigma^4}} = \frac{\sigma^2}{1 - \frac{1}{2b\sigma^2}} = \frac{2b\sigma^2 \sigma^2}{2b\sigma^2 - 1}$

$= \frac{\left(1 + \frac{\sigma^2}{\sigma_P^2}\right)\sigma^2}{\left(1 + \frac{\sigma^2}{\sigma_P^2}\right) - 1} = \frac{\left(1 + \frac{\sigma^2}{\sigma_P^2}\right)\sigma^2}{\frac{\sigma^2}{\sigma_P^2}} = \sigma_P^2\left(1 + \frac{\sigma^2}{\sigma_P^2}\right) = \sigma_P^2 + \sigma^2$

$\frac{A}{2B} = A\left(\sigma_P^2 + \sigma^2\right) = \frac{\hat{\theta}_P}{2b\sigma^2\sigma_P^2}\left(\sigma_P^2 + \sigma^2\right) = \frac{\hat{\theta}_P}{\left(1 + \frac{\sigma^2}{\sigma_P^2}\right)\sigma_P^2}\left(\sigma_P^2 + \sigma^2\right) = \frac{\hat{\theta}_P}{\sigma_P^2 + \sigma^2}\left(\sigma_P^2 + \sigma^2\right)$

$P(X_* \mid X, \sigma^2) = N(\hat{\theta}_P, \sigma_P^2 + \sigma^2)$

$= N\left(\hat{\theta}_P, \left(1 + \frac{1}{n + n_0}\right)\sigma^2\right) \to N(\hat{\theta}, \sigma^2)$

$\hat{\theta}_P \to \theta, \quad \sigma_P^2 + \sigma^2 \to \sigma^2$

An overdispersed normal with known variance is normal.

F: iid $N(\theta, \sigma^2)$ with θ known

$\mathcal{L}(\sigma^2; X, \theta) = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(x_i - \theta)^2}$

$\ell(\sigma^2; X, \theta) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum(x_i - \theta)^2 = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum(x_i - \theta)$

$\dot{\ell}(\sigma^2; X, \theta) = -\frac{n}{2} \cdot \frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum(x_i - \theta)^2 \overset{set}{=} 0 \Rightarrow -n + \frac{\sum(x_i - \theta)^2}{\sigma^2} = 0$

$\Rightarrow \hat{\sigma}_{MLE}^2 = \frac{\sum(x_i - \theta)^2}{n} \overset{similar}{\sim} s^2 = \frac{\sum(x_i - \bar{x})^2}{n - 1} \quad \sigma^2 \sum(x_i - \theta)^2 = n\hat{\sigma}_{MLE}^2$

Laplace prior: $P(\sigma^2 \mid \theta) \propto 1$ improper since $\sigma^2 \in (0, \infty)$

$P(\sigma^2 \mid X, \theta) \propto P(X \mid \theta, \sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}_{MLE}^2}{2\sigma^2}} \propto (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}_{MLE}^2/2}{\sigma^2}}$

$= (\sigma^2)^{-\left(\frac{n}{2} - 1\right) - 1} e^{-\frac{n\hat{\sigma}_{MLE}^2/2}{\sigma^2}}$

$= \text{InvGamma}\left(\frac{n-2}{2}, \frac{n\hat{\sigma}_{MLE}^2}{2}\right)$

$U \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha - 1} e^{-\beta u}$

$V = \frac{1}{U} = g(U) \sim f_U(g^{-1}(v))\left|\frac{d}{dv}\left[g^{-1}(v)\right]\right|$

$g^{-1}(v) = \frac{1}{v} = \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{v}\right)^{\alpha - 1} e^{-\frac{\beta}{v}}\left|-\frac{1}{v^2}\right|$

$= \frac{\beta^\alpha}{\Gamma(\alpha)} v^{-\alpha + 1} v^{-2} e^{-\frac{\beta}{v}} = \frac{\beta^\alpha}{\Gamma(\alpha)} v^{-\alpha - 1} e^{-\frac{\beta}{v}}$

$\propto k(v) = v^{-\alpha - 1} e^{-\frac{\beta}{v}} = \text{Inv Gamma}(\alpha, \beta)$

inverse gamma r.v.

$P(\sigma^2 \mid X, \theta) \propto P(X \mid \theta, \sigma^2) P(\sigma^2 \mid \theta) \propto (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}_{MLE}^2/2}{\sigma^2}} k(\sigma^2 \mid \theta)$

$k(\sigma^2 \mid \theta) = ?$ to get conjugacy? $= (\sigma^2)^{-a} e^{-\frac{b}{\sigma^2}} \checkmark$

$= (\sigma^2)^{-n/2 + a} e^{-\frac{n\hat{\sigma}_{MLE}^2/2 + b}{\sigma^2}}$ $\propto \text{InvGamma}$ is the conjugate prior

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Let $P(\sigma^2 \mid \theta) = \text{Inv Gamma}(\alpha, \beta)$

$\Rightarrow P(\sigma^2 \mid X, \theta) \propto (\sigma^2)^{-n/2} e^{-\frac{n\hat{\sigma}_{MLE}^2/2}{\sigma^2}}(\sigma^2)^{-\alpha - 1} e^{-\frac{\beta}{\sigma^2}} \quad \mathcal{X}(\sigma^2)$

$= (\sigma^2)^{-\left(\frac{n}{2} + \alpha\right) - 1} e^{-\frac{\left(\frac{n\hat{\sigma}_{MLE}^2}{2} + \beta\right)}{\sigma^2}} \propto \text{Inv Gamma}\left(\frac{n}{2} + \alpha, \frac{n\hat{\sigma}_{MLE}^2}{2} + \beta\right)$

let $\alpha = \frac{n_0}{2}, \beta = \frac{n_0 \sigma_0^2}{2} \Rightarrow P(\sigma^2 \mid \theta) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$

$\Rightarrow P(\sigma^2 \mid \theta, X) = \text{Inv Gamma}\left(\frac{n + n_0}{2}, \frac{n\hat{\sigma}_{MLE}^2 + n_0\sigma_0^2}{2}\right)$

Pseudodata: $Y_1, \ldots, Y_{n_0} \sim N(\theta, \sigma_0^2) \Rightarrow n_0\sigma_0^2 = \sum(Y_i - \theta)^2$

n_0 small $\to$ uninformative $\Rightarrow \sigma_0^2 = \frac{\sum(Y_i - \theta)^2}{n_0}$

Haldane: $n_0 = 0, \sigma_0^2 = ? \Rightarrow P(\sigma^2 \mid \theta) = \text{InvGamma}(0, 0)$