Lecture 7

02/18/20

$\tilde{F} = \text{Binomial}(n, \theta)$ with n known

$P(\theta) = \text{Beta}(\alpha, \beta) = U(0,1)$ hyperparameters they represent pseudo data

$\downarrow$ if $\alpha = \beta = 1$

conjugacy

$P(\theta | X) = \text{Beta}(\alpha + \underset{\sim}{X}, \beta + n - \underset{\sim}{X})$

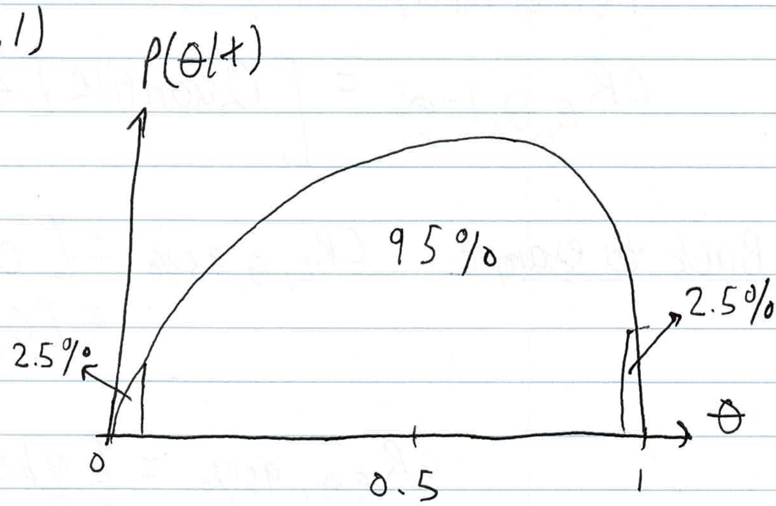
$n_0 = \alpha + \beta$

$\downarrow$ pseudo observations

$P(\theta) = U(0,1) = \text{Beta}(1,1)$

$X = 1, \quad n = 2$

$P(\theta | X) = \text{Beta}(2, 2)$



$\hat{\theta}_{MMSE} = \hat{\theta}_{MMAE} = \hat{\theta}_{MAP} = \frac{1}{2}$

I want a region providing a confidence set for θ

$CR_{\theta, 1-\alpha} = [\text{Quantile}[\theta | X, \frac{\alpha}{2}], \text{Quantile}[\theta | X, 1 - \frac{\alpha}{2}]]$

$\downarrow$

2 Sided Credible Region

$P(\theta \in CR_{\theta, 1-\alpha} | X) = 1 - \alpha$

Ex. $CR_{\theta, 95\%} = [\text{qbeta}(2.5\%, 2, 2), \text{qbeta}(97.5\%, 2, 2)]$
$= [0.09, 0.91]$

Left sided Credible Region:

$$P(\theta \in CR_{L,\theta,1-\alpha} | x) = 1-\alpha$$

$$P(\theta \le L | x) = 1-\alpha$$

$$CR_{L,\theta,1-\alpha} = \left[\underset{\underset{\inf(\Theta)}{\parallel}}{\text{Smallest Value of } \theta \text{ or } -\infty}, \ \text{Quantile}[\theta | X, 1-\alpha] \right]$$

Right sided Credible Region:

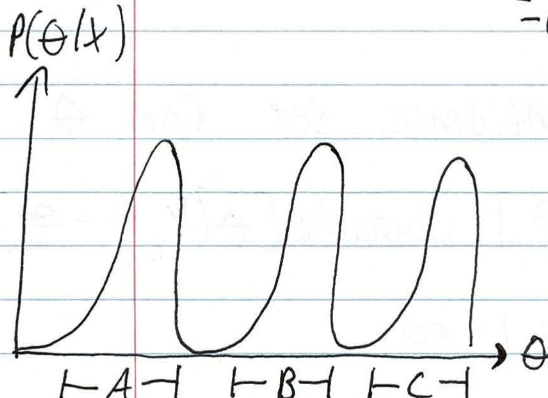
$$P(\theta \in CR_{R,\theta,1-\alpha} | x) = 1-\alpha$$

$$P(\theta \ge R | x) = 1-\alpha$$

$$CR_{R,\theta,1-\alpha} = \left[\text{Quantile}[\theta | X, \alpha], \ \text{largest value of } \theta \text{ or } \infty \right]$$

<u>Back to example</u>: $CR_{L,\theta,95\%} = [0, \ qbeta[95\%, 2, 2]]$
$= [0, .86]$

$CR_{R,\theta,95\%} = [qbeta(5\%, 2, 2), 1]$
$= [.135, 1]$



High Density Region

$HDR = A \cup B \cup C$

Smallest Possible region that goes to $1-\alpha$

$$P(\theta \in HDR_{\theta,1-\alpha}) = 1-\alpha$$

Disadvantages of HDR

1) Computationally Intense

2) Non-contiguous, makes little sense

Third goal of Inference: Theory Testing

You wish to convince someone of something (H_a) but people continue to believe a business-as-usual idea (H_0)

Example: H_0: UFO's do NOT exist and Aliens have NOT visited Earth
H_a: UFO's exist and ~~Alien's~~ Aliens have visited Earth

Two ways of "Proving" H_a:

I. Assume H_a is true and demand evidence to the contrary (wait for people to disprove you) If there is not sufficient evidence, H_a stands.

II. Assume your theory is wrong and demonstrate contradictory evidence to H_0 until people see that you are right. In other words, I believe H_a and I'm very confident that it's true, so I'm willing to ~~sup~~ provide contradictory evidence to the opposite (H_0), until everyone sees H_0 is wrong and they are forced to believe in H_a.

In Strategy II, everyone has a level of skepticism with evidence, we call that α. If the evidence doesn't meet or beat this level, we retain H_0.

In inference, we wish to test theories about θ. We would like to demonstrate the following:

A) $H_A: \theta \neq \theta_0 \Rightarrow H_0: \theta = \theta_0$ (Two-Sided Test)

B) $H_A: \theta < \theta_0 \Rightarrow H_0: \theta \geq \theta_0$ (Left-Sided Test)

C) $H_A: \theta > \theta_0 \Rightarrow H_0: \theta \leq \theta_0$ (Right-Sided Test)

Bayesian Hypothesis Testing

$P(H_0 | X) < \alpha \Rightarrow$ Reject H_0, Accept H_A

$P(H_0 | X) \geq \alpha \Rightarrow$ Retain H_0

Coin Toss Example: Flip a coin 100 times, get 61 heads. Is the coin unfair/weighted towards heads? $\alpha = 5\%$ is the scientific standard

$H_0: \theta < 0.5$

$H_A: \theta \geq 0.5$

$F: Bin(n, \theta)$ n is known } In general, n is always known

$n = 100, \; X = 61$

$= P(H_0 | X)$

$P(\theta) = U(0,1) \Rightarrow P(\theta | X) = Beta(62, 40)$

$$= P(\theta \leq 0.5 | X) = \int_0^{0.5} \frac{1}{B(62,40)} \theta^{61}(1-\theta)^{31} \, d\theta$$

$P(\theta \leq .5 | X) = pbeta(0.5, 62, 40) = 0.014 < 5\%$

Reject H_0, Accept H_A. The coin is unfairly weighted towards heads

Notation for integrals of beta distribution:

$P(X \leq x) = F(x) = pbeta(x, \alpha, \beta)$

$P(X > x) = 1 - F(x) = 1 - pbeta(x, \alpha, \beta)$

Ride-Share Example

A driver does 200 rides and gets 37 non-5 star ratings. Do we fire the driver?

θ : proportion of non-5 star rides

If $\theta > 25\% \longrightarrow$ fire the driver

$H_0 : \theta \leq 25\%$ $\quad n = 200, \; X = 37$
$H_A : \theta > 25\%$

F : Binomial(n, θ) ← n is known
$P(\theta) = U(0,1) \Longrightarrow P(\theta | x) = \text{Beta}(38, 164)$

$$P(\theta \leq 25\% | X) = \int_0^{.25} \frac{1}{B(38,164)} \theta^{37} (1-\theta)^{163} \, d\theta$$

$$= \text{pbeta}(.25, 38, 164) = .98$$

Retain H_0 : Do NOT fire the driver

What if: $H_0 : \theta = \theta_0$
$H_A : \theta \neq \theta_0$

$$P(\theta = \theta_0 | X) = 0$$

We have a problem with 2-sided tests.