Math 341 Spring 2020
Final Examination

Professor Adam Kapelner

Thursday, May 21, 2020

Code of Academic Integrity

Since the college is an academic community, its fundamental purpose is the pursuit of knowledge. Essential to the success of this educational mission is a commitment to the principles of academic integrity. Every member of the college community is responsible for upholding the highest standards of honesty at all times. Students, as members of the community, are also responsible for adhering to the principles and spirit of the following Code of Academic Integrity.

Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

Instructions

This exam is 120 minutes with a bathroom break (with time limit varying for each question). The exam has 145 points and your final score will be scaled out of 100%. The exam is closed book but you are allowed **three** pages (front and back) of a "cheat sheet", one table of reference and scrap paper and a graphing calculator. Please read the questions carefully. No food is allowed, only drinks.

Problem 1 [4min] We return the topic of inference for a baseball batting averages, the θ in a binomial model for a fixed number of n at bats where the realization x is the # of hits.

- [8 pt / 8 pts] Record the letter(s) of all the following that are **true**.

  (a) $\mathbb{P}(\theta) = \text{Binomial}(n, \infty)$ is conjugate.

  (b) $\mathbb{P}(\theta) = \text{Binomial}(n, \infty)$ is proper.

  (c) $\mathbb{P}(\theta) = \text{Beta}(0, 0)$ is conjugate.

  (d) $\mathbb{P}(\theta) = \text{Beta}(0, 0)$ is proper.

  (e) $\mathbb{P}(\theta) = \text{Beta}(1, 1)$ is uninformative.

  (f) $\mathbb{P}(\theta) = \text{Beta}(1, 1)$ is proper.

  (g) $\mathbb{P}(\theta) = \text{Beta}(42.3, 127.7)$ is uninformative.

  (h) $\mathbb{P}(\theta) = \text{Beta}(42.3, 127.7)$ is proper.

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

$\boxed{\text{CEFH}}$

Problem 2 [22min] Imagine the batter bats both leftie and rightie (both sides of the oncoming ball). The propensity to get a hit when batting leftie is θ_L and the propensity to get a hit on when batting rightie is θ_R. He bats leftie ρ proportion of the games. Let I_i be the indicator variable indicating if he bats leftie for the ith game (it is equal to 1 when he bats leftie and equal to 0 if he bats rightie). There are n independent at bats per game and G independent total games per season and both n and G are known. The data $x_1, \ldots, x_G$ are number of hits per game. All other quantities are unknown.

- [17 pt / 25 pts] Record the letter(s) of all the following that are **true**.

(a) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho) = \text{Binomial}(G,\, \theta_L + \theta_R)$

(b) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho) = \text{Binomial}(G,\, \rho\theta_L + (1-\rho)\theta_R)$

(c) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho) = \prod_{i=1}^G (\rho\theta_L(1-\theta_L) + (1-\rho)\theta_R(1-\theta_R))$

(d) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho) = \prod_{i=1}^G (\rho\theta_L^{x_i}(1-\theta_L)^{n-x_i} + (1-\rho)\theta_R^{x_i}(1-\theta_R)^{n-x_i})$

(e) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho) = \prod_{i=1}^G \left(\rho\binom{n}{x_i}\theta_L^{x_i}(1-\theta_L)^{n-x_i} + (1-\rho)\binom{n}{x_i}\theta_R^{x_i}(1-\theta_R)^{n-x_i}\right)$

(f) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \prod_{i=1}^G (\rho\theta_L^{x_i}(1-\theta_L)^{n-x_i} + (1-\rho)\theta_R^{x_i}(1-\theta_R)^{n-x_i})$

(g) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G) =$
$\prod_{i=1}^G \left(\left(\rho\binom{n}{x_i}\theta_L^{x_i}(1-\theta_L)^{n-x_i}\right)^{I_i} + \left((1-\rho)\binom{n}{x_i}\theta_R^{x_i}(1-\theta_R)^{n-x_i}\right)^{1-I_i}\right)$

(h) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \prod_{i=1}^G \left(\rho\binom{n}{x_i}\theta_L^{x_i}(1-\theta_L)^{n-x_i}\right)^{I_i} \left((1-\rho)\binom{n}{x_i}\theta_R^{x_i}(1-\theta_R)^{n-x_i}\right)^{1-I_i}$

(i) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \prod_{i=1}^G \left(\binom{n}{x_i}\theta_L^{x_i}(1-\theta_L)^{n-x_i}\right)^{I_i} \left(\binom{n}{x_i}\theta_R^{x_i}(1-\theta_R)^{n-x_i}\right)^{1-I_i}$

(j) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \prod_{i=1}^G \binom{n}{x_i} \prod_{i=1}^G (\theta_L^{x_i}(1-\theta_L)^{n-x_i})^{I_i} (\theta_R^{x_i}(1-\theta_R)^{n-x_i})^{1-I_i}$

(k) $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G \mid X) = \mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G)$

(l) $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G \mid X) = \mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G)\,\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G)$

(m) $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G \mid X) \propto \mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G)\,\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G)$

(n) $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \mathbb{P}(\theta_L, \theta_R)\,\mathbb{P}(\rho, I_1, \ldots, I_G)$ if θ_L, θ_R are independent of $\rho, I_1, \ldots, I_G$.

(o) $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \mathbb{P}(\theta_L)\,\mathbb{P}(\theta_R)\,\mathbb{P}(\rho)\,\mathbb{P}(I_1, \ldots, I_G \mid \rho)$ if θ_L, θ_R are independent of $\rho, I_1, \ldots, I_G$ and θ_L, θ_R are independent of each other.

(p) $\mathbb{P}(I_1, \ldots, I_G \mid \rho) = \text{Binomial}(G,\, \rho)$

(q) $\mathbb{P}(I_1, \ldots, I_G \mid \rho) = \rho^{\sum_{i=1}^G I_i}(1-\rho)^{\sum_{i=1}^G (1-I_i)}$

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

$\boxed{\text{EIJMNOQ}}$

Problem 3 [15min] Imagine the batter bats both leftie and rightie (both sides of the oncoming ball). The propensity to get a hit when batting leftie is θ_L and the propensity to get a hit on when batting rightie is θ_R. He bats leftie ρ proportion of the games. Let I_i be the indicator variable indicating if he bats leftie for the ith game (it is equal to 1 when he bats leftie and equal to 0 if he bats rightie). There are n independent at bats per game and G independent total games per season and both n and G are known. The data $x_1, \ldots, x_G$ are number of hits per game. All other quantities are unknown. All sum signs are indexed from $1, \ldots, G$. The full likelihood is

$$\mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \prod_{i=1}^{G} \left(\theta_L^{x_i}(1-\theta_L)^{n-x_i}\right)^{I_i} \left(\theta_R^{x_i}(1-\theta_R)^{n-x_i}\right)^{1-I_i} \prod_{i=1}^{G} \binom{n}{x_i}$$

and the prior is

$$\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \mathbb{P}(\theta_L)\,\mathbb{P}(\theta_R)\,\mathbb{P}(\rho)\,\rho^{\sum I_i}(1-\rho)^{\sum(1-I_i)}$$

Assume the following about the components in the overall prior:

$$\mathbb{P}(\theta_L) \propto 1,\ \mathbb{P}(\theta_R) \propto 1,\ \mathbb{P}(\rho) \propto 1.$$

- [14 pt / 39 pts] Record the letter(s) of all the following that are **true**.

(a) The prior is uninformative.

(b) The prior is improper.

(c) $\mathbb{P}(\theta_L) = \text{Beta}(1,\ 1)$

(d) $\mathbb{P}(\theta_L) = \text{Gamma}(1,\ 0)$

(e) $\mathbb{P}(I_i) = \text{U}(0,\ 1)$ for all i

(f) $\mathbb{P}(X) = \text{U}(0,\ 1)$.

(g) $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \rho^{\sum I_i}(1-\rho)^{\sum(1-I_i)}$

(h) $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G) \propto \rho^{\sum I_i}(1-\rho)^{\sum(1-I_i)}$

(i) The conjugate model for $\theta_L, \theta_R, \rho, I_1, \ldots, I_G$ under the likelihood above is a standard distribution we have studied before.

(j) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G) \propto 1$

(k) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \left(\theta_L^{\sum x_i}(1-\theta_L)^{n-\sum x_i}\right)^{I_i} \left(\theta_R^{\sum x_i}(1-\theta_R)^{n-\sum x_i}\right)^{1-I_i} \prod_{i=1}^{G} \binom{n}{x_i}$

(l) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \theta_L^{\sum I_i x_i}(1-\theta_L)^{\sum(1-I_i)x_i}\theta_R^{\sum(1-I_i)x_i}(1-\theta_R)^{n-\sum(1-I_i)x_i} \prod_{i=1}^{G} \binom{n}{x_i}$

(m) $\mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \theta_L^{\sum I_i x_i}(1-\theta_L)^{\sum I_i(n-x_i)}\theta_R^{\sum(1-I_i)x_i}(1-\theta_R)^{\sum(1-I_i)(n-x_i)} \prod_{i=1}^{G} \binom{n}{x_i}$

(n) There are constants that can be removed when finding the kernel of (m).

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

ACGHM

Problem 4 [15min] Imagine the batter bats both leftie and rightie (both sides of the oncoming ball). The propensity to get a hit when batting leftie is θ_L and the propensity to get a hit on when batting rightie is θ_R. He bats leftie ρ proportion of the games. Let I_i be the indicator variable indicating if he bats leftie for the ith game (it is equal to 1 when he bats leftie and equal to 0 if he bats rightie). There are n independent at bats per game and G independent total games per season and both n and G are known. The data $x_1, \ldots, x_G$ are number of hits per game. All other quantities are unknown. All sum signs are indexed from $1, \ldots, G$. The full likelihood is

$$\mathbb{P}(X \mid \theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \theta_L^{\sum I_i x_i}(1-\theta_L)^{\sum I_i(n-x_i)}\theta_R^{\sum(1-I_i)x_i}(1-\theta_R)^{\sum(1-I_i)(n-x_i)}\prod_{i=1}^{G}\binom{n}{x_i}$$

and assume the prior is

$$\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G) = \rho^{\sum I_i}(1-\rho)^{\sum(1-I_i)}$$

- [16 pt / 55 pts] Record the letter(s) of all the following that are **true**.

(a) $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G \mid X) \propto \theta_L^{\sum I_i x_i}(1-\theta_L)^{\sum I_i(n-x_i)}\theta_R^{\sum(1-I_i)x_i}(1-\theta_R)^{\sum(1-I_i)(n-x_i)} \times \rho^{\sum I_i}(1-\rho)^{n-\sum I_i}$

(b) $\mathbb{P}(\theta_L \mid X, \theta_R, \rho, I_1, \ldots, I_G) \propto 1$

(c) $\mathbb{P}(\theta_L \mid X, \theta_R, \rho, I_1, \ldots, I_G) \propto \theta_L^{\sum I_i x_i}(1-\theta_L)^{\sum I_i(n-x_i)}$

(d) $\mathbb{P}(\theta_L \mid X, \theta_R, \rho, I_1, \ldots, I_G) \propto \text{Binomial}(\theta_L, \sum I_i)$

(e) $\mathbb{P}(\theta_L \mid X, \theta_R, \rho, I_1, \ldots, I_G) \propto \text{Beta}(1 + \sum I_i x_i, 1 + \sum I_i(n-x_i))$

(f) $\mathbb{P}(\theta_R \mid X, \theta_L, \rho, I_1, \ldots, I_G) \propto 1$

(g) $\mathbb{P}(\theta_R \mid X, \theta_L, \rho, I_1, \ldots, I_G) \propto \theta_R^{\sum(1-I_i)x_i}(1-\theta_R)^{\sum(1-I_i)(n-x_i)}$

(h) $\mathbb{P}(\theta_R \mid X, \theta_L, \rho, I_1, \ldots, I_G) \propto \text{Binomial}(\theta_R, n - \sum I_i)$

(i) $\mathbb{P}(\theta_R \mid X, \theta_L, \rho, I_1, \ldots, I_G) \propto \text{Beta}(1 + \sum(1-I_i)x_i, 1 + \sum(1-I_i)(n-x_i))$

(j) $\mathbb{P}(\rho \mid X, \theta_L, \theta_R, I_1, \ldots, I_G) \propto 1$

(k) $\mathbb{P}(\rho \mid X, \theta_L, \theta_R, I_1, \ldots, I_G) \propto \rho^{\sum I_i}(1-\rho)^{\sum(1-I_i)}$

(l) $\mathbb{P}(\rho \mid X, \theta_L, \theta_R, I_1, \ldots, I_G) \propto \text{Binomial}(\rho, \sum(1-I_i))$

(m) $\mathbb{P}(\rho \mid X, \theta_L, \theta_R, I_1, \ldots, I_G) \propto \text{Beta}(1 + \sum I_i, 1 + \sum(1-I_i))$

(n) $\mathbb{P}(I_1 \mid X, \theta_L, \theta_R, I_2, \ldots, I_G, \rho) = \text{Bernoulli}(\cdot)$ with a parameter that is a function of $X, \theta_L, \theta_R, I_2, \ldots, I_G, \rho$.

(o) $\mathbb{P}(I_1 \mid X, \theta_L, \theta_R, I_2, \ldots, I_G, \rho) = \text{Bernoulli}(\cdot)$ with a parameter that is a function of $X, \theta_L, \theta_R, \rho$.

(p) $\mathbb{P}(I_1 \mid X, \theta_L, \theta_R, I_2, \ldots, I_G, \rho) = \text{Bernoulli}(\cdot)$ with a parameter that is a function of only X and ρ.

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

ACEGIKMO

Problem 5 [8min] Imagine the batter bats both leftie and rightie (both sides of the oncoming ball). The propensity to get a hit when batting leftie is θ_L and the propensity to get a hit on when batting rightie is θ_R. He bats leftie ρ proportion of the games. Let I_i be the indicator variable indicating if he bats leftie for the ith game (it is equal to 1 when he bats leftie and equal to 0 if he bats rightie). There are n independent at bats per game and G independent total games per season and both n and G are known. The data $x_1, \ldots, x_G$ are number of hits per game. All other quantities are unknown. All sum signs are indexed from $1, \ldots, G$. Consider a Gibbs sampler using the following conditional distributions:

$$\begin{aligned}
\mathbb{P}(\theta_L \mid X, \theta_R, \rho, I_1, \ldots, I_G) &= \text{Beta}\left(1 + \sum I_i x_i,\ 1 + \sum I_i (n - x_i)\right) \\
\mathbb{P}(\theta_R \mid X, \theta_L, \rho, I_1, \ldots, I_G) &= \text{Beta}\left(1 + \sum (1 - I_i) x_i,\ 1 + \sum (1 - I_i)(n - x_i)\right) \\
\mathbb{P}(\rho \mid X, \theta_L, \theta_R, I_1, \ldots, I_G) &= \text{Beta}\left(1 + \sum I_i,\ 1 + \sum (1 - I_i)\right) \\
\forall i \ \ \mathbb{P}(I_i \mid X, \theta_L, \theta_R, I_{-i}, \rho) &= \text{Bernoulli}\left(\frac{\rho \theta_L^{x_i}(1-\theta_L)^{n-x_i}}{\rho \theta_L^{x_i}(1-\theta_L)^{n-x_i} + (1-\rho)\theta_R^{x_i}(1-\theta_R)^{n-x_i}}\right)
\end{aligned}$$

- [13 pt / 68 pts] Record the letter(s) of all the following that are **true**.

(a) The first conditional distribution can be sampled using the `rbeta` function.

(b) This Gibbs sampler will converge after a sufficient burn-in period.

(c) This Gibbs sampler will converge after thinning.

(d) This Gibbs sampler does not include all the conditional distributions necessary to provide samples from the posterior.

(e) This Gibbs sampler will not converge if we pick a starting position that is far from the center of the posterior's density.

(f) This Gibbs sampler requires us to specify the upper and lower bounds of θ_L, θ_R and ρ before we begin.

(g) This Gibbs sampler can get stuck in local modes rendering our entire inferential procedure inaccurate.

(h) This Gibbs sampler will provide $\overset{iid}{\sim}$ samples from the posterior $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G \mid X)$ after a sufficient burn-in number of samples are dropped.

(i) This Gibbs sampler will provide $\overset{iid}{\sim}$ samples from the posterior $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G \mid X)$ after a sufficient burn-in number of samples are dropped and the samples are thinned to eliminate autocorrelation among the iterations of the sampler.

(j) The conditional distributions tell us how many total samples to run in the sampler.

(k) This Gibbs sampler requires one or more grid sampling step(s).

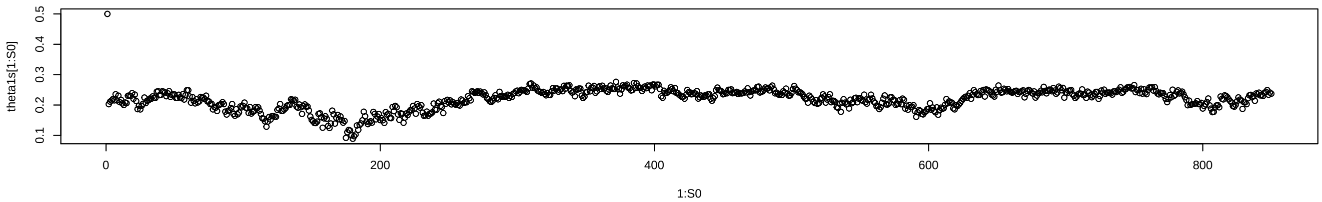
(l) Sampling I_i from its conditional distribution is impossible since it is not a continuous random variable.

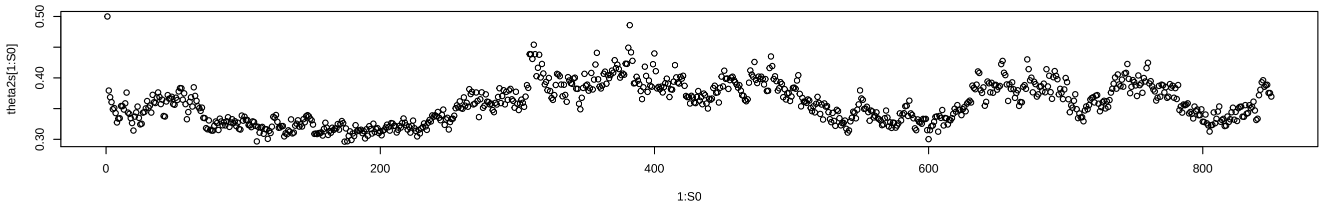
(m) Since the dimension of the parameter space (i.e. the number of parameters) is large, this Gibbs sampler cannot provide accurate inference.

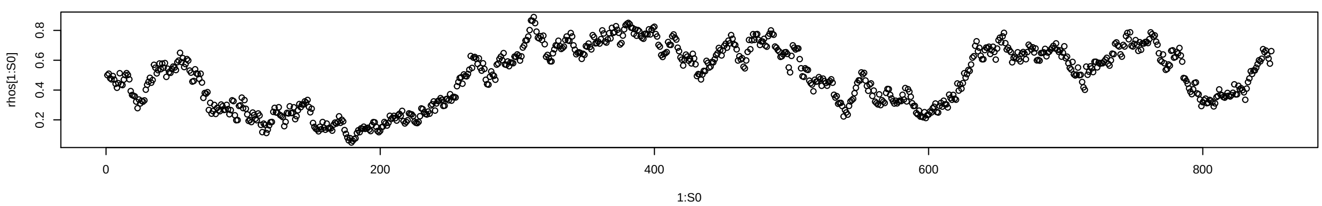
Your answer will consist of a string (e.g. **aebgd**) where the order of the letters does not matter nor does upper / lowercase.

ABI

Problem 6 [4min] We employ the Gibbs sampler specified previously to provide $\overset{iid}{\sim}$ samples from the posterior $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G \mid X)$. We run $S = 100,000$ samples. We plot the first 850 below where the top plot is the chain for $\theta_L \mid X$, the middle plot is the chain for $\theta_R \mid X$ and the bottom plot is the chain for $\rho \mid X$. The chains for $I_1 \mid X, \ldots, I_G \mid X$ are not displayed.







- [4 pt / 72 pts] Where should we burn-in the chain? Select the best answer below:

  (a) 10

  (b) 380

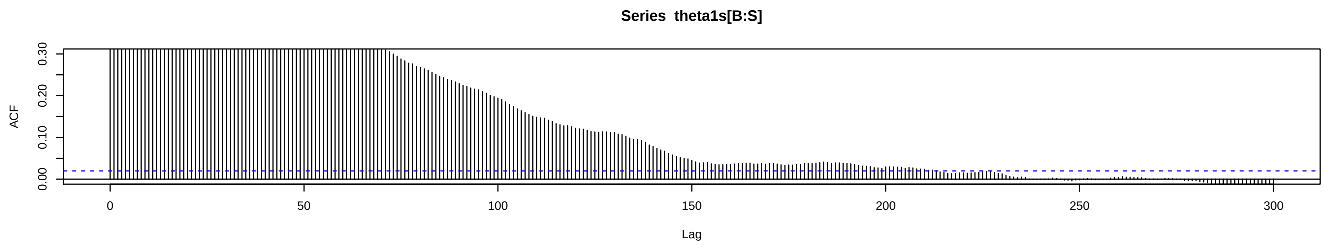
  (c) 750

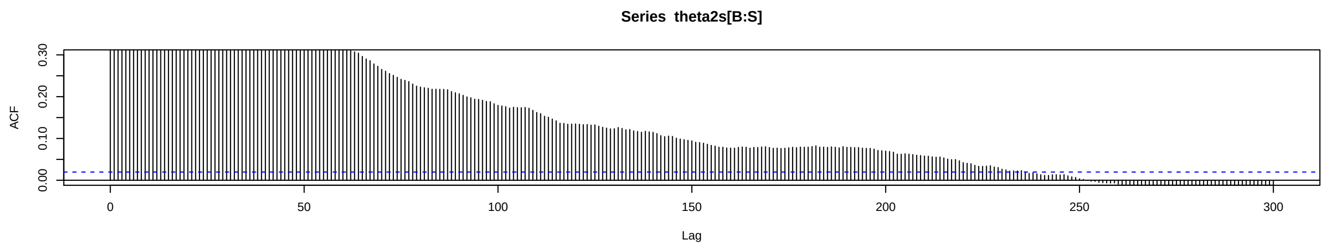
  (d) never

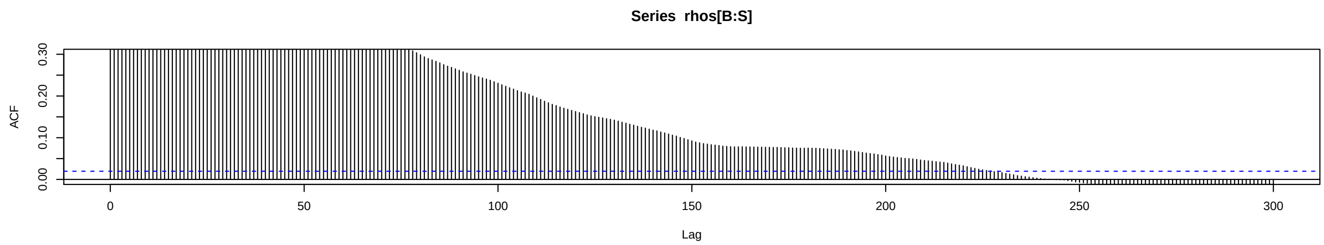
Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

B

Problem 7 [4min] We employ the Gibbs sampler specified previously to provide $\stackrel{iid}{\sim}$ samples from the posterior $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G \mid X)$. We run $S = 100,000$ samples. We appropriately burned in. Now we wish to thin the chain. Below are autocorrelation plots. The top plot is the chain for $\theta_L \mid X$, the middle plot is the chain for $\theta_R \mid X$ and the bottom plot is the chain for $\rho \mid X$. The chains for $I_1 \mid X, \ldots, I_G \mid X$ are not displayed.







- [4 pt / 76 pts] What multiple of the chain samples should we thin? Select the best answer below:

  (a) 10

  (b) 100

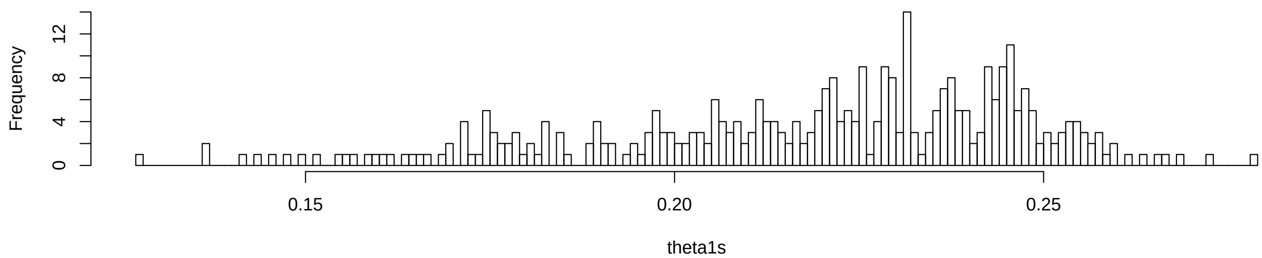
  (c) 250

  (d) never

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

C

Problem 8 [6min] We employ the Gibbs sampler specified previously to provide $\stackrel{iid}{\sim}$ samples from the posterior $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G \mid X)$. We run $S = 100,000$ samples. We appropriately burned and thinned and now we provide histograms for the marginal posteriors. Below is a plot for θ_L.



- [12 pt / 88 pts] Record the letter(s) of all the following that are **true**.

(a) The above approximates $\mathbb{P}(\theta_L \mid X, \theta_R, \rho, I_1, \ldots, I_G)$

(b) The above approximates $\mathbb{P}(\theta_L \mid X, \theta_R, \rho)$

(c) The above approximates $\mathbb{P}(\theta_L \mid X)$

(d) The MMSE for θ_L is approximately 0.15

(e) The MMSE for θ_L is approximately 0.20

(f) The MMSE for θ_L is approximately 0.22

(g) The MMSE for θ_L is equal to the MMAE for θ_L.

(h) The MMSE for θ_L is larger than the MMAE for θ_L.

(i) A 95% credible region for θ_L is impossible to estimate from this plot alone.

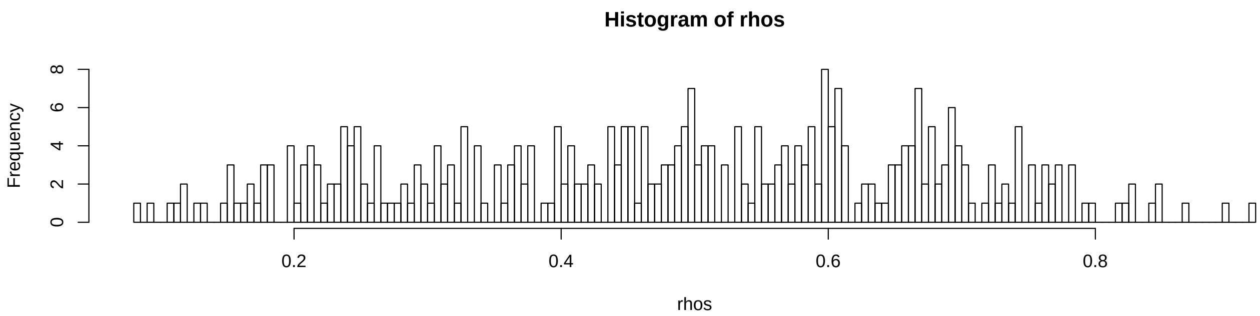
(j) This plot alone will allow us to test if $\theta_L \neq \theta_R$.

(k) If you are testing if $\theta_L > 0.15$, the p value would be small and the null likely rejected.

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

CFK

Problem 9 [6min] We employ the Gibbs sampler specified previously to provide $\stackrel{iid}{\sim}$ samples from the posterior $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G \mid X)$. We run $S = 100,000$ samples. We appropriately burned and thinned and now we provide histograms for the marginal posteriors. Below is a plot for ρ.



- [5 pt / 93 pts] Record the letter(s) of all the following that are **true**.

(a) The above approximates $\mathbb{P}(\rho \mid X)$

(b) A 95% credible region for ρ would range from 0.1 to 0.9.

(c) The main conclusion here is that ρ cannot be estimated well.

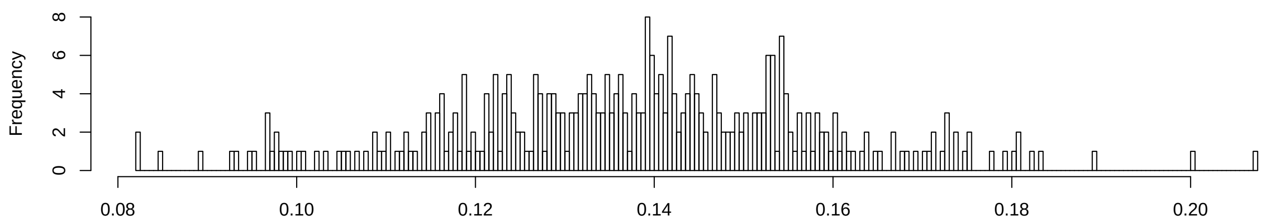
(d) The MMSE for ρ is nearly equal to the MMAE for ρ.

(e) Based on the plot above, estimates for most of the I_i's will be ambiguous.

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

ABCDE

Problem 10 [10min] We employ the Gibbs sampler specified previously to provide $\stackrel{iid}{\sim}$ samples from the posterior $\mathbb{P}(\theta_L, \theta_R, \rho, I_1, \ldots, I_G \mid X)$. We run $S = 100,000$ samples. We appropriately burned and thinned and now we provide histograms for the marginal posteriors. Below is a plot for each θ_L sample subtracted from each θ_R sample.



- [10 pt / 103 pts] Record the letter(s) of all the following that are **true**.

(a) The above plot approximates $\mathbb{P}(\theta_R - \theta_L \mid X)$.

(b) The above plot tells us that more likely than not $\rho < 50\%$.

(c) The main conclusion here is that ρ cannot be estimated well.

(d) This plot alone will allow us to test if $\theta_R \neq \theta_L$.

(e) The null hypothesis that $\theta_R = \theta_L$ will result in a p-value estimated to be ≈ 0.

(f) The null hypothesis that $\theta_R = \theta_L$ will result in a p-value estimated to be $= 0$.

(g) One can conclude from this plot that this batter is better at batting rightie than leftie.

(h) One can conclude from this plot that this batter has about a 14% higher probability of getting a hit when batting rightie than leftie.

The I_1 distribution after burning and thinning is 14 0's and 343 1's.

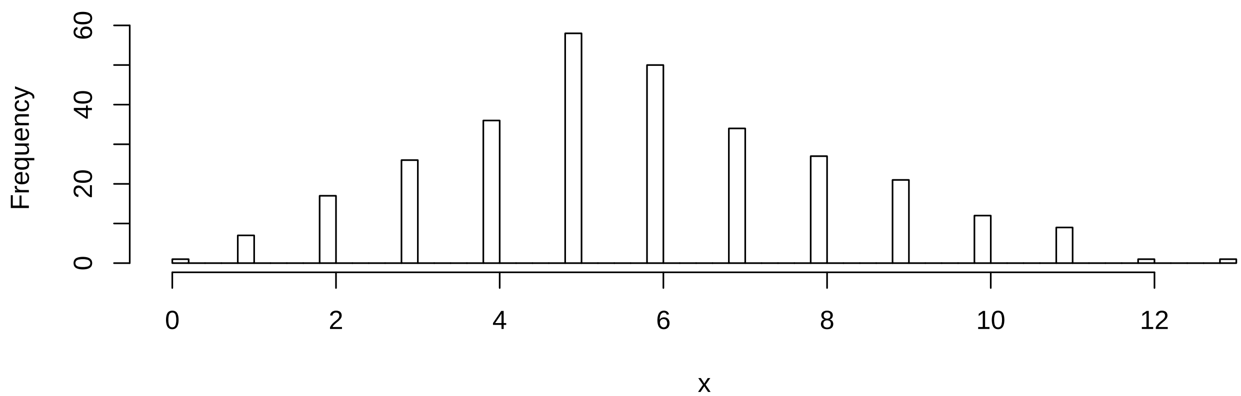
(i) The probability that the batter bat leftie for game #1 is about 96%.

(j) The best guess as to his batting orientation for game #1 is leftie.

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

$\boxed{\text{ADFGHIJ}}$

Problem 11 [4min] Imagine the batter bats both leftie and rightie (both sides of the oncoming ball). The propensity to get a hit when batting leftie is θ_L and the propensity to get a hit on when batting rightie is θ_R. He bats leftie ρ proportion of the games. Let I_i be the indicator variable indicating if he bats leftie for the ith game (it is equal to 1 when he bats leftie and equal to 0 if he bats rightie). There are n independent at bats per game and G independent total games per season and both n and G are known. The data $x_1, \ldots, x_G$ are number of hits per game. All other quantities are unknown. All sum signs are indexed from $1, \ldots, G$. Here is a histogram of the original data for the $G = 300$ games:



- [10 pt / 113 pts] Record the letter(s) of all the following that are **true**.

  (a) This data indicates that our model formulation $\mathcal{F}$ must be incorrect.

  (b) This data indicates that a betabinomial model may be appropriate.

  (c) If the model is correctly formulated, this data shows that θ_L and θ_R are similar.

  (d) If the model is correctly formulated, this data shows that ρ is difficult to estimate.

  (e) If the model is correctly formulated, this data shows that the I_i's are difficult to estimate.

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

BCDE

Problem 12 [17min] Let $\mathcal{F} : \stackrel{iid}{\sim} \mathcal{N}(\theta,\, \sigma^2)$ with both θ and σ^2 unknown.

- [20 pt / 133 pts] Record the letter(s) of all the following that are **true**.

(a) The prior $\mathbb{P}(\theta, \sigma^2)$ being normal is conjugate to the posterior which is normal.

(b) The prior $\mathbb{P}(\theta, \sigma^2)$ being normal-inverse-gamma is conjugate to the posterior which is normal.

(c) The prior $\mathbb{P}(\theta, \sigma^2)$ being normal-inverse-gamma is conjugate to the posterior which is normal-inverse-gamma.

(d) The prior $\mathbb{P}(\theta, \sigma^2) \propto 1$ is conjugate.

(e) The prior $\mathbb{P}(\theta, \sigma^2) \propto 1/\sigma^2$ is conjugate.

(f) The prior $\mathbb{P}(\theta \mid \sigma^2)\, \mathbb{P}(\sigma^2)$ is always conjugate.

(g) The prior $\mathbb{P}(\theta)\, \mathbb{P}(\sigma^2)$ is always semi-conjugate.

(h) The prior $\mathbb{P}(\theta \mid \sigma^2)\, \mathbb{P}(\sigma^2)$ is conjugate only if $\mathbb{P}(\theta \mid \sigma^2) = \mathcal{N}(\mu_0,\, \sigma^2/n_0)$ and $\mathbb{P}(\sigma^2)$ is inverse gamma.

(i) The prior $\mathbb{P}(\theta \mid \sigma^2)\, \mathbb{P}(\sigma^2)$ is conjugate only if $\mathbb{P}(\theta \mid \sigma^2) = \mathcal{N}(\mu_0,\, \tau^2)$ and $\mathbb{P}(\sigma^2)$ is inverse gamma where τ^2 is a constant.

(j) A conjugate prior will yield a posterior kernel $k(\theta, \sigma^2 \mid X) = (\sigma^2)^{-\alpha-1}\, e^{-(\beta+\lambda(\theta-\mu_0)^2)/\sigma^2}$ where $\alpha, \beta, \lambda, \mu_0$ are constants which do not depend on θ or σ^2.

Regardless of whether (j) was true, assume this kernel for the rest of the problem.

(k) If $k(\theta, \sigma^2 \mid X)$ is factored into $k(\theta \mid X, \sigma^2) k(\sigma^2 \mid X)$ then $k(\theta \mid X, \sigma^2) \propto$ a normal.

(l) If $k(\theta, \sigma^2 \mid X)$ is factored into $k(\theta \mid X, \sigma^2) k(\sigma^2 \mid X)$ then $k(\theta \mid X, \sigma^2)$ is a kernel for no known random variable.

(m) If $k(\theta, \sigma^2 \mid X)$ is factored into $k(\theta \mid X, \sigma^2) k(\sigma^2 \mid X)$ then $k(\sigma^2 \mid X) \propto$ an inverse gamma distribution.

(n) If $k(\theta, \sigma^2 \mid X)$ is factored into $k(\theta \mid X, \sigma^2) k(\sigma^2 \mid X)$ then $k(\sigma^2 \mid X)$ is a kernel for no known random variable.

(o) It can be shown that $\int_0^\infty k(\theta, \sigma^2 \mid X) d\sigma^2 \propto$ a normal distribution.

(p) It can be shown that $\int_0^\infty k(\theta, \sigma^2 \mid X) d\sigma^2 \propto$ a students T distribution.

(q) It can be shown that $\int_0^\infty k(\theta, \sigma^2 \mid X) d\sigma^2 \propto$ an inverse gamma distribution.

(r) It can be shown that $\int_{\mathbb{R}} k(\theta, \sigma^2 \mid X) d\theta \propto$ a normal distribution.

(s) It can be shown that $\int_{\mathbb{R}} k(\theta, \sigma^2 \mid X) d\theta \propto$ a students T distribution.

(t) It can be shown that $\int_{\mathbb{R}} k(\theta, \sigma^2 \mid X) d\theta \propto$ an inverse gamma distribution.

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

$\boxed{\text{CDEHJKMQS}}$

Problem 13 [7min] Let $\mathcal{F} : \overset{iid}{\sim} \text{Poisson}(\theta)$. Let $\mathbb{P}(\theta) = \text{Gamma}(\alpha, \beta)$, X_* represent the future observation(s) and n_* represent the number of future observation(s).

- [12 pt / 145 pts] Record the letter(s) of all the following that are **true**.

(a) $\mathbb{P}(X \mid \theta) = \mathbb{P}(X_1 \mid \theta) \cdot \mathbb{P}(X_2 \mid \theta) \cdot \ldots \cdot \mathbb{P}(X_n \mid \theta)$.

(b) The prior is conjugate.

(c) The prior is proper regardless of the values of α and β.

(d) The posterior predictive distribution $\mathbb{P}(X_* \mid X)$ is always negative binomial.

(e) $\mathbb{P}(X_* \mid X)$ is only negative binomial if $n_* = 1$.

(f) $\mathbb{P}(X_* \mid X)$ is only negative binomial if $n_* > 1$.

Assume $n_* = 10$ for the remainder of this problem.

(g) To sample one X_*, we first need to draw θ_{samp}, a sample from $\mathbb{P}(\theta \mid X)$.

(h) To sample one X_*, we first need to draw θ_{samp}, a sample from $\mathbb{P}(\theta)$.

(i) To sample one X_*, we then draw $X_{\text{samp},1}, X_{\text{samp},2}, \ldots, X_{\text{samp},n}$ values independently using `rpois`(θ_{samp}).

(j) To sample one X_*, we then draw $X_{\text{samp},1}, X_{\text{samp},2}, \ldots, X_{\text{samp},10}$ values independently using `rpois`(θ_{samp}).

(k) Regardless of the method used to sample one X_*, the components of X_* are all independent of each other.

(l) To sample many X_*'s, we repeat the procedure of correctly drawing θ_{samp} and then correctly drawing X_* over and over again.

Your answer will consist of a string (e.g. `aebgd`) where the order of the letters does not matter nor does upper / lowercase.

$\boxed{\text{ABEGJL}}$