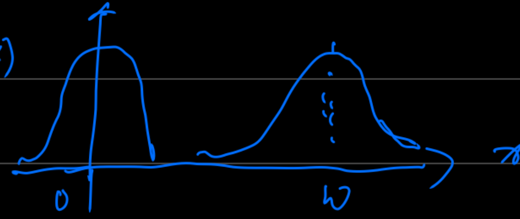
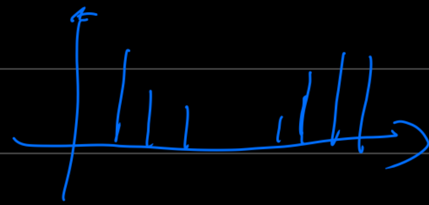
$$X \sim \begin{cases} N(0, 1^2) & wp\ \frac{1}{2} \\ N(10, 2^2) & wp\ \frac{1}{2} \end{cases}$$

$f_X(x)$



$$x \sim \begin{cases} Bin(10, 0.1) & wp\ \frac{1}{4} \\ Bin(10, 0.6) & wp\ \frac{3}{4} \end{cases}$$

$\uparrow$ F



$$p(x) = \int_{\Theta} p(x|\theta)p(\theta)\,d\theta$$
$$= \sum_{\theta \in \Theta} P(x|\theta)P(\theta)\,d\theta$$

$$P(\theta|x) = \frac{P(x|\theta)\cdot P(\theta)}{p(x)} \longrightarrow \int_{\Theta} \underbrace{P(x|\theta)}_{\text{likelihood}}\underbrace{P(\theta)}_{\text{prior}}d\theta$$

$\uparrow$ F Binomial $p(\theta) = Beta(\alpha, \beta) \Rightarrow p(\theta|x) =$

$Beta(x+\alpha, n-x+\beta)$

$$p(x) = \int_0^1 \left[\binom{n}{x}\theta^x(1-\theta)^{n-x}\right]\left(\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\right)d\theta = \binom{n}{x}\frac{1}{B(\alpha,\beta)}\int_0^1 \theta^{\alpha+x-1}(1-\theta)^{n-x+\beta-1}\,d\theta = \binom{n}{x}\frac{B(\alpha+x, n-x+\beta)}{B(\alpha,\beta)}$$

$= $ Beta Binomial (n, α, β)

$$= \frac{n!}{x!(n-x)!}\frac{\Gamma(\alpha+x)\Gamma(n-x+\beta)}{\Gamma(n+\alpha+\beta)}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

Supp $x = \{0, 1, \dots, n\}$

$$= \frac{n!}{x!(n-x)!}\frac{(\alpha+x-1)!(n-x+\beta-1)!}{(n+\alpha+\beta-1)!}\frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!}$$

$$\boxed{E(x) = \sum_{x=0}^{n} x\,p(x) = \frac{n\alpha}{\alpha+\beta}}$$

c.

$$= \frac{\binom{\alpha+x-1}{x}\binom{n-x+\beta-1}{n-x}}{\binom{\alpha+\beta+n-1}{n}} = c$$

$$Var(x) = n\frac{\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)} \quad n \in \mathbb{N}\ \alpha, \beta > 0$$

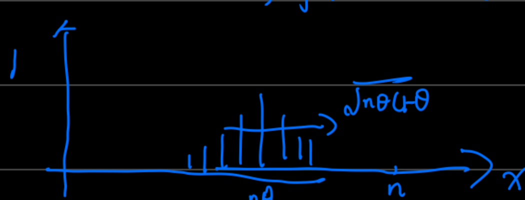
$$\alpha = \frac{\beta\theta}{(1-\theta)}$$

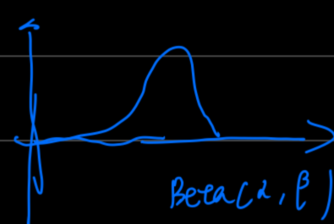
$\Leftrightarrow \frac{\alpha}{\alpha+\beta}\ \frac{\beta}{\alpha+\beta}$

A difference paramtrization let: $\theta = \frac{\alpha}{\alpha+\beta} \Rightarrow \beta = \frac{\alpha(1-\theta)}{\theta}$ $x \sim BetaBin(n, \alpha, \beta) = BetaBin(n, \theta, \alpha)$

$$E(x) = n\theta \qquad Var(x) = \cdots = n\theta(1-\theta)\frac{\frac{\alpha}{\theta}+n}{\frac{\alpha}{\theta}+1} \qquad \lim_{\alpha\to\infty} Var(x) = n\theta(1-\theta)\lim_{\alpha\to\infty}\frac{\frac{\alpha}{\theta}+n}{\frac{\alpha}{\theta}+1} = n\theta(1-\theta)$$

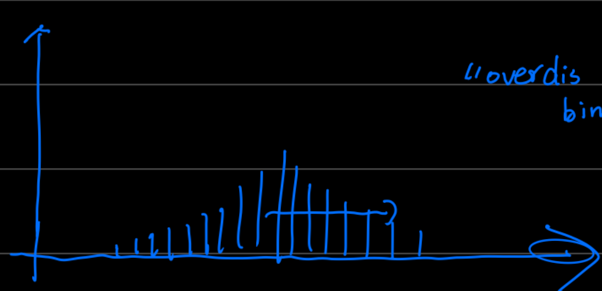
$\theta = 0.7 = \frac{\alpha}{\alpha+\beta}$, if $\alpha = 3.5 \Rightarrow \beta = 1.5$ $\quad x \sim Bin(n, \theta)$





Beta(α, β)

"overdis binomial"



BetaBin(n, α, β)

A model : Birth Gender Data example , $P(\text{Male}) = 0.51$ 6,115 women will more than ≥ 13 children.

in 12 Baby

# Boy	0	1	2	3	4	5	6	7	8	9	10	11	12	
x	3	24	104	286	670	1033	1340	1112	829	478	181	45	7	6115
Mod 1 pr $\hat{F}_1$	1	12	72	239	628	1085	1367	1266	854	410	152	26	2	6115
Mod 2 $\hat{F}_2$	2	23	105	311	656	1036	1258	1182	854	462	178	94	5	

$X|\theta$ 日有 BetaBin (12, 34, 32) 分布.

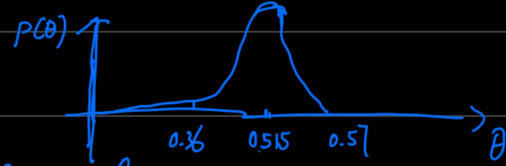
Model 1: $\text{Bin}(12, 0.51)$

Model 2: $\text{BetaBin}(12, 34.32)$

$\alpha_{MLE}: 34 \quad \beta_{MLE}: 32 \quad E(\theta) = 0.515$

qbeta (0.005, 34, 32) = 0.36

qbeta (0.995, 34, 32) = 0.57



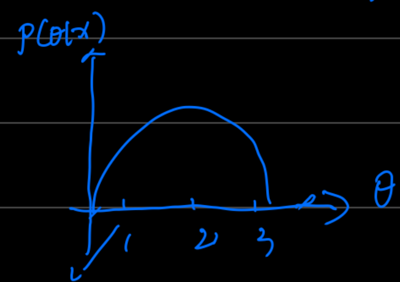
$\hat{F}$ Binomial fixed n $\quad p(\theta) = \text{Beta}(\alpha, \beta) \Rightarrow p(\theta|x) = \text{Beta}(x+\alpha, n-x+\beta) \Rightarrow \hat{\theta}_{MMSE} = \dfrac{x+\alpha}{n+\alpha+\beta}$

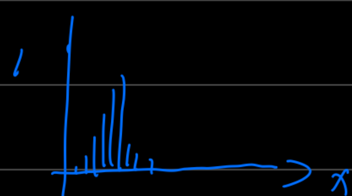
Imagine n_α finite observation $n_x \geq 1$, X_α be # of success if θ was known.

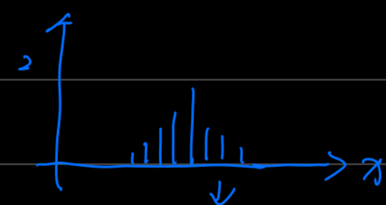
$X_\alpha \sim \text{Bin}(n_\alpha, \theta)$

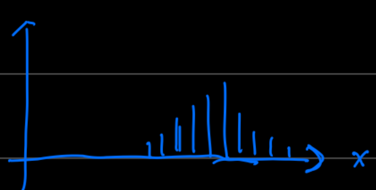
In real life θ is unknown let's Baye's $\Rightarrow p(\theta|x)$

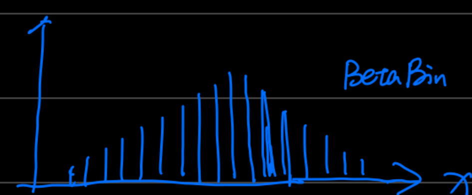
$$P(X_\alpha | x) = \int_\Theta \underbrace{P(X_\alpha|\theta)}_{\text{Bin}(n_\alpha \theta)} \underbrace{p(\theta|x)}_{\text{Beta}(\alpha+x, \beta+n-x)} d\theta \quad = \text{Beta Bin}(n_\alpha, \alpha+x, \beta+n-x) \text{ posterior prediction dist.}$$











For $\hat{F}$ Binomial indifferent prior (laplace), an example of an uninformative prior

$p(\theta): \text{Beta}(\frac{1}{2}, \frac{1}{2})$

Haldane Prior (1932) $p(\theta) = \text{Beta}(0, 0)$ not a legal dist ("improper") don't care.

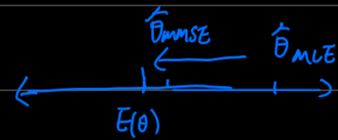
$\Rightarrow p(\theta|x) = \text{Beta}(x, n-x)$ will be proper if $x \neq 0$ and $x \neq n$

$\hookrightarrow \hat{\theta}_{MMSE} = \dfrac{x}{n} = \hat{\theta}_{MLE}$ (no shrinkage)

Objectist: the date must speak for itself. (Haldane)

$$\overset{E(\theta)=0.5}{p(\theta) = U(0,1) = Beta(1,1)} \Rightarrow n_0 = 2 \Rightarrow \rho = \frac{\alpha+\beta}{\alpha+\beta+n} > 0 \checkmark$$

Informative prior: $\theta \sim Beta(\alpha, \beta)$ $\quad$ α, β sth "large" related to n



$E(\theta) = \frac{\alpha}{\alpha+\beta}$ $\quad$ $\rho = \frac{\alpha+\beta}{n+\alpha+\beta}$ is large