Lecture -10 03/05/2020

Informative Priors.

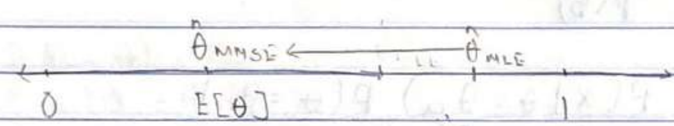
Let θ be the career prob. of getting a hit for a batter in baseball.

$$\hat{\theta}_{MLE} = \frac{X \leftarrow \text{\# of hits}}{n \leftarrow \text{\# of bets}}$$ bad estimator (b/c small n)

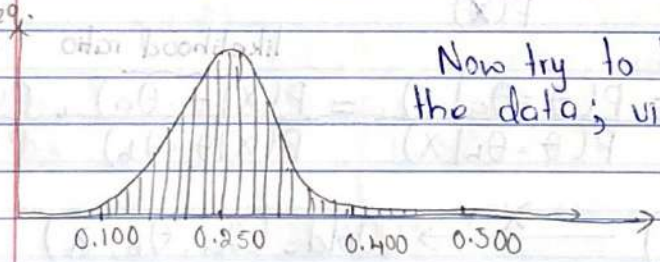
Bein θ in history is 0.366
Averoge is 0.260

$n = 3,\ x = 2$
$\hat{\theta}_{MLE} = \hat{\theta}_{MMSE} = 0.667$ if $\theta \sim Beta(0,0)$
If $\theta \sim Beta(1,1)$
$$\hat{\theta}_{MMSE} = \frac{1+2}{7+1+1} = \frac{3}{9} = \frac{1}{3} = 0.600$$

1) Design a prior s.t. pick α, β
$E[\theta] = 0.260.$



Look at previous data, e.g all players $\geq$ 500 at bats and we examine $\bar{x}$'s



Now try to fit a Beta distribution to the data; via maximum likelihood,
$\hat{\alpha}_{MLE} = 78.7$ &
$\hat{\beta}_{MLE} = 224.8$
$\Rightarrow E[\theta] = 0.260$
$\Rightarrow n_0 = 303.5$

This prior is called "empirical Bayes"
$$\Rightarrow \rho = \frac{303.5}{303.5+3} = 99\%$$

$$\hat{\theta}_{MMSE} = \underset{(1-\rho)}{(1\%)}\underset{\hat{\theta}_{MLE}}{(0.667)} + \underset{\rho}{(99\%)}\underset{E[\theta]}{(0.260)} = 0.263$$

$\theta \sim Beta(1/2, 1/2)$ is called the Jeffery's prior. (uninformative).

$$Odds(A) := \frac{P(A)}{P(A^c)} = \frac{P(A)}{1-P(A)} \in [0, \infty)$$

where $P(A) \in [0,1]$

$$Odds\ Against(A) = Odd(A)^{-1} = \frac{1-P(A)}{P(A)} \in [0, \infty)$$

Example

$$X = \begin{cases} 5 & \text{wp } 1/6 \\ -1 & \text{wp } 5/6 \end{cases}$$

$$E[\theta] = 5(1/6) + (-1)(5/6) = 5/6 - 5/6 = 0.$$

$$Odds(A, B) = \frac{P(A)}{P(B)}$$

$$P(\theta = \theta_a | X) = \frac{P(X|\theta = \theta_a)\, P(\theta = \theta_a)}{P(X)}$$

$$P(\theta = \theta_b | X) = \frac{P(X|\theta = \theta_b)\, P(\theta = \theta_b)}{P(X)}$$

$$Odds(\theta_a, \theta_b | X) = \frac{P(\theta = \theta_a | X)}{P(\theta = \theta_b | X)} = \underbrace{\frac{P(X|\theta = \theta_a)}{P(X|\theta = \theta_b)}}_{\text{likelihood ratio}} \cdot \underbrace{\frac{P(\theta = \theta_a)}{P(\theta = \theta_b)}}_{\substack{Odds(\theta_a, \theta_b) \\ \text{Prior odds}}}$$

$$\Rightarrow Odds(\theta_a, \theta_b) \xrightarrow{X} Odds(\theta_a, \theta_b, X)$$

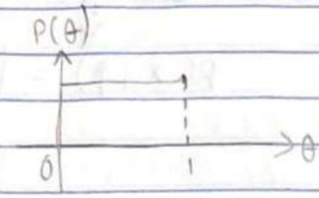
posterior odds

e.g. $1:1 \xrightarrow{X} 5:1$

F: Binomial; fixed n

Let $\phi(\theta)$ be odds θ.

$$\phi(\theta) = \frac{\theta}{1-\theta}$$



$$P(\theta) = U(0,1)$$

Fisher's idea → What is prior of indifferent on ϕ?

$P(\phi) \overset{?}{=} U(0, \infty) = 0 \neq$ not a valid PDF $\quad \int_0^\infty 0 \, d\phi \neq 1$

If $P(\theta) \sim U(0,1)$; What is $P(\phi) = ?$

For a continuous random variable X,

If $y = t(x)$ where t is invertible and $f_x(x)$ known

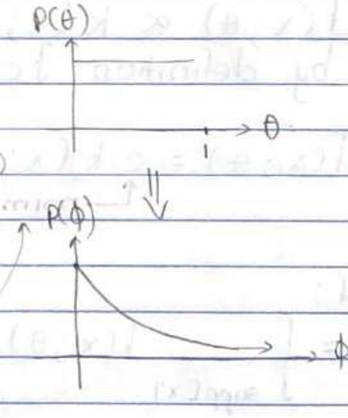
$\Rightarrow f_y(y) = f_x(t^{-1}(y)) \left| \frac{d}{dy}[t^{-1}(y)] \right|$ (derived in Math 368)

$$\phi = \phi(\theta) = \frac{\theta}{1-\theta} = t(\theta)$$

$$f_\phi(\phi) = f_\theta\left(\frac{\phi}{1+\phi}\right) \left| \frac{1}{(1+\phi)^2} \right|$$

$$= \frac{1}{(1+\phi)^2}$$

$$\phi(1-\theta) = \theta$$
$$\phi - \theta\phi = \theta$$
$$\phi = \theta + \theta\phi$$
$$\phi = \theta(1+\phi)$$
$$\theta = \frac{\phi}{(1+\phi)} = t^{-1}(\phi)$$



$$\frac{d}{d\phi}[t^{-1}(\phi)] = \frac{(1+\phi)(1) - \phi(1)}{(1+\phi)^2}$$
$$= \frac{1}{(1+\phi)^2}$$

$$\mathcal{F}: p(x\mid\theta) \xrightarrow{\;t\;} p(x\mid\phi)$$

$$p(x\mid\theta) = \binom{n}{x}\theta^{x}(1-\theta)^{n-x}$$

$$\Updownarrow$$

$$p(x\mid\phi) = \binom{n}{x}\left(\frac{\phi}{1+\phi}\right)^{x}\left(1-\frac{\phi}{1+\phi}\right)^{n-x}$$

$$= \binom{n}{x}\frac{\phi^{x}}{(1+\phi)^{n}}$$

Binomial parametrization with odds.

$$\phi = \phi(\theta) = \frac{\theta}{1-\theta} = t(\theta)$$

$$\Longrightarrow \theta = \frac{\phi}{1+\phi} = t^{-1}(\phi)$$

$$\begin{tikzcd}
\mathcal{F}: p(x\mid\theta) \arrow[r, "\text{Jeffrey's}", "\text{protocol}"'] \arrow[d, "t"] & p(\theta) \\
p(x\mid\phi) \arrow[r, "\text{Jeffrey's}", "\text{protocol}"'] & p(\phi) \arrow[u, "\text{transformation of variables formula works}"']
\end{tikzcd}$$

Let X be continuous with density $f(x;\theta)$

$f(x;\theta) \propto k(x;\theta)$ is unique
by definition $\exists c>0$ not a function of x

$$f(x;\theta) = \underbrace{c}_{\text{normalization constant}}\cdot k(x;\theta)$$

def:

$$\left(1 = \int_{\text{supp}[x]} f(x;\theta)\,dx\right) = \int_{\text{supp}[x]} ck(x;\theta)\,dx$$

$$\Longrightarrow \left(\int_{\text{supp}[x]} k(x;\theta)\,dx\right)^{-1} = c$$

$P(x;\theta) \propto k(x;\theta)$

which means $\exists c > 0$.

$$P(x;\theta) = ck(x;\theta)$$

$$c = \left(\sum_{x \in supp[x]} k(x;\theta) \right)^{-1}$$

$$y \sim \beta eta(\alpha, \beta) = \underbrace{\overset{\text{if}}{\overbrace{\frac{1}{\beta(\alpha,\beta)}}} \overset{K(y;\alpha,\beta)}{\overbrace{y^{\alpha-1}(1-y)^{\beta-1}}}}_{f_y(y)}$$

$$\propto y^{\alpha-1}(1-y)^{\beta-1}$$

F: Binomial, $P(\theta) = \beta eta(\alpha, \beta)$

$$\Longrightarrow P(\theta|x) = \beta eta(x+\alpha, n-x+\beta)$$

$$P(\theta|x) = \frac{P(x|\theta)\, P(\theta)}{P(x)}$$

$$\propto P(x|\theta)\, p(\theta) = \left(\binom{n}{x} \theta^{x} (1-\theta)^{n-x} \right) \left(\frac{1}{\beta(\alpha,\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \right)$$

$$\propto \theta^{x+\alpha-1} (1-\theta)^{n-x+\beta-1}$$

$$\propto \beta eta(x+\alpha, n-x+\beta)$$

$$y \sim N(\theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-1/2\sigma^2 (y-\theta)^2}$$

$$\propto e^{-1/2\sigma^2 (y-\theta)^2}$$

$$= e^{-1/2\sigma^2 (y^2 - 2\theta y + \theta^2)}$$

$$= e^{-y^2/2\sigma^2 + \theta y/\sigma^2 - \theta^2/2\sigma^2}$$

$$= e^{-y^2/2\sigma^2 + \theta y/\sigma^2} e^{-\theta^2/2\sigma^2}$$

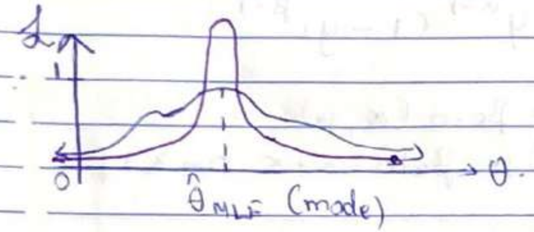
$$\propto e^{\theta y/\sigma^2 - y^2/2\sigma^2} = k(y;\theta,\sigma^2)$$

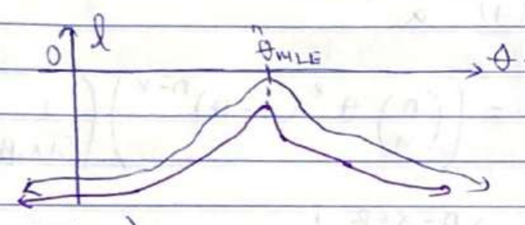
$$C = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\theta^2/2\sigma^2}$$

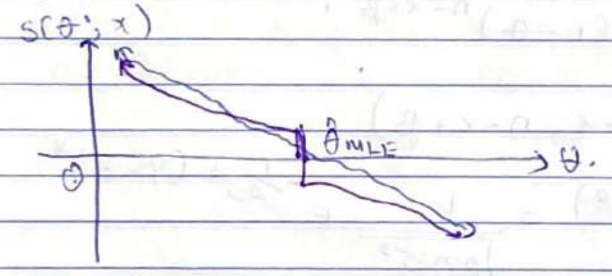
Remark: $\underbrace{\mathcal{L}(\theta;x)}^{\text{likelihood}} = P[x;\theta]$, $\quad \overbrace{\ell(\theta;x)}^{\log} = \ln(\mathcal{L})$

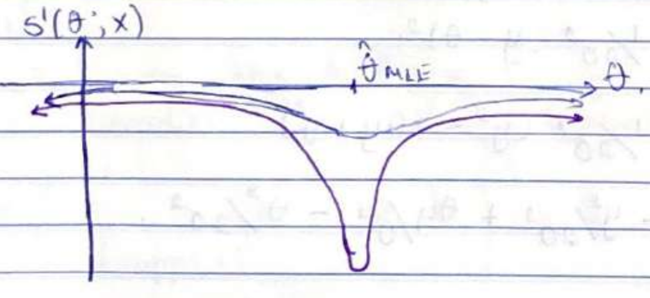
score function $S(\theta, x) := \ell'(\theta;x)$

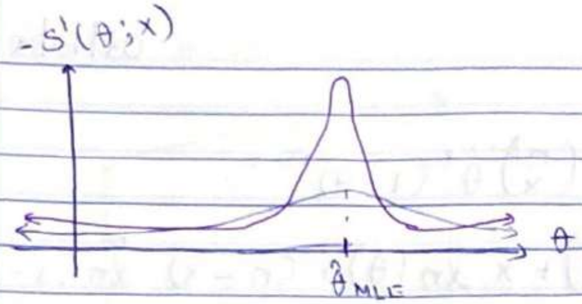
$$\underset{\text{fisher Informative}}{I(\theta)} = \mathrm{Var}_x[S(\theta;x)] = \underset{\text{MA 369}}{\_\_\_} = -E_x[\underbrace{\ell''(\theta;x)}_{S'(\theta;x)}]$$











$X \sim \text{Bin}(n, \theta)$

$P(X; \theta) = \binom{n}{x} \theta^{x} (1-\theta)^{n-x}$

$\ell(\theta; x) = \ln\left(\binom{n}{x}\right) + x \ln(\theta) + (n-x) \ln(1-\theta)$

$$S(\theta; x) = \ell'(\theta; x) = \frac{x}{\theta} - \frac{n-x}{1-\theta}.$$

$$S'(\theta; x) = \ell''(\theta; x) = -\frac{x}{\theta} - \frac{n-x}{(1-\theta)^2}.$$

$$-\ell''(\theta; x) = \frac{x}{\theta} + \frac{n-x}{(1-\theta)^2}.$$

$$\Longrightarrow I(\theta) = E_x\left[x/\theta^2 + \underbrace{n-x}_{n(1-\theta)} \big/ (1-\theta)^2 \right]$$

$$= \frac{n\cancel{\theta}}{\theta^2} + \frac{n - n\theta}{(1-\theta)^2}.$$

$$= \frac{n}{\theta} + \frac{n}{1-\theta} = \boxed{n\left(\frac{1}{\theta(1-\theta)}\right) = I(\theta)}$$