Math 341 / 650 Spring 2020
Midterm Examination One

Solutions

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Full Name ____

Code of Academic Integrity

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Activities that have the effect or intention of interfering with education, pursuit of knowledge, or fair evaluation of a student's performance are prohibited. Examples of such activities include but are not limited to the following definitions:

Cheating Using or attempting to use unauthorized assistance, material, or study aids in examinations or other academic work or preventing, or attempting to prevent, another from using authorized assistance, material, or study aids. Example: using an unauthorized cheat sheet in a quiz or exam, altering a graded exam and resubmitting it for a better grade, etc.

I acknowledge and agree to uphold this Code of Academic Integrity.

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signature date

Instructions

This exam is seventy five minutes and closed-book. You are allowed **one** page (front and back) of a "cheat sheet." You may use a graphing calculator of your choice. Please read the questions carefully. If the question reads "compute," this means the solution will be a number otherwise you can leave the answer in *any* widely accepted mathematical notation which could be resolved to an exact or approximate number with the use of a computer. I advise you to skip problems marked "[Extra Credit]" until you have finished the other questions on the exam, then loop back. I also advise you to use pencil. The exam is 100 points total plus extra credit. Partial credit will be granted for incomplete answers on most of the questions. $\boxed{\text{Box}}$ in your final answers. NO FOOD but drinks okay. Good luck!

Distribution of r.v.	Quantile Function	PMF / PDF function	CDF function	Sampling Function
beta	qbeta(p, α, β)	d-(x, α, β)	p-(x, α, β)	r-(α, β)
betabinomial	qbetabinom(p, n, α, β)	d-(x, n, α, β)	p-(x, n, α, β)	r-(n, α, β)
betanegativebinomial	qbeta_nbinom(p, r, α, β)	d-(x, r, α, β)	p-(x, r, α, β)	r-(r, α, β)
binomial	qbinom(p, n, θ)	d-(x, n, θ)	p-(x, n, θ)	r-(n, θ)
exponential	qexp(p, θ)	d-(x, θ)	p-(x, θ)	r-(θ)
gamma	qgamma(p, α, β)	d-(x, α, β)	p-(x, α, β)	r-(α, β)
geometric	qgeom(p, θ)	d-(x, θ)	p-(x, θ)	r-(θ)
inversegamma	qinvgamma(p, α, β)	d-(x, α, β)	p-(x, α, β)	r-(α, β)
negative-binomial	qnbinom(p, r, θ)	d-(x, r, θ)	p-(x, r, θ)	r-(r, θ)
normal (univariate)	qnorm(p, θ, σ)	d-(x, θ, σ)	p-(x, θ, σ)	r-(θ, σ)
poisson	qpois(p, θ)	d-(x, θ)	p-(x, θ)	r-(θ)
T (standard)	qt(p, ν)	d-(x, ν)	p-(x, ν)	r-(ν)
uniform	qunif(p, a, b)	d-(x, a, b)	p-(x, a, b)	r-(a, b)

Table 1: Functions from R (in alphabetical order) that can be used on this exam with their arguments. The hyphen in colums 3, 4 and 5 is shorthand notation for the full text of the r.v. which can be found in column 2.

Problem 1 Let $\mathcal{F}$ be binomial with known sample size $n = 3$. The data is all "successes" i.e. $x = 3$. For all questions that have numerical answers, use three significant digits e.g. 0.123 and 1.23×10^{-5} or fractions.

(a) [2 pt / 2 pts] Find the maximum likelihood estimate for θ. $\bar{x} = \frac{3}{3} = 1 = \hat{\theta}_{mle}$

(b) [4 pt / 6 pts] What is the main problem with your estimate in (a)?

It implies that a realization of 0 is impossible. This is imprudent to say after only seeing $n=3$ observations!

(c) [3 pt / 9 pts] Find the $CI_{\theta,99\%}$. $\{1\}$

(d) [4 pt / 13 pts] Does the interval in (c) fulfill the second goal of statistical inference? Yes / no and explain your answer.

No. The second goal of inference (confidence set) is to provide a set of possible likely values of θ. This set only has one value in it! And that value is problematic as we mentioned in (b).

(e) [2 pt / 15 pts] We will now conduct Bayesian inference. Consider the reduced parameter space $\Theta_0 = \{0.50, 0.99\} \subset \Theta = (0, 1)$. We believe strongly in $\theta = 0.5$ but we want to give some credence to the alternate theory. Thus we establish a prior of

$$\mathbb{P}(\theta) = \begin{cases} 0.50 & \text{w.p. } 0.9 \\ 0.99 & \text{w.p. } 0.1 \end{cases}$$

Is this the "prior of indifference" for the reduced parameter space? Yes / no and explain.

No. The prior of indifference is $\theta \sim U(\{0.5, 0.99\})$

(f) [5 pt / 20 pts] Find $\hat{\theta}_{\text{MAP}}$.

$$\left.\begin{aligned} P(X=3 \mid \theta=0.5)\, P(\theta=0.5) &= \left(\tfrac{1}{2}\right)^3 0.9 = 0.1125 \\ P(X=3 \mid \theta=0.99)\, P(\theta=0.99) &= 0.99^3 \cdot 0.1 = 0.0970 \end{aligned}\right\} \Rightarrow \hat{\theta}_{map} = 0.5$$

(g) [5 pt / 25 pts] Find $\mathbb{P}(X = x)$.

$$P(X=3) = \sum_{\theta \in \Theta_0} P(X=3 \mid \theta) = P(X=3 \mid \theta=0.5) P(\theta=0.5) + P(X=3 \mid \theta=0.99) P(\theta=0.99) = 0.1125 + 0.0970 = 0.210$$

(h) [5 pt / 30 pts] Find the posterior predictive probability $\mathbb{P}(X_* = 1 \mid X = x)$ where X_* denotes the next observation.

$$P(X_* = 1 \mid X=3) = \sum_{\theta \in \Theta_0} P(X_1^* \mid \theta)\, P(\theta \mid X=3) = \frac{1}{P(X=3)} \sum_{\theta \in \Theta_0} P(X^*=1 \mid \theta)\, P(X=3 \mid \theta)\, P(\theta)$$

$$= \frac{1}{P(X=3)} \left(P(X^*=1 \mid \theta=0.5) P(X=3 \mid \theta=0.5) P(\theta=0.5) + P(X^*=1 \mid \theta=0.99) P(X=3 \mid \theta=0.99) P(\theta=0.99) \right)$$

$$= \frac{1}{0.210} \left(0.5 \cdot 0.1125 + 0.99 \cdot 0.0970 \right) = 0.727$$

(i) [3 pt / 33 pts] We will now consider the entire parameter space for the binomial model i.e. $\Theta = (0, 1)$. We will use the prior $\theta \sim \text{Beta}\left(\frac{1}{2}, \frac{1}{2}\right)$. We will see later in class that this is called the "Jeffrey's Prior". Is this an uninformative prior? Yes / no and explain.

Yes, $n_0 = \frac{1}{2} + \frac{1}{2} = 1$ which is not a lot of prior data. Thus, this prior is uninformative.

(j) [2 pt / 35 pts] Is this the "prior of indifference"? Yes / no and explain.

No. The prior of indifference is $\theta \sim U(0,1) = \text{Beta}(1,1)$.

(k) [2 pt / 37 pts] How many pseudosuccesses and pseudofailures is within this prior?

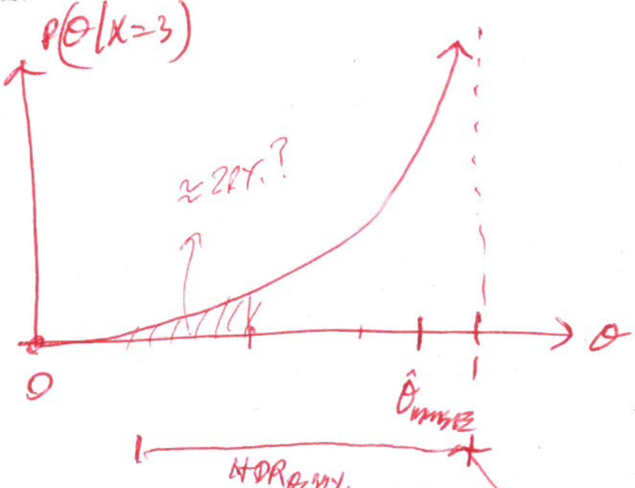
0.5 pseudosuccesses and 0.5 pseudofailures.

(l) [3 pt / 40 pts] What is $\mathbb{E}[\theta]$? $0.5 = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}}$

(m) [5 pt / 45 pts] Find $\mathbb{P}(\theta \mid X = x)$.

$P(\theta|X) = \text{Beta}(\alpha+x, \beta+n-x)$, $P(\theta|X=3) = \text{Beta}(3.5, 0.5)$

(n) [6 pt / 51 pts] Draw $\mathbb{P}(\theta \mid X = x)$ to the best of your ability. Label all axes and critical points.



(o) [2 pt / 53 pts] Does $\hat{\theta}_{\text{MAP}}$ exist? Yes / No.

(p) [4 pt / 57 pts] Find $\hat{\theta}_{\text{MMSE}}$ and denote it in the illustration in (n).

$\hat{\theta}_{MMSE} = \frac{3.5}{3.5+0.5} = 0.875$

(q) [4 pt / 61 pts] What is the proportion of shrinkage towards the prior expectation if you employ the posterior expectation as your point estimate?

$$\rho = \frac{\alpha+\beta}{n+\alpha+\beta} = \frac{0.5+0.5}{3+0.5+0.5} = 0.25$$

(r) [5 pt / 66 pts] Find the $CR_{\theta,99\%}$.

$$CR_{\theta,99\%} = [\text{qbeta}(0.005, 3.5, 0.5), \text{qbeta}(.995, 3.5, 0.5)]$$

(s) [5 pt / 71 pts] Find the $HDR_{\theta,99\%}$ and denote it in the illustration in (n).

$$HDR_{\theta,99\%} = [\text{qbeta}(0.01, 3.5, 0.5), 1]$$

(t) [10 pt / 81 pts] Test if $\theta > 0.5$. Write out the hypotheses and declare the α level ~~you are comfortable~~ with. Estimate the Bayesian p-value from the illustration of the posterior distribution in (n) and provide the conclusion of the test.

$H_0: \theta \le 0.5$

$H_a: \theta > 0.5$

let $\alpha = 5\%$

$p_{val} = P(H_0 \mid X=3) = P(\theta \le 0.5 \mid X=3) = \text{pbeta}(0.5, 3.5, 0.5) \approx 20\% \Rightarrow$ Fail to reject H_0.

There is no compelling evidence to suggest $\theta > 0.5$.

(u) [4 pt / 85 pts] Find the posterior predictive ~~distribution~~ probability $\mathbb{P}(X_* = 1 \mid X = x)$ where X_* denotes the next observation.

$$P(X_* \mid X) = \text{Bern}\left(\frac{\alpha+x}{n+\alpha+\beta}\right) \Rightarrow P(X_* = 1 \mid X=3) = \frac{0.5+3}{3+0.5+0.5} = \boxed{0.875}$$

(v) [3 pt / 88 pts] What is your best guess of X_*?

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Problem 2 Consider $X_1, \ldots, X_n \overset{iid}{\sim} \text{Beta}(1, \theta)$.

(a) [6 pt / 94 pts] Find $\mathcal{L}(\theta; X_1, \ldots, X_n)$. Simplify so that your answer does not include the $B(\cdot,\cdot)$ function or the $\Gamma(\cdot)$ function.

$$\mathcal{L}(\theta; x) = \prod_{i=1}^{n} \frac{1}{B(1,\theta)} x_i^{(1)-1} (1-x_i)^{\theta-1} = \prod_{i=1}^{n} \theta (1-x_i)^{\theta-1} = \theta^n \prod_{i=1}^{n} (1-x_i)^{\theta-1}$$

$$\frac{1}{B(1,\theta)} = \frac{\Gamma(1+\theta)}{\underset{\overset{\shortparallel}{1}}{\Gamma(1)}\Gamma(\theta)} = \frac{\theta\,\cancel{\Gamma(\theta)}}{\cancel{\Gamma(\theta)}} = \theta$$

(b) [3 pt / 97 pts] Find $\ell(\theta; X_1, \ldots, X_n)$. Simplify as much as possible.

$$\ell(\theta; x) = n \ln(\theta) + (\theta - 1) \sum_{i=1}^{n} \ln(1-x_i) = n \ln(\theta) + \theta \sum_{i=1}^{n} \ln(1-x_i) - \sum_{i=1}^{n} \ln(1-x_i)$$

(c) [3 pt / 100 pts] Find $\hat{\theta}_{\text{MLE}}$.

$$\ell'(\theta; x) = \frac{n}{\theta} + \sum_{i=1}^{n} \ln(1-x_i) \overset{set}{=} 0 \implies \hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^{n} \ln(1-x_i)}$$

(d) [8 pt / 108 pts] [Extra Credit] Consider $X_1, \ldots, X_n \overset{iid}{\sim} \text{Beta}(\theta_1, \theta_2)$. Find the MLE for θ_1 and the MLE for θ_2. Partial credit is given.