Informative Priors

Let θ be the career prob of getting a hit for a battle of baseball.

$\hat{\theta}_{MLE} = \frac{x}{n}$ $\frac{\text{\# hits}}{\text{\# beats}}$ Bern θ in history : 0.366 average 0.260

$n=3$, $x=2$ $\hat{\theta}_{MLE} = 0.667 = \hat{\theta}_{MMSE}$ if $\theta \sim Beta(0,0)$ If $\theta \sim Beta(1,1)$

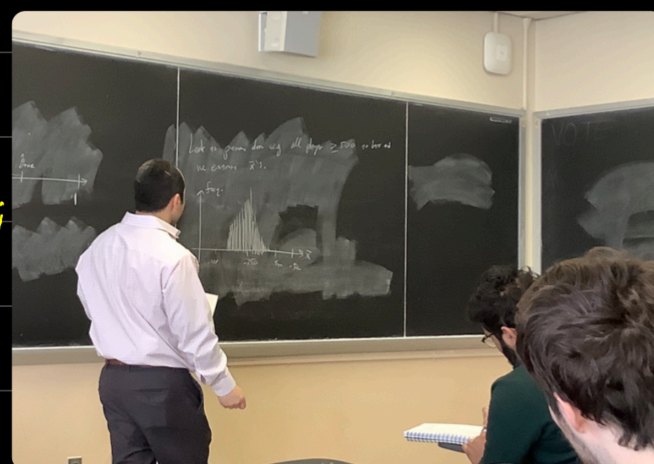
$\hat{\theta}_{MMSE} = \frac{x+\alpha}{n+\alpha+\beta} = \frac{1+2}{3+1+1} = 0.6$

Design a prior s.t $E(\theta) = 0.260$

$\hat{\theta}_{MMSE}$ $\hat{\theta}_{MLE}$

0 $E(\theta)$ 1

look at ______ all plays ≥ 500



try to fit Beta same way as exam1 last question -04

use MLE. $\hat{\alpha}_{MLE} = 78.7$ $\hat{\beta}_{MLE} = 224.8$ $E(\theta) = 0.26$

$n_0 = 303.5$

The prior called: empirical Baye"

$e = \frac{303.5}{303.5+3} = 99\%$ $\hat{\theta}_{MMSE} = \underset{1-e}{1\%}\ \underset{\hat{\theta}_{MLE}}{(0.667)} + 99\%\ \underset{E(\theta)}{(0.26)} = 0.263$

Odds $\theta \sim Beta(\frac{1}{2}, \frac{1}{2})$ is called the Jeffery's prior

Odds $(A) = \frac{P(A)}{P(A^c)} = \frac{P(A)}{1-P(A)} \in (0, +\infty)$ ① Odd against $(A) = \frac{1-P(A)}{P(A)} \in (0, +\infty)$

The odds A and B or Odds $(A, B) = \frac{P(A)}{P(B)}$

$P(\theta = \theta_a | x) = \frac{P(x|\theta=\theta_a) P(\theta=\theta_a)}{P(x)}$ $P(\theta=\theta_b|x) = \frac{P(x|\theta=\theta_b) P(\theta=\theta_b)}{P(x)}$

$\text{Odds}(\theta_a, \theta_b | x) = \frac{P(\theta=\theta_a|x)}{P(\theta=\theta_b|x)} = \underbrace{\frac{P(x|\theta=\theta_a)}{P(x|\theta=\theta_b)}}_{\text{likelihood ratio}} \cdot \underbrace{\frac{P(\theta=\theta_a)}{P(\theta=\theta_b)}}_{\substack{\text{Odds}(\theta_a,\theta_b) \\ \text{prior odds}}} \Rightarrow \text{Odd}(\theta_a, \theta_b) \xrightarrow{x} \text{Odds}(\theta_a, \theta_b | x)$

e.g $1:1 \xrightarrow{x} 5:1$

F: Binomial, n. let $\phi(\theta)$ be odds θ $\phi(\theta) = \frac{\theta}{1-\theta}$ $P(\theta) = U(\theta)\ U(0,1)$

What is prior of indifference a ϕ $P(\phi) \stackrel{?}{=} U(0, +\infty) = 0 \mp$ not a valid PDF

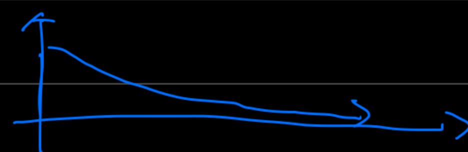
If $P(\theta) = U(0,1)$ $\quad P(\phi) = ?$

For a constant r'v X, If $Y = t(X)$ when t is invertible and $f_X(x)$ known. $\quad X = t^{-1}(Y)$

$$f_Y(y) = f_X(t^{-1}(y)) \left\| \frac{d}{dy}[t^{-1}(y)] \right\|$$

$\phi = \phi(\theta) = \frac{\theta}{1-\theta} = t(\theta)$ $\quad \phi \cdot (1-\theta) = \theta$ $\quad \phi = \theta(1+\phi) \Rightarrow \theta = \frac{\phi}{1+\phi} = t^{-1}(\phi)$

$\frac{d}{d\phi}[t^{-1}(\phi)] = \frac{1}{(1+\phi)^2}$ $\quad \therefore f_\phi(\phi) = \underbrace{f_\theta\left(\frac{\phi}{1+\phi}\right)}_{1} \left|\frac{1}{(1+\phi)^2}\right| = \frac{1}{(1+\phi)^2}$



F: $p(x|\theta) \xrightarrow{t} p(x|\phi)$ $\quad P(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} \iff P(x|\phi) = \binom{n}{x}\left(\frac{\phi}{1+\phi}\right)^x\left(\frac{1}{1+\phi}\right)^{n-x} = \frac{\phi^x}{(1+\phi)^n}$

Binomial parametrized in odds.

$$\begin{tikzcd} p(x|\theta) \arrow[r, "\text{Jeffery protocol}"] \arrow[d, "t"'] & p(\theta) \arrow[d, leftrightarrow, "\text{transform variable}"] \\ p(x|\phi) \arrow[r, "\text{Jeffery protocol ?}"] & p(\phi) \end{tikzcd}$$

about kernel $\quad f(x) = c \cdot g(x) \leftarrow$ take out all possible constant factor

$g(x) \overset{\text{proportional to}}{\propto} f(x)$

Let X be a continuous r'v's with density $f(x;\theta) \propto k(x;\theta)$ is unique. if $\exists c > 0$ not a function of x $\quad f(x;\theta) = c\,k(x;\theta)$ $\quad$ normalization constant

$$c = \begin{cases} \left(\sum_{supp(x)} k(x;\theta)\right)^{-1} \\ \left(\int_{supp(x)} k(x;\theta)\,d\theta\right)^{-1} \end{cases}$$

e.g. $Y = \underbrace{\frac{1}{B(\alpha,\beta)}}_{c} \underbrace{y^{\alpha-1}(1-y)^{\beta-1}}_{k(y;\alpha,\beta)}$

F Binomial: $P(\theta) \sim Bin(n,\theta)$ $\quad p(\theta|x) = Beta(x+\alpha, n-x+\beta)$

$p(\theta|x) \propto \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}$

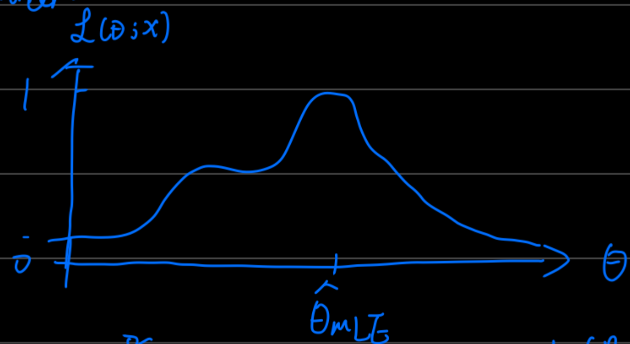
$\propto Beta(x+\alpha, n-x+\beta)$

$Y \sim N(\theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} \propto e^{-\frac{y^2-2\mu y}{2\sigma^2}}$ $\quad C = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}}$

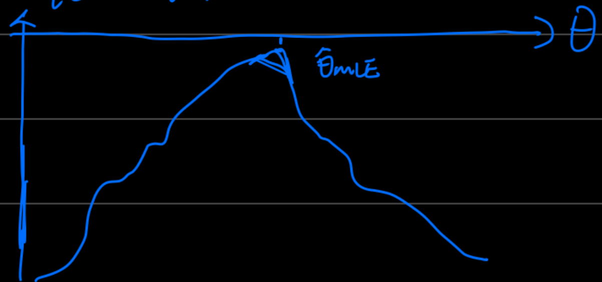
Remark: $\underset{\text{likelihood}}{\mathcal{L}(\theta;x)} = P(X;\theta)$ $\quad \underset{\text{log likelihood}}{l(\theta;x)} = \ln \mathcal{L}$ $\quad$ Score function $= l'(\theta;x)$

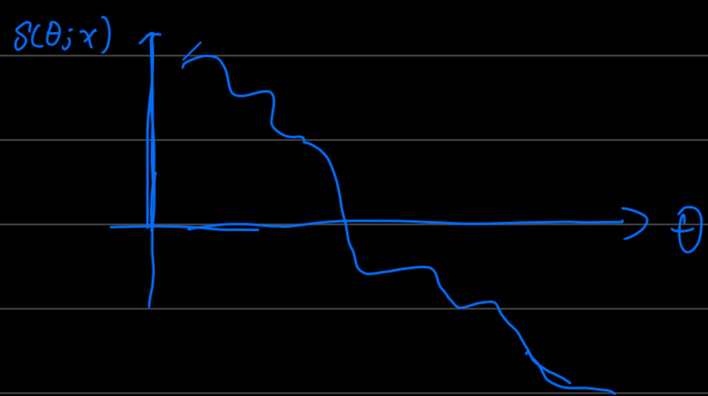
$I(\theta) := Var_x[s(\theta;x)] = \underset{MB69}{\cdots\cdots} = -E_x[l''(\theta;x)]$

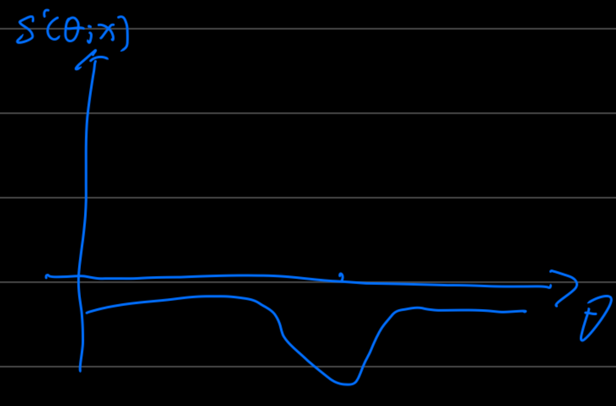
Fisher Information



$l(\theta;x)$ b/c $\mathcal{L}(\theta;x) < 1$ $\therefore$ $l(\theta;c) = \ln(\mathcal{L}(\theta;x)) < 0$







$X \sim Bin(n,\theta)$ $\quad P(x;\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$ $\quad l(\theta;x) = \ln(\binom{n}{x}) + x\ln(\theta) + (n-x)\ln(1-\theta)$

$\frac{d}{d\theta}$

$s(\theta;x) = l'(\theta;x) = \frac{x}{\theta} - \frac{n-x}{1-\theta}$ $\quad l''(\theta;x) = -\frac{x}{\theta^2} - \frac{n-x}{(1-\theta)^2}$ $\quad -l''(\theta;x) = \frac{x}{\theta^2} + \frac{n-x}{(1-\theta)^2}$

$I(\theta) = E_x\left[\frac{x}{\theta^2} + \frac{n-x}{(1-x)^2}\right] = \frac{n\theta}{\theta^2} + \frac{n-n\theta}{(1-\theta)^2} = n\left(\frac{1}{\theta(1-\theta)}\right) = I(\theta)$