Lecture - 15 04/02/2020

F : iid Normal (θ, σ^2) with σ^2 known

$P(\theta/\sigma^2) = \mathbb{N}(\mu_0, \tau^2) = \mathbb{N}(\mu_0, \sigma^2/n_0)$

Imagine pseudodata $y_1, \ldots, y_{n_0} \overset{iid}{\sim} N(\mu_0, \sigma^2)$

$\Rightarrow \bar{y} \sim N(\mu_0, \sigma^2/n_0)$

$\Rightarrow P(\theta | x, \sigma^2) = \mathbb{N}(\hat{\theta}_p, \sigma_p^2)$ where $\hat{\theta}_p = \dfrac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{n/\sigma^2 + 1/\tau^2} = \dfrac{n\bar{x} + n_0\mu_0}{n + n_0}$

$$\sigma_p^2 = \frac{1}{n/\sigma^2 + 1/\tau^2} = \frac{\sigma^2}{n + n_0}$$

$n_* = 1$

$$P(x_* | x, \sigma^2) = \int_{\Theta} P(x_* | \theta, \sigma^2)\, P(\theta | x, \sigma^2)\, d\theta.$$

$$= \int_{\mathbb{R}} \mathbb{N}(\theta, \sigma^2)\, \mathbb{N}(\hat{\theta}_p, \sigma_p^2)\, d\theta.$$

$$= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-1/2\sigma^2 (x_* - \theta)^2}\right)\left(\frac{1}{\sqrt{2\pi\sigma_p^2}} e^{-1/2\sigma_p^2 (\theta - \hat{\theta}_p)^2}\right) d\theta.$$

$$\propto \int_{\mathbb{R}} e^{-1/2\sigma^2 (x_* - \theta)^2}\, e^{-1/2\sigma_p^2 (\theta - \hat{\theta}_p)^2}\, d\theta.$$

$$= \int_{\mathbb{R}} e^{-x_*^2/2\sigma^2 + 2x_*\theta/2\sigma^2 - \theta^2/2\sigma^2}\, e^{-\theta^2/2\sigma_p^2 + 2\theta\hat{\theta}_p/2\sigma_p^2 - \hat{\theta}_p^2/2\sigma_p^2}$$

$$= \int_{\mathbb{R}} e^{-x_*^2/2\sigma^2 + x_*\theta/\sigma^2 - \theta^2/2\sigma^2}\, e^{-\theta^2/2\sigma_p^2 + \cancel{2}\theta\hat{\theta}_p/\cancel{2}\sigma_p^2 - \hat{\theta}_p^2/2\sigma_p^2}.$$

$$\propto e^{-x_*^2/2\sigma^2} \int_{\mathbb{R}} e^{x_*\theta/\sigma^2}\, e^{-\theta^2/2\sigma^2}\, e^{-\theta^2/2\sigma_p^2}\, e^{\theta\hat{\theta}_p/\sigma_p^2}\, d\theta.$$

$$= e^{-x_*^2/2\sigma^2} \int_{\mathbb{R}} e^{\overbrace{\left(x_*/\sigma^2 + \hat{\theta}_p/\sigma_p^2\right)}^{a}\theta - \overbrace{\left(1/2\sigma^2 + 1/2\sigma_p^2\right)}^{b}\theta^2} d\theta$$

Review for kernal

$c \cdot k(\theta) \Downarrow PDF$

$$\theta \sim \mathbb{N}\left(a/2b, 1/2b\right) = \frac{1}{\sqrt{2\pi(1/2b)}} e^{-1/2(1/2b)(\theta - a/2b)^2}$$

$$= \sqrt{b/\pi}\, e^{-b(\theta^2 - a\theta/b + a^2/4b^2)}$$

$$= \sqrt{b/\pi}\, e^{-b\theta^2 + a\theta - a^2/4b}$$

$$= \sqrt{b/\pi}\, \underbrace{e^{-a^2/4b}}_{c} \underbrace{e^{a\theta - b\theta^2}}_{k(\theta)} \propto e^{a\theta - b\theta^2}$$

$$= e^{-x_*^2/2\sigma^2} \sqrt{\pi/b}\, e^{a^2/4b} \underbrace{\int_{\mathbb{R}} \sqrt{b/\pi}\, e^{-a^2/4b} e^{a\theta - b\theta^2} d\theta}_{1}.$$

$$= e^{-x_*^2/2\sigma^2} \sqrt{\pi/b}\, e^{a^2/4b}$$

$$\propto e^{-x_*^2/2\sigma^2} e^{\left(x_*/\sigma^2 + \hat{\theta}_p/\sigma_p^2\right)^2/4b}$$

$$= e^{-x_*^2/2\sigma^2} e^{x_*^2/4\sigma^{2^2} b} e^{x_* \hat{\theta}_p/2\sigma_p^2 \sigma^2 b} \overbrace{e^{\hat{\theta}_p^2/4\sigma_p^{2^2} b}}^{\text{constant}}$$

$$\propto e^{\underbrace{\left(\hat{\theta}_p/2\sigma_p^2\sigma^2 b\right)}_{A} x_* - \underbrace{\left(1/2\sigma^2 - 1/4\sigma^{2^2} b\right)}_{B} x_*^2}$$

$$= e^{Ax_* - Bx_*^2} \propto \mathbb{N}\left(A/2B, 1/2B\right)$$

$$2b\sigma^2 = 2\left(\frac{1}{2\sigma^2} + \frac{1}{2\sigma_p^2}\right)\sigma^2 = \boxed{1 + \frac{\sigma^2}{\sigma_p^2}} \Longrightarrow \frac{1}{2B} = \frac{1}{2\left(\frac{1}{2\sigma^2} - \frac{1}{4\sigma^{2^2} b}\right)}$$

$$= \frac{1}{1/\sigma^2 - 1/2b\sigma^{2^2}} = \frac{\sigma^{2^2}}{\sigma^2 - 1/2b} = \frac{2b\sigma^2\sigma^2}{2b\sigma^2 - 1} =$$

$$\frac{\left(1+\sigma^2/\sigma_p^2\right)\sigma^2}{\left(1+\sigma^2/\sigma_p^2\right)-1} = \frac{\left(1+\sigma^2/\sigma_p^2\right)\sigma^2}{\sigma^2/\sigma_p^2} = \sigma_p^2\left(1+\sigma^2/\sigma_p^2\right)$$

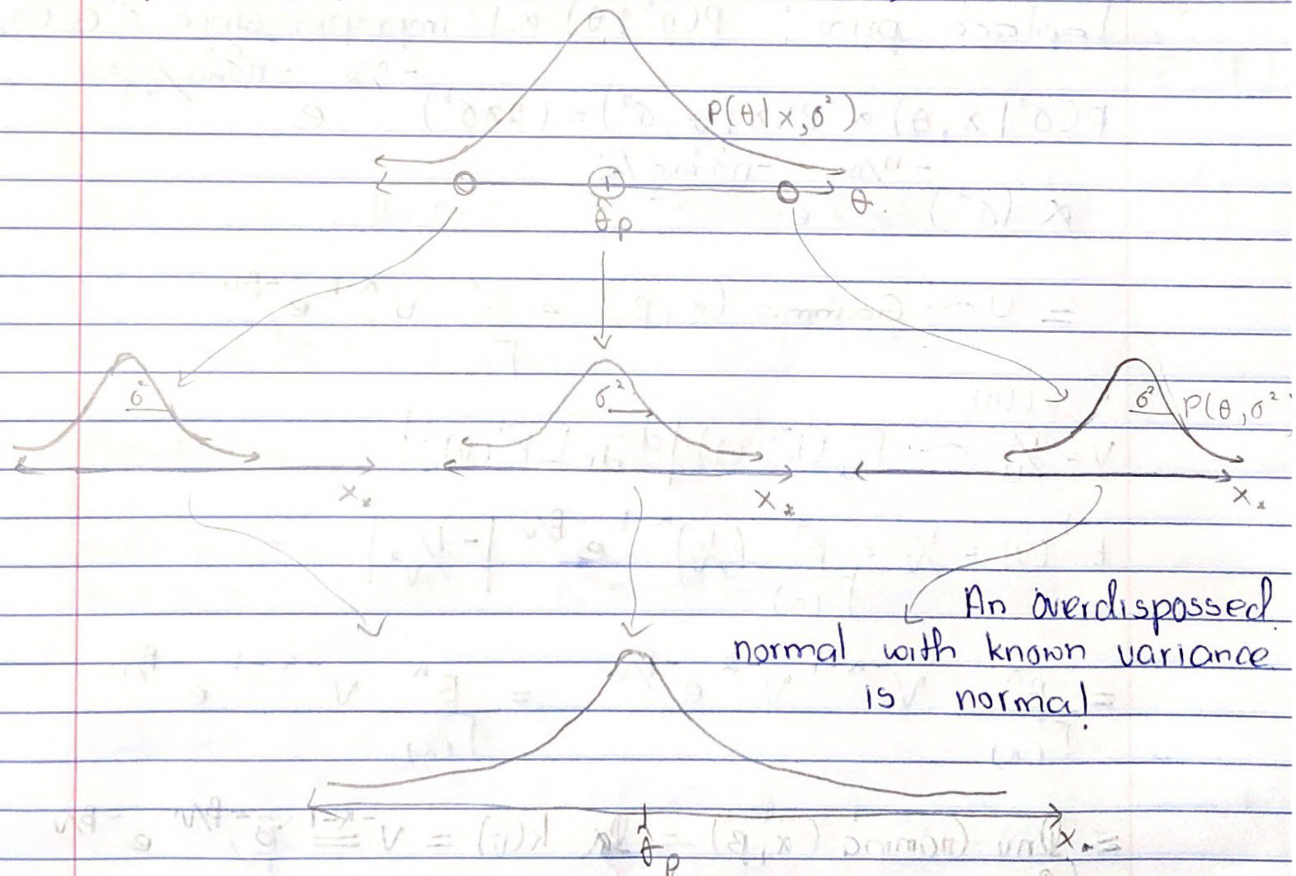
$$= \sigma_p^2+\sigma^2$$

$$A/_{2B} = A\left(\sigma_p^2+\sigma^2\right) = \frac{\hat{\theta}_p}{2\sigma^2\sigma_p^2 b}\left(\sigma_p^2+\sigma^2\right) = \frac{\hat{\theta}_p}{\left(1+\sigma^2/\sigma_p^2\right)\sigma_p^2}\left(\sigma_p^2+\sigma^2\right)$$

$$= \frac{\hat{\theta}_p}{\left(\sigma_p^2+\sigma^2\right)}\left(\sigma_p^2+\sigma^2\right) = \hat{\theta}_p$$

$$P(x_*|x,\sigma^2) = \mathbb{N}(\hat{\theta}_p, \sigma_p^2+\sigma^2) = \mathbb{N}\left(\hat{\theta}_p, \left(1+\tfrac{1}{n+n_0}\right)\sigma^2\right)$$

$$\left[\hat{\theta}_p \longrightarrow \theta,\ \sigma_p^2+\sigma^2 \longrightarrow \sigma^2\right] \Longrightarrow \mathbb{N}(\theta,\sigma^2)$$



An overdisposed normal with known variance is normal

F: iid $N(\theta, \sigma^2)$ with θ known

$$\mathcal{L}(\sigma^2; x, \theta) = (2\pi\sigma^2)^{-n/2} e^{-1/2\sigma^2 \Sigma(x_i - \theta)^2}$$

$$l(\sigma^2; x, \theta) = -n/2 \ln(2\pi\sigma^2) - 1/2\sigma^2 \Sigma(x_i - \theta)^2$$

$$= -n/2 \ln(2\pi) - n/2 \ln(\sigma^2) - 1/2\sigma^2 \Sigma(x_i - \theta)^2$$

$$l'(\sigma^2; x, \theta) = 0 - n/2\sigma^2 + \frac{1}{2\sigma^{2^2}} \Sigma(x_i - \theta)^2 \overset{set}{=} 0$$

$$\Longrightarrow -n + \frac{\Sigma(x_i - \theta)^2}{\sigma^2} = 0$$

$\rightarrow \Sigma(x_i - \theta)^2 = n\hat{\sigma}^2_{MLE}$

sample variance

$$\Longrightarrow \hat{\sigma}^2_{MLE} = \frac{\Sigma(x_i - \theta)^2}{n} \quad \text{similar to} \quad s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1} \longrightarrow \sigma^2$$

Laplace prior: $p(\sigma^2 | \theta) \propto 1$ improper since $\sigma^2 \in (0, \infty)$

$$p(\sigma^2 | x, \theta) \propto p(x | \theta, \sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-n\hat{\sigma}^2_{MLE}/2\sigma^2}$$

$$\propto (\sigma^2)^{-n/2} e^{\frac{-n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}$$

$$U \sim \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u}$$

$= t(u)$

$$V = 1/u \sim f_u(t^{-1}(v)) \left| d/dv [t^{-1}(v)] \right|$$

$$t^{-1}(v) = 1/v = \frac{\beta^\alpha}{\Gamma(\alpha)} (1/v)^{\alpha-1} e^{-\beta/v} \left| -1/v^2 \right|$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} v^{-\alpha+1} v^{-2} e^{-\beta/v} = \frac{\beta^\alpha}{\Gamma(\alpha)} v^{-\alpha-1} e^{-\beta/v}$$

$$\propto K(v) = v^{-\alpha-1} e^{-\beta/v} = \text{Inv Gamma}(\alpha, \beta)$$

inverse gamma r.v

$$= (\sigma^2)^{\underbrace{-(n/2-1)}_{\alpha}-1} e^{\frac{-n\hat{\sigma}^2_{MLE}/2}{\sigma^2}} \leftarrow \beta$$

$$= \text{Inv Gamma}\left(n/2 - 1, n\hat{\sigma}^2_{MLE}/2\right)$$

$$= \text{Inv Gamma}\left(n-2/2, n\hat{\sigma}^2_{MLE}/2\right)$$

$$P(\sigma^2 | x, \theta) \propto P(x|\theta, \sigma^2)\, P(\sigma^2|\theta)$$

$$\propto (\sigma^2)^{-n/2} e^{\frac{-n\hat{\sigma}^2_{MLE}/2}{\sigma^2}} k(\sigma^2|\theta)$$

$k(\sigma^2|\theta) = ?$ to get conjugacy?

$= (\sigma^2)^a e^{-b/\sigma^2}$ ✓, $\propto$ Inv Gamma is the conjugate prior

$= (\sigma^2)^{-n/2+a} e^{-\frac{n\hat{\sigma}^2/2+b}{\sigma^2}}$

let $p(\sigma^2|\theta) = \text{Inv Gamma}(\alpha, \beta)$

$$\Longrightarrow P(\sigma^2|x,\theta) \propto \left((\sigma^2)^{-n/2} e^{\frac{-n\hat{\sigma}^2_{MLE}/2}{\sigma^2}}\right)\left((\sigma^2)^{-\alpha-1} e^{-\beta/\sigma^2}\right)$$

$\Sigma(x_i-\theta)^2$ $=$ $n\hat{\sigma}^2_{MLE}$

$$= (\sigma^2)^{-(\frac{n}{2}+\alpha)-1} e^{-\frac{(n\hat{\sigma}^2_{MLE}/2+\beta)}{\sigma^2}} \propto \text{Inv Gamma}\left(\frac{n}{2}+\alpha, \frac{n\hat{\sigma}^2_{MLE}}{2}+\beta\right)$$

let $\alpha = \frac{n_0}{2}$, $\beta = \frac{n_0\sigma_0^2}{2}$ $\Longrightarrow P(\sigma^2|\theta) = \text{Inv Gamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$

$$\Longrightarrow P(\sigma^2|\theta, x) = \text{Inv Gamma}\left(\frac{n+n_0}{2}, \frac{n\hat{\sigma}^2_{MLE}+n_0\sigma_0^2}{2}\right)$$

Pseudodata: $y_1, \ldots, y_{n_0} \sim N(\underset{\text{known}}{\theta}, \underset{\text{belief}}{\sigma_0^2})$

$\Longrightarrow n_0\sigma_0^2 = \Sigma(y_i-\theta)^2$

$\Longrightarrow \sigma_0^2 = \frac{\Sigma(y_i-\theta)^2}{n_0}$ $\quad$ n_0 small $\longrightarrow$ uninformative

Haldane: $n_0 = 0$, $\sigma_0^2 = ?$

$\Longrightarrow P(\sigma^2|\theta) = \text{Inv Gamma}(0,0)$