Lecture10.

03/05 목 [MATH341]

Informative Priors

Let θ be the career prob. of getting a hit for a batter in baseball.

$\hat{\theta}_{MLE} = \dfrac{x}{n}$ ← # hits, ← # at bats   bad estimator (b/c small n).

Best θ in history is 0.366

Average is 0.260

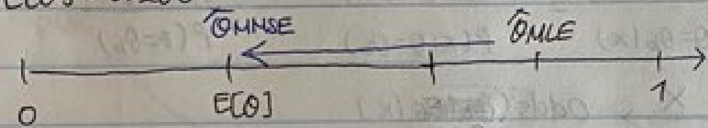
- $n=3$, $x=2$.

  $\hat{\theta}_{MLE} = 0.667 = \hat{\theta}_{MMSE}$ if $\theta \sim Beta(0,0)$

  If $\theta \sim Beta(1,1)$, $\hat{\theta}_{MMSE} = \dfrac{1+2}{3+1+1} = 0.600$.

Design a prior. → Pick α, β

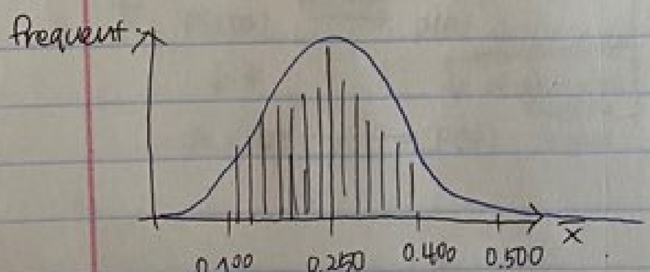
$E[\theta] = 0.260$



$\hat{\theta}_{MMSE} = \underset{n-\rho}{(1\%)}\ \underset{\hat{\theta}_{MLE}}{(0.667)} + \underset{\rho}{(99\%)}\ \underset{E[\theta]}{(0.260)} = 0.263$.

Look at previous data. eg. all players $\geq$ 500 at bats.

We examine $\bar{x}$'s.



Now try to fit a Beta distribution to the data.

via maximum likelihood $\hat{\alpha}_{MLE} = 78.7$ &

$\hat{\beta}_{MLE} = 224.8$

$\Rightarrow E[\theta] = 0.260$

$\Rightarrow n_0 = 303.5$

This process is called "empirical Bayes".

$\Rightarrow \rho = \dfrac{303.5}{303.5+3} = 99\%$.

cf. $n = \alpha+\beta$이었음

n_0 : [illegible]

$\therefore \rho = \dfrac{\alpha+\beta}{\alpha+\beta+n}$.

$\theta \sim Beta(\frac{1}{2}, \frac{1}{2})$ is called the Jeffery's prior (uninformative).

$Odds(A) := \frac{\overset{\in[0,1]}{P(A)}}{P(A^c)} = \frac{P(A)}{1-P(A)} \in [0, \infty)$

· $Odds\ Against(A) := odds(A)^{-1} = \frac{1-P(A)}{P(A)} \in [0, \infty]$

· $Odds(A, B) := \frac{P(A)}{P(B)}$

· $P(\theta=\theta_a|X) = \frac{P(X|\theta=\theta_a)\,P(\theta=\theta_a)}{P(X)}$

· $P(\theta=\theta_b|X) = \frac{P(X|\theta=\theta_b)\,P(\theta=\theta_b)}{P(X)}$

$\rightarrow Odds(\theta_a,\theta_b|X) = \frac{P(\theta=\theta_a|X)}{P(\theta=\theta_b|X)} = \underbrace{\frac{P(X|\theta=\theta_a)}{P(X|\theta=\theta_b)}}_{\text{likelihood}} \cdot \underbrace{\frac{P(\theta=\theta_a)}{P(\theta=\theta_b)}}_{\text{prior; prior odds } (odds(\theta_a,\theta_b))}$

$\Rightarrow Odds(\theta_a,\theta_b) \xrightarrow{X} Odds(\theta_a,\theta_b|X)$

posterior odds

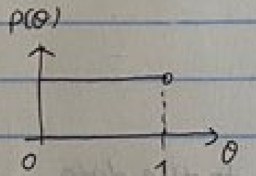
e.g. $1:1 \xrightarrow{X} 5:1$

Let $\phi(\theta)$ be odds θ.

F: Binomial, fixed n

$\phi(\theta) = \frac{\theta}{1-\theta}$

$P(\theta) = U(0,1)$



Fisher's idea → Q) What is prior of indifference of ϕ?

A) $P(\phi) \overset{?}{=} U(0,\infty) = 0 \neq$ not a valid PDF. $\int_0^\infty 0\, d\phi \neq 1$.

* If $P(\theta) = U(0,1)$, what is $P(\phi) = ?$

For a continuous random variable x.

If $Y=t(X)$ where t is invertible and $f_X(x)$ known

$\Rightarrow f_Y(y) = f_X(t^{-1}(y)) \left| \frac{d}{dy}[t^{-1}(y)] \right|$ (derived in Math 368)

$Q = \phi(\theta) = \frac{\theta}{1-\theta} = t(\theta)$

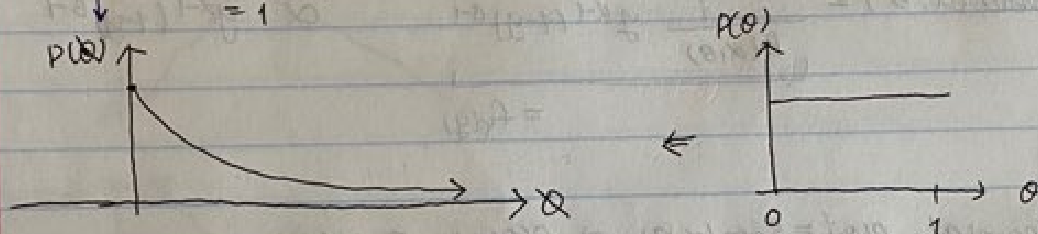
$Q(1-\theta)=\theta$.

$Q-\theta Q=\theta$.

$\Rightarrow \theta = \frac{Q}{1+Q} = t^{-1}(Q)$

$\frac{d}{dQ}[t^{-1}(Q)] = \frac{(1+Q)\cdot 1 - Q\cdot 1}{(1+Q)^2} = \frac{1}{(1+Q)^2}$

$f_Q(Q) = \underbrace{f_\theta\left(\frac{Q}{1+Q}\right)}_{=1} \left| \frac{1}{(1+Q)^2} \right| = \frac{1}{(1+Q)^2}$



$$\begin{array}{ccc} F: P(X|\theta) & \xrightarrow{\text{Jeffery's protocol}} & p(\theta) \\ \downarrow t & & \updownarrow \text{transformation of variables formula works} \\ P(X|Q) & \xrightarrow{\quad '' \quad} & p(Q) \end{array}$$

$P(X|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$.

$\Updownarrow$

$P(X|Q) = \binom{n}{x}\left(\frac{\theta}{1+\theta}\right)^x\left(1-\frac{\theta}{1+\theta}\right)^{n-x}$

$= \binom{n}{x}\frac{\phi^x}{(1+Q)^n}$

$\Rightarrow$ Binomial Parametrization with odds.

$Q = Q(\theta) = \frac{\theta}{1-\theta} = t(\theta) \Rightarrow \theta = \frac{Q}{1+Q} = t^{-1}(Q)$

Let X be continuous with density $f(x;\theta)$.

$f(x;\theta) \propto k(x;\theta)$ is unique.

by definition $\exists c>0$ not a function of x.

$f(x;\theta) = c\cdot k(x;\theta)$

$\uparrow$

norm constant

def. 이용.

$$\left(1 = \int_{Supp[X]} f(x, \theta)\, dx\right) = \int_{Supp[X]} c \cdot k(x;\theta)\, dx$$

$$\Rightarrow \left(\int_{Supp[X]} k(x;\theta)\, dx\right)^{-1} = c$$

$P(x;\theta) \propto k(x;\theta)$

which means $\exists\, c > 0$.

$p(x|\theta) = c\,k(x;\theta)$

$$c = \left(\sum_{x \in Supp[X]} k(x;\theta)\right)^{-1}$$

$k(\alpha;\theta)$ 구하기

ex) $$Y \sim Beta(\alpha, \beta) = \underbrace{\overbrace{\frac{1}{B(\alpha,\beta)}}^{=c} \overbrace{y^{\alpha-1}(1-y)^{\beta-1}}^{=k(y;\alpha,\beta)}}_{=f_Y(y)} \quad \propto y^{\alpha-1}(1-y)^{\beta-1}$$

F: Binomial, $P(\theta) = Beta(\alpha,\beta) \Rightarrow P(\theta|X) = Beta(x+\alpha, n-x+\beta)$

$$P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)} \propto P(X|\theta)P(\theta) = \left(\binom{n}{x}\theta^{x}(1-\theta)^{n-x}\right)\left(\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\right)$$

$$\propto \theta^{x+\alpha-1}(1-\theta)^{n-x+\beta-1}$$

space food처럼 dehydrated → 우리내

$$\propto Beta(x+\alpha, n-x+\beta)$$

$$Y \sim N(\theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y-\theta)^2}$$

$$\propto e^{-\frac{1}{2\sigma^2}(y-\theta)^2}$$

$$= e^{-\frac{1}{2\sigma^2}(y^2-2\theta y+\theta^2)}$$

$$= e^{-\frac{y^2}{2\sigma^2}+\frac{\theta y}{\sigma^2}}\, e^{-\frac{\theta^2}{2\sigma^2}}$$

$$\propto e^{\frac{\theta y}{\sigma^2}-\frac{y^2}{2\sigma^2}} = k(y;\theta,\sigma^2)$$

$$\therefore c = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}}$$

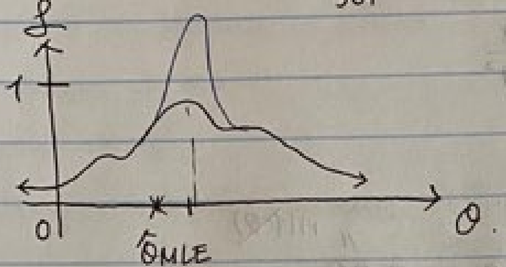
Remember. $\underset{\text{likelihood}}{\mathcal{L}(\theta;x)} = p(x;\theta)$. $\underset{\text{log}}{\ell(\theta;x)} := \ln(\mathcal{L})$.

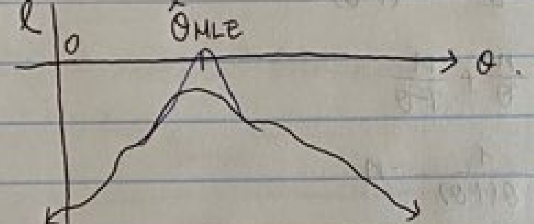
Score function $s(\theta,x) := \ell'(\theta;x)$.

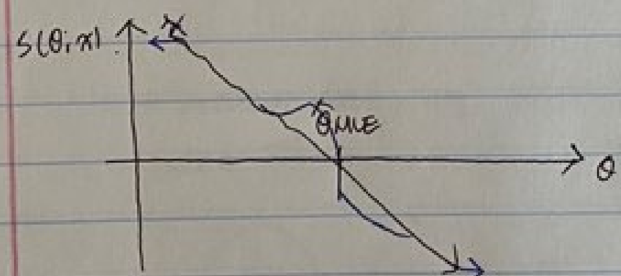
$I(\theta) := Var_X[s(\theta;X)] = \underset{\text{MATH 369}}{\ldots} = -E_X[\ell''(\theta;x)]$

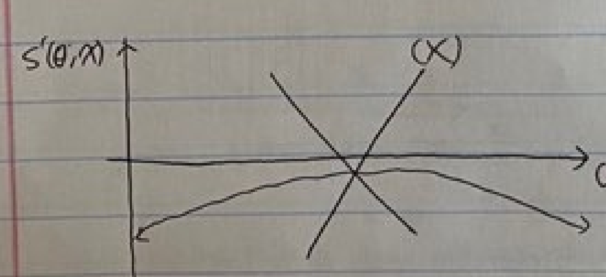
$\ell''(\theta;x) = s'(\theta;x)$.

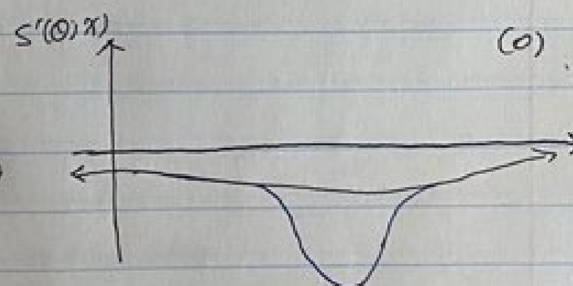
↑ Fisher Information

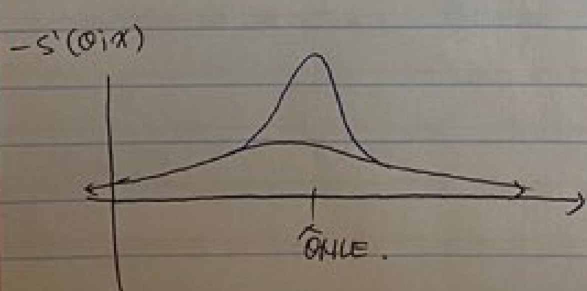












$X \sim Bin(n, \theta)$

$P(x; \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$

$\ell(\theta; x) = \ln\left(\binom{n}{x}\right) + x \ln(\theta) + (n-x)\ln(1-\theta)$

$s(\theta; x) = \ell'(\theta; x) = \frac{x}{\theta} - \frac{n-x}{1-\theta}$

$\ell''(\theta; x) = -\frac{x}{\theta^2} - \frac{n-x}{(1-\theta)^2}$

$-\ell''(\theta; x) = \frac{x}{\theta^2} + \frac{n-x}{(1-\theta)^2}$

$$\Rightarrow I(\theta) = E_X\left[\frac{x}{\theta^2} + \frac{n-x}{(1-\theta)^2}\right] = \frac{n\theta}{\theta^2} + \frac{\overset{n(1-\theta)}{\overset{\shortparallel}{n - n\theta}}}{(1-\theta)^2}$$

$$= \frac{n}{\theta} + \frac{n}{1-\theta}$$

$$= \frac{1}{\theta(1-\theta)} \cdot n$$