F: Poisson$(\theta) \overset{iid}{}$, $\theta \sim \text{Gamma}(\alpha, \beta)$ (const.) $\Rightarrow \theta|x \sim \text{Gamma}\left(\sum x_i + \alpha, n+\beta\right)$

$n_* = 1$ $\quad p(x_*|x) = \int \underbrace{p(x_*|\theta)}_{\text{poisson}} p(\theta|x)\, d\theta$

$$= \int_0^\infty \left(\frac{e^{-\theta}\theta^{x_*}}{x_*!}\right)\left(\frac{(n+\beta)^{\sum x_i+\alpha}}{\Gamma(\sum x_i+\alpha)}\theta^{\sum x_i+\alpha-1} e^{-(n+\beta)\theta}\right) d\theta$$

$$= \frac{(n+\beta)^{\sum x_i+\alpha}}{x_*!\,\Gamma(\sum x_i+\alpha)} \int_0^\infty \theta^{\sum x_i + x_* + \alpha - 1} e^{-(n+\beta+1)\theta}\, d\theta = \frac{(n+\beta)^{\sum x_i+\alpha}}{x_*!\,\Gamma(\sum x_i+\alpha)} \cdot$$

let $t = (n+\beta+1)\theta \Rightarrow \frac{dt}{d\theta} = n+\beta+1$, $\theta = \frac{t}{n+\beta+1}$, $d\theta = \frac{dt}{n+\beta+1}$

$$\int_0^\infty \left(\frac{t}{n+\beta+1}\right)^{\sum x_i + x_* + \alpha - 1} e^{-t}\frac{dt}{n+\beta+1} = \frac{(n+\beta)^{\sum x_i+\alpha}}{x_*!\,\Gamma(\sum x_i+\alpha)} \cdot \frac{1}{(n+\beta+1)^{\sum x_i + x_* + \alpha - 1}} \cdot \frac{1}{(n+\beta+1)}$$

$$\underbrace{\int_0^\infty t^{\sum x_i + x_* + \alpha - 1} e^{-t}\, dt}_{\Gamma(\sum x_i + x_* + \alpha)} = \frac{(n+\beta)^{\sum x_i+\alpha}}{x_*!\,\Gamma(\sum x_i+\alpha)} \cdot \frac{\Gamma(\sum x_i + x_* + \alpha)}{\underbrace{(n+\beta+1)^{\sum x_i + x_* + \alpha}}_{(n+\beta+1)^{\sum x_i+\alpha}(n+\beta+1)^{x_*}}}$$

$$\left(\frac{n+\beta}{n+\beta+1}\right)^{\sum x_i+\alpha}\left(\frac{1}{n+\beta+1}\right)^{x_*}\frac{\Gamma(\sum x_i + x_* + \alpha)}{x_*!\,\Gamma(\sum x_i+\alpha)}$$

let $p := \frac{n+\beta}{n+\beta+1} \in (0,1)$

$\Rightarrow 1-p = \frac{1}{n+\beta+1} \in (0,1)$

let $r := \sum x_i + \alpha$

$$= \frac{\Gamma(x_*+r)}{x_*!\,\Gamma(r)} p^r (1-p)^{x_*} = \text{ExtNegBin}(r,p)$$

extended negative binomial model

If $r = \sum x_i + \alpha \in \mathbb{N}_0 \Rightarrow \alpha \in \mathbb{N}_0$

Recall $\Gamma(x) = (x-1)$ if $x \in \mathbb{N}$

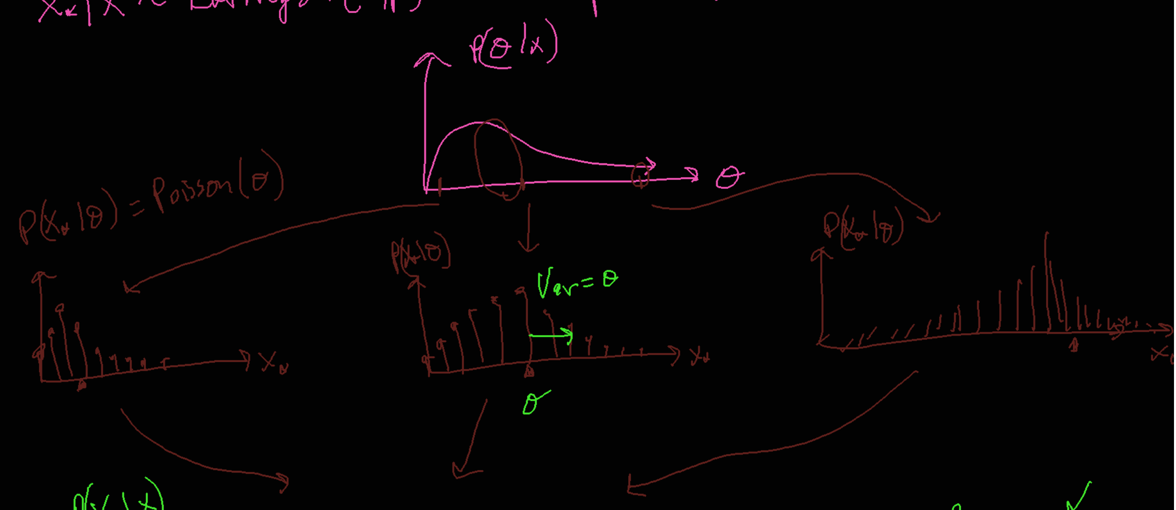
$$= \binom{x_*+r-1}{x_*} p^r (1-p)^{x_*} = \text{NegBin}(r,p)$$

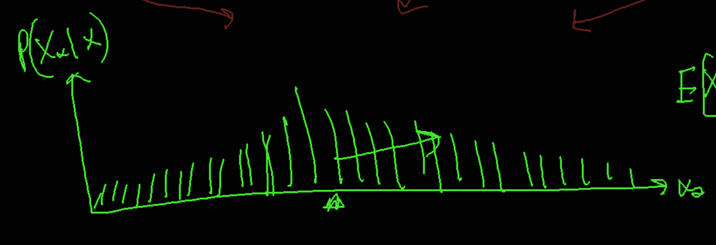
$X_1, \ldots, X_r; p \overset{iid}{\sim} \text{Geom}(p) := (1-p)^x p \quad E[X_i] = \frac{p}{1-p}$

$\sum_{i=1}^{r} X_i \sim \text{NegBin}(r,p) \Rightarrow E\left[\sum X_i\right] = r\frac{p}{1-p}$

This is a mistake. We will fix next class.

$X_*|X \sim \text{ExtNegBin}(r,p)$ Overdispersed Poisson





$E[X_*|X] = r\frac{p}{1-p} = \mu$

$\text{Var}[X_*|X] = \frac{pr}{(1-p)^2} = \frac{1}{1-p}\mu$

$\frac{1}{1-p} \in (1, \infty)$

Normal model $X \sim N(\theta, \sigma^2) = N(\theta_1, \theta_2)$ $\quad \dim[\vec{\theta}] = 2$

Pretend we know σ^2. We want inference for the mean, θ.

$X \sim N(\theta, \sigma^2)$ Kernel practice...

$$p(x|\theta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} \propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2} = e^{-\frac{x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}}$$

$$\propto e^{-\frac{x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}} = e^{ax - bx^2} \text{ where } a = \frac{\theta}{\sigma^2} \text{ and } b = \frac{1}{2\sigma^2} > 0$$

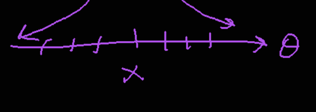
"posterior" $\quad \propto N\left(\underbrace{\frac{a}{2b}}_{=\theta}, \underbrace{\frac{1}{2b}}_{=\sigma^2}\right) \Rightarrow \sigma^2 = \frac{1}{2b}, \theta = \frac{a}{2b}$

$$p(\theta|x,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} \propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2} = e^{-\frac{x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}}$$

$$\propto e^{\frac{x\theta}{\sigma^2} - \frac{\theta^2}{2\sigma^2}} = e^{a\theta - b\theta^2} \propto N\left(\frac{a}{2b}, \frac{1}{2b}\right) = N(x, \sigma^2)$$

where $a = \frac{x}{\sigma^2}$, $b = \frac{1}{2\sigma^2} \Rightarrow \frac{a}{2b} = \frac{\frac{x}{\sigma^2}}{2\frac{1}{2\sigma^2}} = x$

$$\Rightarrow p(\theta|x,\sigma^2) \overset{\theta}{=} N(x, \sigma^2)$$



F: $X_1, \ldots, X_n; \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$

$$p(x|\theta,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2} = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\theta)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\left(\sum x_i^2 - 2\theta n\bar{x} + n\theta^2\right)}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{\sum x_i^2}{2\sigma^2}} e^{\frac{\theta n\bar{x}}{\sigma^2}} e^{-\frac{n\theta^2}{2\sigma^2}}$$

$= \sum (x_i^2 - 2\theta x_i + \theta^2)$

$= \sum x_i^2 - \sum 2\theta x_i + \sum \theta^2$

$= \sum x_i^2 - 2\theta \sum x_i + n\theta^2$

$= \sum x_i^2 - 2\theta n\bar{x} + n\theta^2$

$$p(\theta|x,\sigma^2) \propto e^{\overbrace{\frac{n\bar{x}}{\sigma^2}}^{a}\theta - \overbrace{\frac{n}{2\sigma^2}}^{b}\theta^2} \propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

$\text{Var} = \frac{1}{2b} = \frac{1}{2\left(\frac{n}{2\sigma^2}\right)} = \frac{1}{\frac{n}{\sigma^2}} = \frac{\sigma^2}{n}$

$\text{mean} = \frac{a}{2b} = \frac{\frac{n\bar{x}}{\sigma^2}}{2\left(\frac{n}{2\sigma^2}\right)} = \bar{x}$

Laplace uninformative prior $\quad p(\theta) = U(\Theta)$

$p(\theta) \propto 1$

if $\Theta = (0,1)$

$\Rightarrow p(\theta) = 1$

$\Theta = (0, 100)$

$p(\theta) = \frac{1}{100} \propto 1$

Amazing trick

Poisson-Gamma

$$p(\theta|x) = \frac{p(x|\theta)\,p(\theta)}{p(x)} \propto p(x|\theta)\,\overset{\propto 1}{p(\theta)} \propto p(x|\theta) = \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod x_i!} \propto e^{-n\theta}\theta^{\sum x_i + 1 - 1}$$

$\propto \text{Gamma}(\sum x_i + 1, n)$

$\Rightarrow \theta \sim \text{Gamma}(1, 0)$

Generally $p(\theta) = \text{Gamma}(\alpha, \beta)$

$\Rightarrow p(\theta|x) = \text{Gamma}(\sum x_i + \alpha, n+\beta)$

F: $X_1, \ldots, X_n; \theta, \sigma^2 \overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 known

prior on θ is Laplace $\Rightarrow p(\theta) \propto 1$ improper!

$$p(\theta|x,\sigma^2) \propto p(x|\theta,\sigma^2) \propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$