Lecture : 7 02-18-20

$\mathcal{F}$: Bin (n, θ) with n known

IF $\alpha = \beta = 1$

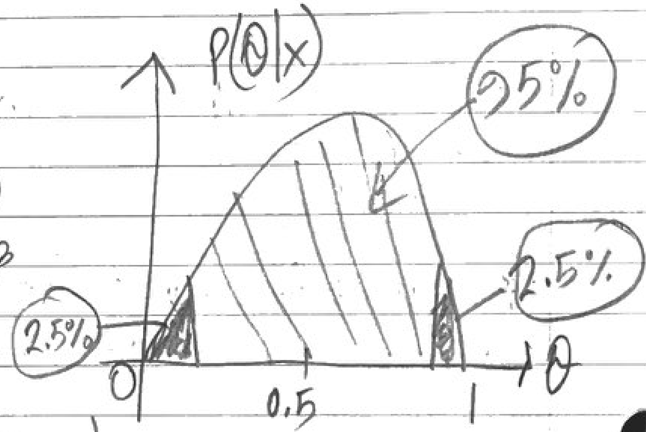
$P(\theta) = \text{Beta}(\alpha, \beta) = U(0,1)$

hyperparameters

$P(\theta|x) = \text{Beta}(\alpha + x, \beta + n - x)$

$n_0 = \alpha + \beta$

Conjugacy



Ex) $P(\theta) = U(0,1) = \text{Beta}(1,1)$, $\alpha = 1$, $\beta = 1$

$x = 1$, $n = 2$

$\Rightarrow P(\theta|x) = \text{Beta}(2,2)$

$\hat{\theta}_{MMSE} = \hat{\theta}_{MMAE} = \hat{\theta}_{MAP} = \frac{1}{2}$

I want a region providing a confidence set for θ.

$$CR_{\theta, 1-\alpha} = \left[\text{Quantile}\left[\theta|x, \tfrac{\alpha}{2}\right], \text{Quantile}\left[\theta|x, 1-\tfrac{\alpha}{2}\right]\right]$$

$\boxed{\text{2-sided credible region}}$

$$P(\theta \in CR_{\theta, 1-\alpha} \mid x) = 1 - \alpha$$

$$CR_{\theta, 95\%} = \left[\text{qbeta}(2.5\%, 2, 2), \text{qbeta}(97.5\%, 2, 2)\right]$$

$$= [0.09, 0.91]$$

Left sided credible region:

$$P(\theta \in CR_{L, \theta, 1-\alpha} \mid x) = 1 - \alpha$$

$\Rightarrow P(\theta \leq L \mid x) = 1-\alpha$

$$CR_{L,\theta,1-\alpha} = \left[\underbrace{\text{smaller value of } \theta \text{ or } -\infty}_{= \inf(\Theta)}, \text{Quantile}[\theta \mid x, 1-\alpha]\right]$$

$CR_{L,\theta,95\%} = [0, \text{qbeta}(95\%, 2, 2)]$

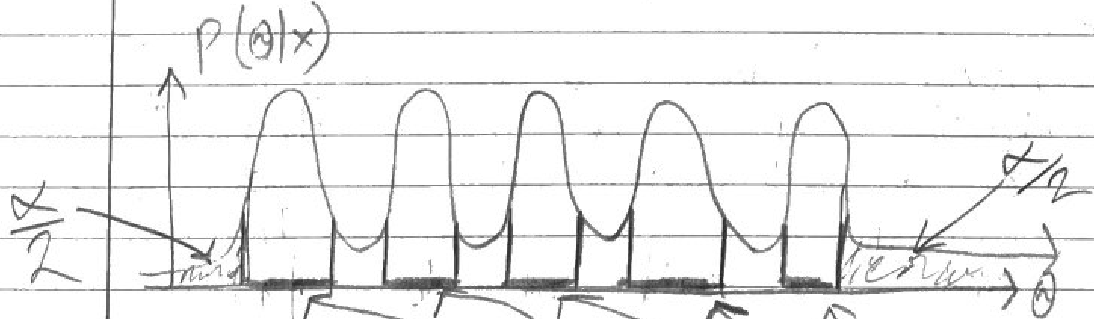
<u>Right sided credible region:</u>

$P(\theta \in CR_{R,\theta,1-\alpha} \mid x) = 1-\alpha$

$\Rightarrow P(\theta \geq R \mid x) = 1-\alpha$

$$CR_{R,\theta,1-\alpha} = \left[\text{Quantile}[\theta \mid x, \alpha], \underbrace{\text{Largest value of } \theta \text{ or } \infty}_{= \sup(\Theta)}\right]$$

$CR_{R,\theta,95\%} = [\text{qbeta}(5\%, 2, 2), 1]$



$HDR_{\theta,1-\alpha} = \ \cup \ \cup \ \cup \ \cup$

$= [0.1, 0.3] \cup [0.5, 0.6] \cup [0.7, 0.8] \cup [0.9 \cup 1.0]$

High-density region.
Smallest posible region
this given $(1-\alpha)$ prob.

$P(\theta \in HDR_{\theta,1-\alpha}) = 1-\alpha$

Ho: UFO's don't exist and alliens have not visited earth.
Ha: UFO's exist and alliens have visited earth.

Disadvantage of HDR:

(1) Computationally Intense.

(2) Non-contiguous in strange.

(4) 3rd goal of inference: theory testing

You wish someone to convience someone of something (Ha) but people currently believe a business-as-usual idea (Ho).

Two ways of "proving" Ha:

(I) Assume Ha is true and defend evidence to the contrrcy. If you cannot provide evidence, Ha stands.

(II) Even though I believe Ha, I'm so confident that it's true that I'm willing to suppose the opposite (Ho) and evidence until everyone sees Ho is wrong and they will conclude Ha.

In stage II, everyone has a level of skepticision with evidence. We call that α. If the evidence doesn't meet or beat this level, we retain Ho. In science at large, we've agreed upon a communal α-level.

In inference, we wish to test theories about θ. We would like to demonstrate the following :-

(A) $H_a : \theta \neq \theta_0 \Rightarrow H_0 : \theta = \theta_0$ (two-sided test)

(B) $H_a : \theta < \theta_0 \Rightarrow H_0 : \theta \geq \theta_0$ (left-sided test)

(C) $H_a : \theta > \theta_0 \Rightarrow H_0 : \theta \leq \theta_0$ (right-sided test)

Baysian Hypothesis testing:

p-val $= P(H_0 | x) < \alpha \Rightarrow$ rejects H_0 / accepts H_a.

$P(H_0 | x) \geq \alpha \Rightarrow$ retain H_0.

$\alpha = 5\%$ is the scientific standard.

(e.g.) $H_0 : \theta < 0.5$
$H_a : \theta \geq 0.5$

$f : Bin(n, \theta)$, n known

$n = 100$, $x = 61$

$P(\theta) = U(0,1)$

$\Rightarrow P(\theta | x) = Beta(62, 40)$

$$P(\theta \leq 0.5 | x) = \int_0^{0.5} \frac{1}{B(62,40)} \theta^{61} (1-\theta)^{31} \, d\theta$$

$= pbeta(0.5, 62, 40)$

$= 0.014 < 5\%$

Rejects H_0. "The coin is unfairly weighed toward heads" Accept H_a.

Notation for integrals of beta dist:

$P(X \leq x) = F(x) = \text{pbeta}(x, \alpha, \beta)$

$P(X > x) = 1 - F(x) = 1 - \text{pbeta}(x, \alpha, \beta)$

(4) θ: prop. of non-5-star rides

IF $\theta > 25\% \Rightarrow$ fire the driver

Bob does 200 rides and gets 37 non-5-star ratings. Do they fire Bob?

$H_0 : \theta \leq 25\%$

$H_a : \theta > 25\%$

$f : \text{Bin}(n, \theta)$, n known

$n = 200$, $x = 37$

$P(\theta) = U(0,1)$

$\Rightarrow P(\theta | x) = \text{Beta}(38, 164)$

$$\text{P-val} = P(\theta \leq 25\% \mid x)$$

$$= \int_0^{0.25} \frac{1}{B(38, 164)} \theta^{37} (1-\theta)^{163} \, d\theta$$

$$= \text{pbeta}(0.25, 38, 164)$$

$$= 0.98.$$

Retain H_0. Don't fire Bob.

(2) $H_0: \theta = \theta_0$
$H_a: \theta \neq \theta_0$

$P\text{-val} = P(\theta = \theta_0 | x) = 0$

We have a problem with 2-sided testing.