Math 341 lec 14 03/26

$\hat{F}$: i.i.d poisson(θ) $x_1, \dots, x_n$; $\theta \sim$ Poisson θ, $\theta \sim$ Gamma(α, β)

posterior Predictive $P(x^* | x) = $ Ext NegBin(r, p) where $r = \alpha + \Sigma x_i$ $p = \frac{n+\beta}{n+\beta+1}$

if $\alpha \in \mathbb{N}_0 \Rightarrow$ NegBin$(r, p) = \binom{x^* + r - 1}{r-1} (1-p)^{x^*} p^r$

if $x_1, \dots, x_n \overset{iid}{\sim}$ Geom$(p) \Rightarrow \sum_{i=1}^{r} x_i \sim$ NegBin(r, p) $E[x_i] = \frac{1-p}{p} \neq \frac{p}{1-p} \Rightarrow E(\Sigma x_i) = \frac{r(1-p)}{p}$

$E(x^* | x) = \frac{r(1-p)}{p} = (\alpha + \Sigma x_i) \frac{\frac{1}{n+\beta+1}}{(n+\beta)/(n+\beta+1)} = \frac{\alpha + \Sigma x_i}{n+\beta} = E(\theta | x)$ (the posterior expectation)

$\hat{F}$: iid $N(\theta, \sigma^2)$ with σ^2 known. For the likelihood $\propto P(x | \theta, \sigma^2) \propto e^{ax - bx^2} \propto N(\frac{a}{2b}, \frac{1}{2b})$

$P(\theta | x, \sigma^2) \propto e^{a\theta - b\theta^2} \propto N(\frac{a}{2b}, \frac{1}{2b})$ $a = \frac{n\bar{x}}{\sigma^2}$, $b = \frac{n}{2\sigma^2}$ so $\Rightarrow N(\bar{x}, \frac{\sigma^2}{n})$

under Laplace prior i.e $p(\theta) \propto 1$

$P(\theta | x, \sigma^2) \propto P(x | \theta, \sigma^2) \propto N(\bar{x}, \frac{\sigma^2}{n})$

Conjugatecy

$P(\theta | \sigma^2) \propto k(\theta | \sigma^2)$

$P(\theta | x, \sigma^2) = \frac{\cancel{P(x|\theta)}\ P(x | \theta, \sigma^2) P(\theta | \sigma^2)}{P(x | \sigma^2)} \propto P(x | \theta, \sigma^2) P(\theta | \sigma^2) \propto k(x | \theta, \sigma^2) k(\theta | \sigma^2)$

$= e^{a\theta - b\theta^2} k(\theta | \sigma^2)$ ① ~~if~~

if $k(\theta | \sigma^2) = e^{\alpha\theta - \beta\theta^2}$ then ① $= e^{a\theta - b\theta^2} e^{\alpha\theta - \beta\theta^2} = e^{(a+\alpha)\theta - (b+\beta)\theta^2}$ ②

So the prior of this $k(\theta | \sigma^2)$ is $N(\frac{\alpha}{2\beta}, \frac{1}{2\beta})$ (conjugate prior)

② $\propto N(\frac{\alpha + a}{2(\beta + b)}, \frac{1}{2(\beta + b)})$ But a, b could represent by $n, \bar{x}, \sigma^2$

From above we know $a = \frac{n\bar{x}}{\sigma^2}$, $b = \frac{n}{2\sigma^2}$, $\alpha =$, $\beta =$

τ^2 tell square

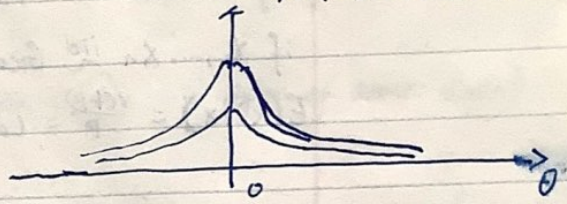
let $N(\frac{\alpha}{2\beta}, \frac{1}{2\beta}) = N(\mu_0, \tau^2)$ $\alpha = \frac{\mu_0}{\tau^2}$, $\beta = \frac{1}{2\tau^2}$

F: iid $N(\theta, \sigma^2)$ σ^2 known, $p(\theta) = N(\nu_0, \tau^2)$

$\frac{1}{2(b+\theta)} = 1/2(\frac{1}{2\sigma^2}+\frac{1}{2\tau^2}) = \frac{1}{n/\sigma^2+1/\tau^2}$ $\frac{a+d}{2(b+\theta)} = \left(\frac{n\bar{x}}{\sigma^2}+\frac{\nu_0}{\tau^2}\right)\left(\frac{1}{n/\sigma^2+1/\tau^2}\right) = \frac{n\bar{x}\tau^2+\nu_0\sigma^2}{\sigma^2+\tau^2 \cdot n}$

$p(\theta|x) = N\left(\frac{\frac{n\bar{x}}{\sigma^2}+\frac{\nu_0}{\tau^2}}{n/\sigma^2+1/\tau^2}, \frac{1}{n/\sigma^2+1/\tau^2}\right) \xlongequal[\text{Laplace}]{\text{under}} N(\bar{x}, \frac{\sigma^2}{n}) \Rightarrow \nu_0, \tau^2$

$p(\theta|\sigma^2): N(\nu_0, \tau^2)$



$P(\theta) \propto 1 \propto N(0, \infty)$

find $\hat{\theta}_{MMSE} = E[\theta|x] = \frac{n\bar{x}/\sigma^2+\nu_0/\tau^2}{n/\sigma^2+1/\tau^2}$

normal is symmetric by y-axis $\Rightarrow \hat{\theta}_{med}$ $\hat{\theta}_{MAE} = Med(\theta|x) = \frac{n\bar{x}/\sigma^2+\nu_0/\tau^2}{n/\sigma^2+1/\tau^2}$, if $\tau^2 \to \infty \Rightarrow N(\nu_0, \tau^2) \propto 1$

3 point estimates are the same

$\hat{\theta}_{MAP} = Mode[\theta|x]$ = same as above

credible region $CR_{\theta,1-\alpha} = [qnorm(\frac{\alpha}{2}, \frac{n\bar{x}/\sigma^2+\nu_0/\tau^2}{n/\sigma^2+1/\tau^2}, \frac{1}{n/\sigma^2+1/\tau^2}), qnorm(1-\frac{\alpha}{2}, \quad, \quad)]$

$H_0: \theta \le \theta_0$, $H_a: \theta > \theta_0$ p-value $= p(H_0|x) = pnorm(\theta_0, \quad, \quad) = \int_{-\infty}^{\theta_0} PDF\, d\theta$

Jeffrey's prior: likelihood $\xrightarrow{\ln(L)}$ first derivative $\Rightarrow$ second.

$P_J(\theta) \propto \sqrt{I(\theta)}$ $P_J(\theta|\sigma^2) \propto \sqrt{I(\theta;\sigma^2)}$

$L(\theta; x, \sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum x_i^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}}$ ①

$\ln$ ① $= l(\theta; x, \sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{\sum x_i^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}$ ②

$(d\theta)$ ②' $: l'(\theta; x, \sigma^2) = \frac{n\bar{x}}{\sigma^2} - \frac{n\theta}{\sigma^2}$ ③ $\Rightarrow -$③' $: -l''(\theta; x, \sigma^2) = \frac{n}{\sigma^2}$

$I(\theta; \sigma^2) = E_x[-l''(\theta; x, \sigma^2)] = E_x[\frac{n}{\sigma^2}] = \frac{n}{\sigma^2}$

$P_J(\theta|\sigma^2) \propto \sqrt{n/\sigma^2} = \frac{\sqrt{n}}{\sigma}$ is a constant. $\propto 1 \propto N(0, +\infty)$ Jeffrey's prior is the same as laplace's prior.

Holdane: pure ? let n_0 be number of pseudo observations

$\hat{\theta} = \frac{n\bar{x}/\sigma^2+\nu_0/\tau^2}{n/\sigma^2+1/\tau^2}$ let $\tau^2 = \frac{\sigma^2}{n_0}$ $\Rightarrow \frac{n\bar{x}/\sigma^2+n_0\nu_0/\sigma^2}{n/\sigma^2+n_0/\sigma^2} = \frac{n\bar{x}+n_0\nu_0}{n+n_0}$

Since σ^2 known $= \frac{\sigma^2}{n_0}$

if $p(\theta, \sigma^2) = N(\nu_0, \tau^2)$
$= N(\nu_0, \sigma^2/n_0)$

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$P(\theta|\sigma^2) = N(\nu_0, \tau^2) = N(\nu_0, \sigma^2/n_0)$. ~~if $n_0 \to \infty$ then $N(\nu_0, 0)$~~

ν_0: the average of pseudo datas. convention is 0.

So Holdane: $n_0 = 0 \Rightarrow \tau^2 = \infty$, $\nu_0 = 0$ $P(\theta|\sigma^2) = N(0, \infty)$ same as laplace.

all principle of uninformative prior is the same in this model

let $y_1, y_2, \dots, y_{n_0}$ be the pseudo data $\nu_0 = \bar{y} = \frac{\Sigma y_i}{n_0}$

then $\frac{n\bar{x} + n_0\nu_0}{n+n_0} = \frac{\Sigma x_i + \Sigma y_i}{n+n_0}$ ($\hat{\theta}$ could be regarded as a mean of some data)

why $\sigma_0^2 = \sigma$ since σ^2 known you could consider τ^2 as a function of n_0, σ^2.

predictive posterior distribution $P(x^*|x, \sigma^2)$

if $n^* = 1$ $P(x^*|x) = \int_{\mathbb{R}} P(x^*|\theta, \sigma^2) P(\theta|x, \sigma^2) d\theta = \int_{\mathbb{R}} \underset{\uparrow}{\text{PDF}} \cdot \underset{\uparrow}{\text{PDF}} \, d\theta$

$N(\theta, \sigma^2)$ $N(\theta_p, \sigma_p^2)$ (posterior)

$$\theta_p = \frac{n\bar{x} + \nu_0/\tau^2}{n/\sigma^2 + 1/\tau^2}, \quad \sigma_p^2 = \frac{1}{n/\sigma^2 + 1/\tau^2} \qquad = \sigma_p^2\left(\frac{n\bar{x}}{\sigma^2} + \frac{\nu_0}{\tau^2}\right)$$

shrinkage: $$\hat{\theta}_{MMSE} = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\nu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} = \frac{n\bar{x}/\sigma^2}{n/\sigma^2 + 1/\tau^2} + \frac{\nu_0/\tau^2}{n/\sigma^2 + 1/\tau^2} = \frac{n/\sigma^2 \, \hat{\theta}_{MLE}}{n/\sigma^2 + 1/\tau^2} + \frac{1/\tau^2 \, E(\theta)}{n/\sigma^2 + 1/\tau^2}$$

$$= \underbrace{\frac{1}{1 + \sigma^2/n\tau^2}}_{1-e} \hat{\theta}_{MLE} + \underbrace{\frac{1}{1 + \frac{n\tau^2}{\sigma^2}}}_{e} E(\theta)$$

$$= \frac{n\tau^2}{n\tau^2 + \sigma^2} \hat{\theta}_{MLE} + \frac{\sigma^2}{\sigma^2 + n\tau^2} E(\theta) \dots$$