Lecture 11

03/10 ... [Math 441]

$X \sim Bin(n,\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$

$\ell(\theta;x) = \ln(\binom{n}{x}) + x\ln(x) + (n-x)\ell(1-\theta)$

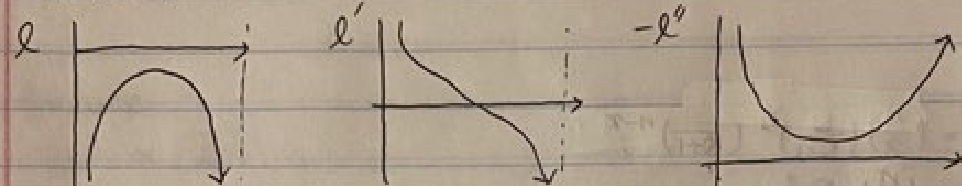
$s(\theta;x) = \ell'(\theta;x) = \frac{x}{\theta} - \frac{n-x}{1-\theta}$

$-\ell''(\theta;x) = \frac{x}{\theta^2} + \frac{n-x}{(1-\theta)^2}$

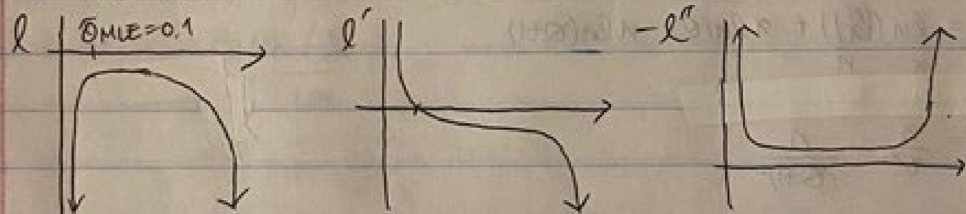
$I(\theta) = E_X[-\ell''(\theta;x)] = n\left(\frac{1}{\theta(1-\theta)}\right)$

$= n I_1(\theta)$

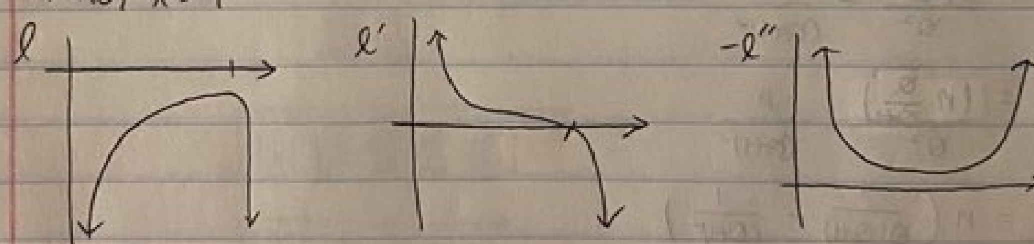
$n=10,\ x=5$



$n=10,\ x=1$

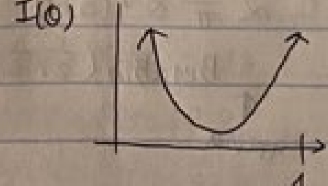


$n=10,\ x=9$



prob. weighted average

Supp[X] = {0, 1, ... 10}



Q) How much "information" is in X?

$I_1(\theta=\frac{1}{2}) = 4$

$I_1(\theta=\frac{1}{100}) \approx 101.$

let $\overset{\text{Jeffery's}}{p_J(\theta)} \propto \sqrt{I(\theta)}$

If τ: Bin, fixed n

★ Jeffery's prior

$p(\theta) \propto \sqrt{n\frac{1}{\theta(1-\theta)}} \propto \sqrt{\frac{1}{\theta(1-\theta)}} = \theta^{-\frac{1}{2}}(1-\theta)^{-\frac{1}{2}} = \theta^{\frac{1}{2}-1}(1-\theta)^{\frac{1}{2}-1} \propto Beta(\frac{1}{2},\frac{1}{2})$

odds $\quad \varnothing = \frac{\theta}{1-\theta}$

$\Leftrightarrow$

$\theta = \frac{\varnothing}{\varnothing+1}$

$\mathcal{L}(\varnothing;x) = \binom{n}{x}\left(\frac{\varnothing}{\varnothing+1}\right)^x\left(1-\frac{\varnothing}{\varnothing+1}\right)^{n-x}$

$= \binom{n}{x}\frac{\varnothing^x}{(\varnothing+1)^n}$

$\ell(\varnothing;x) = \ln(\binom{n}{x}) + x\ln(\varnothing) - n\ln(\varnothing+1)$

$\ell'(\varnothing;x) = \frac{x}{\varnothing} - \frac{n}{\varnothing+1}$

$-\ell''(\varnothing;x) = \frac{x}{\varnothing^2} - \frac{n}{(\varnothing+1)^2}$

$I(\varnothing) = E_X[-\ell''(\varnothing;x)]$

$= \frac{E_X[x]}{\varnothing^2} - \frac{n}{(\varnothing+1)^2}$

$= \frac{n\left(\frac{\varnothing}{\varnothing+1}\right)}{\varnothing^2} - \frac{n}{(\varnothing+1)^2}$

$= n\left(\frac{1}{\varnothing(\varnothing+1)} - \frac{1}{(\varnothing+1)^2}\right)$

$= n\left(\frac{1}{\varnothing(\varnothing+1)^2}\right)$

$p_J(\varnothing) \propto \sqrt{I(\varnothing)} = \sqrt{n\frac{1}{\varnothing(\varnothing+1)^2}} \propto \varnothing^{-\frac{1}{2}}(\varnothing+1)^{-1} \propto \frac{1}{\pi}\varnothing^{-\frac{1}{2}}(\varnothing+1)^{-1}$

$= \text{BetaBin}(\frac{1}{2},\frac{1}{2}) = F_{1,1}$

↑
trivia

$$
\begin{array}{ccc}
P(X|\theta) & \xrightarrow{\text{Jeffery's}} & p_J(\theta) \\
\updownarrow \varnothing = t(\theta) & & \updownarrow \text{respect } \varnothing = t(\theta) \\
P(X|\varnothing) & \xrightarrow[\text{Jeffery's}]{} & p_J(\varnothing)
\end{array}
$$

memorize change of variables

$$* \; P(\varnothing) = P_\theta(t^{-1}(\varnothing)) \left| \frac{d}{d\varnothing}[t^{-1}(\varnothing)] \right|$$

$$= \frac{1}{2\pi \, B(\frac{1}{2},\frac{1}{2})} \cdot \left(\frac{\varnothing}{\varnothing+1}\right)^{-\frac{1}{2}} \left(\frac{1}{\varnothing+1}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{\pi} \cdot \varnothing^{-\frac{1}{2}} (\varnothing+1) \cdot (\varnothing+1)^{-2}$$

$$= \frac{1}{\pi} \cdot \varnothing^{-\frac{1}{2}} (\varnothing+1)^{-1} = \text{BetaBTn}(\tfrac{1}{2},\tfrac{1}{2})$$

$$\cdot \frac{d}{d\varnothing}\left[\frac{\varnothing}{\varnothing+1}\right] = \frac{\varnothing+1-\varnothing}{(\varnothing+1)^2}$$

$$= \frac{1}{(\varnothing+1)^2}$$

$$= (\varnothing+1)^{-2}$$

Assume

$P_J(\theta) \propto \sqrt{I(\theta)}$

Prove

$P_J(\varnothing) \propto \sqrt{I(\varnothing)}$ using $P(\varnothing) = P_\theta(\overset{\theta}{\overset{\shortparallel}{t^{-1}(\varnothing)}}) \left| \frac{d}{d\varnothing}[t^{-1}(\varnothing)] \right|$

$$\rightarrow \; P(\varnothing) = P_\theta(\theta) \left| \frac{d\theta}{d\varnothing} \right|$$

$$\propto \sqrt{I(\theta)} \cdot \left| \frac{d\theta}{d\varnothing} \right|$$

$$= \sqrt{I(\theta) \frac{d\theta}{d\varnothing} \frac{d\theta}{d\varnothing}}$$

$$= \sqrt{E_x[\underbrace{\ell'(\theta;x)^2}_{\frac{d\ell}{d\theta}\frac{d\ell}{d\theta}}] \frac{d\theta}{d\varnothing} \frac{d\theta}{d\varnothing}}$$

$$= \sqrt{E_x\left[\frac{d\ell}{d\theta} \frac{d\ell}{d\theta} \frac{d\theta}{d\varnothing} \frac{d\theta}{d\varnothing}\right]}$$

$$= \sqrt{E_x\left[\frac{d\ell}{d\varnothing} \frac{d\ell}{d\varnothing}\right]}$$

$$= \sqrt{E_x[\ell'(\varnothing;x)]} = \sqrt{I(\varnothing)} \propto P_J(\varnothing)$$

Laplace $P(\theta) = \text{Beta}(1,1)$

H____ $P(\theta) = \text{Beta}(0,0)$

Jeffery's $P_J(\theta) = \text{Beta}(\frac{1}{2},\frac{1}{2})$

} Principle of uninformative priors.

$P(\theta) = \text{Beta}(\alpha, B)$

pick α, B

} empirical Bayes informative prior.

b

$X \sim Bin(n, \theta) = \binom{n}{x} \theta^{x} (1-\theta)^{n-x}$.

$n \to \infty, \ \theta \to 0$ But $\lambda = n\theta$. $\Rightarrow \theta = \frac{\lambda}{n}$

써티벵크제각 | 써티벵크에 접근하는수.

$$\lim_{n\to\infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^{x} \left(1-\frac{\lambda}{n}\right)^{n-x} = \lim_{n\to\infty} \frac{n!}{x!(n-x)!} \cdot \frac{\lambda^{x}}{n^{x}} \left(1-\frac{\lambda}{n}\right)^{n} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^{x}}{x!} \lim_{n\to\infty} \frac{\overbrace{n(n-1)\cdots 1}^{n \text{ terms}}}{\underbrace{(n-x)(n-x-1)\cdots 1}_{n-x \text{ terms}} \cdot \underbrace{n \cdot n \cdots n}_{x \text{ terms}}} \left(1-\frac{\lambda}{n}\right)^{n} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^{x}}{x!} \underbrace{\lim_{n\to\infty} \overset{1}{\cancel{\frac{n}{n}}} \cdot \lim_{n\to\infty} \overset{1}{\cancel{\frac{n-1}{n}}} \cdots \lim_{n\to\infty} \overset{1}{\cancel{\frac{n-x-1}{n}}}}_{x \text{ terms}} \cdot \lim_{n\to\infty} \overset{1}{\cancel{\frac{n-x}{n-x}}} \cdots \lim_{n\to\infty} \overset{1}{\cancel{\frac{1}{1}}}$$

$$\underbrace{\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{n}}_{=e^{-\lambda}} \cdot \lim_{n\to\infty} \overset{1}{\cancel{\left(1-\frac{\lambda}{n}\right)^{-x}}}$$

$$= \frac{\lambda^{x} e^{-\lambda}}{x!} = \text{Poisson}(\lambda)$$

F: Poisson, $n=1$.

We want to find the conjugate prior.

$$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)}$$

$$K(\theta|x) \propto P(x|\theta)P(\theta) \propto k(x|\theta)k(\theta)$$

$$K(\theta|x) \propto \left(\frac{\theta^{x} e^{-\theta}}{x!}\right) P(\theta) \propto \theta^{x} e^{-\theta} k(\theta) = \theta^{x} e^{-\theta} \overbrace{\left(\theta^{a} e^{-b\theta}\right)}^{k(\theta)}$$

prior func. for Poisson

Pattern match to conjecture but the conjugate prior looks ____

$$= \theta^{x+a} e^{-\theta(b+1)}$$

to get $P(\theta)$, I need the normalization constant c.

$K(\theta) = \theta^{a} e^{-b\theta}$

$X \sim \text{Poisson}(\lambda)$. $\lambda = n\theta$

- $\text{Supp}[X] = \{0, 1 \ldots 10\} = \mathbb{N}_0$
- $\lambda \in (0, \infty)$

$$c = \frac{1}{\int_0^\infty \theta^a e^{-b\theta} d\theta} = \frac{1}{\int_0^\infty \left(\frac{t}{b}\right)^a e^{-b \cdot \frac{t}{b}} \cdot \frac{1}{b} \, dt}$$

let $t = b\theta$. $\frac{dt}{d\theta} = b$

- $\Gamma(y) = \int_0^\infty t^{y-1} e^{-t} dt$

$$= \frac{1}{\frac{1}{b^{a+1}} \int_0^\infty t^{(a+1)-1} e^{-t} dt}$$

$$= \frac{b^{a+1}}{\Gamma(a+1)}$$

$$\Rightarrow P(\theta) = \frac{b^{a+1}}{\Gamma(a+1)} \theta^a e^{-b\theta} = \text{Gamma}(a+1, b).$$

[conjugate prior for Poisson model]

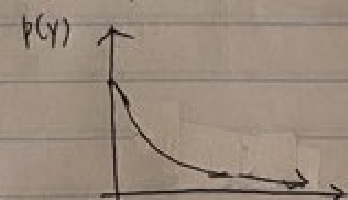
C ______ .

$$P(\theta) = \text{Gamma}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

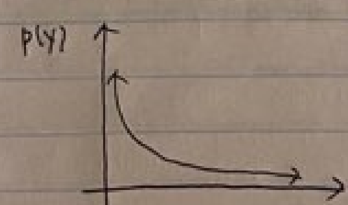
$Y \sim \text{Gamma}(\alpha, \beta)$

- $\text{Supp}[Y] = (0, \infty)$
- parameter space : $\alpha, \beta > 0$
- $E[Y] = \frac{\alpha}{\beta}$
- $\text{Var}[Y] = \frac{\alpha}{\beta^2}$,
- $\text{Mode}[Y] = \frac{\alpha - 1}{\beta}$ if $\alpha \geq 1$
- $\text{Med}[Y] = \text{qgamma}(0.5, \alpha, \beta)$
  no closed form.

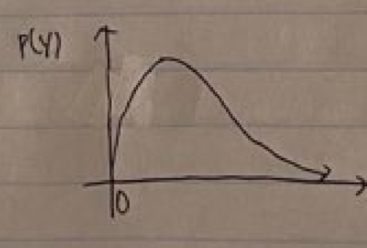
(3 shapes)



$\alpha = 1$.



$\alpha < 1$
· left sided.



$\alpha > 1$

$\mathcal{F}$ = Poisson, $n=1$.

we want to find the conjugate prior.

$$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)} = \text{Gamma}(x+\alpha, 1+\beta)$$

$\uparrow$

if $P(\theta) = \text{Gamma}(\alpha, \beta)$

Poisson - Gamma conjugate model.